

Inviscid Fluid Flows, Incompressible Potential Flows, Elementary Flows, Superposition, von Karman Method, Conformal Mapping

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2D Incompressible Potential Flows

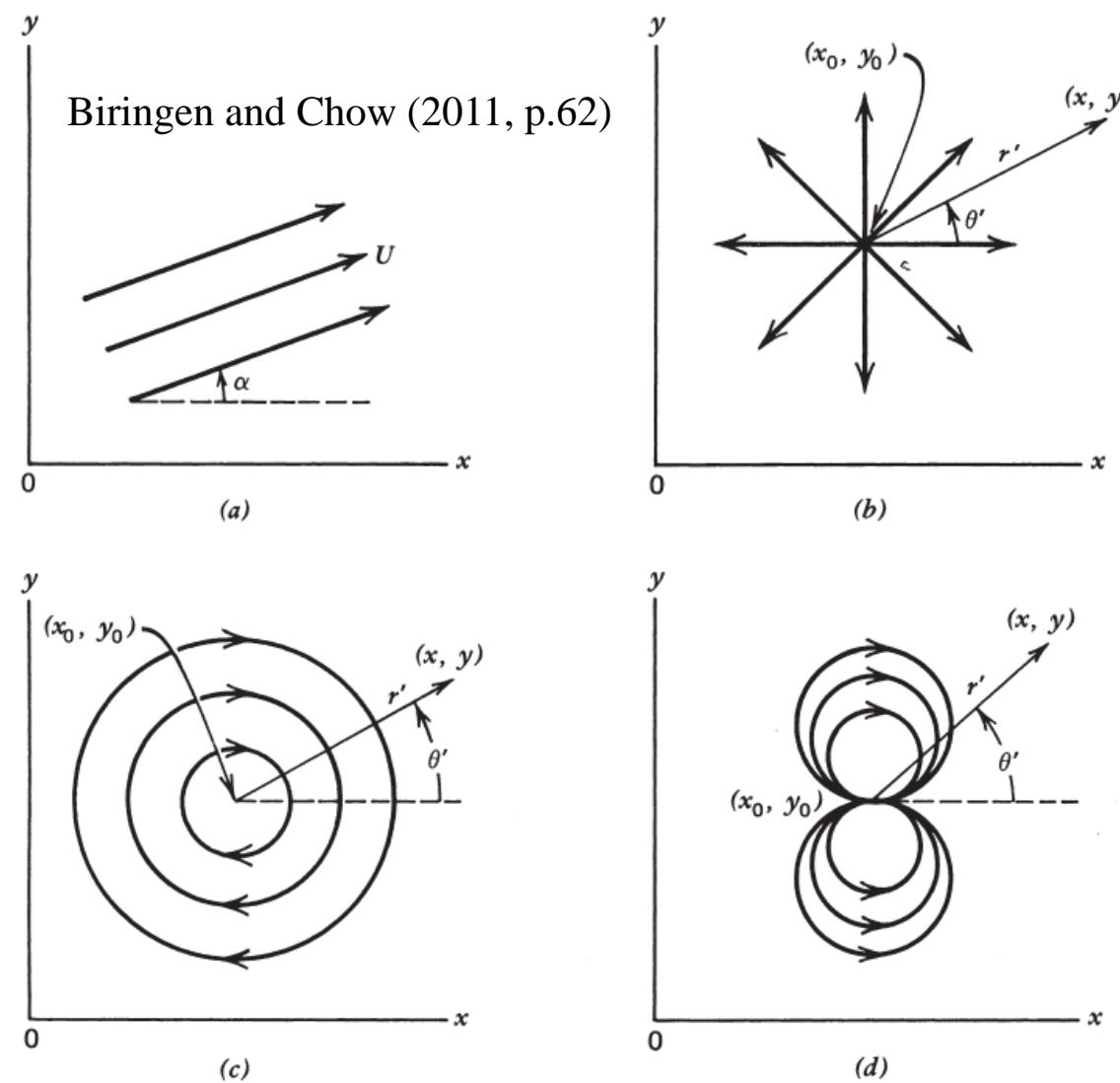
- Inviscid flow is the flow of a fluid with zero viscosity.
- Inviscid flows can be classified as potential flows (irrotational flows) and rotational inviscid flows.
- Potential or irrotational flows has zero vorticity, i.e., $\nabla \times \mathbf{v} = \mathbf{0}$ and the **velocity potential** ϕ can be defined such that $\mathbf{v} = -\nabla \phi$
- If a potential flow is also incompressible, i.e., $\nabla \cdot \mathbf{v} = 0$, the potential ϕ satisfies the Laplace equation $\nabla^2 \phi = 0$
- 2D planar flows: $v_x = \frac{\partial \phi}{\partial x}, \quad v_y = \frac{\partial \phi}{\partial y}, \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$
- 2D axisymmetric flows: $v_r = \frac{\partial \phi}{\partial r}, \quad v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \quad \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0$

2D Incompressible Potential Flows

- The stream function ψ can also be used instead of the velocity potential.
- The continuity equation $\nabla \cdot \mathbf{v} = 0$ suggests that the velocity can also be expressed interm of the stream function as $\mathbf{v} = \nabla \times (\psi \hat{\mathbf{k}})$.
- A line along which $\psi = \text{constant}$ is called a streamline.
- Fluid velocities are always tangential to streamlines.
- The irrotational condition becomes $\nabla \times \nabla \times (\psi \hat{\mathbf{k}}) = 0$
- 2D planar flows: $v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x}, \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$
- 2D axisymmetric flows: $v_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad v_\theta = -\frac{\partial \psi}{\partial r}, \quad \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0$

Elementary Flows

Stream functions corresponding to 4 elementary flows are as follows.



(a) Uniform flow with angle α

$$\psi = U(y \cos \alpha - x \sin \alpha)$$

(b) Line source with strength Λ

$$\psi = \frac{\Lambda}{2\pi} \tan^{-1} \left(\frac{y - y_0}{x - x_0} \right) = \frac{\Lambda \theta}{2\pi}$$

(c) Line vortex with circulation Γ :

$$\psi = \frac{\Gamma}{2\pi} \ln \left[(x - x_0)^2 + (y - y_0)^2 \right]^{\frac{1}{2}} = \frac{\Gamma}{2\pi} \ln r$$

(d) Doublet with strength κ :

$$\psi = -\frac{\kappa}{2\pi} \frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2} = -\frac{\kappa \sin \theta}{2\pi r}$$

Velocity Potentials of Elementary Flows

(a) Uniform flow with angle α :

$$\phi(x, y) = -U(x \cos \alpha + y \sin \alpha), \quad \phi(r, \theta) = -Ur \cos(\theta - \theta_0)$$

(b) Line source with strength Λ :

$$\phi(x, y) = \Lambda \ln(x^2 + y^2)/4\pi, \quad \phi(r, \theta) = \Lambda \ln r/2\pi$$

(c) Line vortex with circulation Γ :

$$\phi(x, y) = -\frac{\Gamma}{2\pi} \tan^{-1}\left(\frac{y - y_0}{x - x_0}\right), \quad \phi(r, \theta) = -\frac{\Gamma \theta}{2\pi}$$

(d) Doublet with strength κ :

$$\phi(x, y) = \frac{\kappa}{2\pi} \frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2}, \quad \phi(r, \theta) = \frac{\kappa}{2\pi} \frac{\cos \theta}{r}$$

Velocities of Elementary Flows

(a) Uniform flow with angle α :

$$\mathbf{v}(x, y) = U \cos \alpha \hat{\mathbf{i}} + U \sin \alpha \hat{\mathbf{j}}$$

(b) Line source with strength Λ

$$\mathbf{v}(x, y) = \frac{\Lambda}{2\pi} \left(\frac{(x - x_0)}{(x - x_0)^2 + (y - y_0)^2} \hat{\mathbf{i}} + \frac{(y - y_0)}{(x - x_0)^2 + (y - y_0)^2} \hat{\mathbf{j}} \right)$$

(c) Line vortex with circulation Γ :

$$\mathbf{v}(x, y) = \frac{\Gamma}{2\pi} \left(\frac{(y - y_0)}{(x - x_0)^2 + (y - y_0)^2} \hat{\mathbf{i}} - \frac{(x - x_0)}{(x - x_0)^2 + (y - y_0)^2} \hat{\mathbf{j}} \right)$$

(d) Doublet with strength κ :

$$\mathbf{v}(x, y) = -\frac{\kappa}{2\pi} \left(\frac{(y - y_0)^2 - (x - x_0)^2}{[(x - x_0)^2 + (y - y_0)^2]^2} \hat{\mathbf{i}} - \frac{2(x - x_0)(y - y_0)}{[(x - x_0)^2 + (y - y_0)^2]^2} \hat{\mathbf{j}} \right)$$

Velocities of Elementary Flows

(a) Uniform flow with angle α :

$$\mathbf{v}(r, \theta) = U \cos(\theta - \alpha) \hat{\mathbf{r}} - U \sin(\theta - \alpha) \hat{\theta}$$

(b) Line source with strength Λ

$$\mathbf{v}(r, \theta) = \frac{\Lambda}{2\pi r} \hat{\mathbf{r}}$$

(c) Line vortex with circulation Γ :

$$\mathbf{v}(r, \theta) = -\frac{\Gamma}{2\pi r} \hat{\theta}$$

(d) Doublet with strength κ :

$$\mathbf{v}(r, \theta) = -\frac{\kappa}{2\pi} \frac{\cos \theta}{r^2} \hat{\mathbf{r}} - \frac{\kappa}{2\pi} \frac{\sin \theta}{r^2} \hat{\theta}$$

Superposition of Elementary Flows

- If ψ_1 and ψ_2 are solution to $\nabla^2\psi = 0$, $a\psi_1 + b\psi_2$ is also a solution.
- "When a source of strength Λ at $(x_0 - \Delta x, y_0)$ is added to a sink of strength $-\Lambda$ at $(x_0 + \Delta x, y_0)$, a new flow field is obtained."
- "By letting Δx approach zero while keeping the product $2\Delta x\Lambda$ a constant κ , the stream function for a doublet at (x_0, y_0) is obtained."
- "The flow pattern of a double can also be produced by superimposing a vortex at $(x_0, y_0 - \Delta y)$ to a vortex of opposite circulation at $(x_0, y_0 + \Delta y)$, and then letting Δy approach zero."
- The velocities at the center of a line source, a line vortex, and a doublet are infinitely large. The center of these flows are called **singularities**.
- Singularities do not cause a problem when they are within the boundary of a rigid body.

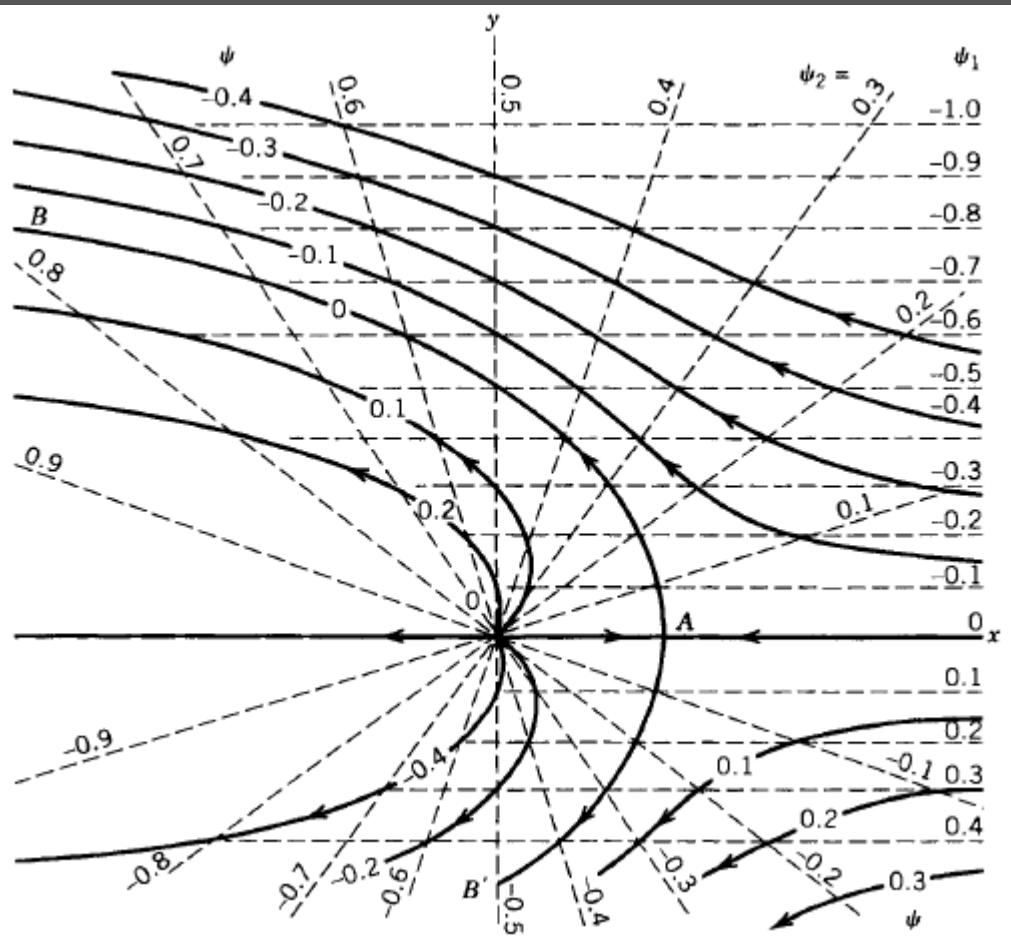
Source in a Uniform Flow

- The stream function for a uniform flow with $\alpha = 0$ is $\psi_1 = -Uy$.
- If a source of strength Λ with $(x_0 = 0, y_0 = 0)$ and stream function $\psi_2 = \frac{\Lambda\theta}{2\pi}$ is superimposed on the uniform flow, the resultant stream function is

$$\psi = Uy + \frac{\Lambda\theta}{2\pi} = U\left(\frac{h\theta}{2\pi} - y\right)$$

