RC lattice analysis

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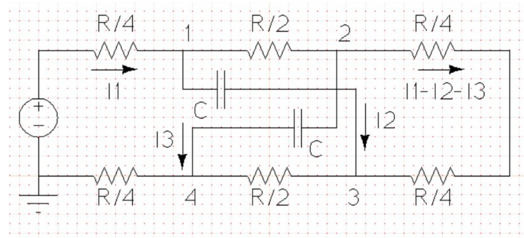
June 20, 2024

General Assumptions

- The total resistance of the circuit is chosen as **400** Ω.
- Considering a 1:2:1 ratio of resistors used in the lattice, the middle resistance is $\frac{R}{2}$.
- The time delay T_d is given by $T_d = 2 * \frac{R}{2} * C$.
- We choose C =**750** fF to give a delay of $T_d =$ **300 ps**.
- The ideal delay function is given by $sinc(f * T_d) * e^{-j2\pi f * T_d}$.
- The Bandwidth is taken to be $\frac{1.1 * f_s}{8} = 4.583 * 10^8 Hz$

Transfer Function Derivation



From the above circuit diagram, Between points 1 and 4,

$$\frac{i_2}{sC} + \frac{R}{2} * (i_1 - i_3) = \frac{R}{2} * (i_1 - i_2) + \frac{i_3}{sC} \tag{1}$$

$$i_2 = i_3 \tag{2}$$

Between points 1 and 3,

$$\frac{i_2}{sC} = \frac{R}{2} * (i_1 - i_2) + \frac{R}{2} * (i_1 - i_2 - i_3) \tag{3}$$

$$R * i_1 = i_2 * (\frac{1}{sC} + \frac{3R}{2}) \tag{4}$$

Now considering the entire loop,

$$0 = V_i - \frac{R}{4} * (i_1) - \frac{i_2}{sC} - \frac{R}{2} * (i_1 - i_3) - \frac{R}{4} * (i_1) \tag{5}$$

$$V_i = \frac{R}{2} * (2 * i_1 - i_2) + \frac{i_2}{sC} \tag{6}$$

$$V_i = i_2 * \frac{2}{sC} + i_2 * R \tag{7}$$

The output current is,

$$I_{out} = i_1 - i_2 - i_3 \tag{8}$$

$$= i_1 - 2 * i_2 \tag{9}$$

The transfer function is

$$\frac{I_{out}}{V_i} = \frac{i_1 - 2 * i_2}{V_i} \tag{10}$$

$$= \frac{1}{2R} * \frac{2 - RsC}{2 + RsC} \tag{11}$$

$$= \frac{1}{2R} * \frac{1 - (\frac{R}{2}sC)}{1 + (\frac{R}{2}sC)} \tag{12}$$

This gives a phase response of,

$$-2 * tan^{-1}(\omega * \frac{R}{2} * C) = -\omega * R * C \tag{13}$$

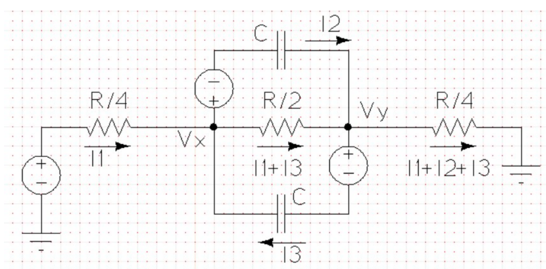
$$\tag{14}$$

The time delay is approximately,

$$-\frac{(-\omega * R * C)}{\omega} = R * C \tag{15}$$

$$\tag{16}$$

For the modified circuit,



$$0 = V_i - i_1 * R - i_3 * \frac{3R}{4} - i_2 * \frac{R}{4} \tag{17}$$

$$V_y = -V_x - \frac{i_2}{sC} \tag{18}$$

$$V_x = -V_y - \frac{i_3}{sC} \tag{19}$$

This gives,

$$i_2 = i_3 \tag{20}$$

$$V_i = R * (i_1 + i_2) \tag{21}$$

Now,

$$V_y = \frac{R}{4} * (i_1 + i_2 + i_3) \tag{22}$$

$$V_x = V_i - \frac{R}{4} * (i_1) \tag{23}$$

$$V_x + V_y = \frac{R}{4} * (i_1 + 2 * i_2) + V_i - \frac{R}{4} * (i_1) \tag{24}$$

Using the previously derived equations,

$$-\frac{i_2}{sC} = \frac{R}{4} * (i_1 + 2 * i_2) + V_i - \frac{R}{4} * (i_1) \tag{25}$$

$$V_i = -i_2 * (\frac{1}{sC} + \frac{R}{2}) \tag{26}$$

The output current is,

$$I_{out} = i_1 + i_2 + i_3 \tag{27}$$

$$= i_1 + 2 * i_2 \tag{28}$$

The transfer function is given by,

$$\frac{I_{out}}{V_i} = \frac{i_1 + 2 * i_2}{V_i} \tag{29}$$

$$= \frac{1}{R} + \frac{-1}{\frac{1}{sC} + \frac{R}{2}} \tag{30}$$

$$= \frac{1}{R} * \frac{1 - (\frac{R}{2}sC)}{1 + (\frac{R}{2}sC)} \tag{31}$$

This gives a phase response of,

$$-2 * tan^{-1}(\omega * \frac{R}{2} * C) = -\omega * R * C \tag{32}$$

$$\tag{33}$$

The time delay is approximately,

$$-\frac{(-\omega * R * C)}{\omega} = R * C \tag{34}$$

$$\tag{35}$$

General Procedure

To find the optimal circuit parameters the following objective function is minimized over the bandwidth.

$$objective = (angle(H(f)) - angle(T(f)))^2 \tag{36}$$

where, H(f) is the transfer function of the circuit and T(f) is the ideal delay line transfer function.

Old Approach : The objective function was determined at 250 points per decade and the sum was minimised to find the optimal transfer function. But, on observing the value of the objective function it was determined that the value was too small and multiple transfer functions gave identical values upto 4 decimal places.

New Approach : The objective function is now integrated over the range of frequencies of the bandwidth and this integral is now minimised to obtain the optimal transfer function.

