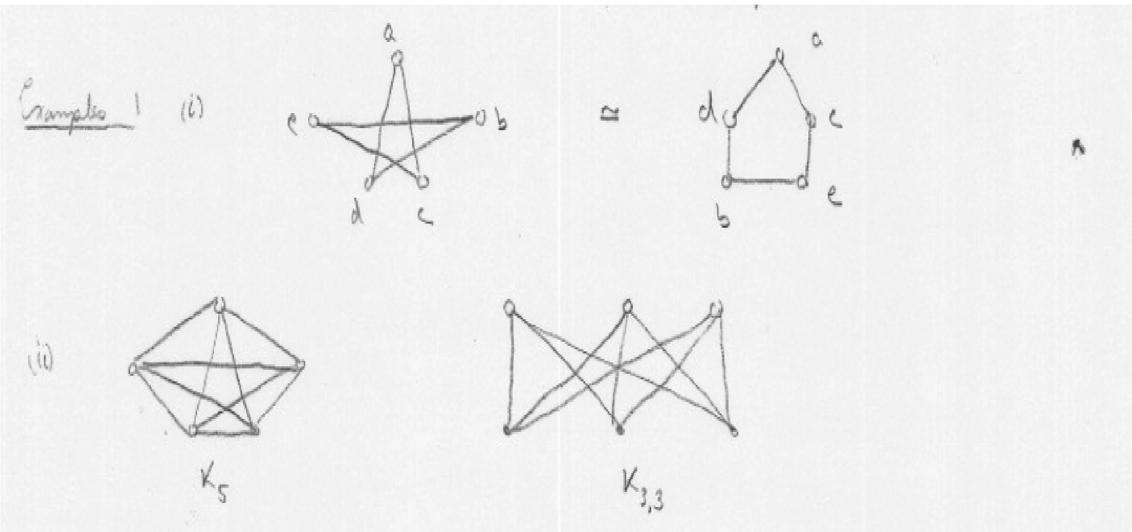
Planarity

Problem: Let G be a graph. Can G be drawn in such a manner so that no two edges intersect?

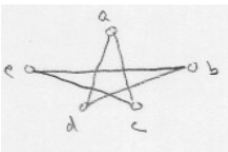
Example 1. *Any drawing of K_5 or $K_{3,3}$ have (at least) 2 edges which cross (Proof to come)*



Definition 1. *A graph G is said to be planer if it can be drawn in $\mathbb{R}^2$ so that no two edges cross. Such a drawing is called a plane drawing. The graph associated with a plane drawing is usually refered to as a plane graph.*

Remark.

1. *Any subgraph of a planer graph is planer.*
2. *Every plane graph is planar.*



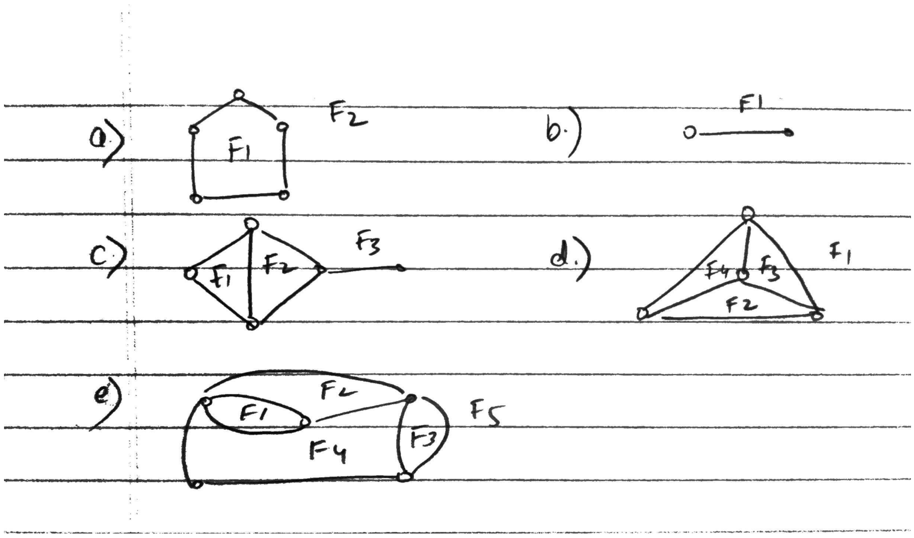
is a planer but not plane.

Our aim is to determine condition which ensures that a graph G is planar. Let G be a plane graph. Consider the set S obtained from $\mathbb{R}^2$ by deleting (the vertices and edges) of G. We observe that S is the disjoint union of finitely many subsets $F_1, F_2, \ldots F_l$ of $\mathbb{R}^2$ having the following two properties:

1. Any two points of F_i can be joined by a curve not crossing G.
2. Any curve in $\mathbb{R}^2$ which joins a point of F_i to one of $F_j, i \neq j$, must cross G

Definition 2. *Let G be a plane graph. The sets $F_1, \ldots F_l$ described above are called the faces of G.*

Example 2. *Faces*



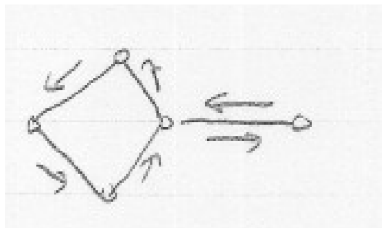
Remark. *One face is always unbounded, with the remaining faces all bounded.*

Definition 3. *Let F be a face of a plane graph G. The boundary of F consists of a finite # of vertices and edges of G*

The length of a closed walk around the boundary of F is called the degree of F, *usually denoted* $\deg F$

Example 2. *(Cont.)*

1. $\deg F_1 = \deg F_2 = 5$ *for graph a)*
2. $\deg F_2 = 2$ *the closed walk is* $x \to y \to x$ *for graph b)*
3. $\deg F_1 = \deg F_2 = 3$
   $\deg F_3 = 6$ *for the graph below*



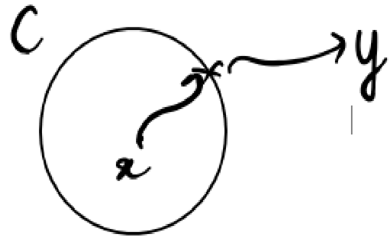
Theorem 1. *(Euler's Formula, 1780) Let G be a connected plane graph. If G has n nodes and f faces then*

$$n - m + f = 2$$

*proof of **Theorem 1** requires a couple of preliminarily lemmas.*

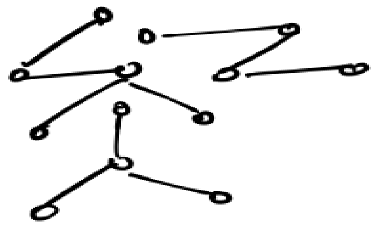
Lemma 2. *Let G be a plane graph. G contains a cycle if and only if the number of faces of $G \geq 2$.*

Proof. $\Rightarrow$ Let C be a cycle of G. Let x and y be in the interior and exterior of C, respectively, then any curve in $\mathbb{R}^2$ connecting x and y must cross C, have cross G



Choosing x, y not lying on G, we deduce x and y belong to different faces of G.

