Inviscid Fluid Flows

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Outline

- Inviscid fluid flows
- Potential flows
- Elementary flows
- Superposition of elementary flows
- von Karman method
- Conformal mapping
- Finite-difference method for potential flows
- Iterative methods for linear systems

2D Incompressible Potential Flows

- Inviscid flow is the flow of a fluid with zero viscosity.
- Inviscid flows can be classified as potential flows (irrotational flows) and rotational inviscid flows.
- Potential or irrotational flows has zero vorticity, i.e., $\nabla \times \mathbf{v} = \mathbf{0}$ and the velocity potential ϕ can defined such that $\mathbf{v} = -\nabla \phi$
- If a potential flow is also incompressible, i.e., $\nabla \cdot \mathbf{v} = 0$, the potential ϕ satisfies the Laplace equation $\nabla^2 \phi = 0$
- 2D planar flows: $v_x = \dfrac{\partial \phi}{\partial x}, \quad v_y = \dfrac{\partial \phi}{\partial y}, \quad \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0$
- 2D axisymmetric flows: $v_r = \dfrac{\partial \phi}{\partial r}, \quad v_\theta = \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta}, \quad \dfrac{\partial^2 \phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2} = 0$

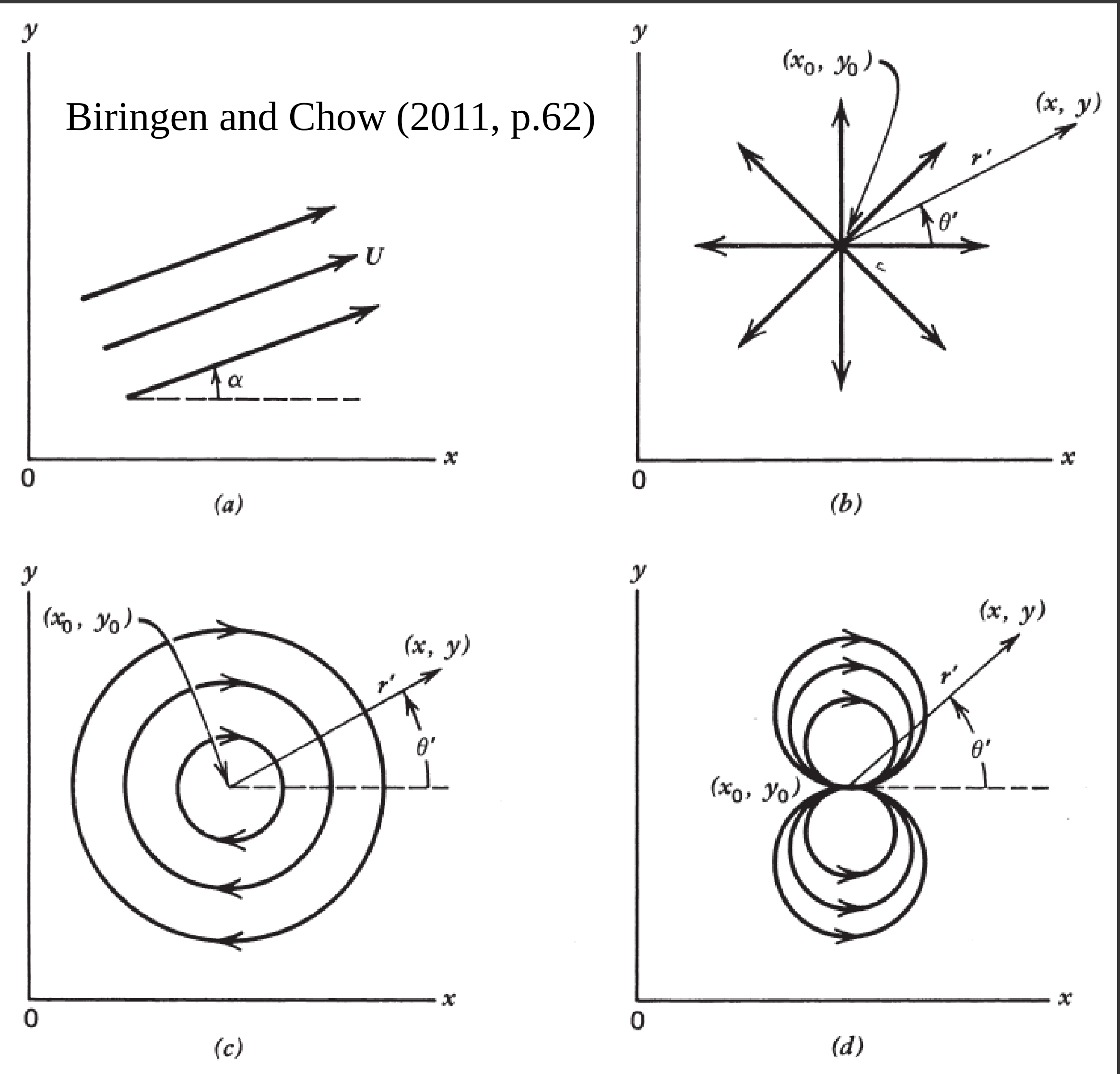
2D Incompressible Potential Flows

- The stream function ψ can also be used instead of the velocity potential.
- The continuity equation $\nabla \cdot \mathbf{v} = 0$ suggests that the velocity can also be expressed interm of the stream function as $\mathbf{v} = \nabla \times \left(\psi \hat{\mathbf{k}}\right)$.
- A line along which ψ = constant is called a streamline.
- Fluid velocities are always tangential to streamlines.
- The irrotational condition becomes $\nabla \times \nabla \times \left(\psi \hat{\mathbf{k}}\right) = \mathbf{0}$
- 2D planar flows: $v_x = \dfrac{\partial \psi}{\partial y}, \qquad v_y = -\dfrac{\partial \psi}{\partial x}, \qquad \dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2} = 0$
- 2D axisymmetric flows: $v_r = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta}, \quad v_\theta = -\dfrac{\partial \psi}{\partial r}, \quad \dfrac{\partial^2 \psi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \psi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \psi}{\partial \theta^2} = 0$

Biringen and Chow (2011, p.53-54)

Elementary Flows

Stream functions corresponding to 4 elementary flows are as follows.



(a) Uniform flow with angle α

$$\psi = U\left(y\cos\alpha - x\sin\alpha\right)$$

(b) Line source with strength Λ

$$\psi = \frac{\Lambda}{2\pi}\tan^{-1}\left(\frac{y-y_0}{x-x_0}\right) = \frac{\Lambda\theta}{2\pi}$$

(c) Line vortex with circulation Γ:

$$\psi = \frac{\Gamma}{2\pi}\ln\left[\left(x-x_0\right)^2+\left(y-y_0\right)^2\right]^{\frac{1}{2}} = \frac{\Gamma}{2\pi}\ln r$$

(d) Doublet with strength κ:

$$\psi = -\frac{\kappa}{2\pi}\frac{y-y_0}{\left(x-x_0\right)^2+\left(y-y_0\right)^2} = -\frac{\kappa}{2\pi}\frac{\sin\theta}{r}$$

Velocity Potentials of Elementary Flows

(a) Uniform flow with angle α :

$$\phi(x,y) = -U\left(x\cos\alpha + y\sin\alpha\right), \qquad \phi(r,\theta) = -Ur\cos\left(\theta - \theta_0\right)$$

(b) Line source with strength Λ:

$$\phi(x,y) = \Lambda\ln\left(x^2 + y^2\right)/4\pi, \qquad \phi(r,\theta) = \Lambda\ln r/2\pi$$

(c) Line vortex with circulation Γ:

$$\phi(x,y) = -\frac{\Gamma}{2\pi}\tan^{-1}\left(\frac{y-y_0}{x-x_0}\right), \qquad \phi(r,\theta) = -\frac{\Gamma\theta}{2\pi}$$

(d) Doublet with strength κ:

$$\phi(x,y) = \frac{\kappa}{2\pi}\frac{x-x_0}{\left(x-x_0\right)^2 + \left(y-y_0\right)^2}, \qquad \phi(r,\theta) = \frac{\kappa}{2\pi}\frac{\cos\theta}{r}$$

Velocities of Elementary Flows

(a) Uniform flow with angle α :

$$\mathbf{v}(x,y) = U\cos\alpha\,\hat{\mathbf{i}} + U\sin\alpha\,\hat{\mathbf{j}}$$

(b) Line source with strength Λ

$$\mathbf{v}(x,y) = \frac{\Lambda}{2\pi}\left(\frac{(x-x_0)}{(x-x_0)^2+(y-y_0)^2}\hat{\mathbf{i}} + \frac{(y-y_0)}{(x-x_0)^2+(y-y_0)^2}\hat{\mathbf{j}}\right)$$

(c) Line vortex with circulation Γ:

$$\mathbf{v}(x,y) = \frac{\Gamma}{2\pi}\left(\frac{(y-y_0)}{(x-x_0)^2+(y-y_0)^2}\hat{\mathbf{i}} - \frac{(x-x_0)}{(x-x_0)^2+(y-y_0)^2}\hat{\mathbf{j}}\right)$$

(d) Doublet with strength κ:

$$\mathbf{v}(x,y) = -\frac{\kappa}{2\pi}\left(\frac{(y-y_0)^2-(x-x_0)^2}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}\hat{\mathbf{i}} - \frac{2(x-x_0)(y-y_0)}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}\hat{\mathbf{j}}\right)$$

Velocities of Elementary Flows

(a) Uniform flow with angle α :

$$\mathbf{v}(r,\theta) = U\cos(\theta-\alpha)\hat{\mathbf{r}} - U\sin(\theta-\alpha)\hat{\boldsymbol{\theta}}$$

(b) Line source with strength Λ

$$\mathbf{v}(r,\theta) = \frac{\Lambda}{2\pi r}\hat{\mathbf{r}}$$

(c) Line vortex with circulation Γ:

$$\mathbf{v}(r,\theta) = -\frac{\Gamma}{2\pi r}\hat{\boldsymbol{\theta}}$$

(d) Doublet with strength κ:

$$\mathbf{v}(r,\theta) = -\frac{\kappa}{2\pi}\frac{\cos\theta}{r^2}\hat{\mathbf{r}} - \frac{\kappa}{2\pi}\frac{\sin\theta}{r^2}\hat{\boldsymbol{\theta}}$$

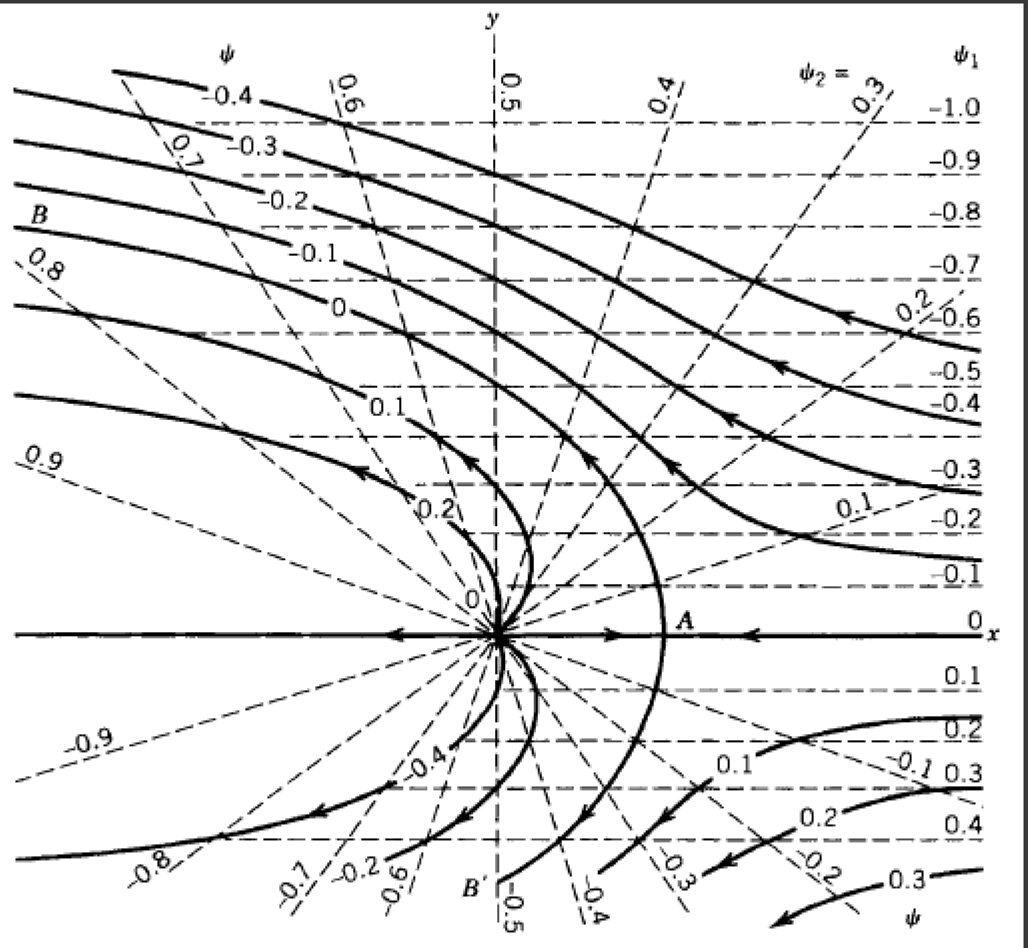
Superposition of Elementary Flows

- If ψ_1 and ψ_2 are solution to $\nabla^2\psi = 0$, $a\psi_1 + b\psi_2$ is also a solution.
- "When a source of strength Λ at $(x_0 - \Delta x, y_0)$ is added to a sink of strength -Λ at $(x_0 + \Delta x, y_0)$, a new flow field is obtained."
- "By letting Δx approach zero while keeping the product $2\Delta x\Lambda$ a constant κ, the stream function for a doublet at (x_0, y_0) is obtained."
- "The flow pattern of a double can also be produced by superimposing a vortex at $(x_0, y_0 - \Delta y)$ to a vortex of opposite circulation at $(x_0, y_0 + \Delta y)$, and then letting Δy approach zero."
- The velocities at the center of a line source, a line vortex, and a doublet are infinitely large. The center of these flows are called singularities.
- Singularities do not cause a problem when they are within the boundary of a rigid body.

Biringen and Chow (2011, p.64)

Source in a Uniform Flow

- The stream function for a uniform flow with $\alpha = 0$ is $\psi_1 = -Uy$.
- If a source of strength Λ with ($x_0 = 0$, $y_0 = 0$) and stream function $\psi_2 = \dfrac{\Lambda\theta}{2\pi}$ is superimposed on the uniform flow, the resultant stream function is



$$\psi = Uy + \frac{\Lambda\theta}{2\pi} = U\left(\frac{h\theta}{2\pi} - y\right)$$

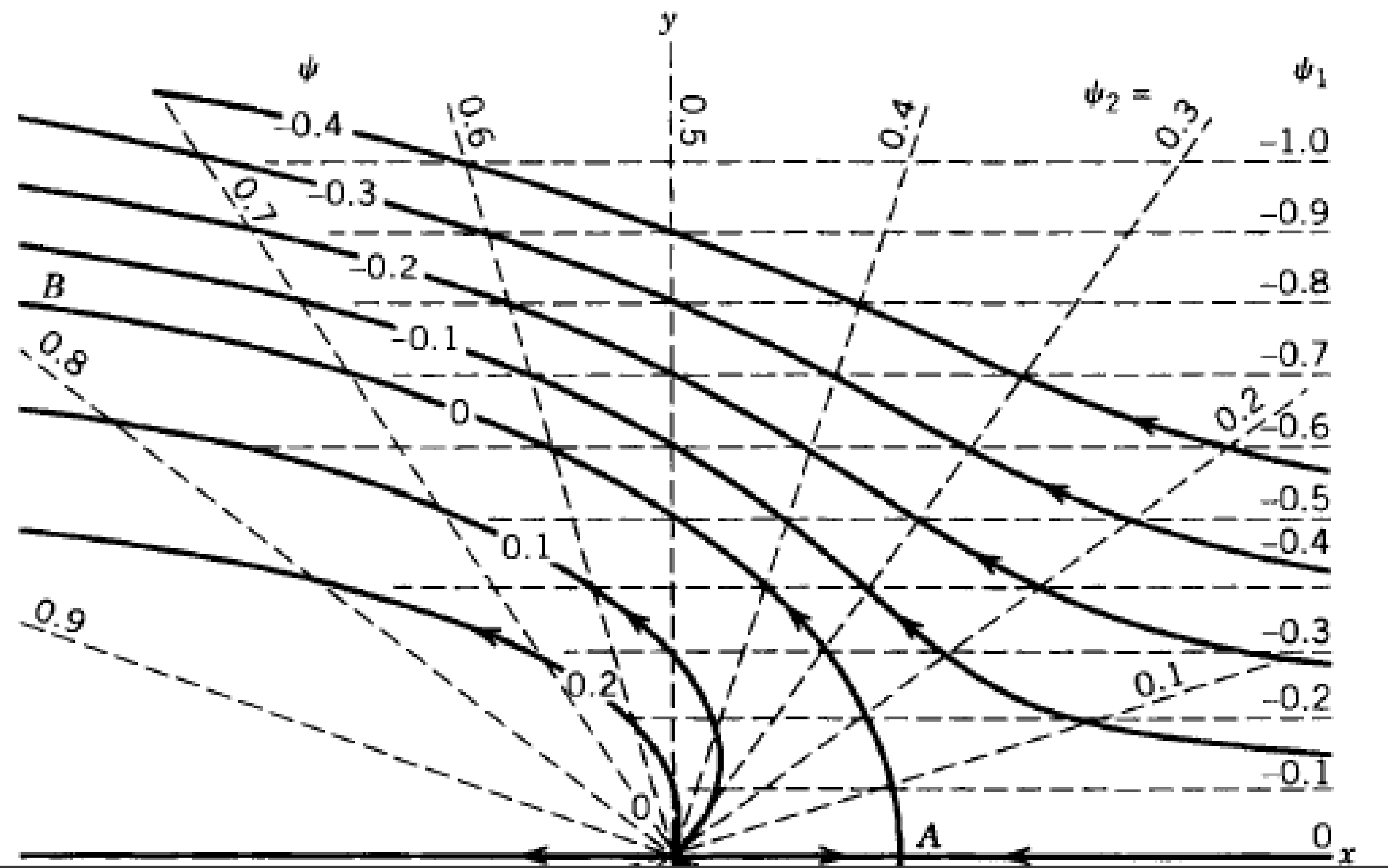
where $h = \Lambda/2U$ is a characteristic length.

- The streamline BAB' with $\psi = 0$ is considered a solid surface enclosing the source.
- "The flow exterior to the surface satisfies the continuity equation and is irrotational."

Kuethe and Chow (1998, p.97)

Source in a Uniform Flow

- "The flow field may be interpreted as that of a horizontal wind past a cliff, whose shape (y_0, θ) is described by the equation $\psi = 0$, that is,



$$y_0 = r_0 \sin\theta = \frac{h\theta}{\pi}, \qquad 0 \le \theta \le \pi$$

where r_0 is the radial distance of a point on the cliff at height y_0 above the x axis."

When $\theta \to -\infty$, $y_0 \to h = \Lambda/2U$.

Kuethe and Chow (1998, p.97)

Source in a Uniform Flow

- With $\theta = \tan^{-1}(y/x)$, the velocity components are

$$v_x = \frac{\partial \psi}{\partial y} = -U + \frac{Uh}{\pi}\frac{x}{x^2+y^2}$$

$$v_y = -\frac{\partial \psi}{\partial x} = \frac{Uh}{\pi}\frac{y}{x^2+y^2}$$

- According to these equation, the velocity vanishes ($v_x = v_y = 0$) at point (h/π,0) = ($\Lambda/2\pi U$,0).
- "In other words, the velocity vanishes at point A on the *x* axis where the velocity from the source, $\Lambda/2\pi x$, cancels the velocity *U* from the uniform flow."