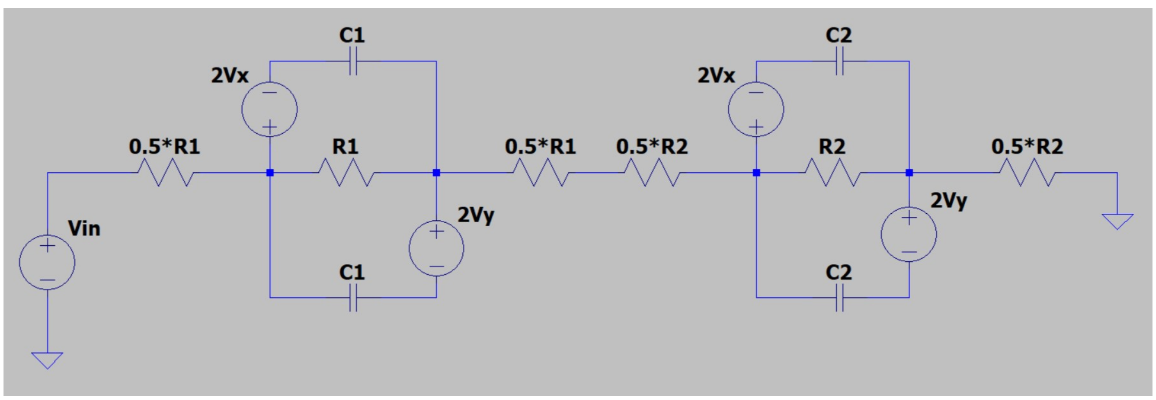
1 2x- cascaded RC lattice - 1

1.1 Aim of the experiment

For a general 2 stage RC lattice, the total resistance is fixed and when **equal resistors** are used for each stage what is the best ratio of capacitors to get close to ideal response.
The capacitors are also varied so that our approximation of delay (= 2*R*C) is satisfied.

1.2 Design

The circuit diagram is given below



We ensure the following conditions,

$$R_1 = \frac{R}{4} = 100\Omega \tag{37}$$
$$R_2 = \frac{R}{4} = 100\Omega \tag{38}$$
$$2 * R/2 * C = 2R_1 * C_1 + 2R_2 * C_2 \tag{39}$$
$$2C = C_1 + C_2 \tag{40}$$
$$\tag{41}$$

1.3 Simulation results

Integral of squares of the errors in angles was minimised over the bandwidth. The transfer function of the circuit is given by,

$$H(s) = \frac{(1 - sR_1C_1)(1 - sR_2C_2)}{(1 + sR_1C_1)(1 + sR_2C_2)} \tag{42}$$

The above conditions were ensured and using the quasi-newton algorithm in the fmincon function of MATLAB the following results were obtained.

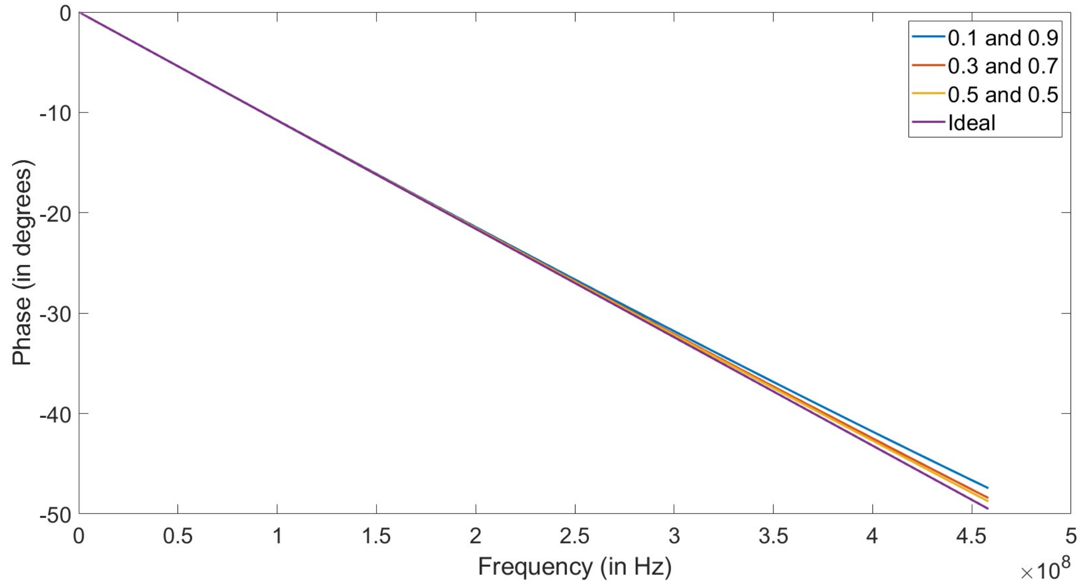
Table 1: Simulation results

C_1	C_2
750 fF	750 fF

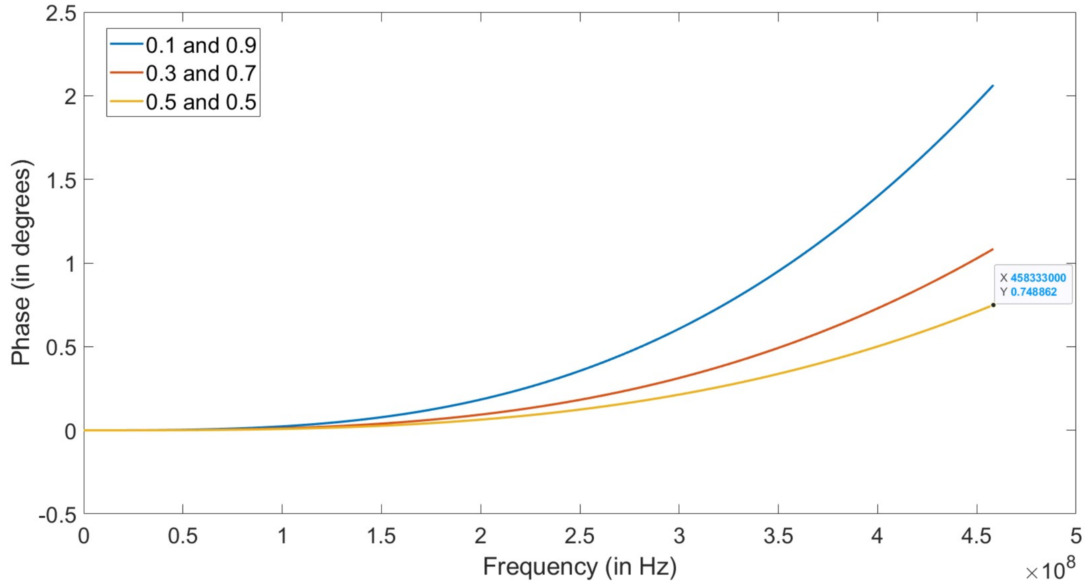
```
<stopping criteria details>
    0.5000
```

