Smoothed Particle Hydrodynamics (SPH)

Chaiwoot Boonyasiriwat

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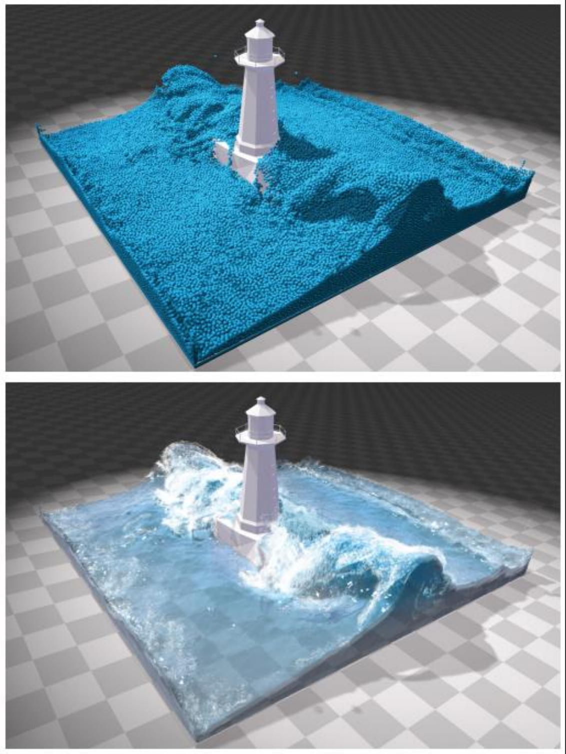
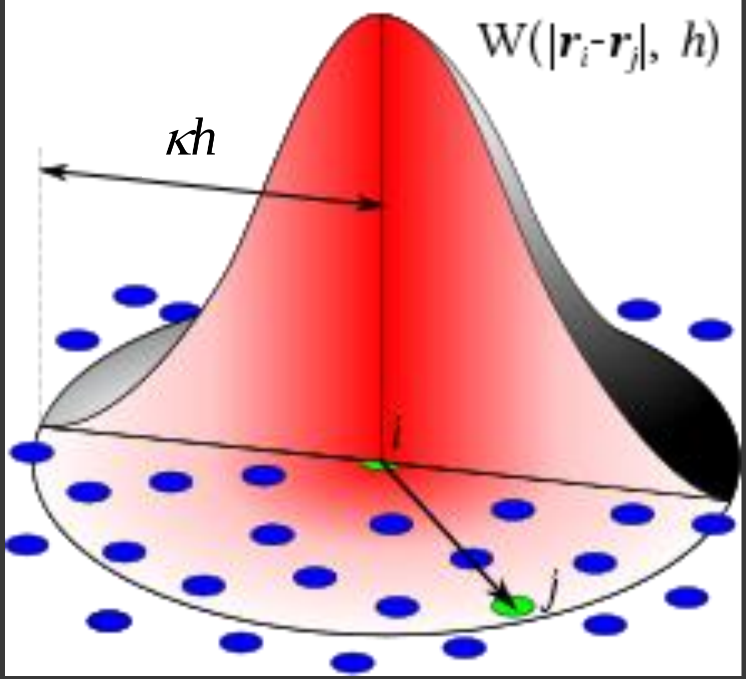
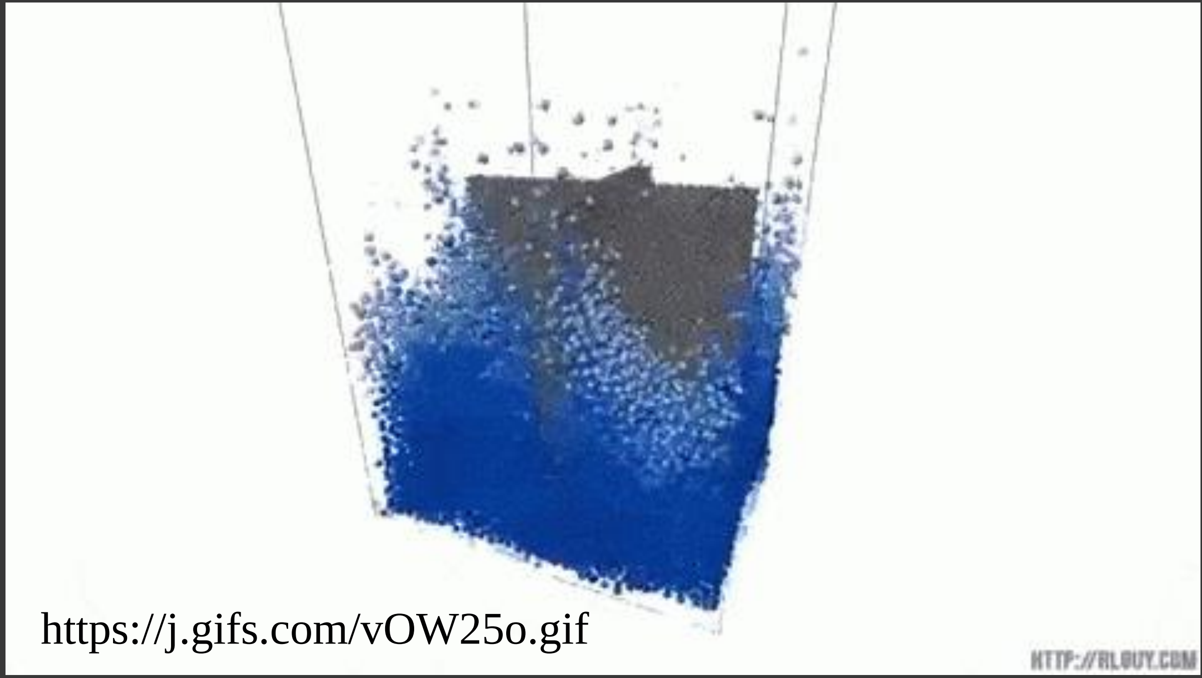
Introduction to SPH

