Econ C142 Problem Set 4

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1. a) By the FOC, $\sum \hat{u}_i = 0$ $\left(\frac{\partial Q}{\partial \hat{\beta}_0} = 0 \right)$
$\sum x_i \hat{u}_i = 0$

$$\Rightarrow \sum y_i = \sum \hat{\beta}_0 + \sum \hat{\beta}_1 x_i$$

$$\Rightarrow \sum y_i = N \hat{\beta}_0 + \hat{\beta}_1 \sum x_i$$

$$\Rightarrow \frac{\sum y_i}{N} = \hat{\beta}_0 + \hat{\beta}_1 \sum x_i$$

$$\Rightarrow \bar{y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x}$$

b) Now, $y_i - \bar{y} = (\hat{\beta}_0 + \hat{\beta}_1 x_i + \hat{u}_i) - (\hat{\beta}_0 + \hat{\beta}_1 \bar{x})$

$$= \hat{\beta}_1 (x_i - \bar{x}) + \hat{u}_i$$

c) $\frac{1}{N} \sum (y_i - \bar{y}) \hat{u}_i$

$$= \frac{1}{N} \sum (\hat{\beta}_1 (x_i - \bar{x}) + \hat{u}_i) \hat{u}_i$$

$$= \frac{1}{N} \sum \hat{u}_i^2 + \hat{\beta}_1 \frac{1}{N} \sum \hat{u}_i (x_i - \bar{x})$$

$$= \frac{1}{N} \sum \hat{u}_i^2 + \hat{\beta}_1 \Big(\underbrace{\sum \hat{u}_i x_i}_{=0 \text{ by FOC 2}} + \underbrace{\bar{x} \sum \hat{u}_i}_{=0 \text{ by FOC 1}} \Big)$$

$$= \frac{1}{N} \sum \hat{u}_i^2$$

d) $\hat{\beta}_1 \frac{1}{N} \sum (y_i - \bar{y})(x_i - \bar{x})$

$$= \frac{1}{N} \sum (y_i - \bar{y}) \hat{\beta}_1 (x_i - \bar{x}) \qquad (y_i - \bar{y} = \hat{\beta}_1 (x_i - \bar{x}) + \hat{u}_i)$$

$$= \frac{1}{N} \sum (y_i - \bar{y})(y_i - \bar{y} - \hat{u}_i)$$

$$= \frac{1}{N} \sum (y_i - \bar{y})(y_i - \bar{y}) - \frac{1}{N} \sum (y_i - \bar{y}) \hat{u}_i$$

$$= \frac{1}{N} \sum (y_i - \bar{y})^2 - \frac{1}{N} \sum \hat{u}_i^2 \qquad (\text{by c)})$$

e) Note that $R^2 = 1 - \dfrac{\frac{1}{N} \sum \hat{u}_i^2}{\frac{1}{N} \sum (y_i - \bar{y})^2}$

$$= \frac{\frac{1}{N} \sum (y_i - \bar{y})^2 - \frac{1}{N} \sum \hat{u}_i^2}{\frac{1}{N} \sum (y_i - \bar{y})^2}$$

By d), we get that:

$$\hat{\beta}_1 \frac{1}{N} \sum (y_i - \bar{y})(x_i - \bar{x}) = \frac{1}{N} \sum (y_i - \bar{y})^2 - \frac{1}{N} \sum \hat{u}_i^2$$

$$\Rightarrow \frac{\left(\frac{1}{N} \sum (y_i - \bar{y})(x_i - \bar{x})\right)^2}{\frac{1}{N} \sum (x_i - \bar{x})^2} = \frac{1}{N} \sum (y_i - \bar{y})^2 - \frac{1}{N} \sum \hat{u}_i^2$$

$$(*) \quad \Rightarrow \left(\frac{1}{N} \sum (y_i - \bar{y})(x_i - \bar{x})\right)^2 = \frac{1}{N} \sum (x_i - \bar{x})^2 \left(\frac{1}{N} \sum (y_i - \bar{y})^2 - \frac{1}{N} \sum \hat{u}_i^2\right)$$

Now, $\rho_{xy}^2 = \dfrac{\left(\frac{1}{N} \sum (y_i - \bar{y})(x_i - \bar{x})\right)^2}{\frac{1}{N} \sum (y_i - \bar{y})^2 \frac{1}{N} \sum (x_i - \bar{x})^2}$

$$= \frac{\frac{1}{N} \sum (x_i - \bar{x})^2 \cdot \left(\frac{1}{N} \sum (y_i - \bar{y})^2 - \frac{1}{N} \sum \hat{u}_i^2\right)}{\frac{1}{N} \sum (x_i - \bar{x})^2 \frac{1}{N} \sum (y_i - \bar{y})^2} \quad \text{(via } (*)\text{)}$$

$$= \frac{\frac{1}{N} \sum (y_i - \bar{y})^2 - \frac{1}{N} \sum \hat{u}_i^2}{\sum (y_i - \bar{y})^2}$$

$$= R^2$$