$\Leftarrow$ The absence of a cycle, G is a forest. On induction on the number of components of G shows G has only one (unbounded) face.

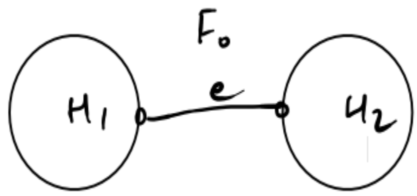


$\square$

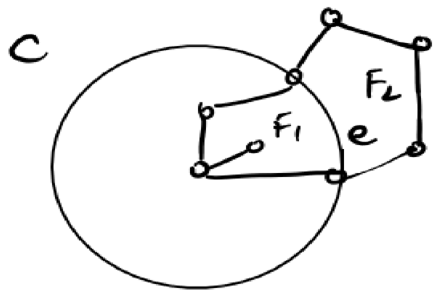
Lemma 3. *Let e be an edge of a plane graph G.*

1. *If e is a bridge then it lies on boundary of exactly one face.*
2. *If e is not a bridge then it lies on the boundary of exactly two faces of G*

Proof. 1. Let H_1 and H_2 be the components of $G \backslash e$. The edge e lies in face F_1 of H_1, as well as a face F_2 of H_2. The intersetion $F_1 \cap F_2$ contains a unique face F_o of G. e lies on the boundary of F_o and is the unique face of G.



2. Let C be a cycle containing e. The edge e lies on the boundary of one face lying in the interior of C and one face lying in the exterior of C. Thus, e lies on the boundary of at least 2 faces of G.



The fact the faces are disjoint can be used to show that e lies on the boundary of at most 1 face lying in the interor of C, and one face lying in the exterior of C. Then e lies in the boundary of at most 2 faces.

$\square$

Corollary. *(Handshaking Lemma for Planar Graphs) If G is a plane graph of size m then*

$$2m = \Sigma_{faces\ F} \deg F$$

Proof. Each edge e of G contributes 2 to the sum on the right:

If e is a bridge lying on the face F_o then it contributes 2 to $\deg F_o$ and 0 to the remaining $\deg F$.

If e is not a bridge then it contributes 1 to the degree of two distinct faces of G, and 0 to the remaining. $\square$

Proof. Proof of Euler's Formula

We proved by induction on # of faces of G. In the case $f = 1$. **Lemma 2** asserts G is a forest, where G is a tree.

Thus, $m = n - 1$, and hence

$$n - m + f = n - (n - 1) + 1 = 2$$

Assume the result is true for connected plane graph H having f faces, and let G have $f + 1$ faces, say $F_1, F_2, \ldots, F_l, F_{l+1}$. If $n = |G|, m = e(G)$ then we are required to show

$$n - m + (f + 1) = 2$$

Since $f + 1 \geq 2$, G contains a cycle C (**Lemma 2**). Fix an edge e lying on C. Since e is not a bridge of G. **Lemma 2** ensures that e lies on the boundary of 2 distince faces of G, say F_l and F_{l+1}.

Construst the subgraph $H = G \backslash e$. We note H is a connected plane graph of order n and size $m-1$. Furthermore, the number of faces of H is f, denoted each of the faces $F_1, f_2, \ldots F_{l-1}$ of G occur on faces of H. Since e does not appear in the boundary of any of these faces.

The remaining faces of H is obtained by joining F_l and F_{l+1} along the edge e.

$$\hat{F}_l = F_l \cup F_{l+1} \cup e$$

Since H has f faces, the indcuntion hypotheses allows in G to conclude that

$$2 = n - (m-1) + f = n - m + (f+1)$$

as required

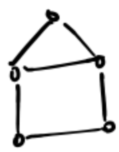
$\square$

Corollary. *Let G be a connected planer graph. Each plane drawing of G has the same number of faces, namely $2 + m - n$*

Euler's formula can be used to obtain a necessary condition for a simple connected graph to be planar.

Definition 4. *Let G be a graph, If G contains a cycle then the* ***girth*** *$gr(G)$ is defined as the length of the smallest cycle in G. If G is a forest then we set $gr(G) = \infty$*

Example 3. *Girth*

a.)  has girth 3

b.) Peterson has girth 5

c.) K_5 has girth 3, $K_{3,3}$ has girth 4

Remark. *If G is simple then $gr(G) \geq 3$*

Theorem 4. *Let G be a connected simple planar graph. If G has order n and size m then*

$$m \leq \begin{cases} n-1 & \text{if } gr(G) = \infty \\ \frac{gr(G)}{gr(G)-2}(n-2) & \text{if } gr(G) \text{ is finite} \end{cases} \tag{1}$$

Proof. If $gr(G) = \infty$ then G is a tree, hence $m = n - 1$. If $gr(G)$ is finite then G has ≥ 2 faces. In this case, the boundary of each face of G contains a cycle, hence

$$\deg(F) \geq gr(G)$$

for each face F of G. Therfore, if $f = \#$ of faces of G then the handshaking lemma of planer graphs yields

$$2m = \Sigma_{\text{faces} F} \deg F \geq \Sigma gr(G) = gr(G) f$$

From Euler's formula, $f = 2 + m - n$, substitues f_i in the preceding yields

$$2m \geq gr(G)(2 + m - n)$$

Solving for m yields the requied identity $\square$

Example 4.

