

Planarity

Problem: Let G be a graph. Can G be drawn in such a manner so that no two edges intersect?

Example 1. *Example Graph here*

Any drawing of K_5 or $K_{3,3}$ have (at lease) 2 edges which cross (*Proof to come*)

Definition 1. A graph G is said to be planer if it can be drawn in \mathbb{R}^2 so that no two edges cross. Such a drawing is called a plane drawing. The graph associated with a plane drawing is usually refered to as a plane graph.

Remark. 1. Any subgraph of a planer graph is planer.

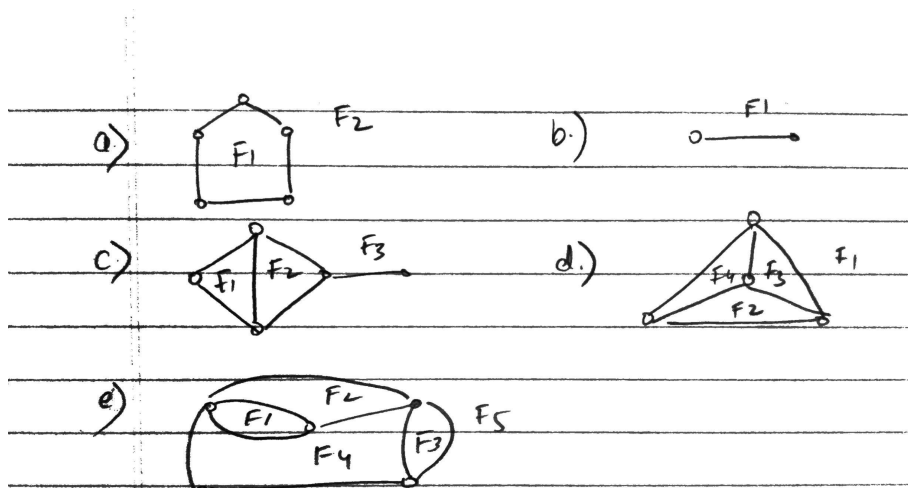
2. Every plane graph is planar.

graph here is a planer but not plane.

Our aim is to determine condition which ensures that a graph G is planar. Let G be a plane graph. Consider the set S obtained from \mathbb{R}^2 by deleting (the vertices and edges) of G . We observe that S is the disjoint union of finitely many subsets F_1, F_2, \dots, F_l of \mathbb{R}^2 having the following two properties:

1. Any two points of F_i can be joined by a curve not crossing G .
2. Any curve in \mathbb{R}^2 which joins a point of F_i to one of $F_j, i \neq j$, must cross G

Definition 2. Let G be a plane graph. The sets F_1, \dots, F_l described above are called the faces of G .



Example 2.

Remark. One face is always unbounded, with the remaining faces all bounded.

Definition 3. Let F be a face of a plane graph G . The boundary of F consists of a finite # of vertices and edges of G

The length of a closed walk around the boundary of F is called the degree of F , usually denoted $\deg F$

Example 2. (Cont.)

1. $\deg F_1 = \deg F_2 = 5$

2. $\deg F_2 = 2$ the closed walk is $x \rightarrow y \rightarrow x$

3. $\deg F_1 = \deg F_2 = 3$
 $\deg F_3 = 6$

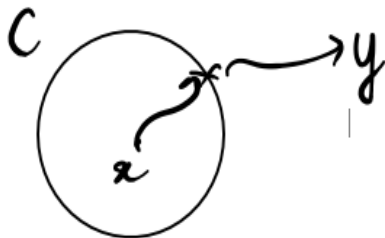
Theorem 1. (Euler's Formula, 1780) Let G be a connected plane graph. If G has n nodes and f faces then

$$n - m + f = 2$$

proof of **Theorem 1** requires a couple of preliminary lemmas.