Kuethe and Chow (1998, p.98)

Flow Pattern of a Source-Sink Pair

- Consider a source of strength Λ at ($-x_0$, 0) and a sink of strength $-\Lambda$ at (x_0, 0)
- The stream function of the combine flow at (x, y) is

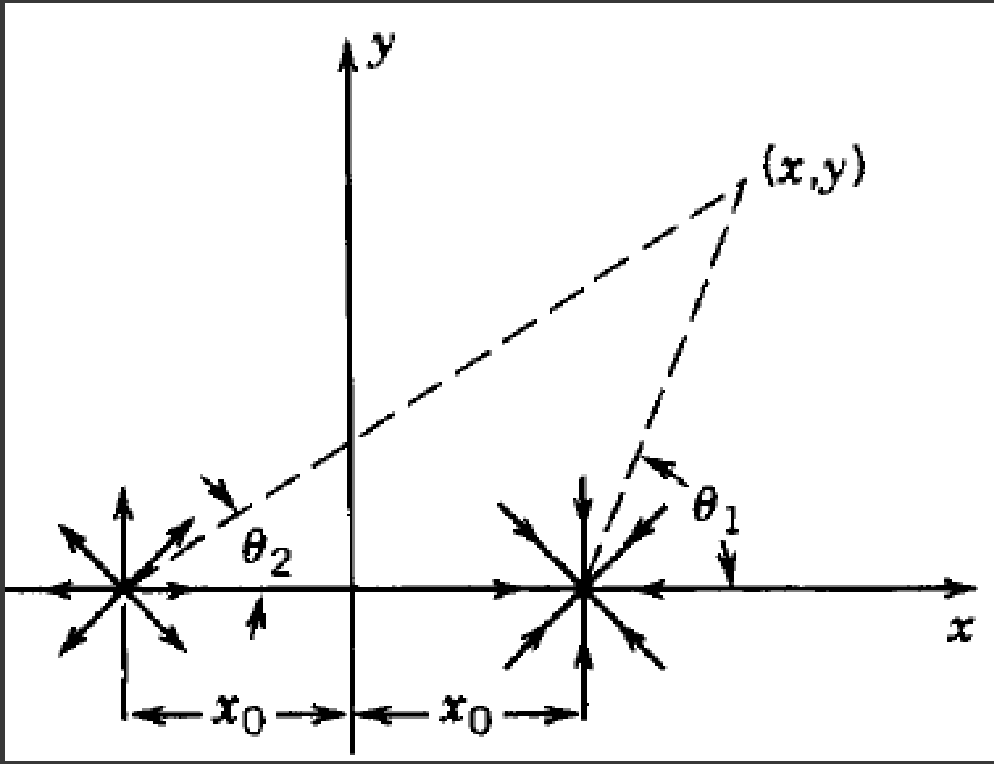
$$\psi = \frac{\Lambda\theta_2}{2\pi} - \frac{\Lambda\theta_1}{2\pi} = \frac{\Lambda\theta_2}{2\pi}\left(\tan^{-1}\frac{y}{x+x_0} - \tan^{-1}\frac{y}{x-x_0}\right)$$

- Using the trigonometric relation

$$\tan^{-1}A - \tan^{-1}B = \tan^{-1}\frac{A-B}{1+AB}$$

we then obtain

$$\psi = -\frac{\Lambda}{2\pi}\tan^{-1}\frac{2x_0 y}{x^2+y^2-x_0^2}$$

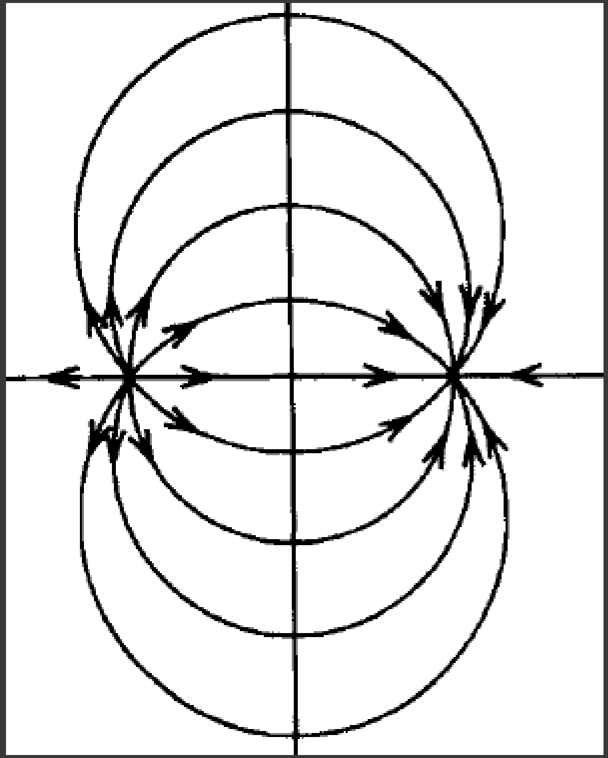


Kuethe and Chow (1998, p.99)

Flow Pattern of a Source-Sink Pair

- Rearranging the equation, we obtain

$$-\tan\frac{2\pi\psi}{\Lambda}=\frac{2x_0 y}{x^2+y^2-x_0^2}$$



$$x^2+y^2+2x_0 y\cot\frac{2\pi\psi}{\Lambda}=x_0^2$$

$$x^2+\left(y+x_0\cot\frac{2\pi\psi}{\Lambda}\right)^2=x_0^2+x_0^2\cot^2\left(\frac{2\pi\psi}{\Lambda}\right)$$

$$=x_0^2\csc^2\left(\frac{2\pi\psi}{\Lambda}\right)$$

- "This equation represents a family of circles with centers on the *y* axis."
- "When $y = 0$, $x = \pm x_0$ for all values of ψ."
- The flow pattern is shown in the figure.

Kuethe and Chow (1998, p.99-100)

Flow Pattern of a Source-Sink Pair

- The flow pattern of a doublet can be obtained "when the distance between the source and sink approaches zero while their strengths approach infinity in such a way that their product remains a constant value of $\kappa = 2x_0\Lambda$."
- As x_0 approaches zero, we have

$$\psi = \lim_{x_0 \to 0}\left[-\frac{\Lambda}{2\pi}\tan^{-1}\frac{2x_0 y}{x^2+y^2-x_0^2}\right] = \lim_{x_0 \to 0}\left[-\frac{1}{2\pi}\frac{(2x_0\Lambda)y}{x^2+y^2-x_0^2}\right]$$

$$= -\frac{\kappa}{2\pi}\frac{y}{x^2+y^2} = -\frac{\kappa}{2\pi}\frac{\sin\theta}{r}$$

- In the limit, the velocity potential is $\phi = \frac{\kappa}{2\pi}\frac{\cos\theta}{r}$

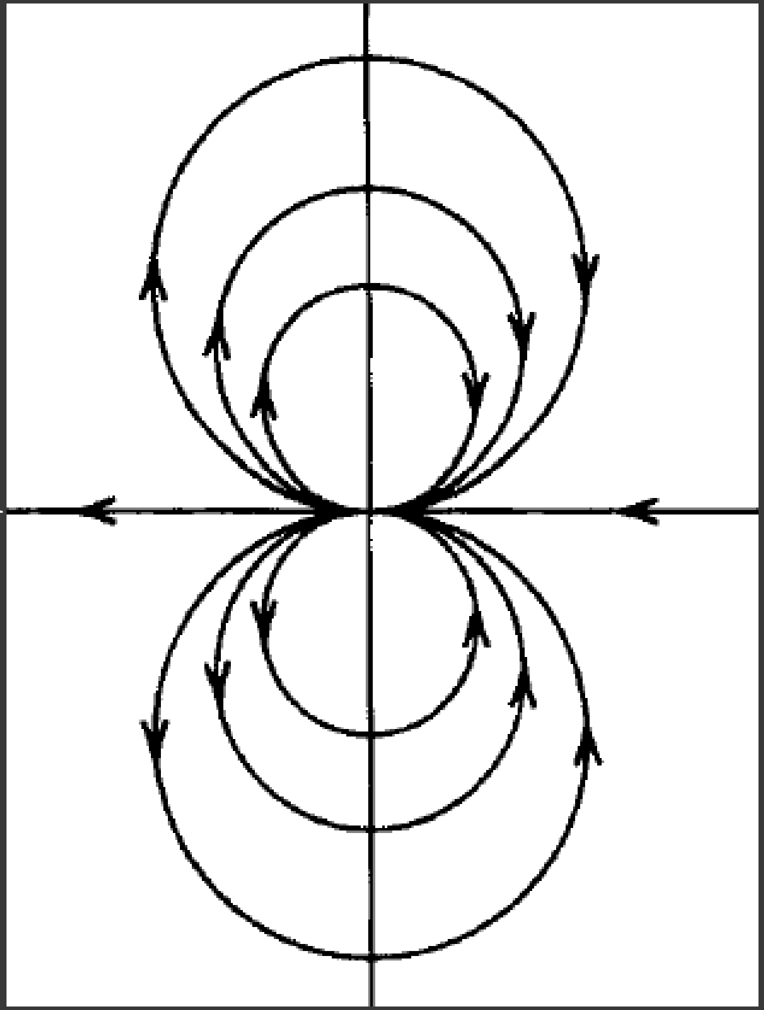
Kuethe and Chow (1998, p.100)

Flow Pattern of a Source-Sink Pair

- The streamlines of a doublet flow of strength κ (lines of constant ψ) are circles as can be seen by rearranging the equation in the form

$$x^2 + \left(y + \frac{\kappa}{4\pi\psi} \right)^2 = \left(\frac{\kappa}{4\pi\psi} \right)^2$$

- Each circle has a center are at (0, -$\kappa/4\pi\psi$) and a radius of $\kappa/4\pi\psi$.
- All streamlines (circles) pass through the origin.



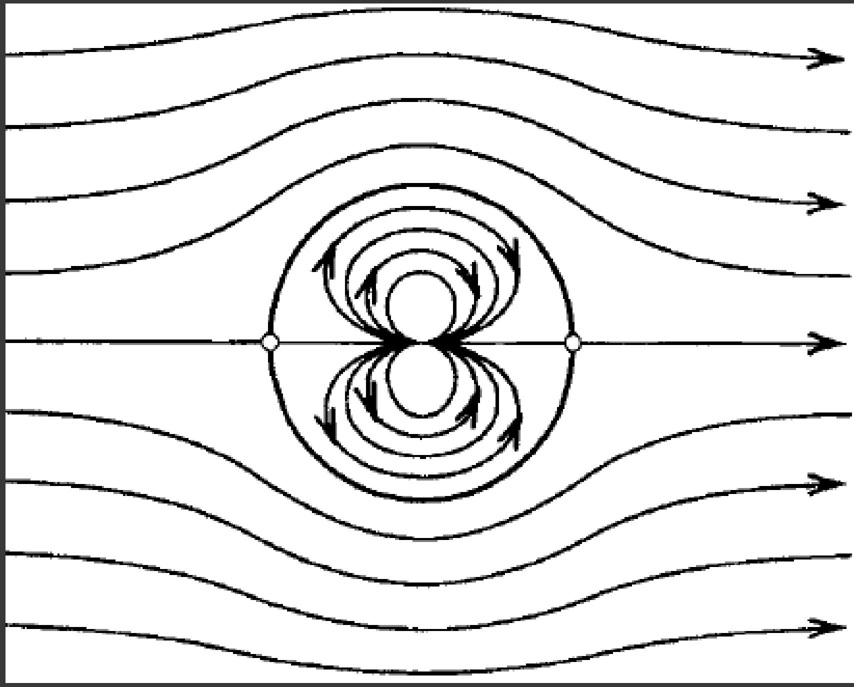
Kuethe and Chow (1998, p.101)

Flow Past a Circular Cylinder

- "The stream function for a uniform flow with velocity U in the direction of the positive x axis is $\psi = Uy$."
- "If the uniform flow is added to a doublet, the flow about a circular cylinder in a uniform stream is obtained."
- The resulting stream function is $\psi = Uy - \dfrac{\kappa y}{2\pi r^2} = Uy\left(1 - \dfrac{\kappa}{2\pi U r^2}\right)$
- Let $\kappa/2\pi U = a^2$. Then, we have

$$\psi = Uy\left(1 - \frac{a^2}{r^2}\right)$$

- "The zero streamline consists of the x axis and a circle of radius $r = a$."



Kuethe and Chow (1998, p.101)

Flow Past a Circular Cylinder

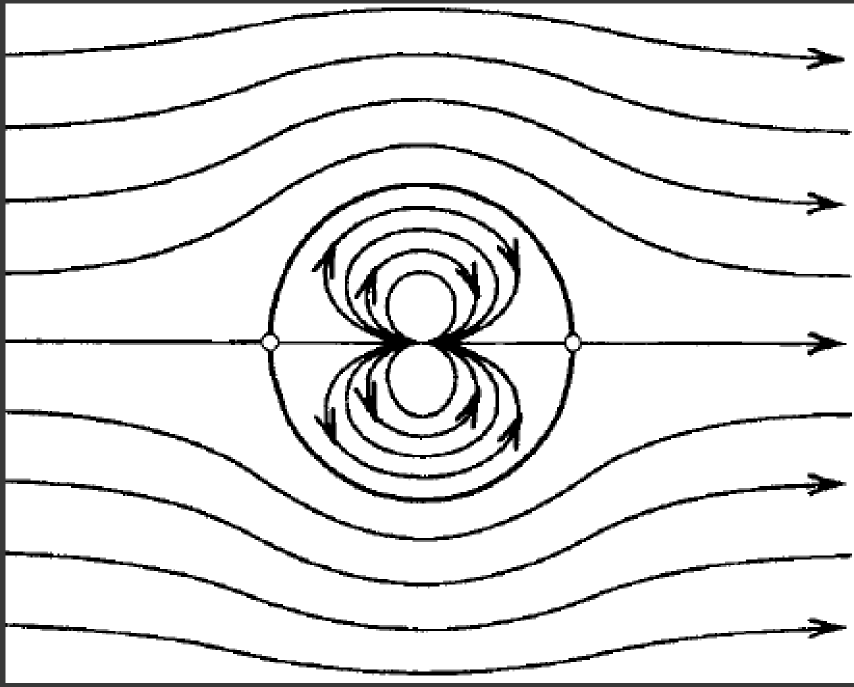
- With $y = r \sin\theta$, the velocity components are

$$v_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \qquad v_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

- On the cylinder surface ($r = a$), $u_r = 0$ and $u_\theta = -2U \sin\theta$.
- The pressure distribution on the surface is given by the Bernoulli equation

$$C_p = \frac{p - p_\infty}{q_\infty} = 1 - \left(\frac{u_\theta}{U}\right)^2 = 1 - 4\sin^2\theta \quad \text{for } r = a$$

where C_p is called the pressure coefficient.

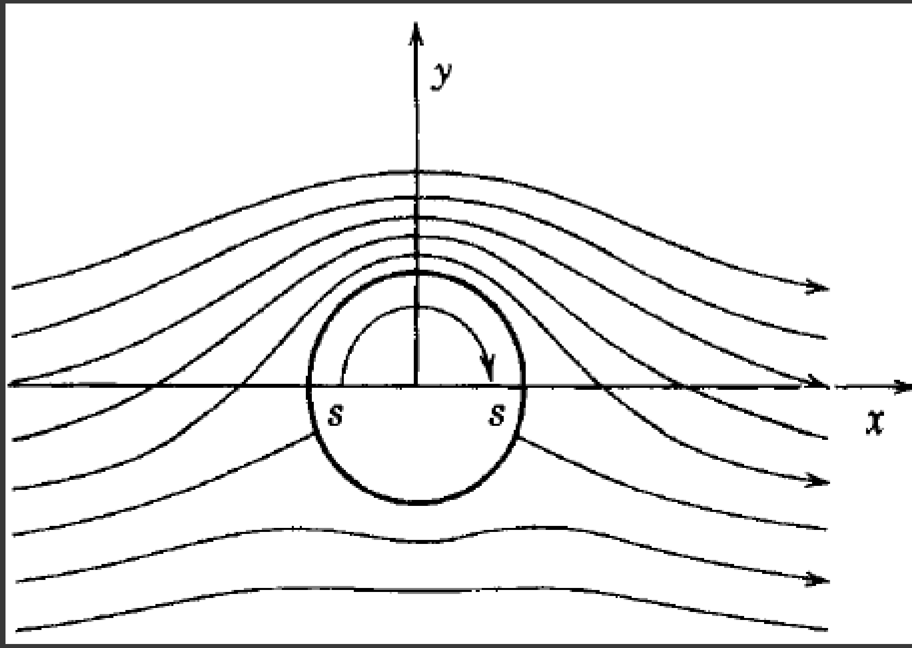


Kuethe and Chow (1998, p.101-102)

Circulatory Flow about a Cylinder

- "If a stream function for a vortex at origin is added to $\psi = Uy\left(1 - a^2/r^2\right)$, the resulting stream function will satisfy the continuity, irrotationality, the boundary conditions for the circulatory flow about a circular cylinder in a uniform stream:"

$$\psi = Uy\left(1 - \frac{a^2}{r^2}\right) + \frac{\Gamma}{2\pi}\ln\left(\frac{r}{a}\right)$$



- "The uniform stream is in the direction of the positive x axis, and the circulatory flow is clockwise."
- The zero streamline corresponds to a cylinder of radius $r = a$.

Kuethe and Chow (1998, p.102)

Circulatory Flow about a Cylinder

- The velocity components for this flow is as follows.

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \qquad u_\theta = -\frac{\partial\psi}{\partial r} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r}$$

- On the cylinder surface ($r = a$), we have

$$u_r = 0, \qquad u_\theta = -2U\sin\theta - \frac{\Gamma}{2\pi a}$$

- u_θ vanishes when $\theta = \theta_s$: $\quad \sin\theta_s = -\dfrac{\Gamma}{4\pi a U}$

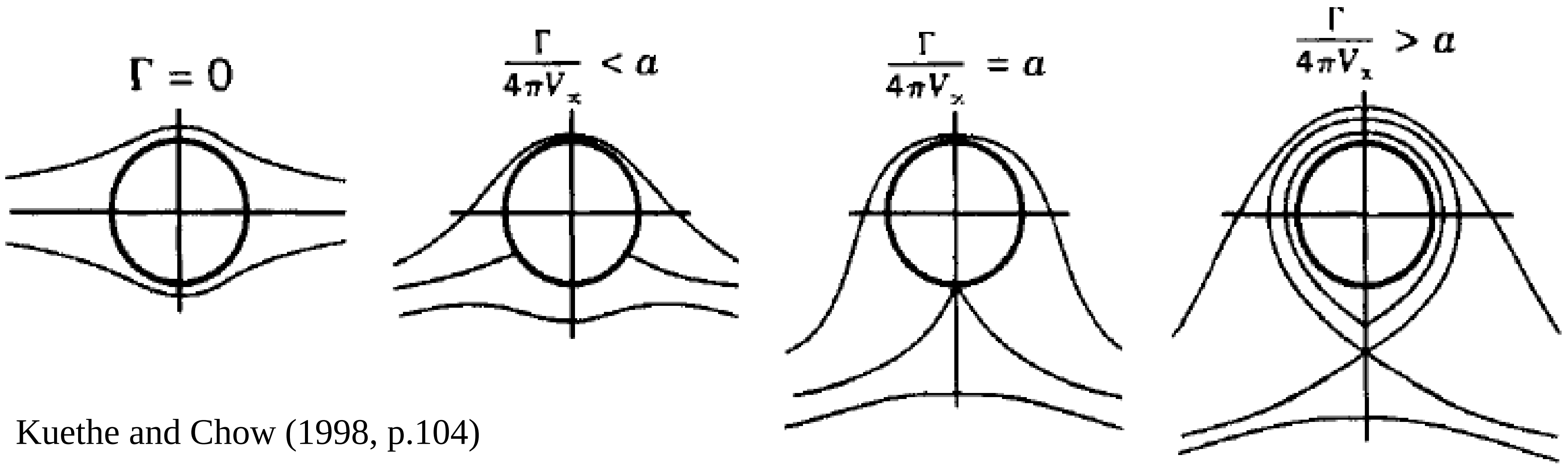
- Since $\sin\theta = y/r$, the stagnation points are

$$x_s = \pm\sqrt{a^2 - y_s^2}, \qquad y_s = -\frac{\Gamma}{4\pi U}$$

Kuethe and Chow (1998, p.102-103)

Circulatory Flow about a Cylinder

- "As Γ becomes large, the stagnation points move downward until $(\Gamma/4\pi U)^2$ equals a^2; for this condition, the stagnation points coincide on the y axis at (0, -a)."
- When $(\Gamma/4\pi U)^2 > a^2$, the stagnation points leave the body and the equations $x_s = \pm\sqrt{a^2 - y_s^2}$, $y_s = -\dfrac{\Gamma}{4\pi U}$ no longer hold.



Kuethe and Chow (1998, p.104)

Exercise

- Use a linear combination of the 4 elementary flows of your choice to generate a flow pattern.
- Plot streamlines and equipotential lines for the flow.
- Also plot the velocity fields.

von Karman Method

- The drawback of the method of superposition of elementary flows is that the positions and strengths of elementary flows must manually adjusted so that the shape of the resulting body is close to what is desired.
- In 1927, von Karman developed a systematic means of calculating the strengths of a group of sources placed at fixed locations for axisymmetric flows. The method can also be applied to 2D planar flows.
- For 3D axisymmetric flows, the cylindrical coordinate system is used and the velocity components can be written as

$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$$

- Here, we will use the von Karman method to approximate flow past a body of revolution.

