Computational Fluid Dynamics

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Fluids

- A fluid is a substance with no shear strength, i.e., its shear modulus is zero.
- Although a fluids is composed of a large number of molecules, we can approximately treat a fluid as a continuous substance.
- A flow of a fluid is due to an external force such as a pressure difference, gravity, wind, and surface tension.
- External forces can be classified as surface forces and body forces.
- The most important fluid properties are mass density and viscosity.
- Other fluid properties affect fluid flows only under some conditions.

Ferziger et al. (2020)

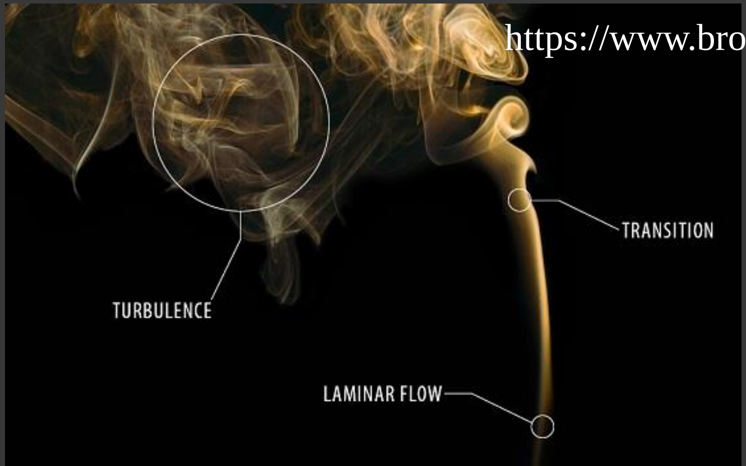
Creeping Flows or Stokes Flows

- The flow speed is low enough that inertia forces are small compared to viscous forces, i.e., the Reynolds number Re << 1.
- The Reynolds number is the ratio of inertial forces to viscous forces.
- The Reynolds number is defined as $\mathrm{Re} = uL/\nu = \rho uL/\mu$ where ρ is the fluid density, u is the flow speed, L is the characteristic length, μ is dynamic viscosity, and $\nu = \mu/\rho$ is kinematic viscosity.
- This flow regime is important in flows with small particles or flows in porous media.
- Creeping flows are governed by the Stokes equations, a linearized, steady-state version of the Navier-Stokes equations.

Ferziger et al. (2020)

Laminar and Turbulent Flows

- Laminar flows: At a larger flow speed such that the inertia is not negligible and fluid particles still follow smooth trajectories.
- In the laminar flow regime the Reynolds number is smaller than a critical value beyond which the flow becomes turbulent: $Re < Re_{critical}$.
- Turbulent flows: When the flow speed is so large that an instability occurs, various random flows could happen.
- Laminar-turbulent transition: A transition from laminar flows to turbulent flows occur when the Reynolds number is in a certain range specific to the situation.

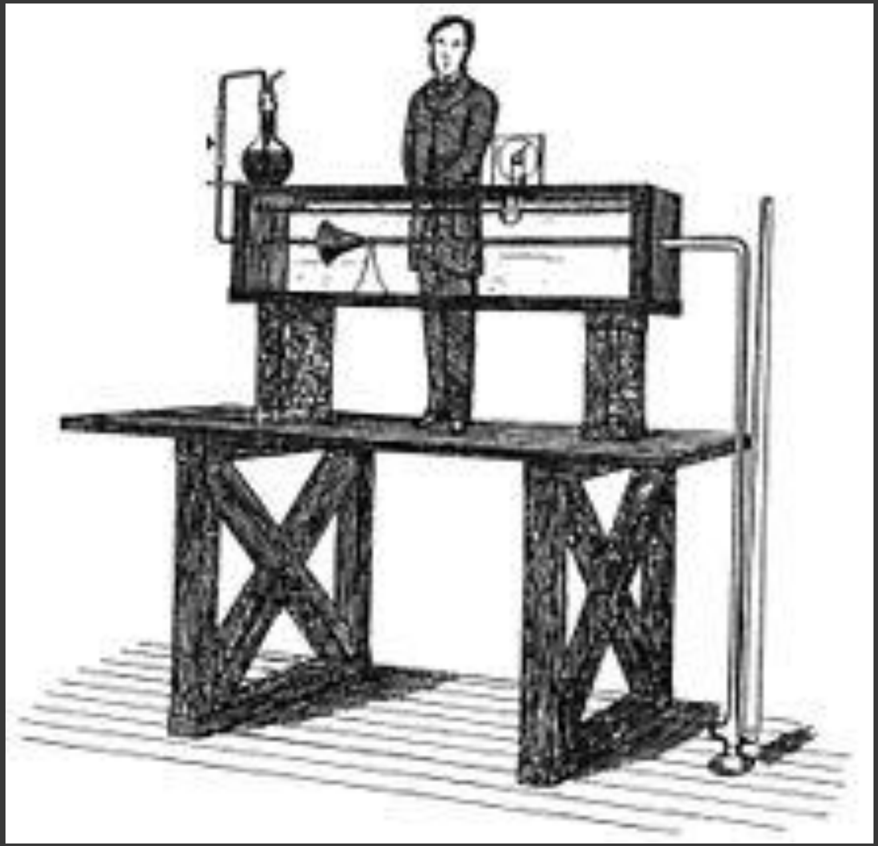


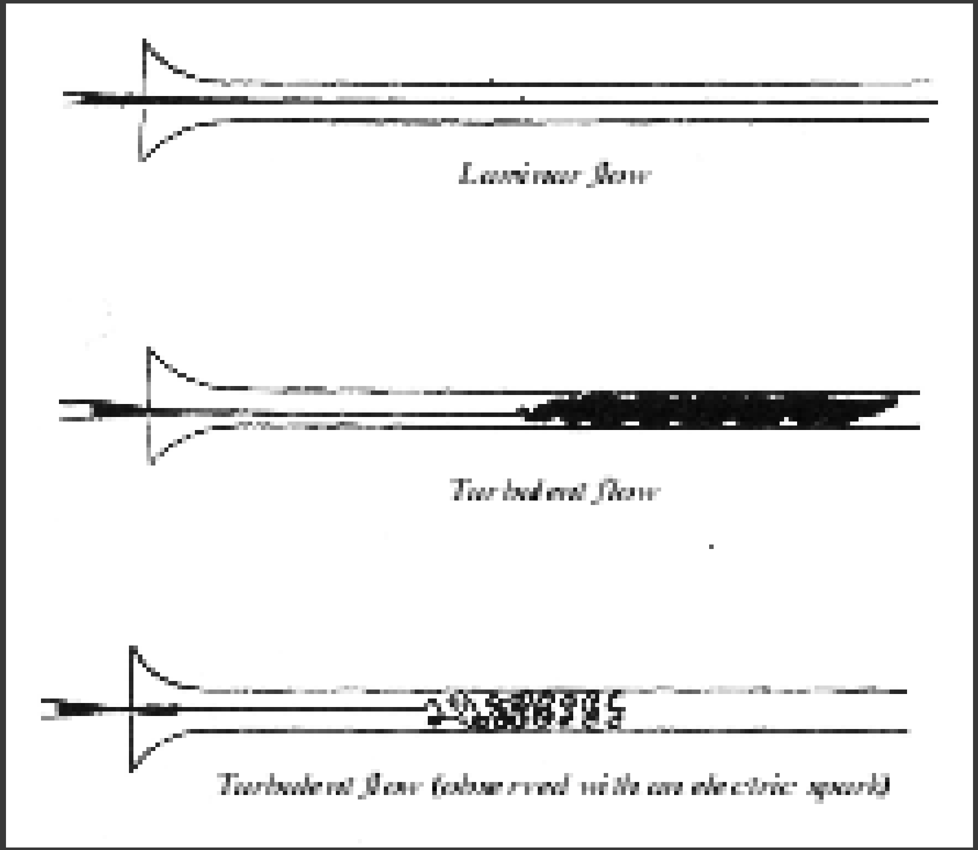
https://www.bronkhorst.com/int/blog-1/what-is-the-difference-between-laminar-flow-and-turbulent-flow/

Ferziger et al. (2020)

Reynolds' Experiment

- In 1883, Osborne Reynolds varied the flow rate of a dyed water jet to study the behavior of water flow.
- The laminar-turbulent transition occurs when 2000 < Re < 13000.

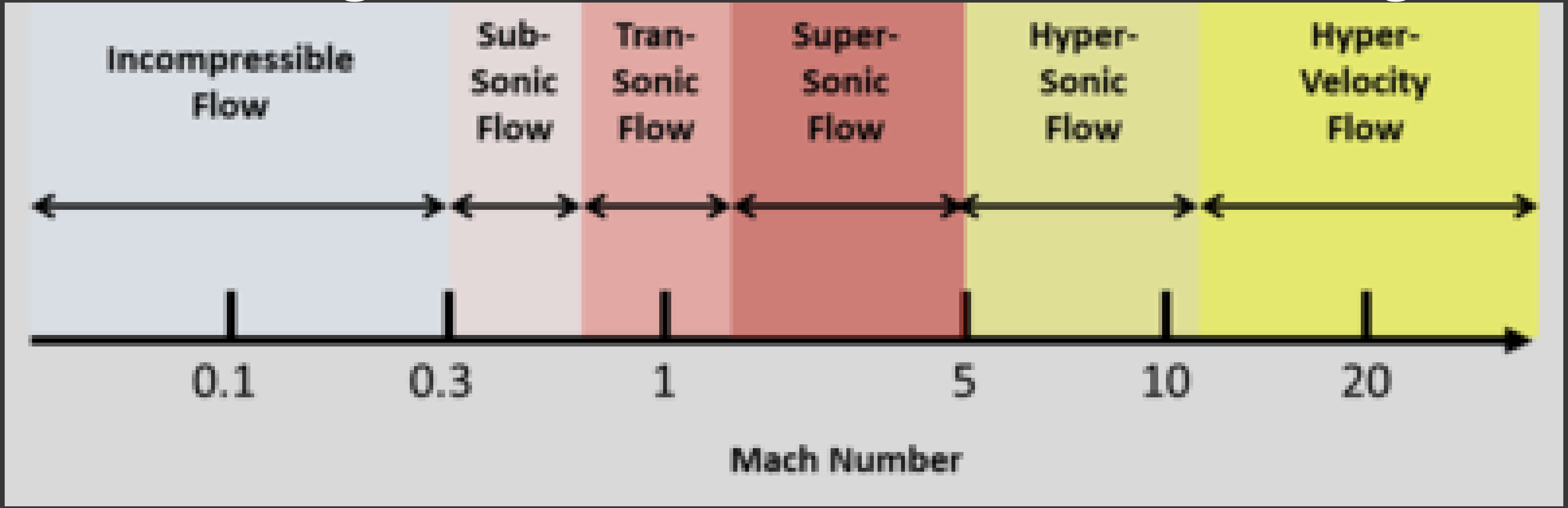




https://en.wikipedia.org/wiki/Laminar%E2%80%93turbulent_transition

Mach Number Flow Regimes

- The Mach number Ma is the ratio of the local flow speed u to the local sound speed c: Ma = u/c.
- Incompressible flows (fluid density is considered a constant): Ma < 0.3
- Compressible flows: Ma ≥ 0.3
- Hypersonic flows: When Ma > 5, "the compression may create high enough temperatures to change the chemical nature of fluid." Ferziger et al. (2020)



https://en.wikipedia.org/wiki/Compressible_flow#Mach_number,_wave_motion,_and_sonic_speed

Free Falling of a Spherical Body

- A sphere of diameter d, density ρ, and mass m is dropped from rest in a fluid of density ρ_f and kinematic viscosity ν.
- Let the origin be at the center of the sphere and the z-axis is the vertical axis pointing downward in the direction of the gravitational acceleration g.
- While the sphere is moving, various forces acting on it include:
  - the buoyant force $-m_f g$ where $m_f = \pi d^3 \rho_f / 6$
  - the force on an accelerating body: a body moving in a fluid induces a flow field leading to a drag even if the fluid is frictionless. This drag was derived from an inviscid theory and its value is $-\frac{1}{2} m_f \, dv/dt$
  - the forces caused by viscosity are equal to $-\frac{1}{8}\pi c_d \rho_f d^2 |v| v$ where c_d is the drag coefficient which depends on the velocity v.
  - the wave drag due to shock waves developed ahead of the body.