1. *Does there exist a simple connected planar graph G of order 12 and size 40?*

   Solution 1. *Suppose such a graph G exists, Note that G cannot be a tree, hence $gr(G)$ is finite. Thus,* ***Theorem 3*** *asserts*

   $$40 \geq \frac{gr(G)}{gr(G)-2}(12-2)$$

   Solving for $gr(G)$

   $$gr(G) \leq \frac{8}{3} < 3$$

   This contradicts that $gr(G) \geq 3$ No such G exists.

2. *Let G be a planar graph of size 14 and girth 5. What can one say about $n = |G|$*

   Solution 2. *From* ***Theorem 2***

   $$14 \leq \frac{5}{5-2}(n-2)$$

   solving f n yields $n \geq 52/5$ some n is an integer $n \geq 11$.

Theorem 4 are a number of interesting corollories

Corollary. *K_5 and $K_{3,3}$ are non-planar*

Proof. K_5 has order 5, size 10 and girth 3. Since

$$\frac{3}{3-2}(5-2) = 9 \leq 10$$

Theorem 3 asserts K_5 is non-planar.

$K_{3,3}$ has nodes 6, size 9 and girth 4.

$$\frac{4}{4-2}(6-2) = 8 \leq 9$$

Theorem 3 asserts that $K_{3,3}$ is non-planar. □

Exercise Show that Peterson is non-planar

Corollary. *Let G be a simple connected planar graph.*

1. *if G has order $n \geq 3$ then $e(G) \leq 3n-6$*
2. *Furthermore, if G contains no triangles then $e(G) \leq 2n-4$*

Proof. if G is a tree then $e(G) = n-1 \leq 3n-6 \leq 2n-4$

Suppose G has finite girth $gr(G) \geq 3$. Observe the function $f(x) = \frac{x}{x-2}$ is decreasing on $[3, \infty](f'(x) = \frac{-2}{(x-2)^2})$ which proves 1.)

For 2.) note the hypothesis implies $gr(G) \geq 4$ when

$$e(G) \leq \frac{gr(G)}{gr(G)-2}(n-2) \leq \frac{4}{4-2} = 2n-4$$

$$\frac{gr(G)}{gr(G)-2} \leq \frac{3}{3-2} = 3$$

Thus, from Theorem 3,

$$e(G) \leq \frac{gr(G)}{gr(G)-2}(n-2) \leq 3(n-2) = 3n-6$$

□

Remark. *The preceding two results are often used as tools for non-planarity, they are weakter then the full theorem 4.*

Corollary. *Every simple planar graph G contains a vertex of degree at most 5 (i.e $\delta(G) \leq 5$)*

Proof. WLOG G can eb assumed to be connected and has at least 7 vertices. If each vertex has degree ≥ 2, the handshaking lemma would imply

$$2m = \Sigma \deg v \geq 6n, m = e(G), n = |G|$$

hence $m \geq 3n$

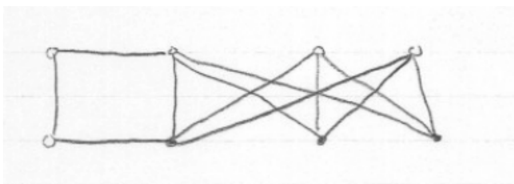
On the other hand, G is planar, so the preceding corollary yields $m \leq 3n-6$. Corollay with the above inequality, we get

$$3n \leq m \leq 3n-6$$

□

Example 5. *K_n is non-planar of $n \geq 7$. Each vertex of K_n has degree $n-1 \geq 6$*

Example 6. *The condition of theorem 4 (as well as it corollories) is only necessary for planarity, not sufficient. For example, the following graph G is non-planar, as it contains a copy of $K_{3,3}$*



On the other hand, $|G| = 8, e(G) = 12$ and $gr(G) = 4$

$$\frac{4}{4-2}(8-2) = 12$$

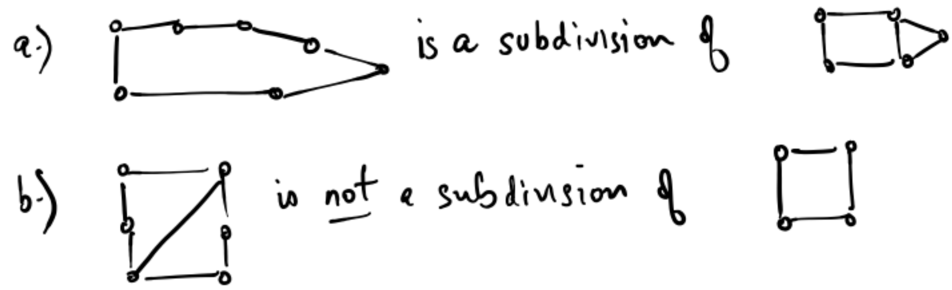
Remark. *Euler's formula can be extended to disconnected plane graphs using induction on the # of components. If G is a planar graph of order n size m with f faces and k components then*

$$n - m + f = k + 1$$

There are also analogue for Theorem 3 and its corollories.

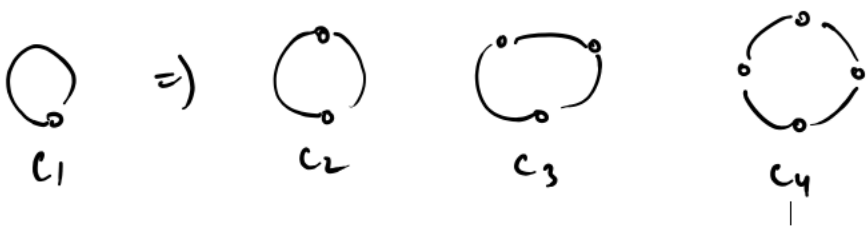
We have already observed that Theorem 3 and its corollories present many necessary conditions for a graph to be planar.

Definition 5. *Let G and H be graphs. H is said to be a <u>subdivion</u> of G if the formal graph can be constructed from the latter by introduction a finite # of new vertices along existing edges.*



Example 7.

c.) *Each cycle graph C_n is a subdivision of C_1.C_n can be obtained from C_1 by the addition of $n-1$ new vertices on the single edge of C_1*



requires the addition of an extra edge to the latter

Remark.