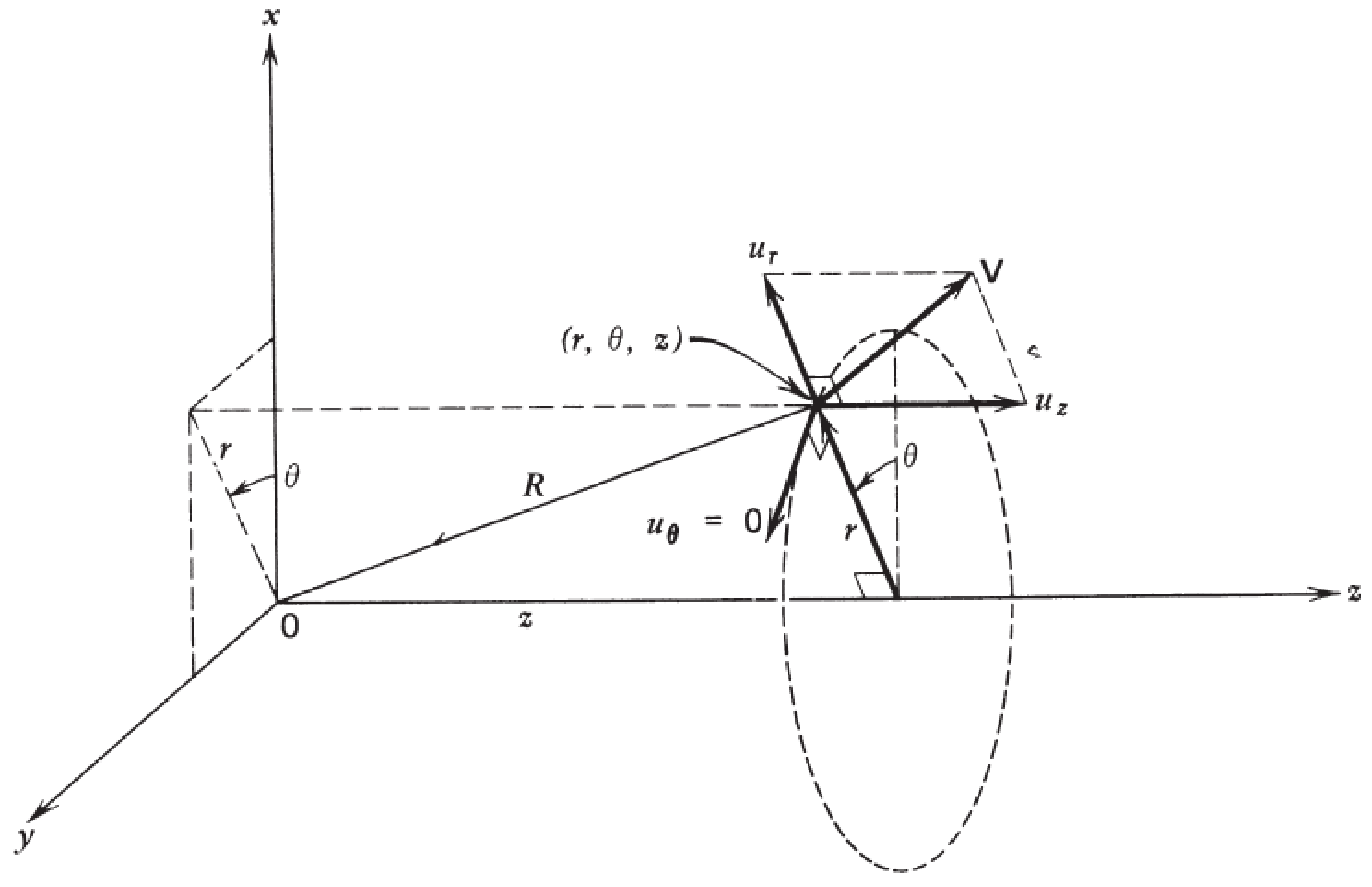
Biringen and Chow (2011, p.69)

von Karman Method

- The irrotational condition $\nabla \times \mathbf{v} = 0$ can be represented in term of the stream function as

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = 0$$

- Although this is not the Laplace equation, it is still a linear equation.
- So, the superposition of elementary flows is still valid in this case.



Biringen and Chow (2011, p.69)

von Karman Method

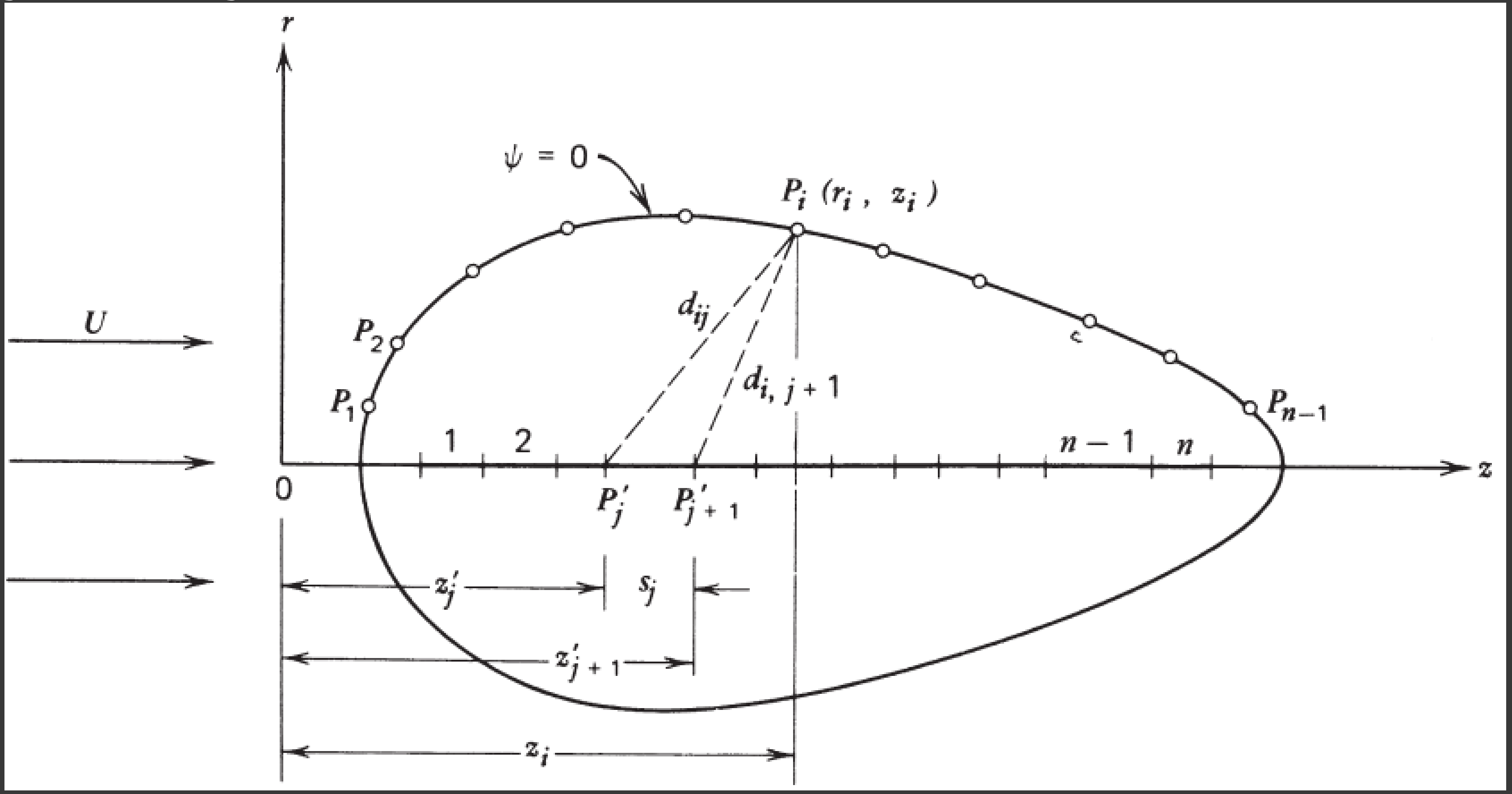
- For a stream of speed U flowing along the z axis, $\psi = \frac{1}{2}Ur^2$
- For a point source at the origin, $\psi = -mz/R,\, R = \sqrt{r^2 + z^2}$ and the corresponding velocity **v** is pointed radially outward from the origin and its magnitude is equal to m/R^2.
- The constant $m = vR^2$, $v = |\mathbf{v}|$, is called the strength of the point source.
- "A doublet can be constructed for the axisymmetric flow by letting a source-sink pair approach each other along the z axis while keeping the product of the strength and the distance in between a constant μ."
- "The doublet formed at the origin has the stream function"

$$\psi = -\frac{\mu r^2}{\left(r^2 + z^2\right)^{3/2}}$$

Biringen and Chow (2011, p.70)

von Karman Method

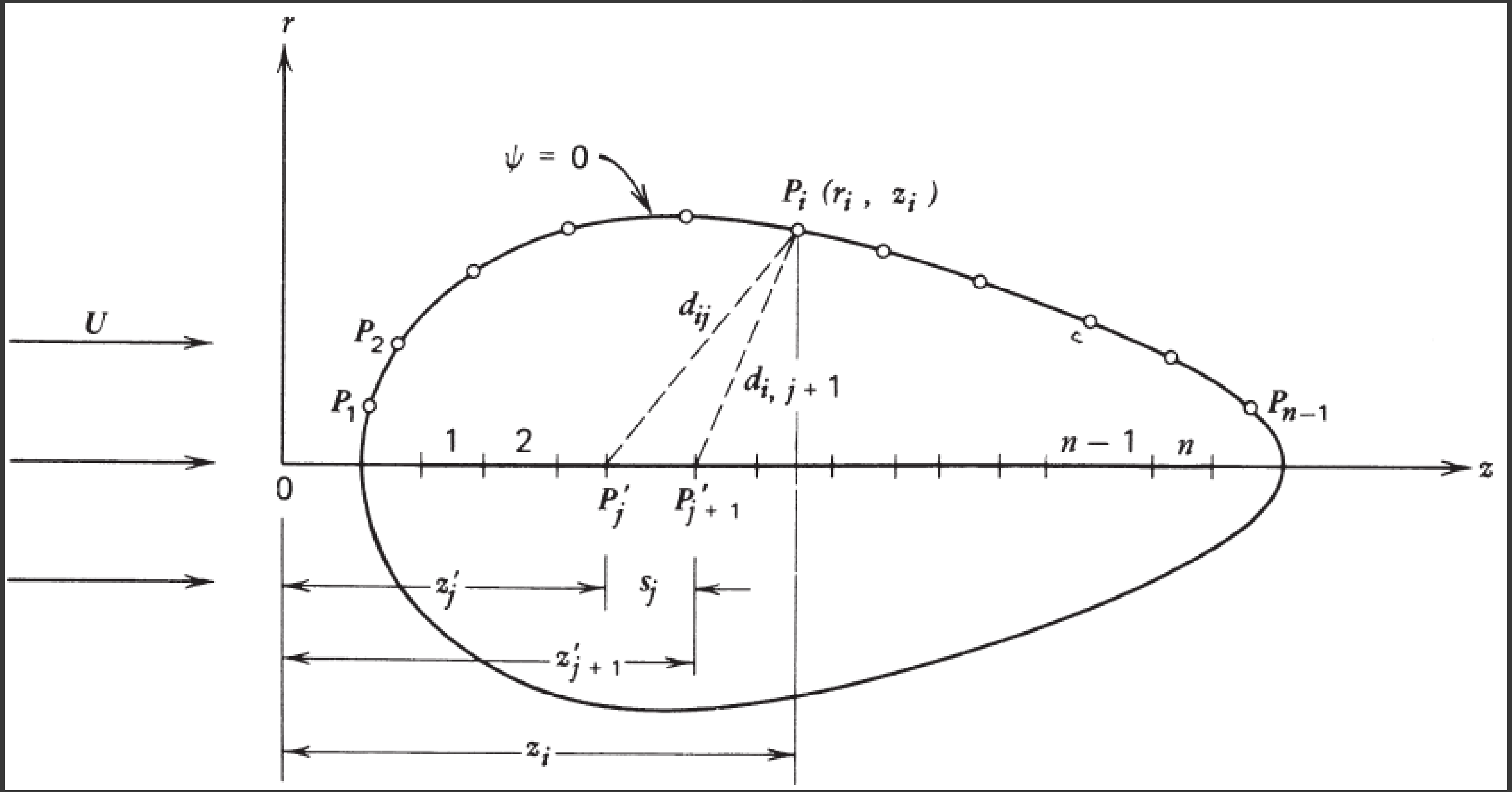
- "To approximate the flow resulting from a uniform axial stream of speed *U* past a given body of revolution, point sources or sinks are distributed uniformly within each of the *n* properly chosen segments inside the body along the axis of symmetry."
- Segments, labeled 1, ..., *n*, have different lengths and contain sources or sinks of different densities.



Biringen and Chow (2011, p.70)

von Karman Method

- "The number of segments depends on the desired degree of accurary."
- "The end points of segment j are denoted as $P'_j\left(0, z'_j\right)$ and $P'_{j+1}\left(0, z'_{j+1}\right)$, and the segment length is $s_j = z'_{j+1} - z'_j$."
- Along segment j, source of constant strength q_j per unit length are distributed."



Biringen and Chow (2011, p.70)

von Karman Method

- "The stream function of the flow induced at any point (r, z) by the sources within a small interval dζ located at (0, ζ) on this segment is"

$$d\psi = -\frac{q_j\left(z-\zeta\right)d\zeta}{\sqrt{r^2+\left(z-\zeta\right)^2}}$$

- "Integration from z'_j to z'_{j+1} gives the induced stream function at (r, z) caused by the whole source distribution on segment j"

$$q_j\left[\sqrt{r^2+\left(z-z'_{j+1}\right)^2}-\sqrt{r^2+\left(z-z'_j\right)^2}\right]$$

- So, the stream function due to the uniform stream and source distributions is

$$\psi\left(r,z\right)=\frac{1}{2}Ur^2+\sum_{j=1}^{n}q_j\left[\sqrt{r^2+\left(z-z'_{j+1}\right)^2}-\sqrt{r^2+\left(z-z'_j\right)^2}\right]$$

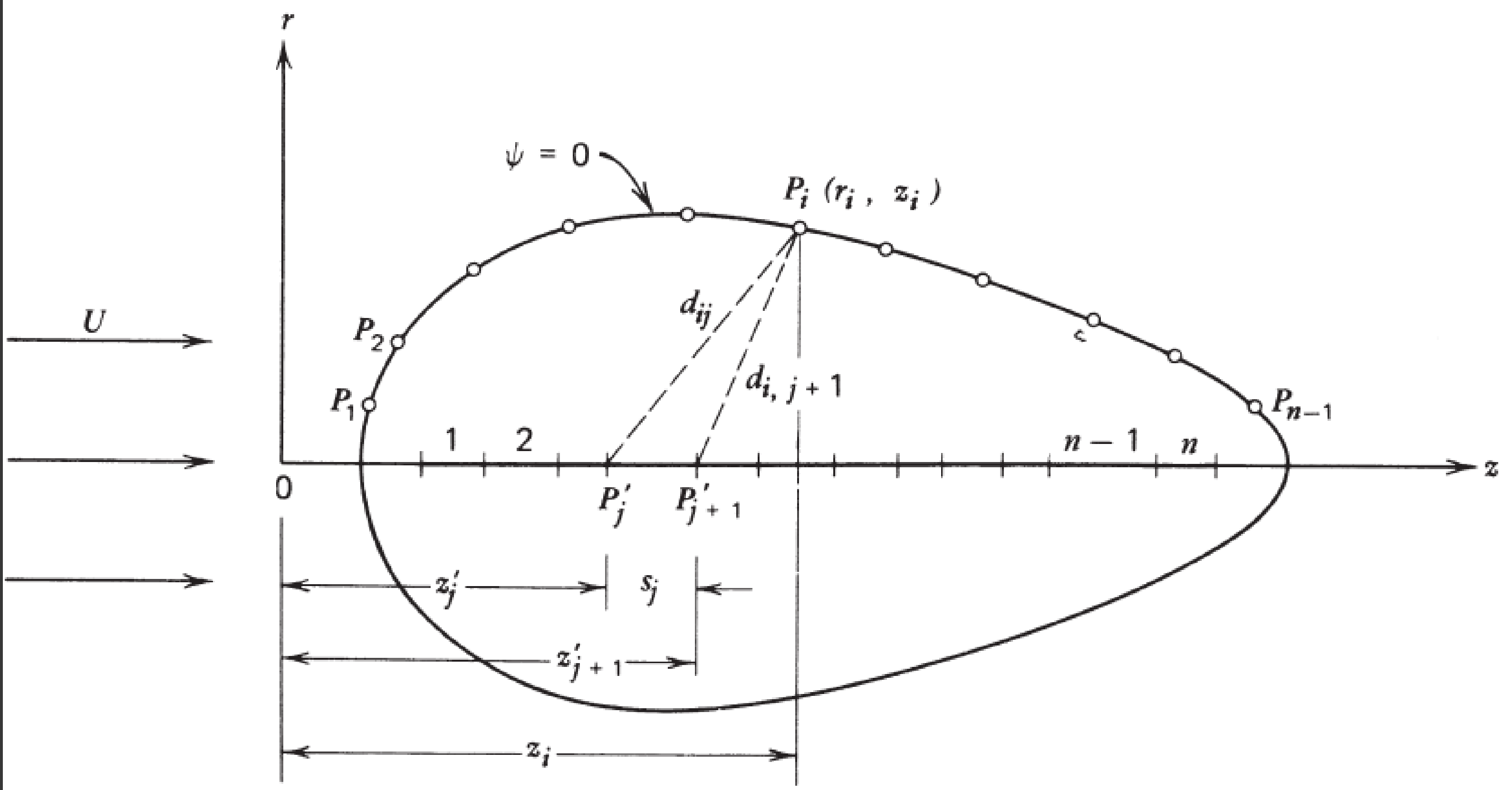
Biringen and Chow (2011, p.71)

von Karman Method

- On the surface of a prescribed body, n-1 points P_1, P_2, ..., $P_{n\text{-}1}$ are selected.
- At a surface point $P_i(r_i, z_i)$ the stream function is

$$\psi(r_i, z_i) = \frac{1}{2}Ur_i^2 + \sum_{j=1}^{n} q_j \left[\sqrt{r_i^2 + \left(z - z'_{j+1}\right)^2} - \sqrt{r_i^2 + \left(z - z'_j\right)^2} \right] = 0$$

and its value is zero since P_i is on the body surface. The two terms in the bracket are the distances from P_i to the end points of segment j (P'_j, P'_{j+1}).



Biringen and Chow (2011, p.72)

von Karman Method

- That is $$d_{i,j+1}=\sqrt{r_i^2+\left(z_i-z'_{j+1}\right)^2},\qquad d_{i,j}=\sqrt{r_i^2+\left(z_i-z'_j\right)^2}$$
- Let $\Delta d_{ij}=d_{i,j}-d_{i,j+1}$, and $Q_j=q_j/U$
- We then obtain $$\sum_{j=1}^{n}\Delta d_{ij}Q_j=\frac{1}{2}r_i^2,\qquad i=1,2,\ldots,n-1$$
- There are n unknowns (Q_j , j = 1, 2, ..., n) but n-1 equations.
- The last equation can be obtained using the fact that, to form a closed body, the total strength of the sources/sinks must vanish, that is,

$$\sum_{j=1}^{n}s_jQ_j=0$$

Biringen and Chow (2011, p.72)

von Karman Method

- We then obtain the exactly determined linear system

$$\begin{bmatrix} \Delta d_{11} & \Delta d_{12} & \cdots & \Delta d_{1n} \\ \Delta d_{21} & \Delta d_{22} & \cdots & \Delta d_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \Delta d_{n-1,1} & \Delta d_{n-1,2} & \cdots & \Delta d_{n-1,n} \\ s_1 & s_2 & \cdots & s_n \end{bmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{n-1} \\ Q_n \end{pmatrix} = \begin{pmatrix} r_1^2/2 \\ r_2^2/2 \\ \vdots \\ r_{n-1}^2/2 \\ 0 \end{pmatrix}$$

- After $Q_j = q_j/U$ are computed, the stream function can be computed by

$$\psi(r,z) = \frac{1}{2}Ur^2 + \sum_{j=1}^{n} q_j \left[\sqrt{r^2 + \left(z - z'_{j+1}\right)^2} - \sqrt{r^2 + \left(z - z'_j\right)^2} \right]$$

Biringen and Chow (2011, p.72)

von Karman Method

- The velocity components can be computed directly using Q_j as follows.

$$u_r(r,z) = -\sum_{j=1}^{n} \frac{UQ_j}{r}\left[\frac{z - z'_{j+1}}{\sqrt{r^2 + \left(z - z'_{j+1}\right)^2}} - \frac{z - z'_j}{\sqrt{r^2 + \left(z - z'_j\right)^2}}\right]$$

$$u_z(r,z) = U\left\{1 + \sum_{j=1}^{n} Q_j\left[\frac{1}{\sqrt{r^2 + \left(z - z'_{j+1}\right)^2}} - \frac{1}{\sqrt{r^2 + \left(z - z'_j\right)^2}}\right]\right\}$$

- Note that the flow speed induced by the source distributions decreases with the distance away from the body.

