Viscous Fluid Flows

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Governing Equations for Viscous Flows

- The continuity equation remain unchanged:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

- The momentum equation is the Navier-Stokes equation

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p - \nabla \times [\mu (\nabla \times \mathbf{v})] + \nabla [(2\mu + \lambda) \nabla \cdot \mathbf{v}]$$

where λ is the Lamé first parameter and μ is the Lamé second parameter.

- The last two terms on the right-hand side are the viscous forces.
- When there exists a solid surface, the flow velocity on the surface must vanish, i.e., $\mathbf{v} = \mathbf{0}$.
- The equation of state for an ideal gas is $p = \rho RT$ where R is the gas constant.

Governing Equations for Viscous Flows

The energy equation written in term of specific enthalpy $h = c_p T$ is

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \nabla \cdot \left(k \nabla T \right) + \Phi$$

where c_p is the constant-pressure specific heat, k is the thermal conductivity of the gas, T is the absolute temperature, and Φ is the dissipation function defined as

$$\Phi = 2\mu \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] + \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2$$

$$+ \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2 + \lambda \left(\nabla \cdot \mathbf{v} \right)^2$$

which is the time rate at which energy the fluid motion per unit volume is dissipated into heat through the action of viscosity.

Governing Equations for Viscous Flows

- These from a system of 6 scalar equations with 6 variables ρ, p, T, u, v, w.
- These nonlinear system is difficult to solve.
- Assuming that the fluid is incompressible and that the temperature variation is so small that fluid properties are constant greatly simplifies the system.
- Under such assumptions, ρ is constant, the equation of state is not needed, the energy equation is uncoupled from the continuity equation and the momentum equation which become

$$\nabla \cdot \mathbf{v} = 0, \qquad \rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{v}$$

- The stream function can be defined but the velocity potential cannot be defined since the flow is rotational due to the viscous forces.

Biringen and Chow (2011, p. 147)

Laminar Boundary-Layer Flows

- In a flow with a high Reynolds number past a body, the effect of viscosity is confined within a thin boundary layer next to the body surface.
- "The governing equations for boundary-layer flows can be deduced from Navier-Stokes equations under the assumption that the boundary-layer thickness δ is small compared to the characteristic length L of the body."
- "Considering a thin, 2D boundary layer around a body having its surface parallel to the x axis, Kuethe and Chow (1998) found that $v << u$ and $\partial/\partial x << \partial/\partial y$ when operating on velocity and temperature."
- "If the x component of the equation of motion is of order unity, then the y component is of order δ/L."

$$\nabla \cdot \mathbf{v} = 0, \qquad \rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{v}$$

Biringen and Chow (2011, p. 147)

Laminar Boundary-Layer Flows

- The Navier-Stokes equation and the energy equation become

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$$

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2$$

- These are the boundary-layer equations for 2D compressible flow.
- Together with the continuity equation and the equation of state, they forms a system of 4 equations and 4 unknowns ρ, T, u, and v.
- The pressure p is treated as constant across the boundary layer.
- These approximations form the basis of Prandtl's boundary-layer theory.

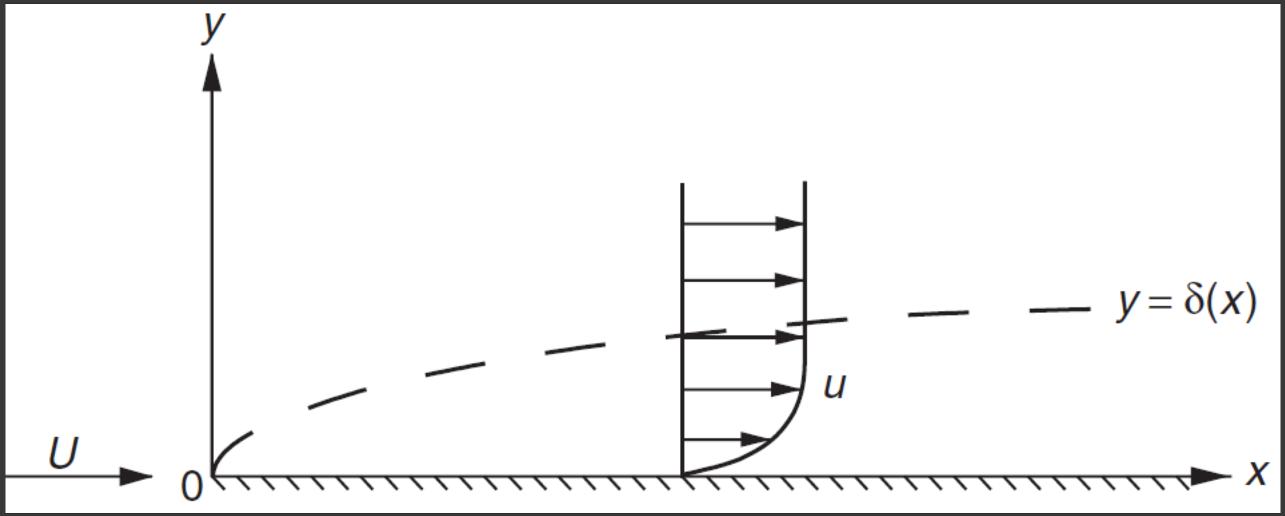
Biringen and Chow (2011, p. 147-148)

Laminar Boundary-Layer Flows

- "Consider a semi-infinite flat plate aligned with a uniform flow of constant speed U and of constant physical properties."
- The governing equations for a steady flow are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad \nu=\mu/\rho$$

- "This formulation describes the thin laminar boundary layer on the surface of a 2D streamlined body that moves in an incompressible fluid."