- SPH is a particle, meshfree, Lagrangian method for fluid simulations.
- The fluid is represented by a finite number of particles.
- A physical quantity at a point in the domain are computed from particles within a compact support of a Gaussian-like kernel centered at the point.







https://j.gifs.com/vOW25o.gif

https://en.wikipedia.org/wiki/Smoothed-particle_hydrodynamics

https://philip-mocz.medium.com/create-your-own-smoothed-particle-hydrodynamics-simulation-with-python-76e1cec505f1

Kernel Approximation of Function

- A spatial function $f(\mathbf{r})$ can be written as the volume integral

$$f(\mathbf{r}) = \int_{\Omega} f(\mathbf{r}')\delta(\mathbf{r}' - \mathbf{r})\,d\mathbf{r}'$$

  where $\delta(\mathbf{r})$ is the Dirac delta function and Ω is the domain.
- The integral representation of function $f(\mathbf{r})$ comes from the sifting property of the Dirac delta function.
- A crucial step of SPH is to approximate the Dirac delta function by a compactly supported kernel function $W(\mathbf{r} - \mathbf{r}', h)$ where h is a parameter that defines the area of influence of the kernel.
- This is called the kernel approximation and the integral representation then becomes

$$f(\mathbf{r}) \approx \int_{\Omega} f(\mathbf{r}')W(\mathbf{r}' - \mathbf{r}, h)\,d\mathbf{r}'$$

Liu and Liu (2003, p. 36)

Properties of the Kernel Function

- The kernel function must satisfy the normalization condition:

$$\int_{\Omega} W\left(\mathbf{r}' - \mathbf{r}, h\right) d\mathbf{r}' = 1$$

- The kernel function is evenly symmetric:

$$W\left(\mathbf{r}' - \mathbf{r}, h\right) = W\left(\mathbf{r} - \mathbf{r}', h\right)$$

- The kernel function must converge to the Dirac delta function when h approaches zero:

$$\lim_{h \to 0} W\left(\mathbf{r}' - \mathbf{r}, h\right) = \delta\left(\mathbf{r} - \mathbf{r}'\right)$$

- The kernel function must satisfy the compact condition:

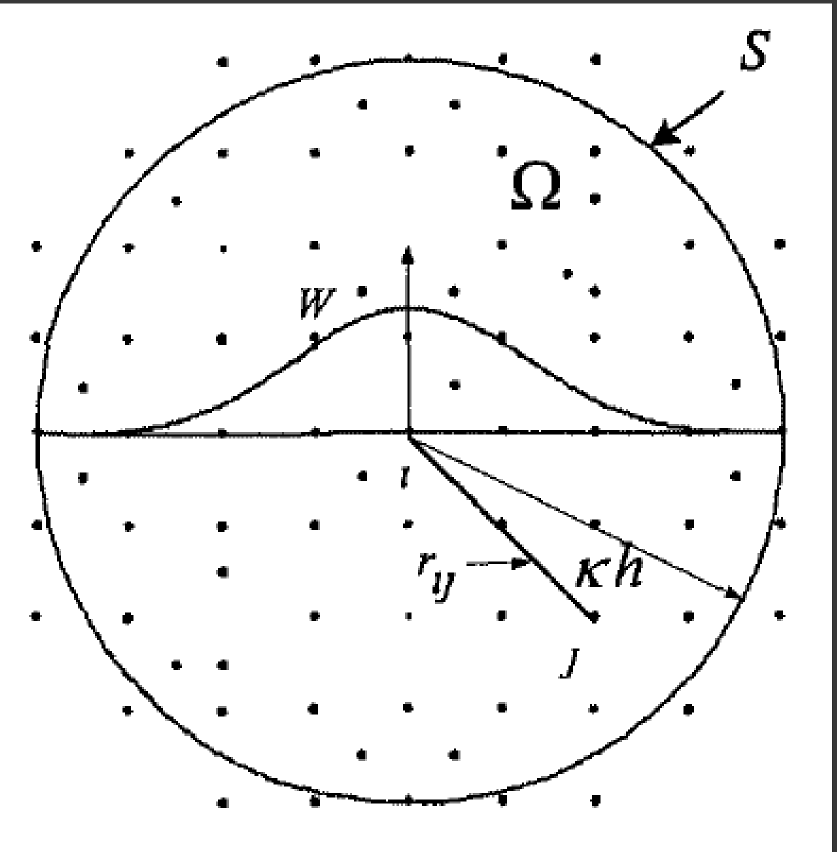
$$W\left(\mathbf{r}' - \mathbf{r}, h\right) = 0 \quad \text{when } \left|\mathbf{r}' - \mathbf{r}\right| > \kappa h$$

where κ defines the support domain of the kernel at point **r**.

Liu and Liu (2003, p. 37)

Particle Approximation of Function

- In SPH, the fluid is represented by particles.
- The integral representation can be approximated by a finite sum over all the particles in the support domain. This is known as particle approximation.