Biringen and Chow (2011, p.72-73)

von Karman Method

- The pressure field can be computed from the steady-state Bernoulli equation $p+\frac{1}{2}\rho v^2 = \text{constant}$
- If P is the pressure far away from the body, we can then calculate the pressure p from $$p+\tfrac{1}{2}\rho\left(u_r^2+u_z^2\right)=P+\tfrac{1}{2}\rho U^2$$
- This equation can be rewritten in term of the pressure coefficient c_p as

$$c_p = 1-\left(\frac{u_r}{U}\right)^2-\left(\frac{u_z}{U}\right)^2, \qquad c_p \equiv \frac{p-P}{\frac{1}{2}\rho U^2}$$

Biringen and Chow (2011, p.73)

Exercise

- Adding a uniform flow to a point doublet yields a uniform flow past a circular cylinder whose stream function is $\psi = \tfrac{1}{2}Ur^2 - \mu r^2 \big/ \left(r^2 + z^2\right)^{3/2}$
- When the doublet strength $\mu = Ua^3/2$, the stream function can be written as

$$\psi = \tfrac{1}{2}Ur^2\left(1 - \frac{a^3}{R^3}\right), \qquad R = \sqrt{r^2 + z^2}$$

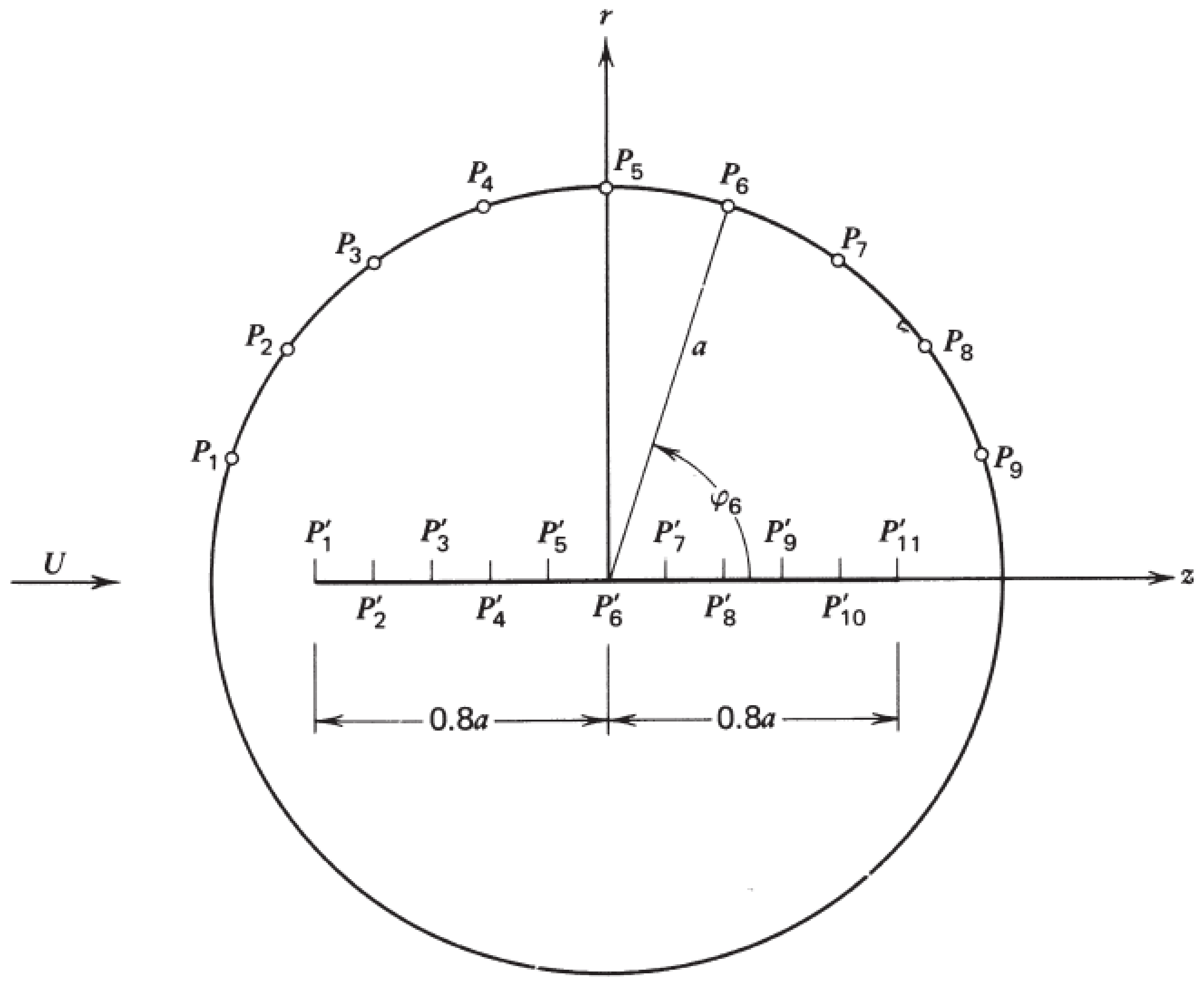
- The stream surface $\psi = 0$ consists of the z axis (along which $r = 0$) and the surface of a sphere of radius a.
- The flow speed and the pressure coefficient at point (r, z) on the sphere are

$$v = \frac{3}{2}U\frac{r}{a}, \qquad c_p = 1 - \frac{9}{4}\left(\frac{r}{a}\right)^2$$

- Apply the von Karman method to this case using 10 source segments.

Biringen and Chow (2011, p.73)

Exercise



- Sources are distributed within 10 segments of equal lengths on the z-axis between $z \pm 0.8a$.
- 9 equally spaced points are chosen on the upper surface of the sphere.
- Let θ_i be the angle between the z-axis and the radial position of the surface point $P_i(r_i, z_i)$.

$$\theta_i = (10 - i)\pi/10, \qquad i = 1, 2, \ldots, 9$$

$$r_i = a \sin\theta_i, \quad z_i = a\cos\theta_i$$

- Compute the velocity components and compare with exact solution.

Biringen and Chow (2011, p.74)

Steady Potential Flows in 2D

- The velocity components can be computed from the velocity potential ϕ and stream function ψ as follows.

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

- Both ϕ and ψ satisfy the Laplace equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \qquad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

- Curves $\phi =$ constant are called equipotential lines.
- Curves $\psi =$ constant are called streamlines.
- In 2D flows, the equipotential lines and streamlines form two families of mutually orthogonal curves since

$$\begin{aligned} \nabla \phi \cdot \nabla \psi &= \phi_x \psi_x + \phi_y \psi_y \\ &= (u)(-v) + (u)(v) = 0 \end{aligned}$$

Biringen and Chow (2011, p.76)

Analytic Function

- Let's construct a complex function from the velocity potential and the stream function as
$$w(z)=\phi(x,y)+i\psi(x,y), \qquad z=x+iy, \quad i=\sqrt{-1}$$
- Since the real and imaginary parts of *w* satisfy the Cauchy-Riemann conditions, i.e.,
$$\frac{\partial \phi}{\partial x}=\frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y}=-\frac{\partial \psi}{\partial x},$$

  w(*z*) is complex-differentiable through out the region occupied by the fluid.
- In other words, *w*(*z*) is an analytic function of *z*.
- "If we choose an arbitrary analytic function *w*(*z*), the real and imaginary parts of the function are then qualify as the velocity potential and stream function of a potential flow in the *x*-*y* plane."

Biringen and Chow (2011, p.76-77)

Complex Potential

- The complex function $w(z)$ is called the complex potential, whose derivative is related to the velocity components as $dw/dz = u - iv$
- So, the magnitude of the velocity vector is $|\mathbf{v}| = |dw/dz| = \sqrt{u^2 + v^2}$
- The complex potential corresponding to the uniform flow is

$$w(z) = Uze^{-i\alpha} = U(x\cos\alpha + y\sin\alpha) + iU(y\cos\alpha - x\sin\alpha)$$

- The complex potential of the source is $w(z) = (\Lambda/2\pi)\log(z - z_0)$
- The complex potential of the vortex is $w(z) = (i\Gamma/2\pi)\log(z - z_0)$
- The complex potential of the doublet is $w(z) = \kappa/2\pi(z - z_0)$
- Here, $z_0 = x_0 + iy_0$.
- The complex potential of the source, vortex, and doublet has a singular point at z_0, where the first derivative of the function is unbounded.

Biringen and Chow (2011, p.77)

Exercise

- Using the complex potentials given in the previous slide, plot the equipotential lines and streamlines of the 4 elementary flows and a linear combination of elementary flows.

Conformal Mapping

- The principle of superposition of elementary flows can be applied to the complex potential to generate new flows.
- For example, the sum of Uz (uniform horizontal flow) and $\kappa/2\pi z$ (doublet at origin) represents the complex potential of a uniform flow past a circular cylinder of radius $\sqrt{\kappa/2\pi U}$
- The method of conformal mapping can generate new flow patterns using coordinate transformations.
- Let $z = f(z')$ where f is an analytic function of z'.
- We then have

$$\frac{dw}{dz'} = \frac{dw}{dz}\frac{dz}{dz'} = \frac{dw}{dz}\frac{df}{dz'}$$

Biringen and Chow (2011, p.78)

Conformal Mapping

- "After being transformed into the z' plane, the complex potential can be written in terms of the new coordinates as"

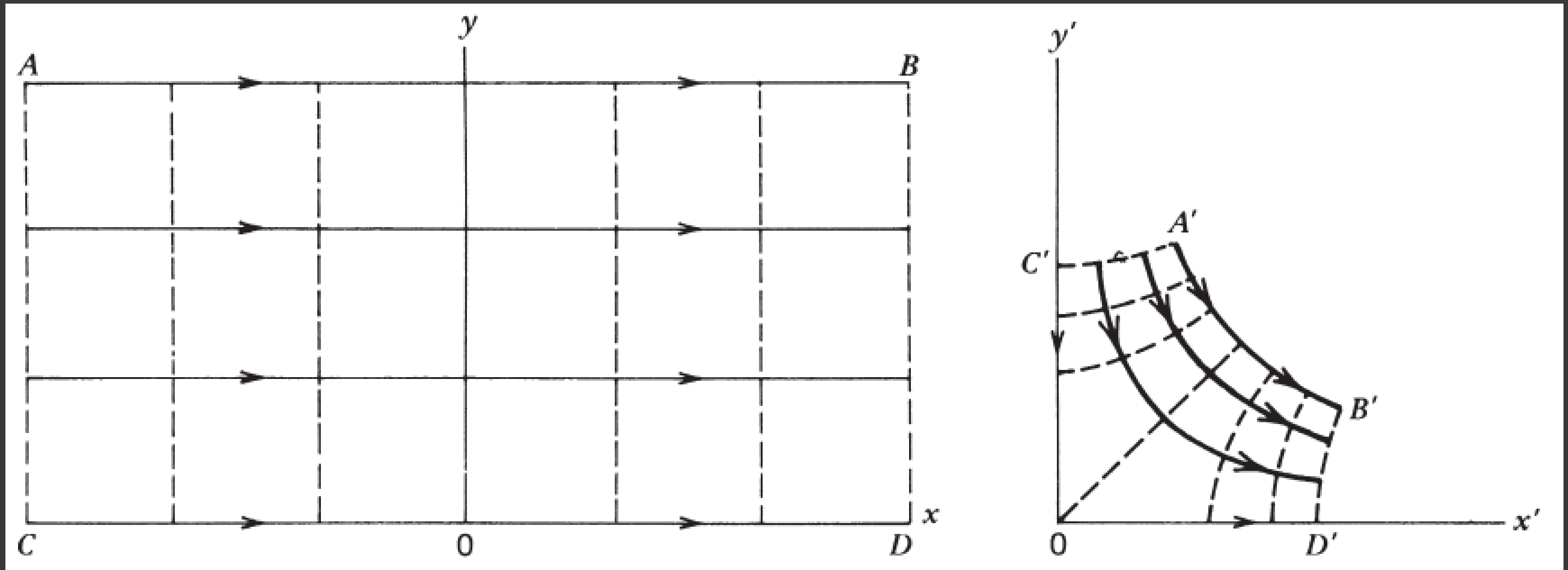
$$w\left[f(z')\right] = \phi'\left(x', y'\right) + i\psi'\left(x', y'\right)$$

- The curves ϕ' = constant and ψ' = constant remain mutually orthogonal in the x'-y' plane after the transformation or mapping.
- Thus, this is called conformal mapping.

Biringen and Chow (2011, p.78)

Conformal Mapping: Example

- Consider the complex potential $w(z) = Uz = Ux + iUy$ of a uniform flow with speed U in the positive x direction. Let the mapping be $z = z'^2$.
- The complex potential becomes $w(z') = Uz'^2 = U\left(x'^2 - y'^2\right) + i2Ux'y'$
- "The equipotential lines $x = c$ (dashed lines) and streamlines $y = k$ (solid lines) in the x-y plane are mapped into equipotential lines $x'^2 - y'^2 = c$ and streamlines $2x'y' = k$ in the x'-y' plane, respectively."



Biringen and Chow (2011, p.80)

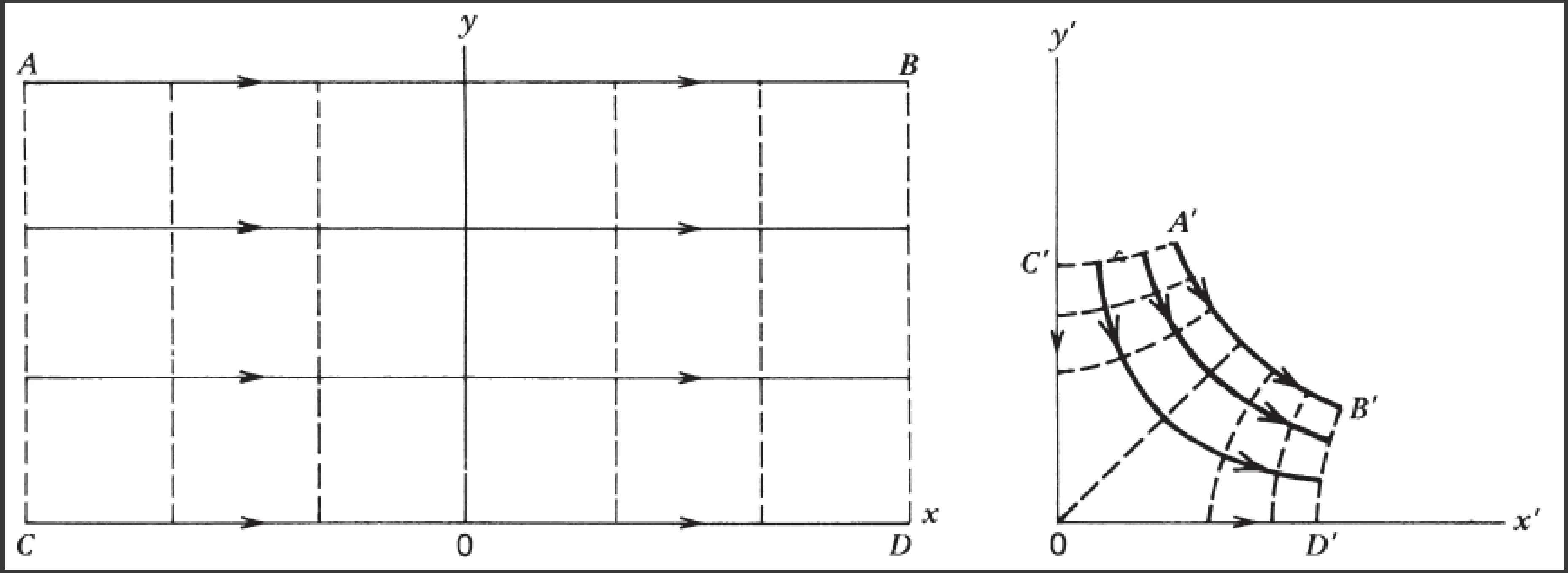
Exercise

Plot the equipotential lines and streamlines of the uniform flow

$$w(z) = Uz = Ux + iUy$$

in the original domain, and those of the corresponding flow in the transformed domain

$$w(z') = Uz'^2 = U\left(x'^2 - y'^2\right) + i2Ux'y'$$



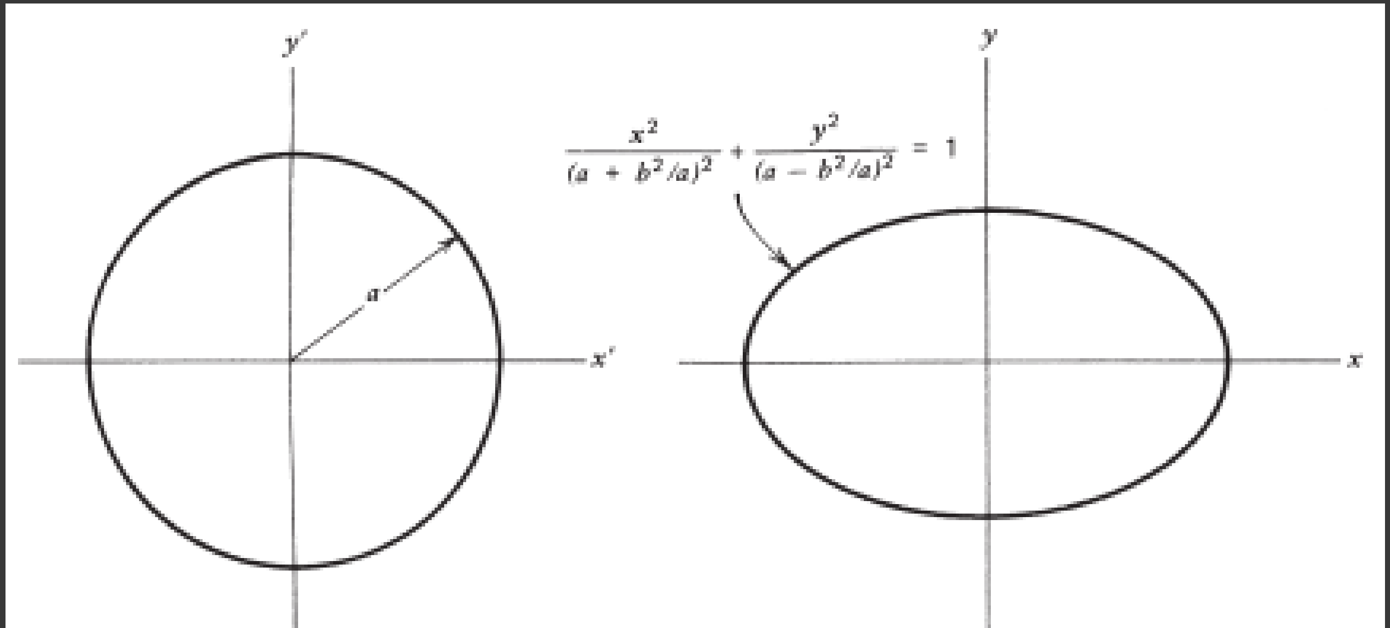
Biringen and Chow (2011, p.80)

Exercise

Plot the equipotential lines and streamlines of the source flow, vortex flow, and doublet flow after using the conformal mapping $z = z'^2$.