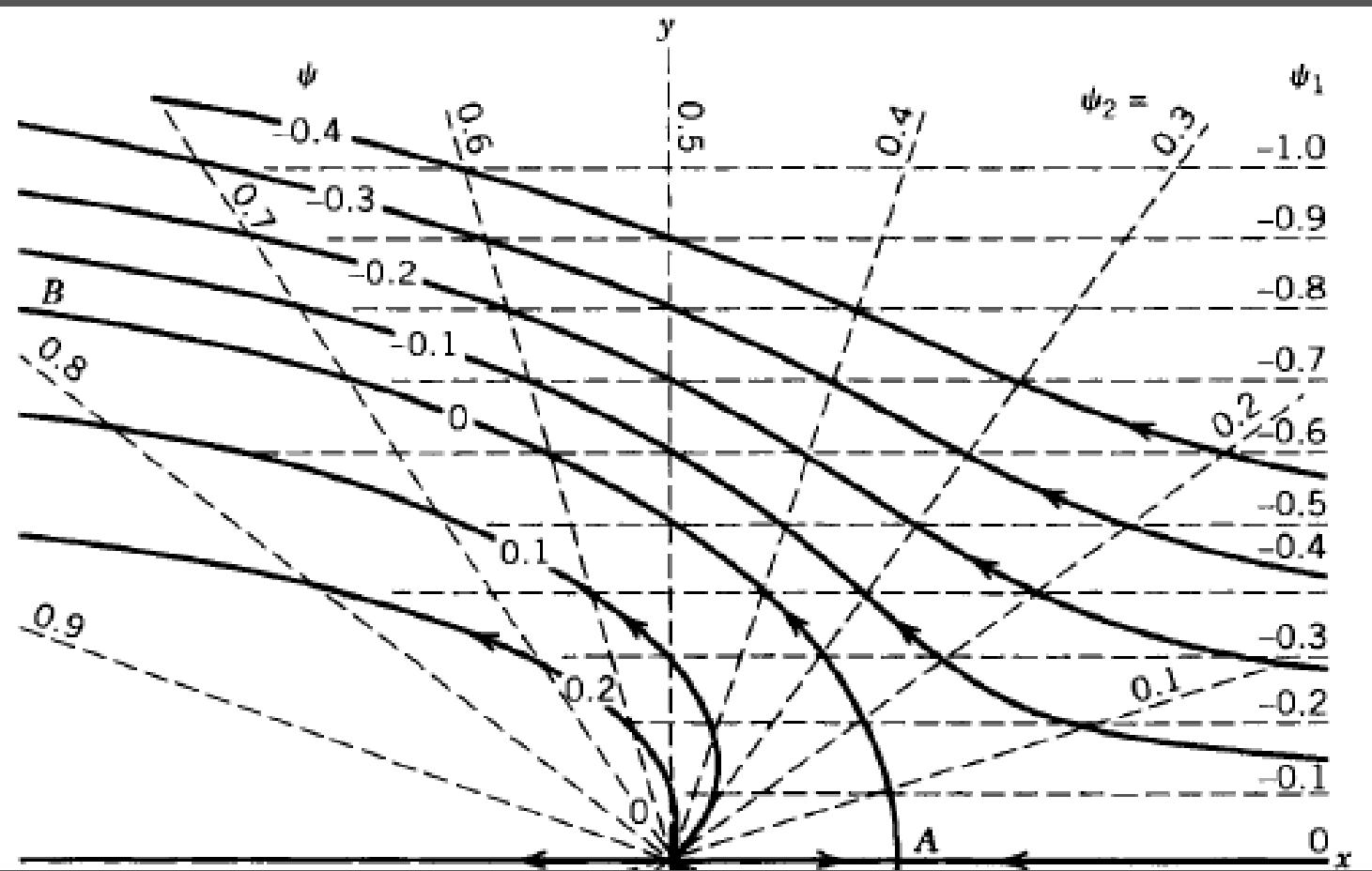
where $h = \Lambda/2U$ is a characteristic length.

- The streamline BAB' with $\psi = 0$ is considered a solid surface enclosing the source.
- "The flow exterior to the surface satisfies the continuity equation and is irrotational."



Source in a Uniform Flow

- "The flow field may be interpreted as that of a horizontal wind past a cliff, whose shape (y_0 , θ) is described by the equation $\psi = 0$, that is,



$$y_0 = r_0 \sin \theta = \frac{h\theta}{\pi}, \quad 0 \leq \theta \leq \pi$$

where r_0 is the radial distance of a point on the cliff at height y_0 above the x axis."

When $\theta \rightarrow -\infty$, $y_0 \rightarrow h = \Lambda/2U$.

Source in a Uniform Flow

- With $\theta = \tan^{-1}(y/x)$, the velocity components are

$$v_x = \frac{\partial \psi}{\partial y} = -U + \frac{Uh}{\pi} \frac{x}{x^2 + y^2}$$

$$v_y = -\frac{\partial \psi}{\partial x} = \frac{Uh}{\pi} \frac{y}{x^2 + y^2}$$

- According to these equations, the velocity vanishes ($v_x = v_y = 0$) at point $(h/\pi, 0) = (\Lambda/2\pi U, 0)$.
- "In other words, the velocity vanishes at point A on the x axis where the velocity from the source, $\Lambda/2\pi x$, cancels the velocity U from the uniform flow."

Flow Pattern of a Source-Sink Pair

- Consider a source of strength Λ at $(-x_0, 0)$ and a sink of strength $-\Lambda$ at $(x_0, 0)$
- The stream function of the combine flow at (x, y) is

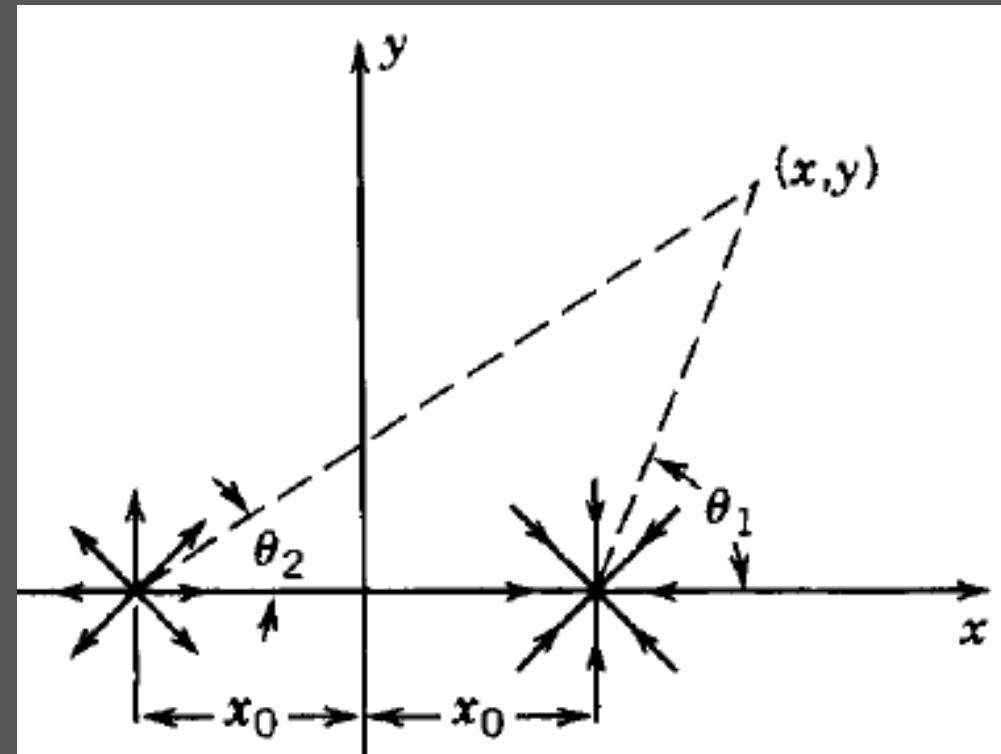
$$\psi = \frac{\Lambda \theta_2}{2\pi} - \frac{\Lambda \theta_1}{2\pi} = \frac{\Lambda \theta_2}{2\pi} \left(\tan^{-1} \frac{y}{x+x_0} - \tan^{-1} \frac{y}{x-x_0} \right)$$

- Using the trigonometric relation

$$\tan^{-1} A - \tan^{-1} B = \tan^{-1} \frac{A - B}{1 + AB}$$

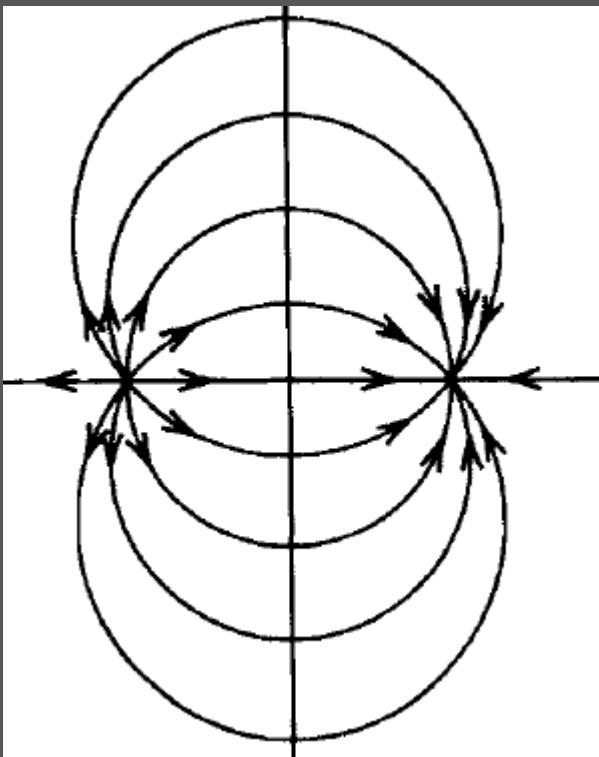
we then obtain

$$\psi = -\frac{\Lambda}{2\pi} \tan^{-1} \frac{2x_0 y}{x^2 + y^2 - x_0^2}$$



Flow Pattern of a Source-Sink Pair

- Rearranging the equation, we obtain



$$-\tan \frac{2\pi\psi}{\Lambda} = \frac{2x_0y}{x^2 + y^2 - x_0^2}$$

$$x^2 + y^2 + 2x_0y \cot \frac{2\pi\psi}{\Lambda} = x_0^2$$

$$x^2 + \left(y + x_0 \cot \frac{2\pi\psi}{\Lambda} \right)^2 = x_0^2 + x_0^2 \cot^2 \left(\frac{2\pi\psi}{\Lambda} \right)$$

$$= x_0^2 \csc^2 \left(\frac{2\pi\psi}{\Lambda} \right)$$

- "This equation represents a family of circles with centers on the y axis."
- "When $y = 0$, $x = \pm x_0$ for all values of ψ ."
- The flow pattern is shown in the figure.

Flow Pattern of a Source-Sink Pair

- The flow pattern of a doublet can be obtained "when the distance between the source and sink approaches zero while their strengths approach infinity in such a way that their product remains a constant value of $\kappa = 2x_0\Lambda$."
- As x_0 approaches zero, we have

$$\begin{aligned}\psi &= \lim_{x_0 \rightarrow 0} \left[-\frac{\Lambda}{2\pi} \tan^{-1} \frac{2x_0 y}{x^2 + y^2 - x_0^2} \right] = \lim_{x_0 \rightarrow 0} \left[-\frac{1}{2\pi} \frac{(2x_0\Lambda)y}{x^2 + y^2 - x_0^2} \right] \\ &= -\frac{\kappa}{2\pi} \frac{y}{x^2 + y^2} = -\frac{\kappa}{2\pi} \frac{\sin \theta}{r}\end{aligned}$$

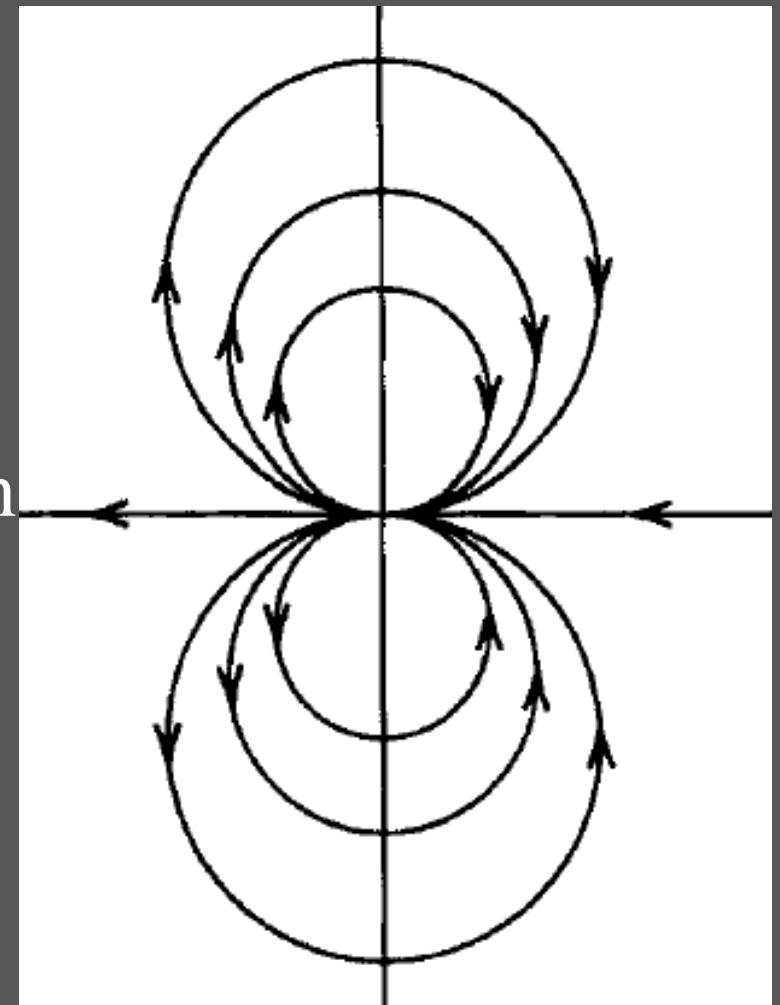
- In the limit, the velocity potential is $\phi = \frac{\kappa}{2\pi} \frac{\cos \theta}{r}$

Flow Pattern of a Source-Sink Pair

- The streamlines of a doublet flow of strength κ (lines of constant ψ) are circles as can be seen by rearranging the equation in the form

$$x^2 + \left(y + \frac{\kappa}{4\pi\psi} \right)^2 = \left(\frac{\kappa}{4\pi\psi} \right)^2$$

- Each circle has a center at $(0, -\kappa/4\pi\psi)$ and a radius of $\kappa/4\pi\psi$.
- All streamlines (circles) pass through the origin.

