Numerical Solution of the Incompressible Navier-Stokes Equation

Chaiwoot Boonyasiriwat

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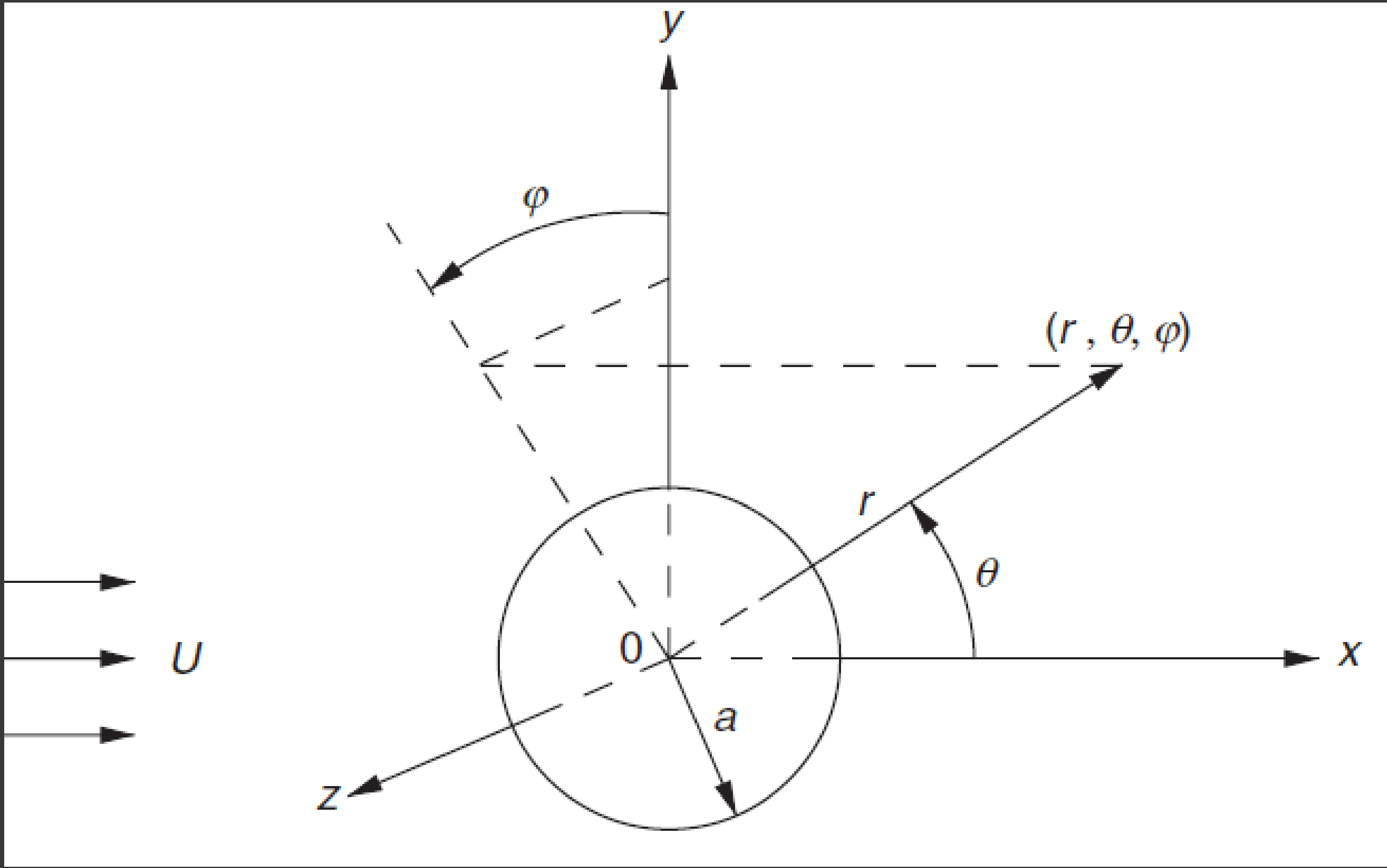
Flow around A Sphere

- Consider an incompressible flow past a sphere.



- "Far away from the sphere the flow is of a constant speed U along the x axis."
- Due to the axisymmetry, the flow field is independent of the azimuth coordinate φ.

Biringen and Chow (2011, p.216)

Flow around A Sphere

- The velocity components u_r and u_θ can be written in term of the stream function ψ as follows.

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

- The Navier-Stokes equation in the general form is given by

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t} - \mathbf{v}\times(\nabla\times\mathbf{v})\right] = -\nabla\left(p + \frac{1}{2}\rho v^2\right) - \mu\nabla\times\nabla\times\mathbf{v}, \quad v = |\mathbf{v}|$$

- Taking the curl of the last equation and using the steady-flow assumption, we obtain

$$\frac{1}{r^2\sin\theta}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta} + 2\cot\theta\frac{\partial\psi}{\partial r} - \frac{2}{r}\frac{\partial\psi}{\partial\theta}\right)D^2\psi = \frac{2}{\mathrm{Re}}D^4\psi$$

Biringen and Chow (2011, p.217)

Flow around A Sphere

- The Reynolds number Re = $2\rho Ua/\mu$ is based on the sphere diameter a.
- The stream function ψ and radial coordinate r are nondimensionalized by reference to Ua^2 and a, respectively.
- The operator D^2 is defined as

$$D^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$

- The non-slip boundary condition on the sphere surface leads to

$$\psi(1,\theta) = 0, \qquad \frac{\partial\psi}{\partial r}(1,\theta) = 0$$

- Far away from the sphere the flow is uniform, i.e.,

$$\psi(r,\theta) = \frac{1}{2}r^2\sin^2\theta \quad \text{as } r\to\infty$$

Biringen and Chow (2011, p.217)

Flow around A Sphere

- In term of $\zeta = \cos\theta$, the equation

$$\frac{1}{r^2 \sin\theta}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta} + 2\cot\theta\frac{\partial\psi}{\partial r} - \frac{2}{r}\frac{\partial\psi}{\partial\theta}\right)D^2\psi = \frac{2}{\mathrm{Re}}D^4\psi$$

can be written as

$$\frac{2}{\mathrm{Re}}D^4\psi + \frac{1}{r^2}\left(\frac{\partial\psi}{\partial\zeta}\frac{\partial}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial}{\partial\zeta} - \frac{2\zeta}{1-\zeta^2}\frac{\partial\psi}{\partial r} - \frac{2}{r}\frac{\partial\psi}{\partial\zeta}\right)D^2\psi = 0$$

where $D^2 \equiv \dfrac{\partial^2}{\partial r^2} + \dfrac{1-\zeta^2}{r^2}\dfrac{\partial^2}{\partial\zeta^2}$

- The boundary condition at points far away from the sphere becomes

$$\psi(r,\zeta) = \frac{1}{2}r^2\left(1-\zeta^2\right) \quad \text{as } r \to \infty$$

Biringen and Chow (2011, p.217)

Flow around A Sphere

- Using the Galerkin method, the trial solution is written as a linear combination of basis functions Φ_i: $\psi = \sum_{i=1}^{n} \Phi_i(r,\zeta)$
- The basis functions Φ_i must be linear independent and differentiable.
- For this problem the basis functions are chosen to be

$$\Phi_i(r,\zeta) = f_i(r)\left[P_{i-1}(\zeta) - P_{i+1}(\zeta)\right], \qquad i = 1, 2, \ldots$$

where $P_m(\zeta)$ are the Legendre polynomials defined by the Rodrigues' formula

$$P_m(\zeta) = \frac{1}{2^m m!}\frac{d^m}{d\zeta^m}\left(\zeta^2 - 1\right)^m$$

with the orthogonality $\int_{-1}^{1} P_m(\zeta) P_n(\zeta)\, d\zeta = \frac{2}{2n+1}\delta_{mn}$

Biringen and Chow (2011, p.218)

Flow around A Sphere

- Using $n = 2$ and $f_i(r)$ is chosen such that

$$\psi = \Phi_1(r,\zeta) + \Phi_2(r,\zeta)$$

where

$$\Phi_1(r,\zeta) = \left(\frac{1}{2}r^2 + \frac{A_1}{r} + \frac{A_2}{r^2} + \frac{A_3}{r^3} + \frac{A_4}{r^4}\right)\left(1-\zeta^2\right)$$

$$\Phi_2(r,\zeta) = \left(\frac{B_1}{r} + \frac{B_2}{r^2} + \frac{B_3}{r^3} + \frac{B_4}{r^4}\right)\zeta\left(1-\zeta^2\right)$$

- To determine the 8 coefficients, A_i and B_i, we need 8 equations.
- The derivation of these equations are given in the reference below.

Biringen and Chow (2011, p.218)

Rayleigh-Benard Instability

- When a layer of fluid is heated from below, the hotter fluid at the bottom has a lower density than the cooler fluid at the top.
- The buoyant force pushes the hotter fluid from the bottom to the top while the viscous force opposes the buoyant force.
- The dimensionless number called Rayleigh number Ra is the ratio between the buoyant force and the viscous force:

$$\mathrm{Ra} = \frac{g\alpha}{\nu\kappa}\left(T_{\mathrm{bottom}} - T_{\mathrm{top}}\right)H^3$$

where α = thermal expansion coefficient of the fluid

κ = thermal diffusivity of the fluid

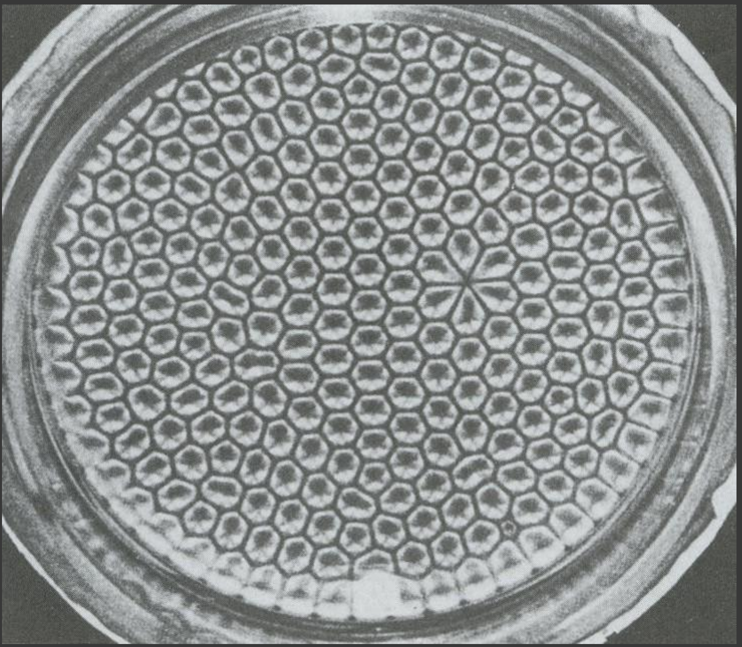
ν = kinematic viscosity of the fluid

H = height of fluid layer

T_{bottom}, T_{top} are temperatures at the bottom and top of the fluid

Rayleigh-Benard Instability

- At a large Rayleigh number, the buoyant force is dominant.
- At a critical Rayleight number, the fluid flow becomes unstable and this instability leads to the formation of Benard cells.
- The onset of Rayleigh-Benard instability can be predicted by a linearized theory presented in Section 3.8 of Biringen and Chow (2011).



- Once the flow becomes unstable, the fluid motion is governed by the nonlinear Navier-Stokes equation.

https://people.duke.edu/~av8/vandongen_lab/selforganization/SO_Figures/figure%201%20-%20Benard%20cells.html

Rayleigh-Benard Instability

The governing equations in the case are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p - \mu \nabla^2 \mathbf{v} + \Delta \rho \mathbf{g}$$

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \nabla \cdot (k \nabla T) + \Phi$$

$$\rho = \rho_0 \left[1 - \alpha (T - T_0) \right]$$

where T_0 is the temperature at which the fluid density is ρ_0.