Biringen and Chow (2011, p. 148)

Laminar Boundary-Layer Flows

- "At any fixed x on the plate, 3 boundary conditions are needed, 2 for the first equation and 1 for the second equation."
- These boundary conditions are the no-slip condition at the surface and the condition of uniform flow at infinity:

$$u = v = 0, \qquad y = 0$$

$$u \to U, \qquad y \to \infty$$

- This boundary-value problem is called the Blasius problem.

Biringen and Chow (2011, p. 148)

Pipe Flow and Open-Channel Flow

- Pipe flow is a fluid flow within a closed conduit such as a pipe.
- Open-channel flow is a fluid flow within a conduit with a free surface.
- Consider a steady incompressible flow along an infinitely long conduit which is parallel to the x axis along which $\partial \mathbf{v}/\partial x = 0$.
- In this situation, only velocity component u is nonzero while $v = w = 0$.
- So, the continuity equation is satisfied automatically.
- The momentum equation is

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right)=-\nabla p+\mu\nabla^2\mathbf{v}+\mathbf{f}$$

where $\mathbf{f}$ is the gravitational force per unit volume.

- In the current situation, all the terms on the left-hand side vanish. Why?

Biringen and Chow (2011, p. 164)

Pipe Flow and Open-Channel Flow

- The x component of the momentum equation becomes the Poisson equation

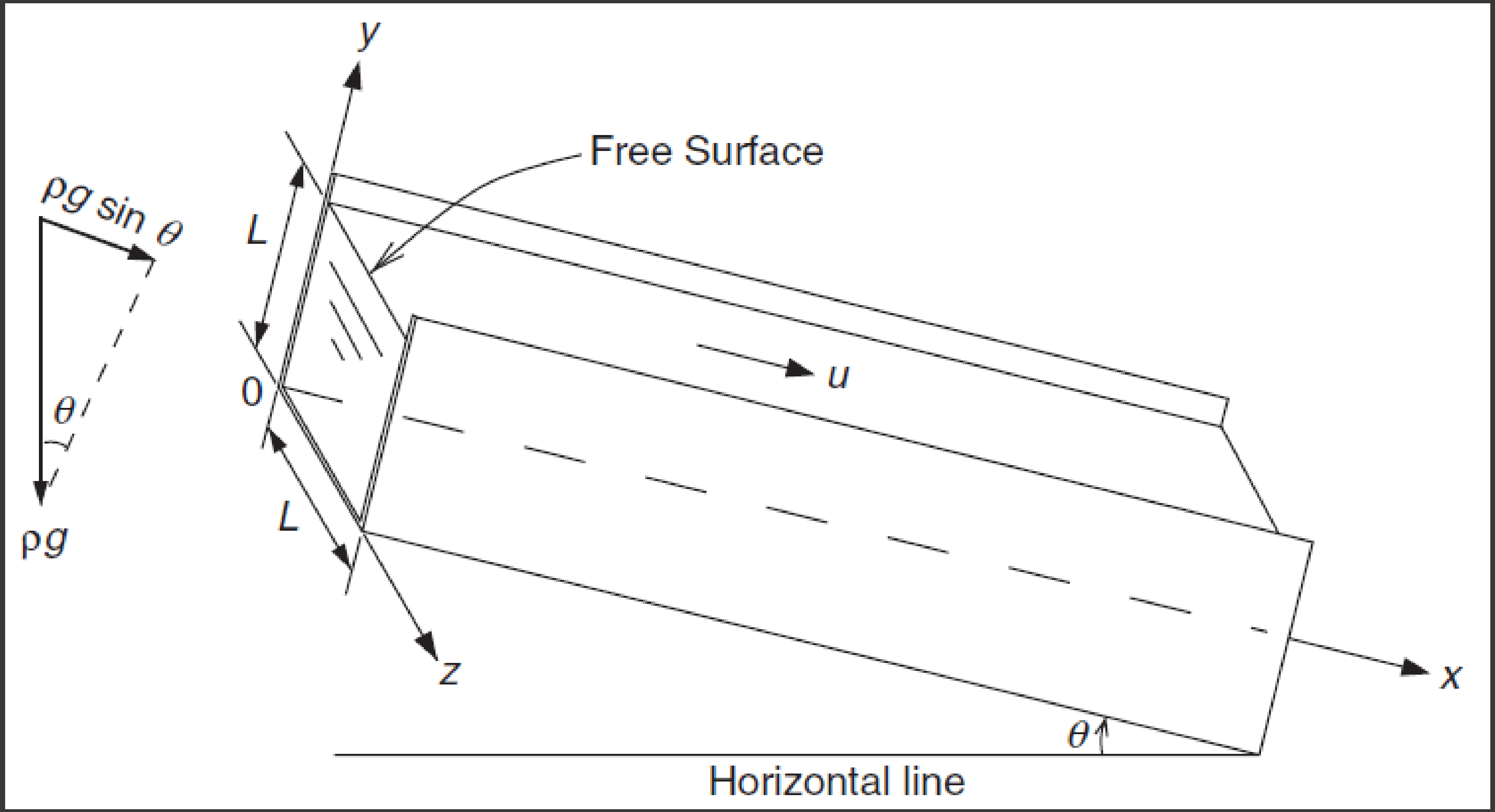
$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\mu}\left(\frac{\partial p}{\partial x} - f_x\right)$$