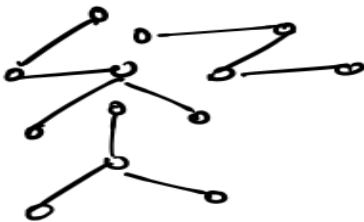
Lemma. Let G be a plane graph. G contains a cycle if and only if the number of faces of $G \geq 2$.

Proof. \rightarrow Let C be a cycle of G . Let x and y be in the interior and exterior of C , respectively, then any curve in \mathbb{R}^2 connecting x and y must cross C , have cross G



Choosing x, y not lying on G x and y belong to different faces of G .

\leftarrow The absence of a cycle, G is a forest. On induction on the number of components of G shows G has only one (unbounded) face.

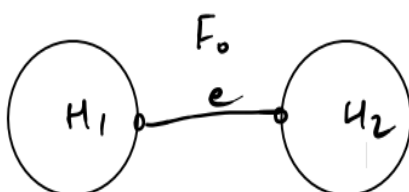


□

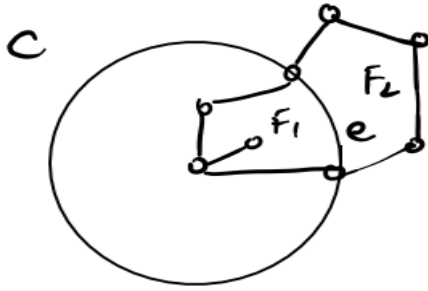
Lemma. Let e be an edge of a plane graph G .

1. If e is a bridge then it lies on boundary of exactly one face.
2. if e is not a bridge then it lies on the boundary of exactly two faces of G

Proof. 1. Let H_1 and H_2 be the components of $G \setminus e$. The edge e lies in face F_1 of H_1 , as well as a face F_2 of H_2 . The intersection $F_1 \cap F_2$ contains a unique face F_o of G . e lies on the boundary of F_o and is the unique face of G .



2. Let C be a cycle containing e . The edge e lies on the boundary of one face lying in the interior of C and one face lying in the exterior of C . Thus, e lies on the boundary of at least 2 faces of G .



The fact faces are disjoint can be used to show that e lies on the boundary of at most 1 face lying in the interior of C , and one face lying in the exterior of C . Then e lies in the boundary of at most 2 faces.

□

Corollary. (*Handshaking Lemma for Planar Graphs*) If G is a plane graph of size m then

$$2m = \sum_{\text{faces } F} \deg F$$

Proof. Each edge e of G contributes 2 to the sum on the right:

If e is a bridge lying on the face F_o then it contributes 2 to $\deg F_o$ and 0 to the remaining $\deg F$.

If e is not a bridge then it contributes 1 to the degree of two distinct faces of G , and 0 to the remaining. □

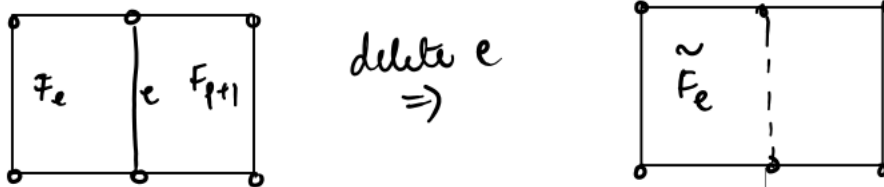
Proof. Proof of Euler's Formula

We proved by induction on # of faces of G on the case $f = 1$. **Lemma 2** asserts G is a forest, where G is a tree. Thus, $m = n - 1$, and hence $n - m + f = n - (n - 1) + 1 = 2$

Assume the result is true f connected plane graph H having f faces, and let G have $f + 1$ faces, say $F_1, F_2, \dots, F_l, F_{l+1}$. If $n = |G|, m = e(G)$ then we are required to show

$$n - m + (f + 1) = 2$$

Since $f + 1 \geq 2$, G contains a cycle C (**Lemma 2**). Fix an edge e lying on C . Assume e is not a bridge of G . **Lemma 2** ensures that e lies on the boundary of 2 distinct faces of G , say F_l and F_{l+1} .