Biringen and Chow (2011, p.234)

Rayleigh-Benard Instability

- For ordinary liquids, α is of order 10^{-4}.
- So the Boussinesq approximation can be used, i.e., density can be consider constant everywhere except in the buoyant force term.
- The continuity equation and the Navier-Stokes equation become

$$\nabla \cdot \mathbf{v} = 0, \qquad \rho_0 \frac{D\mathbf{v}}{Dt} = -\nabla p - \mu \nabla^2 \mathbf{v} - \left(\rho - \rho_0\right) g\mathbf{j}$$

  where **j** is the unit vector along the y-axis opposite to the gravity.
- It can be shown that the energy equation reduces to

$$\frac{DT}{Dt} = \kappa \nabla^2 T$$

Biringen and Chow (2011, p.234)

Rayleigh-Benard Instability in 2D

- Consider a 2D flow in the x-y plane.
- Let the height of the fluid layer be H.
- Initially, the fluid has a uniform temperature of T_0.
- The fluid is then heated from below such that the temperature of the fluid at bottom is fixed at T_1.
- Any fluid rotation is in the x-y plane. So, the vorticity is in the z direction.
- Let the magnitude of vorticity at any location be denoted by ζ.
- For this problem, the reference length is H, reference velocity is $\mu/\rho_0 H$, and the reference temperature difference is T_1-T_0.

Biringen and Chow (2011, p.235)

Rayleigh-Benard Instability in 2D

- Dimensionless variables are defined as follows.

$$X = \frac{x}{H}, \quad Y = \frac{y}{H}, \quad T = \frac{t}{\rho_0 H^2 / \mu}, \quad U = \frac{u}{\mu / \rho_0 H}, \quad V = \frac{v}{\mu / \rho_0 H}$$

$$\Psi = \frac{\psi}{\mu / \rho_0}, \quad \Omega = \frac{\zeta}{\mu / \rho_0 H^2}, \quad \theta = \frac{T - T_0}{T_1 - T_0}$$

- From now on T is the dimensionless time.

Biringen and Chow (2011, p.235)

Rayleigh-Benard Instability in 2D

In terms of the dimensionless variables the governing equations are

$$U = \frac{\partial \Psi}{\partial Y}, \qquad V = -\frac{\partial \Psi}{\partial X}, \qquad \Omega = -\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \right) \Psi$$

$$\frac{\partial \Omega}{\partial T} + \frac{\partial (U\Omega)}{\partial X} + \frac{\partial (V\Omega)}{\partial V} = \mathrm{Gr}\frac{\partial \theta}{\partial X} + \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \right) \Omega$$

$$\frac{\partial \theta}{\partial T} + \frac{\partial (U\theta)}{\partial X} + \frac{\partial (V\theta)}{\partial V} = \mathrm{Pr}^{-1} \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \right) \theta$$

where $\mathrm{Gr} = \alpha g H^3 (T_1 - T_0) / \nu^2$ is the Grashof number

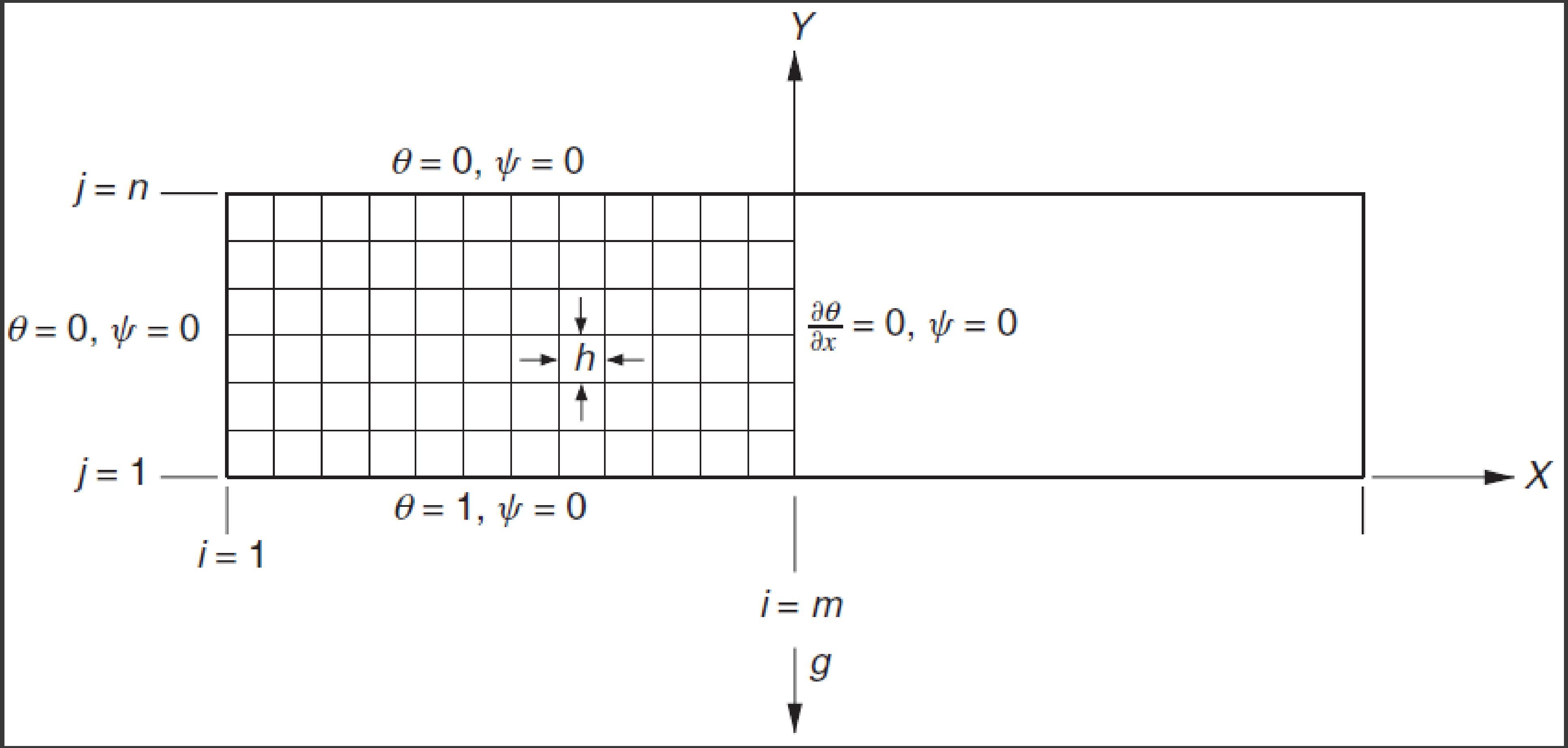
$\mathrm{Pr} = \nu / \kappa$ is the Prandtl number.

The Rayleigh number is Ra = Gr×Pr.

Biringen and Chow (2011, p.236)

Rayleigh-Benard Instability in 2D

Let the fluid be water (Pr = 6.75) enclosed by a rectangular box shown below. Due to the symmetry about the *Y* axis, only the left half needs to be computed.



Biringen and Chow (2011, p.236)

Rayleigh-Benard Instability in 2D

- At time $T = 0$, the temperature distribution is $\theta = 0$ everywhere, except at the bottom where $\theta = 1$.
- The boundary condition for temperature θ is shown in the previous figure.
- Along the Y axis $d\theta/dX = 0$ due to the symmetry.
- The governing equations are approximated by FD approximations as follows.

$$U = \frac{\partial \Psi}{\partial Y} \qquad \rightarrow U_{i,j} = \frac{\Psi_{i,j+1} - \Psi_{i,j-1}}{2h}$$

$$V = -\frac{\partial \Psi}{\partial X} \qquad \rightarrow V_{i,j} = -\frac{\Psi_{i+1,j} - \Psi_{i-1,j}}{2h}$$

Biringen and Chow (2011, p.236)

Rayleigh-Benard Instability in 2D

- The FD and SOR method can be used to solve the Poisson equation

$$\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}\right)\Psi=-\Omega$$

- The boundary conditions for Ψ are $\Psi = 0$ on the three solid walls since there is no net flow across these boundaries.
- In addition, $\Psi = 0$ along the Y-axis to make the fluid motion to its right the mirror image of that to its left.

Biringen and Chow (2011, p.237)

Rayleigh-Benard Instability in 2D

- The symmetric distribution of θ and the antisymmetric distribution of Ψ about the Y axis result in the following boundary conditions for all values of j.

$$\theta_{m+1,j} = \theta_{m-1,j}$$

$$\Psi_{m,j} = 0$$

$$\Psi_{m+1,j} = -\Psi_{m-1,j}$$

$$U_{m+1,j} = -U_{m-1,j}$$

$$U_{m,j} = 0$$

$$V_{m,j} = \Psi_{m-1,j}/h$$

$$\Omega_{m,j} = 0$$

Biringen and Chow (2011, p.237)

Rayleigh-Benard Instability in 2D

- To solve these equations

$$\frac{\partial \Omega}{\partial T}+\frac{\partial (U\Omega)}{\partial X}+\frac{\partial (V\Omega)}{\partial Y}=\mathrm{Gr}\frac{\partial \theta}{\partial X}+\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}\right)\Omega$$

$$\frac{\partial \theta}{\partial T}+\frac{\partial (U\theta)}{\partial X}+\frac{\partial (V\theta)}{\partial Y}=\mathrm{Pr}^{-1}\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}\right)\theta$$

  the upwind FD scheme is used to approximate the nonlinear convection terms.
- We can write a program to solve the generic equation

$$\frac{\partial P}{\partial T}=-\frac{\partial (UP)}{\partial X}-\frac{\partial (VP)}{\partial Y}+A\frac{\partial Q}{\partial X}+B\left(\frac{\partial^2 P}{\partial X^2}+\frac{\partial^2 P}{\partial Y^2}\right)$$

Biringen and Chow (2011, p.237)

Rayleigh-Benard Instability in 2D

- Let's define U_f and U_b as the average x-directional velocities evaluated at half a grid point forward and backward from the point (X_i, Y_j)

$$U_f = \frac{1}{2}\left(U_{i+1,j} + U_{i,j}\right), \qquad U_b = \frac{1}{2}\left(U_{i,j} + U_{i-1,j}\right)$$

- Similarly, V_f and V_b are defined as

$$V_f = \frac{1}{2}\left(V_{i,j+1} + V_{i,j}\right), \qquad V_b = \frac{1}{2}\left(V_{i,j} + V_{i,j-1}\right)$$

Biringen and Chow (2011, p.238)

Rayleigh-Benard Instability in 2D

- For the convection terms, the upwind FD scheme is used to define

$$P1 = \left(U_f - \left|U_f\right|\right)P_{i+1,j} + \left(U_f + \left|U_f\right| - U_b + \left|U_b\right|\right)P_{i,j} - \left(U_b + \left|U_b\right|\right)P_{i-1,j}$$

$$P2 = \left(V_f - \left|V_f\right|\right)P_{i,j+1} + \left(V_f + \left|V_f\right| - V_b + \left|V_b\right|\right)P_{i,j} - \left(V_b + \left|V_b\right|\right)P_{i,j-1}$$

- The terms multiplied by *A* and *B* are approximated by central FD schemes:

$$P3 = Q_{i+1,j} - Q_{i-1,j}$$

$$P4 = P_{i+1,j} + P_{i-1,j} + P_{i,j+1} + P_{i,j-1} - 4P_{i,j}$$

- The time derivative is approximated by the forward FD scheme:

$$\left(\frac{\partial P}{\partial T}\right)_{i,j} = \frac{P'_{i,j} - P_{i,j}}{\Delta T}$$