Joukowski Transformation

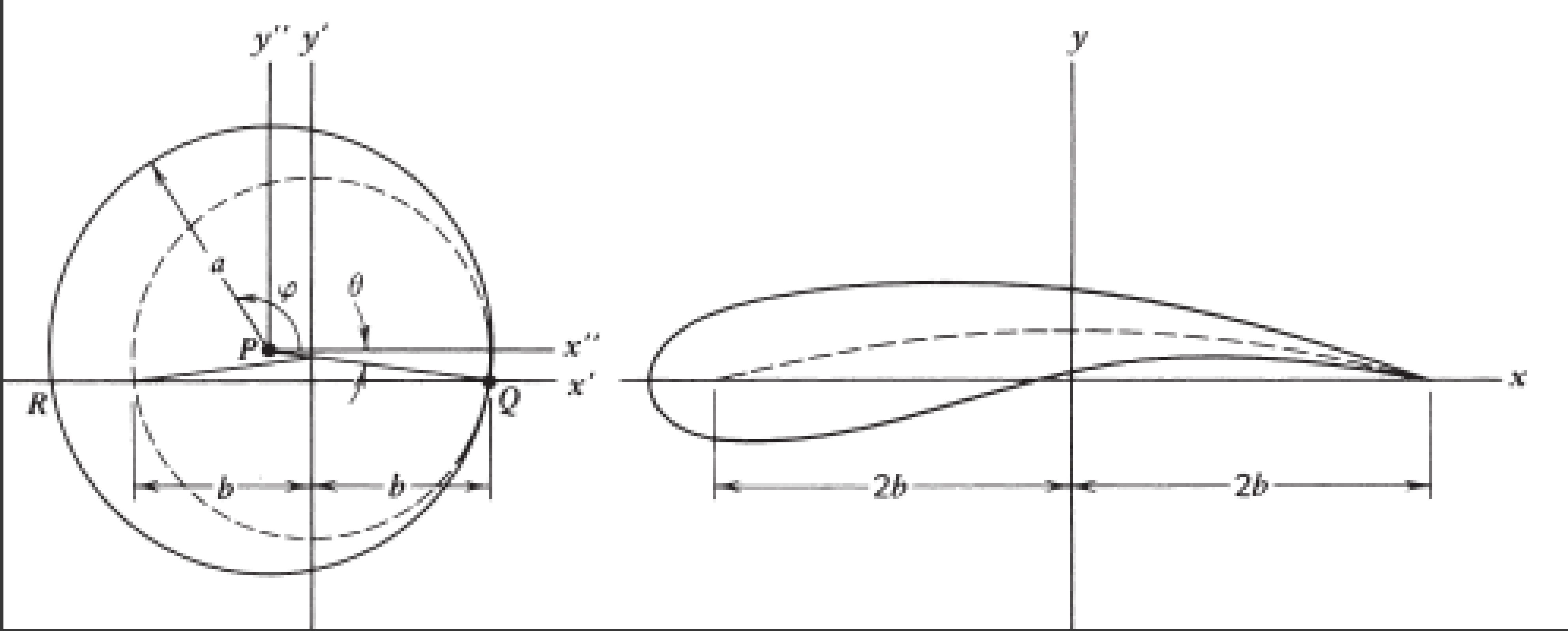
- The mapping $z = z' + a^2/z'$ is called the Joukowski transformation.
- After the transformation, the complex exponential of a uniform flow, $w(z) = Uz$, becomes $w(z') = U\left(z' + a^2/z'\right)$ which is the complex potential for a uniform flow past a circular cylinder of radius a.
- The Joukowski transformation $z = z' + b^2/z'$, $b^2 < a^2$, maps a circle of radius a with center at the origin of the x'-y' plane into an ellipse in the x-y plane.



Biringen and Chow (2011, p.81)

Joukowski Transformation

- The Joukowski transformation $z = z' + b^2 / z'$, $b^2 < a^2$, also maps 2 circles into a circular arc and a so-called Joukowski airfoil in the *x*-*y* plane.



Biringen and Chow (2011, p.82)

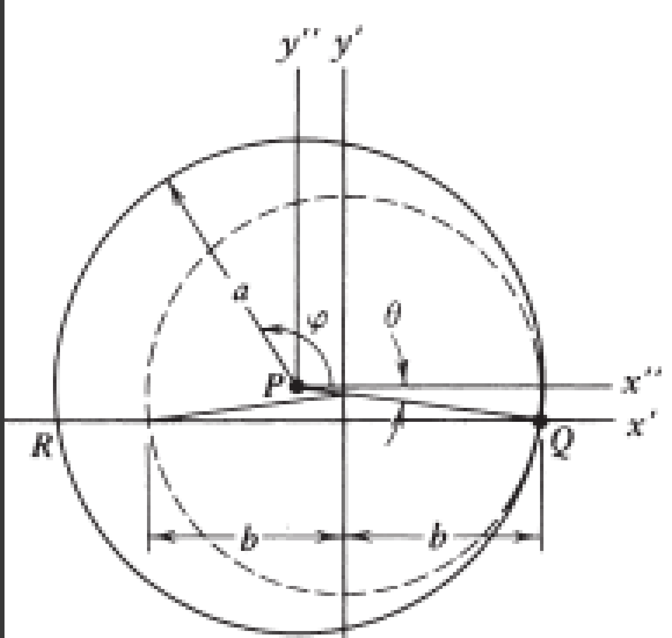
Flow past a Joukowski Airfoil

- Applying the Joukowski transformation to the flow of uniform speed U past a circular cylinder results in the uniform flow past a Joukowski airfoil.
- The flow speed $|\mathbf{v}|$ in the x-y plane is related to the flow speed $|\mathbf{v}'|$ in the x'-y' plane as $$|\mathbf{v}| = \left|\frac{dw}{dz}\right|, |\mathbf{v}'| = \left|\frac{dw}{dz'}\right| \rightarrow \left|\frac{dw}{dz}\right| = \left|\frac{dw}{dz'}\right|\left|\frac{dz'}{dz}\right|$$
- Since we have the transform $z = z' + b^2/z'$, it is easier to compute dz/dz'.
- So, $$|\mathbf{v}| = |\mathbf{v}'| \Big/ \left|\frac{dz}{dz'}\right| = |\mathbf{v}'| \Big/ \left|1 - \left(\frac{b}{z'}\right)^2\right|$$
- Point Q ($x' = b$) in the x'-y' plane corresponds to the trailing edge ($x = 2b$) in the x-y plane. When the flow speed at Q is finite, the flow speed at the trailing edge becomes infinite.

Biringen and Chow (2011, p.83)

Flow past a Joukowski Airfoil

- The infinite flow speed is not allowed and does not occur physically.
- This problem can be avoided by requiring that *Q* is a stagnation point.
- Since *Q* is below the origin of the circle, there must be a circulation due to a line vortex.
- Kuethe and Chow (1998) stated that the circulation is $\Gamma = 4\pi a U \sin\theta$ where θ is the angle between line *PQ* and the direction of the unifom flow.
- Since $\sin\theta = y'_P / a$, $\Gamma = 4\pi y'_P U$
- The flow about the circle is then corresponding to a combination of a uniform flow, a doublet, and a vortex.



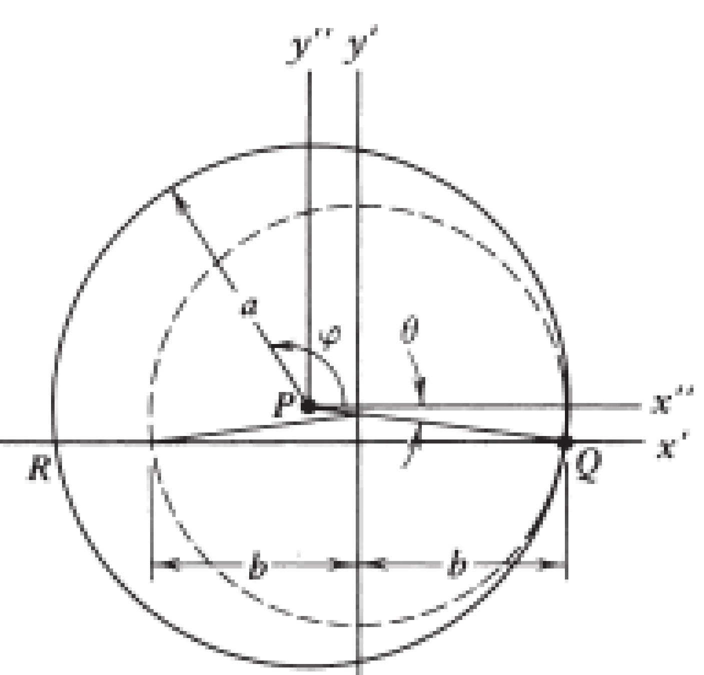
Biringen and Chow (2011, p.83)

Flow past a Joukowski Airfoil

- The complex potential of the resultant flow in the *x''-y''* plane whose origin is at point *P* is

$$w = U\left[z'' + \frac{a^2}{z''} + i2y'_P \log\left(\frac{z''}{a}\right)\right]$$

- "The constant $-i2y'_P \log a$ is added so that the value of the stream function on the circle remains unchanged after the introduction of the vortex."

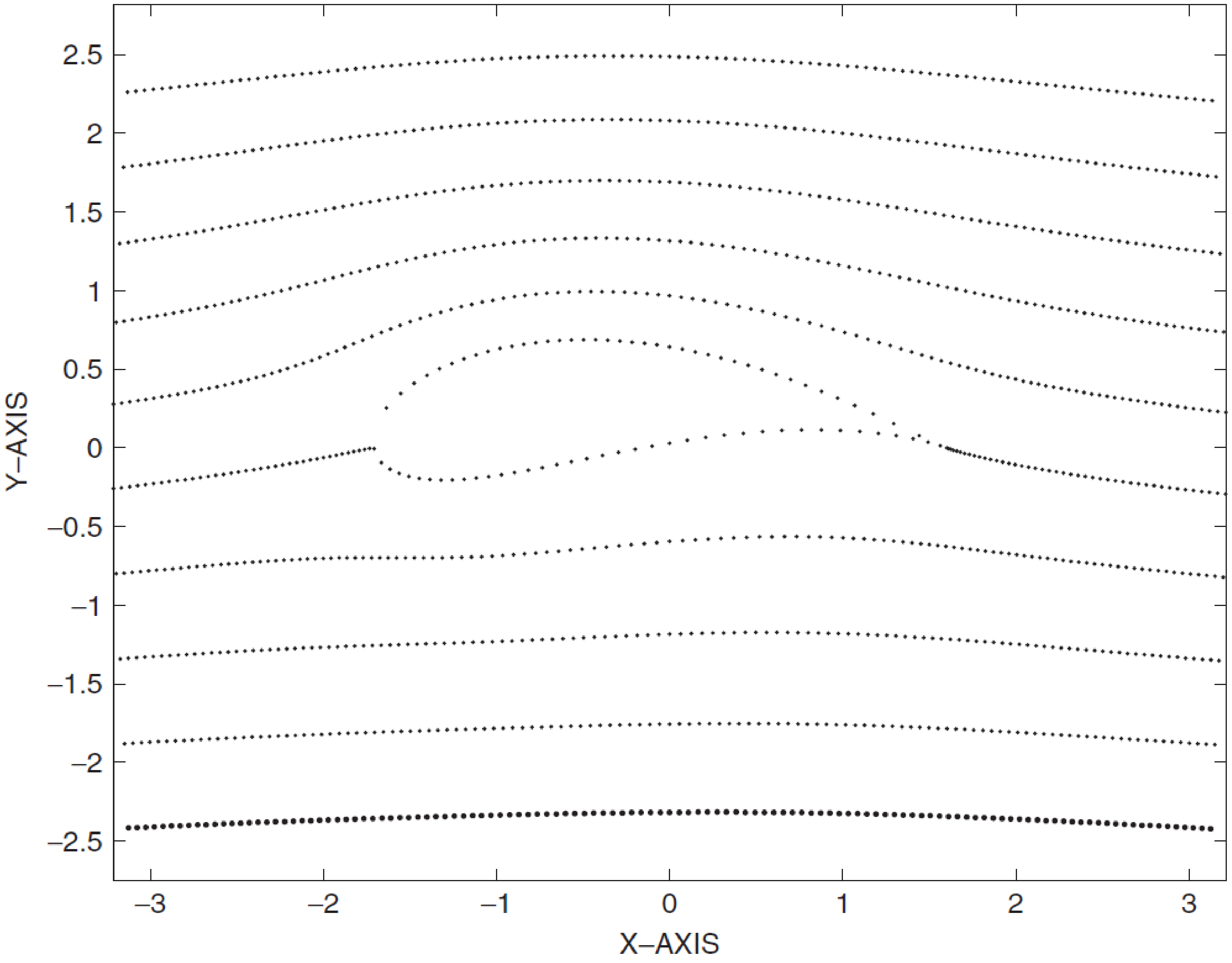


- The complex potential can be written in term of z' by using the transformation $z'' = z' - z'_P$.
- The airfoil shape is controlled by the values of *a*, *b*, and the coordinate of point *P* (x'_P, y'_P).
- They are related by $a^2 = y'^2_P + (b - x'_P)^2$

$$x'_P = b - \sqrt{a^2 - y'^2_P}$$

Biringen and Chow (2011, p.84)

Flow past a Joukowski Airfoil

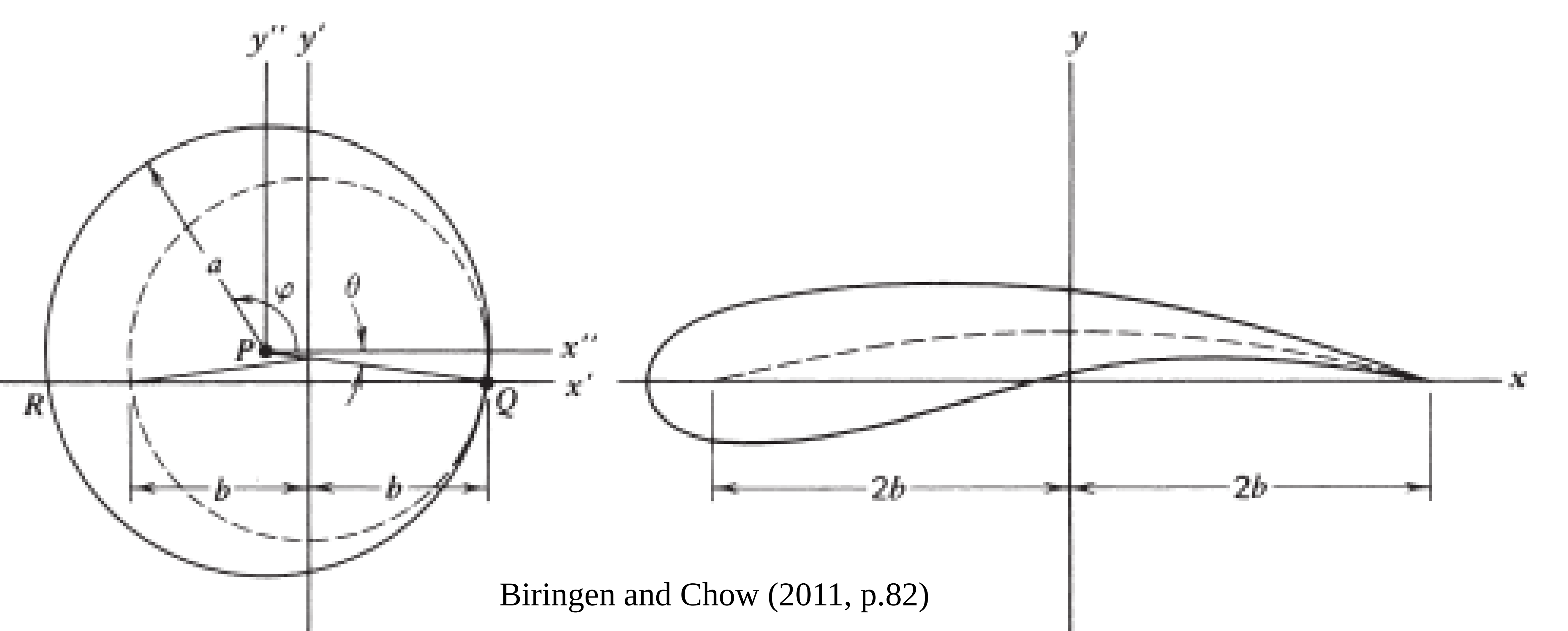


Parameters: $U = 1$ m/s
$a = 1$ m, $b = 0.8$ m
$x'_P = -0.18$ m, $y'_P = 0.199$ m
The stream function was computed in the x'-y' plane using the imaginary part of the complex potential.
Points on some stream lines are then mapped to the x-y plane using the Joukowski transformation.

Biringen and Chow (2011, p.86)

Exercise

Use the Joukowski transformation $z = z' + b^2/z'$ to map a circle of radius a centered at point P (see the figure below) into a Joukowski airfoil.
Here, $b < a$.



Biringen and Chow (2011, p.82)

Exercise

- Compute the complex potential using

$$w = U\left[\left(z' - z'_P\right) + \frac{a^2}{z' - z'_P} + i2y'_P \log\left(\frac{z' - z'_P}{a}\right)\right]$$

- Then compute the stream function from the imaginary part of the complex potential.
- Then compute the coordinates (x', y') of points on a stream line.
- Use the Joukowski transformation to map those points to the x-y plane.

$$z = z' + b^2 / z'$$

FD Solution to Elliptic PDEs

- Consider the 2D Poisson equation, an elliptic PDE,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = q(x, y)$$

- Approximating the derivative terms using the centered FD approximations yields the algebraic equations

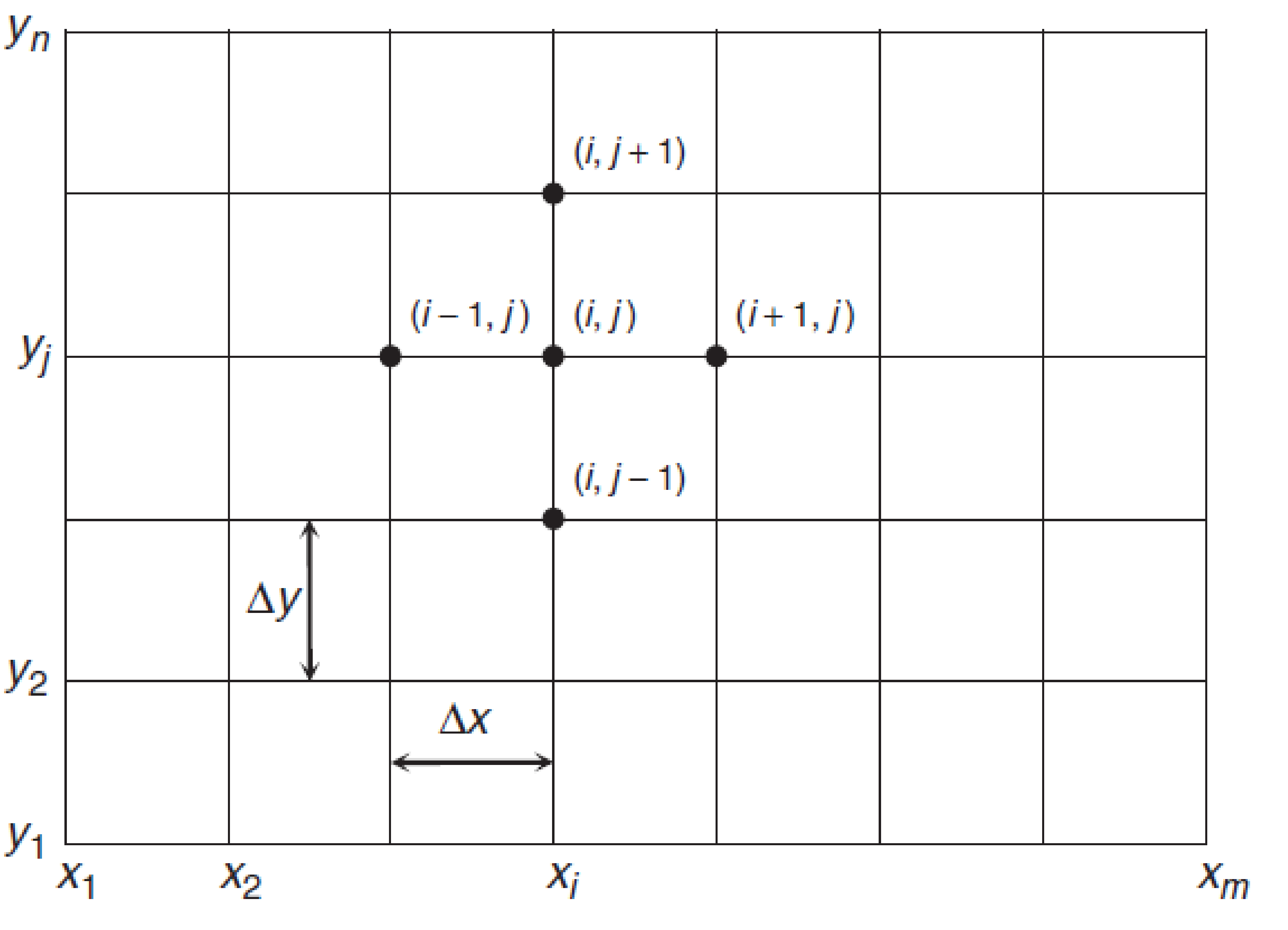
$$\frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta x)^2} + \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta y)^2} = q_{i,j}, \quad i = 2, \ldots, m; j = 2, \ldots, n$$

  where $f_{i,j} = f(x_i, y_j)$.
- Using a square grid, i.e., $\Delta x = \Delta y = h$, we can rearrange the equation as

$$f_{i-1,j} + f_{i+1,j} - 4f_{i,j} + f_{i,j-1} + f_{i,j+1} = h^2 q_{i,j}$$

Biringen and Chow (2011, p.90)

FD Grid



Biringen and Chow (2011, p.91)

Iterative Linear Solvers

- An iterative scheme can also be written as

$$\mathbf{x}^{(k+1)} = \mathbf{G}\mathbf{x}^{(k)} + \mathbf{c}$$

  where **G** and **c** are chosen so that a **fixed point** of the equation

$$\mathbf{x} = \mathbf{G}\mathbf{x} + \mathbf{c}$$

  is a solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$.
- "If **G** and **c** are constant over all iterations, such a scheme is said to be **stationary**."