Construct the subgraph $H = G - e$. We note H is a connected plane graph of order n and size $m - 1$. Furthermore, the number of faces of H is f . denoted each of the faces F_1, f_2, \dots, F_{l-1} of G occur on faces of H . Assume e does not appear in the boundary of any of these faces.

The remaining faces of H is obtained by joining F_l and F_{l+1} along the edge e .

$$\hat{F}_l = F_l \cup F_{l+1} \cup e$$

Since H has f faces, the induction hypotheses allows in G to conclude that

$$2 = n - (m - 1) + f = n - m + (f + 1)$$

as required

□

Corollary. Let G be a connected planar graph. Each plane drawing of G has the same number of faces, namely $2 + m - n$

Euler's formula can be used to obtain a necessary condition for a simple connected graph to be planar.

Definition 4. Let G be a graph, If G contains a cycle then the **girth** $gr(G)$ is defined as the length of the smallest cycle in G . If G is a forest then we set $gr(G) = \infty$

a.)  has girth 3

b.) Peterson has girth 5

c.) K_5 has girth 3, $K_{3,3}$ has girth 4

Example 3.

Remark. If G is simple then $gr(G) \geq 3$

Theorem 2. Let G be a connected simple planar graph. If G has order n and size m then

$$m \leq \begin{cases} n - 1 & \text{if } gr(G) = \infty \\ \frac{gr(G)}{gr(G)-2}(n - 2) & \text{if } gr(G) \text{ is finite} \end{cases} \quad (1)$$

Proof. If $gr(G) = \infty$ then G is a tree, hence $m = n - 1$ if $gr(G)$ is finite then G has ≥ 2 faces. In this case, the boundary of each face of G contains a cycle, hence

$$\deg(F) \geq gr(G)$$

If each face F of G . therefore, if $f = \#$ of faces of G then the handshaking lemma of planar graphs

$$2m = \sum_{\text{faces } F} \deg F \geq \sum gr(G) = gr(G)f$$

From Euler's formula, $f = 2 + m - n$, substitutes f_i in the preceding yields

$$2m \geq gr(G)(2 + m - n)$$

□

Example 4. 1. Does there exist a simple connected planar graph G of order 12 and size 40?

Solution 1. Suppose such a graph G exists, Note that G cannot be a tree, hence $gr(G)$ is finite. Thus, **Theorem 3** asserts

$$40 \geq \frac{gr(G)}{gr(G) - 2}(12 - 2)$$

Solving for $gr(G)$

$$gr(G) \leq \frac{8}{3} < 3$$

This contradicts that $gr(G) \geq 3$ No such G exists.

2. Let G be a planar graph of size and girth 5. What can one say about $n = |G|$

Solution 2. From **Theorem 2**

$$14 \leq \frac{S}{S - 2}(n - 2)$$

solving for n yields $n \geq 52/5$ some n is an integer $n \geq 11$.

Corollary. K_5 and $K_{3,3}$ are non-planar

Proof here

Exercise Show that Peterson is non-planar

Corollary. Let G be a simple connected planar graph.

1. if G has order $n \geq 3$ then $e(G) \leq 3n - 6$
2. Furthermore, if G contains no triangles then $e(G) \leq 2n - 4$

proof here

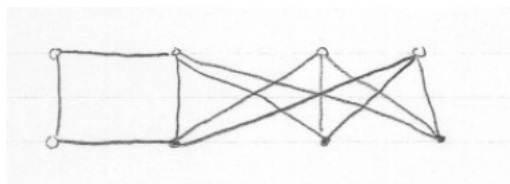
Remark. The preceding two results are often used as tools for non-planarity, they are weaker than the full theorem 2.