Biringen and Chow (2011, p.238)

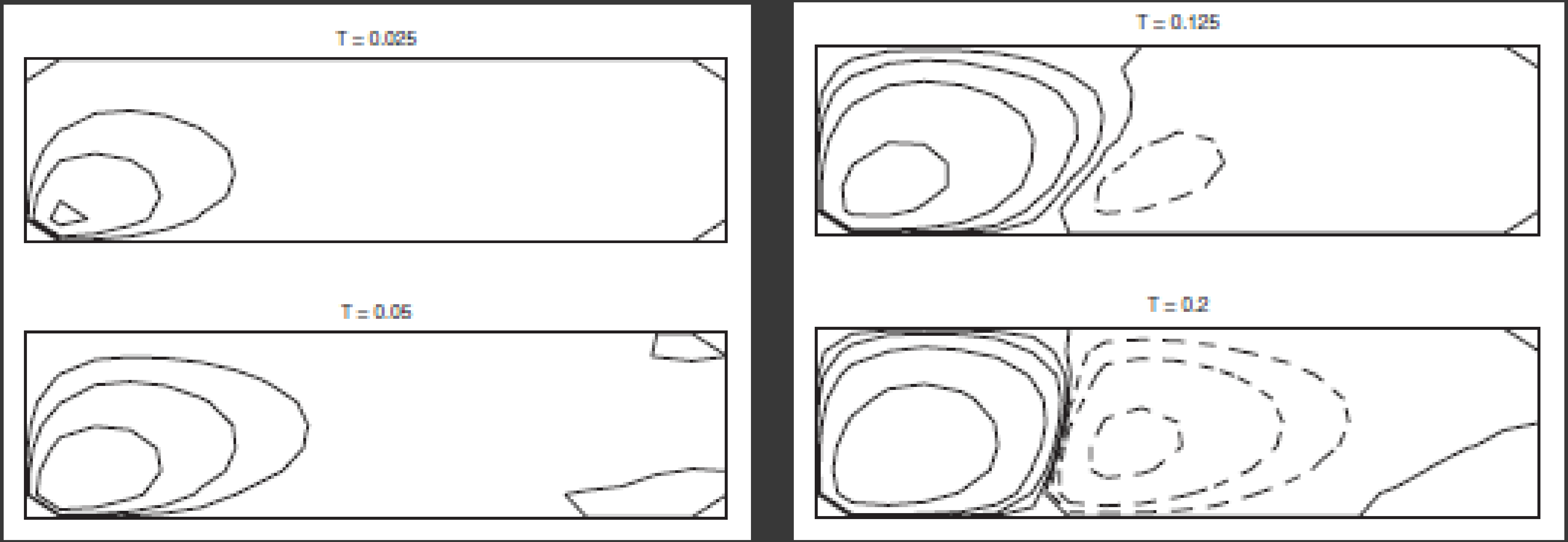
Rayleigh-Benard Instability in 2D

- Here P' denote the value of P evaluated at time $T + \Delta T$.

$$P'_{i,j} = P_{i,j} + \frac{\Delta T}{2h}\left(-P1 - P2 + A \cdot P3 + 2B\frac{P4}{h}\right)$$

Biringen and Chow (2011, p.238)

Rayleigh-Benard Instability in 2D



Incompressible Navier-Stokes Eq.

- 2D incompressible Navier-Stokes equations are given as

$$u_t + p_x = -\left(u^2\right)_x - \left(uv\right)_y + \frac{1}{\text{Re}}\left(u_{xx} + u_{yy}\right)$$

$$v_t + p_y = -\left(uv\right)_x - \left(v^2\right)_y + \frac{1}{\text{Re}}\left(v_{xx} + v_{yy}\right)$$

$$u_x + v_y = 0$$

incompressibility condition $\nabla \cdot \mathbf{v} = 0$

where Re is the Reynolds number.

No-Slip Boundary Conditions

- Consider the case in which all boundary conditions are no-slip.

$$u\left(x, L_y\right) = u_N(x) \qquad v\left(x, L_y\right) = 0$$

$$u(x,0) = u_S(x) \qquad v(x,0) = 0$$

$$u(0,y) = 0 \qquad v(0,y) = v_W(y)$$

$$u(L_x, y) = 0 \qquad v(L_x, y) = v_E(y)$$

Numerical Solution of N-V eq

- Using a splitting method, we can separately treat the advection terms, the diffusion terms, and the pressure gradient.
- The nonlinear advection is solved using an explicit scheme

$$\frac{u^* - u^n}{\Delta t} = -\left[\left(u^n\right)^2\right]_x - \left[u^n v^n\right]_y$$

$$\frac{v^* - v^n}{\Delta t} = -\left[u^n v^n\right]_x - \left[\left(u^n\right)^2\right]_y$$

- The diffusion equation is solved using an implicit (backward Euler) scheme

$$\frac{u^{**} - u^*}{\Delta t} = \left(u_{xx}^{**} + u_{yy}^{**}\right)/\mathrm{Re}$$

$$\frac{v^{**} - v^*}{\Delta t} = \left(v_{xx}^{**} + v_{yy}^{**}\right)/\mathrm{Re}$$

Pressure Correction

- The pressure is the Lagrange multiplier used to enforce the incompressibility condition.

$$\frac{u^{n+1}-u^{**}}{\Delta t}=-\left(p^{n+1}\right)_x \qquad \frac{v^{n+1}-v^{**}}{\Delta t}=-\left(p^{n+1}\right)_y \quad \Rightarrow \quad \frac{1}{\Delta t}\left(\mathbf{v}^{n+1}-\mathbf{v}^{**}\right)=-\nabla p^{n+1}$$

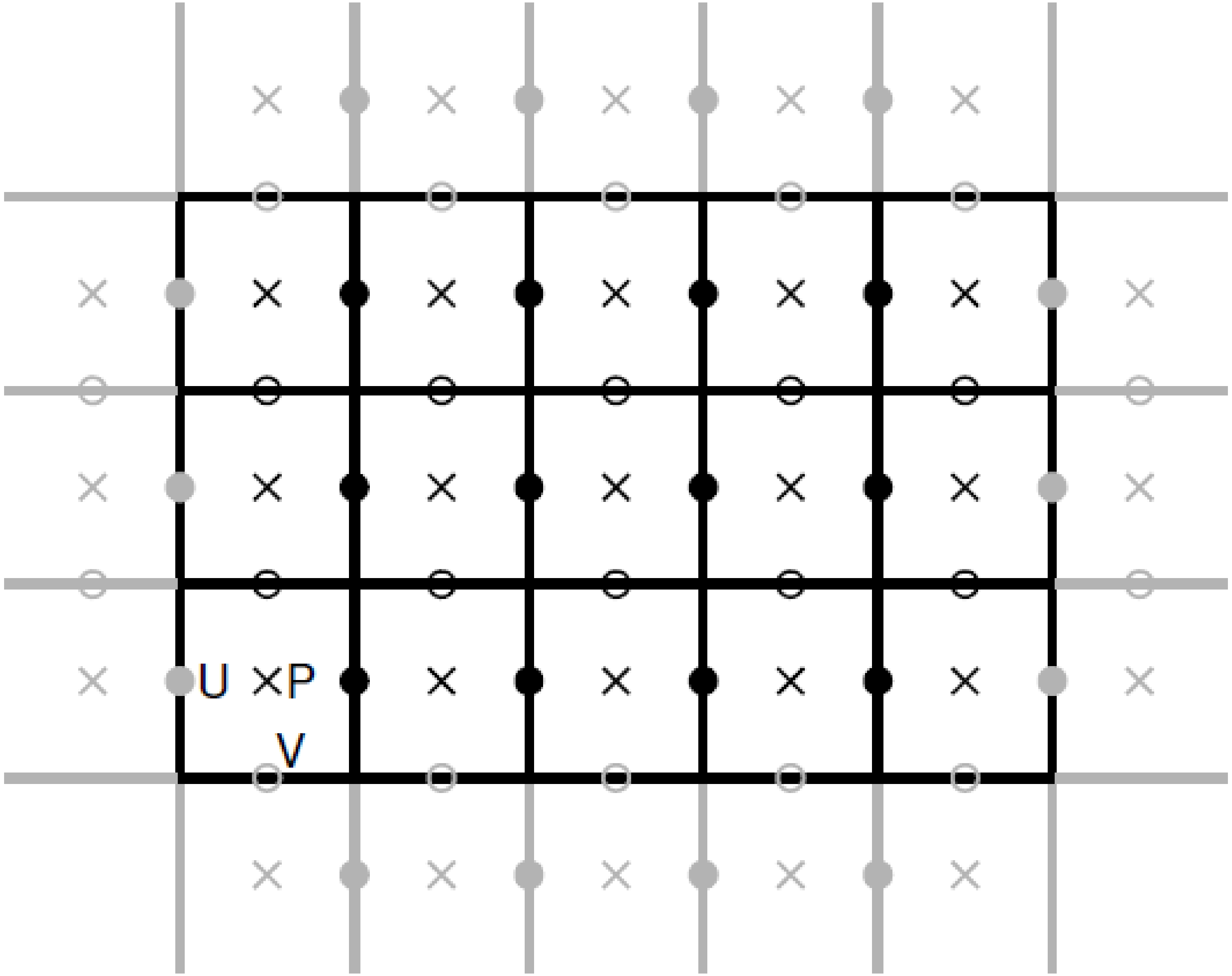
- Applying the divergence yields $-\nabla^2 p^{n+1}=\frac{1}{\Delta t}\left(\nabla\cdot\mathbf{v}^{n+1}-\nabla\cdot\mathbf{v}^{n}\right)=-\frac{1}{\Delta t}\nabla\cdot\mathbf{v}^{n}$

- Solve the Poisson equation for p^{n+1} and then update the velocity field by

$$\mathbf{v}^{n+1}=\mathbf{v}^{**}-\Delta t\nabla p^{n+1}$$

- This is called the pressure projection method.

Staggered Grid



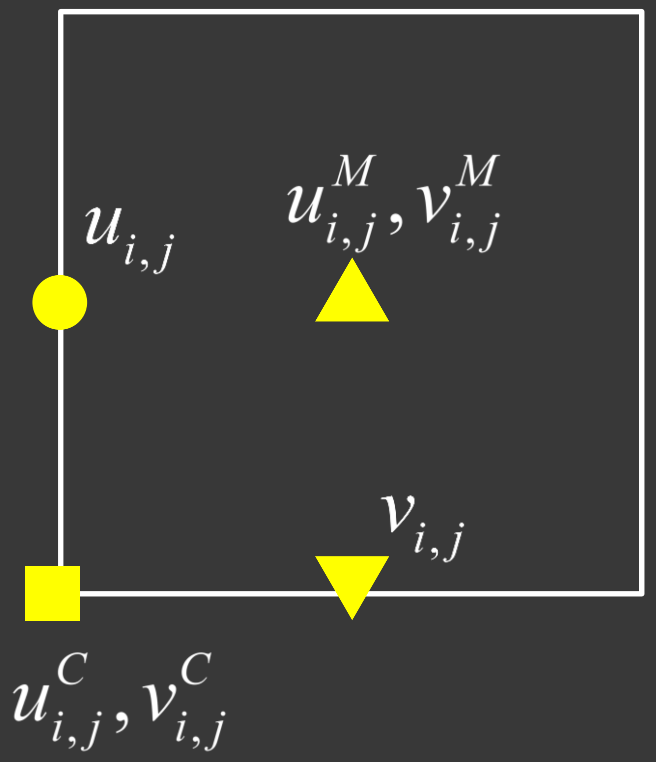
The stream function q used for visualizing streamlines is also computed by solving the Poisson equation

$$\nabla^2 q^n = -\nabla \times \mathbf{v}^n$$

where q^n are defined on the cell corners.