Biringen and Chow (2011)

Free Falling of a Spherical Body

- Suppose the sphere only moves at low subsonic speeds.
- In this case, the wave drag can be omitted and the equation of motion becomes

$$m\frac{dv}{dt} = mg - m_f g - \frac{1}{2}m_f\frac{dv}{dt} - \frac{1}{8}\pi c_d(v)\rho_f d^2|v|v$$

$$\left(m + \frac{1}{2}m_f\right)\frac{dv}{dt} = \left(m - m_f\right)g - \frac{1}{8}\pi c_d(v)\rho_f d^2|v|v$$

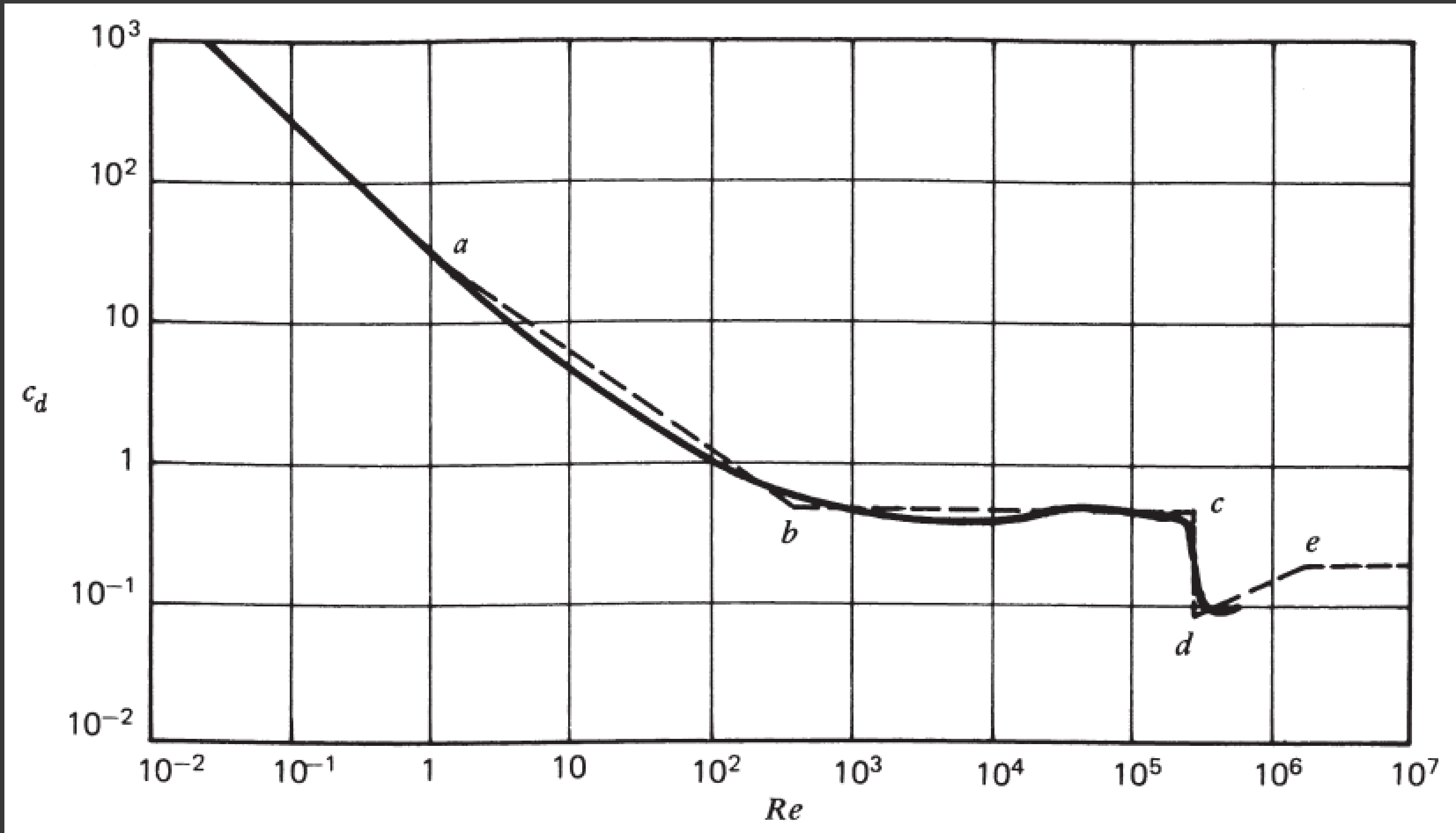
- While the sphere moves with a nonzero acceleration, it behaves as if it mass was increased. The term $\frac{1}{2}m_f$ is referred to as the added mass.
- Substituting $m = \frac{1}{6}\pi d^3\rho$ into the equation of motion and rearranging it yield

$$\frac{dv}{dt} = \frac{1}{A}\left[B - Cc_d(v)|v|v\right], A = 1 + \frac{1}{2}\bar{\rho}, B = \left(1 - \bar{\rho}\right)g, C = \frac{3\bar{\rho}}{4d}, \bar{\rho} = \frac{\rho_f}{\rho}$$

Biringen and Chow (2011)

Free Falling of a Spherical Body

- The drag coefficient c_d is usually determined through experiment.
- The graph of c_d for smooth spheres at various Re is shown in the figure.



- To perform numerical calculations, c_d is approximated to be piecewise linear.
- When Re $\leq$ 1, c_d = 24/Re
- The Reynolds number is

$$\mathrm{Re} = \frac{ud}{\nu}, u = \left| v_f - v \right|$$

where v_f is fluid velocity.

Biringen and Chow (2011)

Free Falling of a Spherical Body

- Here, we consider the dynamics of the sphere only.
- So, the Reynolds number is approximated as Re = $|v|d/\nu$.

$$c_d = \frac{24}{\text{Re}} = \frac{24\nu}{|v|d}, \qquad \text{Re} \le 1$$

$$c_d = \frac{24}{\text{Re}^{0.646}} = 24\left(\frac{\nu}{|v|d}\right)^{0.646}, \qquad 1 < \text{Re} \le 400$$

$$c_d = 0.5, \qquad 400 < \text{Re} \le 3\times10^5$$

$$c_d = 0.000366\,\text{Re}^{0.4275}, \qquad 3\times10^5 < \text{Re} \le 2\times10^6$$

$$c_d = 0.18, \qquad \text{Re} > 3\times10^5$$

Biringen and Chow (2011)

Method for Solving Initial-Value Problem

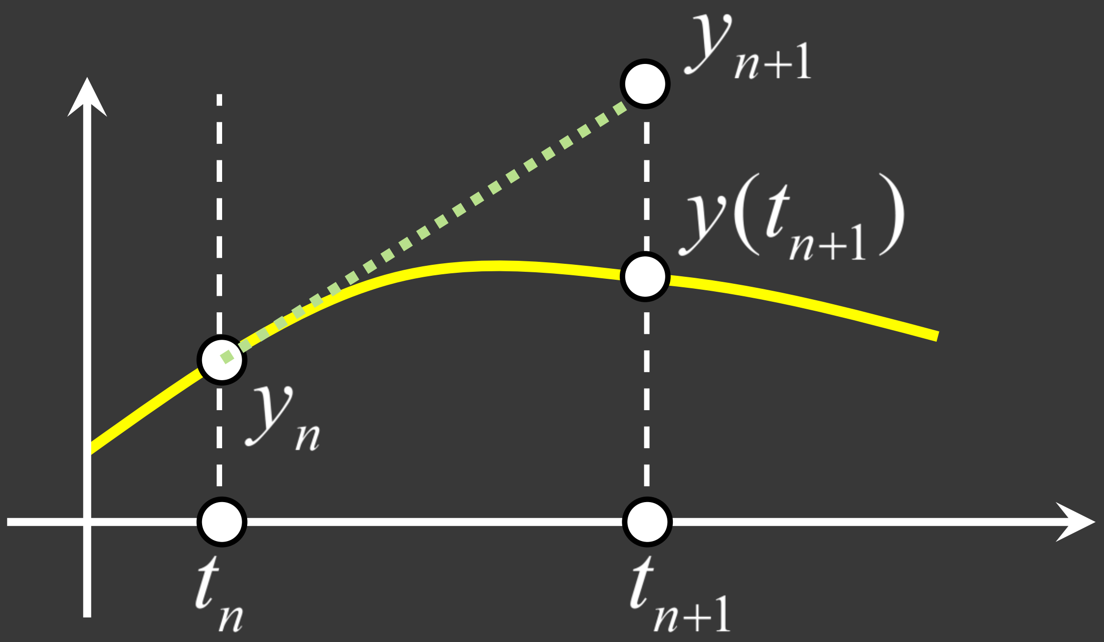
- Consider an initial-value problem $y'(t) = f(t, y), \quad y(0) = y_0$
- Integrating the ODE: $y_{n+1} - y_n = \int_{t_n}^{t_{n+1}} f(t, y)dt, \;\; y_n = y(t_n), y_{n+1} = y(t_n + h)$
- Let's approximate the integral using the slope function at time t_1 as

$$\int_{t_n}^{t_{n+1}} f(t, y)dt \approx \int_{t_n}^{t_{n+1}} f(t_n, y_n)dt = hf(t_n, y_n), \quad t_{n+1} - t_n = h$$

- We then obtain the recursive scheme $y_{n+1} = y_n + hf(t_n, y_n)$
- When $n = 0$, $y(h) = y_1 = y_0 + hf(t_0, y_0)$
- When $n = 1$, $y(2h) = y_2 = y_1 + hf(t_1, y_1)$
- The recursive scheme can be used to compute the solution forward in time.
- This recursive scheme is called the Euler method.

Euler Method

- The Euler method can be illustrated using the diagram below.



$$y_{n+1} = y_n + h_n f\left(t_n, y_n\right)$$

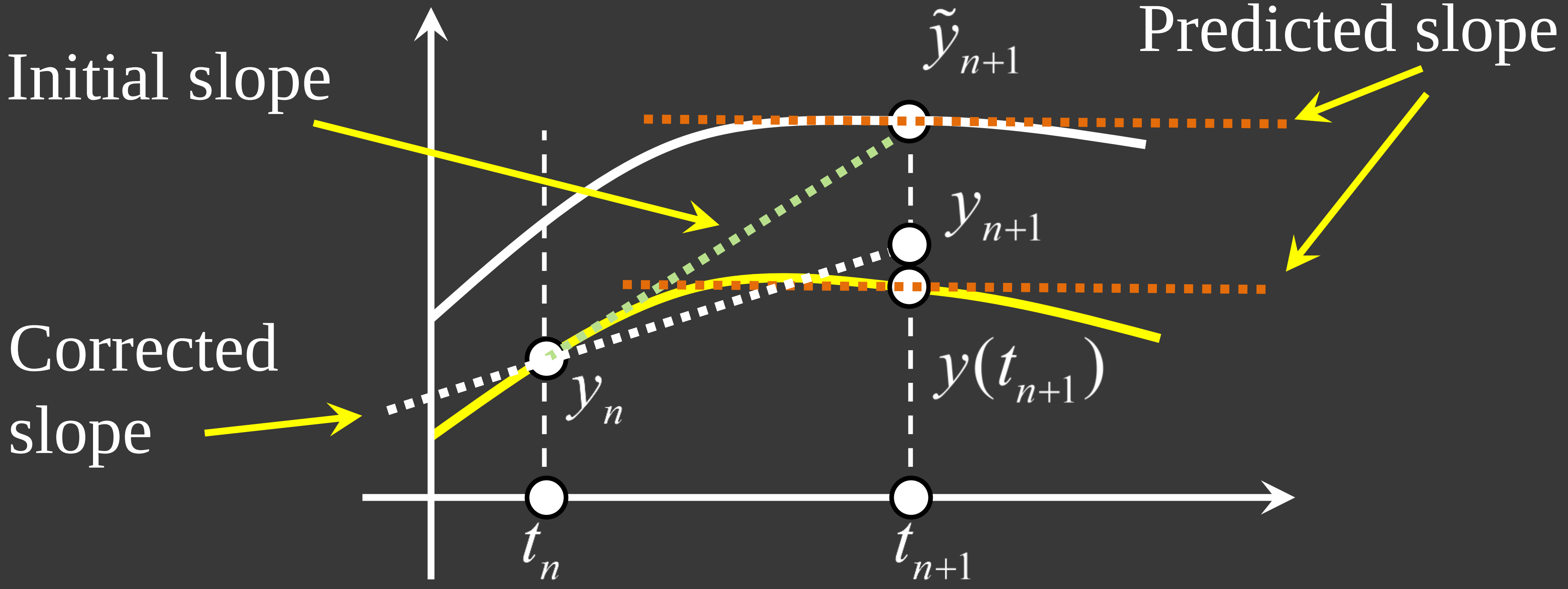
- The Euler method is easy to apply but only provides a first-order accurary.
- The Heun method presented in the next slide has a second-order accurary and; thus, provides a more accurate solution.

Heun Method

The Heun method is given by

$$\tilde{y}_{n+1} = y_n + h_n f\left(t_n, y_n\right) \qquad \text{Predictor}$$

$$y_{n+1} = y_n + h_n \frac{f\left(t_n, y_n\right) + f\left(t_{n+1}, \tilde{y}_{n+1}\right)}{2} \qquad \text{Corrector}$$



Exercise

- Suppose the sphere has d = 0.01 m and ρ = 8000 kg/m^3, and the fluid is the air with ρ_f = 1.22 kg/m^3 and ν = 1.49×10^{-5} m^2/s, and g = 9.8 m/s^2.
- Solve the equations of motion

$$\frac{dv}{dt} = \frac{1}{A}\left[B - Cc_d(v)\left|v\right|v\right], A = 1 + \frac{1}{2}\bar{\rho}, B = \left(1 - \bar{\rho}\right)g, C = \frac{3\bar{\rho}}{4d}, \bar{\rho} = \frac{\rho_f}{\rho}$$

