DESIGN AND ANALYSIS OF ALGORITHMS

Tutorial - 1

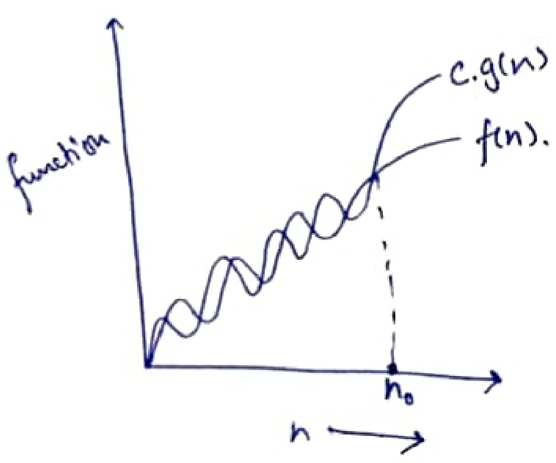
1. What do you understand by Asymptotic Notations. Define different Asymptotic notation with examples.

Ans:- **Asymptotic Notation:-** These notations are used to tell the complexity of an algorithm, when input is very large. These are mathematical notations used to describe running time of an algorithm when the input tends towards a particular value or a limiting value.

- **Different Asymptotic Notations:**

i) Big - Oh (O):-

$f(n) = O(g(n))$



g(n) is "tight" upper bound.

$f(n) = O(g(n))$

iff

$f(n) \leq c.g(n)$

$\forall\ n \geq n_0$ and some constant, $c > 0$.

E.g.:

```
for (i=1; i<=n; i++)
{
  print (i);   —— O(1).
}
```

i
1
2
3
⋮

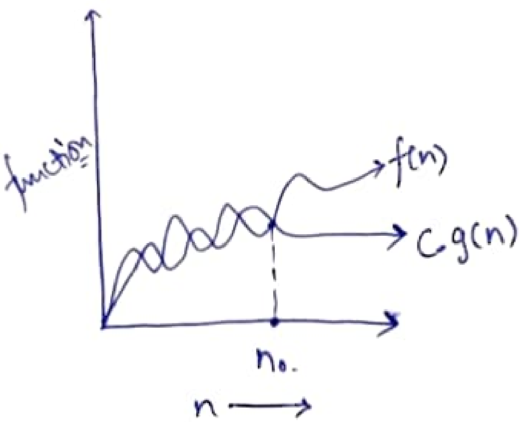
~~n times~~

n times $\Rightarrow O(n)$

$\Rightarrow \quad T(n) = O(n)$.

ii) Big Omega (Ω):-

$$f(n) = \Omega(g(n))$$



$g(n)$ is "tight" lower bound.

$f(n) = \Omega(g(n))$

iff.

$f(n) \geq c.g(n)$

$\forall \; n \geq n_0$, and some constant $c > 0$.

E.g. $f(n) = 2n^2 + 3n + 5$, $\quad g(n) = n^2$.

$\Rightarrow \quad \because \quad 0 \leq c.g(n) \leq f(n)$

$\Rightarrow \quad 0 \leq c.n^2 \leq 2n^2 + 3n + 5$

$\Rightarrow \quad c \leq 2 + \frac{3}{n} + \frac{5}{n^2}$

On putting $n = \infty$, $\quad \Rightarrow \frac{3}{n} \to \infty, \; \frac{5}{n^2} \to \infty$.

$\Rightarrow \quad c = 2$,

$\Rightarrow \quad 2n^2 \leq 2n^2 + 3n + 5$

On putting $n = 1$,

$2 \leq 2 + 3 + 5$

$2 \leq 10$ True.