- The infinitesimal volume $d\mathbf{r}'$ in

$$f(\mathbf{r}) \approx \int_{\Omega} f(\mathbf{r}') W(\mathbf{r}' - \mathbf{r}, h) d\mathbf{r}'$$

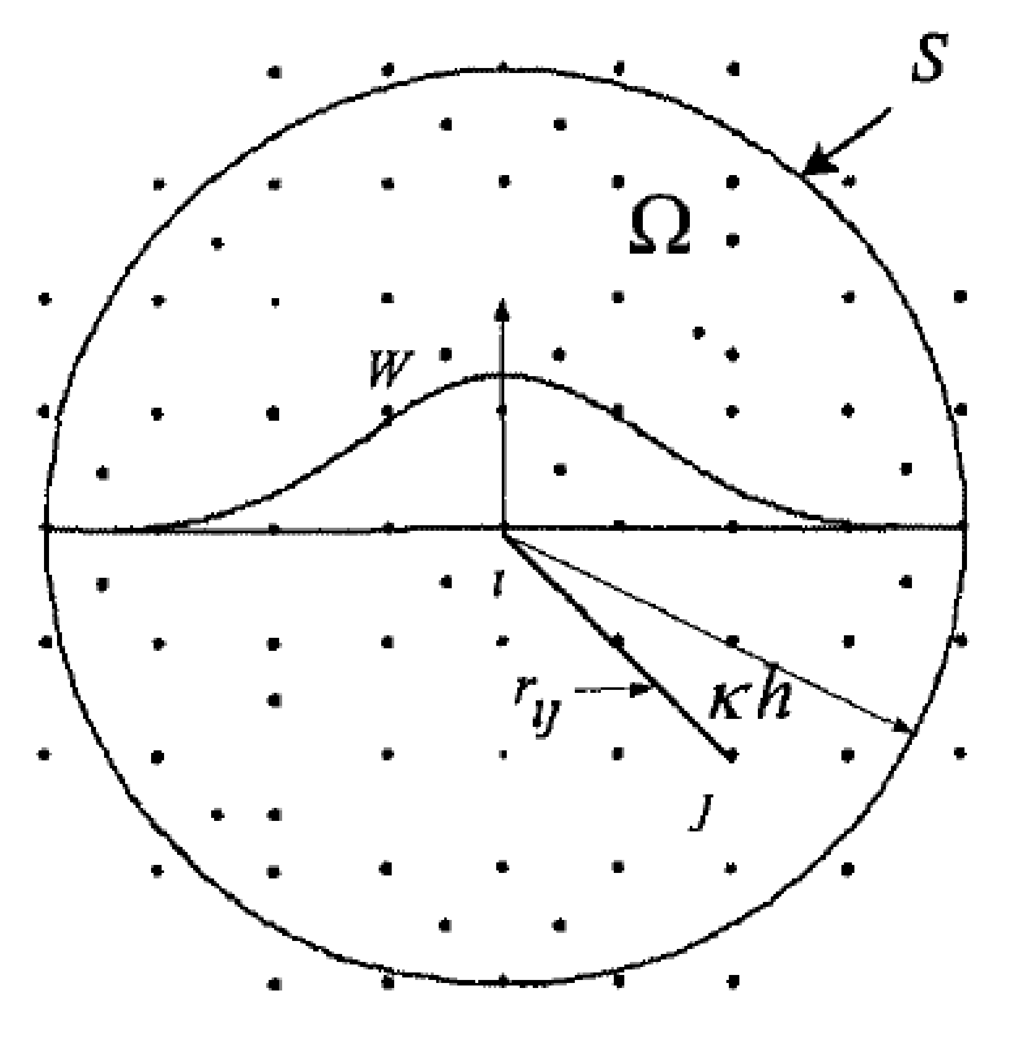
is approximated by the finite volume of the j^{th} particle ΔV_j. which is related to the mass and density by

$$\rho_j = \frac{m_j}{\Delta V_j}$$

Liu and Liu (2003, p. 41)

Particle Approximation of Function

- Using the particle approximation, the integral representation becomes



Liu and Liu (2003, p. 41)

$$
\begin{aligned}
f(\mathbf{r}) &\approx \int_{\Omega} f(\mathbf{r}')W(\mathbf{r}-\mathbf{r}',h)\,d\mathbf{r}' \\
&\approx \sum_{j=1}^{n} f(\mathbf{r}_j)W(\mathbf{r}-\mathbf{r}_j,h)\Delta V_j \\
&\approx \sum_{j=1}^{n} \frac{m_j}{\rho_j} f(\mathbf{r}_j)W(\mathbf{r}-\mathbf{r}_j,h)
\end{aligned}
$$

where n is the number of particles within the support domain of point $\mathbf{r}$.

Particle Approximation of Function

- In SPH, the approximated value of $f(\mathbf{r})$ is denoted as $\langle f(\mathbf{r})\rangle$.
- The SPH approximation of $f(\mathbf{r})$ at the i^{th} particle is then written as

$$\langle f(\mathbf{r}_i)\rangle = \sum_{j=1}^{n} \frac{m_j}{\rho_j} f\left(\mathbf{r}_j\right) W_{ij}$$

where

$$W_{ij} = W\left(\mathbf{r}_i - \mathbf{r}_j, h\right) = W\left(\left|\mathbf{r}_i - \mathbf{r}_j\right|, h\right) = W\left(r_{ij}, h\right)$$

- The SPH approximation of the density ρ is then

$$\rho_i = \sum_{j=1}^{n} m_j W_{ij}$$

- This is called the summation density approach to obtaining density in SPH.