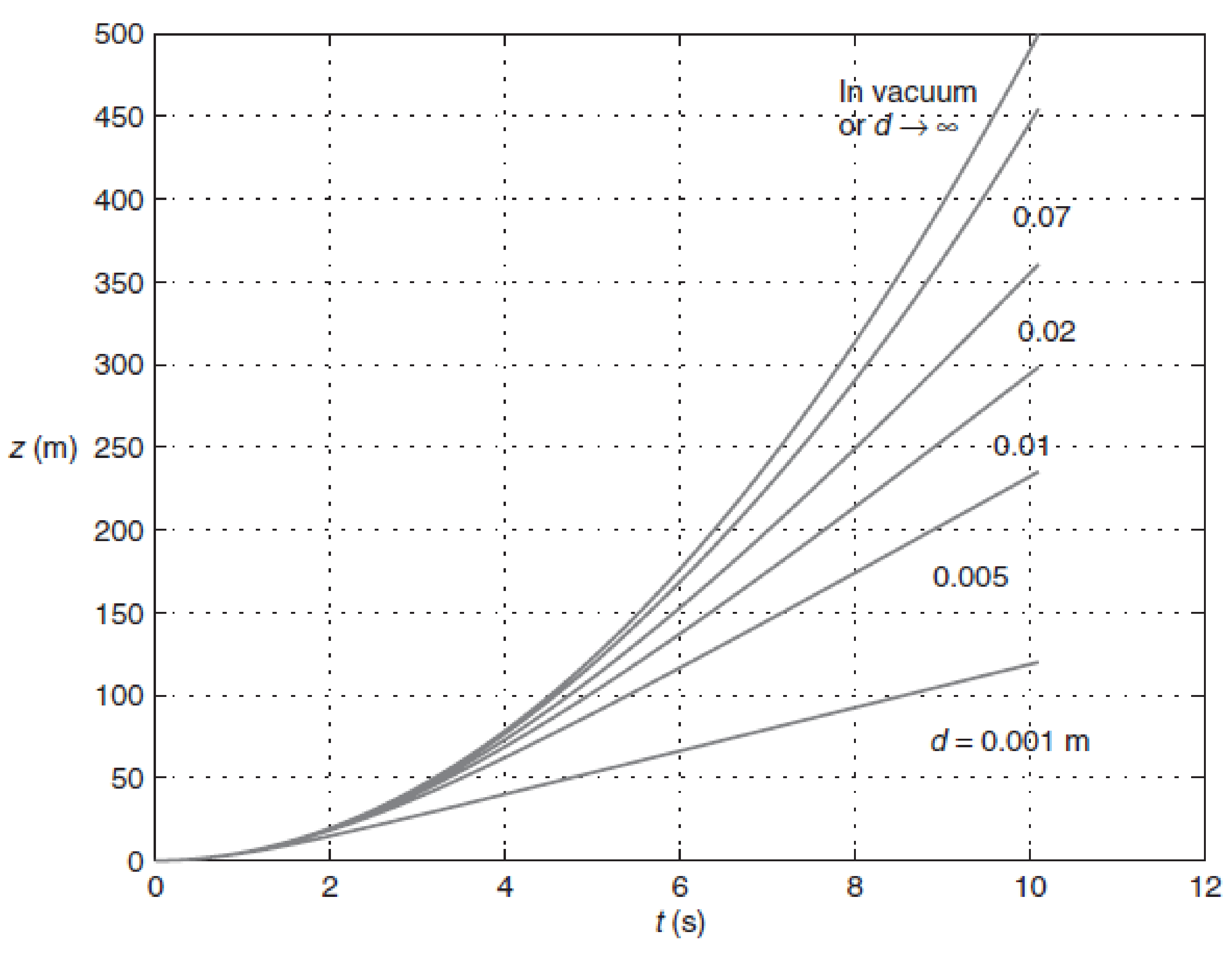
$$\frac{dz}{dt} = v$$

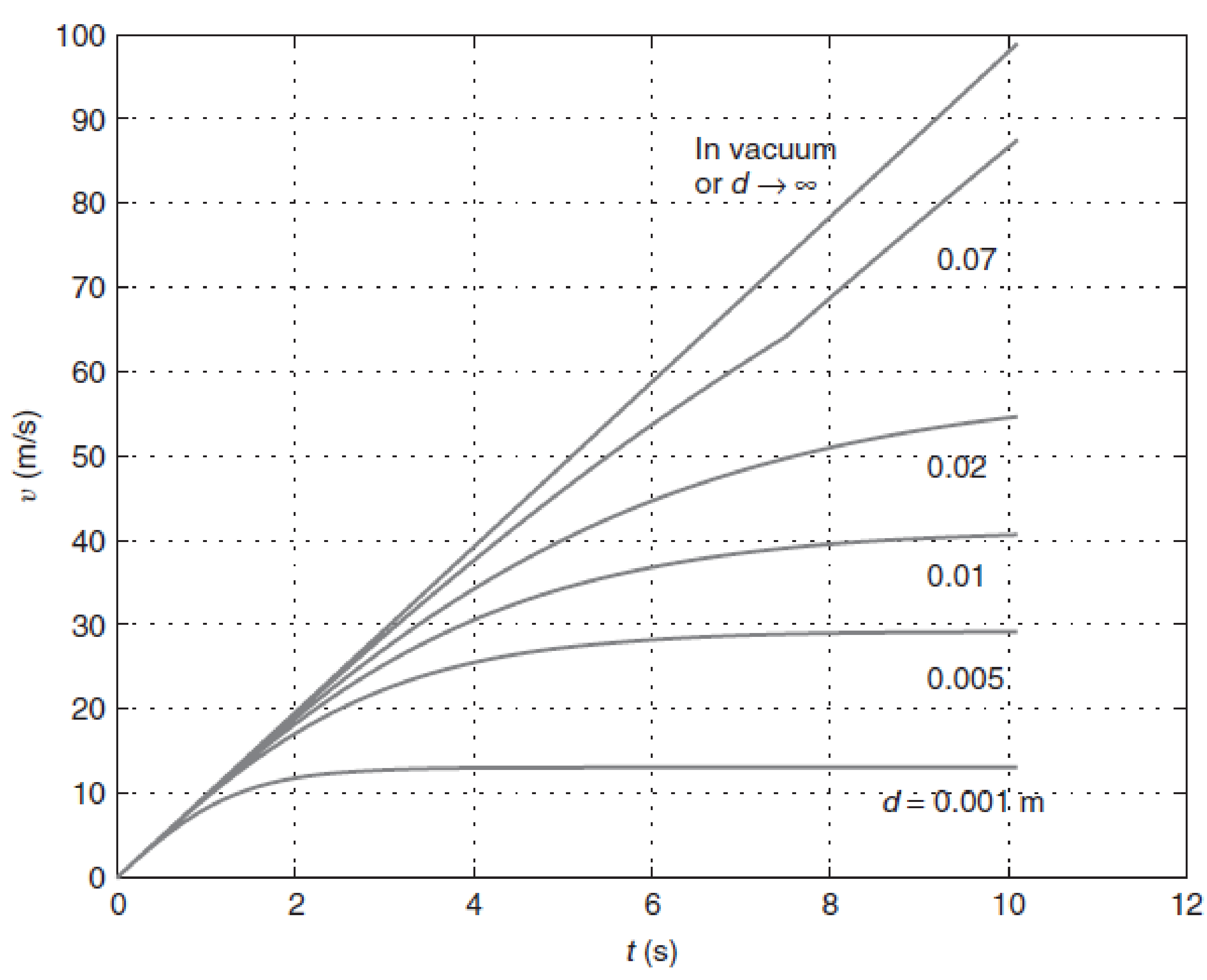
  using the Euler method and the Heun method.
- Compare your result against that obtained using the MATLAB function `ode45`.

Biringen and Chow (2011)

Exercise

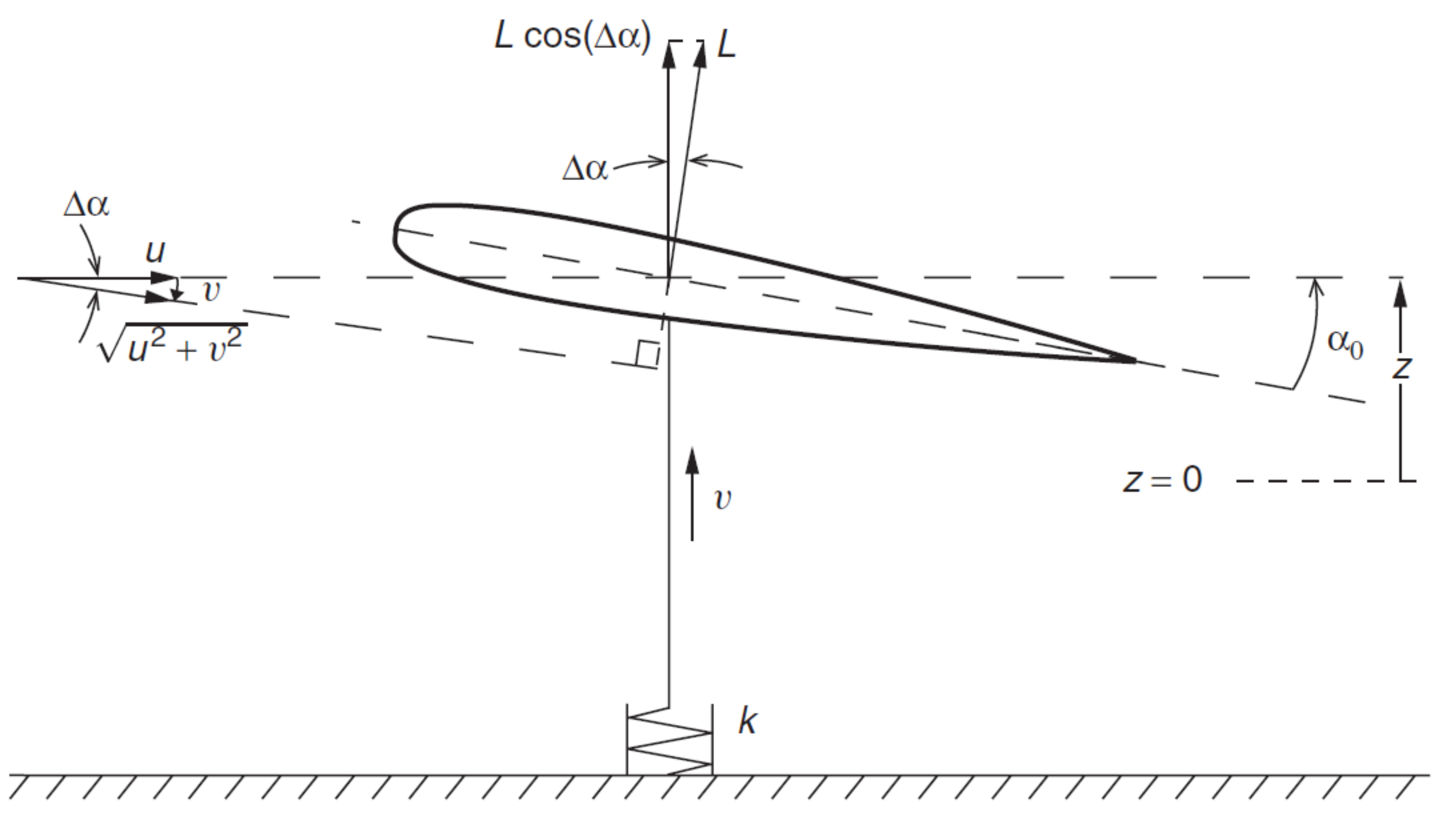
Graphs of $z(t)$ and $v(t)$ for various values of d.





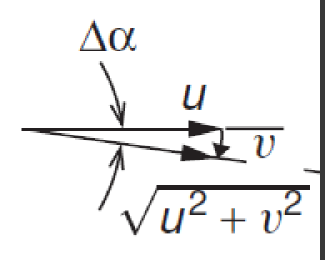
Aeroelastic System

- A symmetric airfoil of mass m is installed in a wind tunnel and supported by a spring of spring constant k.
- Without a wind, the airfoil makes an angle α_0 with the horizontal and its center of mass under a state of equilibrium is at a height $z = 0$.
- The airfoil can only move vertically.



Biringen and Chow (2011, p. 17)

Aeroelastic System



- "The wind tunnel supplies a uniform horizontal wind of speed *u*."
- "When the airfoil moves upward at a speed *v*, the surrounding air moves downward relative to the wind at the same speed."
- "To the oncoming air flow, the attack angle of the airfoil is"

$$\alpha = \alpha_0 - \Delta\alpha = \alpha_0 - \tan^{-1}(v/u)$$

- "The lift of the airfoil, in the direction normal to the flow, has the expression

$$L = \frac{1}{2} c_l \rho \left(u^2 + v^2\right) S$$

where ρ_f is air density, S the projected wing area, and c_l the lift coefficient of the airfoil."
- "According to the thin-airfoil theory, the lift coefficient is a linear function of the attack angle."

Biringen and Chow (2011, p. 17)

Aeroelastic System

- For symmetric airfoil, the lift coefficient is $c_l = 2\pi\alpha$
- "The theory agrees with the experiment if α is within a certain limit."
- "Beyond the limit the airfoil no longer behaves like a thin body, and the flow separates from the surface."
- Suppose that the limit is $-18^\circ \le \alpha \le 18^\circ$ and that beyond this the lift is zero.
- The equation of motion for the center of mass is

$$\frac{dz}{dt} = v$$

$$m\frac{dv}{dt} = -kz + L\cos(\Delta\alpha) = -kz + \frac{1}{2}\rho_f S c_l \frac{u}{\sqrt{u^2+v^2}}$$

- It is more economical to solve this system in a nondimensionalized form.