Heath (2002)

Splitting

- Matrix **G** can be obtained by splitting **A** as **A** = **M** - **N**.
- Consequently,

$$\mathbf{Ax} = \mathbf{b}$$
$$(\mathbf{M} - \mathbf{N})\mathbf{x} = \mathbf{b}$$
$$\mathbf{Mx} = \mathbf{Nx} + \mathbf{b}$$
$$\mathbf{x} = \mathbf{M}^{-1}\mathbf{Nx} + \mathbf{M}^{-1}\mathbf{b}$$

- This is a fixed-point iteration with iteration function

$$\mathbf{g}(\mathbf{x}) = \mathbf{M}^{-1}\mathbf{Nx} + \mathbf{M}^{-1}\mathbf{b}$$

whose Jacobian matrix is $\mathbf{G}(\mathbf{x}) = \mathbf{M}^{-1}\mathbf{N}$

Heath (2002)

Convergence of Iterative Methods

- A stationary iteration scheme is convergent if the spectral radius

$$\rho(\mathbf{G}) = \rho\left(\mathbf{M}^{-1}\mathbf{N}\right) < 1$$

- Spectral radius of an $n \times n$ matrix **G** is defined as

$$\rho(\mathbf{G}) = \max\left\{|\lambda| : \lambda \in \lambda\left(\mathbf{G}\right)\right\}$$

  where λ's are the eigenvalues of **G**.

- The smaller $\rho(\mathbf{G})$, the faster the convergence.

Heath (2002), Golub and van Loan (1996)

Jacobi Method

- The Jacobi method is a stationary iterative method with

$$\mathbf{M} = \mathbf{D}, \qquad \mathbf{N} = -\left(\mathbf{L} + \mathbf{U}\right), \qquad \mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}$$

  where **L** and **U** are strict lower and upper triangular matrices (diagonal elements are zero) and **D** is diagonal.

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j \neq i} a_{ij} x_i^{(k)}\right), \quad i = 1, \ldots, n$$

- The Jacobi method is convergent if $\rho(\mathbf{G}) = \rho\left(\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})\right) < 1$
- The Jacobi method is convergent if **A** is strictly diagonally dominant.
- The Jacobi method does not converge for every symmetric positive-definite matrix since the spectral radius of such a matrix always exceeds 1.

Heath (2002), https://en.wikipedia.org/wiki/Jacobi_method

Gauss-Seidel Method

- "The Gauss-Seidel (GS) method improves the convergence rate of the Jacobi method by using each new component of the solution as soon as it has been computed."
- The Gauss-Seidel method has **M** = **D** + **L** and **N** = -**U**.

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij} x_i^{(k+1)} - \sum_{j>i} a_{ij} x_i^{(k)}\right), \quad i = 1, \dots, n$$

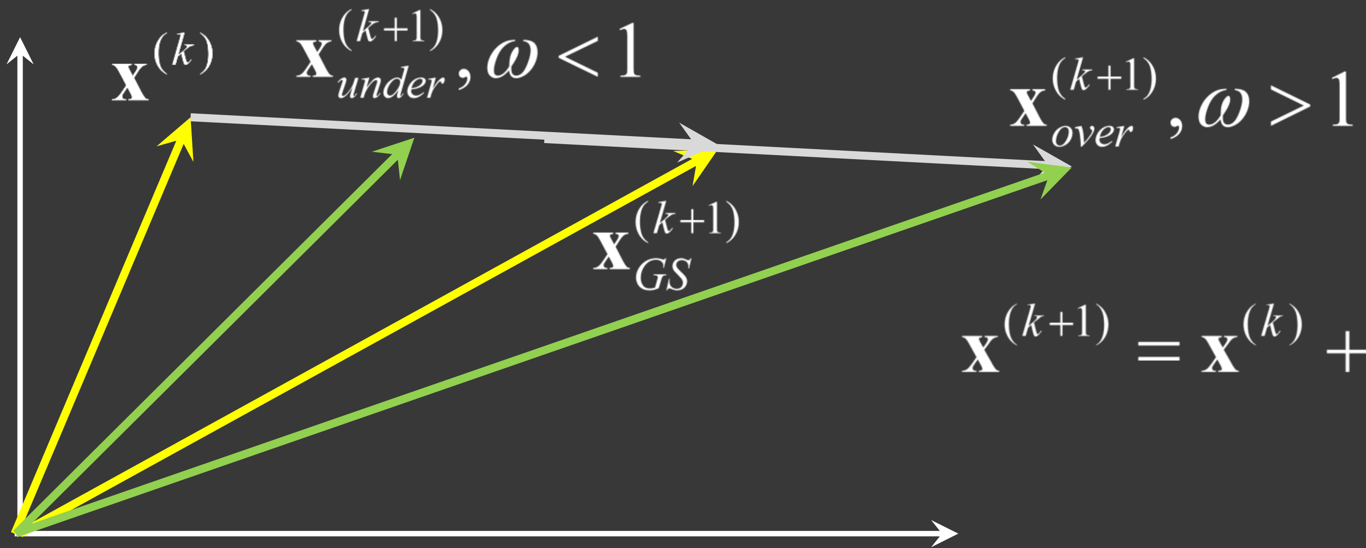
- The Gauss-Seidel method is convergent if $\rho(\mathbf{G}) = \rho\left((\mathbf{D}+L)^{-1}\mathbf{U}\right) < 1$
- The Gauss-Seidel method is convergent if **A** is symmetric positive-definite or strictly diagonally dominant.

Heath (2002), https://en.wikipedia.org/wiki/Gauss%E2%80%93Seidel_method

Successive Over-Relaxation (SOR)

- The SOR method improves the convergence rate of GS method through optimization with respect to the relaxation parameter ω.
- The SOR method as $\mathbf{M} = \frac{1}{\omega}\mathbf{D} + \mathbf{L}, \quad \mathbf{N} = \left(\frac{1}{\omega} - 1\right)\mathbf{D} - \mathbf{U}$

$$x_i^{(k+1)} = \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_i^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_i^{(k)}\right) + (1-\omega) x_i^{(k)}, \quad i = 1, \ldots, n$$



SOR converges only when $0 < \omega < 2$.

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \omega\left(\mathbf{x}_{GS}^{(k+1)} - \mathbf{x}^{(k)}\right)$$

Iterative Linear Solvers

- The linear system can be efficiently solved using iterative methods in which the values of f at all interior points are initialized as $f_{i,j}^{(0)}$.
- The Jacobi iterative formula is given by

$$f_{i,j}^{(k+1)} = \tfrac{1}{4}\left(f_{i-1,j}^{(k)} + f_{i+1,j}^{(k)} + f_{i,j-1}^{(k)} + f_{i,j+1}^{(k)}\right) - \tfrac{1}{4}h^2 q_{i,j}$$

- The Gauss-Seidel iterative formula is given by

$$f_{i,j}^{(k+1)} = \tfrac{1}{4}\left(f_{i-1,j}^{(k+1)} + f_{i+1,j}^{(k)} + f_{i,j-1}^{(k+1)} + f_{i,j+1}^{(k)}\right) - \tfrac{1}{4}h^2 q_{i,j}$$

- The successive overrelaxation (SOR) iterative formula is given by

$$f_{i,j}^{(k+1)} = f_{i,j}^{(k)} + \frac{\beta}{4}\left(f_{i-1,j}^{(k+1)} + f_{i+1,j}^{(k)} + f_{i,j-1}^{(k+1)} + f_{i,j+1}^{(k)} - 4f_{i,j}^{(k)} - h^2 q_{i,j}\right), \quad 1 \le \beta < 2$$

- The optimal value of β is

$$\beta_{\text{optimal}} = \frac{8 - 4\sqrt{4-\alpha^2}}{\alpha^2}, \qquad \alpha = \cos\left(\frac{\pi}{m}\right) + \cos\left(\frac{\pi}{n}\right)$$

Biringen and Chow (2011, p.93-94)

Exercise: Flow due to A Line Vortex

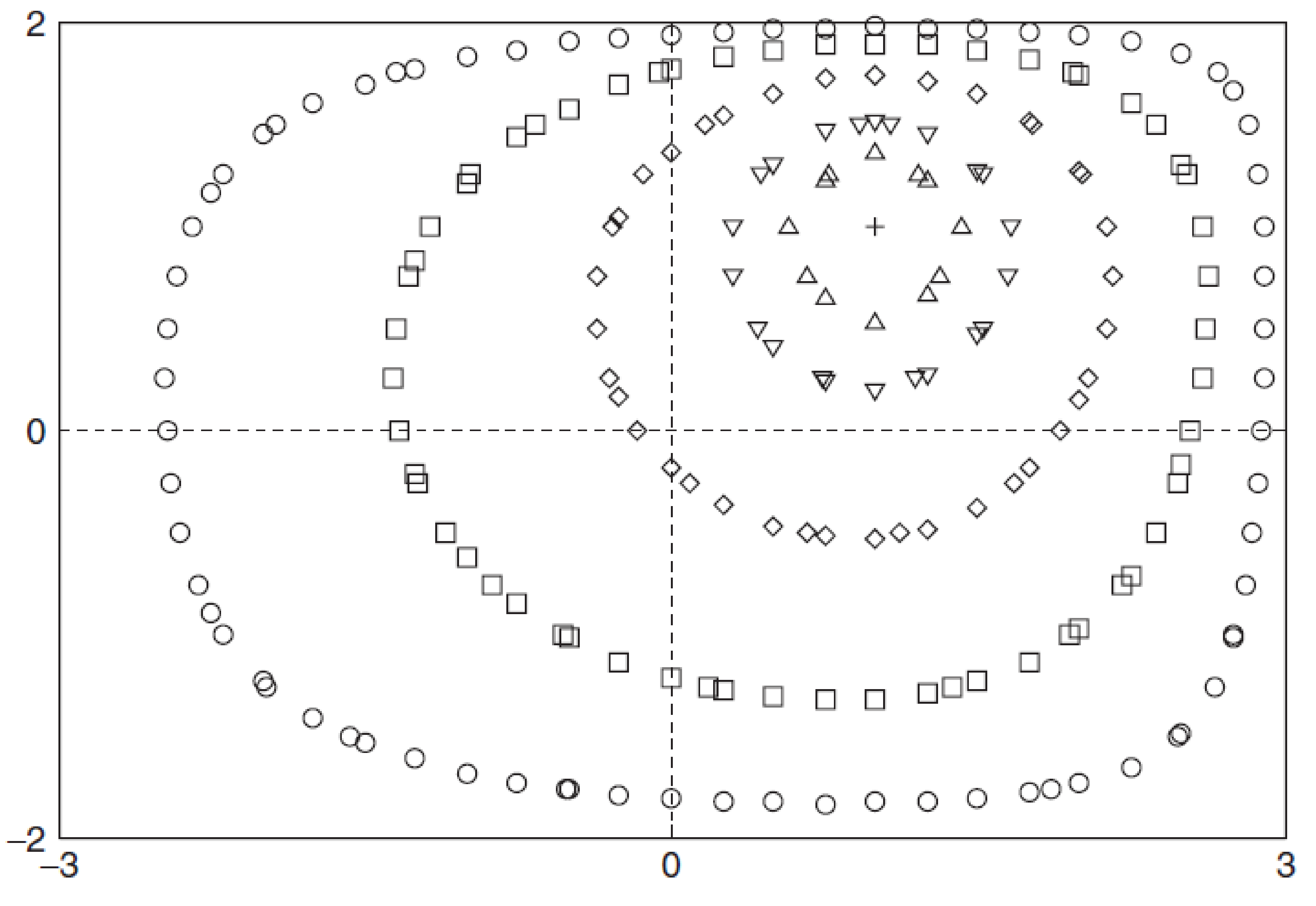
- The vorticity $\boldsymbol{\omega}$ is the curl of the flow velocity $\mathbf{v}$, i.e., $\boldsymbol{\omega} = \nabla \times \mathbf{v}$
- The stream function is defined as $\mathbf{v} = \nabla \times \left(\psi \hat{\mathbf{k}}\right)$
- In 2D, this corresponds to the Poisson equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega(x, y)$$

- The boundary condition is that $\psi = 0$ on the boundary.
- Let the domain be $\Omega = \left\{(x, y) : -3 \le x \le 3, -2 \le y \le 2\right\}$
- Let the grid spacing be $h = 0.25$.
- Suppose that the line vortex of vorticity $\omega = 1$ is located at (1,1).

Biringen and Chow (2011, p.95)

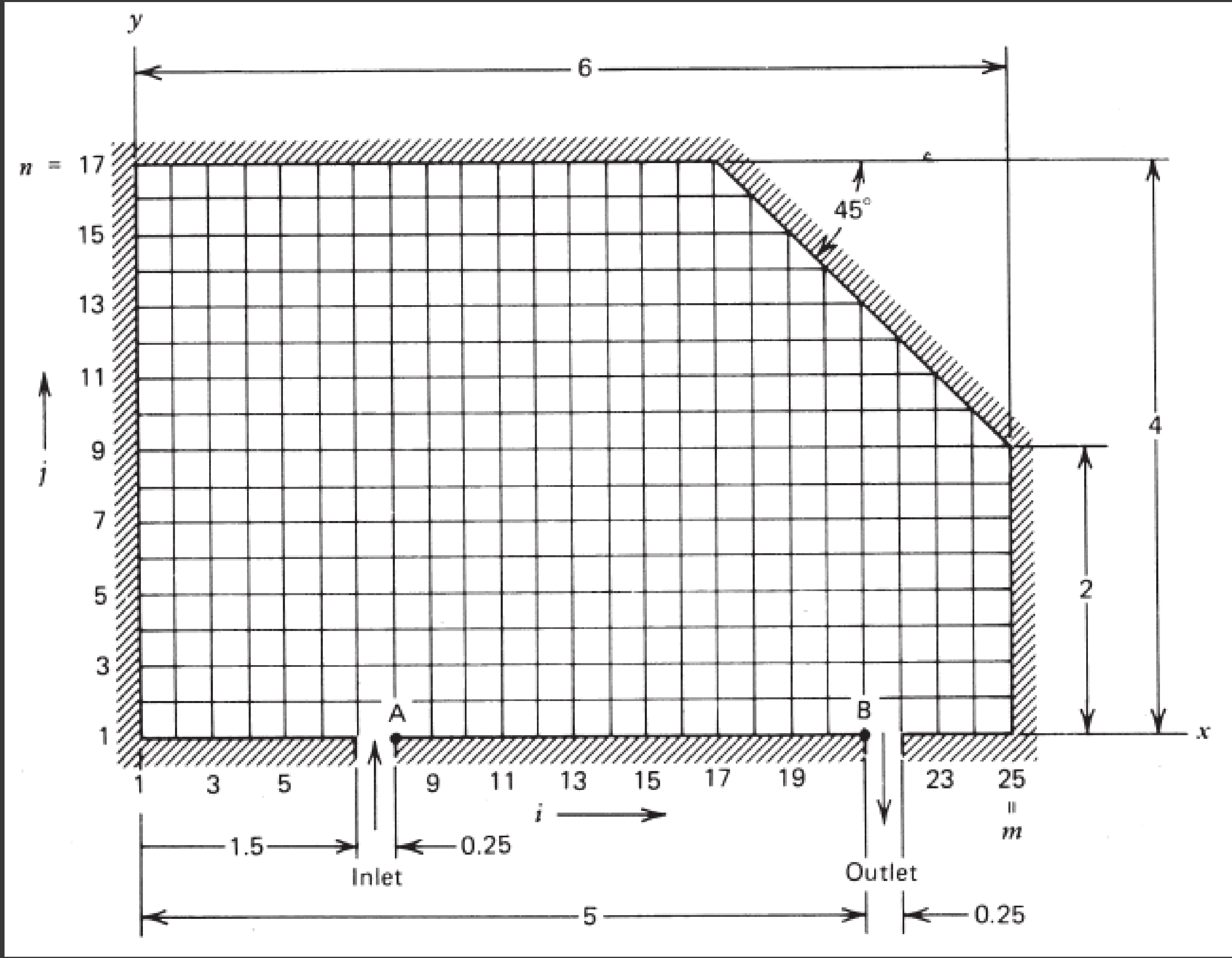
Flow due to a Line Vortex



Biringen and Chow (2011, p.96)

Potential Flows in Ducts

- "An inviscid incompressible fluid is flowing steadily through the chamber between an inlet and an outlet."
- The upper right corner is blocked off by a plate making a 45° angle with the x axis.
- Using grid spacing $h = 0.25$, the slant surface intersects exactly at some grid points.



Biringen and Chow (2011, p.97)

Potential Flows in Ducts

- Suppose there is no vorticity: $\dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2} = 0$

- Using the second-order centered finite-difference approximation yields

$$\frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{(\Delta x)^2} + \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{(\Delta y)^2} = 0$$

- Using a square grid with $h = \Delta x = \Delta y$, we obtain

$$\psi_{i,j} = \frac{1}{4}\left(\psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1}\right)$$

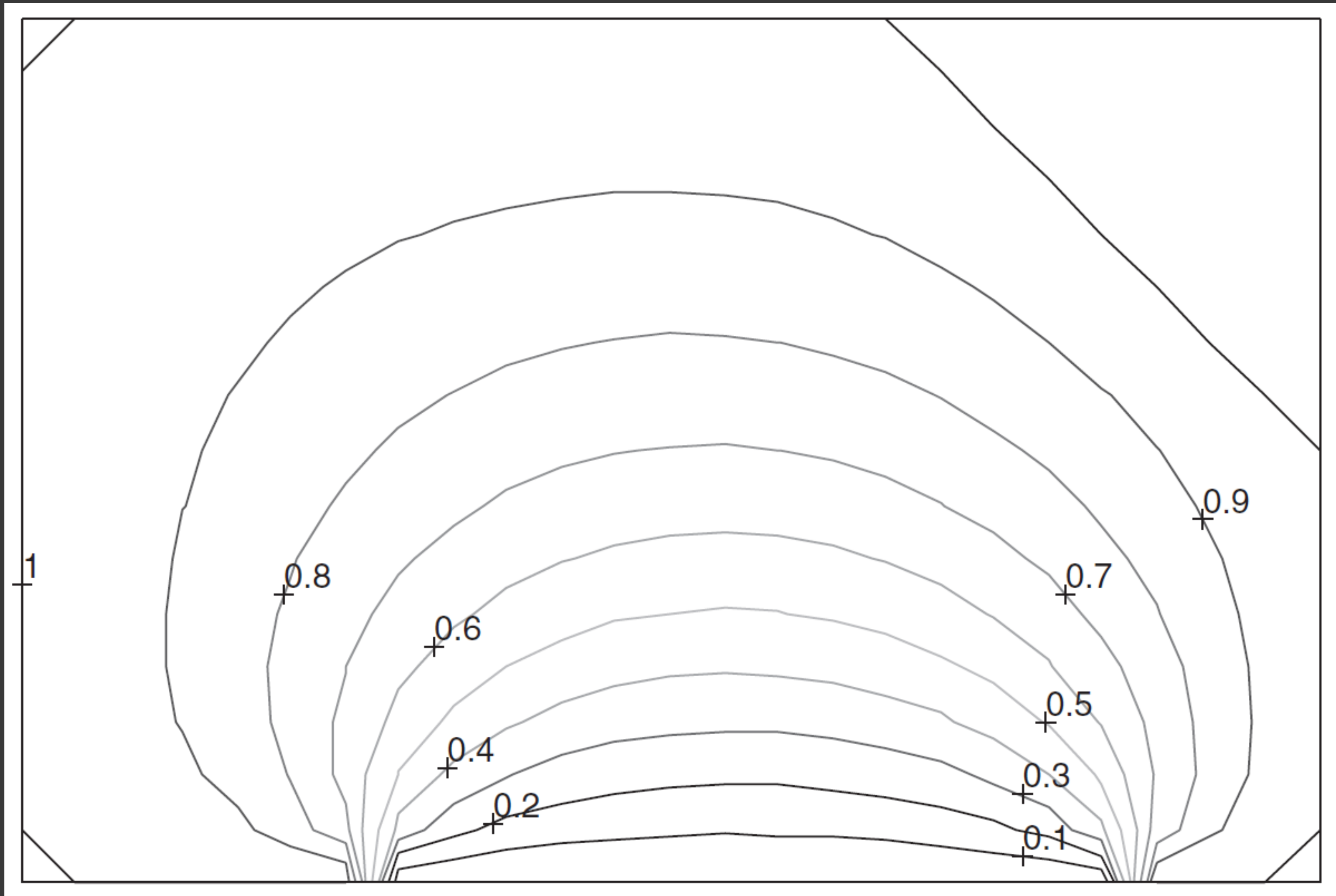
- The SOR scheme is then

$$\psi_{i,j}^{(k+1)} = (1-\beta)\psi_{i,j}^{(k)} + \frac{\beta}{4}\left(\psi_{i-1,j}^{(k+1)} + \psi_{i+1,j}^{(k)} + \psi_{i,j-1}^{(k+1)} + \psi_{i,j+1}^{(k)}\right)$$

Biringen and Chow (2011, p.98)