1. *The process of subdivion only introduces new vertices of degree 2*
2. *Note that if H is a subdivion of G then H has the same shape as G.*

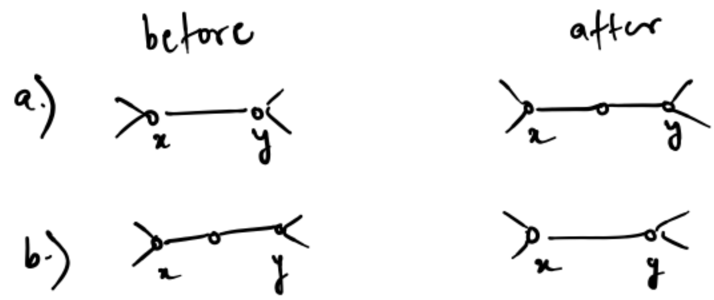
Definition 6. *Two graphs G and H are said to be homeomorphism if they are both subdivision of a common graph.*

Example 8.

1. *If $n, m \geq 1$ then C_n is homeomorphism to C_m as both are subdivion of C_1.*
2. *If $n, m \geq 2$ then P_n and P_m are homoeomorphism as both are subdivion of P_2*

Remark.

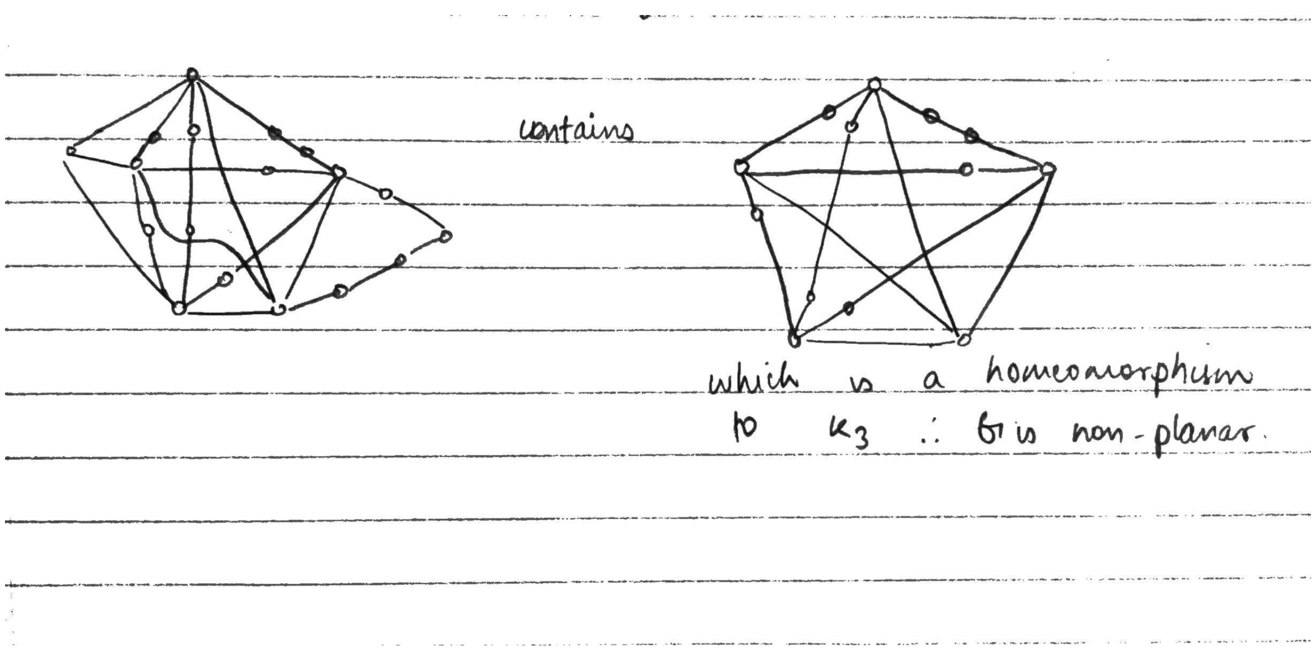
1. *G and H are homeomorphic if the latter can be obtained from the former by a finite sequence of following 2 operations:*
   (a) *addition of new vertex along an existing edge.*
   (b) *filling in of an existing verted of degree 2.*



2. homeomorphic graphs have the same shape.

Theorem 5. *(Kuratowski, 1930) A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.*

Example 9. *The graph*

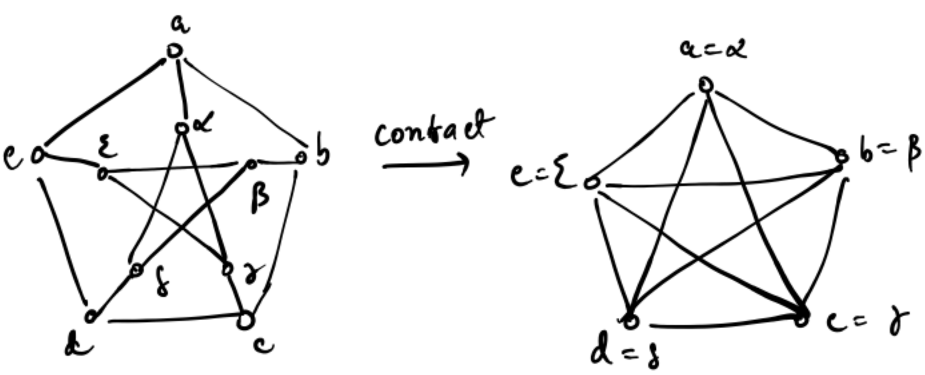


an alternate characterization of non-planar graphs can be obtained using the notion of contradictions.

Definition 7. *Let G and H be graphs. G is said to be* <u>*contractible to H*</u> *if H can be obtained from G by successivily contracting a finite number of edges.*

Theorem 6. *A graph G is planar if and only if it contains no subgraph which is contractible to K_5 or $K_{3,3}$.*

