Econ C142 Problem Set 3

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1.a) Let $x_i = \begin{pmatrix} a \\ x_{1i} \\ \vdots \end{pmatrix}$

Then the FOC is a system of equations:

$$\sum \begin{pmatrix} a \\ \vdots \end{pmatrix} (y_i - x_i' \hat{\beta}) = 0$$

$$\Rightarrow \sum a (y_i - x_i' \hat{\beta}) = 0, \quad \sum x_{1i} (y_i - x_i' \hat{\beta}) = 0, \text{ etc}$$

Considering the first equation:

$$a \sum (y_i - x_i' \hat{\beta}) = 0$$

$$\Rightarrow \sum (y_i - x_i' \hat{\beta}) = 0$$

$$\Rightarrow \sum_{i=1}^{N} y_i = \sum_{i=1}^{N} x_i' \hat{\beta}$$

$$\Rightarrow \sum y_i = \hat{\beta} \sum x_i'$$

$$\Rightarrow \frac{1}{N} \sum y_i = \hat{\beta} \frac{1}{N} \sum x_i'$$

$$\Rightarrow \frac{1}{N} \sum y_i = \hat{\beta} \left(\frac{1}{N} \sum x_i\right)' \qquad (\vec{x}' + \vec{y}' \Leftrightarrow (\vec{x} + \vec{y})')$$

$$\Rightarrow \bar{y} = \hat{\beta} (\bar{x})'$$

b) By FOC: $\sum D_g (y_i - x_i' \hat{\beta}) = 0$ (Let $x_i = \begin{pmatrix} \vdots \\ D_g \\ \vdots \end{pmatrix}$)

$$\Rightarrow \sum_{i=1}^{N} D_g y_i = \sum_{i=1}^{N} D_g x_i' \hat{\beta}$$

$$\Rightarrow \sum_{i \in g} D_g y_i + \sum_{i \notin g} \underbrace{D_g}_{=0} y_i = \sum_{i \in g} D_g x_i' \hat{\beta} + \sum_{i \notin g} \underbrace{D_g}_{=0} x_i' \hat{\beta}$$

$$\Rightarrow \sum_{i \in g} 1 \cdot y_i + \sum_{i \notin g} 0 \, y_i = \sum_{i \in g} 1 \cdot x_i' \beta + \sum_{i \notin g} 0 \, x_i' \hat{\beta}$$

$$\Rightarrow \sum_{i \in g} y_i = \hat{\beta} \sum_{i \in g} x_i'$$

$$\Rightarrow \frac{1}{N_g} \sum_{i \in g} y_i = \hat{\beta} \frac{1}{N_g} \sum_{i \in g} x_i'$$

$$\Rightarrow \bar{y}_g = \hat{\beta} \left(\frac{1}{N_g} \sum_{i \in g} x_i\right)'$$

$$\Rightarrow \bar{y}_g = \hat{\beta} (\bar{x}_g)'$$

c) Consider the Full Regression $\overset{OLS}{:}$ $y_i = x_i'\hat{\beta} + \hat{u}_i$
with FOC : $\frac{1}{N} \sum x_i \hat{u}_i = 0$ ①

OLS Aux. regression : $x_{ji} = x'_{(-j)i} \hat{\pi} + \hat{\xi}_i$ with
FOC : $\frac{1}{N} \sum x_{(-j)i} \hat{\xi}_i = 0$ ②

Note $y_i = x_i'\hat{\beta} + \hat{u}_i = \hat{\beta}_1 x_{1i} + \ldots + \hat{\beta}_K x_{Ki} + \hat{u}_i$

Now, $\frac{1}{N} \sum_{i=1}^{N} \hat{\xi}_i y_i = \frac{1}{N} \sum_{i=1}^{N} \hat{\xi}_i (\hat{\beta}_1 x_{1i} + \ldots + \hat{\beta}_K x_{Ki} + \hat{u}_i)$

$= \frac{1}{N}(\hat{\beta}_1 \sum \hat{\xi}_i x_{1i} + \ldots + \hat{\beta}_K \sum \hat{\xi}_i x_{Ki} + \sum \hat{\xi}_i \hat{u}_i)$

Notice $\sum \hat{\xi}_i x_{mi} = 0$ unless $m = j$ via ② $(\sum x_{(-j)i} \hat{\xi}_i = 0)$
Moreover, $\sum \hat{\xi}_i \hat{u}_i$
$= \sum (x_{ji} - x'_{(-j)i} \hat{\pi}) \hat{u}_i = 0$
via ① because $\hat{u}_i$ is orthogonal to all the x's $(\sum x_i \hat{u}_i = 0)$

Hence $\frac{1}{N} \sum \hat{\xi}_i y_i = \frac{1}{N}(0 + \ldots + \hat{\beta}_j \sum \hat{\xi}_i x_{ji} + \ldots + 0 + 0)$

Observe $\sum \hat{\xi}_i x_{ji} = \sum \hat{\xi}_i (x'_{(-j)i} \hat{\pi} + \hat{\xi}_i)$
$= 0 + \sum \hat{\xi}_i^2$ (via ②)

$\Rightarrow \frac{1}{N} \sum \hat{\xi}_i y_i = \frac{1}{N} \hat{\beta}_j \sum \hat{\xi}_i^2$

$\Rightarrow \hat{\beta}_j = \left(\frac{1}{N} \sum_{i=1}^{N} \hat{\xi}_i^2\right)^{-1} \left(\frac{1}{N} \sum_{i=1}^{N} \hat{\xi}_i y_i\right)$

2.a) Model (5) is $\log(\text{wage}) = \hat{\beta}_1 + \hat{\beta}_2\,\text{Immigrant} + \hat{\beta}_3\,\text{Education} + \hat{u}_i$
Model (1) is $\log(\text{wage}) = \hat{\beta}_1^0 + \hat{\beta}_2^0\,\text{Immigrant} + \hat{\varepsilon}_i$
Aux. Model is $\text{Education} = \hat{\pi}_1 + \hat{\pi}_2\,\text{Immigrant} + \hat{v}_i$

$\Rightarrow \hat{\beta}_2^0 = \hat{\beta}_2 + \hat{\beta}_3 \hat{\pi}_2$

b) $\hat{\beta}_2^0 = -0.18$, $\hat{\beta}_2 = -0.0101$, $\hat{\beta}_3 = 0.1139$, $\hat{\pi}_2 = -1.692$ (Females)

As expected, $-0.18 = -0.0101 + 0.1139 \times -1.692$

$\hat{\beta}_2^0 = -0.2448$, $\hat{\beta}_2 = -0.07453$, $\hat{\beta}_3 = 0.10562$, $\hat{\pi}_2 = -1.612$

As expected, $-0.2448 = -0.07453 + 0.10562 \times -1.612$