Where,

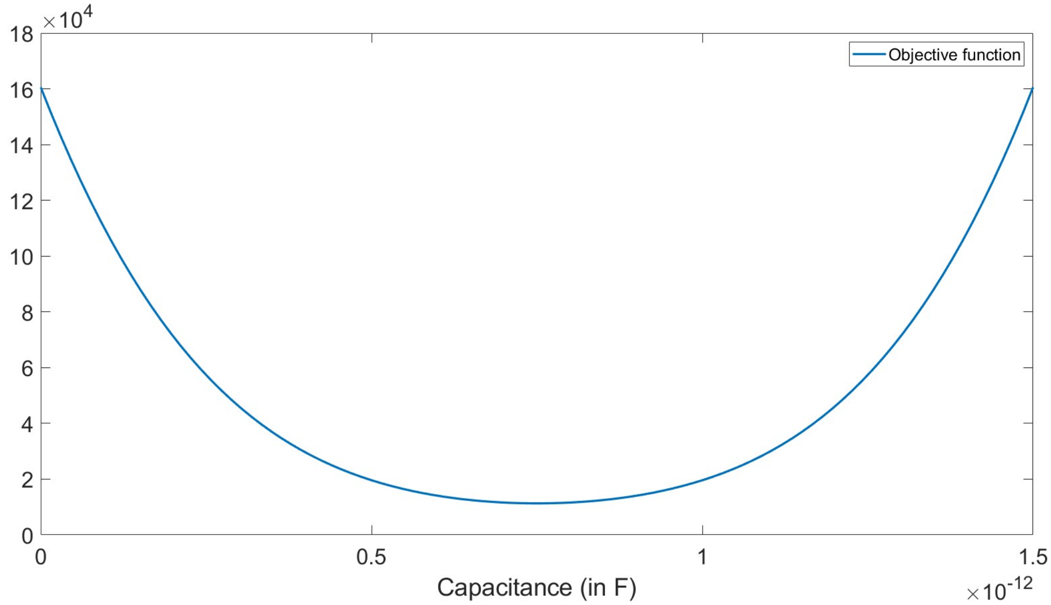
$$C_1 = 2C * 0.5 \tag{43}$$
$$= 750fF \tag{44}$$
$$C_2 = 2C * 0.5 \tag{45}$$
$$= 750fF \tag{46}$$



In the above figure, “Ideal” indicates the ideal delay function modelled using sinc and exponential.
Various other combinations of C_1 and C_2 have also been plotted and it can be observed that the optimized output of $C_1 = C_2 = C$ gives the best result (0.5 and 0.5).
The difference between ideal and optimized is shown below.



The optimized function deviates from ideal by a maximum of **0.75** degrees over the bandwidth.
The objective function is plotted below for varying values of capacitors, the capacitance of the first capacitor is the x axis.



1.4 Experimental results

We observe that when equal resistors are used in the 2 stages, the best result is obtained by using equal capacitors in both stages.

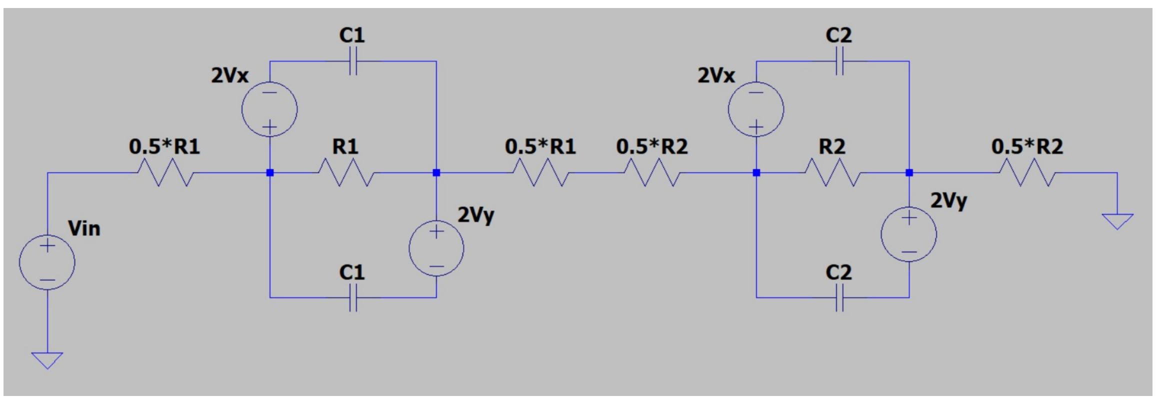
2 2x- cascaded RC lattice - 2

2.1 Aim of the experiment

For a general 2 stage RC lattice, the total resistance is fixed but unequal resistors and capacitors can be used for each stage.
In this case, we find the ratio of resistors and capacitors so as to get a response close to the ideal response.

2.2 Design

The circuit diagram is given below



We ensure the following conditions,

$$R = 2R_1 + 2R_2 \tag{47}$$

2.3 Simulation results

Integral of squares of the errors in angles was minimised over the bandwidth. The transfer function of the circuit is given by,

$$H(s) = \frac{(1 - sR_1C_1)(1 - sR_2C_2)}{(1 + sR_1C_1)(1 + sR_2C_2)} \tag{48}$$

The above conditions were ensured and using the quasi-newton algorithm in the fmincon function of MATLAB the following results were obtained.