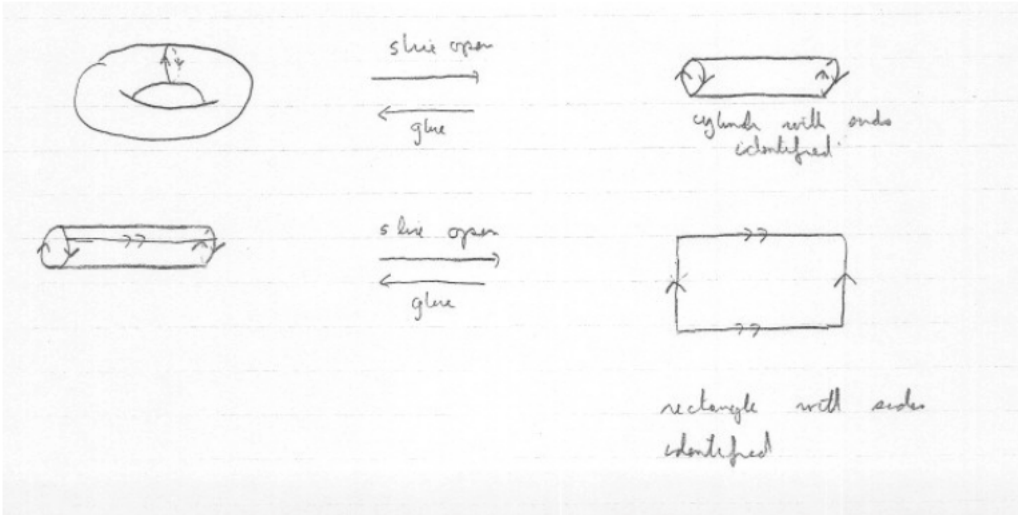
Example 10. *Peterson graph is contractible to K_5.*



Thus peterson is non-planar.

We have observed that K_5 and $K_{3,3}$ are non-planar. What happens if we consult other surface?

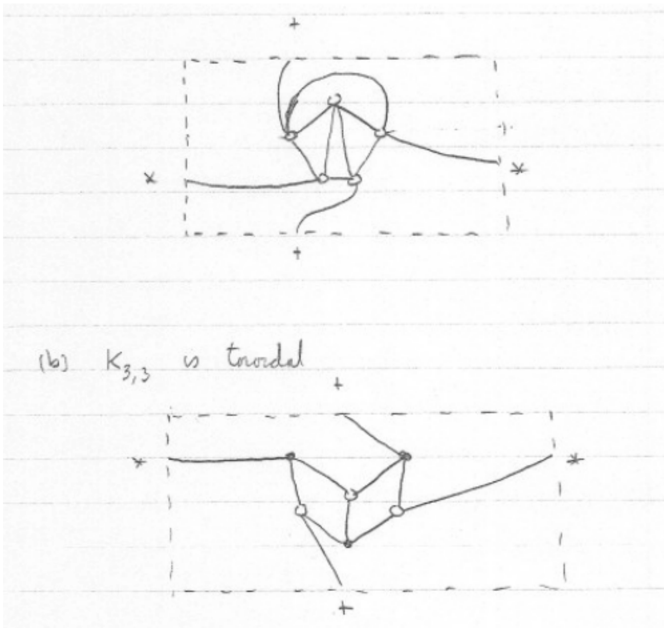
Example 11. *Torous (Donut)*



Definition 8. *A non-planar graph G is said to be* <u>*toroidal*</u> *if it can be drawn on the torus so as no two edges cross.*

Example 12. *1.* *K_5 is torodial*

2. *$K_{3,3}$ is torodial*



Question: Is there analogous of Euler's Formula for the toros?

Theorem 7. *Let G be a simple connected torodial graph of order n, size m and f faces, Then*

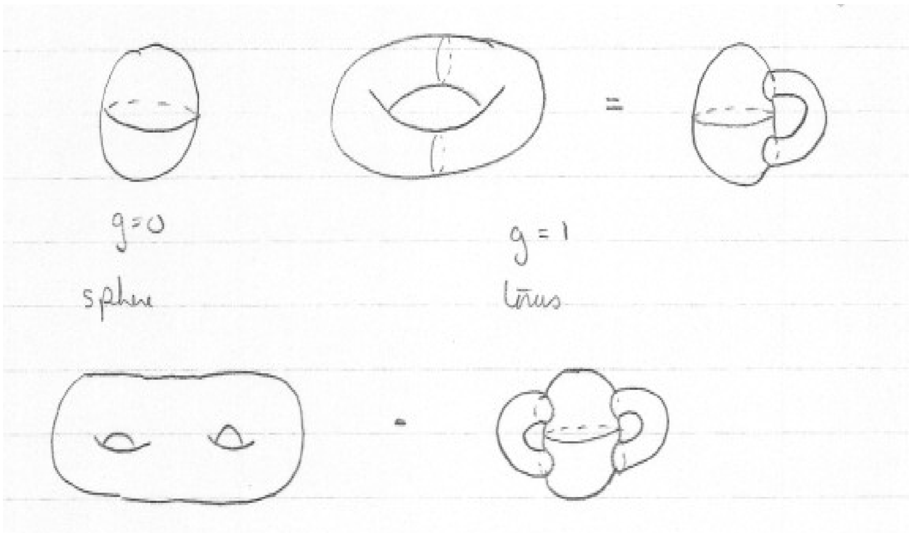
$$n - m + f = 0$$

Furthermore,

$$m \leq \frac{gr(G)}{gr(G) - 2} n$$

The proof of Theorem 7 is esentially the same as that of Theorem 1 (and to corollary Theorem 2). Furthermore, it is also analogus of Kuratowski - there are a finite # of graphs where subdivision cannot appear in torodial graphs.

More genrally, one can conceive a donut with g holes. Thus it is the 'same' thing as a sphere with g handles.



The number of holes (or handles) is called the genus of the surface.

Definition 9. *A graph G is said to have genus g if G can be drawn on a surface of genus g with no edges crossing, but no drawing on a surface of genus $g-1$ exists. (i.e planar = genus 0, torodial = genus 1)*