Corollary. Every simple planar graph G contains a vertex of degree at most 5 ($\delta(G) \leq 5$)

proof here

Example 5. K_n is non-planar of $n \geq 7$. Each vertex of K_n has degree $n - 1 \geq 6$

Example 6. The condition of theorem 4 (as well as its corollaries) is only necessary for planarity, not sufficient. For example, the following graph G is non-planar, as it contains a copy of $K_{3,3}$



On the other hand, $|G| = 8, e(G) = 12$ and $gr(G) = 4$

$$\frac{4}{4 - 2}(8 - 2) = 12$$

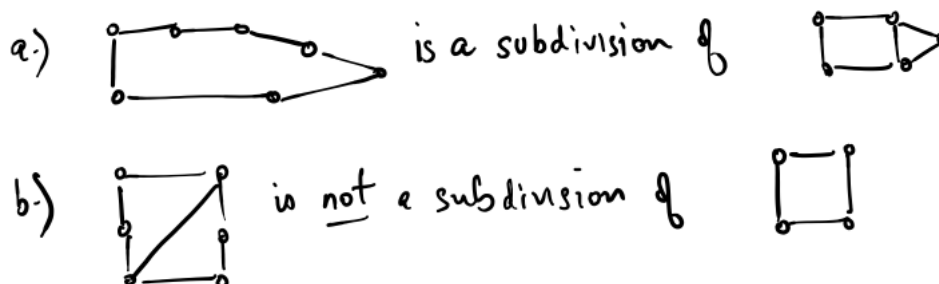
Remark. Euler's formula can be extended to disconnected plane graphs using induction on the # of components. If G is a planar graph of order n , size m , with f faces and k components then

$$n - m + f = k + 1$$

There are also analogues for Theorem 3 and its corollaries.

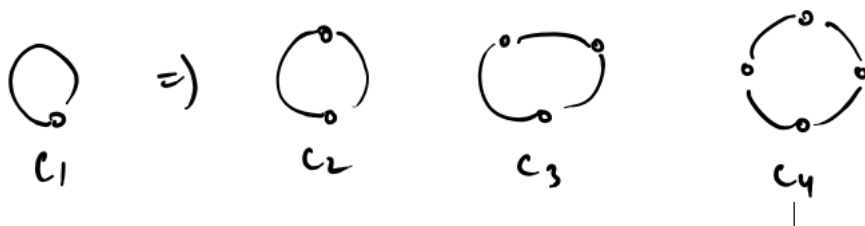
We have already observed that Theorem 3 and its corollaries present many necessary conditions for a graph to be planar.

Definition 5. Let G and H be graphs. H is said to be a subdivision of G if the former graph can be constructed from the latter by introduction of a finite # of new vertices along existing edges.



Example 7.

c.) Each cycle graph C_n is a subdivision of C_1 . C_n can be obtained from C_1 by the addition of $n - 1$ new vertices on the single edge of C_1 .



Remark. 1. The process of subdivision only introduces new vertices of degree 2

2. Note that if H is a subdivision of G then H has the same shape as G .

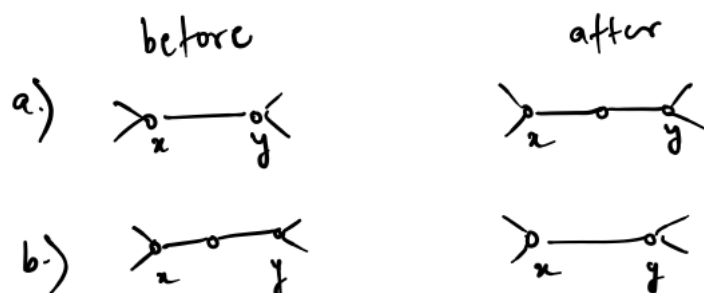
Definition 6. Two graphs G and H are said to be homeomorphic if they are both subdivisions of a common graph.

Example 8. 1. If $n, m \geq 1$ then C_n is homeomorphic to C_m as both are subdivisions of C_1 .