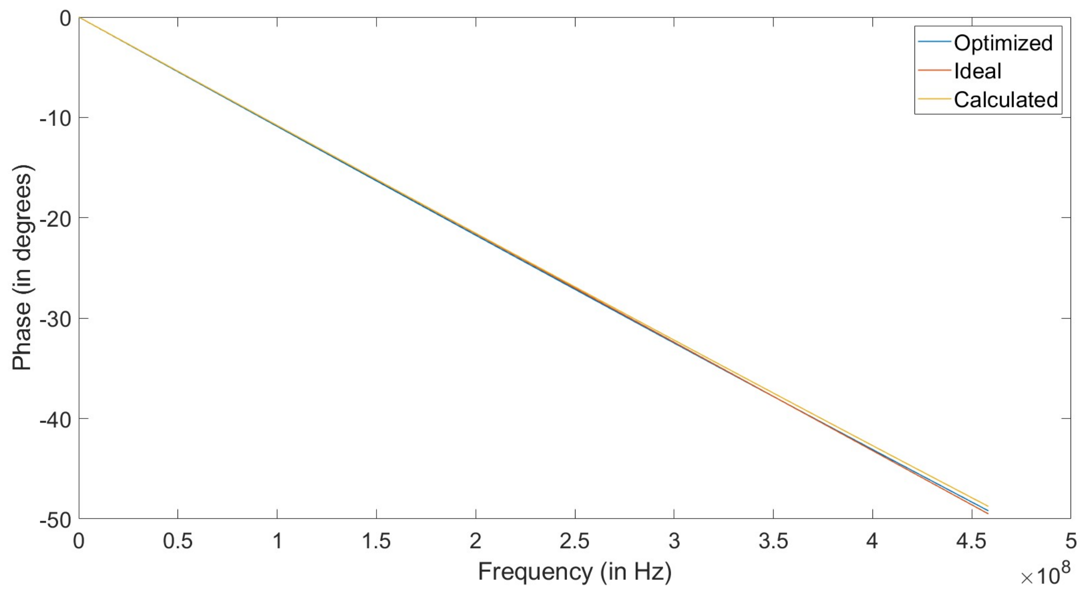
Table 2: Simulation results

R_1	R_2	C_1	C_2
100Ω	100 Ω	757.05 fF	757.05 fF

```
optimal =

    0.5000    1.0094    1.0094
```

Where,

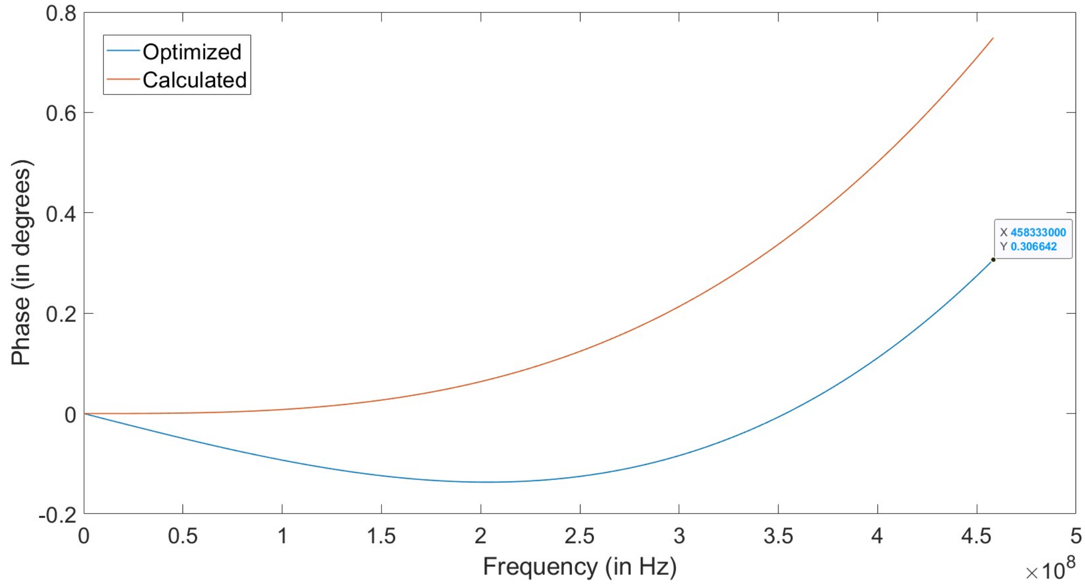
$$\begin{aligned} R_1 &= R/2 * 0.5 && (49)\\ &= 100\Omega && (50)\\ R_2 &= R/2 * (1 - 0.5) && (51)\\ &= 100\Omega && (52)\\ C_1 &= C * 1.0094 && (53)\\ &= 757.05 fF && (54)\\ C_2 &= C * 1.0094 && (55)\\ &= 757.05 fF && (56) \end{aligned}$$



In the above figure, "Ideal" indicates the ideal delay function modelled using sinc and exponential.
"Calculated" is the circuit constructed using the approximation of

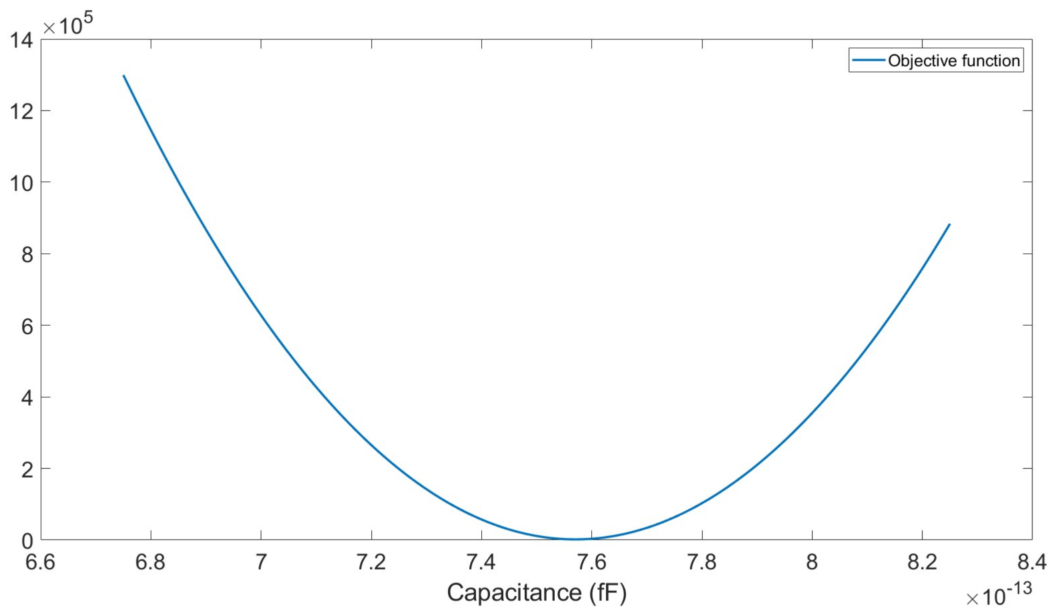
$$2 * \frac{R}{2} * C = 2 * R_1 * C_1 + 2 * R_2 * C_2.$$

"Optimized" is the output from the program to get the best ratio of resistors and capacitors with the same total resistance.
The difference between ideal and optimized is shown below along with the difference between the calculated and optimized.