Theorem 8. *Let G be a connected graph of genus g, order n, size m, and face f. Then*

$$n - m + f = 2 - 2g$$

Furthermore, if G is simple of finite girth then

$$m \leq \frac{gr(G)}{gr(G)-2}(n + 2g - 2)$$

Corollary. *Let G be a connected simple graph of genus g, order $n \geq 3$ and size m then,*

$$m \leq 3(n + 2g - 2)$$

$$m \leq 2(n + 2g - 2) \text{ if no triangle present}$$

Corollary. *Let G be a connected simple graph of order $n \geq 4$ and size m. Then the genus g satisfies*

$$g \geq \lceil \frac{m - 3n}{6} + 1 \rceil$$

$\lceil x \rceil$ = *least integer greater than or equal to x*

Proof. Solving the inequality $m \leq 3(n + 2g - 2)$ for g yields

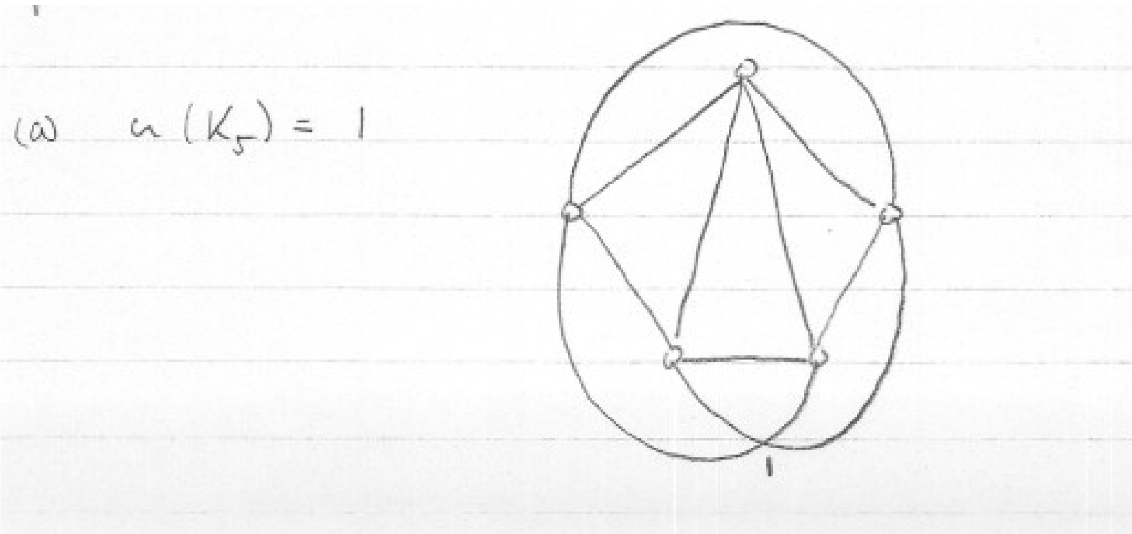
$$\frac{m - 3n}{6} + 1 \leq g$$

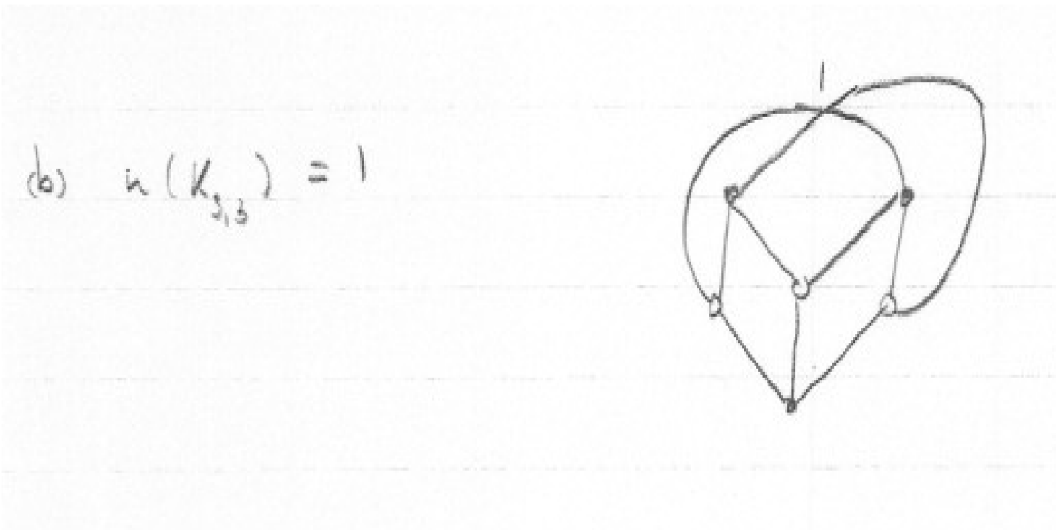
The result follows by nothing g is an integer

□

Definition 10. *Let G be a graph. The* <u>*crossing number*</u> *$cr(G)$ is the minimum # of crossing that can occur when G is drawn in the plane.*

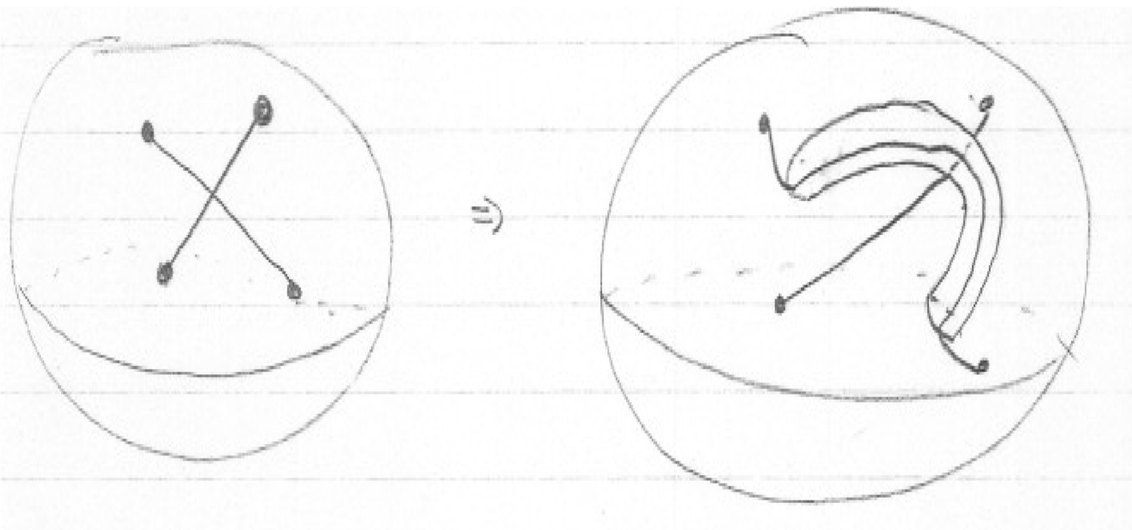
Example 13.





Theorem 9. *The genus of the graph G is $\leq cr(G)$*

Proof. Drawing the graph on the sphere. Suppose we have a crossing



Distribution of a handle allows a bridge with one edge can go under and the second edge go over, Then removing the crossing. Then the addition of $cr(G)$ handles resusts in a surface of genus g on which G can be drawn with no edge crossing.

