Assignment 4

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1. Answer each part TRUE or FALSE

   a. $2^{\log^2 n} \in \mathcal{O}(n^k)$
   b. $n! \in 2^{\mathcal{O}(n \log n)}$
   c. $3^n \in \mathcal{O}(2^{n \log n})$
   d. $n^k \in o(2^{\log n})$
   e. $2^n \in o(n!)$
   f. $\frac{1}{n} \in o(1)$

Solution:

a. $2^{\log^2 n} \in \mathcal{O}(n^k)$

   True
   Proof: $2^{\log^2 n} = n$ and $n \in \mathcal{O}(n^k)$

b. $n! \in 2^{\mathcal{O}(n \log n)}$

c. $3^n \in \mathcal{O}(2^{n \log n})$

   False
   As 3^n grows much faster than $2^{n \log n}$

d. $n^k \in o(2^{\log n})$ True

e. $2^n \in o(n!)$

f. $\frac{1}{n} \in o(1)$ True as $1/n$ is strictly smaller than 1 after $n > 1$

2. Show that P is closed under union, complement, concatenation, and star.

3. Show that NP is closed under union, concatenation, and star

4. We normally assume natural numbers are represented in binary, such that a number $n \in \mathcal{N}$ is represented by the string $b_{\lfloor \log n \rfloor} b_{\lfloor \log n \rfloor - 1} \dots b_0, b_i \in \{0, 1\}$, and $n = \Sigma_{i=0}^{\lfloor \log n \rfloor} b_i 2^i$. We could also write a number in unary, where a number $n \in \mathcal{N}$ is represented by n consecutive 1s. The problem of factoring a number in binary is not known to be in P, but what if the number is given in unary? Prove your answer.

5. Suppose the number k and the graph G are given, and we want to know if there exists a subset S of size k from the vertices of G such that there is no edge between them in G. Prove that this problem is NP-Complete.

6. Show the following languge is NP-Complete:

$$DOUBLE-SAT = \{\langle\phi\rangle | \phi \text{ has at least 2 satisfying assignments } \}$$

7. Let S be a set and let C be a collection of subsets of S. A set $S' \subseteq S$ is called a set hitting set for C if every subset in C contains at least an element in S'. Let

$$HITSET = \{\langle C, k\rangle | C \text{ has a hitting set of size } k\}$$

Prove that $HITSET$ is NP-Complete.

8. a. Prove that NP = coNP iff there is an NP-Complete problem in coNP.
   b. Show that if coNP $\neq$ NP then P $\neq$ NP.

9. Show that P is closed under homomorphism iff P = NP

10. Let $CNF_k = \{\langle\phi\rangle | \phi$ is a satisfiable cnf-formula where each variable appears in at most k places $\}$.
    a. Show that $CNF_2 \in P$
    b. Show that CNF_3 is NP-complete