- If $f_x = 0$ and the pipe cross section is circular, the above equation describes the Poiseuille flow in which an analytical solution exists.
- In other cases, it is difficult to find an analytical solution.
- We will now consider the case in which an open channel with a square cross section is tilted and makes an angle θ with the horizontal.
- The x axis is still chosen to be parallel to the channel.

Biringen and Chow (2011, p. 164)

Pipe Flow and Open-Channel Flow

- Assume that the angle θ is so small that the free surface is parallel to the channel base and that there is no applied pressure gradient.
- Thus, $f_x = \rho g \sin\theta$ is the only force term due to gravity.



Biringen and Chow (2011, p. 164)

Pipe Flow and Open-Channel Flow

- Let's define the dimensionless variables

$$Y = \frac{y}{L}, \qquad Z = \frac{z}{L}, \qquad U = \frac{u}{L^2 \rho g \sin\theta / \mu}$$

where L is the width or height of the square cross section.

- The Poisson equation becomes

$$\frac{\partial^2 U}{\partial Y^2} + \frac{\partial^2 U}{\partial Z^2} = -1$$

- Boundary conditions:
  - velocity is zero at solid walls: $U = 0$ at $Y = 0$
    $U = 0$ at $Z = 0$ and $Z = 1$
  - shear stress is zero at the free surface: $\partial U/\partial Y = 0$ at $Y = 1$

Biringen and Chow (2011, p. 164)

Exercise

- Solve the boundary-value problem in the previous slide using the finite-difference method.

Rayleigh Problem

- In 1911, Rayleigh studied the development of a boundary layer on a body moving through an incompressible fluid.
- He considered an unsteady fluid flow due to a sudden motion of an infinitely flat plate moving along its own plane.
- "If the plate is normal to the *y* axis and the motion is in the *x* direction, the incompressible Navier-Stokes equation is simplified to the diffusion equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

  which is a parabolic PDE.
- The equation can also written in term of the vorticity $\omega = -\partial u/\partial y$ as

$$\frac{\partial \omega}{\partial t} = \nu \frac{\partial^2 \omega}{\partial y^2}$$

Biringen and Chow (2011, p. 168)

Explicit Method for Diffusion Eq

- Using the forward FD for the time derivative and the centered FD for the spatial derivative, the diffusion equation becomes

$$u_{j,n+1} = u_{j,n} + R\left(u_{j-1,n} - 2u_{j,n} + u_{j+1,n}\right)$$

  where $R = \nu\tau/h^2$, τ is the time step, and h is the grid spacing.
- This explicit scheme is stable only when a stability condition is satisfied.
- Using the von Neumann stability analysis, substituting the solution

$$u_{j,n} = U_n e^{ikjh}$$

  into the above equation and rearranging it to obtain

$$U_{n+1} = \left[1 - 2R\left(1 - \cos(kh)\right)\right]U_n$$

- The bracket term is the amplification factor λ.
- For the scheme to be stable, it is required that $\lambda^2 \leq 1$.

Biringen and Chow (2011, p. 169)

Explicit Method for Diffusion Eq

- This is equivalent to $$R \le \frac{1}{1-\cos(kh)}$$
- Since the lower limit of the right-hand side is 1/2, the stability condition is
$$\tau \le \frac{h^2}{2\nu}$$
- When $\tau = h^2/2\nu$ is used, the explicit scheme become the Bender-Schmidt recurrence equation $$u_{j,n+1} = \tfrac{1}{2}\left(u_{j-1,n} + u_{j+1,n}\right)$$
- However, more accurate result is obtained when $R < 1/2$.

Biringen and Chow (2011, p. 170)

Exercise

- Consider water contained between two originally stationary flat plates separated by a distance of 1 m.
- At the initial time $t = 0$, the upper plate has suddenly acquired a constant speed $u_0 = 1$ m/s, i.e., when $t \geq 0$ the upper plate moves at speed u_0.
- The sudden motion of the upper plate creates a sharp velocity change and form a vortex sheet right below the plate.
- The vorticity is diffused downward according to $\frac{\partial \omega}{\partial t} = \nu \frac{\partial^2 \omega}{\partial y^2}$ and the velocity is redistributed accordingly.