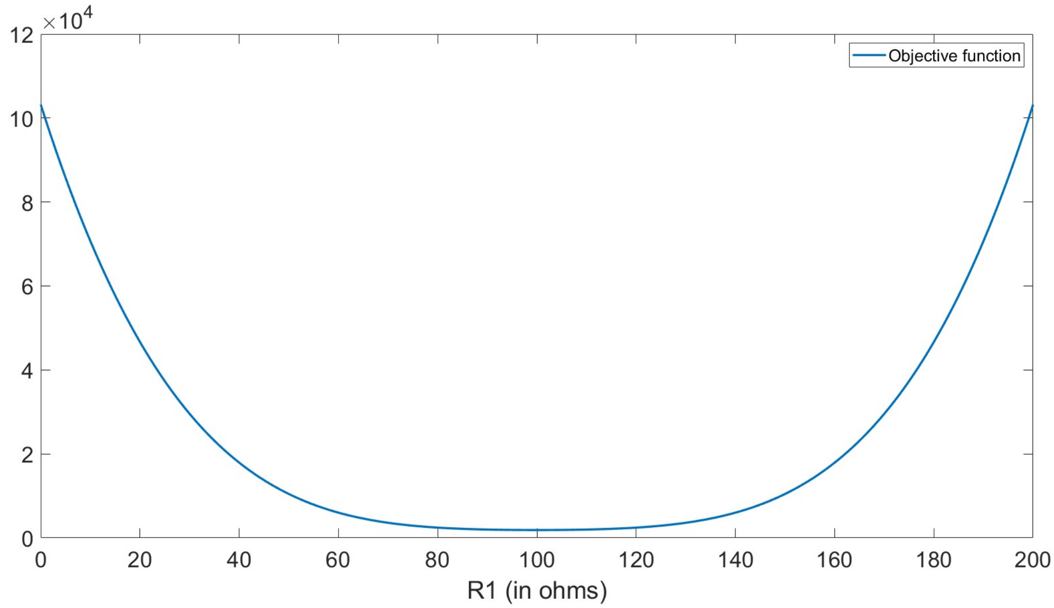


The optimized function deviates from ideal by a maximum of **0.30** degrees over the bandwidth.

Below is the plot of the objective function for equal resistors and varying capacitance, it is observed that the minima is obtained at a value close to 757 fF.



Below is the plot of the objective function for equal capacitors and varying resistors, it is observed that the minima is obtained at a value of 100 ohms.



2.4 Experimental results

Thus we observe that the best circuit consists of equal resistors and capacitors in the 2 sections.
But the capacitors are slightly larger than the approximated values in order to obtain a better delay function.

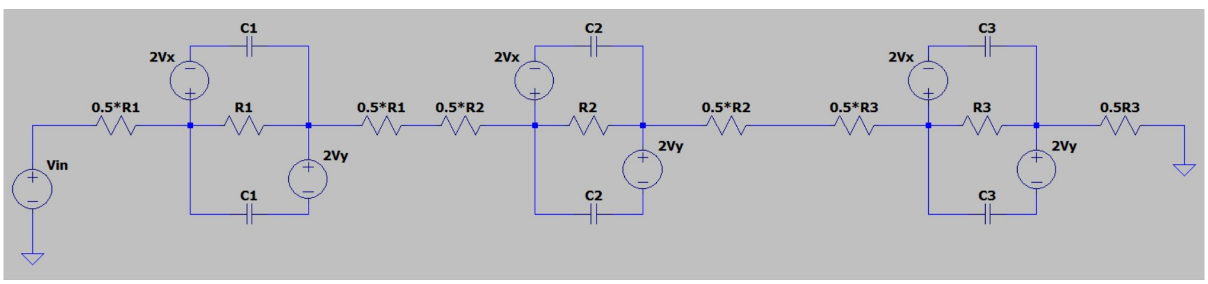
3 3x- cascaded RC lattice - 1

3.1 Aim of the experiment

For a general 3 stage RC lattice, the total resistance is fixed with equal resistors used for each stage.
The capacitors are also varied so that our approximation of delay (= 2*R*C) is satisfied.
In this case, we find the ratio of capacitors so as to get a response close to the ideal response.

3.2 Design

The circuit diagram is given below



We ensure the following conditions,

$$R_1 = \frac{R}{6} \tag{57}$$
$$R_2 = \frac{R}{6} \tag{58}$$
$$R_3 = \frac{R}{6} \tag{59}$$
$$2 * \frac{R}{2} * C = 2R_1 * C_1 + 2R_2 * C_2 + 2R_3 * C_3 \tag{60}$$
$$3C = C_1 + C_2 + C_3 \tag{61}$$

3.3 Simulation results

Integral of squares of the errors in angles was minimised over the bandwidth. The transfer function of the circuit is given by,

$$H(s) = \frac{(1 - sR_1C_1)(1 - sR_2C_2)(1 - sR_3C_3)}{(1 + sR_1C_1)(1 + sR_2C_2)(1 + sR_3C_3)} \tag{62}$$

The above conditions were ensured and using the quasi-newton algorithm in the fmincon function of MATLAB the following results were obtained.