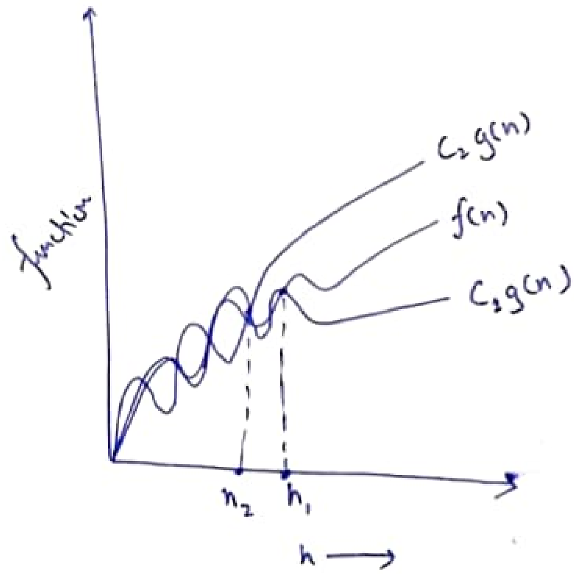
$\Rightarrow \quad \boxed{c = 2, \quad n = n_0 = 1}$

$0 \leq 2n^2 \leq 2n^2 + 3n + 5$

$\Rightarrow f(n) = \Omega(n^2)$.

iii) Big- Theta (Θ) :-

$f(n) = \Theta(g(n))$



$g(n)$ is both, "tight" upper and lower bound of $f(n)$.

$f(n) = \Theta(g(n))$
iff
$c_1 . g(n) \leq f(n) \leq c_2 . g(n)$

$\forall \ n \geq \max(n_1, n_2)$,
and some constant, $c_1 > 0$, $c_2 > 0$.

Eg:- $f(n) = 10 \log_2 n + 4$. , $g(n) = \log_2 n$.

$\Rightarrow f(n) \leq c_2 . g(n)$

$\Rightarrow 10 \log_2 n + 4 \leq 10 \log_2 n + \log_2 n$

$10 \log_2 n + 4 \leq 11 \log_2 n$

$c_2 = 11$

$\Rightarrow$

$4 \leq 11 \log_2 n - 10 \log_2 n$

$4 \leq \log_2 n$

$16 \leq n$

Here, $\forall \ n \geq 16$

$n_2 = 16$

& $c_2 = 11$

$f(n) \geq c_1 g(n)$

$10 \log_2 n + 4 \geq 2 \log_2 n$

$c_1 = 1 \quad , \quad n > 0$

$\Rightarrow \quad n_1 = 1 \quad \Rightarrow \quad n_0 = \max(n_1, n_2) \Rightarrow \quad n_0 = 16.$

$\Rightarrow \quad \log_2 n \leq 10 \log_2 n + 4 \leq 11 \log_2 n$

$c_1 = 1$
$c_2 = 11.$

$\Rightarrow \quad \Theta(\log_2 n).$

iv) Small oh (o):-

$f(n) = o(g(n))$

$g(n)$ is the upper bound of the function $f(n)$.

$f(n) = o(g(n))$

when, $f(n) < c.g(n)$

$\forall \ n > n_0$

and $\forall$ constants, $c > 0$.

v) Small Omega (ω):-

$f(n) = \omega(g(n))$

$g(n)$ is lower bound of the function $f(n)$.

$f(n) = \omega(g(n))$

when

$f(n) > c.g(n)$

$\forall \ n > n_0$

and $\forall \ c > 0$.

2. What should be the complexity of -

for (i = 1 to n) { i = i * 2; }

→

$\Rightarrow$ $i = \underbrace{1, 2, 4, 8, 16 \ldots\ldots n}_{k \text{ terms.}}$

$\Rightarrow$ $a = 1, \quad r = 2.$

$\Rightarrow$ k^{th} term :-

$$t_k = ar^{k-1}$$

$$\Rightarrow n = 1 \cdot 2^{k-1}$$

$$n = 2^{k-1}$$

take $\log_2$ both sides,

$$\Rightarrow \log_2 n = \log_2 2^{k-1}$$

$$\log_2 n = (k-1) \log_2 2$$

$$\log_2 n = k-1 \qquad [\because \log_a a = 1]$$

$$\Rightarrow k = 1 + \log_2 n$$

$$\Rightarrow T(n) = O(k)$$

$$= O(\cancel{1} + \log_2 n)$$

$$= O(\log_2 n).$$

3. $T(n) = \{3T(n-1)$ if $n>0$, otherwise $1\}$

$\rightarrow$

$\because \quad T(n) = 3T(n-1)$ —— (1)

put $n = n-1$ in eq^n (1),

$\Rightarrow \quad T(n-1) = 3T(n-2)$ —— (2)