Liu and Liu (2003, p. 44)

Approximation of Divergence of Vector Field

- The kernel approximation of the divergence of a vector field is

$$\nabla \cdot \mathbf{f}(\mathbf{r}) \approx \int_{\Omega} \nabla \cdot \mathbf{f}\left(\mathbf{r}'\right) W\left(\mathbf{r}'-\mathbf{r}, h\right) d\mathbf{r}'$$

- Using the vector calculus identity

$$\nabla \cdot \mathbf{f}\left(\mathbf{r}'\right) W\left(\mathbf{r}'-\mathbf{r}, h\right) = \nabla \cdot \left[\mathbf{f}\left(\mathbf{r}'\right) W\left(\mathbf{r}'-\mathbf{r}, h\right)\right] - \mathbf{f}\left(\mathbf{r}'\right) \cdot \nabla W\left(\mathbf{r}'-\mathbf{r}, h\right)$$

  we obtain

$$\nabla \cdot \mathbf{f}(\mathbf{r}) \approx \int_{\Omega} \nabla \cdot \left[\mathbf{f}\left(\mathbf{r}'\right) W\left(\mathbf{r}'-\mathbf{r}, h\right)\right] d\mathbf{r}' - \int_{\Omega} \mathbf{f}\left(\mathbf{r}'\right) \cdot \nabla W\left(\mathbf{r}'-\mathbf{r}, h\right) d\mathbf{r}'$$

- Applying the divergence theorem to the first term on the right-hand side yields

$$\nabla \cdot \mathbf{f}(\mathbf{r}) \approx \int_{S} W\left(\mathbf{r}'-\mathbf{r}, h\right) \mathbf{f}\left(\mathbf{r}'\right) \cdot \hat{\mathbf{n}}\, dS - \int_{\Omega} \mathbf{f}\left(\mathbf{r}'\right) \cdot \nabla W\left(\mathbf{r}'-\mathbf{r}, h\right) d\mathbf{r}'$$

Liu and Liu (2003, p. 38)

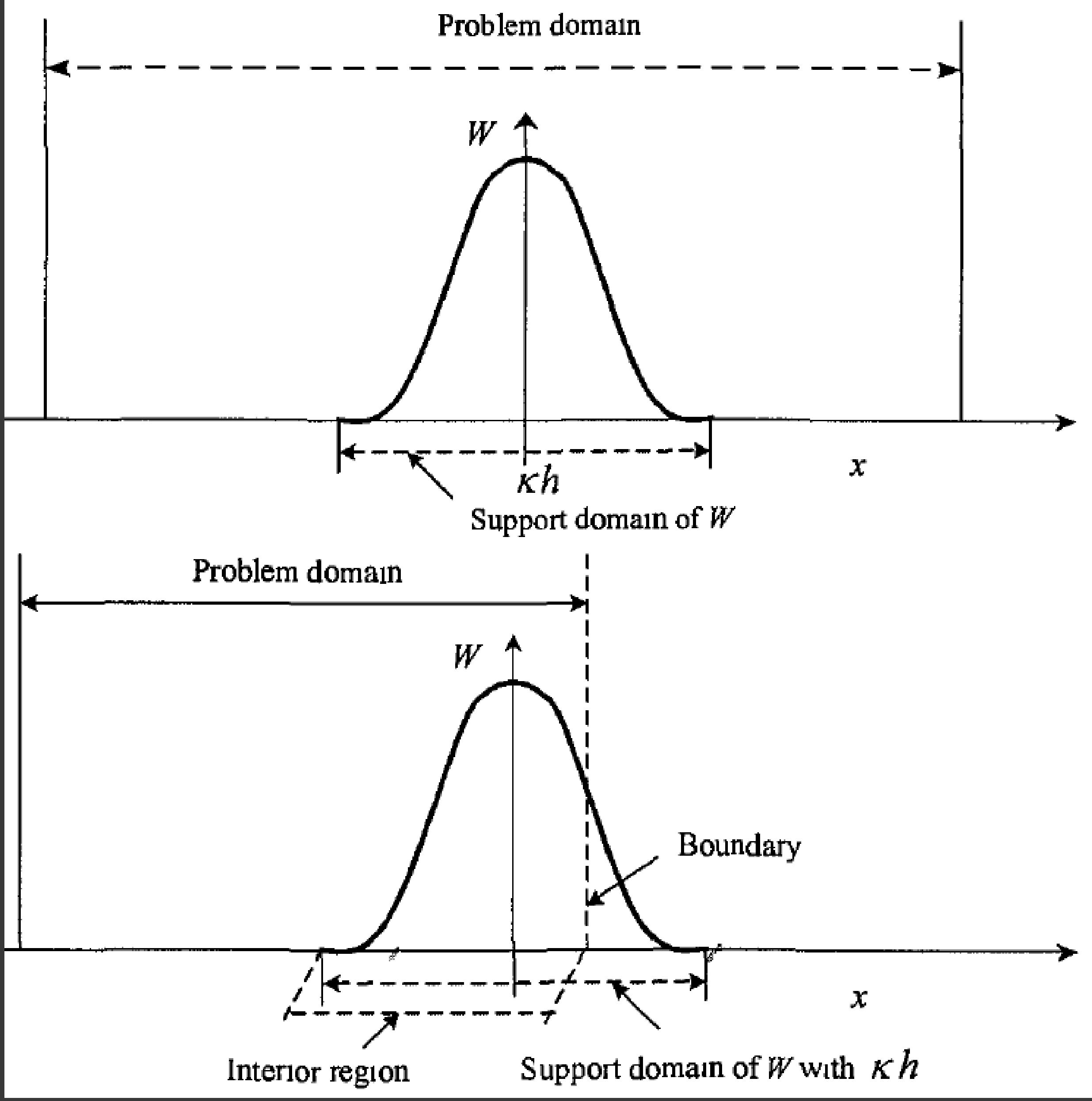
Approximation of Divergence of Vector Field

- When the support domain of W at point $\mathbf{r}$ is located within the problem domain, the surface integral vanishes.
- If the support domain overlaps with the problem domain, the surface integral is nonzero.
- If the support domain at point $\mathbf{r}_i$ is inside the problem domain, we have

$$\nabla\cdot\mathbf{f}(\mathbf{r}_i)\approx-\int_{\Omega}\mathbf{f}\left(\mathbf{r}'\right)\cdot\nabla W\left(\mathbf{r}'-\mathbf{r}_i,h\right)d\mathbf{r}'$$

$$\approx-\sum_{j=1}^{n}\frac{m_j}{\rho_j}\mathbf{f}\left(\mathbf{r}_j\right)\cdot\nabla W_{ij}$$