Nonlinear Advection: Revisited

Let $u^M_{i,j} = \dfrac{u_{i,j} + u_{i+1,j}}{2}, \quad v^M_{i,j} = \dfrac{v_{i,j+1} + v_{i,j}}{2}, \quad u^C_{i,j} = \dfrac{u_{i,j-1} + u_{i,j}}{2}, \quad v^C_{i,j} = \dfrac{v_{i-1,j} + v_{i,j}}{2}$



$$\frac{u^*_{i,j} - u^n_{i,j}}{\Delta t} = -\frac{\left(u^M_{i,j}\right)^2 - \left(u^M_{i-1,j}\right)^2}{\Delta x} - \frac{u^C_{i,j+1} v^C_{i,j+1} - u^C_{i,j} v^C_{i,j}}{\Delta y}$$

$$\frac{v^*_{i,j} - v^n_{i,j}}{\Delta t} = -\frac{u^C_{i+1,j} v^C_{i+1,j} - u^C_{i,j} v^C_{i,j}}{\Delta x} - \frac{\left(v^M_{i,j}\right)^2 - \left(v^M_{i-1,j}\right)^2}{\Delta y}$$

An upwind scheme should be used in this case.
See Seibold (2008) for more details.

Boundary Conditions in 2D Flow

- Suppose that the velocity inlet boundary condition is at the left boundary and the pressure outlet boundary condition is at the right boundary.
- At the velocity inlet BC, we have $u = u_0$ and $v = v_0$.
- At the pressure outlet BC, we have

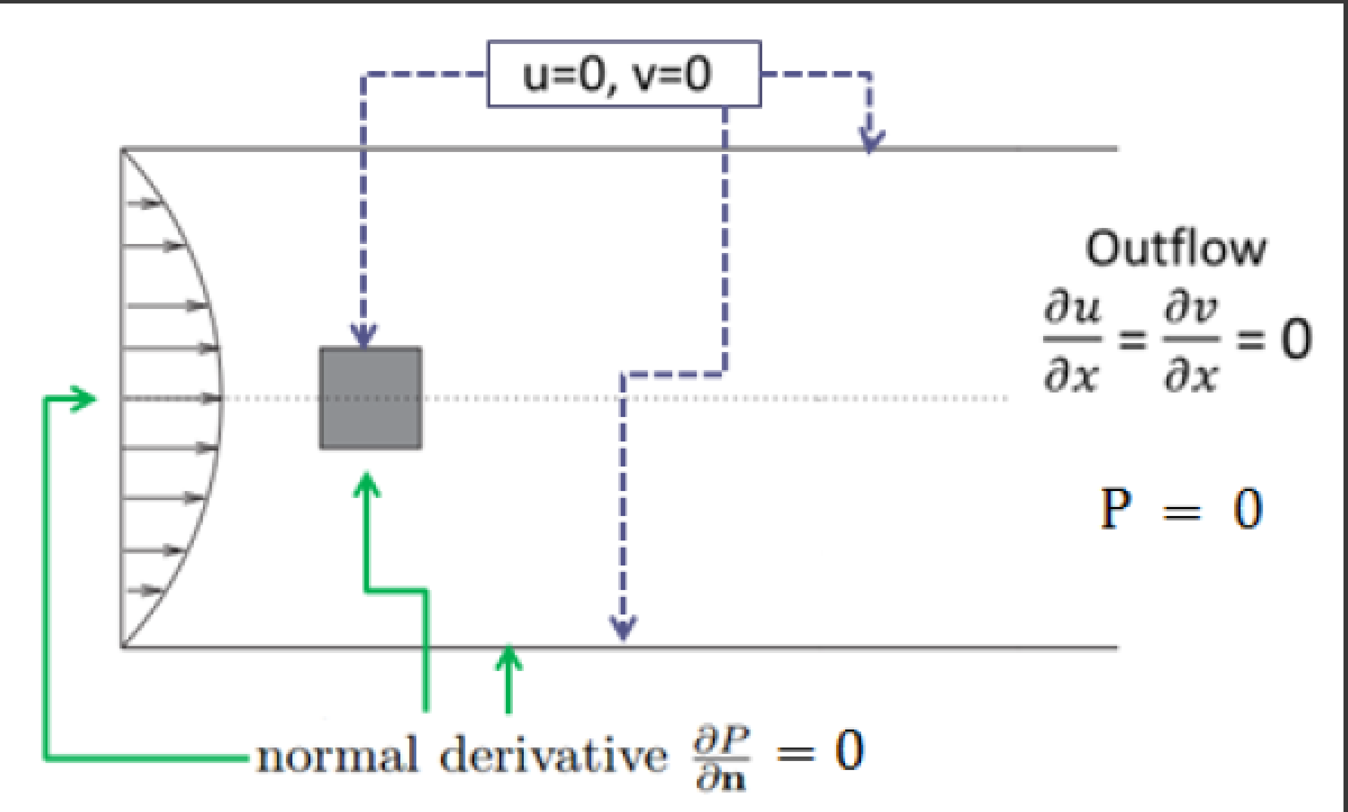
$$\frac{1}{\mathrm{Re}}\frac{\partial u}{\partial x} - p = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} = 0$$

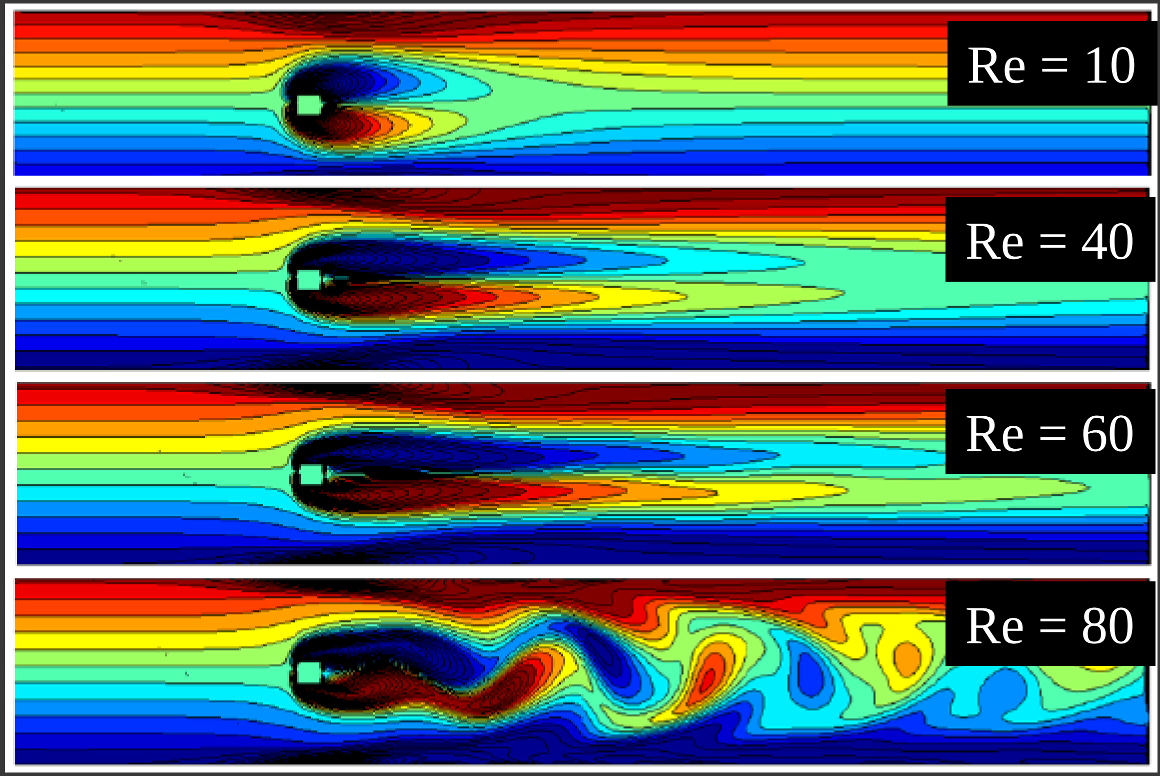
- In a channel flow, no fluid crosses the top and bottom boundaries.
- If the no-slip condition is used, then $u = v = 0$.

Strang (2007)

Flow Past a Square Cylinder

- A flow past a square cylinder can be simulated by solving the 2D incompressible Navier-Stokes equation given previously using the boundary conditions given the below figure.





Pongchanwit and Boonyasiriwat (2015)

2D Smoke Simulation

- A 2D simulation of smoke can be performed by solving the system

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p/\rho_0 + \nu\nabla^2\mathbf{u} + \left(\alpha s - \beta(T - T_{\text{amb}})\right)g$$

$$\nabla\cdot\mathbf{u} = 0$$

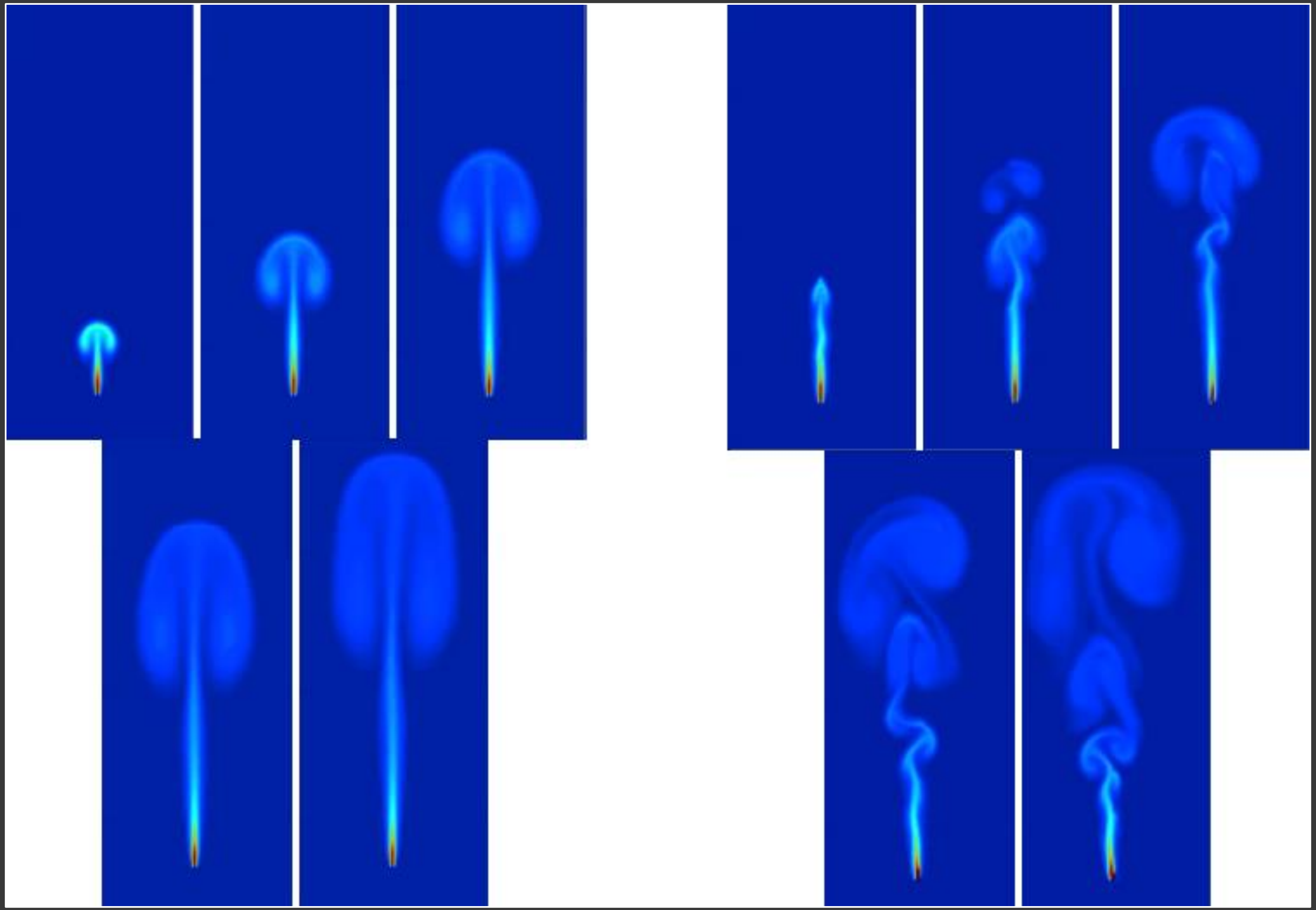
$$T_t + (\mathbf{u}\cdot\nabla)T = D_T\nabla^2 T + r_T\left(T_{\text{source}} - T\right)$$

$$s_t + (\mathbf{u}\cdot\nabla)s = D_s\nabla^2 s + r_s$$

where $\mathbf{u}$ is fluid velocity, p pressure, T temperature, T_{source} temperature at the source point, T_{amb} the ambient temperature, s smoke concentration, ρ_0 the smoke-free air density at ambient temperature, g gravity, D_T diffusion coefficient of temperature, D_s diffusion coefficient of smoke concentration, r_T the rate of heat release, r_s the rate of smoke release.