put this value in eq^n (1),

$T(n) = 3\,[3T(n-2)]$ —— (3)

put $n = n-2$ in eq^n (1),

$T(n-2) = 3T(n-3)$ —— (4)

put this value in eq^n (3),

$\Rightarrow \quad T(n) = 9\,[3T(n-3)]$

$T(n) = 27\,T(n-3)$

$\Rightarrow$ Generalised form:-

$T(n) = 3^k T(n-k)$

put $n-k=0$

$\Rightarrow \quad T(n) = 3^n\,T(0)$

but $T(0) = 1$

$\Rightarrow \quad T(n) = 3^n$

$\Rightarrow \quad O(3^n)$.

1. $T(n) = \{2T(n-1) - 1$ if $n > 0$, otherwise $1\}$

$\Rightarrow$

$$T(n) = 2T(n-1) - 1 \quad \text{——— (1)}$$

put $n-1$ in equation (1)

$$\Rightarrow T(n-1) = 2T(n-2) - 1 \quad \text{——— (2)}$$

put this value in eqn (1)

$$\Rightarrow T(n) = 2\left[2T(n-2) - 1\right] - 1$$

$$T(n) = 4T(n-2) - 2 - 1 \quad \text{——— (3)}$$

put $n = n-2$ in eqn (1),

$$\Rightarrow T(n-2) = 2T(n-3) - 1 \quad \text{——— (4)}$$

put this value in eqn (3),

$$\Rightarrow T(n) = 4\left[2T(n-3) - 1\right] - 2 - 1$$

$$\Rightarrow T(n) = 8T(n-3) - 4 - 2 - 1$$

$\Rightarrow$ Generalised form:-

$$T(n) = 2^K T(n-K) - 2^{K-1} - 2^{K-2} \text{ ------ } -1$$

put $n - K = 0$

$\Rightarrow n = K$, $T(0) = 1$ (Given).

$$\Rightarrow T(n) = 2^n T(0) - 2^{n-1} - 2^{n-2} \text{ ------ } -1$$

$$= 2^n - 2^{n-1} - 2^{n-2} \text{ -------- } -1$$

$$= 2^n - \underbrace{\left[2^{n-1} + 2^{n-2} + \text{ ------- } + 1\right]}_{K \text{ terms.}}$$

$$\Rightarrow a = 2^{n-1}, \quad r = \frac{1}{2}.$$

$$\Rightarrow \text{Sum of G.P} = \frac{2^{n-1}\left[1 - \left(\frac{1}{2}\right)^{n-1}\right]}{1 - \frac{1}{2}} = 2^n - 2.$$

$$\Rightarrow T(n) = 2^n - \left[2^n - 2\right] = 2$$

$$\Rightarrow O(2) = O(1).$$

5. What should be time complexity of -

```
int i=1, s=1;   -> O(1)  -> O(1)
while (s<=n){
  i++; s=s+i;
   printf("#"); -> O(1)
}
```

i	s
1	1
2	3
3	6
⋮	10
n	15
	⋮
	n
	K times.

$\Rightarrow$

$$S = \underbrace{1, 3, 6, 10, 15 \text{-----} n}_{K \text{ terms}}$$

$\Rightarrow$ K_{th} term,

$$t_K = t_{K-1} + K$$

$$\Rightarrow K = t_K - t_{K-1} \quad \text{—} ①$$

Now, from series,

$t_2 - t_1 = 2$

$t_3 - t_2 = 3$

$t_4 - t_3 = 4$

⋮

$\Rightarrow K = n - t_{K-1}$

$\Rightarrow$ loop runs K times.

$\Rightarrow T.C = O(1+1+1+n-t_{K-1})$

but, $t_{K-1} = C$ (constant)

$\Rightarrow T.C = O(3+n-C)$

$= O(n)$.