Potential Flows in Ducts

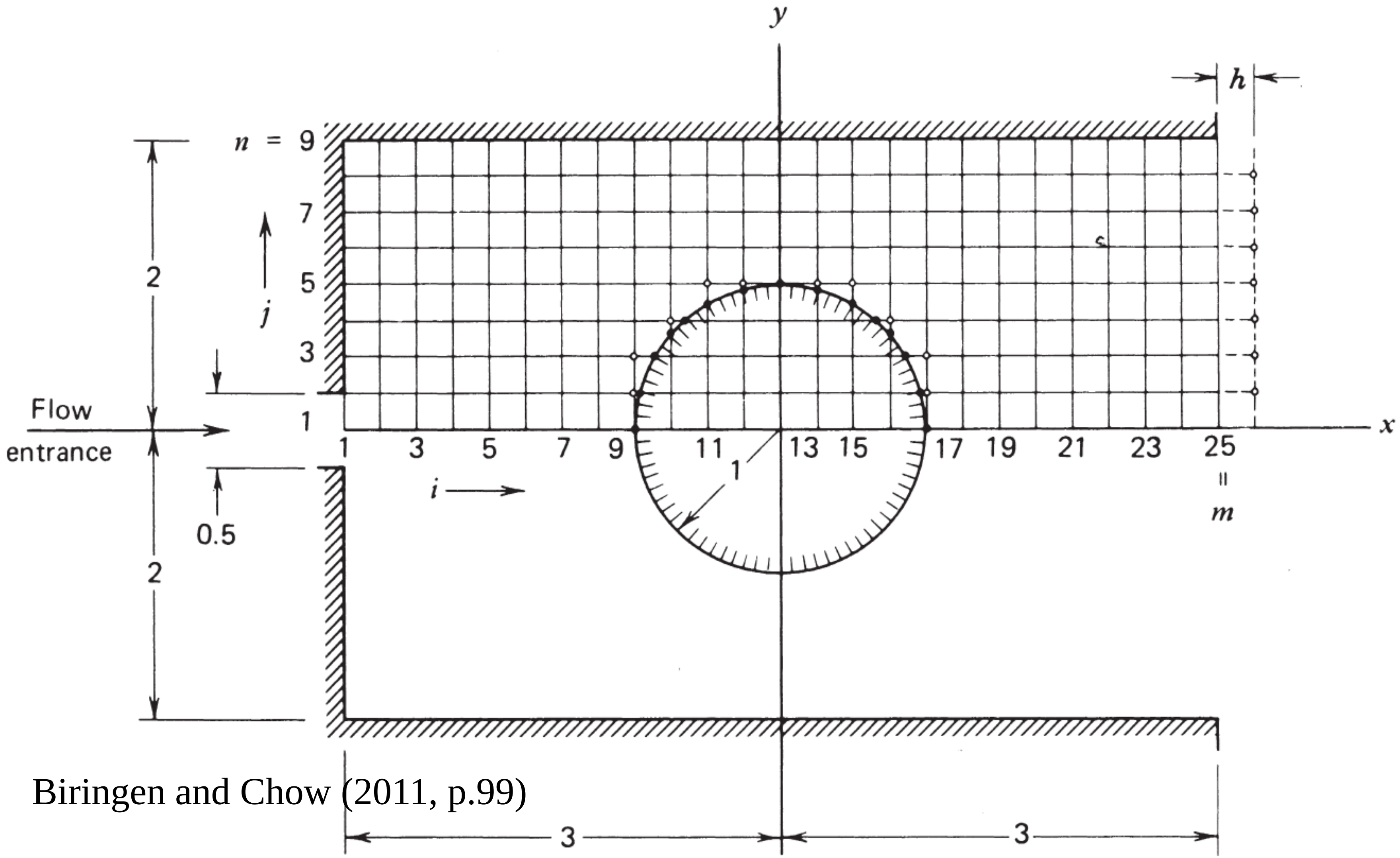
- The boundary conditions are as follows.
- $\psi = 0$ along line AB of wall
- $\psi = 1$ on the rest of the wall
- Solving the corresponding linear system yields the flow pattern shown in the right figure.



Biringen and Chow (2011, p.98)

Channel Flow past a Circular Cylinder

- The flow enters the channel from an opening at the left side.



Biringen and Chow (2011, p.99)

- The flow leaves at the right side at which the flow is assumed to be horizontal.
- The origin is at the center of the cylinder due to the symmetry.

Channel Flow past a Circular Cylinder

- Due to the symmetry about the *x*-axis, only the upper half of the flow will be computed.
- A square mesh of size $h = 0.25$ is used.
- The boundary conditions are as follows.
  - $\psi = 0$ on the cylinder surface and on the x-axis outside the body.
  - $\psi = 1$ on the upper wall of the channel
  - $v = -\partial\psi/\partial x = 0$ at the exit section
- Issue 1: Boundary points on the cylinder do not coincide with grid points.
- This issue can be overcome by using a more general finite-difference approximation.

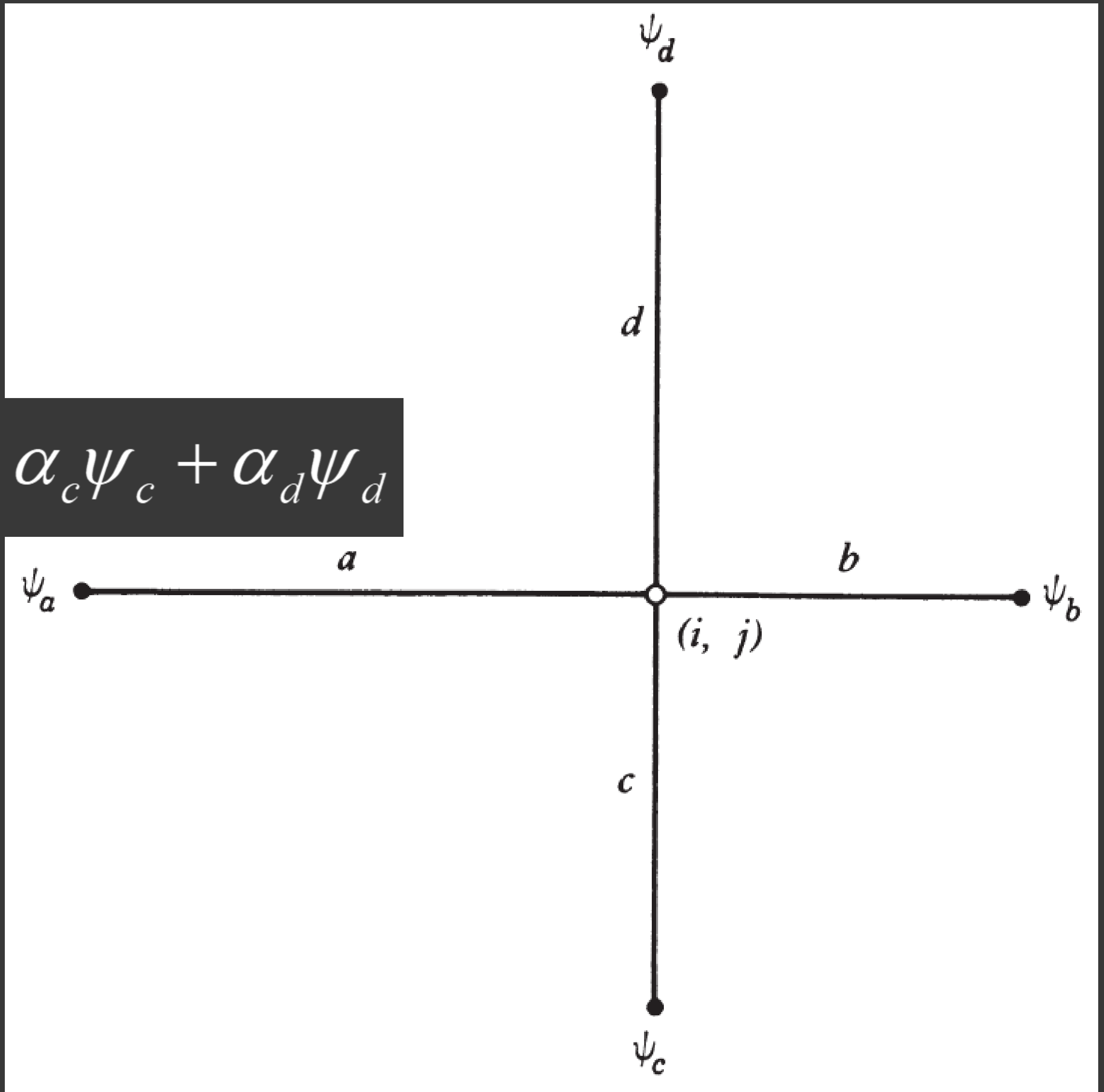
Biringen and Chow (2011, p.99)

Channel Flow past a Circular Cylinder

- Consider the stream functions at various points shown in the right figure.
- The Laplace equation can be approximated as

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \alpha_0 \psi_{i,j} + \alpha_a \psi_a + \alpha_b \psi_b + \alpha_c \psi_c + \alpha_d \psi_d$$

where α's are the coefficients to be determined



Biringen and Chow (2011, p.100)

Channel Flow past a Circular Cylinder

- Using the Taylor's series expansion, we obtain

$$\psi_a = \psi_{i,j} - a\left(\frac{\partial \psi}{\partial x}\right)_{i,j} + \frac{1}{2}a^2\left(\frac{\partial^2 \psi}{\partial x^2}\right)_{i,j} - O\left(a^3\right)$$

$$\psi_b = \psi_{i,j} + b\left(\frac{\partial \psi}{\partial x}\right)_{i,j} + \frac{1}{2}b^2\left(\frac{\partial^2 \psi}{\partial x^2}\right)_{i,j} + O\left(b^3\right)$$

$$\psi_c = \psi_{i,j} - c\left(\frac{\partial \psi}{\partial y}\right)_{i,j} + \frac{1}{2}c^2\left(\frac{\partial^2 \psi}{\partial y^2}\right)_{i,j} - O\left(c^3\right)$$

$$\psi_d = \psi_{i,j} + d\left(\frac{\partial \psi}{\partial y}\right)_{i,j} + \frac{1}{2}d^2\left(\frac{\partial^2 \psi}{\partial y^2}\right)_{i,j} + O\left(d^3\right)$$

Biringen and Chow (2011, p.100)

Channel Flow past a Circular Cylinder

- Rearranging the equations yields

$$\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)_{i,j}=\left(\alpha_0+\alpha_a+\alpha_b+\alpha_c+\alpha_d\right)\psi_{i,j}+\left(b\alpha_b-a\alpha_a\right)\left(\frac{\partial\psi}{\partial x}\right)_{i,j}$$

$$+\left(d\alpha_d-c\alpha_c\right)\left(\frac{\partial\psi}{\partial y}\right)_{i,j}+\frac{1}{2}\left(a^2\alpha_a+b^2\alpha_b\right)\left(\frac{\partial^2\psi}{\partial x^2}\right)_{i,j}+\frac{1}{2}\left(c^2\alpha_c+d^2\alpha_d\right)\left(\frac{\partial^2\psi}{\partial y^2}\right)_{i,j}$$

- Equating the coefficients on both sides, we obtain the linear system

$$\alpha_0+\alpha_a+\alpha_b+\alpha_c+\alpha_d=0$$

$$b\alpha_b-a\alpha_a=0$$

$$d\alpha_d-c\alpha_c=0$$

$$\tfrac{1}{2}\left(a^2\alpha_a+b^2\alpha_b\right)=1$$

$$\tfrac{1}{2}\left(c^2\alpha_c+d^2\alpha_d\right)=1$$

Biringen and Chow (2011, p.100)

Channel Flow past a Circular Cylinder

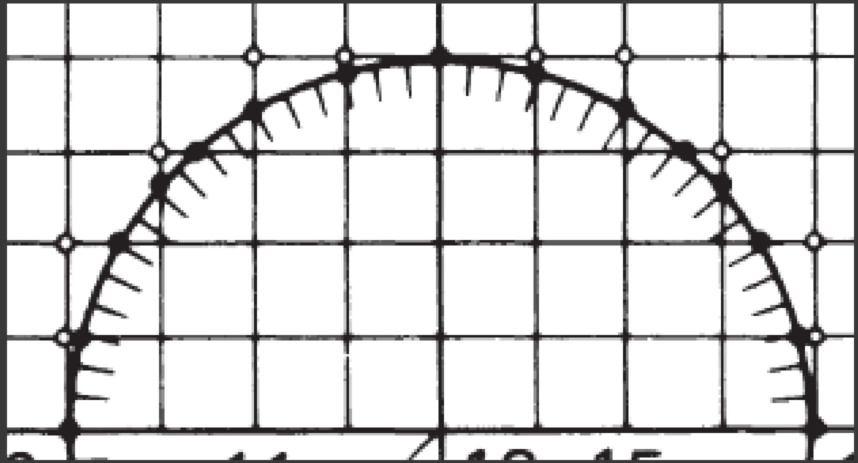
- Solving the linear system yields

$$\alpha_0 = -2\left(\frac{1}{ab}+\frac{1}{cd}\right), \alpha_a = \frac{2}{a(a+b)}, \alpha_b = \frac{2}{b(a+b)}, \alpha_c = \frac{2}{c(c+d)}, \alpha_d = \frac{2}{d(c+d)}$$

- We then obtain the generalized finite-difference scheme

$$\psi_{i,j} = \frac{abcd}{ab+cd}\left[\frac{\psi_a}{a(a+b)}+\frac{\psi_b}{b(a+b)}+\frac{\psi_c}{c(c+d)}+\frac{\psi_d}{d(c+d)}\right]$$

which must be applied at the grid points denoted as hollow circles.



Biringen and Chow (2011, p.101)

Channel Flow past a Circular Cylinder

Applying the FD scheme

$$\psi_{i,j} = \frac{1}{4}\left(\psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1}\right)$$

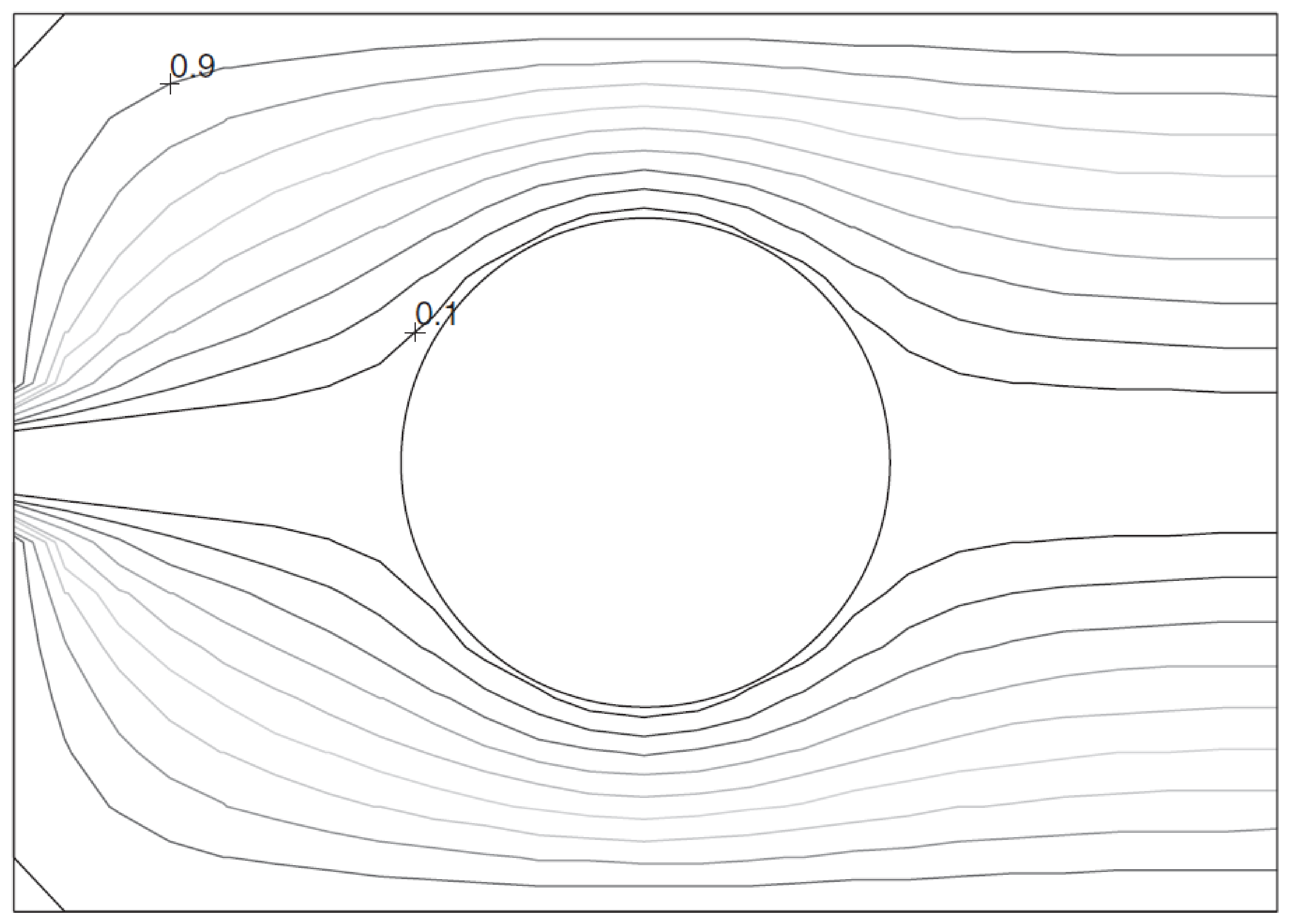
at regular grid points, the generalized FD scheme

$$\psi_{i,j} = \frac{abcd}{ab+cd}\left[\frac{\psi_a}{a(a+b)} + \frac{\psi_b}{b(a+b)} + \frac{\psi_c}{c(c+d)} + \frac{\psi_d}{d(c+d)}\right]$$

to grid points next to the points on the cylinder, and the boundary conditions, we obtain a linear system whose solution provides the values of stream functions at the grid points in the domain.

Biringen and Chow (2011, p.101)

Channel Flow past a Circular Cylinder



Biringen and Chow (2011, p.103)

Propagation of Small-Amplitude Wave

- Consider the wave equation for sound wave of small amplitude

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

  where a is the sound speed and u is the fluid speed.
- The spatial domain considered here is $0 \le x \le L$.
- Suppose we have initial conditions of the form

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

- We consider a flow within a channel of constant cross-sectional area.
- If one end of the channel is a rigid wall, $u = 0$ at all times.
- If one end opens to the atmosphere, pressure = constant or $\dfrac{\partial u}{\partial x} = 0$

Biringen and Chow (2011, p.105)

Explicit Finite-Difference Scheme

- Using the second-order FD approximation, the wave equation becomes

$$\frac{u_{j,n+1} - 2u_{j,n} + u_{j,n-1}}{(\Delta t)^2} = a^2 \frac{u_{j+1,n} - 2u_{j,n} + u_{j-1,n}}{(\Delta x)^2}, \qquad u_{j,n} = u(x_j, t_n) = u(j\Delta x, n\Delta t)$$

  which can be rearranged as

$$u_{j,n+1} = 2u_{j,n} + C^2 \left(u_{j+1,n} - 2u_{j,n} + u_{j-1,n}\right) - u_{j,n-1}$$

  where C is a dimensionless parameter called the Courant number, defined by

$$C = \frac{a\Delta t}{\Delta x}$$

- This numerical scheme is stable (the solution does not grow with time) when a certain condition is satisfied.
- Such a condition can be obtained via von Neumann stability analysis.

Biringen and Chow (2011, p.106)

von Neumann Stability Analysis

- Assume that the spatial and temporal components of the wavefield is separable, and the wavefield can be expanded as a Fourier series.
- The wavefield $u_{j,n}$ is represented as

$$u_{j,n} = U(t_n) e^{ikx_j} = U_n e^{ik(j\Delta x)}, \quad k \text{ is the wavenumber}$$

- Similarly,

$$u_{j\pm1,n} = U_n e^{ik(j\pm1)\Delta x}, \qquad u_{j,n\pm1} = U_{n\pm1} e^{ikj\Delta x}$$

- Substituting these into

$$u_{j,n+1} = 2u_{j,n} + C^2 \left(u_{j+1,n} - 2u_{j,n} + u_{j-1,n} \right) - u_{j,n-1}$$

and eliminating the common factor $e^{ikj\Delta x}$ yields

$$U_{n+1} = 2U_n + C^2 U_n \left(e^{-ik\Delta x} + e^{ik\Delta x} - 2 \right) - U_{n-1}$$

Biringen and Chow (2011, p.106-107)

von Neumann Stability Analysis

- Using the identity $\left(e^{i\theta}+e^{-i\theta}\right)/2=\cos\theta$, we obtain
$$U_{n+1}=AU_n-U_{n-1}$$
where $A\equiv 2\left[1-C^2\left(1-\cos(k\Delta x)\right)\right]$
- An amplification factor λ is defined such that $U_n=\lambda U_{n-1}$, $U_{n+1}=\lambda U_n$.
- So, $U_{n+1}=AU_n-U_{n-1}$ reduces to $\lambda^2-A\lambda+1=0$ whose roots are
$$\lambda=\frac{A}{2}\pm\sqrt{\left(\frac{A}{2}\right)^2-1}$$
- For $|A|\geq 2$, the roots are real but their magnitude are $|\lambda|\geq 1$ causing the wave amplitude to grow with time leading to instability.
- For $|A|<2$, the roots are complex and their magnitude are $|\lambda|=1$ and the wave amplitude does not grow with time, i.e., stable.

von Neumann Stability Analysis

- As a result, $|A| \leq 2$ is the condition for stability leading to

$$1 - C^2\left(1 - \cos(k\Delta x)\right) \leq 1 \quad \rightarrow \quad C^2 \leq \frac{2}{1 - \cos(k\Delta x)}$$

- The lower limit of the right-hand side is 1.
- So, the stability condition is $C^2 < 1$ or $\Delta t < \Delta x / a$.