□

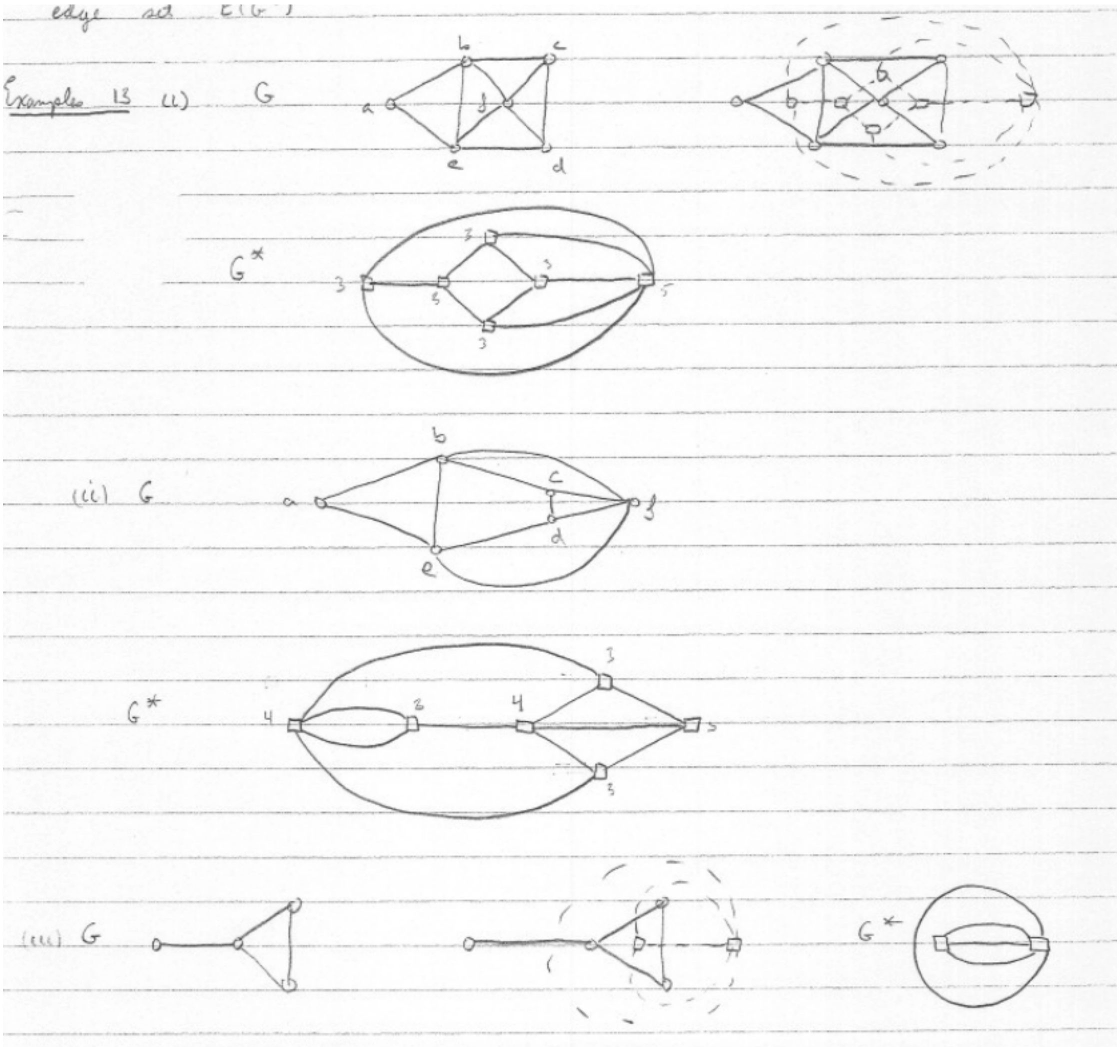
Dual Graphs

Definition 11. *Let G be a planar graph. The* <u>*geometric dual*</u> *G^* of G is the graph constructed as follows:*

1. *denote each face of G, choose a point v^*. The v^* forms the vertex set $V(G^*)$*

2. *For each edge e of G, join the vertices v^* and w^* in the adjacent face by a curve e^* that crosses e and no other edge of G. The collection of e^* froms the edge set $E(G^*)$*

Example 14. *Platonic Graphs*



Graphs	**Geometric Dual**	
tetrahedron	tetrahedron	self-dual
cube	octahedron	
octahedron	cube	
dodecahedron	icosahedron	
icosahedron	dodecahedron	

Remark.

1. *The contrustion of G^* depends on the plane drawing of G. For example, the grphs G in example 13 (i) and (ii) are isomorphic, bnut their geometric duals are not. One has vertex of degree 5 but the other has no such vertex.*

G	G^*
vertices	faces
edges	edge
vertex degree	face degree
handshaking for vertices	handshaking for faces
bridges	loops
deletion of edge	contraction of edge

2. G^* is planar and connected.

Lemma 9. *Let G be a connected planar graph of order n, size m, and face f. If G^* is a geometric dual then the nodes n^* size m^* and number of faces f^* satisfies*

$$n^* = f, m^* = m, f^* = f$$

Proof. By construction, $n^* = f$ and $m^* = m$. Since both G and G^* are connected plane graphs

$$n - m + f = 2 = n^* - m^* + f^*$$

we get $f^* = n$ □

Theorem 10. *Let G be a connected planar graph. Then G^{**} is isomorphic to G.*

Proof. Observe each face of G^* contains at least one vertex of G. The fact $n = f^*$ ensure that there is exactly one in each face. Thus, one use $V(G)$ as the vertex set of G^{**}. Doing so, $E(G)$ can be used as the edge set of G^{**}, where $G^{**} = G$ as required.

The following table shows some of the relations between a plane graph G and its dual.