Biringen and Chow (2011, p. 170)

Exercise

- Use the explit FD scheme to solve the initial-boundary value problem

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2},$$

$$u\left(y < 1, t = 0\right) = 0,$$

$$u\left(y = 1, t = 0\right) = u_0$$

$$u\left(y = 0, t\right) = 0$$

$$u\left(y = 1, t\right) = u_0$$

- The kinematic viscosity of water $\nu = 10^{-6}$ m^2/s.

Biringen and Chow (2011, p. 170)

Implicit Method for Diffusion Eq

- Using the backward FD for the time derivative and the centered FD for the spatial derivative, the diffusion equation becomes

$$\frac{u_{j,n} - u_{j,n-1}}{\tau} = \nu \left(\frac{u_{j-1,n} - 2u_{j,n} + u_{j+1,n}}{h^2} \right)$$

- Rearranging the equation yields the implicit scheme

$$Ru_{j-1,n} - (1+2R)u_{j,n} + Ru_{j+1,n} = -u_{j,n-1}$$

- The von Neumann stability analysis shows that the implicit is stable as long as R is positive.

Biringen and Chow (2011, p. 173)

Thomas Algorithm

- A tridiagonal linear system can be written as

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i, \qquad i = 1, 2, \ldots, n$$

where $a_1 = 0$ and $c_n = 0$.

$$\begin{bmatrix} b_1 & c_1 & & & 0 \\ a_2 & b_2 & c_2 & & \\ & a_3 & b_3 & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ 0 & & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{bmatrix}.$$

- The Thomas algorithm which is the special case of Gaussian elimination can solve this linear system in $O(n)$ operations.

http://en.wikipedia.org/wiki/Tridiagonal_matrix_algorithm

Thomas Algorithm #1

The Thomas algorithm that does not modify the coefficient vectors are given by

$$c_i' = \begin{cases} c_i / b_i & ;\ i = 1 \\ c_i \big/ \left(b_i - a_i c_{i-1}' \right) & ;\ i = 2, 3, \ldots, n-1 \end{cases}$$

$$d_i' = \begin{cases} d_i / b_i & ;\ i = 1 \\ \left(d_i - a_i d_{i-1}' \right) \big/ \left(b_i - a_i c_{i-1}' \right) & ;\ i = 2, 3, \ldots, n \end{cases}$$

$$x_n = d_n'$$

$$x_i = d_i' - c_i' x_{i+1} \quad ; \quad i = n-1, n-2, \ldots, 1$$

http://en.wikipedia.org/wiki/Tridiagonal_matrix_algorithm

Stokes Flow

- In the previous example, a special geometry was chosen so that the nonlinear convection terms vanish.
- Assuming that the Reynolds number is small also leads the same result.
- Let L and U be the characteristic length and speed, respectively.
- We can then define the following dimensionless quantities.

$$t' = \frac{t}{L/U}, \quad (x', y', z') = \left(\frac{x}{L}, \frac{y}{L}, \frac{z}{L}\right), \quad \mathbf{v}' = \frac{\mathbf{v}}{U}, \quad p' = \frac{p}{\rho U^2}$$

- The Navier-Stokes equation can then be written in terms of the dimensionless quantities as

$$\frac{D\mathbf{v}'}{Dt'} = -\nabla' p' + \frac{1}{\mathrm{Re}} \nabla'^2 \mathbf{v}, \qquad \mathrm{Re} = \frac{\rho U L}{\mu}$$

Biringen and Chow (2011, p.179)

Stokes Flow

- The Reynolds number Re is the ratio between the inertial force and the viscous force.
- When Re is small, the viscous force dominates the inertial force.
- For incompressible Newtonian fluids, steady flows are governed by

$$\mu \nabla^2 \mathbf{v} - \nabla p = 0, \qquad \nabla \cdot \mathbf{v} = 0$$

- "Flows for which Re << 1 are called the Stokes flow or creeping flows."
- Taking the curl of the steady-state momentum equation yields

$$\nabla^2 \boldsymbol{\omega} = \mathbf{0}, \qquad \boldsymbol{\omega} = \nabla \times \mathbf{v}$$

- Taking the divergence of the steady-state momentum equation yields

$$\nabla^2 p = 0$$

Biringen and Chow (2011, p.179)

Biharmonic Equation

- Consider a Stokes flow in 2D.
- Recall the stream function defined by $u = \dfrac{\partial \psi}{\partial y}, \qquad v = -\dfrac{\partial \psi}{\partial x}$
- When the flow is in the *x*-*y* plane, the vorticity is $\boldsymbol{\omega} = \nabla \times \mathbf{v} = \omega \hat{\mathbf{k}}$ and satisfies

$$\nabla^2 \omega = 0$$

- The stream function satisfies $\nabla^2 \psi = -\omega$
- Combining the last two equation yields the biharmonic equation