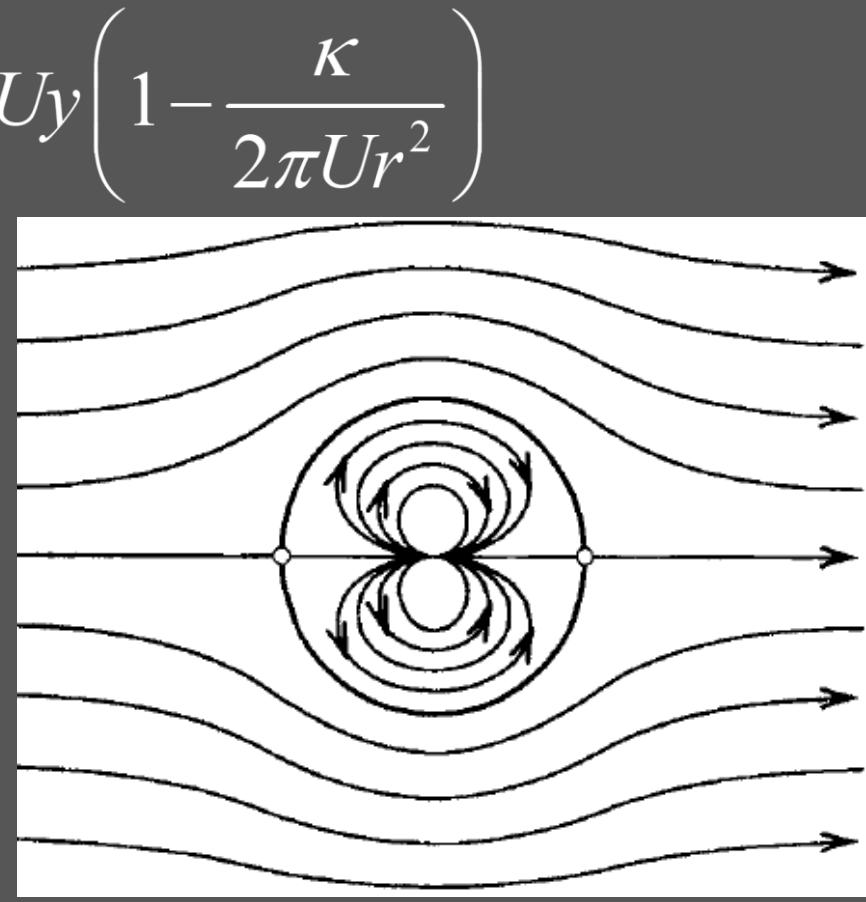


Flow Past a Circular Cylinder

- "The stream function for a uniform flow with velocity U in the direction of the positive x axis is $\psi = Uy$."
- "If the uniform flow is added to a doublet, the flow about a circular cylinder in a uniform stream is obtained."
- The resulting stream function is $\psi = Uy - \frac{\kappa y}{2\pi r^2} = Uy \left(1 - \frac{\kappa}{2\pi Ur^2} \right)$
- Let $\kappa/2\pi U = a^2$. Then, we have

$$\psi = Uy \left(1 - \frac{a^2}{r^2} \right)$$

- "The zero streamline consists of the x axis and a circle of radius $r = a$."



Flow Past a Circular Cylinder

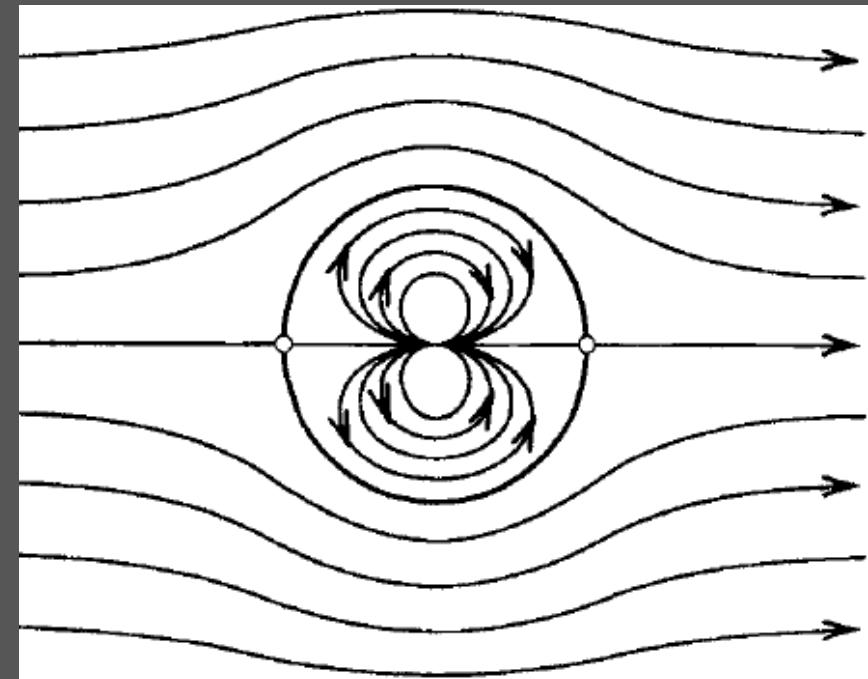
- With $y = r \sin \theta$, the velocity components are

$$v_r = U \left(1 - \frac{a^2}{r^2} \right) \cos \theta, \quad v_\theta = -U \left(1 + \frac{a^2}{r^2} \right) \sin \theta$$

- On the cylinder surface ($r = a$), $u_r = 0$ and $u_\theta = -2U \sin \theta$.
- The pressure distribution on the surface is given by the Bernoulli equation

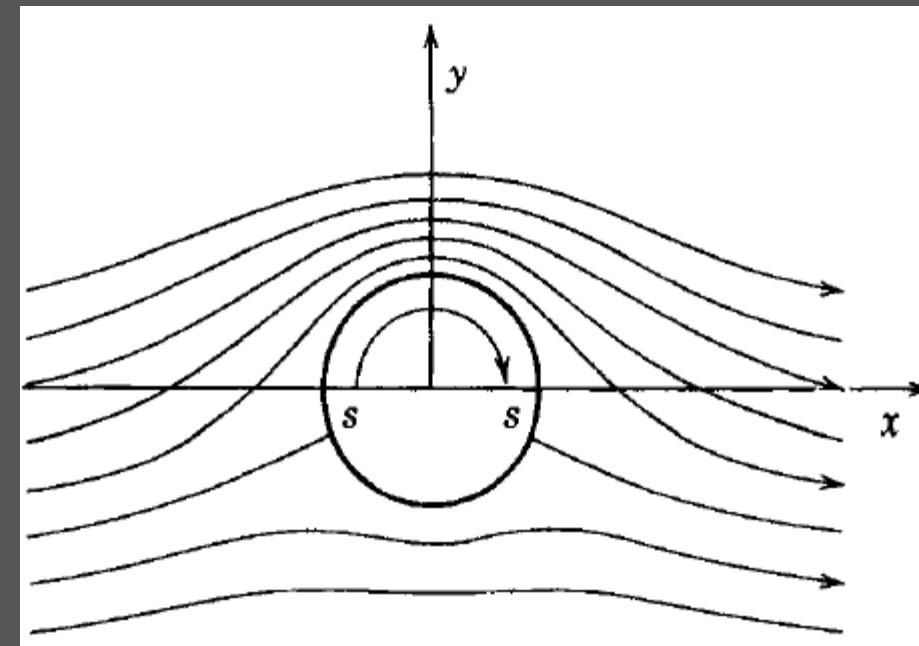
$$C_p = \frac{p - p_\infty}{q_\infty} = 1 - \left(\frac{u_\theta}{U} \right)^2 = 1 - 4 \sin^2 \theta \quad \text{for } r = a$$

where C_p is called the pressure coefficient.



Circulatory Flow about a Cylinder

- "If a stream function for a vortex at origin is added to $\psi = Uy(1 - a^2/r^2)$, the resulting stream function will satisfy the continuity, irrotationality, the boundary conditions for the circulatory flow about a circular cylinder in a uniform stream:"
$$\psi = Uy\left(1 - \frac{a^2}{r^2}\right) + \frac{\Gamma}{2\pi} \ln\left(\frac{r}{a}\right)$$
- "The uniform stream is in the direction of the positive x axis, and the circulatory flow is clockwise."
- The zero streamline corresponds to a cylinder of radius $r = a$.



Circulatory Flow about a Cylinder

- The velocity components for this flow is as follows.

$$u_r = -\frac{1}{r} \frac{\partial \psi}{\partial \theta} = U \left(1 - \frac{a^2}{r^2} \right) \cos \theta, \quad u_\theta = -\frac{\partial \psi}{\partial r} = -U \left(1 + \frac{a^2}{r^2} \right) \sin \theta - \frac{\Gamma}{2\pi r}$$

- On the cylinder surface ($r = a$), we have

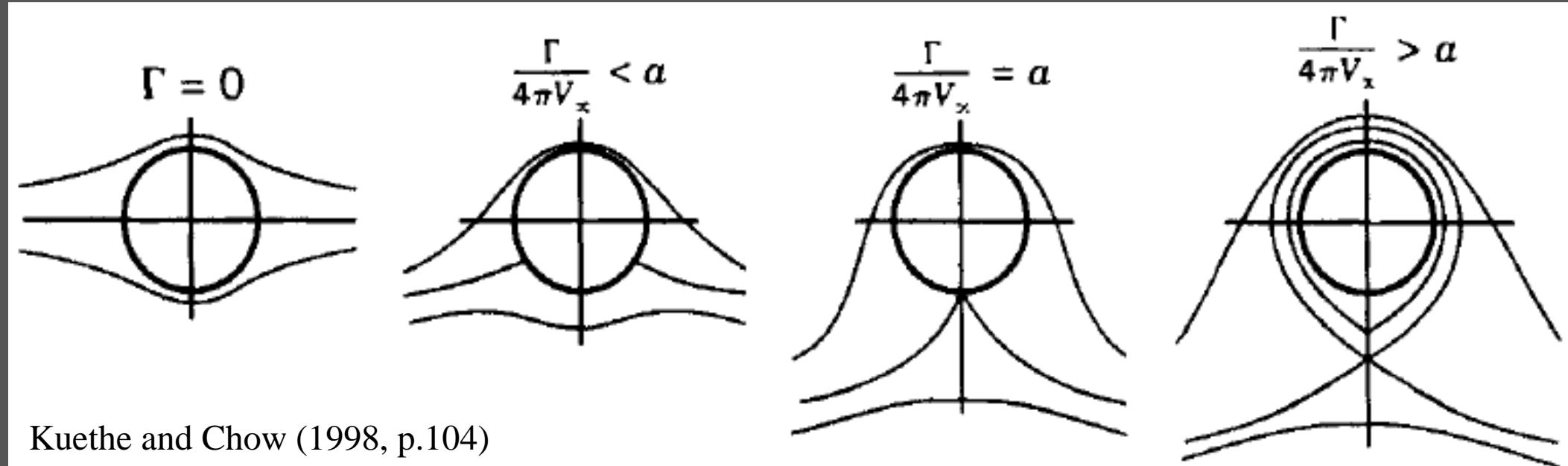
$$u_r = 0, \quad u_\theta = -2U \sin \theta - \frac{\Gamma}{2\pi a}$$

- u_θ vanishes when $\theta = \theta_s$: $\sin \theta_s = -\frac{\Gamma}{4\pi a U}$
- Since $\sin \theta = y/r$, the stagnation points are

$$x_s = \pm \sqrt{a^2 - y_s^2}, \quad y_s = -\frac{\Gamma}{4\pi U}$$

Circulatory Flow about a Cylinder

- "As Γ becomes large, the stagnation points move downward until $(\Gamma/4\pi U)^2$ equals a^2 ; for this condition, the stagnation points coincide on the y axis at $(0, -a)$."
- When $(\Gamma/4\pi U)^2 > a^2$, the stagnation points leave the body and the equations $x_s = \pm\sqrt{a^2 - y_s^2}$, $y_s = -\frac{\Gamma}{4\pi U}$ no longer hold.