Chanthanasaro and Boonyasiriwat (2015)

2D Smoke Simulation



$\nu = 10^{-3}$ $\nu = 10^{-5}$

Chanthanasaro and Boonyasiriwat (2015)

Exercise: 1D Diffusion

Solve the 1D diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

using an implicit FD scheme

$$\frac{u_i^n - u_i^{n-1}}{\Delta t} = D\left(\frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{h^2}\right)$$

$$u_i^n - \frac{D\Delta t}{h^2}\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right) = u_i^{n-1}$$

$$\alpha u_{i-1}^n + \beta u_i^n + \alpha u_{i+1}^n = u_i^{n-1}$$

$$\alpha = -\frac{D\Delta t}{h^2}, \beta = 1 + \frac{2D\Delta t}{h^2}$$

Exercise: 2D Diffusion

Solve the 2D diffusion equation

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

using an implicit FD scheme

$$\frac{u_{i,j}^{n} - u_{i,j}^{n-1}}{\Delta t} = D\left(\frac{u_{i-1,j}^{n} - 2u_{i,j}^{n} + u_{i+1,j}^{n}}{h^2} + \frac{u_{i,j-1}^{n} - 2u_{i,j}^{n} + u_{i,j+1}^{n}}{h^2}\right)$$

$$u_{i,j}^{n} - \frac{D\Delta t}{h^2}\left(u_{i-1,j}^{n} + u_{i+1,j}^{n} + u_{i,j-1}^{n} + u_{i,j+1}^{n} - 4u_{i}^{n}\right) = u_{i}^{n-1}$$

$$\alpha u_{i-1,j}^{n} + \alpha u_{i+1,j}^{n} + \alpha u_{i,j-1}^{n} + \alpha u_{i,j+1}^{n} + \beta u_{i,j}^{n} = u_{i}^{n-1}$$

$$\alpha = -\frac{D\Delta t}{h^2}, \beta = 1 + \frac{4D\Delta t}{h^2}$$

Inviscid Burgers' Equation

- The inviscid Burgers' equation governing a nonlinear convection is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

  where u is the fluid velocity.

- Consider the Taylor series expansion

$$u_i^{n+1} = u_i^n + \left(u_t\right)_i^n \Delta t + \left(u_{tt}\right)_i^n \frac{\Delta t^2}{2} + O\left(\Delta t^3\right)$$

  where $u_t = \dfrac{\partial u}{\partial t}, u_{tt} = \dfrac{\partial^2 u}{\partial t^2}$

- Let's approximate u_{tt} as $\left(u_{tt}\right)_i^n = \dfrac{\left(u_t\right)_i^n - \left(u_t\right)_i^{n-1}}{\Delta t} + O\left(\Delta t\right)$

Inviscid Burgers' Equation

- The Taylor series expansion can then be written as

$$u_i^{n+1} = u_i^n + \left(u_t\right)_i^n \Delta t + \left[\left(u_t\right)_i^n - \left(u_t\right)_i^{n-1}\right]\frac{\Delta t}{2} + O\left(\Delta t^3\right)$$

- According to the inviscid Burgers' equation,

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}$$

- We then obtain an Adams-Bashforth scheme

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{3}{2}\left(-u\frac{\partial u}{\partial x}\right)_i^n - \frac{1}{2}\left(-u\frac{\partial u}{\partial x}\right)_i^{n-1} + O\left(\Delta t^2\right)$$

- The spatial derivatives can be approximated using the second-order centered FD approximation.

Biringen and Chow (2011, p. 250)

Inviscid Burgers' Equation

- We then obtain an explicit second-order numerical scheme

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\frac{3}{2}\left[-u_i^n\left(\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}\right)\right]-\frac{1}{2}\left[-u_i^{n-1}\left(\frac{u_{i+1}^{n-1}-u_{i-1}^{n-1}}{2\Delta x}\right)\right]+O\left(\Delta x^2,\Delta t^2\right)$$

- This scheme is unconditionally unstable but the instability remains bounded if the Courant number satisfies the condition

$$C\equiv\frac{u_{\max}\Delta t}{\Delta x}\le 1 \qquad \leftrightarrow \qquad \Delta t\le\frac{\Delta x}{u_{\max}}$$

- This scheme requires initial velocity field at two time steps.
- We can use the explicit Euler method to obtain the field at the second step as

$$u_i^1=u_i^0-\Delta t\left[u_i^0\left(\frac{u_{i+1}^0-u_{i-1}^0}{2\Delta x}\right)\right]+O\left(\Delta x^2,\Delta t^2\right)$$

Biringen and Chow (2011, p. 251)

Viscous Burgers' Equation

- The viscous Burgers' equation is a nonlinear convection-diffusion equation

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} \equiv H$$

- The corresponding Adams-Bashforth scheme with central FD in space for this case is

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{3}{2}H_i^n - \frac{1}{2}H_i^{n-1} + O\left(\Delta x^2, \Delta t^2\right)$$

- Using von Neumann analysis, the convective and diffusive stability conditions are

$$\frac{u_{\max}\Delta t}{\Delta x} \le 1, \qquad \frac{\nu\Delta t}{\Delta x^2} \le \frac{1}{2}$$

- As a result, the time step must satisfy $\Delta t \le \min\left(\frac{\Delta x}{u_{\max}}, \frac{\Delta x^2}{2\nu}\right)$

Biringen and Chow (2011, p. 252)

Exercise: Inviscid Burger's Eq.

Solve the initial value problem

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \qquad x \in [0,1]$$

$$u(x,0) = 0.1 + 0.1e^{-1000(x-0.2)^2}$$

using the numerical scheme

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{4\Delta x}\left[3u_i^n\left(u_{i+1}^n - u_{i-1}^n\right) - u_i^{n-1}\left(u_{i+1}^{n-1} - u_{i-1}^{n-1}\right)\right]$$

$$u_i^1 = u_i^0 - \Delta t\left[u_i^0\left(\frac{u_{i+1}^0 - u_{i-1}^0}{2\Delta x}\right)\right]$$

$$\Delta t \le \Delta x / u_{\max}$$

Exercise: Viscous Burger's Eq.

Solve the initial value problem

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2},\qquad x\in[0,1],\quad \nu=10^{-4}$$

$$u(x,0)=0.1+0.1e^{-1000(x-0.2)^2}$$

using the numerical scheme

$$u_i^{n+1}=u_i^n+\frac{\Delta t}{2}\left(3H_i^n-H_i^{n-1}\right),\qquad H_i^n=-u_i^n\left(\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}\right)+\nu\left(\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{\Delta x^2}\right)$$

$$u_i^1=u_i^0+\Delta t\left[-u_i^0\left(\frac{u_{i+1}^0-u_{i-1}^0}{2\Delta x}\right)+\nu\left(\frac{u_{i+1}^0-2u_i^0+u_{i-1}^0}{\Delta x^2}\right)\right]$$

$$\Delta t\le\min\left(\frac{\Delta x}{u_{\max}},\frac{\Delta x^2}{2\nu}\right)$$

Diffusion Equation

- Consider the diffusion equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

- Recall the Taylor series expansion

$$f(t_1) = f(t_2) + f'(t_2)(t_1 - t_2) + f''(t_2)\frac{(t_1 - t_2)^2}{2} + f'''(t_2)\frac{(t_1 - t_2)^3}{6} + \cdots$$

- We can then expand u_i^n and u_i^{n+1} as follows.

$$u_i^{n+1} = u_i^n + (u_t)_i^n \Delta t + (u_{tt})_i^n \frac{\Delta t^2}{2} + (u_{ttt})_i^n \frac{\Delta t^3}{6} + \cdots$$

$$u_i^n = u_i^{n+1} - (u_t)_i^{n+1} \Delta t + (u_{tt})_i^{n+1} \frac{\Delta t^2}{2} - (u_{ttt})_i^{n+1} \frac{\Delta t^3}{6} + \cdots$$

Biringen and Chow (2011, p. 252)

Diffusion Equation

- Subtracting the last two equations yield

$$u_i^{n+1} - u_i^n = u_i^n - u_i^{n+1} + (u_t)_i^n \Delta t + (u_t)_i^{n+1} \Delta t +$$

$$(u_{tt})_i^n \frac{\Delta t^2}{2} - (u_{tt})_i^{n+1} \frac{\Delta t^2}{2} + (u_{ttt})_i^n \frac{\Delta t^3}{6} + (u_{ttt})_i^{n+1} \frac{\Delta t^3}{6} + \cdots$$

- Substituting the expansion

$$(u_{tt})_i^{n+1} = (u_{tt})_i^n + (u_{ttt})_i^n \Delta t + O\left(\Delta t^2\right)$$

into the last equation yields

$$2\left(u_i^{n+1} - u_i^n\right) = (u_t)_i^n \Delta t + (u_t)_i^{n+1} \Delta t - (u_{ttt})_i^n \frac{\Delta t^3}{6} + \cdots$$

Biringen and Chow (2011, p. 252-253)

Crank-Nicolson Scheme

- Rearranging the last equation yields

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\frac{1}{2}\left(u_t\right)_i^n+\frac{1}{2}\left(u_t\right)_i^{n+1}\Delta t+O\left(\Delta t^2\right)$$

- Substituting $u_t = \nu u_{xx}$ into the last equation yields

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\frac{1}{2}\left(\nu u_{xx}\right)_i^n+\frac{1}{2}\left(\nu u_{xx}\right)_i^{n+1}\Delta t+O\left(\Delta t^2\right)$$

- Approximating the spatial derivative by second-order central difference yields the Crank-Nicolson scheme

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\frac{\nu}{2\Delta x^2}\left(u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}+u_{i+1}^n-2u_i^n+u_{i-1}^n\right)+O\left(\Delta x^2,\Delta t^2\right)$$

Biringen and Chow (2011, p. 253)

Crank-Nicolson Scheme

- Let $\alpha = \nu\Delta t/\Delta x^2$. The Crank-Nicolson scheme can be written as

$$-\frac{\alpha}{2}u_{i-1}^{n+1}+\left(1+\alpha\right)u_i^{n+1}-\frac{\alpha}{2}u_{i+1}^{n+1}=\frac{\alpha}{2}u_{i-1}^{n}+\left(1-\alpha\right)u_i^{n}+\frac{\alpha}{2}u_{i+1}^{n}$$

- Let's apply the Crank-Nicolson method to the Burgers equation

$$u_t=-uu_x+\nu u_{xx}$$

- Substituting the last equation into the Crank-Nicolson scheme yields

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\frac{1}{2}\left(u_t\right)_i^n+\frac{1}{2}\left(u_t\right)_i^{n+1}\Delta t=\frac{1}{2}\left(-uu_x+\nu u_{xx}\right)_i^n+\frac{1}{2}\left(-uu_x+\nu u_{xx}\right)_i^{n+1}\Delta t$$

- Using the central difference on the convection term

$$\left(uu_x\right)_i^{n+1}=u_i^{n+1}\left(\frac{u_{i+1}^{n+1}-u_{i-1}^{n+1}}{2\Delta x}\right)$$

leads to a nonlinear difference equation which is more difficult to solve.