2. If $n, m \geq 2$ then P_n and P_m are homeomorphic as both are subdivisions of P_2 .

Remark. 1. G and H are homeomorphic if the latter can be obtained from the former by a finite sequence of following 2 operations:

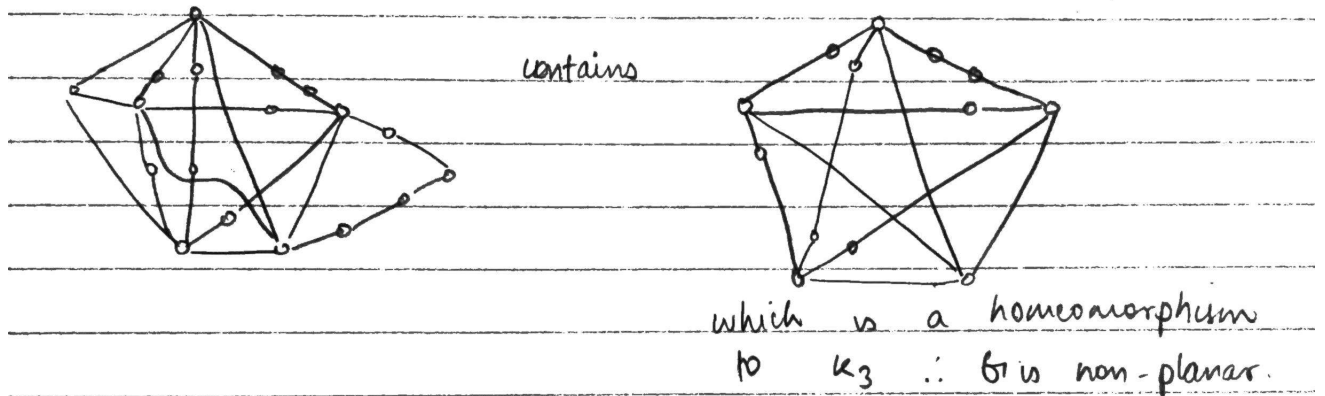
- addition of new vertex along an existing edge.
- filling in of an existing vertex of degree 2.



2. homeomorphic graphs have the same shape.

Theorem 3. A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.

Example 9. The graph

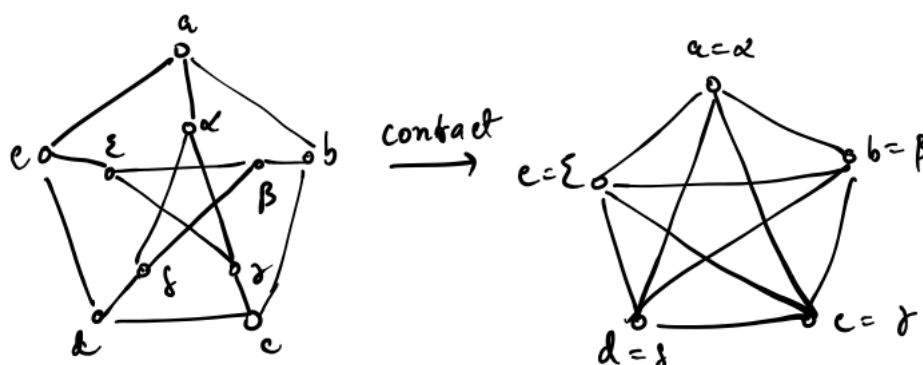


an alternate characterization of non-planar graphs can be obtained using the notion of contractions.

Definition 7. Let G and H be graphs. G is said to be contractible to H if H can be obtained from G by successively contracting a finite number of edges.

Theorem 4. A graph G is planar if and only if it contains no subgraph which is contractible to K_5 or $K_{3,3}$.