6. Time complexity of -

```
void function(int n) {        —— O(1)
    int i, count = 0;         —— O(1)
    for (i = 1; i * i <= n; i++)
        count++;              —— O(1)
}
```

$i*i$
1^2
2^2
3^2
4^2
5^2
⋮
n

$\Rightarrow i*i \Rightarrow \underbrace{1^2, 2^2, 3^2, 4^2, 5^2 - - - - n}_{k \text{ terms.}}$

$\Rightarrow$ K_{th} term:-

$$t_K = K^2$$

$$\Rightarrow K^2 = n$$

$$K = n^{1/2}$$

$$\Rightarrow \quad T.C. = O(1 + 1 + 1 + n^{1/2} + 1)$$

$$= O(n^{1/2}). = O(\sqrt{n}).$$

7. Time complexity of -

```
void function (int n){                 ← O(1)
    int i, j, k, count=0;              —— O(1)
    for (i = n/2; i <= n; i++)
        for (j = 1; j <= n; j = j*2)   —— log(n) times
            for (k = 1; k <= n; k = k*2)   —— log(n) times
                count++;               ← O(1).
}
```

$i \Rightarrow n/2, \frac{n+2}{2}, \frac{n+4}{2}, \frac{n+6}{2}$ ----- upto n

$\Rightarrow \frac{n+0\times 2}{2}, \frac{n+1\times 2}{2}, \frac{n+2\times 2}{2}, \frac{n+3\times 2}{2}$ ----- upto n

$\Rightarrow$ General term:-

$$t_K = \frac{n+K*2}{2}$$

total terms = $K+1$

$$\Rightarrow t_{K+1} = n$$

$$\Rightarrow \frac{n+(K+1)*2}{2} = n$$

$$n + 2K + 2 = 2n$$

$$2K = n-2$$

$$K = \frac{n}{2} - 1$$

$\Rightarrow$

i	j	K
$\frac{n}{2}$	$\log_2 n$ times	$(\log_2 n)^2$
$\frac{n+2}{2}$	$\log_2 n$ times	$(\log_2 n)^2$
$\frac{n+4}{6}$	$\log_2 n$ times	$(\log_2 n)^2$
⋮	⋮	⋮
n	$\log_2 n$ times.	$(\log_2 n)^2$
$(\frac{n}{2}-1)$ times		

$$\Rightarrow \left(\frac{n}{2}-1\right)(\log_2 n)^2$$

$$\Rightarrow O\left(\frac{n}{2}\log_2^2 n - \log^2 n\right)$$

$$\Rightarrow O(n\log^2 n).$$

Q. Time complexity of -

```
function (int n) {
    If (n==1) return;        —— O(1)
    for (i=1 to n) {         —— O(n)
    for (j=1 to n) {         —— O(n)
    printf ("* ");           —— O(1)
    }
  }
  function (n-3);
}
```

for function call,

$$\underbrace{n, n-3, n-6, n-9 \text{-------} 1}_{K \text{ terms.}}$$

$\Rightarrow$ AP with $d = -3$.

$\therefore \Rightarrow l = a + (K-1)d$

$1 = n + (K-1)(-3)$

$\frac{1-n}{(-3)} = K-1$

$\Rightarrow K-1 = \frac{n-1}{3}$

$K = \frac{n-1+3}{3}$

$$\boxed{K = \frac{n+2}{3}}$$

$\Rightarrow$ function gives a recursive call $\frac{n+2}{3}$ times.

$\Rightarrow$ Time complexity $= \left(\frac{n+\cancel{2}}{\cancel{3}}\right)(n)(n)$

$= n^3$

$\Rightarrow O(n^3)$.

9. Time complexity of -

```
void function (int n) {
for (i=1 to n) {
for (j=1; j<=n; j=j+i)
    printf ("*")
}
}
```

i	j
1	n times
2	$(n+1)/2$ times
3	$(n+2)/3$ times.
4	$n+3/4$ times.
n	$\frac{[n+(n-1)]}{n}$ times

$\Rightarrow$ $$T(n) = n + \underbrace{\left(\frac{n+1}{2}\right) + \left(\frac{n+2}{3}\right) + \left(\frac{n+3}{4}\right) + \cdots \left(\frac{2n-1}{n}\right)}_{K \text{ terms}}$$

$\Rightarrow$ General term-

$$T_{K} = \frac{n+K}{K+1}$$

$\Rightarrow$ $\cancel{\frac{2n-A}{n}}$

Sum of k terms.

$$\Rightarrow S_k = \sum_{m=1}^{K} \left(\frac{n+K}{K+1}\right)$$