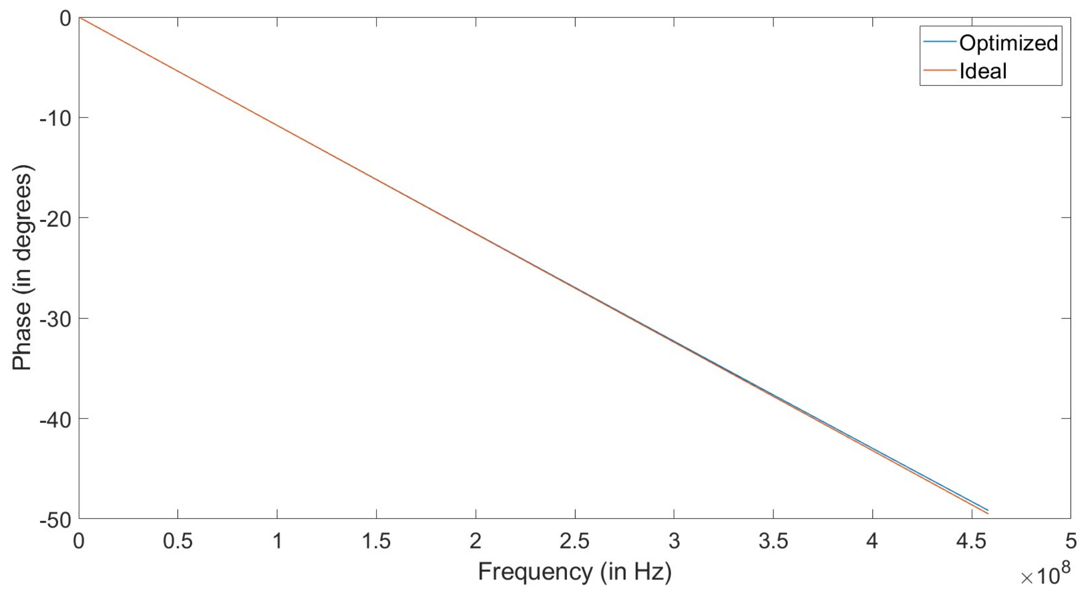
Table 3: Simulation results

C_1	C_2	C_3
750 fF	750 fF	750 fF

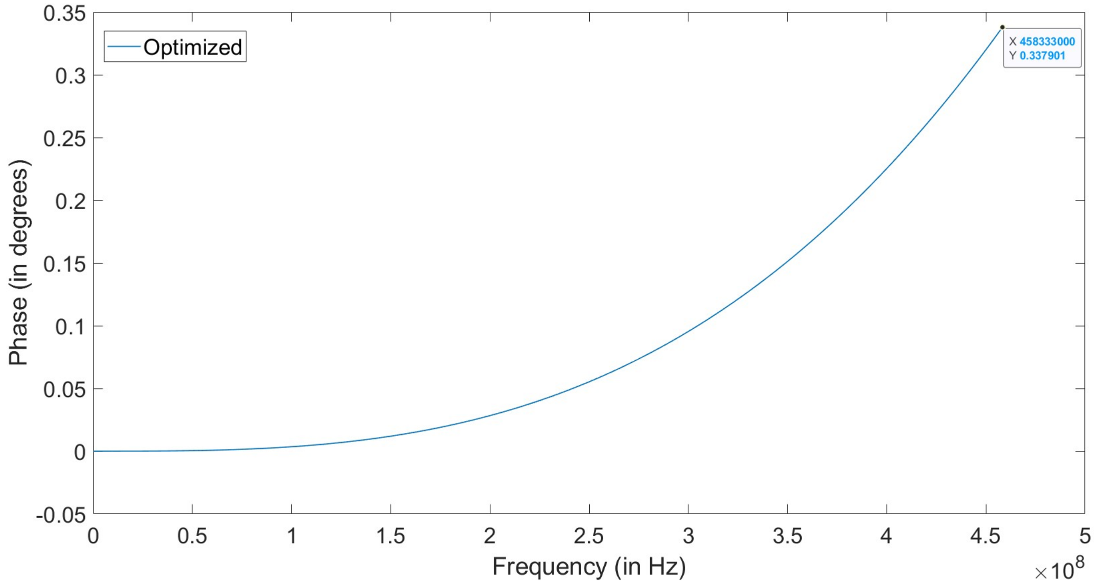
```
opti =

    1.0000    1.0000
```

Where,

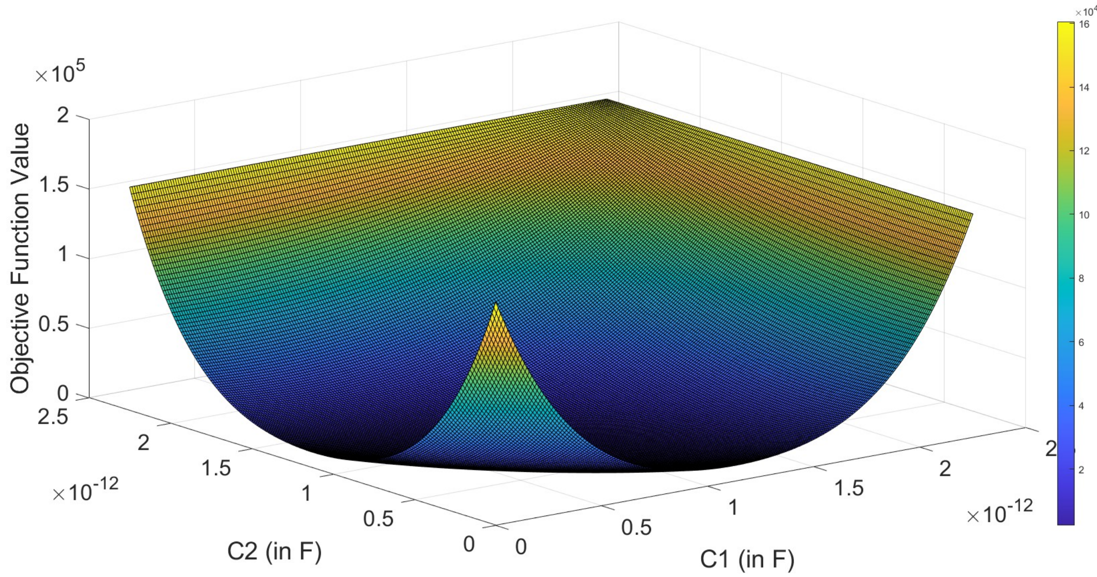
$$\begin{aligned}
C_1 &= C * 1 && (63)\\
&= 750fF && (64)\\
C_2 &= C * 1 && (65)\\
&= 750fF && (66)\\
C_3 &= C * (3 - 1 - 1) && (67)\\
&= 750fF && (68)
\end{aligned}$$