Liu and Liu (2003, p. 39-40)



Approximation of Gradient of Scalar Field

- The kernel approximation of the gradient of a scalar field is

$$\nabla f(\mathbf{r}) \approx \int_{\Omega} \nabla f(\mathbf{r}') W(\mathbf{r}' - \mathbf{r}, h) \, d\mathbf{r}'$$

- Using the vector calculus identity

$$\nabla f(\mathbf{r}') W(\mathbf{r}' - \mathbf{r}, h) = \nabla \left[f(\mathbf{r}') W(\mathbf{r}' - \mathbf{r}, h) \right] - f(\mathbf{r}') \nabla W(\mathbf{r}' - \mathbf{r}, h)$$

we obtain

$$\nabla f(\mathbf{r}) \approx \int_{\Omega} \nabla \left[f(\mathbf{r}') W(\mathbf{r}' - \mathbf{r}, h) \right] d\mathbf{r}' - \int_{\Omega} f(\mathbf{r}') \nabla W(\mathbf{r}' - \mathbf{r}, h) \, d\mathbf{r}'$$

- It can be shown that

$$\int_{\Omega} \nabla f \, dV = \oint_{S} f \mathbf{n} \, dS$$

https://en.wikipedia.org/wiki/Divergence_theorem#Corollaries

Approximation of Gradient of Scalar Field

- As a result, the kernel approximation becomes

$$\nabla \cdot \mathbf{f}(\mathbf{r}) \approx \oint_S f(\mathbf{r}') W(\mathbf{r}' - \mathbf{r}, h)\hat{\mathbf{n}}\,dS - \int_\Omega f(\mathbf{r}')\nabla W(\mathbf{r}' - \mathbf{r}, h)\,d\mathbf{r}'$$

- If the support domain at point $\mathbf{r}_i$ is inside the problem domain, then the surface integral vanishes and the kernel approximation becomes

$$\nabla f(\mathbf{r}_i) \approx -\int_\Omega f(\mathbf{r}')\nabla W(\mathbf{r}' - \mathbf{r}_i, h)\,d\mathbf{r}'$$

- After the particle approximation, we obtain first approximation formula

$$\nabla f(\mathbf{r}_i) = \nabla f_i \approx -\sum_{j=1}^{n} \frac{m_j}{\rho_j} f_j \nabla W_{ij}$$

- The gradient of a scalar field can also be written as $\nabla f = \frac{1}{\rho}\left[\nabla(\rho f) - f\nabla\rho\right]$

Approximation of Gradient of Scalar Field

We then obtain the second approximation formula of gradient:

$$
\begin{aligned}
\nabla f_i &\approx \int_\Omega \frac{1}{\rho(\mathbf{r}')}\left[\nabla\left(\rho(\mathbf{r}') f(\mathbf{r}')\right) - f(\mathbf{r}')\nabla\rho(\mathbf{r}')\right] W(\mathbf{r}'-\mathbf{r}_i, h)\, d\mathbf{r}' \\
&\approx \frac{1}{\rho_i}\int_\Omega \nabla(\rho f) W(\mathbf{r}'-\mathbf{r}_i, h)\, d\mathbf{r}' - \frac{f_i}{\rho_i}\int_\Omega \nabla\rho W(\mathbf{r}'-\mathbf{r}_i, h)\, d\mathbf{r}' \\
&\approx -\frac{1}{\rho_i}\int_\Omega \rho f \nabla W(\mathbf{r}'-\mathbf{r}_i, h)\, d\mathbf{r}' + \frac{f_i}{\rho_i}\int_\Omega \rho \nabla W(\mathbf{r}'-\mathbf{r}_i, h)\, d\mathbf{r}' \\
&\approx -\frac{1}{\rho_i}\sum_{j=1}^{n}\frac{m_j}{\rho_j}\rho_j f_j \nabla W_{ij} + \frac{f_i}{\rho_i}\sum_{j=1}^{n}\frac{m_j}{\rho_j}\rho_j \nabla W_{ij} \\
&= -\frac{1}{\rho_i}\sum_{j=1}^{n} m_j \left(f_j - f_i\right)\nabla W_{ij}
\end{aligned}
$$

Filho (2019, p. 23)

Approximation of Gradient of Scalar Field

- When the previous two formulas are used to compute the pressure gradient, the resulting forces exerting a particle pair may not have the same magnitude.
- Consider the identity

$$\nabla\left(\frac{f}{\rho}\right)=\frac{\nabla f}{\rho}-\frac{f}{\rho^2}\nabla\rho \quad \rightarrow \quad \frac{\nabla f}{\rho}=\nabla\left(\frac{f}{\rho}\right)+\frac{f}{\rho^2}\nabla\rho$$

- Using the first approximation formula for the two terms on the right-hand side, we obtain the third approximation formula

$$\frac{\nabla f_i}{\rho_i}\approx-\sum_{j=1}^{n}\frac{m_j}{\rho_j^2}f_j\nabla W_{ij}-\frac{f_i}{\rho_i^2}\sum_{j=1}^{n}m_j\nabla W_{ij}=-\sum_{j=1}^{n}m_j\left(\frac{f_i}{\rho_i^2}+\frac{f_j}{\rho_j^2}\right)\nabla W_{ij}$$

Filho (2019, p. 24)

Approximation of Laplacian of Scalar Field

- The kernel approximation of the Laplacian of a scalar field is

$$\nabla^2 f(\mathbf{r}) = \nabla \cdot \nabla f(\mathbf{r}) \approx \int_\Omega \nabla \cdot \nabla f(\mathbf{r}') W(\mathbf{r}' - \mathbf{r}, h) d\mathbf{r}'$$