Exercise

- Use a linear combination of the 4 elementary flows of your choice to generate a flow pattern.
- Plot streamlines and equipotential lines for the flow.
- Also plot the velocity fields.

von Karman Method

- The drawback of the method of superposition of elementary flows is that the positions and strengths of elementary flows must manually adjusted so that the shape of the resulting body is close to what is desired.
- In 1927, von Karman developed a systematic means of calculating the strengths of a group of sources placed at fixed locations for axisymmetric flows. The method can also be applied to 2D planar flows.
- For 3D axisymmetric flows, the cylindrical coordinate system is used and the velocity components can be written as

$$u_r = -\frac{1}{r} \frac{\partial \psi}{\partial z}, \quad u_z = \frac{1}{r} \frac{\partial \psi}{\partial r}$$

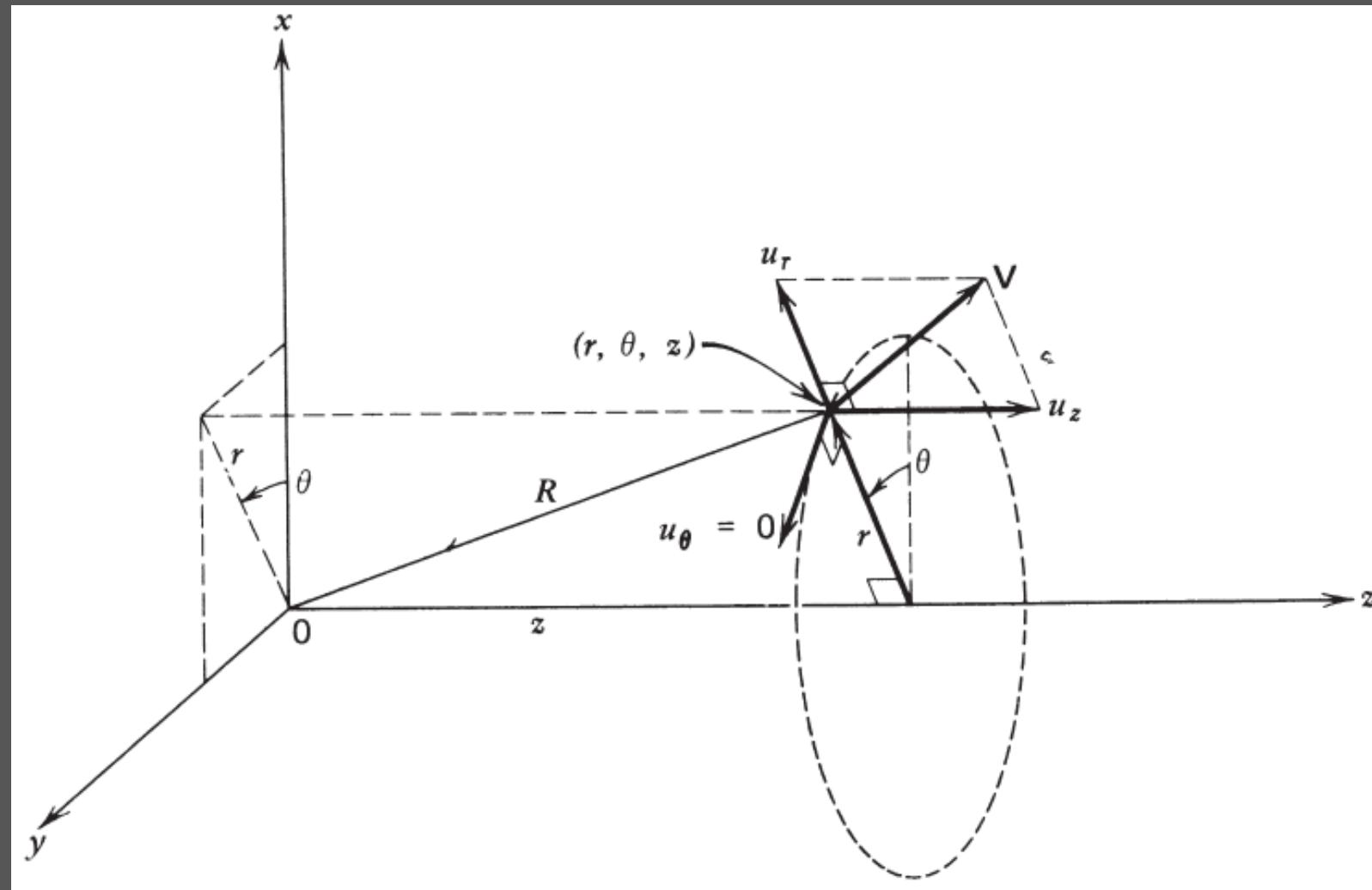
- Here, we will use the von Karman method to approximate flow past a **body of revolution**.

von Karman Method

- The irrotational condition $\nabla \times \mathbf{v} = 0$ can be represented in term of the stream function as

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = 0$$

- Although this is not the Laplace equation, it is still a linear equation.
- So, the superposition of elementary flows is still valid in this case.



von Karman Method

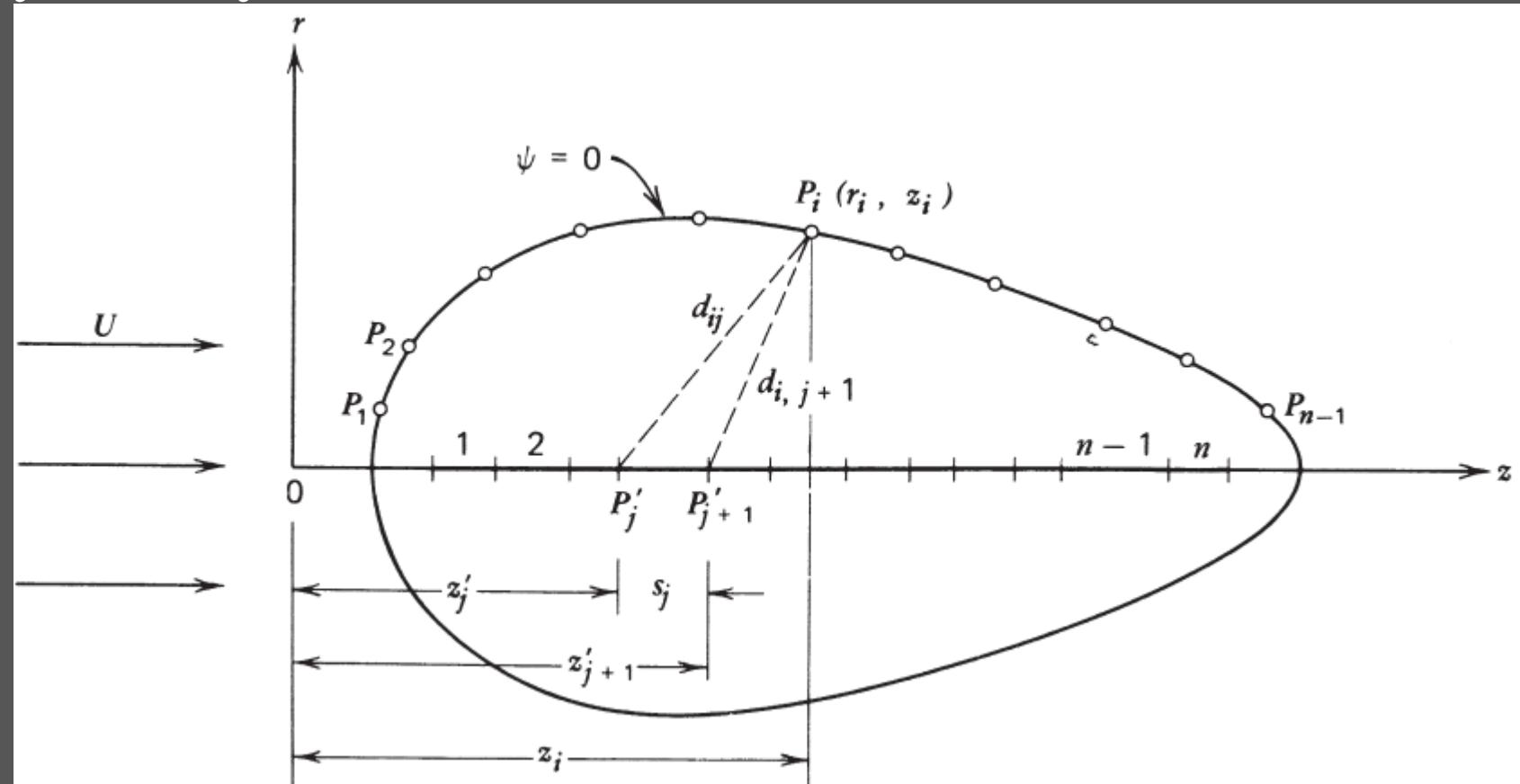
- For a stream of speed U flowing along the z axis, $\psi = \frac{1}{2}Ur^2$
- For a point source at the origin, $\psi = -mz/R$, $R = \sqrt{r^2 + z^2}$ and the corresponding velocity \mathbf{v} is pointed radially outward from the origin and its magnitude is equal to m/R^2 .
- The constant $m = vR^2$, $v = |\mathbf{v}|$, is called the strength of the point source.
- "A doublet can be constructed for the axisymmetric flow by letting a source-sink pair approach each other along the z axis while keeping the product of the strength and the distance in between a constant μ ."
- "The doublet formed at the origin has the stream function"

$$\psi = -\frac{\mu r^2}{(r^2 + z^2)^{3/2}}$$

von Karman Method

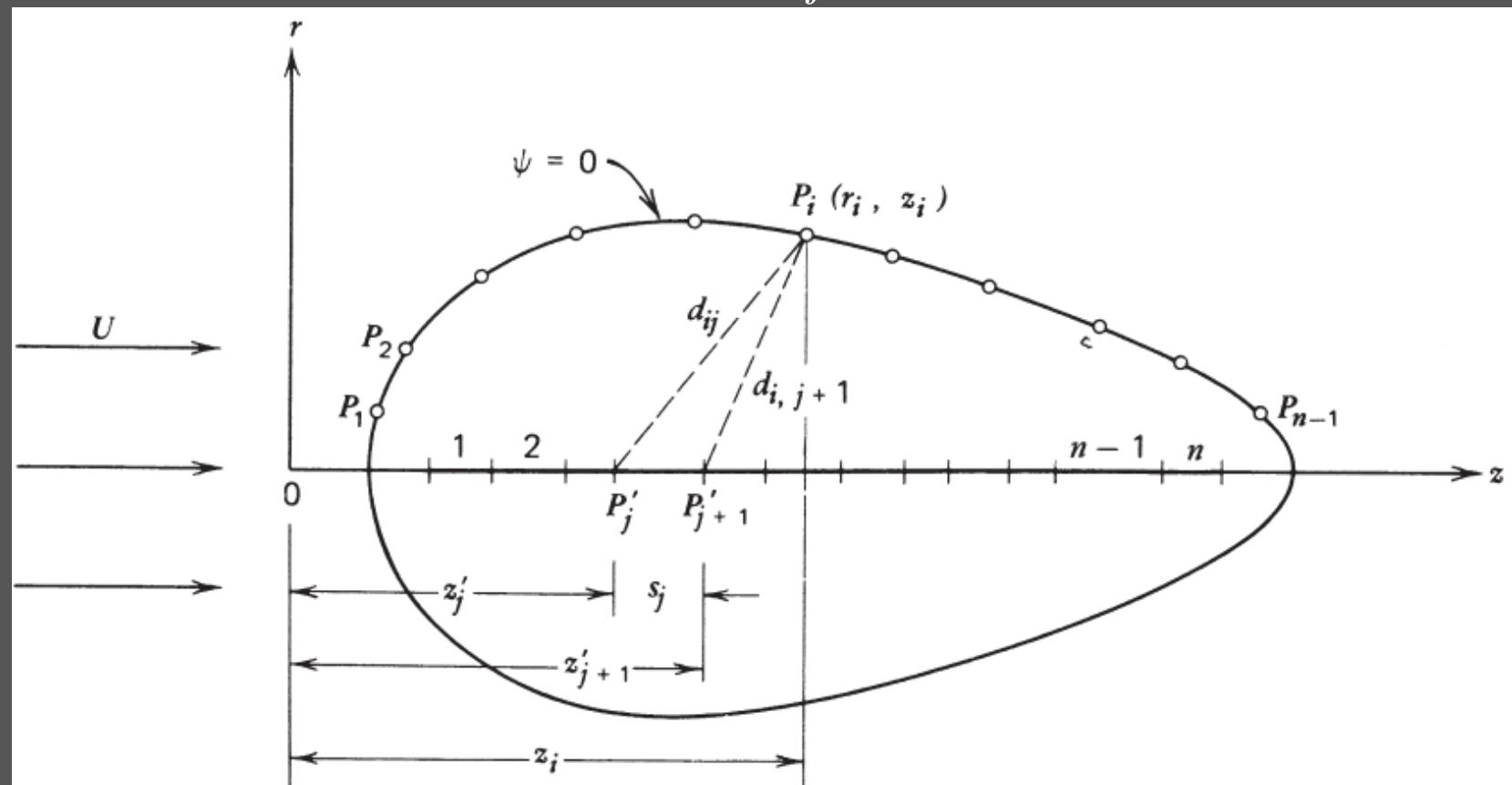
- "To approximate the flow resulting from a uniform axial stream of speed U past a given body of revolution, point sources or sinks are distributed uniformly within each of the n properly chosen segments inside the body along the axis of symmetry."