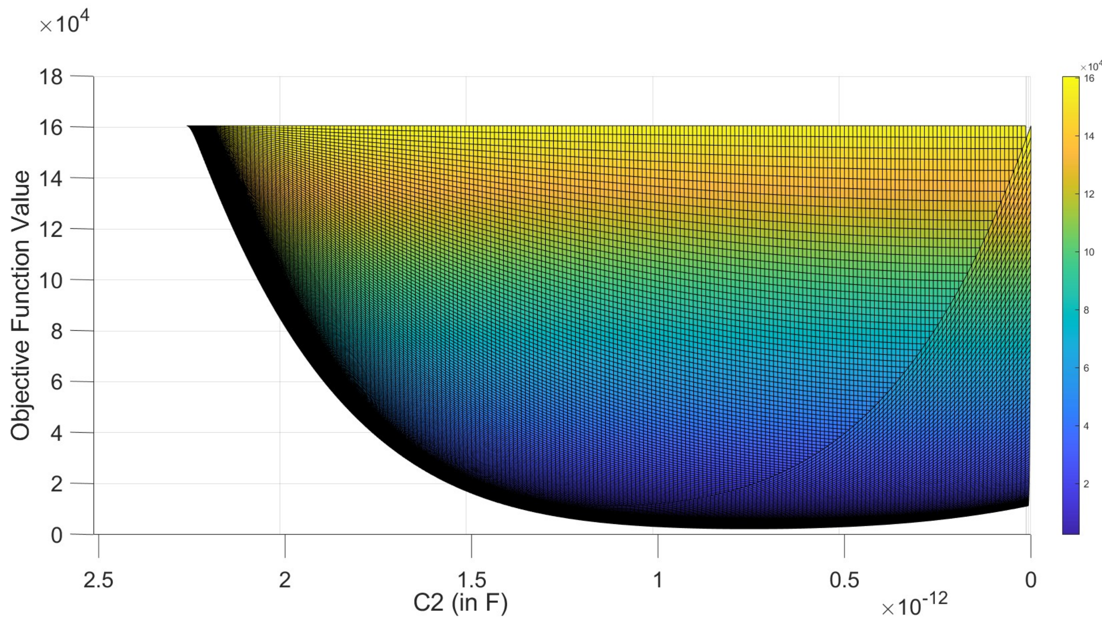


In the above figure, “Ideal” indicates the ideal delay function modelled using sinc and exponential.
“Optimized” is the output from the program to get the best ratio of capacitors when equal resistors are used in the 3 stages.
In this case it turns out that the best circuit is formed by having equal capacitors in the 3 stages. $C_1 = C_2 = C_3 = C$
The difference between ideal and optimized is shown below.



The optimized function deviates from ideal by a maximum of **0.34** degrees over the bandwidth.
Below is the plot of the objective function for equal resistors and varying capacitance of C_1 and C_2, it is observed that the minima is obtained at a value of 750 fF for both C1 and C2.


×10^5
2
1.5
1
0.5
0
Objective Function Value
2.5
2
1.5
1
0.5
0
×10^-12
C2 (in F)
0
0.5
1
1.5
2
×10^-12
C1 (in F)
×10^4
16
14
12
10
8
6
4
2


×10^4
18
16
14
12
10
8
6
4
2
0
Objective Function Value
2.5
2
1.5
1
0.5
0
C2 (in F)
×10^-12
×10^4
16
14
12
10
8
6
4
2

3.4 Experimental results

Thus we observe that the best circuit consisting of equal resistors is obtained when equal capacitors are used in the 3 sections.

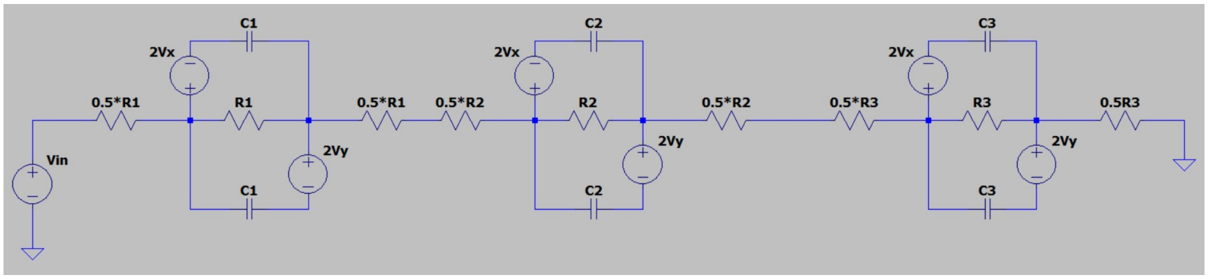
4 3x- cascaded RC lattice - 2

4.1 Aim of the experiment

For a general 3 stage RC lattice, the total resistance is fixed but unequal resistors and capacitors can be used for each stage.
In this case, we find the ratio of capacitors and resistors so as to get a response close to the ideal response.

4.2 Design

The circuit diagram is given below



We ensure the following conditions,

$$R = 2R_1 + 2R_2 + 2R_3 \tag{69}$$

4.3 Simulation results

Integral of squares of the errors in angles was minimised over the bandwidth. The transfer function of the circuit is given by,

$$H(s) = \frac{(1 - sR_1C_1)(1 - sR_2C_2)(1 - sR_3C_3)}{(1 + sR_1C_1)(1 + sR_2C_2)(1 + sR_3C_3)} \tag{70}$$

The above conditions were ensured and using the quasi-newton algorithm in the fmincon function of MATLAB the following results were obtained.