- Consider the identity

$$\nabla \cdot \left[\nabla f(\mathbf{r}') W(\mathbf{r}' - \mathbf{r}, h)\right] = \nabla \cdot \nabla f(\mathbf{r}') W(\mathbf{r}' - \mathbf{r}, h) + \nabla f(\mathbf{r}') \cdot \nabla W(\mathbf{r}' - \mathbf{r}, h)$$

- The SPH approximation of the Laplacian of a scalar field is

$$\nabla^2 f(\mathbf{r}) \approx \int_\Omega \nabla \cdot \left[\nabla f(\mathbf{r}') W(\mathbf{r}' - \mathbf{r}, h)\right] d\mathbf{r}' - \int_\Omega \nabla f(\mathbf{r}') \cdot \nabla W(\mathbf{r}' - \mathbf{r}, h) d\mathbf{r}'$$

$$= -\int_\Omega \nabla f(\mathbf{r}') \cdot \nabla W(\mathbf{r}' - \mathbf{r}, h) d\mathbf{r}'$$

$$\approx -\sum_{j=1}^{n} \frac{m_j}{\rho_j} \nabla f_j \cdot \nabla W_{ij}$$

Approximation of Laplacian of Scalar Field

- Now consider the identity

$$\nabla \cdot \left(f \nabla W \right) = \nabla f \cdot \nabla W + f \nabla^2 W$$

- The kernel approximation then becomes

$$\nabla^2 f(\mathbf{r}) \approx -\int_{\Omega} \nabla \cdot \left[f\left(\mathbf{r}'\right) \nabla W\left(\mathbf{r}' - \mathbf{r}, h\right) \right] d\mathbf{r}' + \int_{\Omega} f\left(\mathbf{r}'\right) \nabla^2 W\left(\mathbf{r}' - \mathbf{r}, h\right) d\mathbf{r}'$$

$$= \int_{\Omega} f\left(\mathbf{r}'\right) \nabla^2 W\left(\mathbf{r}' - \mathbf{r}, h\right) d\mathbf{r}'$$

- Using the particle approximation, we then obtain the SPH approximation formula of the Laplacian of a scalar field as

$$\nabla^2 f_i \approx \sum_{j=1}^{n} \frac{m_j}{\rho_j} f_j \nabla^2 W_{ij}$$

Filho (2019, p. 24)

Approximation of Laplacian of Scalar Field

- Filho (2019, p. 24) presented a formula for approximating the Laplacian of a scalar field in the Cartesian coordinate system as

$$\nabla^2 f_i \approx 2\sum_{j=1}^{n} \frac{m_j}{\rho_j}\left(f_i - f_j\right)\Delta\mathbf{r}_{ij}\cdot\nabla W_{ij}, \qquad \Delta\mathbf{r}_{ij} = \frac{\mathbf{r}_i - \mathbf{r}_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^2}$$

- In the polar coordinate system, the formula is

$$\nabla^2 f_i \approx 2\sum_{j=1}^{n} \frac{m_j}{\rho_j}\left(f_i - f_j\right)\frac{1}{\left|\mathbf{r}_i - \mathbf{r}_j\right|}\frac{\partial W_{ij}}{\partial r}$$

Mass Conservation

- The continuity equation in the Lagrangian viewpoint is

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{v}$$

- Using the SPH approximation only for $\nabla \cdot \mathbf{v}$, the continuity equation becomes the first formula:

$$\left(\frac{D\rho}{Dt}\right)_i = \rho_i \sum_{j=1}^{n} \frac{m_j}{\rho_j} \mathbf{v}_j \cdot \nabla W_{ij}$$

- Consider the SPH approximation of gradient of unity

$$(\nabla 1)_i \approx \int 1 \nabla W(\mathbf{r}_i - \mathbf{r}', h)\, d\mathbf{r}' \approx \sum_{j=1}^{n} \frac{m_j}{\rho_j} \nabla W_{ij} = 0$$

- Since $\rho \mathbf{v} \cdot \nabla 1 = 0$, we have $(\rho \mathbf{v} \cdot \nabla 1)_i = \rho_i \sum_{j=1}^{n} \frac{m_j}{\rho_j} \mathbf{v}_i \cdot \nabla W_{ij} = 0$

Filho (2019, p. 25)

Mass Conservation

- We then obtain the second formula:

$$\left(\frac{D\rho}{Dt}\right)_i = \left(\frac{D\rho}{Dt} - \rho\mathbf{v}\cdot\nabla 1\right)_i = -\rho_i \sum_{j=1}^{n} \frac{m_j}{\rho_j}\left(\mathbf{v}_i - \mathbf{v}_j\right)\cdot\nabla W_{ij}$$

- Using the relative velocity $\mathbf{v}_i$ - $\mathbf{v}_j$ tends to reduce errors from the particle inconsistency problem.
- The most frequently used formula for the continuity equation can be obtained using the identity

$$\rho\nabla\cdot\mathbf{v} = \nabla\cdot\left(\rho\mathbf{v}\right) - \mathbf{v}\cdot\nabla\rho$$

$$= \left[-\sum_{j=1}^{n} m_j \mathbf{v}_j \cdot\nabla W_{ij}\right] - \left[\mathbf{v}_i \cdot\left(-\sum_{j=1}^{n} m_j \nabla W_{ij}\right)\right] = \sum_{j=1}^{n} m_j \left(\mathbf{v}_i - \mathbf{v}_j\right)\cdot\nabla W_{ij}$$

Filho (2019, p. 25)

Mass Conservation

- We then obtain the third formula for the continuity equation

$$\left(\frac{D\rho}{Dt}\right)_i = -\sum_{j=1}^{n} m_j \left(\mathbf{v}_i - \mathbf{v}_j\right) \cdot \nabla W_{ij}$$

- The density approximation $\rho_i = \sum_{j=1}^{n} m_j W_{ij}$ is inaccurate near the domain boundary and solid surfaces. The accuracy can be improved using the normalization

$$\rho_i = \sum_{j=1}^{n} m_j W_{ij} \Bigg/ \sum_{j=1}^{n} \frac{m_j}{\rho_j} W_{ij}$$

- The summation density approach is suitable for flows without discontinuities while the continuity density approach is preferred for flows with discontinuities.