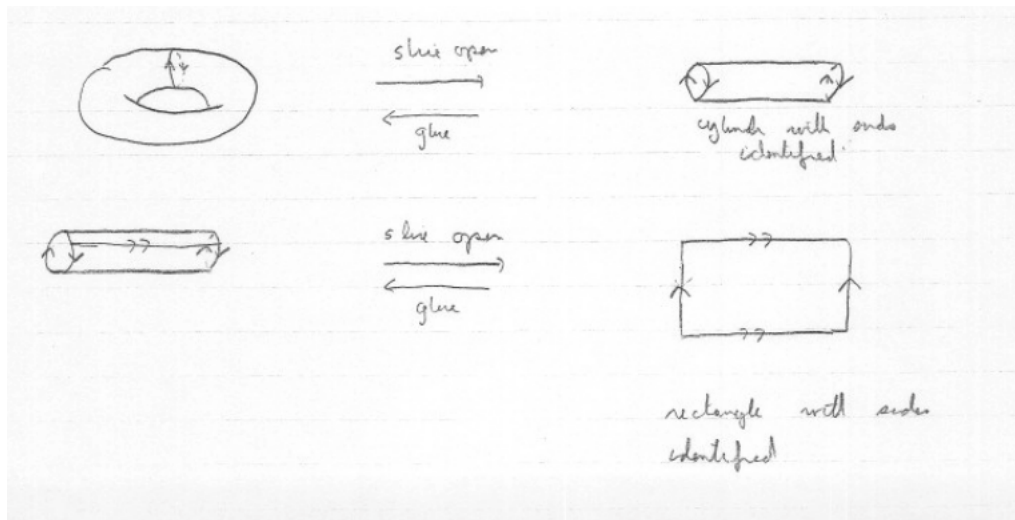
Example 10. Peterson graph is contractible to K_5 .



Thus peterson is non-planar.

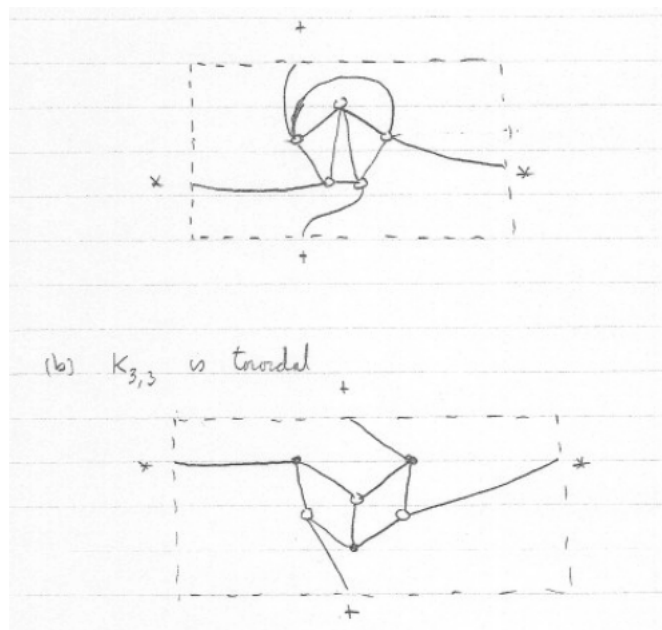
We have observed that K_5 and $K_{3,3}$ are non-planar. What happens if we consult other surface?

Example 11. Torous (Donut)



Definition 8. A non-planar graph G is said to be toroidal if it can be drawn on the torus so as no two edges cross.

Example 12. 1. K_5 is toroidal



2. $K_{3,3}$ is toroidal

Question: Is there analogous of Euler's Formula for the toros?

Theorem 5. Let G be a simple connected torodial graph of order n , size m and f faces, Then

$$n - m + f = 0$$

Furthermore,

$$m \leq \frac{gr(G)}{gr(G) - 2} n$$

Definition 9. A graph G is said to have genus g if G can be drawn on a surface of genus g with no edges crossing, but no drawing on a surface of genus $g - 1$ exists. (i.e planar = genus 0, torodial = genus 1)

Theorem 6. *Let G be a connected graph of genus g , order n , size m , and face f . Then*

$$n - m + f = 2 - 2g$$

Furthermore, if G is simple of finite girth then

$$m \leq \frac{gr(G)}{gr(G) - 2}(n + 2g - 2)$$

Corollary. *Let G be a connected simple graph of genus g , order $n \geq 3$ and size m then,*

$$m \leq 3(n + 2g - 2)$$

$$m \leq 2(n + 2g - 2) \text{ if no triangle present}$$

Corollary. *Let G be a connected simple graph of order $n \geq 4$ and size m . Then the genus g satisfies*

$$g \geq \lceil \frac{m - 3n}{6} + 1 \rceil$$

$\lceil x \rceil = \text{least integer greater than or equal to } x$

Remark. *Let G be a graph. The crossing number $cr(G)$ is the minimum # of crossing that can occur when G is drawn in the plane.*