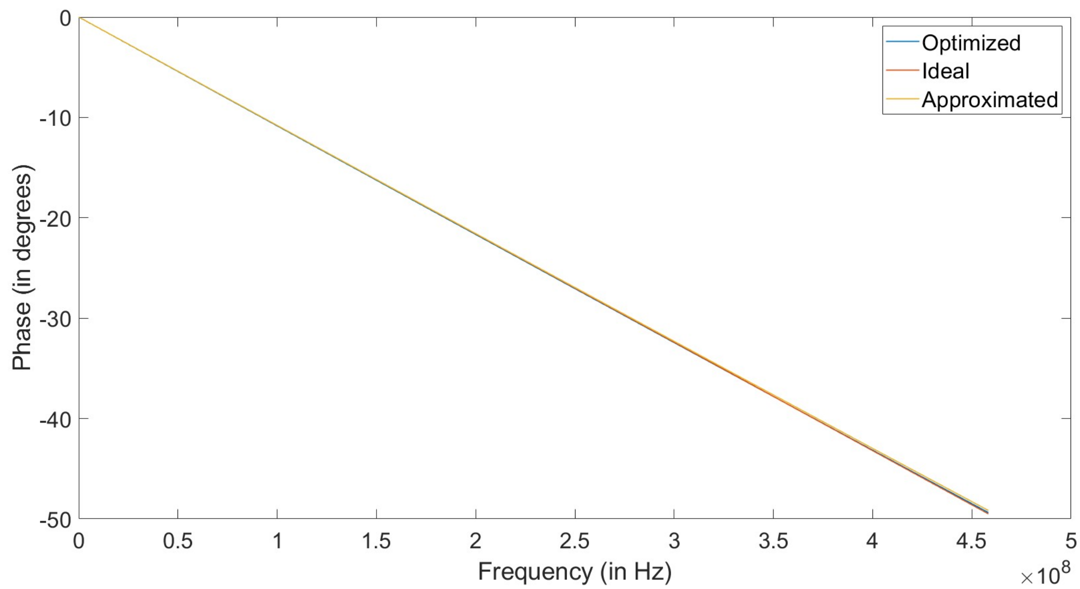
Table 4: Simulation results

R_1	R_2	R_3	C_1	C_2	C_3
67.06Ω	67.06Ω	65.88Ω	748.72 fF	748.66 fF	762.15 fF

```
optimum =

    0.3353    0.3353    0.9983    0.9982    1.0162
```

Where,

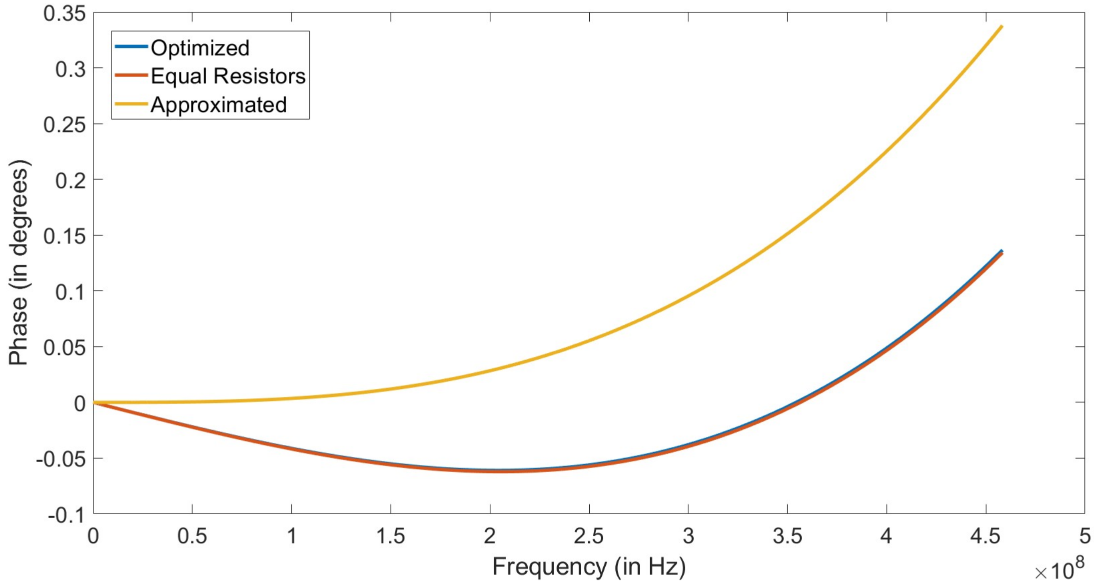
$$
\begin{aligned}
R_1 &= \frac{R}{2} * 0.3353 && (71)\\
&= 67.06\Omega && (72)\\
R_2 &= \frac{R}{2} * 0.3353 && (73)\\
&= 67.06\Omega && (74)\\
R_3 &= \frac{R}{2} * (1 - 0.3353 - 0.3353) && (75)\\
&= 65.88\Omega && (76)\\
C_1 &= C * 0.9983 && (77)\\
&= 748.72fF && (78)\\
C_2 &= C * 0.9982 && (79)\\
&= 748.66fF && (80)\\
C_3 &= C * 1.0162 && (81)\\
&= 762.15fF && (82)
\end{aligned}
$$



In the above figure, "Ideal" indicates the ideal delay function modelled using sinc and exponential.
"Approximated" is the circuit constructed using the approximation of

$$2{*}\frac{R}{2}{*}\mathrm{C} = 2{*}R_1{*}C_1 + 2{*}R_2{*}C_2 + + 2{*}R_3{*}C_3.$$

"Optimized" is the output from the program to get the best ratio of resistors and capacitors with the same total resistance.
The difference between ideal and optimized is shown below along with the difference for the optimized circuit with equal resistors and the approximated circuit.



The optimized function deviates from ideal by a maximum of **0.13** degrees over the bandwidth.
The value of the objective function obtained from the optimized circuit is **373.4307** whereas when identical resistors are used for the 3 stages, the best objective function value is **373.4315**. This is a marginal 0.0002% improvement.
This was obtained when $R_1 = R_2 = R_3 = 66.67\Omega$ and $C_1 = C_2 = C_3 = 753.15fF$.

```
ans =

  372.4307

    1.0042

  372.4315
```

4.4 Experimental results

Thus we observe that there are almost identical resistors and capacitors in the 3 stages of the lattice.

The objective function value obtained using the optimized circuit is marginally better than that with identical capacitors and resistors in the 3 stages.

The graph below consists of the difference of the 4 responses with the ideal response,
2x or 3x indicates the number of stages in the lattice.
"Approximated" indicates the circuit formed using the T_d = 2*R*C approximation.
"Optimized" is the circuit formed by the values obtained after optimization.
Clearly, we observe that the **Optimized 3x RC lattice** gives the best response.