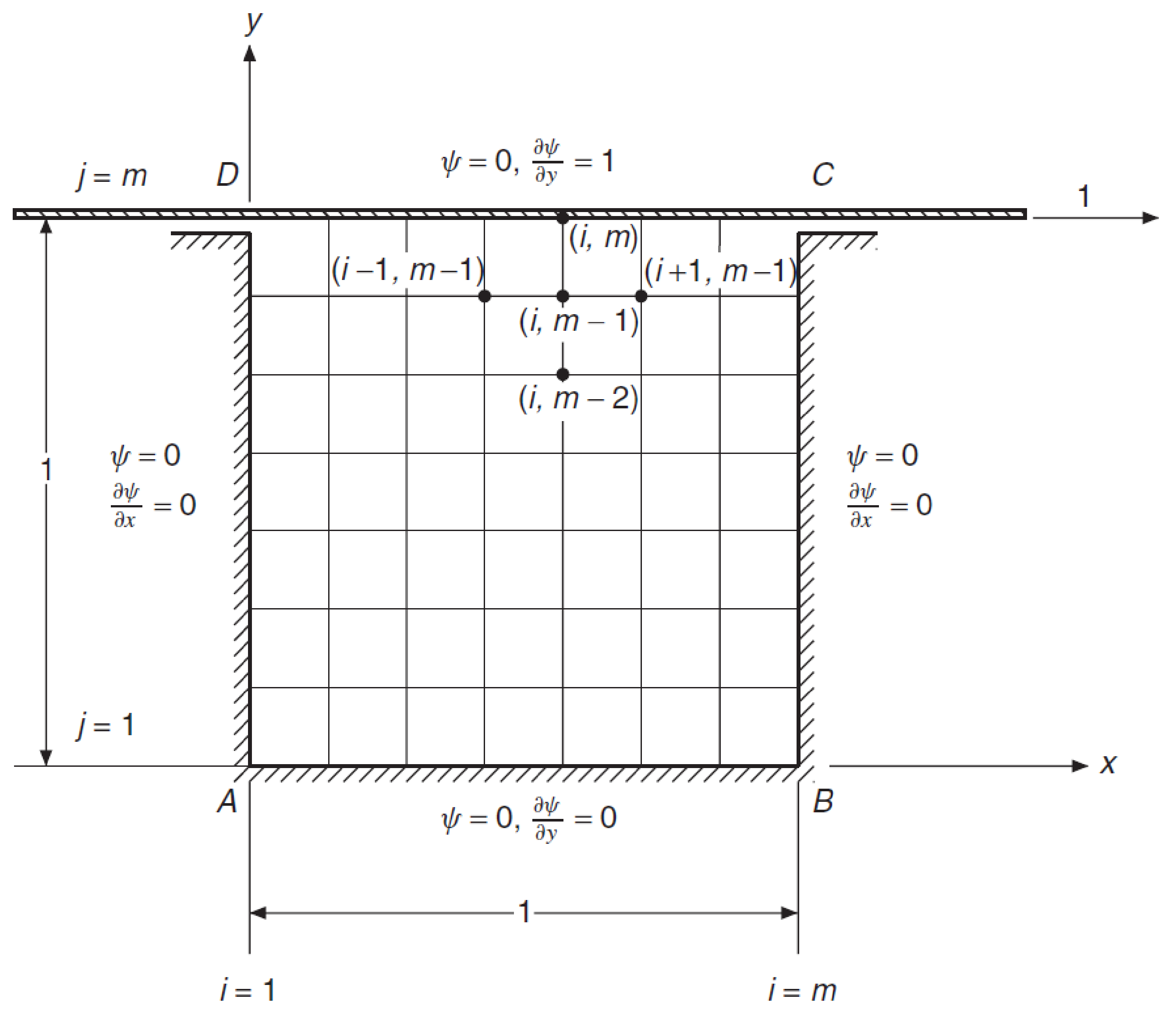
$$\nabla^4 \psi = 0$$

- Here, $\nabla^4 = \dfrac{\partial^4}{\partial x^4} + 2\dfrac{\partial^4}{\partial x^2 \partial y^2} + \dfrac{\partial^4}{\partial y^4}$

Biringen and Chow (2011, p.180)

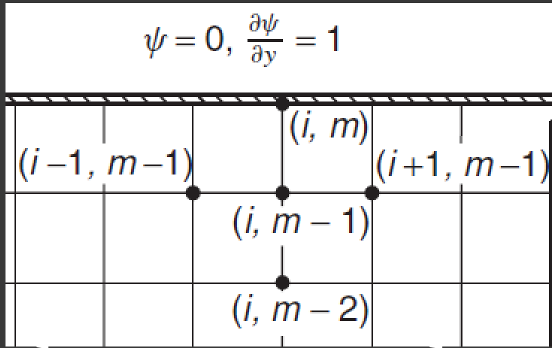
Lid-Driven Cavity Problem

- Consider a square cavity within which a steady fluid motion is generated by sliding an plate lying on top of the cavity.
- Suppose that all variables are normalized to 1 and that the flow is a Stokes flow, i.e., Re << 1.
- The line segment ABCD forms the bounding streamline $\psi = 0$.
- Applying the no-slip boundary condition, we have the Neumann boundary conditions.