Biringen and Chow (2011, p. 18)

Aeroelastic System

- The first step is to find reference quanties: time length, velocity.
- A reference time can be the period of free oscillation of the system: $2\pi\sqrt{m/k}$
- A reference length can be the deformation of spring due to the weight of the airfoil: mg/k.
- A reference velocity can be the ratio of the reference length and reference time: $(g/2\pi)\sqrt{m/k}$
- The following dimensionless variables are defined:

$$T = \frac{t}{2\pi\sqrt{m/k}}, \quad Z = \frac{z}{mg/k}, \quad U = \frac{u}{(g/2\pi)\sqrt{m/k}}, \quad V = \frac{v}{(g/2\pi)\sqrt{m/k}}$$

Biringen and Chow (2011, p. 18)

Aeroelastic System

- Substituting $t = 2\pi T\sqrt{\frac{m}{k}}, z = \frac{Zmg}{k}, u = \frac{g}{2\pi}U\sqrt{\frac{m}{k}}, v = \frac{g}{2\pi}V\sqrt{\frac{m}{k}}$ into the equation of motions yields

$$\frac{dZ}{dT} = V, \qquad \frac{dV}{dt} = -(2\pi)^2 Z + \beta c_l U\sqrt{U^2 + V^2}$$

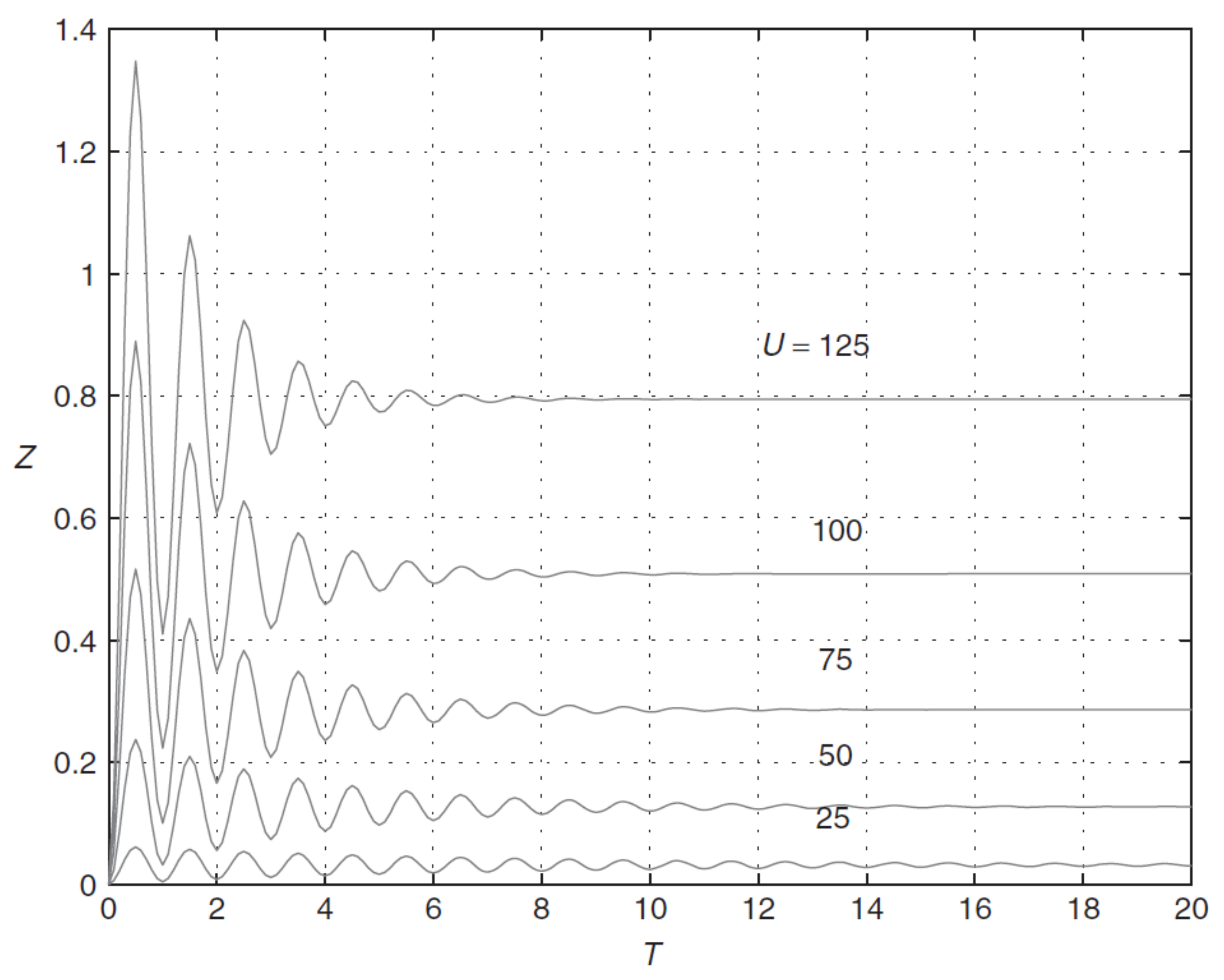
where $\beta = \rho_f gS/2k$ is a dimensionless number.

- The 6 parameters (ρ_f, g, m, S, k, u) reduce to only 2 dimensionless numbers β and U.
- Assume that ρ_f = 1.22 kg/m^3, g = 9.8 m/s^2, m = 3 kg, S = 0.3 m^2, k, = 980 kg/s^2, α_0 = 10°.
- Thus, β = 0.00183, reference time = 0.348 s, reference length = 0.03 m, reference velocity = 0.0862 m/s.

Biringen and Chow (2011, p. 18)

Aeroelastic System

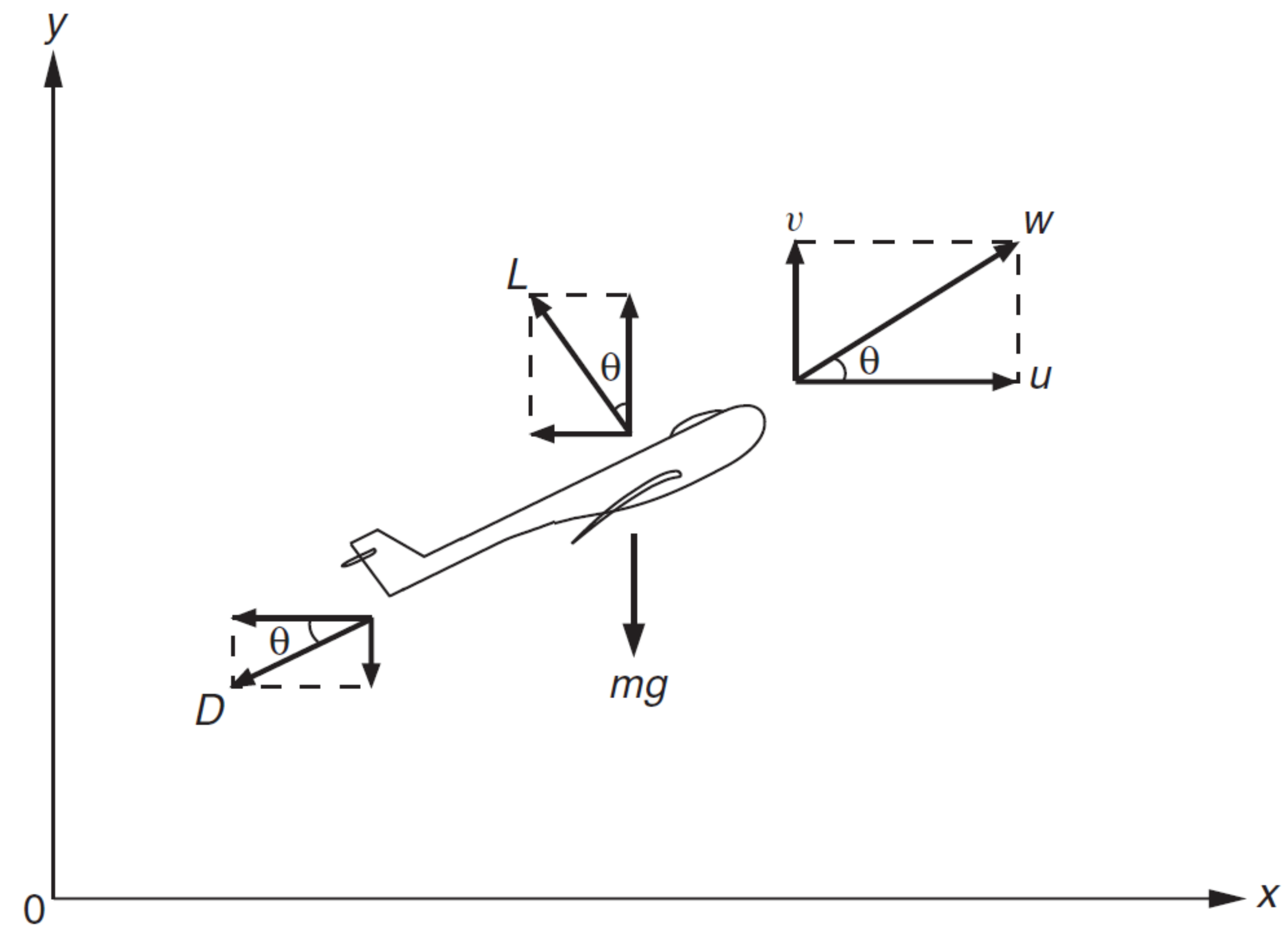
The displacement of the airfoil with 5 different wind speeds are shown below.



Biringen and Chow (2011, p. 19)

Flight Path of a Glider

- "A glider of mass *m* flying at a velocity *w*, which makes an angle θ with the horizontal *x* axis."
- "The aerodynamic forces acting on the glider in the directions normal and parallel to the flight path are called the life *L* and drag *D*, respectively."
- Let's *u* and *v* be the horizontal and vertical components of the velocity *w*, respectively.



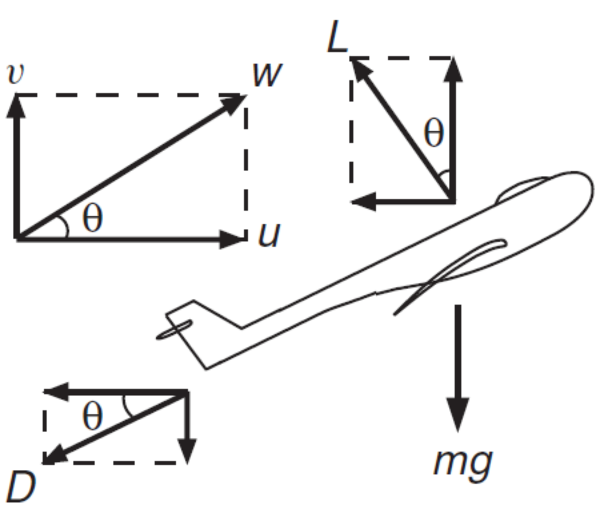
Biringen and Chow (2011, p. 32)

Flight Path of a Glider

- Ignoring the added mass and the angular motion, the equations of motion for the center of mass of the glider are given by

$$m\frac{du}{dt} = -L\sin\theta - D\cos\theta$$

$$m\frac{dv}{dt} = -mg + L\cos\theta - D\sin\theta$$



- Note that $\sin\theta = v/w$ and $\cos\theta = u/w$.
- The life and drag of the glider can be expressed as

$$L = c_l \frac{1}{2}\rho w^2 S, \qquad D = c_d \frac{1}{2}\rho w^2 S$$

where c_l and c_d are the lift and drag coefficients, ρ the air density, S the projected wing area.

Biringen and Chow (2011, p. 32)

Flight Path of a Glider

- "Suppose at an initial instant $t = 0$ the velocity is w_0 and the inclination angle is θ_0."
- "Using w_0 as the reference velocity, w_0/g as the reference time, and w_0^2/g as the reference length, we can construct dimensionless velocity components U, V, dimensionless time T, and dimensionless coordinates X, Y."
- We then obtain the nondimensional form of the equations of motion as

$$\frac{d^2X}{dt^2} = -A\sqrt{U^2+V^2}\left(BU+V\right)$$

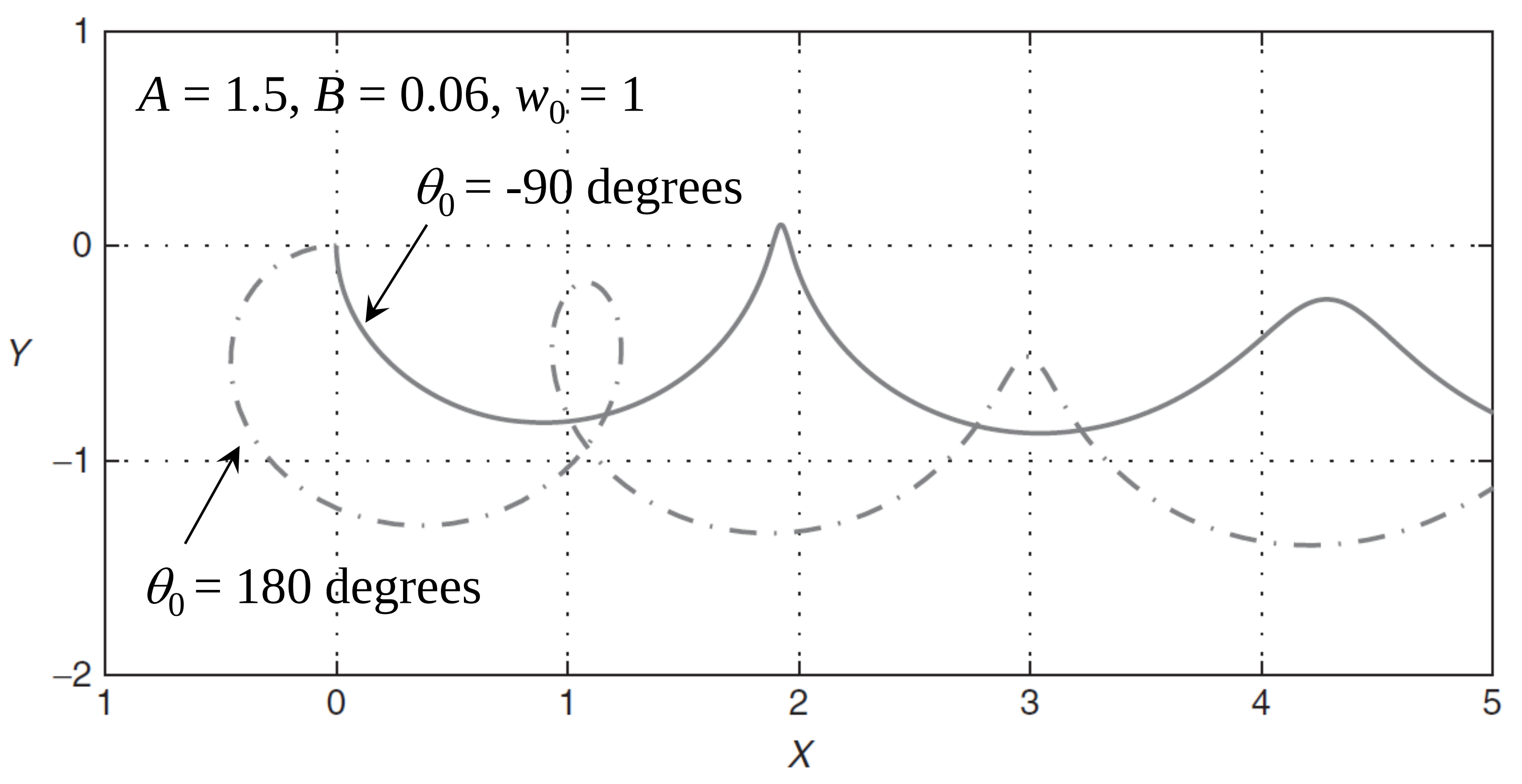
$$\frac{d^2Y}{dt^2} = -1 + A\sqrt{U^2+V^2}\left(U-BV\right)$$