- Segments, labeled $1, \dots, n$, have different lengths and contain sources or sinks of different densities.



von Karman Method

- "The number of segments depends on the desired degree of accuracy."
- "The end points of segment j are denoted as $P'_j(0, z'_j)$ and $P'_{j+1}(0, z'_{j+1})$, and the segment length is $s_j = z'_{j+1} - z'_j$."
- Along segment j , source of constant strength q_j per unit length are distributed."



von Karman Method

- "The stream function of the flow induced at any point (r, z) by the sources within a small interval $d\zeta$ located at $(0, \zeta)$ on this segment is"

$$d\psi = -\frac{q_j(z - \zeta)d\zeta}{\sqrt{r^2 + (z - \zeta)^2}}$$

- "Integration from z'_j to z'_{j+1} gives the induced stream function at (r, z) caused by the whole source distribution on segment j "

$$q_j \left[\sqrt{r^2 + (z - z'_{j+1})^2} - \sqrt{r^2 + (z - z'_j)^2} \right]$$

- So, the stream function due to the uniform stream and source distributions is

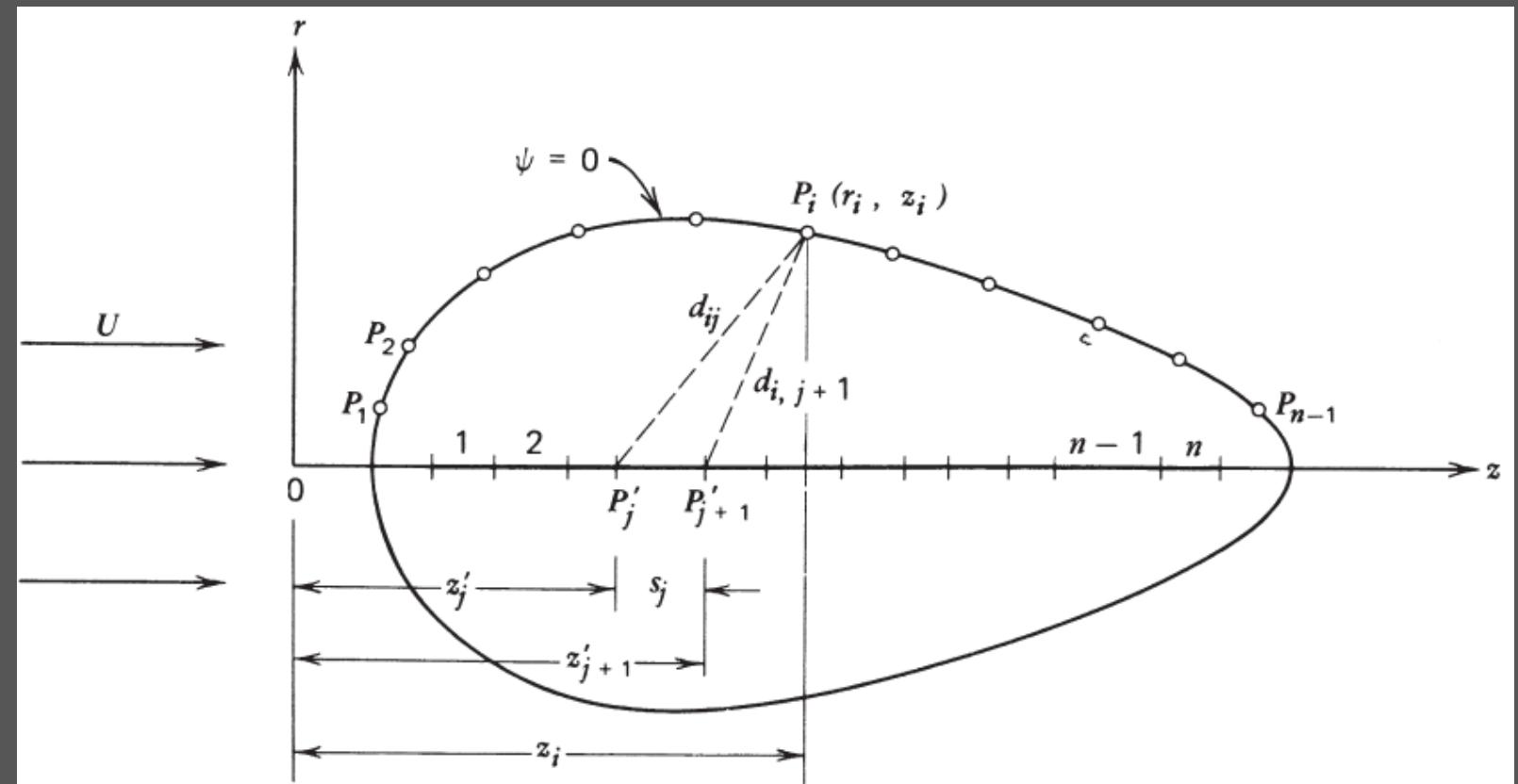
$$\psi(r, z) = \frac{1}{2} Ur^2 + \sum_{j=1}^n q_j \left[\sqrt{r^2 + (z - z'_{j+1})^2} - \sqrt{r^2 + (z - z'_j)^2} \right]$$

von Karman Method

- On the surface of a prescribed body, $n-1$ points P_1, P_2, \dots, P_{n-1} are selected.
- At a surface point $P_i(r_i, z_i)$ the stream function is

$$\psi(r_i, z_i) = \frac{1}{2} U r_i^2 + \sum_{j=1}^n q_j \left[\sqrt{r_i^2 + (z - z'_{j+1})^2} - \sqrt{r_i^2 + (z - z'_j)^2} \right] = 0$$

and its value is zero since P_i is on the body surface. The two terms in the bracket are the distances from P_i to the end points of segment j (P'_j, P'_{j+1}).



von Karman Method

- That is

$$d_{i,j+1} = \sqrt{r_i^2 + (z_i - z'_{j+1})^2}, \quad d_{i,j} = \sqrt{r_i^2 + (z_i - z'_j)^2}$$

- Let $\Delta d_{ij} = d_{i,j} - d_{i,j+1}$, and $Q_j = q_j/U$

- We then obtain

$$\sum_{j=1}^n \Delta d_{ij} Q_j = \frac{1}{2} r_i^2, \quad i = 1, 2, \dots, n-1$$

- There are n unknowns ($Q_j, j = 1, 2, \dots, n$) but $n-1$ equations.
- The last equation can be obtained using the fact that, to form a closed body, the total strength of the sources/sinks must vanish, that is,

$$\sum_{j=1}^n s_j Q_j = 0$$

von Karman Method

- We then obtain the exactly determined linear system

$$\begin{bmatrix} \Delta d_{11} & \Delta d_{12} & \cdots & \Delta d_{1n} \\ \Delta d_{21} & \Delta d_{22} & \cdots & \Delta d_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \Delta d_{n-1,1} & \Delta d_{n-1,2} & \cdots & \Delta d_{n-1,n} \\ s_1 & s_2 & \cdots & s_n \end{bmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{n-1} \\ Q_n \end{pmatrix} = \begin{pmatrix} r_1^2/2 \\ r_2^2/2 \\ \vdots \\ r_{n-1}^2/2 \\ 0 \end{pmatrix}$$

- After $Q_j = q_j/U$ are computed, the stream function can be computed by

$$\psi(r, z) = \frac{1}{2} Ur^2 + \sum_{j=1}^n q_j \left[\sqrt{r^2 + (z - z'_{j+1})^2} - \sqrt{r^2 + (z - z'_j)^2} \right]$$

von Karman Method

- The velocity components can be computed directly using Q_j as follows.

$$u_r(r, z) = -\sum_{j=1}^n \frac{U Q_j}{r} \left[\frac{z - z'_{j+1}}{\sqrt{r^2 + (z - z'_{j+1})^2}} - \frac{z - z'_j}{\sqrt{r^2 + (z - z'_j)^2}} \right]$$

$$u_z(r, z) = U \left\{ 1 + \sum_{j=1}^n Q_j \left[\frac{1}{\sqrt{r^2 + (z - z'_{j+1})^2}} - \frac{1}{\sqrt{r^2 + (z - z'_j)^2}} \right] \right\}$$

- Note that the flow speed induced by the source distributions decreases with the distance away from the body.

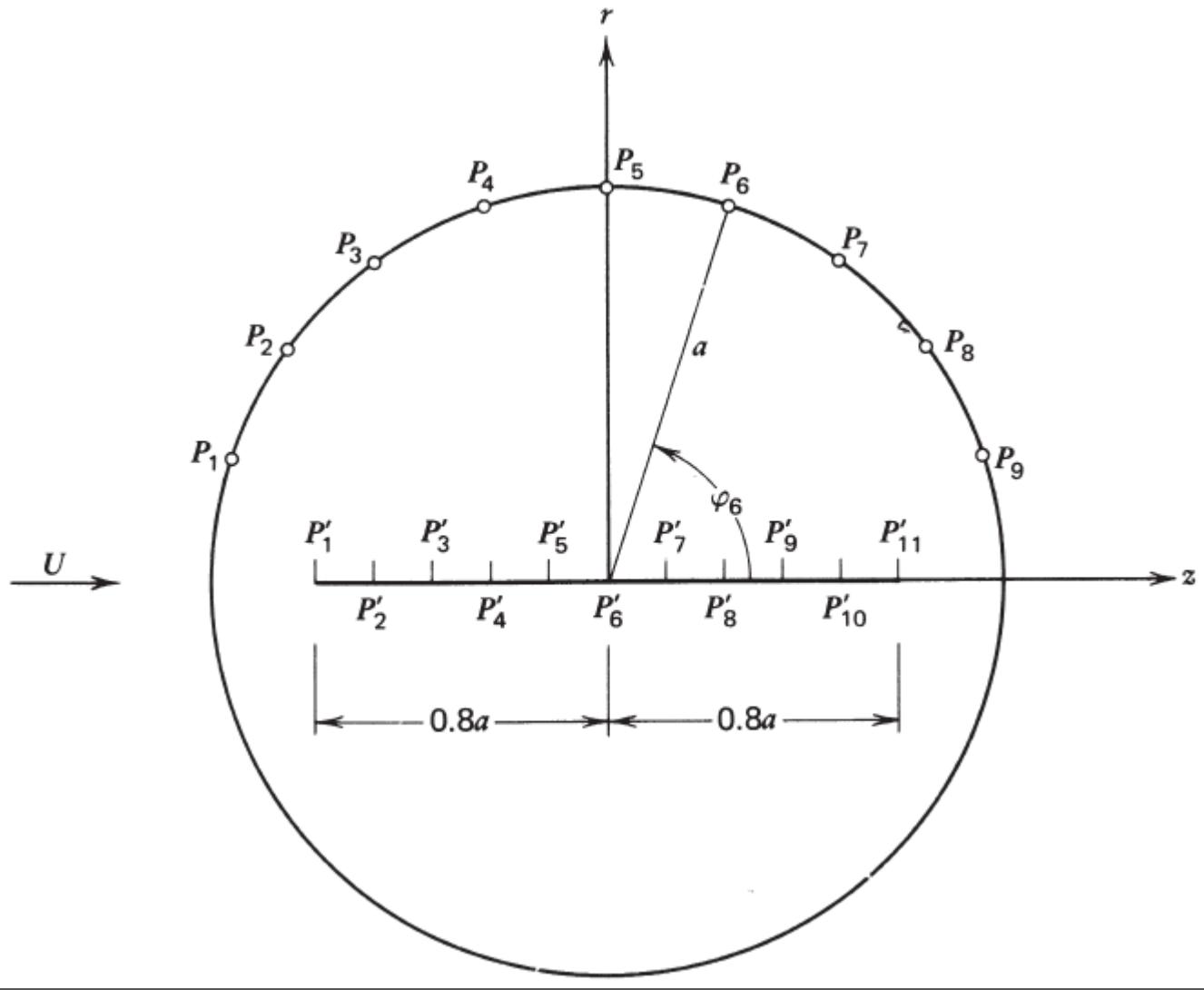
von Karman Method

- The pressure field can be computed from the steady-state Bernoulli equation $p + \frac{1}{2} \rho v^2 = \text{constant}$
- If P is the pressure far away from the body, we can then calculate the pressure p from
$$p + \frac{1}{2} \rho (u_r^2 + u_z^2) = P + \frac{1}{2} \rho U^2$$
- This equation can be rewritten in term of the pressure coefficient c_p as
$$c_p = 1 - \left(\frac{u_r}{U} \right)^2 - \left(\frac{u_z}{U} \right)^2, \quad c_p \equiv \frac{p - P}{\frac{1}{2} \rho U^2}$$

Exercise

- Adding a uniform flow to a point doublet yields a uniform flow past a circular cylinder whose stream function is $\psi = \frac{1}{2}Ur^2 - \mu r^2 / (r^2 + z^2)^{3/2}$
- When the doublet strength $\mu = Ua^3/2$, the stream function can be written as
$$\psi = \frac{1}{2}Ur^2 \left(1 - \frac{a^3}{R^3} \right), \quad R = \sqrt{r^2 + z^2}$$
- The stream surface $\psi = 0$ consists of the z axis (along which $r = 0$) and the surface of a sphere of radius a .
- The flow speed and the pressure coefficient at point (r, z) on the sphere are
$$v = \frac{3}{2}U \frac{r}{a}, \quad c_p = 1 - \frac{9}{4} \left(\frac{r}{a} \right)^2$$
- Apply the von Karman method to this case using 10 source segments.