Biringen and Chow (2011, p.180-181)

Lid-Driven Cavity Problem



- The biharmonic equation is solved by solving

$$\nabla^2\omega = 0, \qquad \nabla^2\psi = -\omega$$

- Boundary values of ω must be expressed in terms of ψ.
- The vorticity at grid point (i, m) on the top moving plate can be expressed as the FD approximation

$$\omega_{i,m} = -\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)_{i,m} = \alpha_1\psi_{i-1,m-1} + \alpha_2\psi_{i,m-1} + \alpha_3\psi_{i+1,m-1} + \alpha_4\psi_{i,m-2} + \alpha_5\left(\frac{\partial\psi}{\partial y}\right)_{i,m}$$

- Using the method of undetermined coefficients to determine the unknown FD coefficients, we then obtain

$$\omega_{i,m} = \frac{1}{h^2}\left(-\psi_{i-1,m-1} + \frac{8}{3}\psi_{i,m-1} - \psi_{i+1,m-1} - \frac{2}{3}\psi_{i,m-2}\right) - \frac{2}{3h}\underbrace{\left(\frac{\partial\psi}{\partial y}\right)_{i,m}}_{=\,1\ \text{(BC)}} \qquad \text{top BC}$$

Biringen and Chow (2011, p.182)

Lid-Driven Cavity Problem

- Similarly, we can express the vorticities at the other 3 boundaries.
- Using the boundary conditions for ψ, we then have

$$\omega_{1,j} = \frac{1}{h^2}\left(-\psi_{2,j-1} + \frac{8}{3}\psi_{2,j} - \psi_{2,j+1} - \frac{2}{3}\psi_{3,j}\right) \qquad \text{left BC}$$

$$\omega_{m,j} = \frac{1}{h^2}\left(-\psi_{m-1,j-1} + \frac{8}{3}\psi_{m-1,j} - \psi_{m-1,j+1} - \frac{2}{3}\psi_{m-2,j}\right) \qquad \text{right BC}$$

$$\omega_{i,1} = \frac{1}{h^2}\left(-\psi_{i-1,2} + \frac{8}{3}\psi_{i,2} - \psi_{i+1,2} - \frac{2}{3}\psi_{i,3}\right) \qquad \text{bottom BC}$$

$$\omega_{i,m} = \frac{1}{h^2}\left(-\psi_{i-1,m-1} + \frac{8}{3}\psi_{i,m-1} - \psi_{i+1,m-1} - \frac{2}{3}\psi_{i,m-2}\right) - \frac{2}{3h} \qquad \text{top BC}$$

Biringen and Chow (2011, p.183)