Biringen and Chow (2011, p. 253)

Adams-Bashforth-Crank-Nicolson Scheme

- A more popular numerical scheme for a convection-diffusion equation is a semi-implicit Adams-Bashforth-Crank-Nicolson scheme.
- The explicit Adams-Bashforth scheme is used for the convection term while the Crank-Nicolsone scheme is used for the diffusion term.
- The Burgers' equation can be written as $u_t = -\frac{1}{2}\left(u^2\right)_x + \nu u_{xx}$
- The explicit Adams-Bashforth scheme for the convection term is

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{3}{2}\left(-\frac{1}{2}\frac{\partial}{\partial x}\left(u^2\right)\right)_i^n - \frac{1}{2}\left(-\frac{1}{2}\frac{\partial}{\partial x}\left(u^2\right)\right)_i^{n-1} + O\left(\Delta t^2\right)$$

- The implicit Crank-Nicolson scheme for the diffusion term is

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{2}\left(\nu u_{xx}\right)_i^n + \frac{1}{2}\left(\nu u_{xx}\right)_i^{n+1} \Delta t + O\left(\Delta t^2\right)$$

Biringen and Chow (2011, p. 254)

Adams-Bashforth-Crank-Nicolson Scheme

- Combining the two schemes yields the semi-implicit second-order scheme

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\frac{3}{2}\left(-\frac{1}{2}\frac{\partial}{\partial x}\left(u^2\right)\right)_i^n-\frac{1}{2}\left(-\frac{1}{2}\frac{\partial}{\partial x}\left(u^2\right)\right)_i^{n-1}$$

$$+\frac{1}{2}\left(\nu u_{xx}\right)_i^n+\frac{1}{2}\left(\nu u_{xx}\right)_i^{n+1}+O\left(\Delta x^2,\Delta t^2\right)$$

where the spatial derivatives are approximated be second-order central FD.

- Rearranging the equation yields

$$u_i^{n+1}-\frac{\nu\Delta t}{2}\left(u_{xx}\right)_i^{n+1}=\underbrace{u_i^n-\frac{3\Delta t}{4}\left(\left(u^2\right)_x\right)_i^n+\frac{\Delta t}{4}\left(\left(u^2\right)_x\right)_i^{n-1}+\frac{\nu\Delta t}{2}\left(u_{xx}\right)_i^n}_{\mathrm{RHS}_i^n}$$

Biringen and Chow (2011, p. 254)

Adams-Bashforth-Crank-Nicolson Scheme

- The semi-implicit scheme can then be written as

$$-\frac{\alpha}{2}u_{i-1}^{n+1}+(\alpha+1)u_i^{n+1}-\frac{\alpha}{2}u_{i+1}^{n+1}=\mathrm{RHS}_i^n$$

where
$$\mathrm{RHS}_i^n=\frac{\alpha}{2}\left(u_{i-1}^n+u_{i+1}^n\right)+(1-\alpha)u_i^n$$
$$-3\beta\left[\left(u_{i+1}^n\right)^2-\left(u_{i-1}^n\right)^2\right]+\beta\left[\left(u_{i+1}^{n-1}\right)^2-\left(u_{i-1}^{n-1}\right)^2\right]$$
$$\alpha=\frac{\nu\Delta t}{\Delta x^2},\beta=\frac{\Delta t}{8\Delta x}$$

- This scheme is stable when $\Delta t\le\Delta x/u_{\max}$.
- The corresponding tridiagonal linear system can be efficiently solved using the Thomas algorithm.

Biringen and Chow (2011, p. 255)

Primitive-Variable Formulation

- Consider the incompressible Navier-Stokes equations

$$\nabla \cdot \mathbf{v} = 0$$

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{v}$$

- Here, $P = p/\rho$, where p is thermodynamic pressure.
- Nondimensionalization was performed using the characteristic length L and characteristic velocity U.
- The Reynolds number is defined as $\mathrm{Re} = UL/\nu$.
- Here, we will use the projection method to solve the incompressible Navier-Stokes equations.

Biringen and Chow (2011, p. 258)

Projection Method

- Let's define **H** as follows.

$$\mathbf{H} = \mathbf{v}\cdot\nabla\mathbf{v} = \nabla\cdot(\mathbf{v}\mathbf{v}) - \mathbf{v}\nabla\cdot\mathbf{v} = \nabla\cdot(\mathbf{v}\mathbf{v}) \quad \text{since } \nabla\cdot\mathbf{v} = 0$$

- The projection method first splits the Navier-Stokes equation into

$$\mathbf{v}_t = -\mathbf{H} + \nabla^2\mathbf{v}/\mathrm{Re}$$

$$\mathbf{v}_t = -\nabla P$$

- The first equation is solved using the semi-implicit scheme presented previously:

$$\frac{\mathbf{v}^* - \mathbf{v}^n}{\Delta t} = -\frac{3}{2}\mathbf{H}^n + \frac{1}{2}\mathbf{H}^{n-1} + \frac{1}{2\,\mathrm{Re}}\nabla^2\left(\mathbf{v}^* + \mathbf{v}^n\right)$$

- Next, the second equation is solved using the implicit scheme

$$\frac{\mathbf{v}^{n+1} - \mathbf{v}^*}{\Delta t} = -\nabla P^{n+1}$$

Biringen and Chow (2011, p. 258)

Projection Method

- Applying divergence to the last equation yields

$$\frac{\nabla \cdot \mathbf{v}^{n+1} - \nabla \cdot \mathbf{v}^{*}}{\Delta t} = -\nabla \cdot \nabla P^{n+1}$$

- We want v^{n+1} to satisfy the incompressibility condition, i.e., $\nabla \cdot \mathbf{v}^{n+1} = 0$
- Thus, we obtain the Poisson equation

$$\nabla^{2} P^{n+1} = \frac{\nabla \cdot \mathbf{v}^{*}}{\Delta t}$$

- After solving the Poisson equation for P^{n+1}, the velocity $\mathbf{v}^{n+1}$ is then updated by

$$\mathbf{v}^{n+1} = \mathbf{v}^{*} - \Delta t \nabla P^{n+1}$$

Biringen and Chow (2011, p. 258)

2D Incompressible Navier-Stokes Eq.

- 2D incompressible Navier-Stokes equations are given as

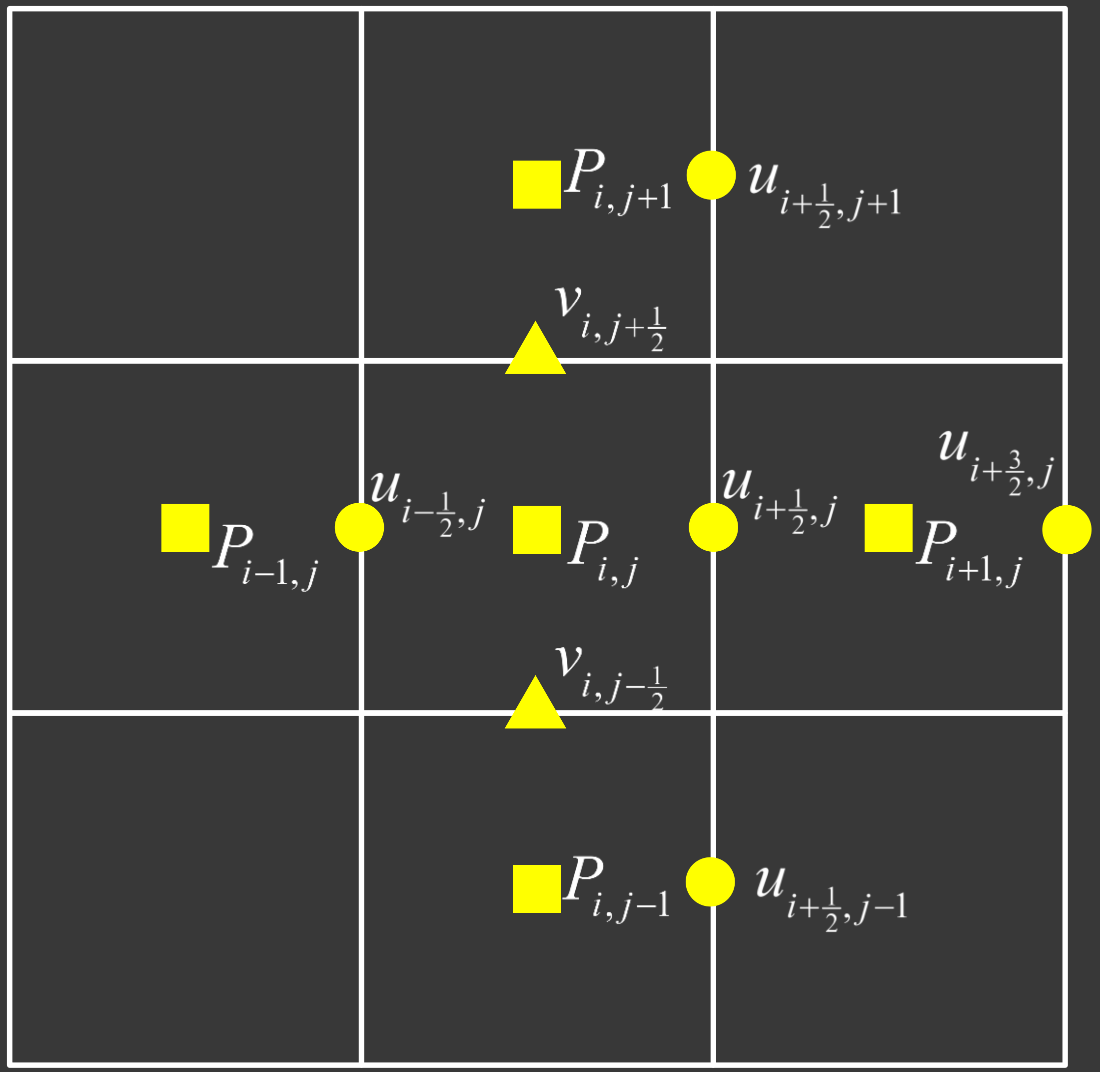
$$u_x + v_y = 0$$

$$u_t = -\left(u^2\right)_x - \left(uv\right)_y + \frac{1}{\mathrm{Re}}\left(u_{xx} + u_{yy}\right) - P_x$$

$$v_t = -\left(uv\right)_x - \left(v^2\right)_y + \frac{1}{\mathrm{Re}}\left(v_{xx} + v_{yy}\right) - P_y$$

Biringen and Chow (2011, p. 265)

x-Momentum Equation



Terms in the x-momentum equation

$$u_t = -\left(u^2\right)_x - \left(uv\right)_y + \frac{1}{\mathrm{Re}}\left(u_{xx} + u_{yy}\right) - P_x$$

are computed as follows.