- Here, $U = \mathrm{d}X/\mathrm{d}T$, $V = \mathrm{d}Y/\mathrm{d}T$, $A = \dfrac{1}{2}\dfrac{c_l \rho w_0^2 S}{mg}, B = \dfrac{c_d}{c_l}$

Biringen and Chow (2011, p. 33)

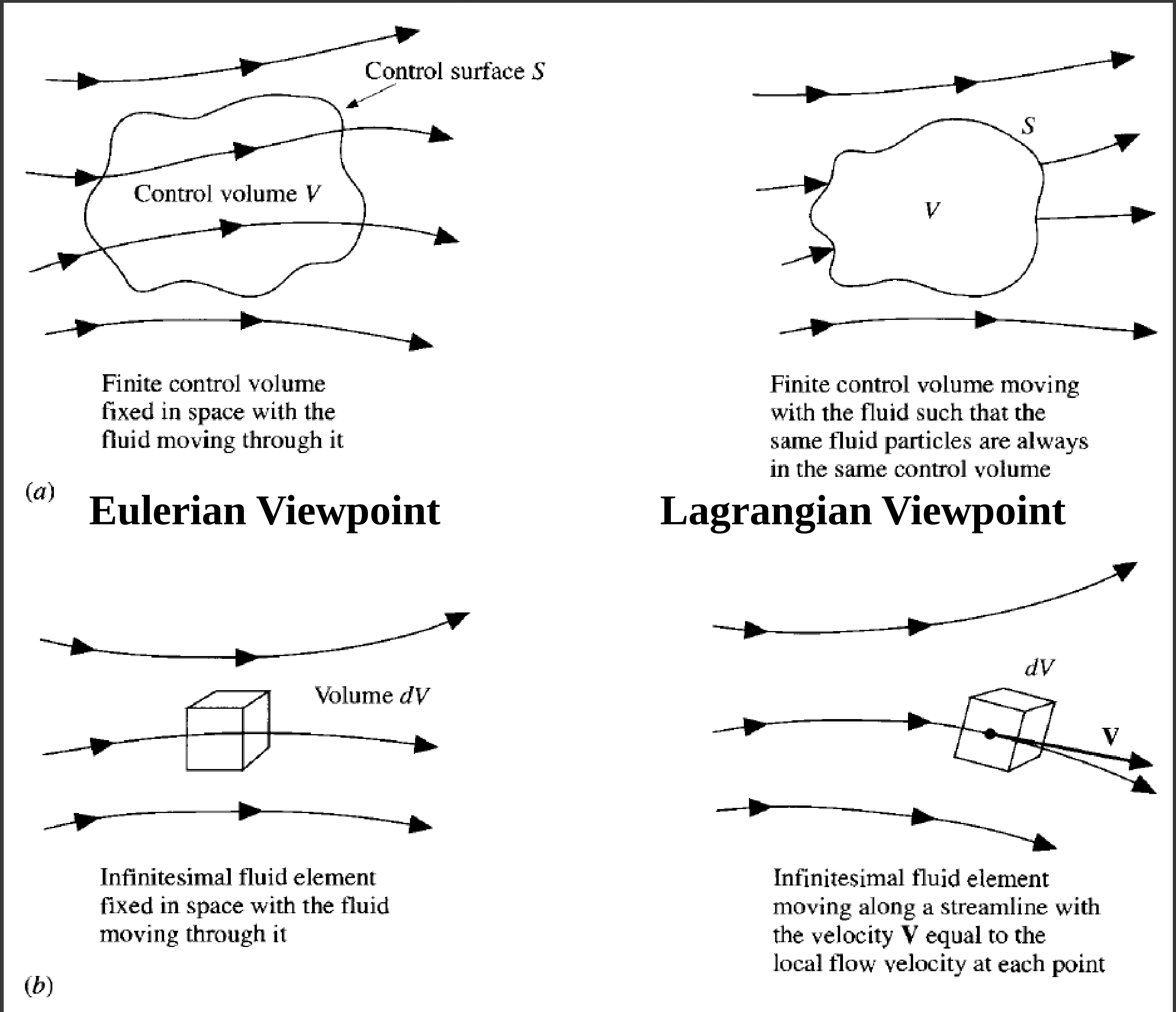
Flight Path of a Glider

- "Since the lift and drag coefficients are functions of the attack angle, both *A* and *B* are generally variables."
- In this example, we consider *A* and *B* to be constant for simplicity.
- This is equivalent to the case where the glider has wings fixed at a constant attack angle.



Biringen and Chow (2011, p. 33)

Eulerian and Lagrangian Viewpoints



Anderson (1995, p. 41)

Material Derivative

- The material derivative is a total derivative that represents time rate of change due to both local change and convective change.

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \rho}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \rho}{\partial z}\frac{\partial z}{\partial t}$$

$$= \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}u + \frac{\partial \rho}{\partial y}v + \frac{\partial \rho}{\partial z}w$$

$$= \frac{\partial \rho}{\partial t} + \left(\mathbf{v}\cdot\nabla\right)\rho$$

$$\frac{D}{Dt}\rho = \left[\frac{\partial}{\partial t} + \left(\mathbf{v}\cdot\nabla\right)\right]\rho$$

Time Rate of Change

Eulerian viewpoint

$$\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)$$

Lagrangian viewpoint

$$\frac{D}{Dt}$$

Local derivative

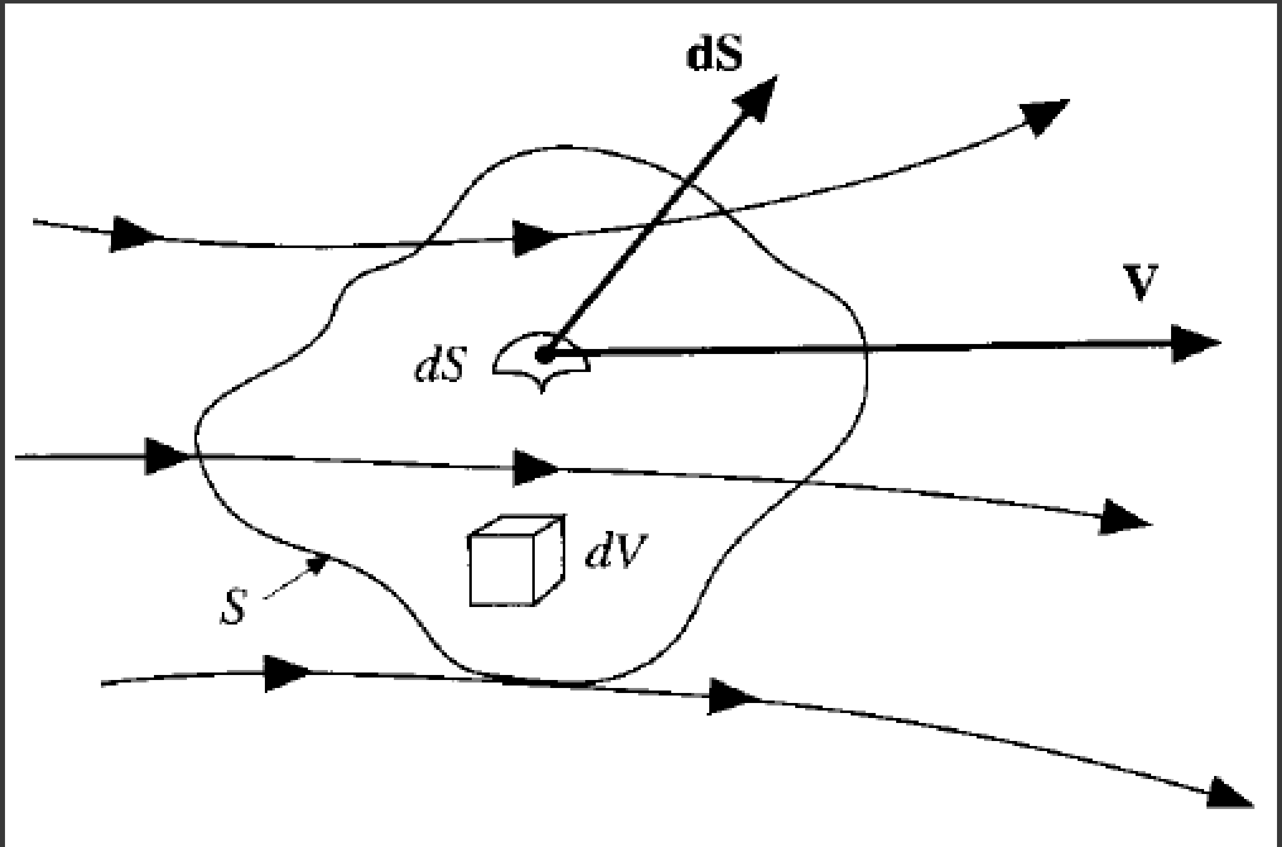
Convective derivative

Scalar quantity: $$\frac{\partial \rho}{\partial t}+(\mathbf{v}\cdot\nabla)\rho=\frac{D\rho}{Dt}$$

Vector quantity: $$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\frac{D\mathbf{v}}{Dt}$$

Continuity Equation: Eulerian Viewpoint

- Consider a finite control volume fixed in space
- Net mass flow out of control volume through surface S is equal to time rate of decrease of mass inside control volume.



Anderson (1995)

Continuity Equation: Eulerian Viewpoint

- Mass flow across a small fixed surface **dS** is $\rho \mathbf{v} \cdot \mathbf{dS}$
- The sign of $\rho \mathbf{v} \cdot \mathbf{dS}$ is positive for an outflow, and is negative for an inflow.

- Net mass flow out of the control volume through surface S is $\iint_S \rho \mathbf{v} \cdot \mathbf{dS}$

- The total mass in the control volume is $\iiint_V \rho dV$

- The time rate of increase of mass is then $\frac{\partial}{\partial t} \iiint_V \rho dV$

- Applying the conservation of mass principle, we obtain the conservative form of the continuity equation

$$\frac{\partial}{\partial t} \iiint_V \rho dV + \iint_S \rho \mathbf{v} \cdot \mathbf{dS} = 0$$

Continuity Equation: Eulerian Viewpoint

- The integral form of the continuity equation is

$$\frac{\partial}{\partial t}\iiint_V \rho dV + \iint_S \rho \mathbf{v}\cdot \mathbf{dS} = 0$$

- Applying the divergence theorem to the second term, the surface integral becomes a volume integral and the continuity equation becomes

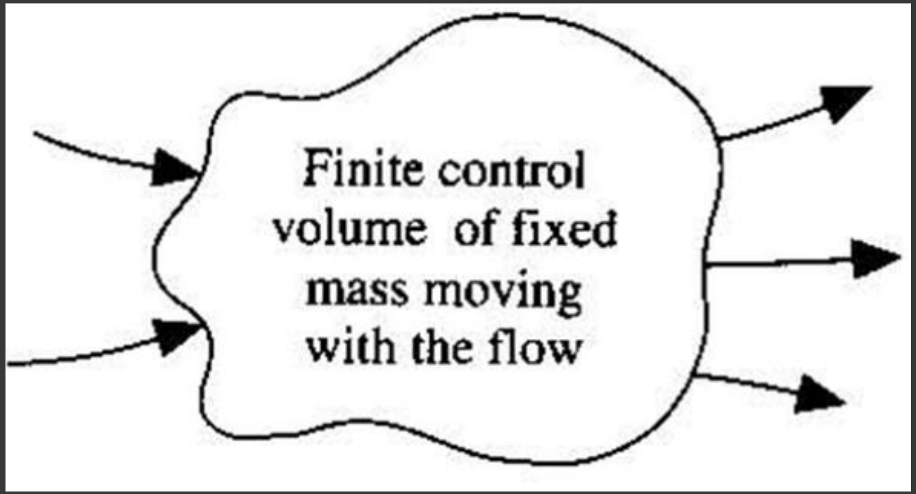
$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right) dV = 0$$

- Since the volume integral is zero, the integrand must vanish.
- So, we obtain the differential form of the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Continuity Equation: Lagrangian Viewpoint

- Let's consider a finite control volume moving with the fluid.
- The total mass of thefinite control volume is $m = \iiint_V \rho dV$
- Since the total mass of finite control volume is always the same, we obtain the nonconservative form of the continuity equation



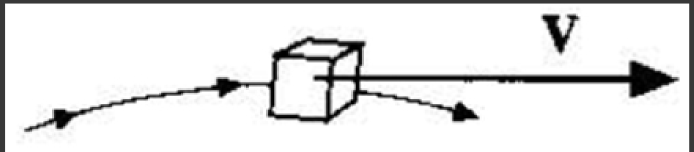
$$\frac{D}{Dt}\iiint_V \rho dV = 0$$

Anderson (1995)

Continuity Equation: Differential Form

Let's consider an infinitesimally small fluid element moving with the flow.