Exercise



- Sources are distributed within 10 segments of equal lengths on the z -axis between $z \pm 0.8a$.
 - 9 equally spaced points are chosen on the upper surface of the sphere.
 - Let θ_i be the angle between the z -axis and the radial position of the surface point $P_i(r_i, z_i)$.
- $$\theta_i = (10 - i)\pi/10, \quad i = 1, 2, \dots, 9$$
- $$r_i = a \sin \theta_i, \quad z_i = a \cos \theta_i$$
- Compute the velocity components and compare with exact solution.

Steady Potential Flows in 2D

- The velocity components can be computed from the velocity potential ϕ and stream function ψ as follows.

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

- Both ϕ and ψ satisfy the Laplace equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

- Curves $\phi = \text{constant}$ are called equipotential lines.
- Curves $\psi = \text{constant}$ are called streamlines.
- In 2D flows, the equipotential lines and streamlines form two families of mutually orthogonal curves since $\nabla \phi \cdot \nabla \psi = \phi_x \psi_x + \phi_y \psi_y$
 $= (u)(-v) + (u)(v) = 0$

Analytic Function

- Let's construct a complex function from the velocity potential and the stream function as

$$w(z) = \phi(x, y) + i\psi(x, y), \quad z = x + iy, \quad i = \sqrt{-1}$$

- Since the real and imaginary parts of w satisfy the **Cauchy-Riemann conditions**, i.e.,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x},$$

$w(z)$ is complex-differentiable through out the region occupied by the fluid.

- In other words, $w(z)$ is an analytic function of z .
- "If we choose an arbitrary analytic function $w(z)$, the real and imaginary parts of the function are then qualify as the velocity potential and stream function of a potential flow in the x - y plane."

Complex Potential

- The complex function $w(z)$ is called the **complex potential**, whose derivative is related to the velocity components as $dw/dz = u - iv$

- So, the magnitude of the velocity vector is $|v| = |dw/dz| = \sqrt{u^2 + v^2}$

- The complex potential corresponding to the uniform flow is

$$w(z) = Uze^{-i\alpha} = U(x \cos \alpha + y \sin \alpha) + iU(y \cos \alpha - x \sin \alpha)$$

- The complex potential of the source is $w(z) = (\Lambda/2\pi) \log(z - z_0)$

- The complex potential of the vortex is $w(z) = (i\Gamma/2\pi) \log(z - z_0)$

- The complex potential of the doublet is $w(z) = \kappa/2\pi(z - z_0)$

- Here, $z_0 = x_0 + iy_0$.

- The complex potential of the source, vortex, and doublet has a singular point at z_0 , where the first derivative of the function is unbounded.

Exercise

- Using the complex potentials given in the previous slide, plot the equipotential lines and streamlines of the 4 elementary flows and a linear combination of elementary flows.

Conformal Mapping

- The principle of superposition of elementary flows can be applied to the complex potential to generate new flows.
- For example, the sum of Uz (uniform horizontal flow) and $\kappa/2\pi z$ (doublet at origin) represents the complex potential of a uniform flow past a circular cylinder of radius $\sqrt{\kappa/2\pi U}$
- The method of conformal mapping can generate new flow patterns using coordinate transformations.
- Let $z = f(z')$ where f is an analytic function of z' .
- We then have

$$\frac{dw}{dz'} = \frac{dw}{dz} \frac{dz}{dz'} = \frac{dw}{dz} \frac{df}{dz'}$$

Conformal Mapping

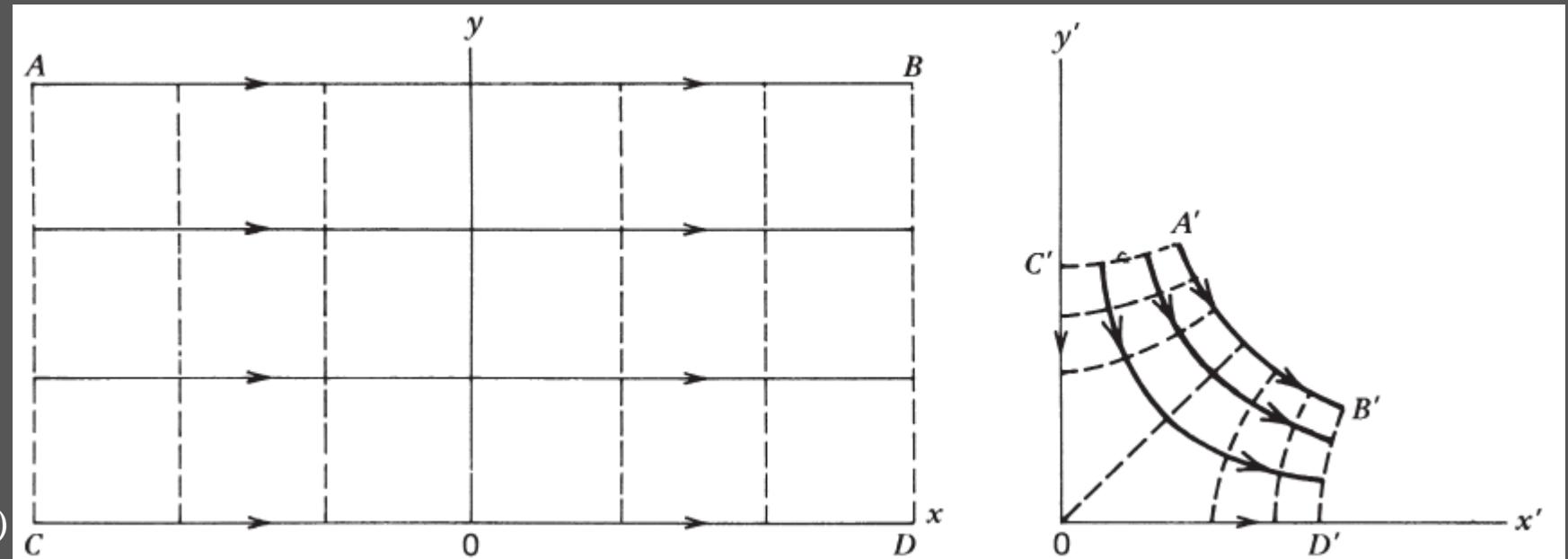
- "After being transformed into the z' plane, the complex potential can be written in terms of the new coordinates as"

$$w[f(z')] = \phi'(x', y') + i\psi'(x', y')$$

- The curves $\phi' = \text{constant}$ and $\psi' = \text{constant}$ remain mutually orthogonal in the x' - y' plane after the transformation or mapping.
- Thus, this is called conformal mapping.

Conformal Mapping: Example

- Consider the complex potential $w(z) = Uz = Ux + iUy$ of a uniform flow with speed U in the positive x direction. Let the mapping be $z = z'^2$.
- The complex potential becomes $w(z') = Uz'^2 = U(x'^2 - y'^2) + i2Ux'y'$
- "The equipotential lines $x = c$ (dashed lines) and streamlines $y = k$ (solid lines) in the x - y plane are mapped into equipotential lines $x'^2 - y'^2 = c$ and streamlines $2x'y' = k$ in the x' - y' plane, respectively."



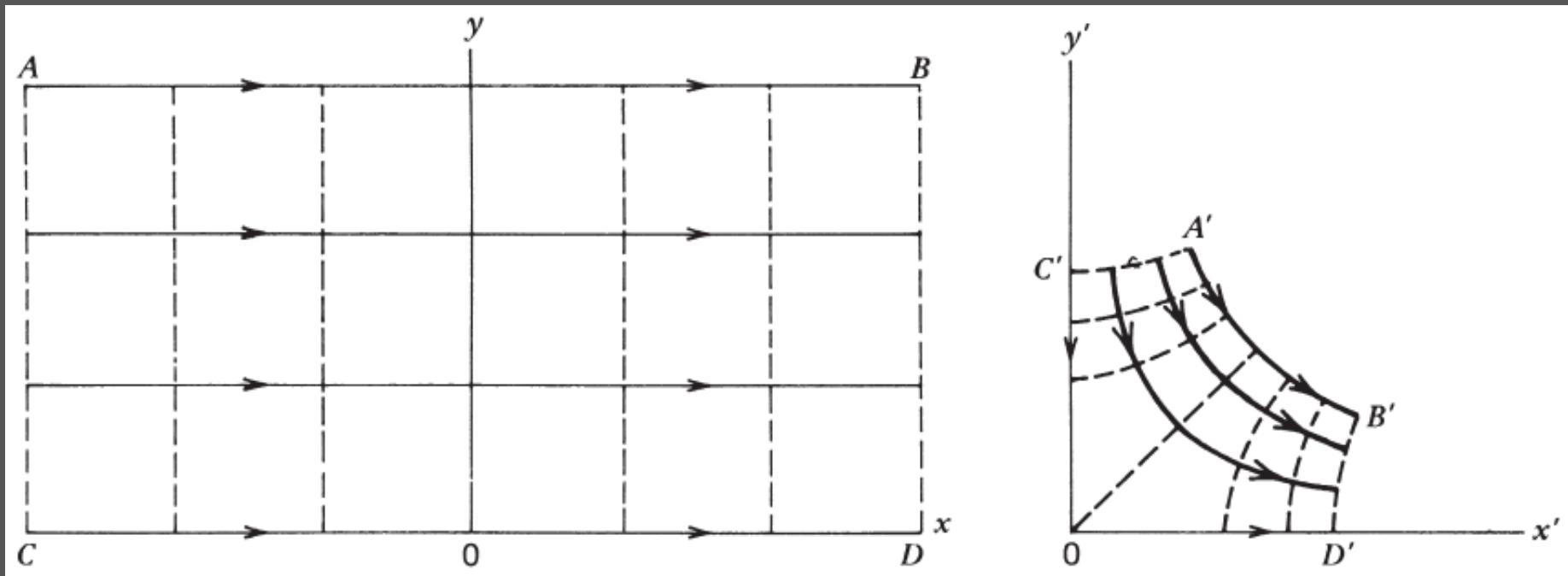
Exercise

Plot the equipotential lines and streamlines of the uniform flow

$$w(z) = Uz = Ux + iUy$$

in the original domain, and those of the corresponding flow in the transformed domain

$$w(z') = Uz'^2 = U(x'^2 - y'^2) + i2Ux'y'$$

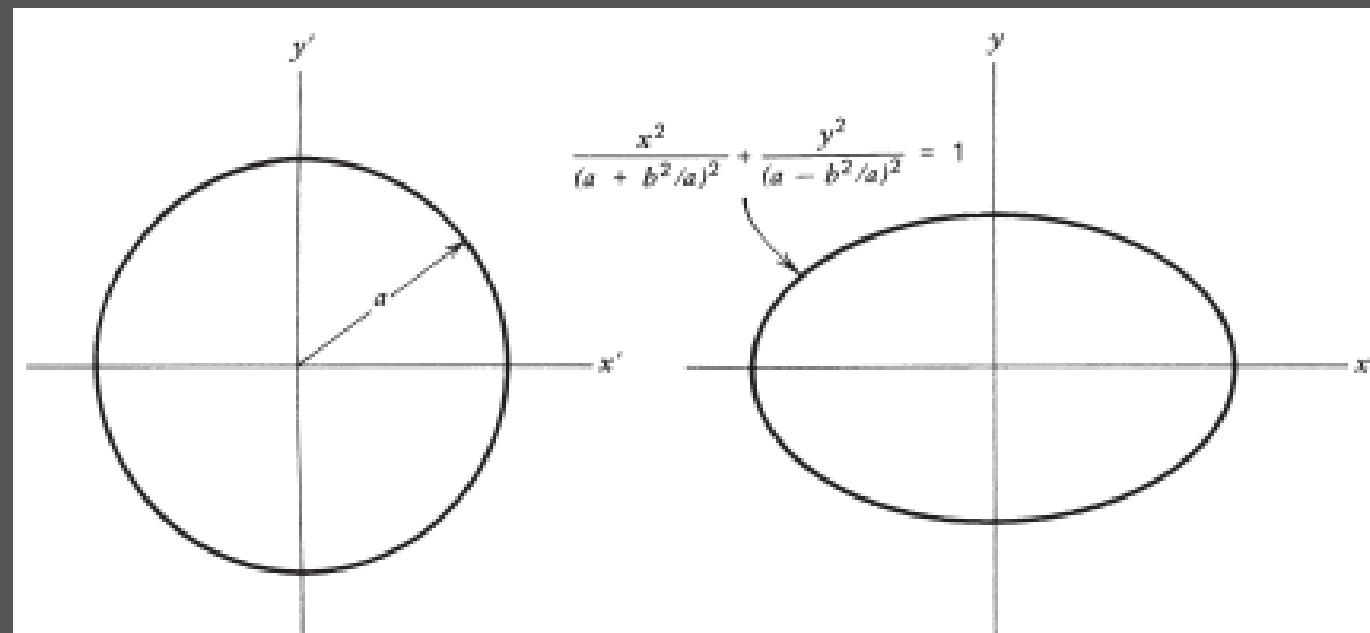


Exercise

Plot the equipotential lines and streamlines of the source flow, vortex flow, and doublet flow after using the conformal mapping $z = z'^2$.