Biringen and Chow (2011, p.107)

Exercise

Use the finite-difference method to solve the 1D wave equation in the spatial domain [0,1] with the initial conditions

$$u(x,0) = \begin{cases} \sin\left(2\pi \dfrac{x-0.2}{0.4}\right), & 0.2 \le x \le 0.6 \\ 0, & \text{otherwise} \end{cases}$$

$$\frac{\partial u}{\partial t}(x,0) = 0$$

and the boundary conditions

$$u(0,t) = 0, \qquad \frac{\partial u}{\partial x}(1,t) = 0$$

Propagation of Finite-Amplitude Wave

- Recall the continuity equation and the Euler equatioin

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v} \right) = 0, \qquad \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p$$

- For 1D flows, the equations become

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{a^2}{\rho} \frac{\partial \rho}{\partial x} = 0$$

where a is the sound speed.

- The process is assumed to be isentropic. So, $\dfrac{\partial p}{\partial x} = \dfrac{\partial p}{\partial \rho} \dfrac{\partial \rho}{\partial x} = a^2 \dfrac{\partial \rho}{\partial x}$

Propagation of Finite-Amplitude Wave

- Under isentropic conditions (adiabatic and reversible process in which entropy is constant), we can eliminate ρ from the previous equations.
- Let the subscript 0 indicate the undisturbed conditions and γ be the ratio of specific heats of the gas.
- In isentropic process, $p \propto \rho^{\gamma} \rightarrow \dfrac{p}{p_0} = \left(\dfrac{\rho}{\rho_0}\right)^{\gamma}$

$$a^2 = \frac{\partial p}{\partial \rho} = \gamma \frac{p}{\rho} = \gamma \frac{p_0}{\rho_0}\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = a_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \rightarrow \rho = \rho_0\left(\frac{a}{a_0}\right)^{2/(\gamma-1)}$$

- We then obtain

$$\frac{2}{\gamma-1}\left(\frac{\partial a}{\partial t} + u\frac{\partial a}{\partial x}\right) + a\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{2a}{\gamma-1}\frac{\partial a}{\partial x} = 0$$

Biringen and Chow (2011, p.120)

Propagation of Finite-Amplitude Wave

- The last 2 equations can be rewritten as

$$\left[\frac{\partial}{\partial t}+(u+a)\frac{\partial}{\partial x}\right]\left(u+\frac{2a}{\gamma-1}\right)=0$$

$$\left[\frac{\partial}{\partial t}+(u-a)\frac{\partial}{\partial x}\right]\left(u-\frac{2a}{\gamma-1}\right)=0$$

- The first equation states that the quantity $P=u+2a/(\gamma-1)$ is constant along a curve in the *x*-*t* plane.
- On this curve $dP=\frac{\partial P}{\partial t}dt+\frac{\partial P}{\partial x}dx=0 \quad \rightarrow \quad \left[\frac{\partial}{\partial t}+\frac{dx}{dt}\frac{\partial}{\partial x}\right]P=0$
- Comparing the first and last equations, we know that the slope of the curve is $dx/dt = u + a$.

Biringen and Chow (2011, p.120)

Propagation of Finite-Amplitude Wave

- Similarly, the second equation

$$\left[\frac{\partial}{\partial t}+(u-a)\frac{\partial}{\partial x}\right]\left(u-\frac{2a}{\gamma-1}\right)=0$$

  states that the quantity $Q=u-2a/(\gamma-1)$ is constant on a curve whose slope is $dx/dt = u - a$.
- These two curves are called the characteristics while P and Q are called the Riemann invariants.
- Since u and a vary with x and t, the characteristics are curved lines.

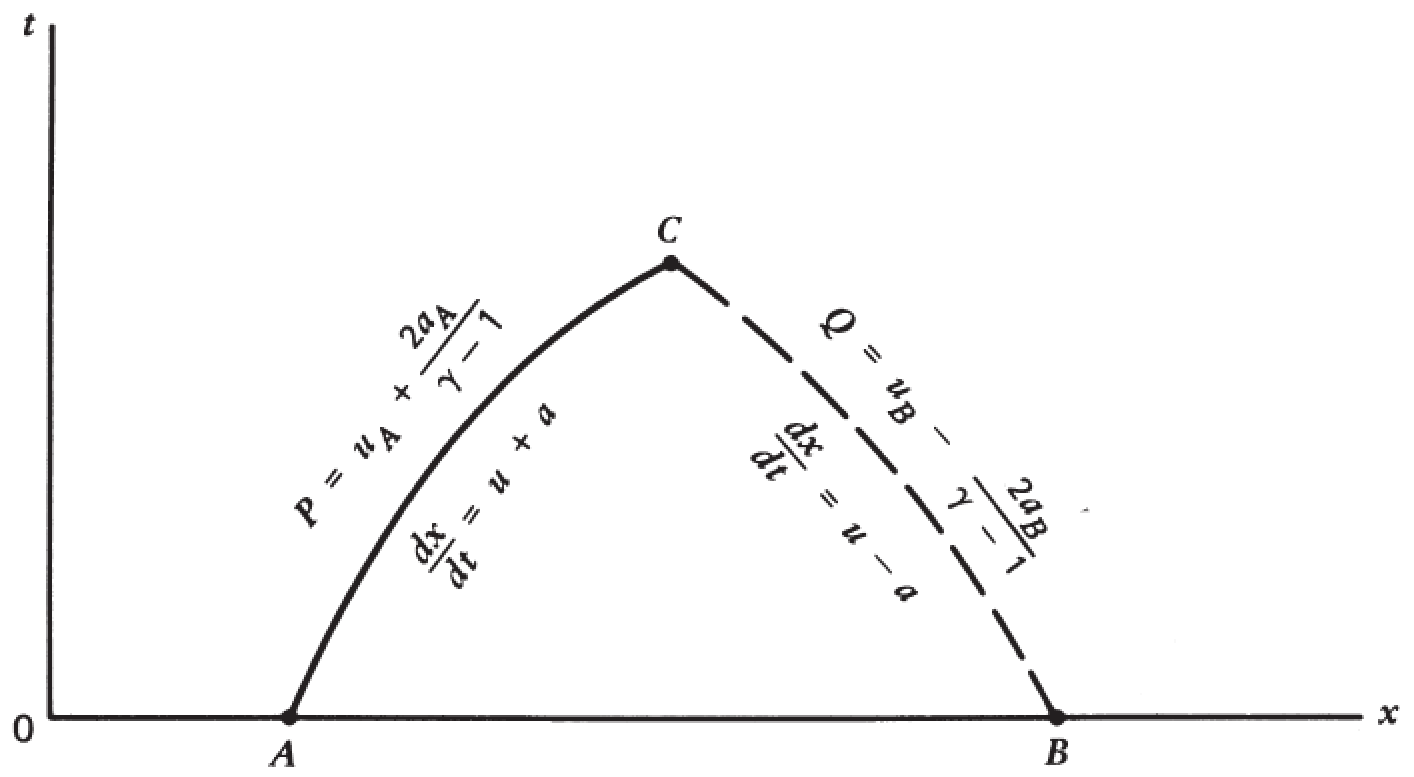
Biringen and Chow (2011, p.120)

Method of Characteristics

- "Suppose the initial data at $t = 0$ are given and the conditions at an arbitrary point C at $t_C > 0$ are to be computed."
- "Through this point there are two characteristics, one of slope $u + a$ and the other of slope $u - a$."
- The characteristics intersect the x-axis at points A and B.
- $P_C = P_A$ and $Q_C = Q_B$. So,

$$u_C + \frac{2a_C}{\gamma - 1} = u_A + \frac{2a_A}{\gamma - 1}$$

$$u_C - \frac{2a_C}{\gamma - 1} = u_B - \frac{2a_B}{\gamma - 1}$$



Biringen and Chow (2011, p.121)

Method of Characteristics

- Solving the equations for u_C and a_C, we obtain

$$u_C = \tfrac{1}{2}\left(u_A + u_B\right) + \tfrac{1}{\gamma-1}\left(a_A - a_B\right)$$

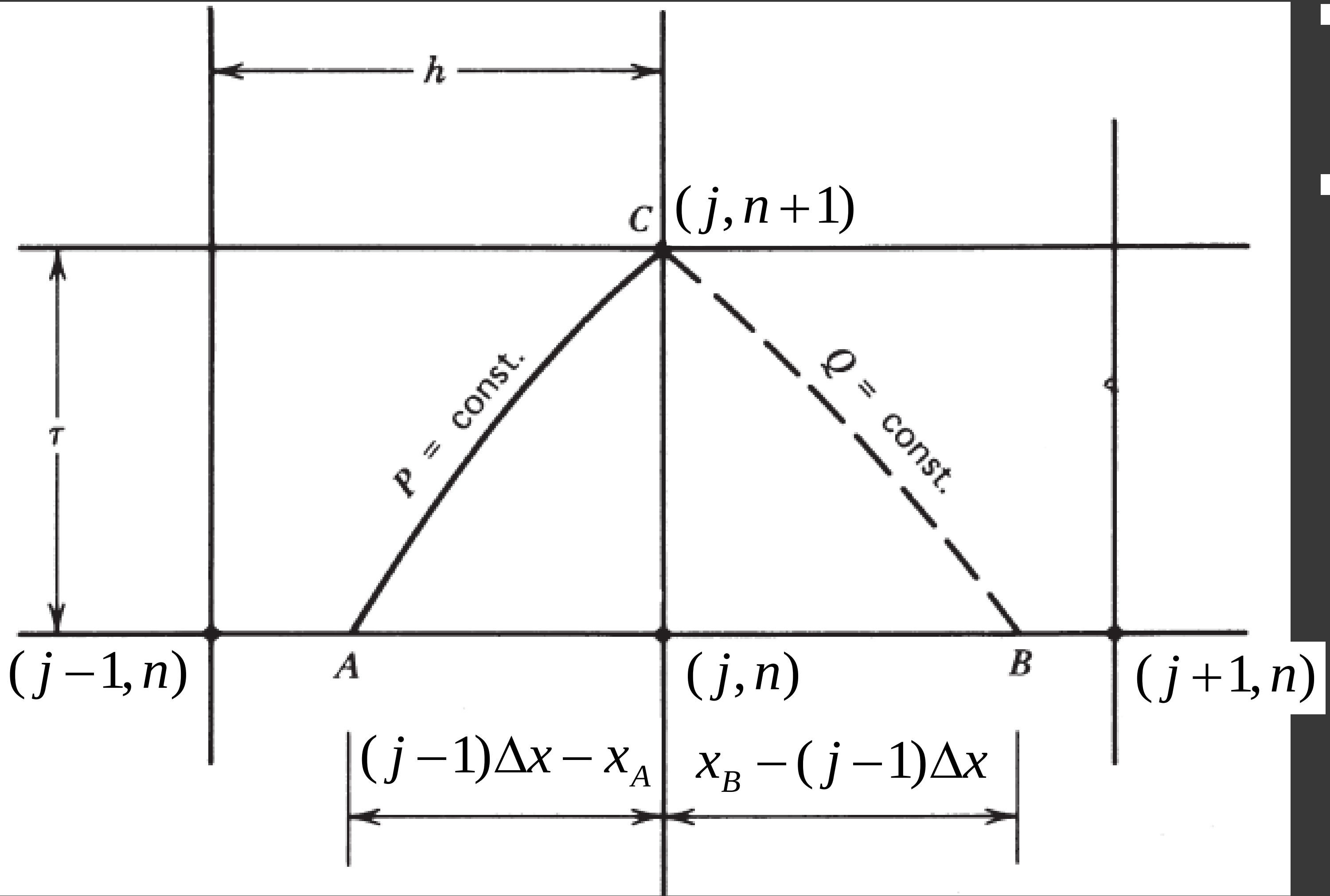
$$a_C = \tfrac{\gamma-1}{4}\left(u_A - u_B\right) + \tfrac{1}{2}\left(a_A + a_B\right)$$

- "The exact shape of the characteristics cannot be determined unless u and a are both known in the region below point C."
- When the time interval is small, the characteristics can be approximated as straight lines of slopes $u_A + a_A$ and $u_B - a_B$, and the above equations can be used.
- We can apply these equations to compute the values at grid points.

Biringen and Chow (2011, p.122)

Method of Characteristics

- Suppose that we have the values of u and a at all grid points at time t_n.
- We want to compute u and a at point C at time t_{n+1}.



- The characteristics intersect at points A and B at time t_n.
- Treating the characteristics as straight lines, we have

$$x_A = (j-1)\Delta x - \Delta t\left(u_{j,n} + a_{j,n}\right)$$

$$x_B = (j-1)\Delta x - \Delta t\left(u_{j,n} - a_{j,n}\right)$$

Biringen and Chow (2011, p.123)

Method of Characteristics

- After we obtain the locations of points A and B, linear interpolations are then used to obtain

$$u_A = u_{j,n} + \frac{\Delta t}{\Delta x}\left(u_{j,n} + a_{j,n}\right)\left(u_{j-1,n} - u_{j,n}\right)$$

$$a_A = a_{j,n} + \frac{\Delta t}{\Delta x}\left(u_{j,n} + a_{j,n}\right)\left(a_{j-1,n} - a_{j,n}\right)$$

$$u_B = u_{j,n} - \frac{\Delta t}{\Delta x}\left(u_{j,n} - a_{j,n}\right)\left(u_{j+1,n} - u_{j,n}\right)$$

$$a_B = a_{j,n} - \frac{\Delta t}{\Delta x}\left(u_{j,n} - a_{j,n}\right)\left(a_{j+1,n} - a_{j,n}\right)$$

- We can then compute u_C and a_C using the formulas on Page 80.

Biringen and Chow (2011, p.123)

Method of Characteristics

- This procedure is also conditionally stable.
- The stability conditions are

$$\frac{\Delta t}{\Delta x}|u+a| \leq 1, \qquad \frac{\Delta t}{\Delta x}|u-a| \leq 1$$

- This first-order method was developed by Courant et al. (1952).
- A second-order method proposed by Hartree (1958) uses the averaged slopes at C and A, and C and B.
- This requires some number of iterations to get the accurate locations of points A and B.

Biringen and Chow (2011, p.123)

Example

- Consider a 2-meter-long tube whose left end is closed and right end is open.
- Let a_0 be the sound speed in the tube in an undisturbed state.
- Let $\Delta x = 0.02$ m.
- At time $t = 0$, the wavefield is

$$u_{j,1} = \begin{cases} \dfrac{a_0}{2}\dfrac{j-1}{12}, & 1 \le j \le 13 \\ \dfrac{a_0}{2}\dfrac{26-j}{13}, & 13 < j \le 39 \\ \dfrac{a_0}{2}\dfrac{j-51}{12}, & 39 < j \le 51 \\ 0, & \text{otherwise} \end{cases}$$

Biringen and Chow (2011, p.124)

Example

- The initial condition for a is determined by

$$a = a_0 \pm \frac{\gamma - 1}{2} u$$

- The negative sign is used when the wave is traveling the negative x direction.
- The boundary condition at the left closed end lead to

$$u_{1,n+1} = 0, \qquad a_{1,n+1} = a_B - \frac{\gamma - 1}{2} u_B$$

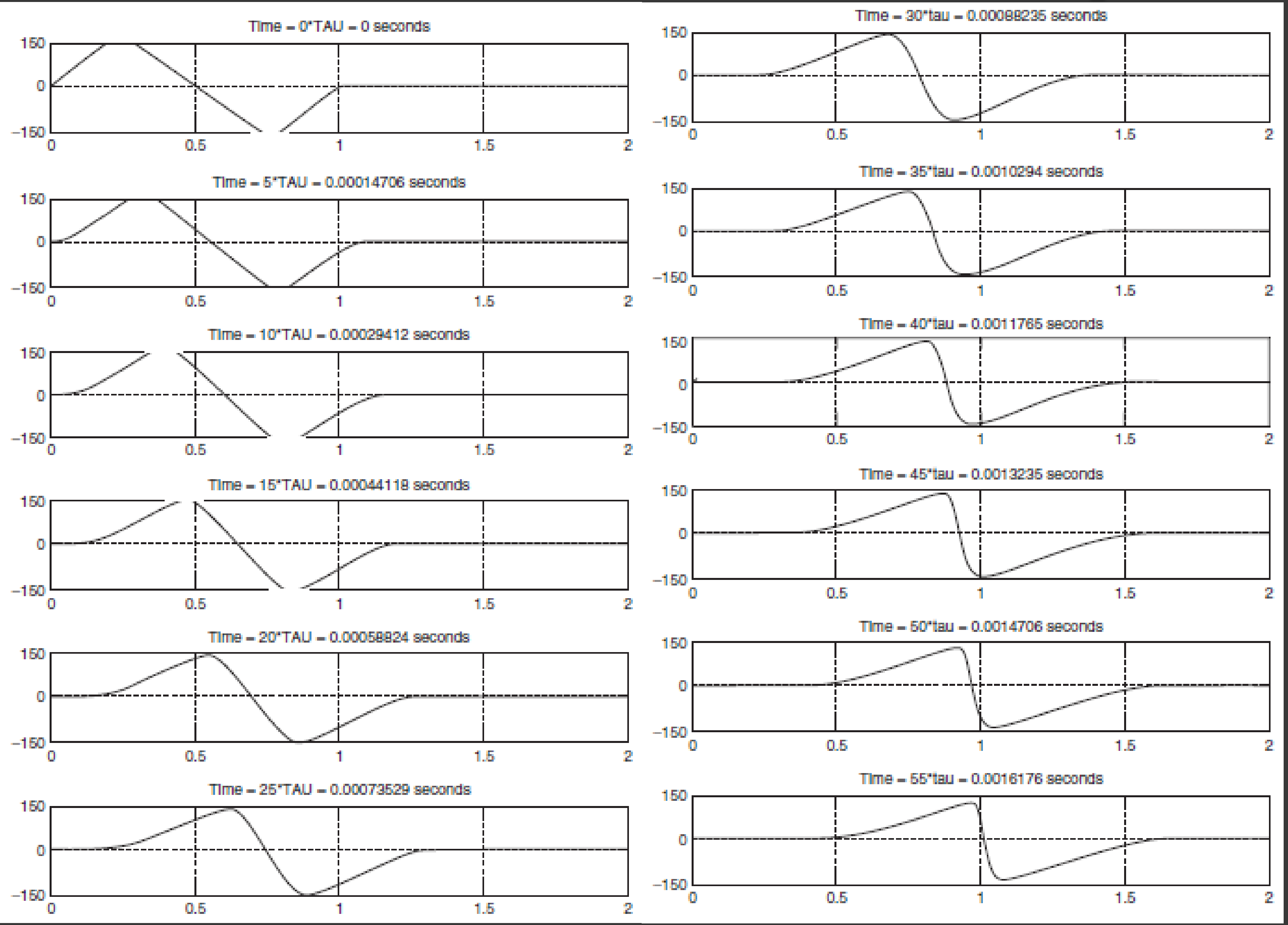
- The boundary condition at the right open end lead to

$$u_{m,n+1} = u_A + \frac{2}{\gamma - 1}\left(a_A - a_0\right), \qquad a_{m,n+1} = a_0$$

where m is the maximum value for j.

Biringen and Chow (2011, p.124)

Example



$\gamma = 1.4$

$\Delta x = 0.02 \text{ m}$

$$\Delta t = \frac{0.5\Delta x}{a_0}$$

Biringen and Chow (2011, p.126-127)

References

- J. D. Anderson, Jr., 1995, Computational Fluid Dynamics: The basics with applications, McGraw-Hill, Singapore.
- S. Biringen and C.Y. Chow, 2011, An Introduction to Computational Fluid Mechanics by Example, John Wiley and Sons.
- J. H. Ferziger, M. Peric, and R. L. Street, 2020, Computational Methods for Fluid Dynamics, Fourth Edition, Springer.
- A. M. Kuethe and C.Y. Chow, 1998, Foundations of Aerodynamics: Bases of Aerodynamic Design, Fifth Edition, John Wiley and Sons.