Liu and Liu (2003, p. 115-117)

Momentum Conservation

- Recall the Navier-Stokes equation in the Lagrangian viewpoint

$$\frac{D\mathbf{v}}{Dt} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{v} + \mathbf{g}$$

- The pressure gradient term can be approximated as

$$\frac{\nabla p_i}{\rho_i} \approx -\sum_{j=1}^{n} m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \nabla W_{ij}$$

- The viscous term can be approximated as

$$\nu_i \nabla^2 \mathbf{v}_i \approx 2\nu_i \sum_{j=1}^{n} \frac{m_j}{\rho_j} \left(\mathbf{v}_i - \mathbf{v}_j \right) \left(\frac{\mathbf{r}_{ij}}{r_{ij}} \cdot \nabla W_{ij} \right), \qquad \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$$

Liu and Liu (2003, p. 115)

Kernel Function

- Cubic spline kernel (Monaghan and Lattanzio, 1985):

$$W(r,h) = W\left(q = \frac{r}{h}\right) = \alpha \begin{cases} 2/3 - q^2 + q^3/2 & 0 \le q \le 1 \\ (2-q)^3/6 & 1 \le q \le 2 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha = 15/7\pi h^2$ in 2D and $\alpha = 3/2\pi h^3$ in 3D.

- Quartic kernel (Lucy, 1977):

$$W(q) = \alpha \begin{cases} (1+3q)(1-q)^3 & 0 \le q \le 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha = 5/\pi h^2$ in 2D and $\alpha = 105/16\pi h^3$ in 3D.

Filho (2019, p. 27)

Kernel Function

- Quartic kernel (Liu and Liu, 2010):

$$W(q) = \alpha \begin{cases} \frac{2}{3} - \frac{9}{8}q^2 + \frac{19}{24}q^3 - \frac{5}{32}q^4 & 0 \le q \le 2 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha = 15/7\pi h^2$ in 2D and $\alpha = 315/208\pi h^3$ in 3D.

- Quintic spline kernel (Morris et al., 1997):

$$W(q) = \alpha \begin{cases} (3-q)^5 - 6(2-q)^5 + 15(1-q)^5 & 0 \le q \le 1 \\ (3-q)^5 - 6(2-q)^5 & 1 \le q \le 2 \\ (3-q)^5 & 2 \le q \le 3 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha = 7/478\pi h^2$ in 2D and $\alpha = 1/120\pi h^3$ in 3D.

Solving the Diffusion Equation using SPH

- Consider the 2D initial-boundary value problem

$$\frac{\partial T}{\partial t} = D\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \qquad (x, y) \in [0,1]\times[0,1]$$

$$T(0, y, t) = 0, \quad T(1, y, t) = 0, \quad T(x, 0, t) = 100, \quad T(x, 1, t) = 0$$

$$T(x, y, 0) = 0$$

- The SPH approximation of the Laplacian is

$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)_i^{(k)} = 2\sum_{j=1}^{n} \frac{m_j}{\rho_j}\left(T_i^{(k)} - T_j^{(k)}\right)\frac{\partial W_{ij}}{\partial r}\frac{1}{r_{ij}}$$

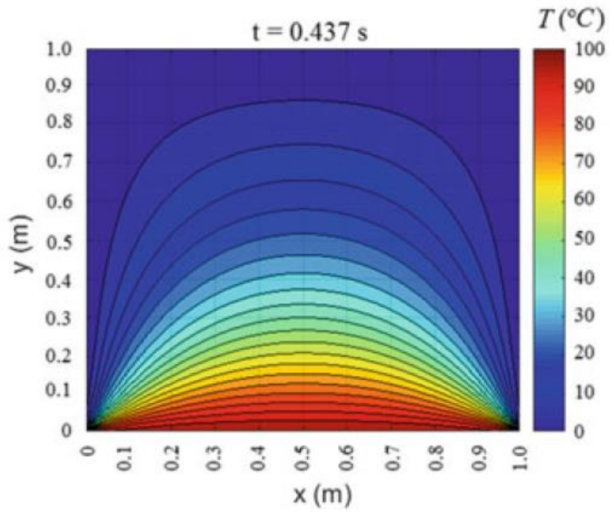
Filho (2019, p. 70)

Solving the Diffusion Equation using SPH

- Using the Euler method for numerical time integration yields

$$T_i^{(k+1)} = T_i^{(k+1)} + D_i \Delta t \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)_i^{(k)}$$

- The domain is then discretized as a set of fixed particles.
- Let n_p be the number of particles in x and y directions.
- The spacing between particles is then $h = 1/n_p$.
- Let's use the kernel radius be 2.5h., time step = 10^{-5} s, and $\alpha = 1$.
- Let's the stopping threshold $\varepsilon = 10^{-6}$.
- The stopping criterion is $\left| T^{(k+1)} - T^{(k)} \right| < \epsilon$
- Let's use $n_p = 50$.