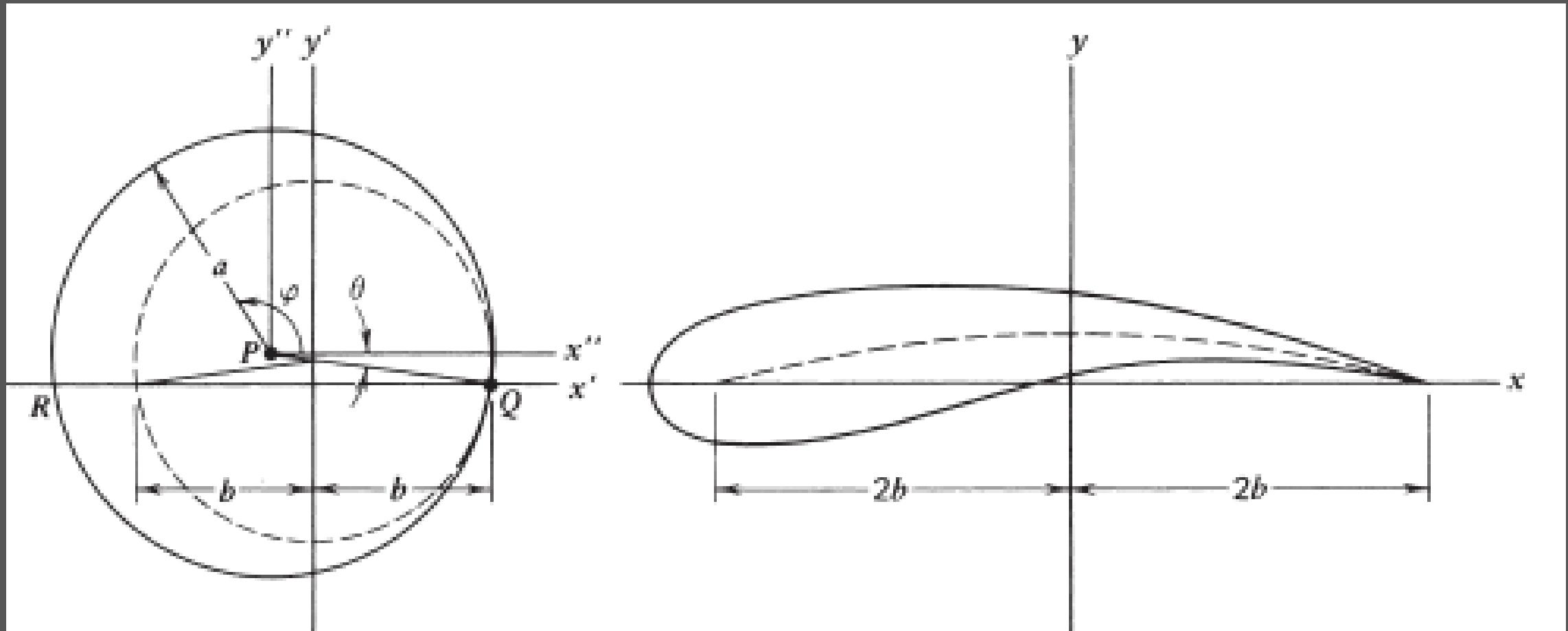
Joukowski Transformation

- The mapping $z = z' + a^2/z'$ is called the Joukowski transformation.
- After the transformation, the complex exponential of a uniform flow, $w(z) = Uz$, becomes $w(z') = U \left(z' + a^2/z' \right)$ which is the complex potential for a uniform flow past a circular cylinder of radius a .
- The Joukowski transformation $z = z' + b^2/z'$, $b^2 < a^2$, maps a circle of radius a with center at the origin of the x' - y' plane into an ellipse in the x - y plane.



Joukowsky Transformation

- The Joukowsky transformation $z = z' + b^2/z'$, $b^2 < a^2$, also maps 2 circles into a circular arc and a so-called Joukowsky airfoil in the x - y plane.



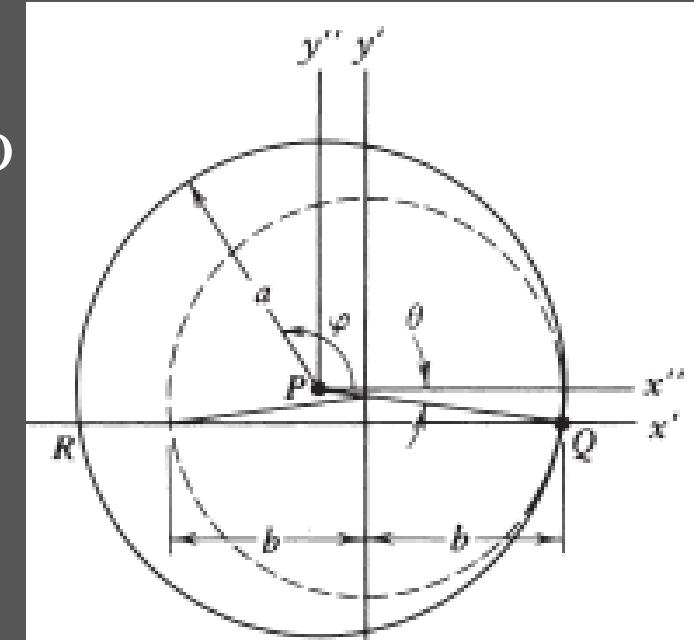
Flow past a Joukowski Airfoil

- Applying the Joukowski transformation to the flow of uniform speed U past a circular cylinder results in the uniform flow past a Joukowski airfoil.
- The flow speed $|\mathbf{v}|$ in the x - y plane is related to the flow speed $|\mathbf{v}'|$ in the x' - y' plane as
$$|\mathbf{v}| = \left| \frac{dw}{dz} \right|, |\mathbf{v}'| = \left| \frac{dw}{dz'} \right| \rightarrow \left| \frac{dw}{dz} \right| = \left| \frac{dw}{dz'} \right| \left| \frac{dz'}{dz} \right|$$

- Since we have the transform $z = z' + b^2/z'$, it is easier to compute dz/dz' .
- So,
$$|\mathbf{v}| = |\mathbf{v}'| \sqrt{\left| \frac{dz}{dz'} \right|} = |\mathbf{v}'| \sqrt{\left| 1 - \left(\frac{b}{z'} \right)^2 \right|}$$
- Point Q ($x' = b$) in the x' - y' plane corresponds to the trailing edge ($x = 2b$) in the x - y plane. When the flow speed at Q is finite, the flow speed at the trailing edge becomes infinite.

Flow past a Joukowsky Airfoil

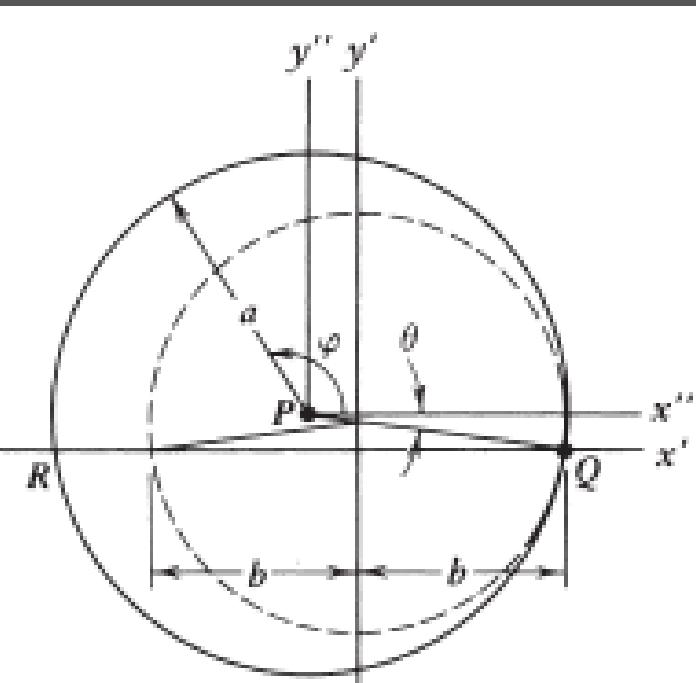
- The infinite flow speed is not allowed and does not occur physically.
- This problem can be avoided by requiring that Q is a stagnation point.
- Since Q is below the origin of the circle, there must be a circulation due to a line vortex.
- Kuethe and Chow (1998) stated that the circulation is $\Gamma = 4\pi aU \sin \theta$ where θ is the angle between line PQ and the direction of the uniform flow.
- Since $\sin \theta = y'_P/a$, $\Gamma = 4\pi y'_P U$
- The flow about the circle is then corresponding to a combination of a uniform flow, a doublet, and a vortex.



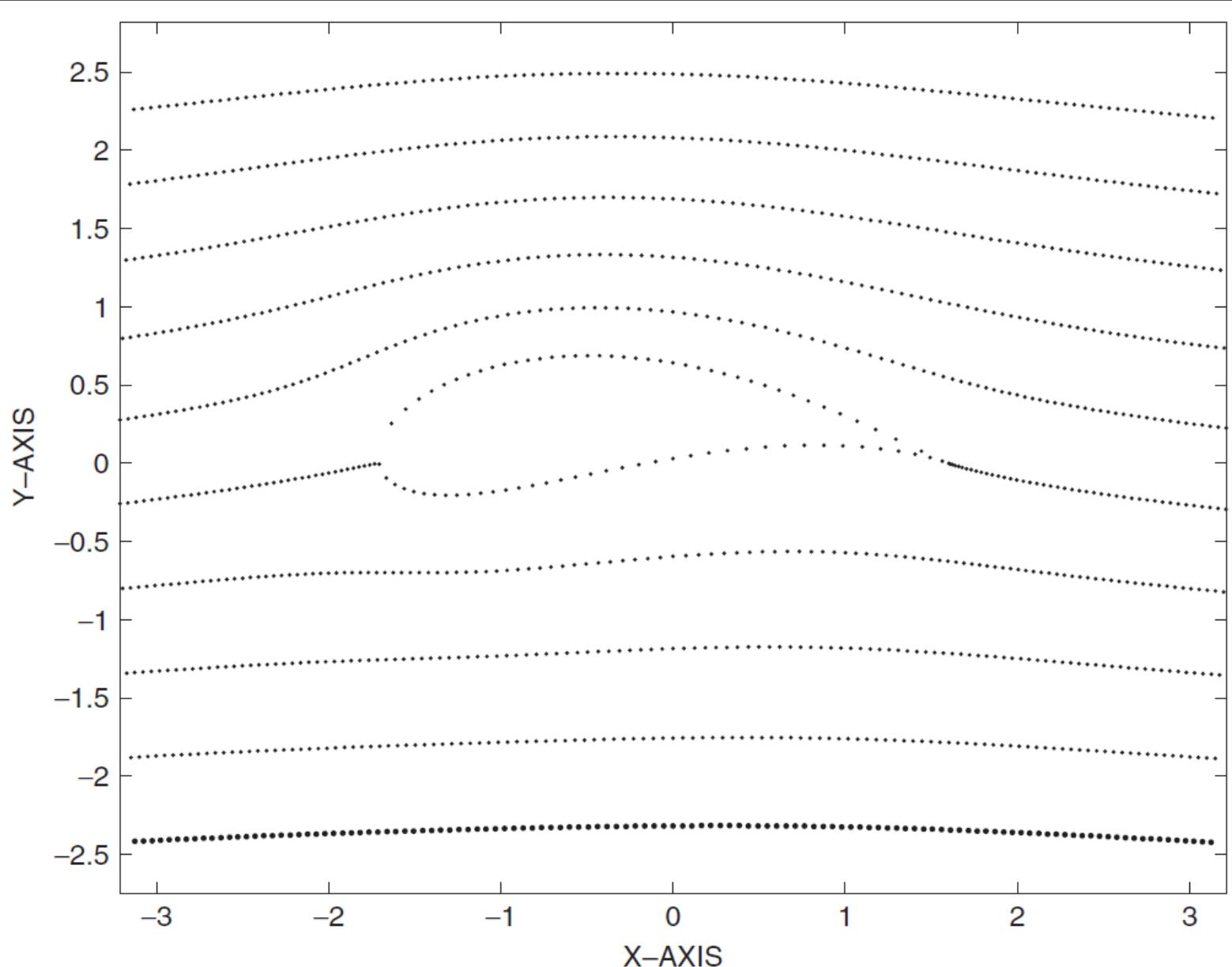
Flow past a Joukowsky Airfoil

- The complex potential of the resultant flow in the $x''-y''$ plane whose origin is at point P is
- "The constant $-i2y'_P \log a$ is added so that the value of the stream function on the circle remains unchanged after the introduction of the vortex."