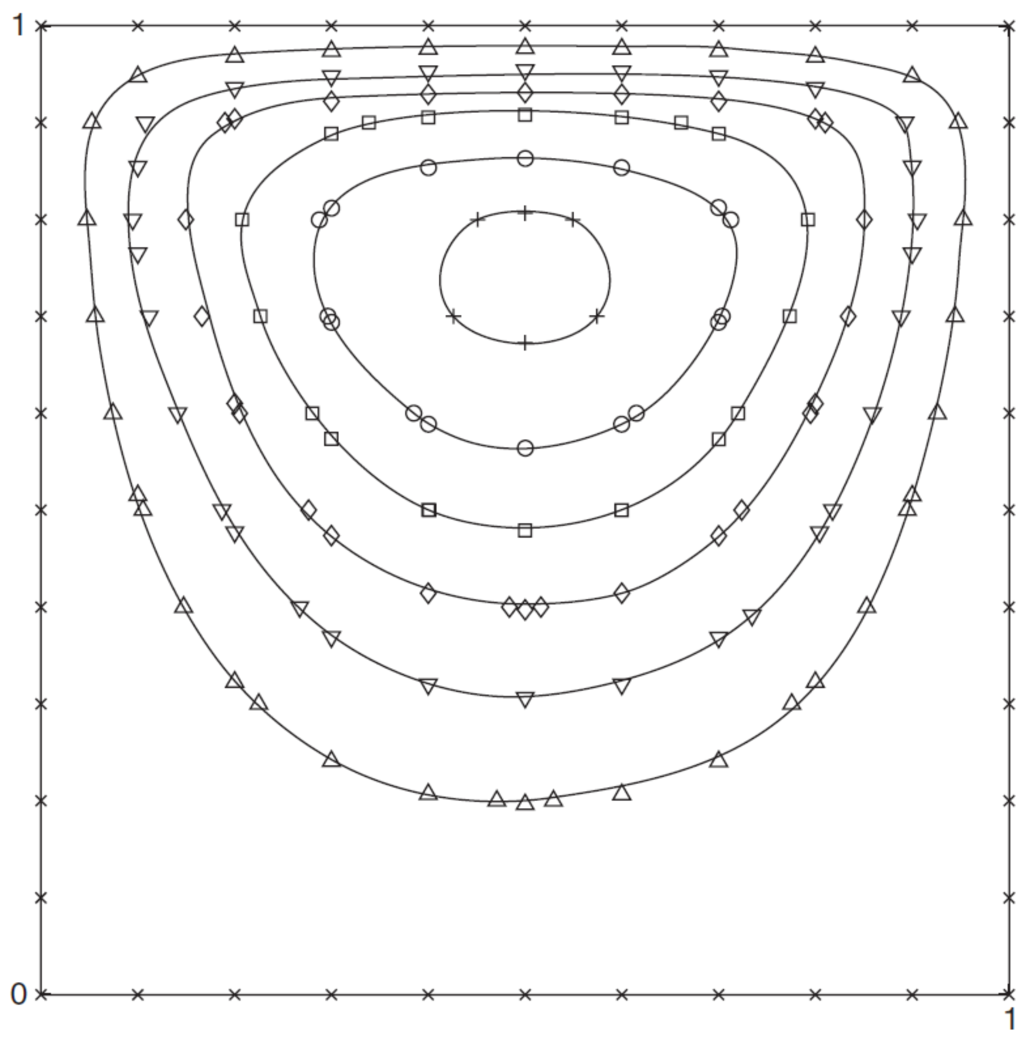
Lid-Driven Cavity Problem

- The vorticity boundary conditions for solving $\nabla^2\omega = 0$ require that we know the value of ψ at some inner nodes.
- However, to solve $\nabla^2\psi = -\omega$ for ψ also requires the distribution of ω within the domain.
- Thus, ψ and ω are coupled and an iterative procedure is needed.
- Assume that the fluid is stationary at the beginning. Thus, $\psi = 0$ everywhere in the cavity.
- The iterative procedure then starts by solving $\nabla^2\omega = 0$ using the boundary conditions in the previous slide with $\psi = 0$. So, the vorticity initially created by the moving plate starts to diffuse into the cavity.
- The vorticity distribution and the condition that $\psi = 0$ on the boundary are then used to solve $\nabla^2\psi = -\omega$.

Biringen and Chow (2011, p.183)

Lid-Driven Cavity Problem

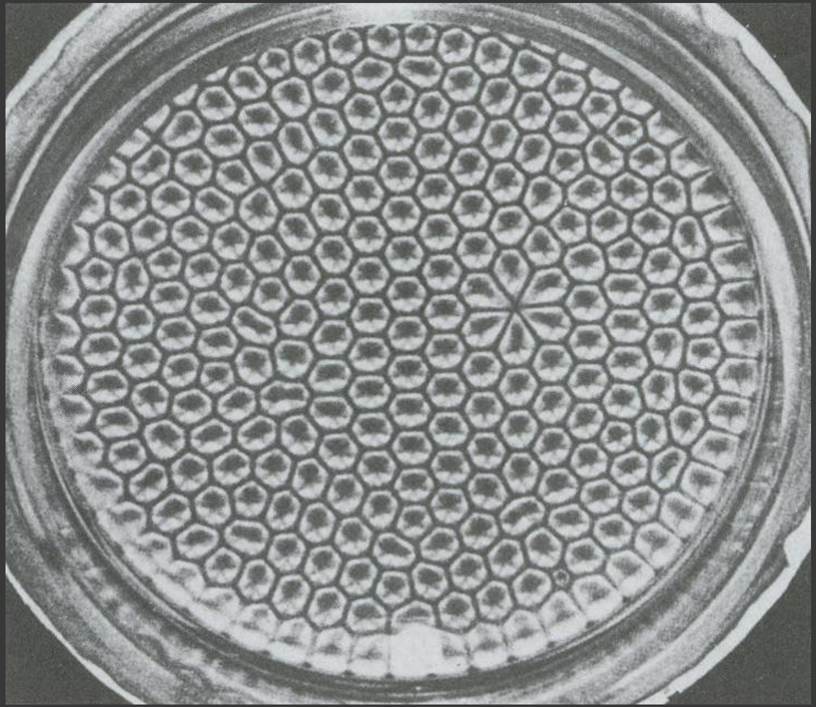
- The values of ψ are then used to update the boundary values of ω.
- These steps are repeated until the changes in both ψ and ω are small.



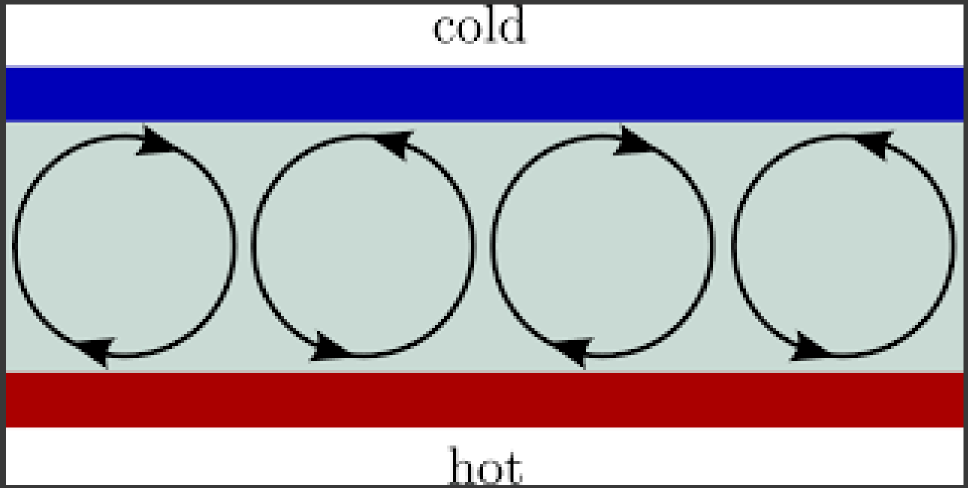
Biringen and Chow (2011, p.184)