Filho (2019, p. 70-73)

Liquid in Immobile Reservoir

- A Newtonian and incompressible fluid is at rest in an immobile reservoir that is open to the atmosphere.
- Since the fluid is at rest, $\mathbf{v} = 0$. The Navier-Stokes equation then becomes

$$\nabla p = \rho \mathbf{g} \rightarrow \frac{\partial p}{\partial x} = 0, \quad \frac{\partial p}{\partial y} = -\rho g, \quad \frac{\partial p}{\partial z} = 0$$

- Integration in the y-direction yields

$$\int dp = -\int \rho g \, dy \rightarrow p(y) = -\rho g y + C$$

$$p(H) = p_0 \rightarrow C = p_0 + \rho g H$$

- Here $y = H$ is the position of the free surface, p_0 is atmospheric pressure.
- We then obtain

$$p(y) = p_0 + \rho g (H - y)$$

Filho (2019, p. 78)

Liquid in Immobile Reservoir

- Filho (2019) proposed to use a modified pressure in the Navier-Stokes equation:

$$p_{\text{mod}} \equiv p - p_0 - \rho g\left(H - y\right)$$

- The Navier-Stokes equation then becomes

$$\frac{D\mathbf{v}}{Dt} = -\frac{\nabla p_{\text{mod}}}{\rho} - \nu\nabla^2\mathbf{v}$$

- When the fluid is in the hydrostatic equilibrium, $\nabla p_{\text{mod}} = \mathbf{0}$

Dam Breaking over a Dry Bed

- The continuity equation and the Navier-Stokes equation are approximated as follows.

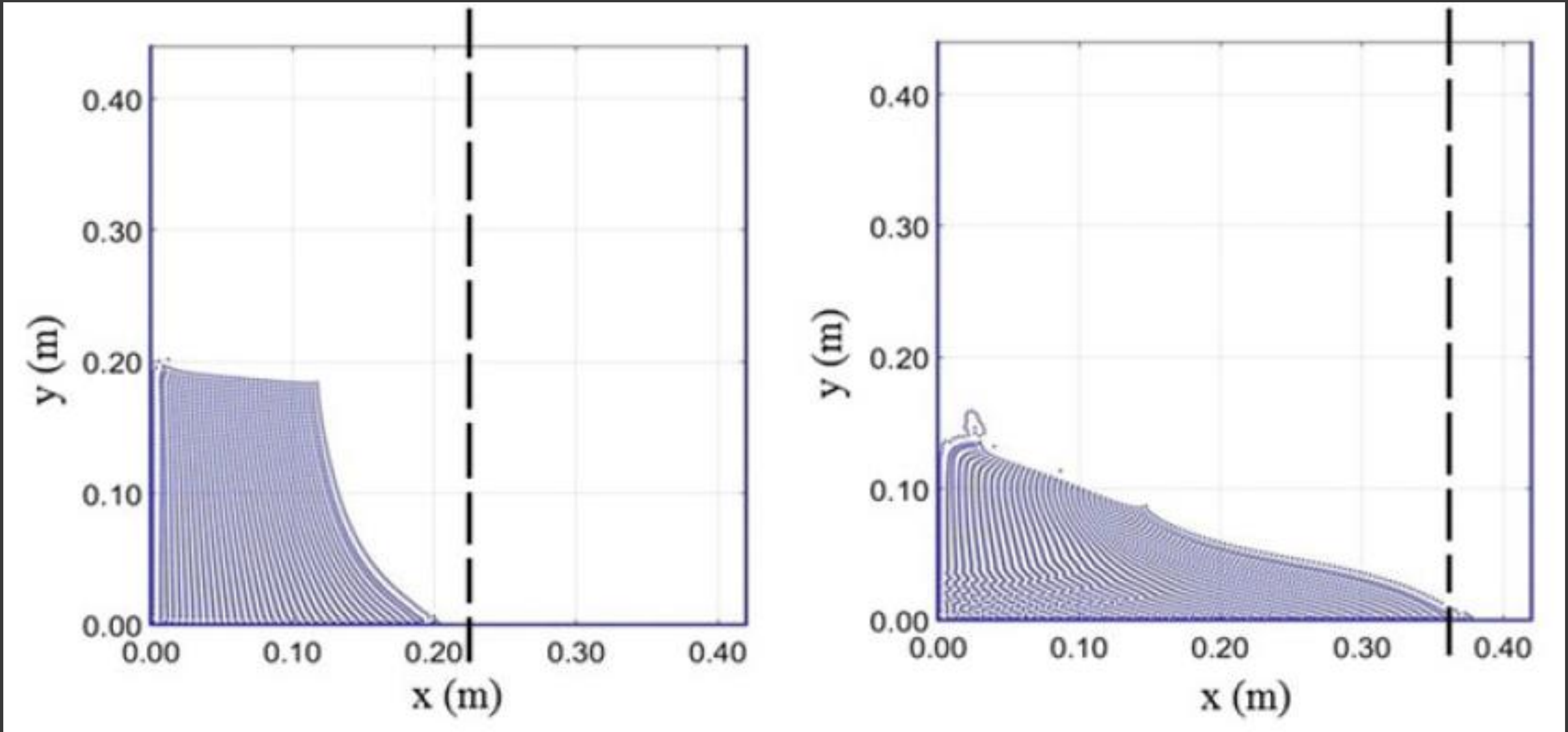
$$\left(\frac{D\rho}{Dt}\right)_i = -\sum_{j=1}^{n} m_j \left(\mathbf{v}_i - \mathbf{v}_j\right)\cdot \nabla W_{ij}$$

$$\frac{D\mathbf{v}}{Dt} = \sum_{j=1}^{n} m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2}\right)\nabla W_{ij}$$

$$+ 2\nu_i \sum_{j=1}^{n} \frac{m_j}{\rho_j}\left(\mathbf{v}_i - \mathbf{v}_j\right)\left(\frac{\mathbf{r}_i - \mathbf{r}_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^2}\cdot \nabla W_{ij}\right) + \mathbf{g}$$

Filho (2019, p. 84)

Dam Breaking over a Dry Bed



Filho (2019, p. 88)

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