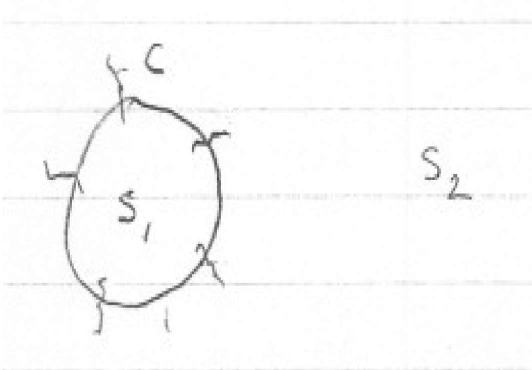
The last is a special case of a more general result. □

Theorem 11. *Let G be a connected planar graph and G^* a geometric dual of G, A subset of $E(G)$ forms a cycle of G if and only if the corresponding subset of $E(G^*)$ is a cutset of G^*.*

Remark. *If e is an edge of G then there is a unique edge e^* of G^* which crosses a*

Proof. (Sketch) $\Rightarrow$ Each cycle C of G partitions $V(G^*)$:

$V(G^*) = S_1 \cup S_2$ where S_1 (or S_2) is the set of vertices lying inside (respectively outside) C.



Both S_1 and S_2 are non-empty, since G has faces lying in the interior and exterior of C, Furthermore, any two vertices of S_1 (or S_2) is a connected by a path in G^* lying inside (resp outside) of C. On the other hand, any path connected a vertec of S_1 with a verted of S_2 must cross C. Hence, certain edge of G^* corresponding to one of C. Thus the set C^* of edges G^* corresponding to ones of C for a disconnecting set, the fact that end edges of C^* join one vertex of S_1 to one of S_2 ensure C^* is a cutset.

$\Leftarrow$ Suppost X^* is a cutset of G^*, say

$$G^* \setminus X^* = G_1^* \cup G_2^*$$

WLOG, assume G_2^* contains the unbounded vertex of G^*. If F denotes the of the faces of G correspong to the vertex of G^* then F is bounded, As the boundary of F must contain a cycle C. If X denotes the edges of G corresponding to X^*, we see $C \subseteq X$. If the was proper, C^* would be a proper subset of X^* which disconnects, contradicting the fact X^* is a cutset, Thus $X = C$, a cycle.

□

Corollary. *Let G be a connected planar graph. A set of edges of G forms a cutset if and only if the corresponding edges of G^* forms a cycle.*

Proof. By duality, applying the preceding Theorem to the case G^*, we get that $G^{**} = G$ since G is connected □

Theorem 12. *Let G be a connected planar graph G is bipartite if and only if G^* is eulerian.*

Proof.

$$G \text{ bipartite } \Leftrightarrow \begin{cases} \text{all cycles of } G \text{ have even length} \\ \text{all cutsets of } G^* \text{ have even size (corollary to therorem 6)} \\ G^* \text{ is a disjoint union of cycles} \\ G^* \text{ is eulerian} \end{cases} \tag{2}$$

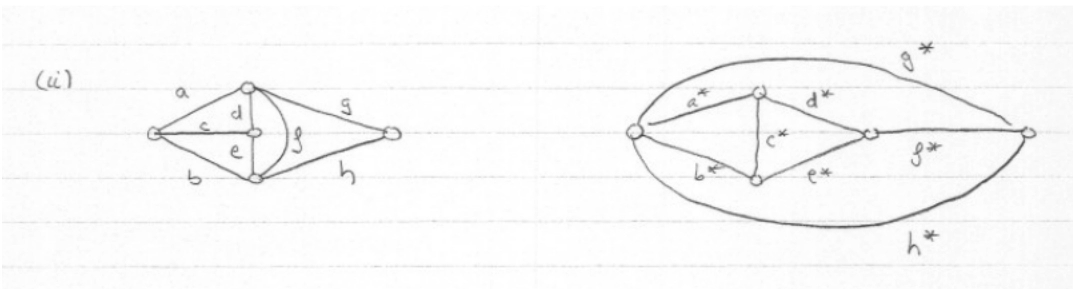
□

Theorem 13. *Let G be a connected planar graph. If G is 3-edge connected then G^* is simple (of order ≥ 3)*

Definition 12. *Let G be a graph. A graph G^* is said to be an abstract dual of G if there exists a one-one correspondence between the edge of G and the of G^* with the property that*

a subset of $E(G)$ forms a cycle $\leftrightarrow$ correspong subset of $E(G^*)$ forms a cutset

Example 15. *If G is plane then its geometric dual G^* is an abstract dual (Theorem 11)*



Theorem 14. *If G^* is an abstract dual of G then G is an abstract dual of G^*.*

Abstract duals provide another characterization of planar graphs.

Theorem 15. *A graph G is planar if and only if G has an abstract dual.*

Proof. Suppose G has a abstract dual G^*. If $e \in E(G)$ has correspoing edge e^* then $G^* \setminus e^*$ is an abstract dual of $G \setminus e$. It follows that every subgraph of G has an abstract dual.

If G' is obtain from G by adding (resp deleting) a vertex of degree 2 along on existing edge then the graph $(G')^*$ obtained from G^* by the addition (resp deleting) of a multiple edge is an abstract dual of G'

As a result, if G admits an abstract dual and G' is homoeomorphic to G then G' admits an abstract dual.

We next observe that neither K_5 or $K_{3,3}$ admit abstract duals. One has $\lambda(K_{3,3}) = 3$ and $gr(K_{3,3}) = 4$. So if $K^*_{3,3}$ existed then $gr(K_{3,3}) = 3$ and $\lambda(K^*_{3,3}) = 4$, The former ocndition ensure $K^*_{3,3}$ is simple, which the latter condition ensure $\delta(K^*_{3,3} \geq 4)$. Thus, $|K^*_{3,3}| \geq 5$ and so

$$e(K^*_{3,3}) = \frac{1}{2}\Sigma \deg v \geq \frac{1}{2}(5 \times 4) = 10$$

$$\leftrightarrow e(K^*_{3,3}) = e(K_{3,3}) = 9$$

For K_5 one has $\lambda(K_5) = 4$ and $gr(K_5) = 3$. So if K^*_5 existed then $gr(K^*_5) = 4$ and $\lambda(K^*_5 = 3)$. The former condition ensure that K^*_5 is simple, while the latter ensure that $\delta(K^*_5) \geq 3$.

$$e(K^*_5) = e(K_5) = 10$$

The face $\delta(K^*_5) \geq 3$ follows $|K^*_5| \leq 6$

Suppose G ensures an abstract dual in light of the preceding discussion, G contains no subgraph homoeomorphic to K_5 or $K_{3,3}$, so Kuratowski ensure G is planar.

$\square$