Example 13. *graph here*

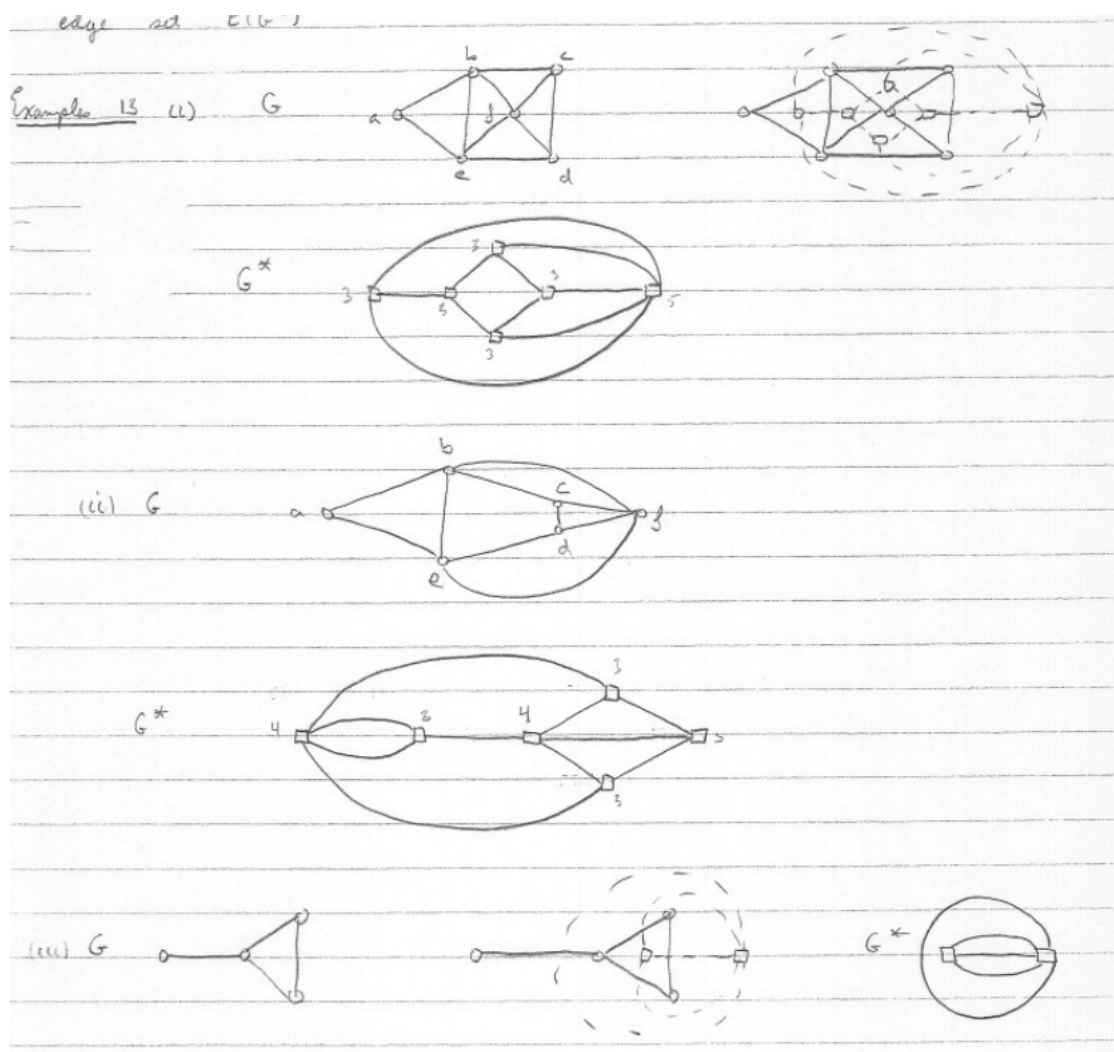
Theorem 7. *The genus of the graph G is $\leq cr(G)$*