The mass of the fluid element is $\delta m = \rho \delta V$



Since the time rate of change of the mass of fluid element is zero, we obtain which leads to

$$\frac{D(\delta m)}{Dt} = 0$$

$$\frac{D(\delta m)}{Dt} = \frac{D(\rho \delta V)}{Dt} = \delta V \frac{D\rho}{Dt} + \rho \frac{D(\delta V)}{Dt} = 0$$

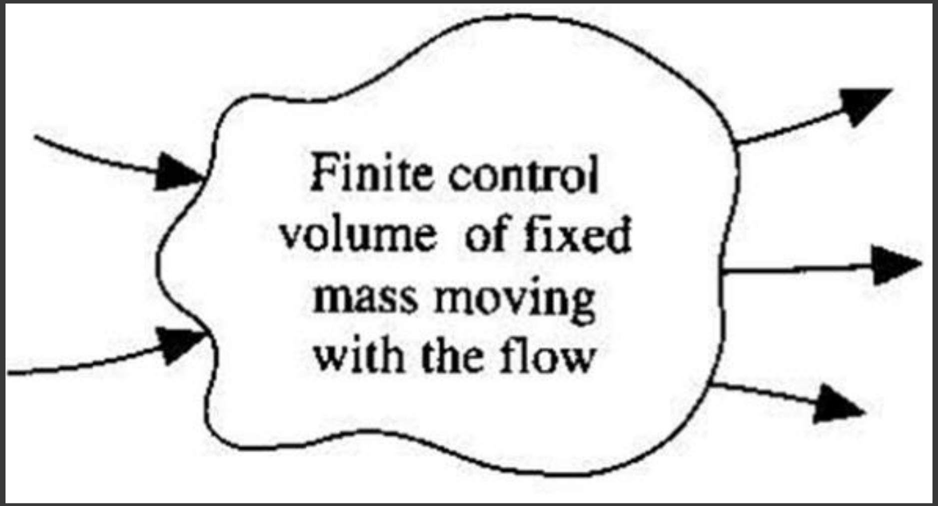
$$\frac{D\rho}{Dt} + \rho \left[\frac{1}{\delta V} \frac{D(\delta V)}{Dt} \right] = 0$$

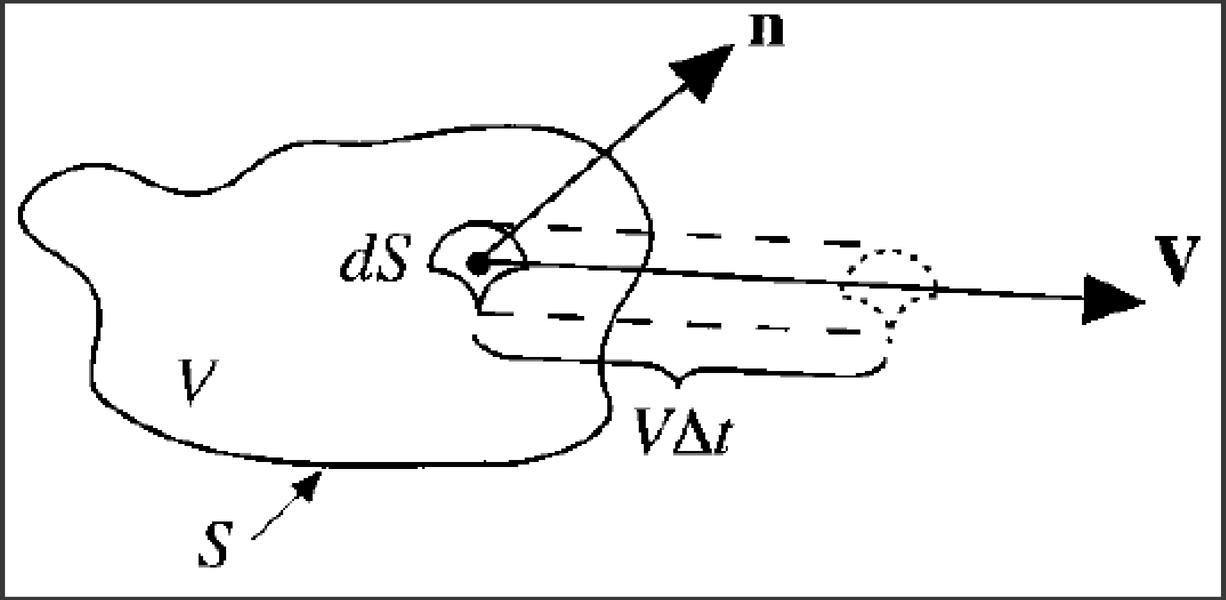
Anderson (1995)

Divergence of Velocity

- Consider a control volume moving with the fluid.
- This control volume has a fixed mass but a changing volume.
- Consider an infinitesimal surface element *dS* moving at local velocity **v**.
- The change in volume due to just the movement of *dS* is

$$\Delta V = \left[\left(\mathbf{v}\Delta t\right)\cdot\mathbf{n}\right]dS = \left(\mathbf{v}\Delta t\right)\cdot\mathbf{dS}$$





Anderson (1995)

Divergence of Velocity

The total change in volume of the whole control volume is the surface integral

$$\Delta V = \iint_S (\mathbf{v}\Delta t) \cdot \mathbf{dS}$$

Dividing by Δt, we obtain the time rate of change of the control volume

$$\frac{DV}{Dt} = \frac{1}{\Delta t} \iint_S (\mathbf{v}\Delta t) \cdot \mathbf{dS} = \iint_S \mathbf{v} \cdot \mathbf{dS} = \iiint_V \nabla \cdot \mathbf{v} \; dV$$

If the moving control volume is shrunk to a very small volume δV becoming an infinitesimal fluid element, we then have

$$\frac{D(\delta V)}{Dt} = \iiint_{\delta V} \nabla \cdot \mathbf{v} \; dV$$

Anderson (1995)

Continuity Equation: Differential Form

Assume that δV is small enough such that $\nabla \cdot \mathbf{v}$ is the same throughout δV. We then obtain

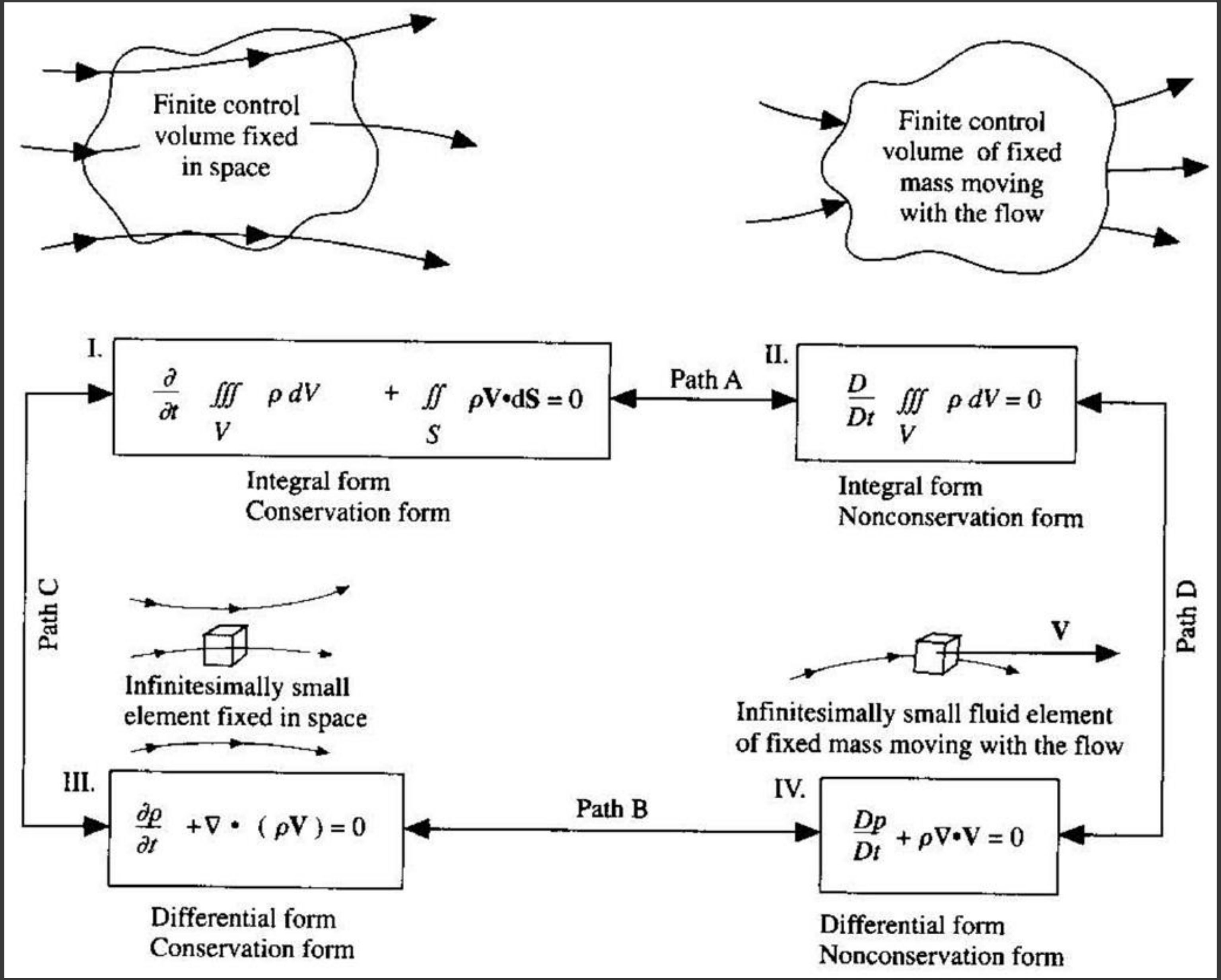
$$\frac{D(\delta V)}{Dt} = (\nabla \cdot \mathbf{v})\,\delta V \rightarrow \nabla \cdot \mathbf{v} = \frac{1}{\delta V}\frac{D(\delta V)}{Dt}$$

Using the previous result, the non-conservative, differential form of continuity equation becomes

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0$$

Anderson (1995)

Continuity Equation

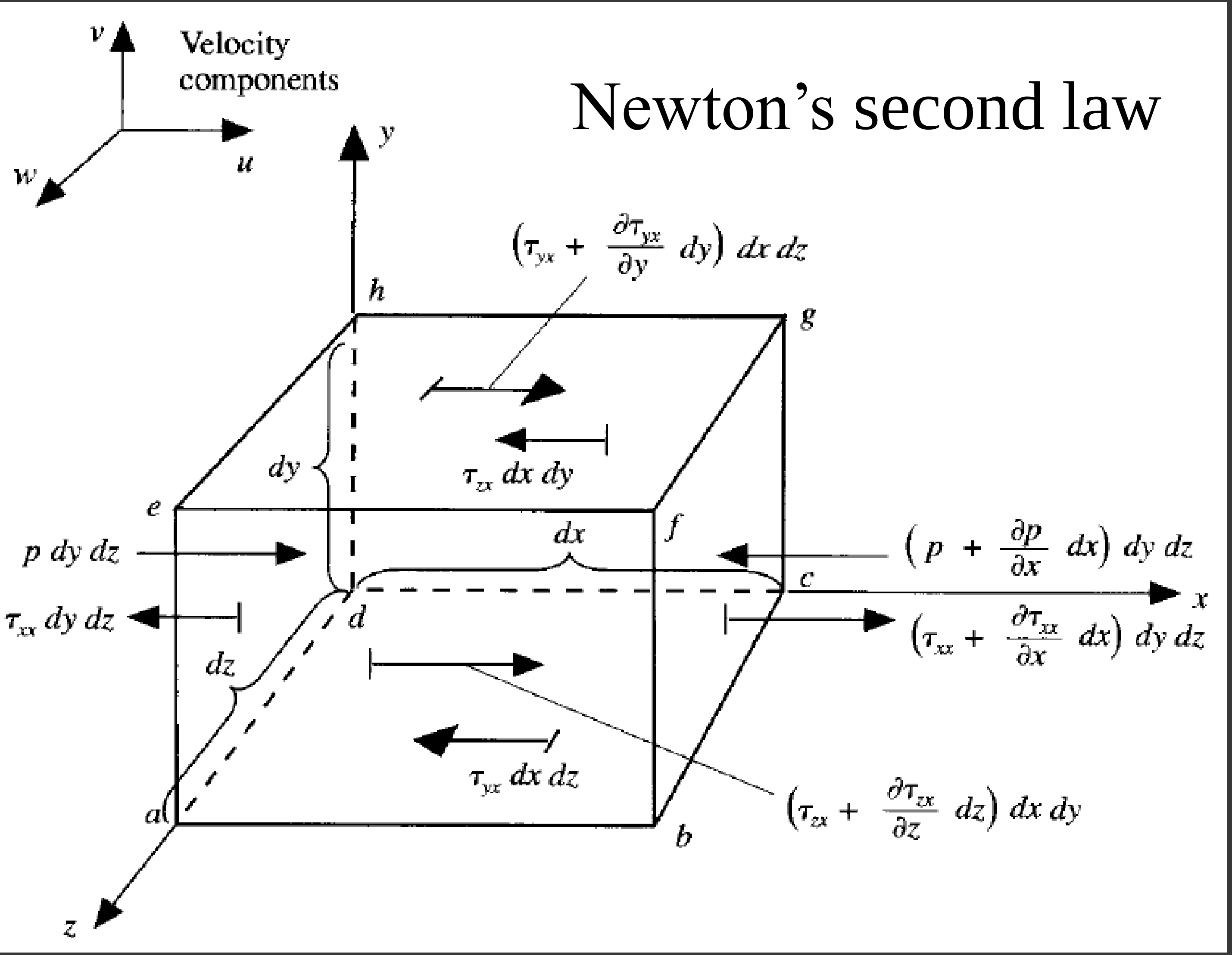


Anderson (1995)

Momentum Equation

Consider a fluid element moving with the flow.

Newton's second law



$$\mathbf{F} = m\mathbf{a}$$

Anderson (1995)

Body and Surface Forces

"**Body forces** act directly on the volumetric mass of the fluid element. These forces act at a distance. Examples: gravitational, electric, and magnetic forces."