$$\left(u_t\right)_{i+\frac{1}{2},j}^{n+1} = \left(u_{i+\frac{1}{2},j}^{n+1} - u_{i+\frac{1}{2},j}^{n}\right)\Big/\Delta t$$

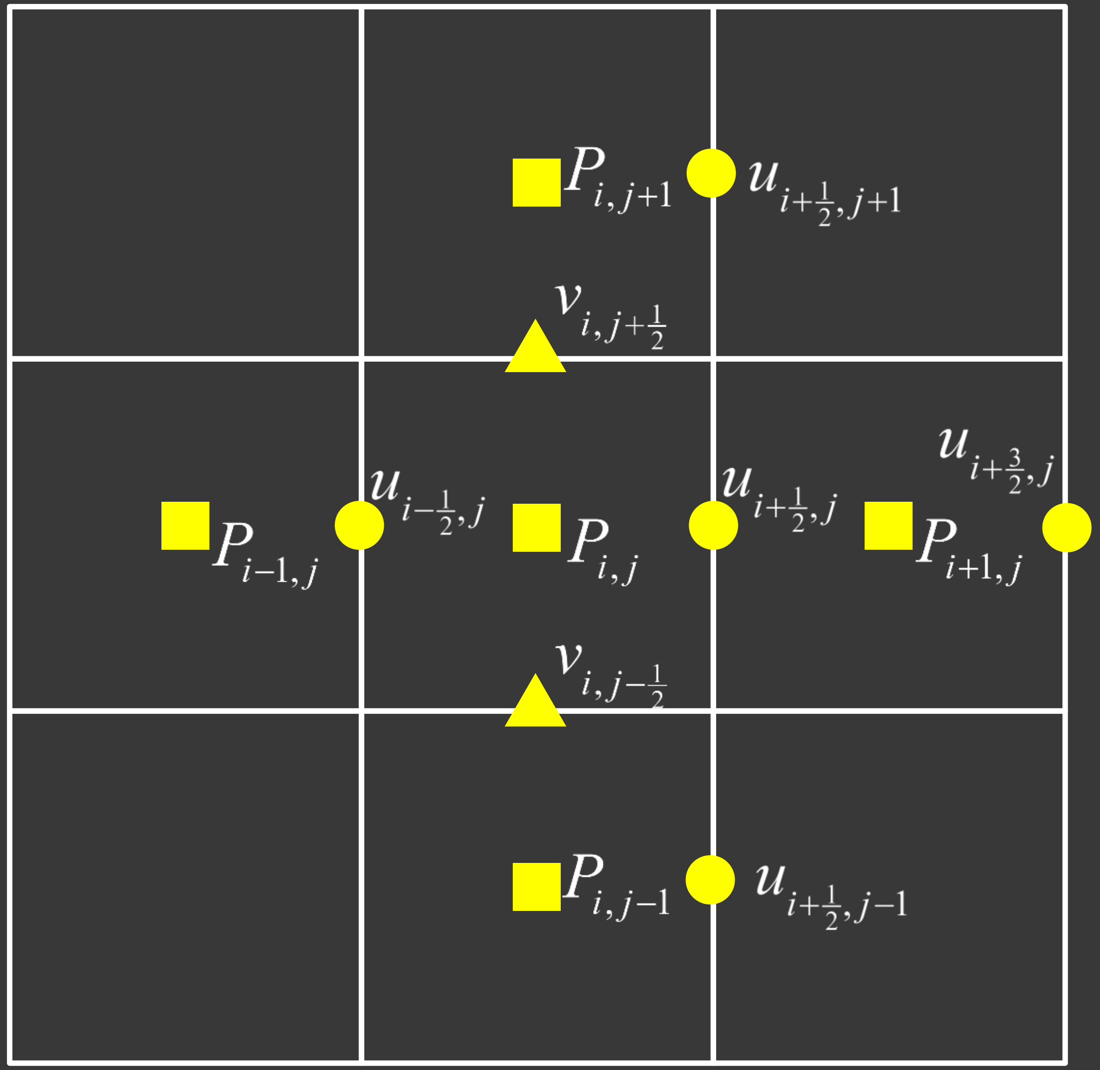
$$\left(u_{xx}\right)_{i+\frac{1}{2},j}^{n+1} = \left(u_{i-\frac{1}{2},j}^{n+1} - 2u_{i+\frac{1}{2},j}^{n+1} + u_{i+\frac{3}{2},j}^{n+1}\right)\Big/\Delta x^2$$

$$\left(u_{yy}\right)_{i+\frac{1}{2},j}^{n+1} = \left(u_{i+\frac{1}{2},j-1}^{n+1} - 2u_{i+\frac{1}{2},j}^{n+1} + u_{i+\frac{1}{2},j+1}^{n+1}\right)\Big/\Delta y^2$$

$$\left(P_x\right)_{i+\frac{1}{2},j}^{n+1} = \left(P_{i+1,j}^{n+1} - P_{i,j}^{n+1}\right)\Big/\Delta x$$

Biringen and Chow (2011, p. 265)

x-Momentum Equation



$$\left(\left(u^2\right)_x\right)^n_{i+\frac{1}{2},j} = \left(\left(u^n_{i+1,j}\right)^2 - \left(u^n_{i,j}\right)^2\right)\Big/\Delta x$$

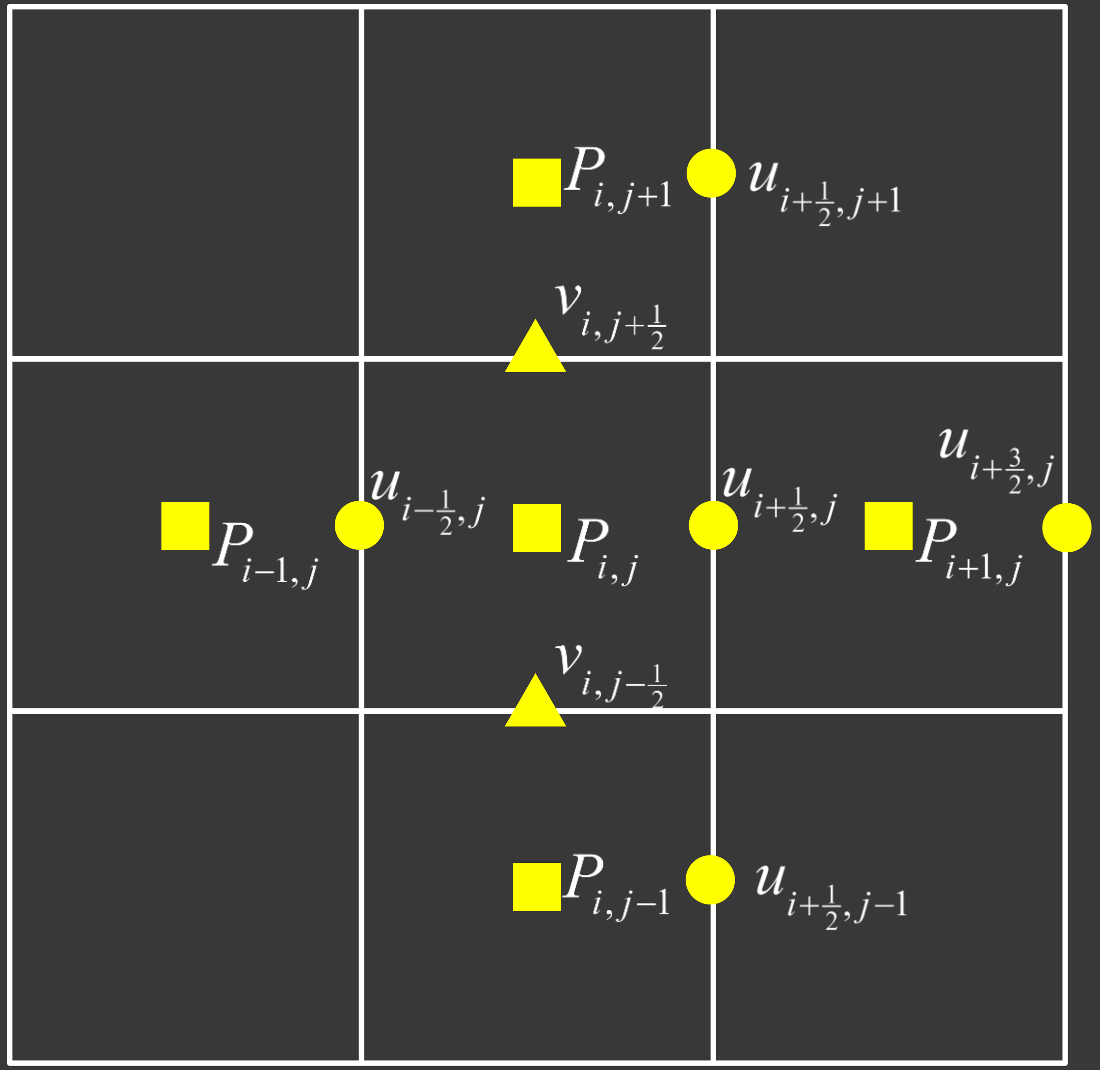
$u_{i+1,j}$ is not defined on a u node. So, it is defined as an average:

$$u_{i+1,j} = \frac{1}{2}\left(u_{i+1/2,j} + u_{i+3/2,j}\right)$$

The term $(uv)_y$ will be evaluated as the product of averages:

$$\left((uv)_y\right)^n_{i+\frac{1}{2},j} = \left[(uv)_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)_{i+\frac{1}{2},j-\frac{1}{2}}\right]\Big/\Delta y$$

x-Momentum Equation



$$(uv)_{i+\frac{1}{2},j+\frac{1}{2}} = u_{i+\frac{1}{2},j+\frac{1}{2}} v_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$(uv)_{i+\frac{1}{2},j-\frac{1}{2}} = u_{i+\frac{1}{2},j-\frac{1}{2}} v_{i+\frac{1}{2},j-\frac{1}{2}}$$

where

$$u_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}\left(u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j+1}\right)$$

$$v_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}\left(v_{i,j+\frac{1}{2}} + v_{i+1,j+\frac{1}{2}}\right)$$

$$u_{i+\frac{1}{2},j-\frac{1}{2}} = \frac{1}{2}\left(u_{i+\frac{1}{2},j-1} + u_{i+\frac{1}{2},j}\right)$$

$$v_{i+\frac{1}{2},j-\frac{1}{2}} = \frac{1}{2}\left(v_{i,j-\frac{1}{2}} + v_{i+1,j-\frac{1}{2}}\right)$$

Biringen and Chow (2011, p. 265)