- The complex potential can be written in term of z' by using the transformation $z'' = z' - z'_P$.
- The airfoil shape is controlled by the values of a , b , and the coordinate of point P (x'_P, y'_P) .
- They are related by $a^2 = y'^2 + (b - x'_P)^2$



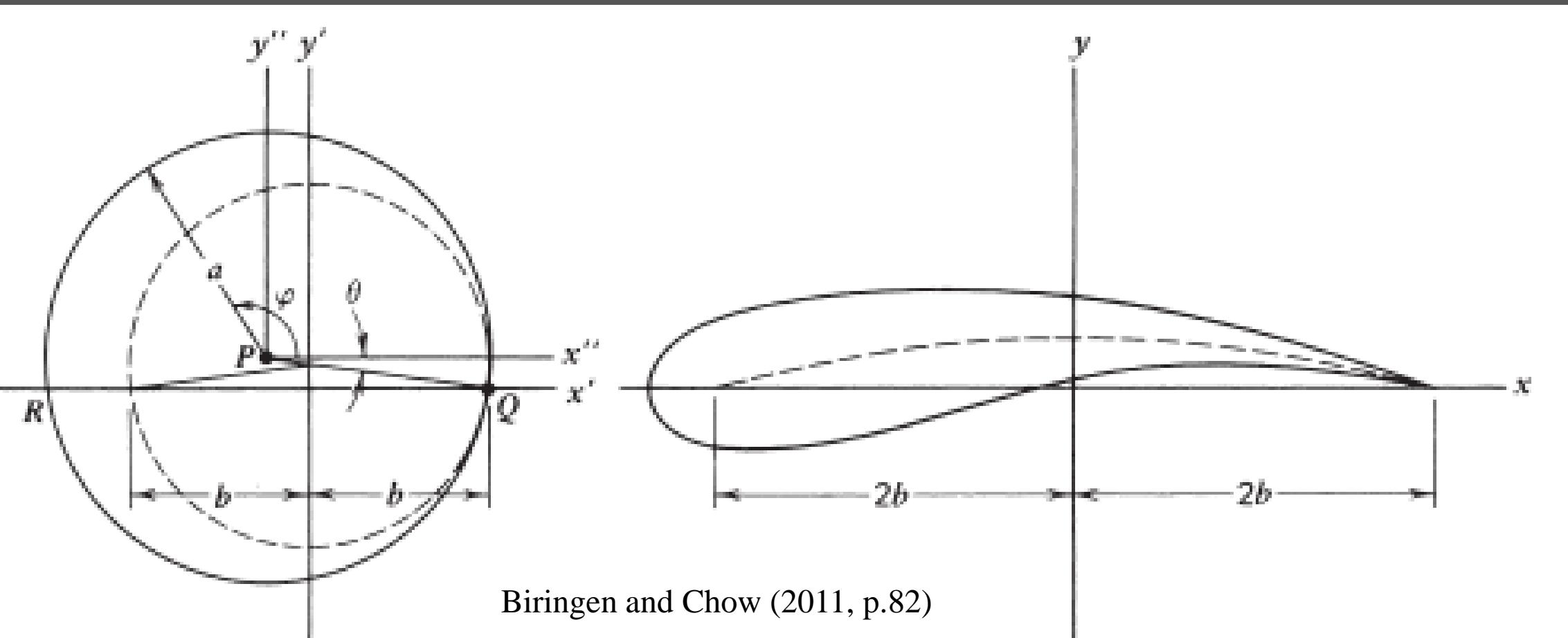
Flow past a Joukowsky Airfoil



Parameters: $U = 1 \text{ m/s}$
 $a = 1 \text{ m}, b = 0.8 \text{ m}$
 $x'_P = -0.18 \text{ m}, y'_P = 0.199 \text{ m}$
The stream function was computed in the x' - y' plane using the imaginary part of the complex potential.
Points on some stream lines are then mapped to the x - y plane using the Joukowski transformation.

Exercise

Use the Joukowsky transformation $z = z' + b^2/z'$ to map a circle of radius a centered at point P (see the figure below) into a Joukowsky airfoil. Here, $b < a$.



Exercise

- Compute the stream function using

$$\psi = 2Uy'_P \log\left(\frac{z' - z'_P}{a}\right)$$

- Then compute the coordinates (x', y') of points on a stream line.
- Use the Joukowski transformation to map those points to the x - y plane.

$$z = z' + b^2/z'$$

FD Solution to Elliptic PDEs

- Consider the 2D Poisson equation, an elliptic PDE,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = q(x, y)$$

- Approximating the derivative terms using the centered FD approximations yields the algebraic equations

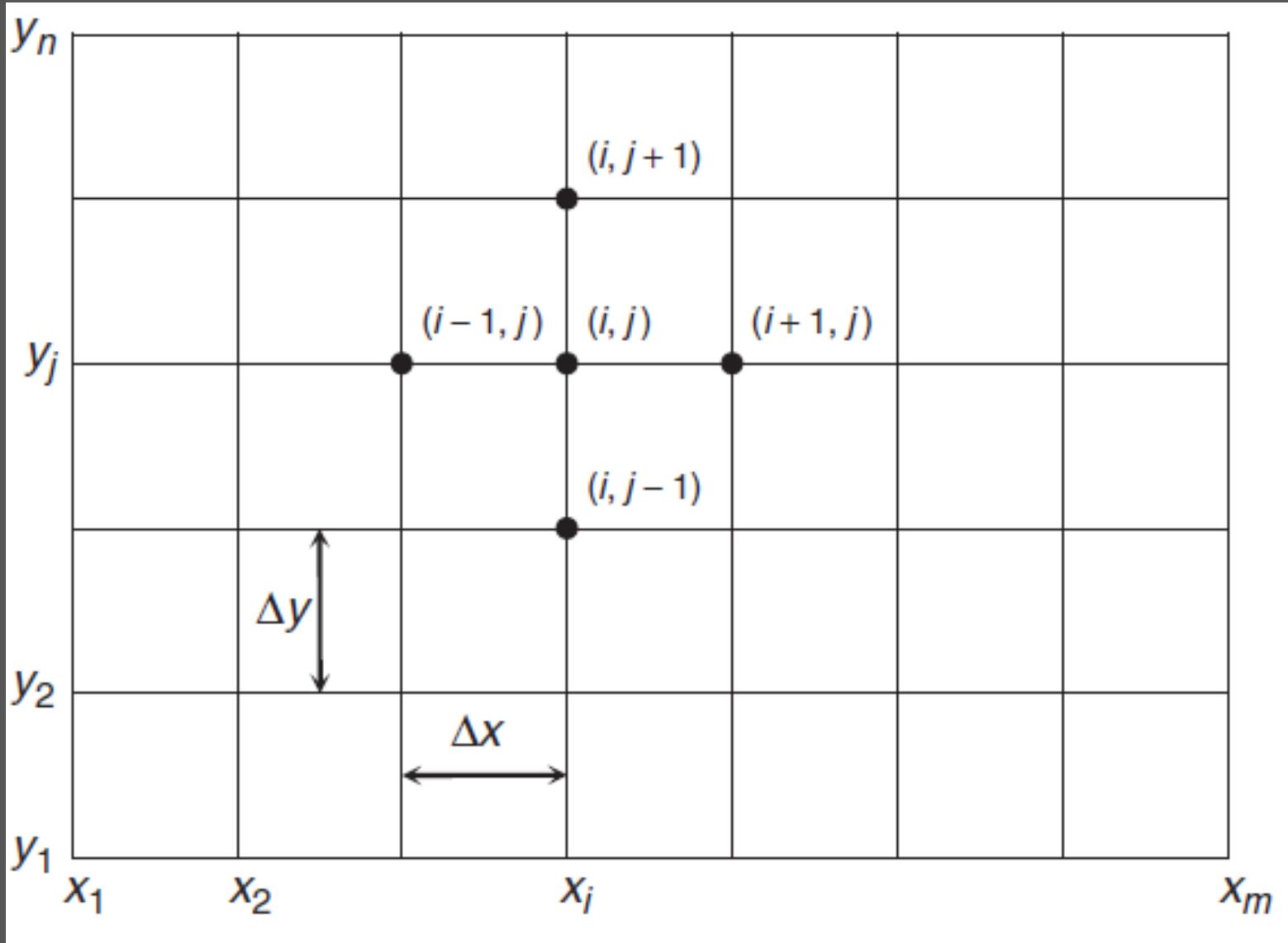
$$\frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta x)^2} + \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta y)^2} = q_{i,j}, \quad i = 2, \dots, m; j = 2, \dots, n$$

where $f_{i,j} = f(x_i, y_j)$.

- Using a square grid, i.e., $\Delta x = \Delta y = h$, we can rearrange the equation as

$$f_{i-1,j} + f_{i+1,j} - 4f_{i,j} + f_{i,j-1} + f_{i,j+1} = h^2 q_{i,j}$$

FD Grid



Iterative Linear Solvers

- The linear system can be efficiently solved using iterative methods in which the values of f at all interior points are initialized as $f_{i,j}^{(0)}$.
- The Jacobi iterative formula is given by

$$f_{i,j}^{(k+1)} = \frac{1}{4} \left(f_{i-1,j}^{(k)} + f_{i+1,j}^{(k)} + f_{i,j-1}^{(k)} + f_{i,j+1}^{(k)} \right) - \frac{1}{4} h^2 q_{i,j}$$

- The Gauss-Seidel iterative formula is given by

$$f_{i,j}^{(k+1)} = \frac{1}{4} \left(f_{i-1,j}^{(k+1)} + f_{i+1,j}^{(k+1)} + f_{i,j-1}^{(k+1)} + f_{i,j+1}^{(k)} \right) - \frac{1}{4} h^2 q_{i,j}$$

- The successive overrelaxation (SOR) iterative formula is given by

$$f_{i,j}^{(k+1)} = f_{i,j}^{(k)} + \frac{\beta}{4} \left(f_{i-1,j}^{(k+1)} + f_{i+1,j}^{(k+1)} + f_{i,j-1}^{(k+1)} + f_{i,j+1}^{(k)} - 4f_{i,j}^{(k)} - h^2 q_{i,j} \right), \quad 1 \leq \beta < 2$$

- The optimal value of β is

$$\beta_{\text{optimal}} = \frac{8 - 4\sqrt{4 - \alpha^2}}{\alpha^2}, \quad \alpha = \cos\left(\frac{\pi}{m}\right) + \cos\left(\frac{\pi}{n}\right)$$

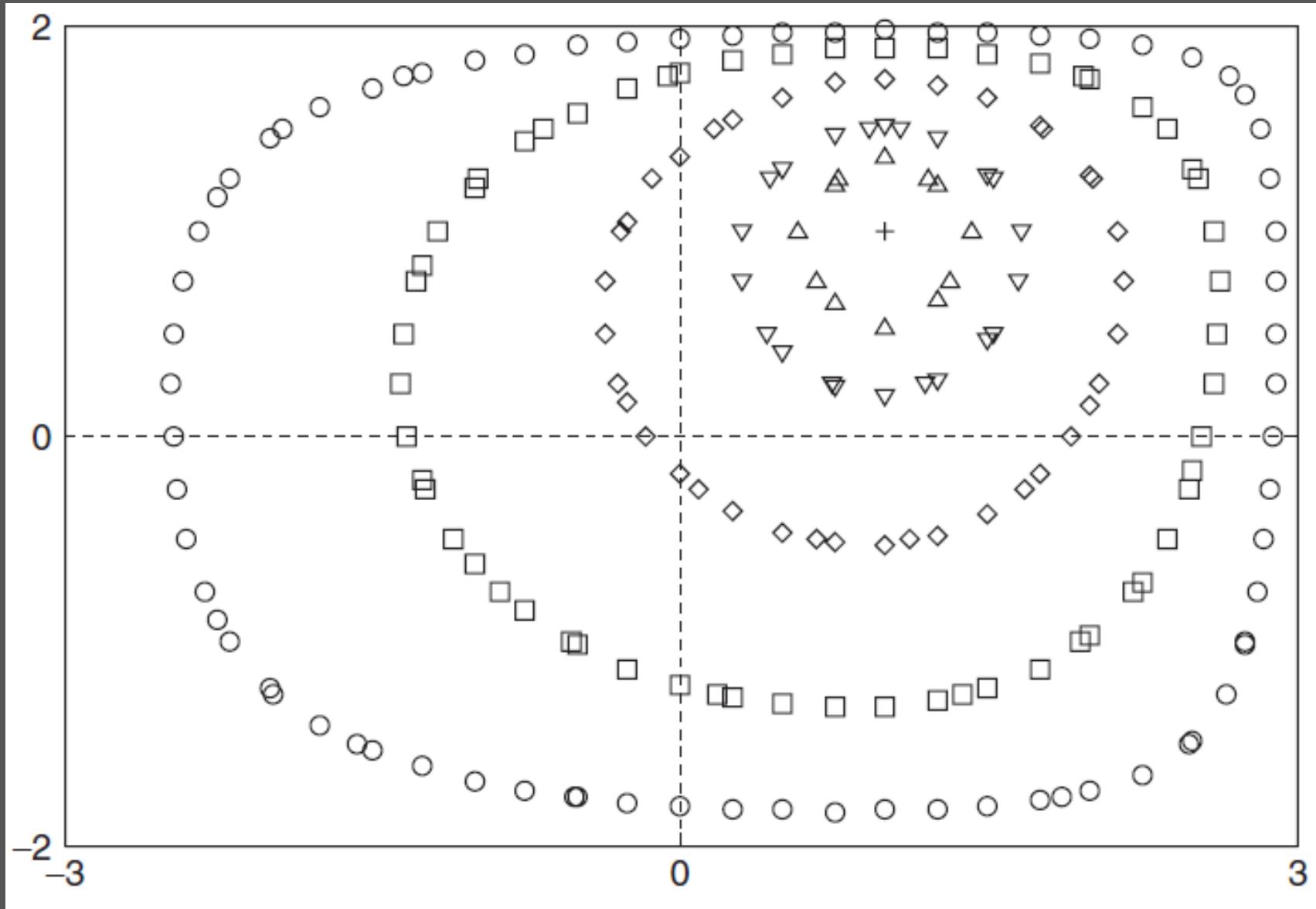
Exercise: Flow due to A Line Vortex

- The vorticity ω is the curl of the flow velocity \mathbf{v} , i.e., $\omega = \nabla \times \mathbf{v}$
- The stream function is defined as $\mathbf{v} = \nabla \times (\psi \hat{\mathbf{k}})$
- In 2D, this corresponds to the Poisson equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega(x, y)$$

- The boundary condition is that $\psi = 0$ on the boundary.
- Let the domain be $\Omega = \{(x, y) : -3 \leq x \leq 3, -2 \leq y \leq 2\}$
- Let the grid spacing be $h = 0.25$.
- Suppose that the line vortex of vorticity $\omega = 1$ is located at $(1, 1)$.

Flow due to a Line Vortex



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