"**Surface forces** acting directly on the surface of the fluid element are due to 2 sources:

- Pressure distribution on the surface imposed by the surrounding fluid
- Shear and normal stress distributions imposed outside fluid by means of friction"

Anderson (1995)

Net Body Force

Body force on fluid element acting in

x direction: $\rho f_x \left(dx\ dy\ dz \right)$

y direction: $\rho f_y \left(dx\ dy\ dz \right)$

z direction: $\rho f_z \left(dx\ dy\ dz \right)$

Anderson (1995)

Surface Force in x direction

Net surface force in *x* direction is

$$\left[p-\left(p+\frac{\partial p}{\partial x}dx\right)\right]dy\ dz+\left[\left(\tau_{xx}+\frac{\partial \tau_{xx}}{\partial x}dx\right)-\tau_{xx}\right]dy\ dz$$

$$+\left[\left(\tau_{yx}+\frac{\partial \tau_{yx}}{\partial y}dy\right)-\tau_{xx}\right]dx\ dz+\left[\left(\tau_{zz}+\frac{\partial \tau_{zz}}{\partial z}dz\right)-\tau_{zz}\right]dx\ dy$$

$$=\left[-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right](dx\ dy\ dz)$$

Anderson (1995)

Net Surface Force

Net surface force in

x direction: $$\left[-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right](dx\ dy\ dz)$$

y direction: $$\left[-\frac{\partial p}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right](dx\ dy\ dz)$$

z direction: $$\left[-\frac{\partial p}{\partial z}+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right](dx\ dy\ dz)$$

Anderson (1995)

Mass and Acceleration

Mass $\rho\left(dx\ dy\ dz\right)$

Acceleration in

x direction: $a_x = \dfrac{Du}{Dt}$

y direction: $a_y = \dfrac{Dv}{Dt}$

z direction: $a_z = \dfrac{Dw}{Dt}$

Anderson (1995)

Navier-Stokes Equations

Combining the previous results yields momentum equation in nonconservative form

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z$$

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}$$

Anderson (1995)

Momentum Conservation

- In the Eulerian viewpoint with a fixed control volume, the momentum conservation gives rise to the equation

$$\frac{\partial}{\partial t}\int_V \rho \mathbf{v} \, \mathrm{d}V + \int_S \rho \mathbf{v}\mathbf{v} \cdot \mathbf{n} \, \mathrm{d}S = \sum \mathbf{F}$$

- The force **F** could be
  - surface forces (pressure, normal and shear stresses, surface tension)
  - body forces (gravity, centrifugal and Coriolis forces, EM forces)
- To make the system of equations close (the number of dependent variables is equal to the number of equations), some assumptions must be made.
- One simple assumption is to assume that the fluid is Newtonian.

Ferziger et al. (2020)

Newtonian Fluids

- The stress tensor T of Newtonian fluids can be written as

$$\mathrm{T} = -\left(p + \frac{2}{3}\mu\nabla\cdot\mathbf{v}\right)I + 2\mu D$$

- where p is static pressure, μ is dynamic viscosity, I is unit tensor, D is the rate of strain tensor:

$$\mathrm{D} = \frac{1}{2}\left[\nabla\mathbf{v} + \left(\nabla\mathbf{v}\right)^T\right]$$

- These two equations can be written in index notation as

$$T_{ij} = -\left(p + \frac{2}{3}\mu\frac{\partial v_j}{\partial x_j}\right)\delta_{ij} + 2\mu D_{ij}$$

$$D_{ij} = \frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right]$$

The Einstein summation convention is used here. The viscous part of the stress tensor is usually denoted as

$$\tau_{ij} = 2\mu D_{ij} - \frac{2}{3}\mu\frac{\partial v_j}{\partial x_j}\delta_{ij}$$

Ferziger et al. (2020)

Momentum Equation

- When there are only stress tensor T and the body force per unit mass b, the momentum conservation equation becomes

$$\frac{\partial}{\partial t}\int_V \rho\mathbf{v}\ \mathrm{d}V + \int_S \rho\mathbf{v}\mathbf{v}\cdot\mathbf{n}\ \mathrm{d}S = \int_S \mathrm{T}\cdot\mathbf{n}\ \mathrm{d}S + \int_V \rho\mathbf{b}\ \mathrm{d}V$$

- Applying the convergence theorem to the surface integrals, we obtain

$$\int_V \left[\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot(\rho\mathbf{v}\mathbf{v} - \mathbf{T}) - \rho\mathbf{b}\right] \mathrm{d}V = 0$$

- The integrand must vanish. So, we obtain the differential form of the momentum equation

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) = \nabla\cdot\mathrm{T} + \rho\mathbf{b}$$

Ferziger et al. (2020)

Conservative Momentum Equation

- The conservative equation for the i^{th} component is

$$\frac{\partial(\rho v_i)}{\partial t} + \nabla\cdot(\rho v_i \mathbf{v}) = \nabla\cdot\mathbf{t}_i + \rho b_i$$

where

$$\mathbf{t}_i = \mu\nabla v_i + \mu(\nabla\mathbf{v})^T\cdot\mathbf{e}_i - \left(p + \frac{2}{3}\mu\nabla\cdot\mathbf{v}\right)\mathbf{e}_i$$

$$= \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\mathbf{e}_j - \left(p + \frac{2}{3}\mu\frac{\partial v_j}{\partial x_j}\right)\mathbf{e}_i$$

and $\mathbf{e}_i$ is the i^{th} Cartesian basis vector.

Ferziger et al. (2020, p. 9)

Nonconservative Momentum Equation

- The nonconservative equation can be obtained as follows.

$$\rho \frac{\partial v_i}{\partial t} + \rho \mathbf{v} \cdot \nabla v_i + v_i \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v} \right) \right] = \nabla \cdot \mathbf{t}_i + \rho b_i$$

- Using the differential form of the continuity equation, the bracket term vanishes.
- Then, we obtain the nonconservative momentum equation

$$\rho \frac{\partial v_i}{\partial t} + \rho \mathbf{v} \cdot \nabla v_i = \nabla \cdot \mathbf{t}_i + \rho b_i$$

- This equation is usually solved using the finite difference method.

Ferziger et al. (2020, p. 9)

Conservative Momentum Equation

The conservative equation

$$\frac{\partial(\rho v_i)}{\partial t} + \nabla \cdot (\rho v_i \mathbf{v}) = \nabla \cdot \mathbf{t}_i + \rho b_i$$

can be written in index notation as

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_j v_i)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho b_i$$

If gravity is the only external force, then we have

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_j v_i)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i$$

where g_i is the i^{th} component of gravitational acceleration **g**.

Ferziger et al. (2020, p. 9)

Navier-Stokes and Euler Equations

- "A viscous flow is one where the transport phenomena of friction, thermal conduction, and/or mass diffusion are included."
- The continuity, momentum, and energy equations previously mentioned are collectively called the **Navier-Stokes equations**.
- "Inviscid flow is a flow where the dissipative, transport phenomena of viscosity, mass diffusion, and thermal conductivity are neglected, resulting to **Euler equations**."

Dimensionless Form of the Equations

- Using normalization the governing equations can be transformed into a dimensionless form.
- Velocities can be normalized by a reference velocity v_0.
- Spatial coordinates can be normalized by a reference length L.
- Time can be normalized by a reference time t_0.
- Pressure can be normalized by a reference velocity ρv_0^2
- Temperature can be normalized by a temperature difference T_1 -T_0.

$$t^* = \frac{t}{t_0}, x_i^* = \frac{x_i}{l_0}, u_i^* = \frac{u_i}{v_0}, p^* = \frac{p}{\rho v_0^2}, T^* = \frac{T - T_0}{T_1 - T_0}$$

Ferziger et al. (2020, p. 9)

Dimensionless Form of the Equations

"If the fluid properties are constant, the dimensionless form of the continuity, momentum, and temperature equations are"

$$\frac{\partial u_i^*}{\partial x_i^*} = 0, \qquad \mathrm{St}\,\frac{\partial u_i^*}{\partial t^*} + \frac{\partial \left(u_j^* u_i^* \right)}{\partial x_j^*} = \frac{1}{\mathrm{Re}}\,\frac{\partial u_i^*}{\partial x_j^{*2}} - \frac{\partial p^*}{\partial x_i^*} + \frac{1}{\mathrm{Fr}^2}\,\gamma_i$$

$$\mathrm{St}\,\frac{\partial T^*}{\partial t^*} + \frac{\partial \left(u_j^* T^T \right)}{\partial x_j^*} = \frac{1}{\mathrm{Re}\ \mathrm{Pr}}\,\frac{\partial^2 T^*}{\partial x_j^{*2}}$$

where the Strouhal number St, the Reynolds number Re, and the Froude number Fr are defined as $\mathrm{St} = \dfrac{l_0}{v_0 t_0}, \quad \mathrm{Re} = \dfrac{\rho v_0 l_0}{\mu}, \quad \mathrm{Fr} = \dfrac{v_0}{\sqrt{l_0 g}}$

and γ_i is the i^{th} component of the normalized gravitational acceleration.

Ferziger et al. (2020, p. 11-12)

Incompressible Flows

- The density of liquids can be considered constant.
- When Ma < 0.3, the density of gases can also be considered constant.
- Flows in such media are said to be incompressible.
- If the flow is isothermal and the viscosity is constant, the continuity and momentum equations reduce to

$$\nabla \cdot \mathbf{v} = 0, \qquad \frac{\partial v_i}{\partial t} + \nabla \cdot \left(v_i \mathbf{v} \right) = \nabla \cdot \left(\nu \nabla v_i \right) - \frac{1}{\rho} \nabla \cdot \left(p \mathbf{e}_i \right) + b_i$$

Ferziger et al. (2020, p. 13)

Inviscid (Euler) Flows

- "In flows far from solid surfaces, the effects of viscosity are usually small."
- "If viscous effects are negligible, i.e., the stress tensor reduces to $\mathrm{T} = -p\mathrm{I}$, the Navier-Stokes equations reduce to the Euler equations."

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \qquad \frac{\partial (\rho v_i)}{\partial t} + \nabla \cdot (\rho v_i \mathbf{v}) = -\nabla \cdot (p \mathbf{e}_i) + \rho b_i$$

- In an inviscid flow, the fluid will not stick to walls and slip will occur at solid boundaries.
- "The Euler equations are often used to study compressible flows at high Mach numbers."
- The Euler equations can be solved using a coarser grid than the Navier-Stokes equations due to the absence of boundary layers in which viscosity effects are important.

Ferziger et al. (2020, p. 13-14)

Potential Flows

- In a potential flow, the fluid is assumed to be inviscid and the flow velocity is irrotational, i.e., $\nabla \times \mathbf{v} = 0$
- As a result, there exists a velocity potential ϕ such that $\mathbf{v} = -\nabla \phi$
- In an incompressible flow, the continuity equation becomes the Laplace equation $\nabla \cdot \mathbf{v} = -\nabla \cdot (\nabla \phi) = -\nabla^2 \phi = 0$
- The velocity vectors are tangential to streamlines which are the lines of constant streamfunction ψ.
- Streamlines are orthogonal to equipotential lines.
- "Potential flows have applications in flows in porous media."
- "The potential theory applied to flow around a body leads to D'Alembert's paradoc, i.e., the body experiences neither drag nor lift."

Ferziger et al. (2020, p. 15)

Creeping (Stokes) Flows

- When Re << 1 (the fluid is very viscous or the object interacting with the fluid is very small), the convection (inertial) terms in the Navier-Stokes equation are very small and can be neglected.
- "The flow is then dominated by the viscous, pressure, and body forces."
- If the fluid properties are constant and the velocities are small, the unsteady terms can be neglected."
- The Navier-Stokes equation becomes the Stokes equations.

$$\nabla \cdot \mathbf{v} = 0, \qquad \nabla \cdot \left(\mu \nabla u_i \right) - \frac{1}{\rho} \nabla \cdot \left(p \mathbf{e}_i \right) + b_i = 0$$

- "Creeping flows are found in porous media, coating technology, micro-devices, etc."

Ferziger et al. (2020, p. 15)

Boussinesq Approximation

- "In flows accompanied by heat transfer, the fluid properties are normally functions of temperature."
- "If the density variation is not large, the density can be treated as constant in the unsteady and convection terms, and treat it as variable only in the gravitational term."
- "This is called the Boussinesq approximation."
- "The density is usually assumed to vary linearly with temperature."

Ferziger et al. (2020, p. 15)

Incompressible Potential Flows

- Air and water have low viscosity, and in many cases their flows have high Reynolds numbers, i.e., the viscous force is small compared to the inertia force.
- "In flows with high Reynolds numbers, the influence of viscosity is confined to a very thin boundary layer in the immediate neighborhood of the solid wall."
- Thus, the flow outside the boundary layer can be considered inviscid flows.
- Vorticity describing a rotational motion of fluid is defined as $\nabla \times \mathbf{v}$
- "Voriticities are generated by the shearing viscous forces."
- So, flows in the boundary layer are rotational flows.
- Outside the boundary layer, the flow can be considered irrotational, i.e., the vorticity vanishes: $\nabla \times \mathbf{v} = \mathbf{0}$

Biringen and Chow (2011)

Incompressible Potential Flows

- The irrotational condition $\nabla \times \mathbf{v} = \mathbf{0}$ is automatically satisfied if a velocity potential ϕ is defined such that $\mathbf{v} = \nabla \phi$.
- As a result, irrotational flows are also called potential flows.
- If the fluid is incompressible, then $\nabla \cdot \mathbf{v} = 0$
- We then have the Laplace equation $\nabla \cdot \mathbf{v} = \nabla \cdot \nabla \phi = \nabla^2 \phi = 0$
- In the problem of an incompressible flow past a body at high Reynolds numbers, the velocity potential outside the boundary layer is computed by solving the Laplace equation with the boundary condition prescribed far upstream and the fluid velocity be tangent to the body surface.
- The fluid velocity within the boundary layer is then obtained by solving the boundary layer equation using the velocity distribution of the external flow along the body surface as the outer boundary condition.