Velocity Boundary Conditions

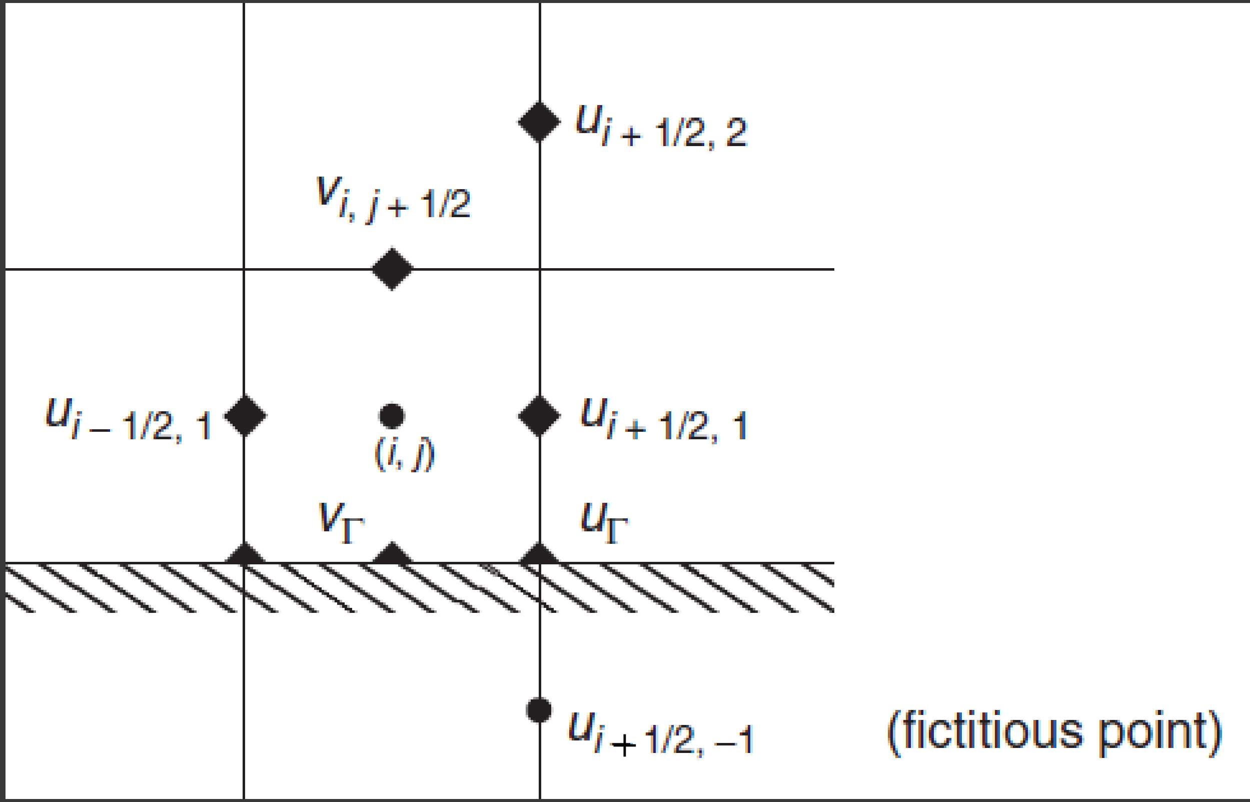
- An explicit boundary condition on the pressure is not needed.
- Suppose we have a boundary condition for u on a horizontal wall as u_Γ.
- The value of u at a fictitious point can be computed from linear extrapolation

$$u_\Gamma = \tfrac{1}{2}\left(u_{i+\frac{1}{2},1} + u_{i+\frac{1}{2},-1}\right)$$

$$u_{i+\frac{1}{2},-1} = 2u_\Gamma - u_{i+\frac{1}{2},1}$$

  or quaratic extrapolation

$$u_{i+\frac{1}{2},-1} = \tfrac{1}{3}\left(8u_\Gamma - 6u_{i+\frac{1}{2},1} + u_{i+\frac{1}{2},2}\right)$$

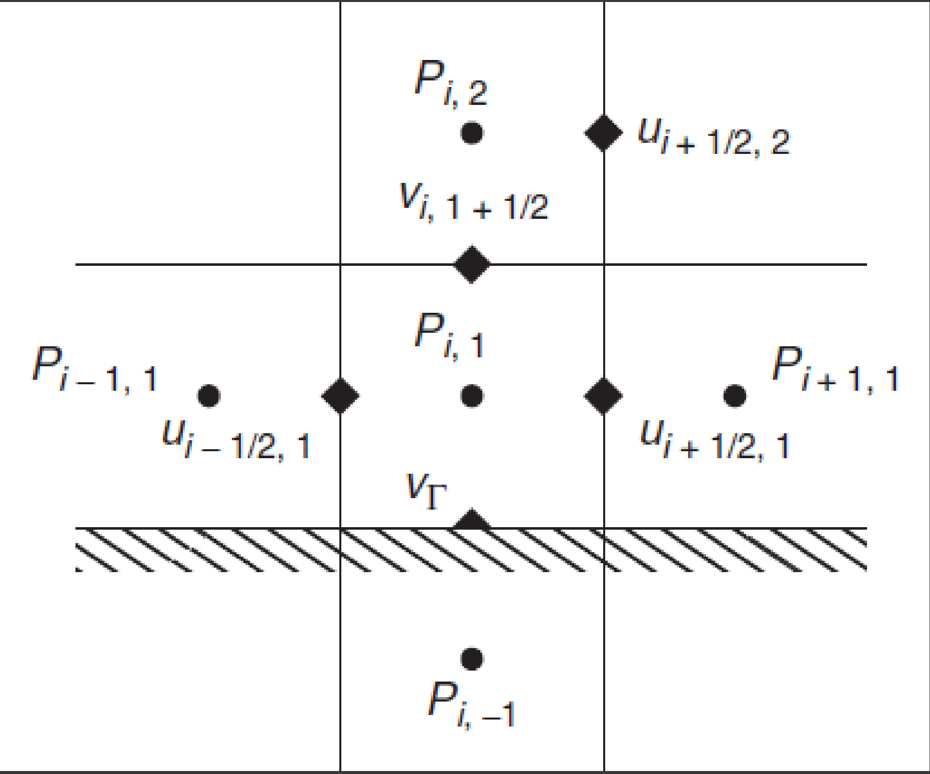


- For a stationary wall, $u_\Gamma = 0$.
- For a vertical wall, extrapolation formulas can be used for v.

Biringen and Chow (2011, p. 266)

Pressure Boundary Conditions

- At a wall, the boundary condition for the pressure is the Neumann boundary condition $\partial P/\partial n = 0$
- The Poisson equation $\nabla^2 P^{n+1} = \dfrac{\nabla \cdot \mathbf{v}^*}{\Delta t}$ becomes



$$\frac{P_{i+1,1}^{n+1} - 2P_{i,1}^{n+1} + P_{i-1,1}^{n+1}}{\Delta x^2} + \frac{P_{i,2}^{n+1} - 2P_{i,1}^{n+1} + P_{i,-1}^{n+1}}{\Delta y^2} =$$

$$\frac{1}{\Delta t}\left[\frac{u_{i+\frac{1}{2},1}^* - u_{i-\frac{1}{2},1}^*}{\Delta x} + \frac{v_{i,1+\frac{1}{2}}^* - v_\Gamma^*}{\Delta y}\right]$$

$$P_{i,-1}^{n+1} = P_{i,1}^{n+1} \quad \text{from } \partial P/\partial n = 0$$

Biringen and Chow (2011, p. 267)

Vorticity-Stream Function Formulation

- Here we will derive the vorticity-stream function formulation of incompressible flows from the Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\left(\frac{p}{\rho}\right) + \nu\nabla^2\mathbf{v}$$

- The convection term can be written as

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \nabla\left(\tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) - \mathbf{v}\times(\nabla\times\mathbf{v}) = \nabla\left(\tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) - \mathbf{v}\times\boldsymbol{\omega}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is the vorticity.

- The Navier-Stokes equation can then be written as

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) - \mathbf{v}\times\boldsymbol{\omega} = -\nabla\left(\frac{p}{\rho}\right) + \nu\nabla^2\mathbf{v}$$

Biringen and Chow (2011, p. 280)

Vorticity-Stream Function Formulation

- Taking curl of $\mathbf{v}_t + \nabla\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) - \mathbf{v}\times\boldsymbol{\omega} = -\nabla\left(p/\rho\right) + \nu\nabla^2\mathbf{v}$, we obtain

$$\boldsymbol{\omega}_t - \nabla\times\left(\mathbf{v}\times\boldsymbol{\omega}\right) = \nu\nabla^2\boldsymbol{\omega}$$

- The gradient terms vanish since $\nabla\times\nabla f = \mathbf{0}$
- Using vector identities, we obtain

$$\nabla\times\left(\mathbf{v}\times\boldsymbol{\omega}\right) = \mathbf{v}\left(\nabla\cdot\boldsymbol{\omega}\right) - \boldsymbol{\omega}\left(\nabla\cdot\mathbf{v}\right) - \left(\mathbf{v}\cdot\nabla\right)\boldsymbol{\omega} + \left(\boldsymbol{\omega}\cdot\nabla\right)\mathbf{v}$$

$\nabla\cdot\boldsymbol{\omega} = \nabla\cdot\nabla\times\mathbf{v} = 0$ Divergence of curl of a vector is zero

$\nabla\cdot\mathbf{v} = 0$ Incompressibility condition

$\left(\boldsymbol{\omega}\cdot\nabla\right)\mathbf{v} = 0$ for 2D flows

- The equation $\boldsymbol{\omega}_t - \nabla\times\left(\mathbf{v}\times\boldsymbol{\omega}\right) = \nu\nabla^2\boldsymbol{\omega}$ then becomes the vorticity transport equation

$$\boldsymbol{\omega}_t + \left(\mathbf{v}\cdot\nabla\right)\boldsymbol{\omega} = \nu\nabla^2\boldsymbol{\omega}$$

Biringen and Chow (2011, p. 281)

Vorticity-Stream Function Formulation

- In 2D flow in the *x*-*y* plane, only the *z*-component of the vorticity, denoted as ω, is nonzero. We then obtain the scalar equation

$$\omega_t + u\omega_x + v\omega_y = \nu\left(\omega_{xx} + \omega_{yy}\right)$$

- The velocities *u* and *v* can be written in terms of the stream function as

$$u = \psi_y, \qquad v = -\psi_x$$

- Substituting these into the definition of vorticity $\omega = v_x - u_y$ yields the Poisson equation

$$\nabla^2\psi = \psi_{xx} + \psi_{yy} = -\omega$$

- For the vorticity-stream function formulation, a collocated grid, in which all variables are at the same nodes, can be used.

Biringen and Chow (2011, p. 281)

Driven Cavity Problem

- The Navier-Stokes equation is nondimensionalized by the lid velocity U and the cavity height L.
- So, the Reynolds number is defined as $\mathrm{Re} = UL/\nu$.
- Using the second-order central differencing for spatial derivatives and the Euler method for time integration, the vorticity transport equation becomes the difference equation

$$\frac{\omega_{i,j}^{n+1} - \omega_{i,j}^{n}}{\Delta t} + u_{i,j}^{n} \frac{\omega_{i+1,j}^{n} - \omega_{i-1,j}^{n}}{2h} + v_{i,j}^{n} \frac{\omega_{i,j+1}^{n} - \omega_{i,j-1}^{n}}{2h} =$$

$$\frac{1}{h^2 \mathrm{Re}} \left(\omega_{i+1,j}^{n} + \omega_{i-1,j}^{n} + \omega_{i,j+1}^{n} + \omega_{i,j-1}^{n} - 4\omega_{i,j}^{n} \right)$$

- Here, a square grid is used, i.e., $h = \Delta x = \Delta y$.

Biringen and Chow (2011, p. 281)

Driven Cavity Problem

- The Poisson equation for the stream function is approximated as the difference equation

$$\frac{\psi_{i+1,j}^{n+1} - 2\psi_{i,j}^{n+1} + \psi_{i-1,j}^{n+1}}{h^2} + \frac{\psi_{i,j+1}^{n+1} - 2\psi_{i,j}^{n+1} + \psi_{i,j-1}^{n+1}}{h^2} = -\omega_{i,j}^{n+1}$$

- After solving the discretized Poisson equation for the stream function, the velocity components are then updated using

$$u_{i,j}^{n+1} = \frac{\psi_{i,j+1}^{n+1} - \psi_{i,j-1}^{n+1}}{2h}, \qquad v_{i,j}^{n+1} = -\frac{\psi_{i+1,j}^{n+1} - \psi_{i-1,j}^{n+1}}{2h}$$

Biringen and Chow (2011, p. 281)

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