Rayleigh-Benard Convection

- When a horizontal layer of fluid is heated from below, the hotter fluid at the bottom starts to flow upward due to the buoyant force and displaces the colder fluid at the top forcing it to flow downward.
- The resulting flow eventually forms a pattern called Benard cells.



https://people.duke.edu/~av8/vandongen_lab/selforganization/SO_Figures/figure%201%20-%20Benard%20cells.html



https://www.mis.mpg.de/applan/research/rayleigh.html

Rayleigh-Benard Convection

- For 2D incompressible flows, the governing equations in nondimensionless form are

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

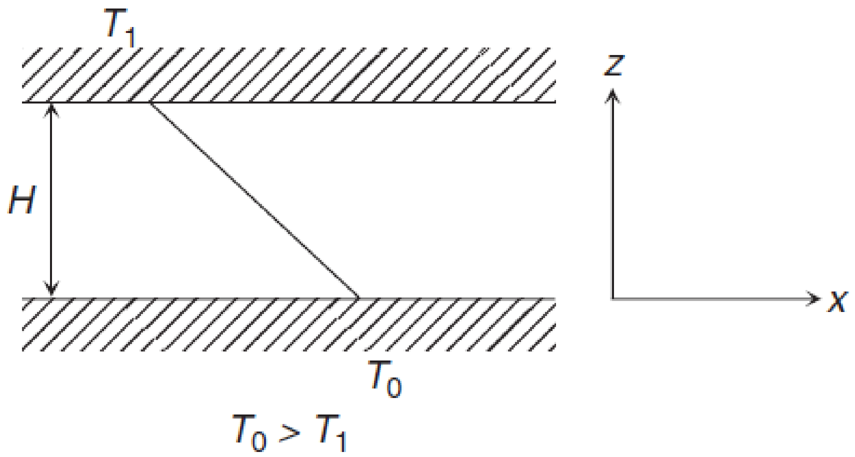
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \mathrm{Pr}\,\nabla^2 u$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \mathrm{Pr}\,\nabla^2 w + \mathrm{Ra}\mathrm{Pr}\,\theta$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} = \nabla^2 \theta$$

$$\mathrm{Ra} = \frac{g\alpha\Delta T H^3}{\kappa\nu}$$

$$\mathrm{Pr} = \frac{\nu}{\kappa}$$



Rayleigh-Benard Convection

- The Rayleigh number Ra is the ratio between buoyant energy production and energy diffusion.
- The Prandtl number Pr is the ratio between the viscous diffusion and thermal diffusion.
- g = gravitational acceleration
- α = thermal expansion coefficient of the fluid
- κ = thermal diffusivity of the fluid
- $\Delta T = T_0 - T_1$ is temperature difference
- ν = kinematic viscosity of the fluid
- H = height of the horizontal fluid

Biringen and Chow (2011, p.185)

Rayleigh-Benard Convection

- The dimensionless quantities u, w, x, z, θ are defined as

$$u = \frac{L}{\kappa}U, \quad w = \frac{L}{\kappa}W, \quad x = \frac{X}{H}, \quad z = \frac{Z}{H}$$

$$\theta = \frac{(T - T_o)}{(T_1 - T_o)}, \quad p = \frac{H^2}{\kappa^2 \rho_o}P, \quad t = \frac{\kappa}{H^2}\tau$$

  where U, W, X, Z, T, P, τ are dimensional quantities and ρ_0 is the reference fluid density.
- Assume that the instabilities due to buoyancy are small and that the initial state is quiescent.
- "The dependent variables can be separated into a mean component that represents the initial conditions (base state), and a time-dependent component that represents the perturbation field."

Biringen and Chow (2011, p.186)

Rayleigh-Benard Convection

- T

$$
\begin{aligned}
u &= 0 + \hat{u}(x, z, t) \\
w &= 0 + \hat{w}(x, z, t) \\
p &= \overline{p}(z) + \hat{p}(x, z, t) \\
\theta &= \overline{\theta}(z) + \hat{\theta}(x, z, t)
\end{aligned}
$$

Biringen and Chow (2011, p.186)

References

- S. Biringen and C.Y. Chow, 2011, An Introduction to Computational Fluid Mechanics by Example, John Wiley and Sons.