Dual Graphs

Definition 10. *Let G be a planar graph. The geometric dual G^* of G is the graph constructed as follows:*

1. *denote each face of G , choose a point v^* . The v^* forms the vertex set $V(G^*)$*
2. *For each edge e of G , join the vertices v^* and w^* in the adjacent face by a curve e^* that crosses e and no other edge of G . The collection of e^* forms the edge set $E(G^*)$*

Example 14.



Platonic Graphs

| Graphs | Geometric Dual | |
|--------------|----------------|-----------|
| tetrahedron | tetrahedron | self-dual |
| cube | octahedron | |
| octahedron | cube | |
| dodecahedron | icosahedron | |
| icosahedron | dodecahedron | |

Remark. 1. The construction of G^* depends on the plane drawing of G . For example, the graphs G in example 13 (i) and (ii) are isomorphic, but their geometric duals are not. One has vertex of degree 5 but the other has no such vertex.

2. G^* is planar and connected.

Lemma. Let G be a connected planar graph of order n , size m , and face f . If G^* is a geometric dual then the nodes n^* size m^* and number of faces f^* satisfies

$$n^* = f, m^* = m, f^* = f$$

Theorem 8. Let G be a connected planar graph. Then G^{**} is isomorphic to G .

Theorem 9. Let G be a connected planar graph and G^* a geometric dual of G , A subset of $E(G)$ forms a cycle of G if and only if the corresponding subset of $E(G^*)$ is a cutset of G^* .

Remark. If e is an edge of G then there is a unique edge e^* of G^* which crosses e .

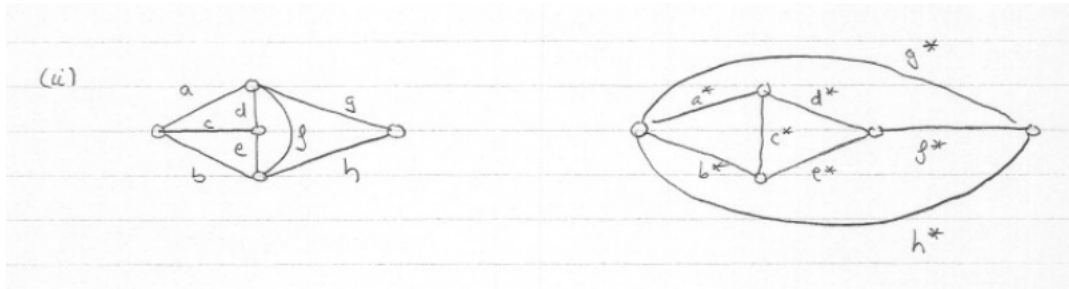
Corollary. Let G be a connected planar graph. A set of edges of G forms a cutset if and only if the corresponding edges of G^* forms a cycle.

Theorem 10. Let G be a connected planar graph. G is bipartite if and only if G^* is eulerian.

Theorem 11. Let G be a connected planar graph. If G is 3-edge connected then G^* is simple (of order ≥ 3)

Definition 11. Let G be a graph. A graph G^* is said to be an abstract dual of G if there exists a one-one correspondence between the edge of G and the edge of G^* with the property that
a subset of $E(G)$ forms a cycle \leftrightarrow corresponding subset of $E(G^*)$ forms a cutset

Example 15. If G is plane then its geometric dual G^* is an abstract dual (Theorem 11)



Theorem 12. If G^* is an abstract dual of G then G is an abstract dual of G^* .
Abstract duals provide another characterization of planar graphs.

Theorem 13. A graph G is planar if and only if G has an abstract dual.