Biringen and Chow (2011)

Incompressible Potential Flows

- "Once the velocity field in the external region is determined, the pressure field p can be computed by solving the Euler equation without body forces"

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p$$

- "It is easier to compute the pressure from the Bernoulli equation, the integrated form of the Euler equation."
- For incompressible, irrotational flows, the Bernoulli equation has the form

$$\rho \frac{d\phi}{dt} + p + \frac{1}{2} \rho v^2 = H$$

where the constant of integration H is the Bernoulli constant and $v = |\mathbf{v}|$.
- For steady flows, the Bernoulli equation reduces to $p + \frac{1}{2} \rho v^2 = H$ where H is the stagnation pressure at a point where $v = 0$.

Biringen and Chow (2011)

2D Incompressible Potential Flows

- In 2D planar flows, the velocity components and the Laplace equation can be expressed in the Cartesian coordinates as

$$v_x = \frac{\partial \phi}{\partial x}, \qquad v_y = \frac{\partial \phi}{\partial y}, \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

- In 2D axisymmetric flows, the velocity components and the Laplace equation are instead expressed in the polar coordinates:

$$v_r = \frac{\partial \phi}{\partial r}, \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}, \qquad \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Biringen and Chow (2011, p.53)

2D Incompressible Potential Flows

- The stream function ψ can also be used instead of the velocity potential.
- The continuity equation $\nabla\cdot\mathbf{v} = 0$ suggests that the velocity can also be expressed interm of the stream function as $\mathbf{v} = \nabla\times\left(\psi\hat{\mathbf{k}}\right)$.
- A line along which ψ = constant is called a streamline.
- Fluid velocities are always tangential to streamlines.
- The irrotational condition becomes $\nabla\times\nabla\times\left(\psi\hat{\mathbf{k}}\right) = \mathbf{0}$
- In 2D planar flows, we then have

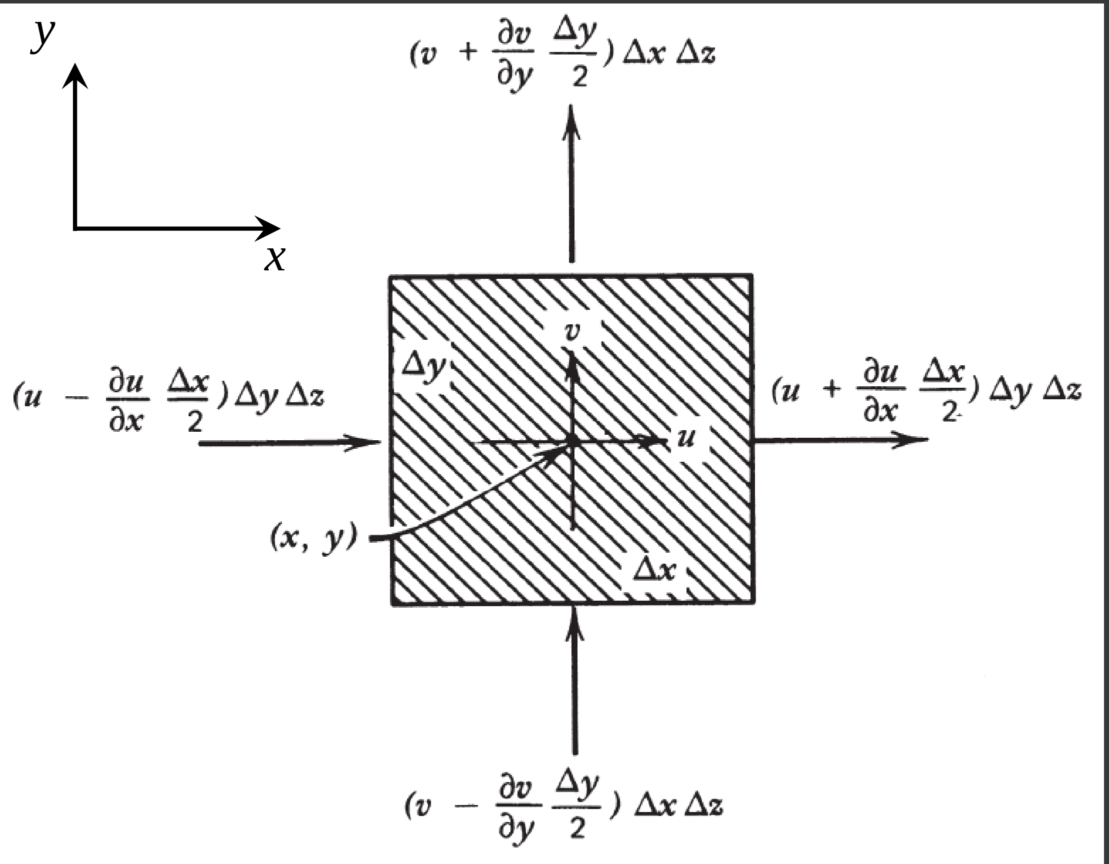
$$v_x = \frac{\partial\psi}{\partial y}, \qquad v_y = -\frac{\partial\psi}{\partial x}, \qquad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad v_\theta = -\frac{\partial\psi}{\partial r}, \qquad \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0$$

Biringen and Chow (2011, p.53)

How to Handle Sources and Sinks

- "The continuity equation $\nabla \cdot \mathbf{v} = 0$ does not apply in regions where fluid mass is created (source) or destroyed (sink)."
- Consider a fluid element $\Delta x\ \Delta y\ \Delta z$ fixed in space.



- "The distribution of sources is represented by $q(x,y,z,t)$, the volume of fluid created per unit time from a unit volume at (x,y,z)."
- "The net volume of fluid flowing out of the fluid element per unit time is"

$$\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \Delta x \Delta y \Delta z$$

Biringen and Chow (2011, p.54)

How to Handle Sources and Sinks

- "For incompressible flows, this amount of fluid must be equal to the amount produced by all the sources contained within the volume: $q\Delta x\Delta y\Delta z$."
- The continuity equation then becomes

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=q(x,y,z,t) \quad \leftrightarrow \quad \nabla\cdot\mathbf{v}=q$$

- Sources are represented by positive values of q while sinks are negative q.
- In the presence of sources or sinks, the equation governing the velocity potential becomes the Poisson equation

$$\nabla^2\phi=q$$

Biringen and Chow (2011, p.54-55)

Radial Flow due to Sources/Sinks

- "Consider a radial flow in the domain $r_0 \le r \le 4r_0$ within which there is an axisymmetric distribution of sources whose strength increases linearly with r." The governing equations for this flow is

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\phi = q_0\frac{r}{r_0}, \qquad \text{where } q_0 \text{ is the source strength at } r = r_0$$

- Let's u_0 be a characteristic velocity. The other dimensionless variables are

$$R=\frac{r}{r_0}, \qquad \Phi=\frac{\phi}{r_0 u_0}, \qquad U=\frac{u_r}{u_0}$$

- The domain then becomes $1 \le R \le 4$.
- The dimensionless equation is $\left(\frac{d^2}{dR^2}+\frac{1}{R}\frac{d}{dR}\right)\Phi = \alpha R, \qquad \text{where } \alpha = \frac{q_0 r_0}{u_0}$

Biringen and Chow (2011, p.60)

Radial Flow due to Sources/Sinks

- The original radial velocity $u_r = \dfrac{d\phi}{dr}$ then becomes $U = \dfrac{d\Phi}{dR}$
- The general solution is given by $\Phi = \dfrac{1}{9}\alpha R^3 + \beta \ln R + \gamma$
- "The first term is the particular solution representing the flow caused by the source distribution while the other two terms comprise the homogeneous solution."
- "Since U is not affected by the constant term in Φ, γ can be set to 0."

Biringen and Chow (2011, p.60)

Radial Flow due to Sources/Sinks

- Consider the boundary-value problem

$$\left(\frac{d^2}{dR^2}+\frac{1}{R}\frac{d}{dR}\right)\Phi = 9R$$

$$\Phi = 1 \qquad \text{at } R = 1$$

$$\Phi = 64 - 24\ln 4 \qquad \text{at } R = 4$$

- The exact solution for this problem is $\Phi = R^3 - 24\ln R$
- The corresponding radial velocity is $U = 3R^2 - 24/R$
- Exercise: Plot the exact solution

Biringen and Chow (2011, p.61)

Finite Difference (FD) Approximation

- The derivative du/dx can be approximated as an arithmetic expression involving only the values of u at various locations.
- For examples,

$$\frac{du}{dx} \approx \frac{u(x+h)-u(x)}{h}$$ forward finite difference

$$\frac{du}{dx} \approx \frac{u(x)-u(x-h)}{h}$$ backward finite difference

$$\frac{du}{dx} \approx \frac{u(x+h)-u(x-h)}{2h}$$ central finite difference

- These are the finite-difference approximations of the first derivative.
- The different between these approximations are the approximation error.

Forward FD and Its Truncation Error

Rearranging the Taylor's series expansion

$$u(x+h) = u(x) + u'(x)h + u''(x)\frac{h^2}{2!} + O(h^3)$$

we obtain

$$u'(x) = \frac{u(x+h) - u(x)}{h} - u''(x)\frac{h}{2!} - \dots$$

$$\approx \frac{u(x+h) - u(x)}{h}$$

The approximation absolute error is

$$\left| u'(x) - \frac{u(x+h) - u(x)}{h} \right| = \left| -u''(x)\frac{h}{2!} - u'''(x)\frac{h^2}{3!} - \dots \right| = O(h)$$

Backward FD and Its Truncation Error

Rearranging the Taylor's series expansion

$$u(x-h) = u(x) - u'(x)h + u''(x)\frac{h^2}{2!} + O(h^3)$$

we obtain

$$u'(x) = \frac{u(x)-u(x-h)}{h} + u''(x)\frac{h}{2!} + \ldots$$

$$\approx \frac{u(x)-u(x-h)}{h}$$

The approximation absolute error is

$$\left| u'(x) - \frac{u(x)-u(x-h)}{h} \right| = \left| u''(x)\frac{h}{2!} + u'''(x)\frac{h^2}{3!} + \ldots \right| = O(h)$$

Central FD and Its Truncation Error

Subtracting the Taylor's series expansions

$$u(x+h) = u(x) + u'(x)h + u''(x)\frac{h^2}{2!} + O(h^3)$$

$$u(x-h) = u(x) - u'(x)h + u''(x)\frac{h^2}{2!} + O(h^3)$$

and rearranging the result yields

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} - u'''(x)\frac{h^2}{3!} - \ldots \approx \frac{u(x+h) - u(x-h)}{2h}$$

The approximation absolute error is

$$\left| u'(x) - \frac{u(x+h) - u(x-h)}{2h} \right| = \left| -u'''(x)\frac{h^2}{3!} - u^{(5)}(x)\frac{h^4}{5!} - \ldots \right| = O\left(h^2\right)$$

Deriving FD Coefficients

- After choosing a set of locations for function evaluations, the corresponding finite difference coefficients can be derived as shown in the below example.
- Suppose we want to approximate $u'(x)$ using the function values $u(x)$, $u(x - h)$, and $u(x - 2h)$. The approximated value is denoted as $Du(x)$.
- First, write a formula as a linear combination of the function values as

$$Du(x) = au(x) + bu(x-h) + cu(x-2h)$$

where a, b, and c are unknowns to be determined.

- Rearranging the Taylor's expansions of $u(x - h)$, and $u(x - 2h)$, we obtain

$$Du(x) = (a+b+c)u(x) - (b+2c)hu'(x)$$

$$+\frac{1}{2}(b+4c)h^2u''(x) - \frac{1}{6}(b+8c)h^3u'''(x) + \ldots$$

LeVeque (2007, p. 7)

Deriving FD Coefficients

To obtain an approximation of $u'(x)$, we need

$$\begin{aligned} a + b + c &= 0 \\ b + 2c &= -1/h \\ b + 4c &= 0 \end{aligned} \longrightarrow \begin{aligned} a &= 3/2h \\ b &= -2/h \\ c &= 1/2h \end{aligned}$$

Thus,

$$Du(x) = \frac{1}{2h}\left[3u(x) - 4u(x-h) + u(x-2h)\right]$$

LeVeque (2007, p. 7)

FD Approximations of Second Derivative

The second derivative $u''(x)$ can be approximated as follows.

$$u''(x) = \frac{u(x) - 2u(x+h) + u(x+2h)}{h^2} + O(h)$$

$$u''(x) = \frac{u(x) - 2u(x-h) + u(x-2h)}{h^2} + O(h)$$

$$u''(x) = \frac{u(x-h) - 2u(x) + u(x+2h)}{h^2} + O\left(h^2\right)$$

Index Notation

To make finite difference expressions concise and simple to handle, we usually use the following index notations.

$$u(x) = u(i\Delta x) = u_i$$

$$u(x+\Delta x) = u\left(\left(i+1\right)\Delta x\right) = u_{i+1}$$

$$u(x, y, z, t) = u(i\Delta x, j\Delta y, k\Delta z, n\Delta t) = u_{ijk}^{n}$$

$$u(x+\Delta x, y-\Delta y, t+\Delta t) = u_{i+1, j-1}^{n+1}$$

Exercise

Use the finite-difference method to solve the boundary-value problem

$$\left(\frac{d^2}{dR^2}+\frac{1}{R}\frac{d}{dR}\right)\Phi = 9R$$

$$\Phi = 1 \qquad \text{at } R = 1$$

$$\Phi = 64 - 24\ln 4 \qquad \text{at } R = 4$$

and compare the result with the exact solution $\Phi = R^3 - 24\ln R$

Biringen and Chow (2011, p.61)

References

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