Bounding sum of first two singular values squared for Kronecker sum of traceless matrices *

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1 Introduction

Below is a solution to Problem 5 as defined in [1]. Let us first state it in its algebraic form: Show that the sum of squares of two largest singular values is bounded by $\frac{1}{2}$ for any Kronecker (tensor) sum, $A \oplus_4 B = A \otimes I_4 + I_4 \otimes B$, where A and B denote traceless matrix of size four satisfying $\text{Tr}(A^\dagger A) + \text{Tr}(B^\dagger B) = \frac{1}{4}$.

For review of properties of Kronecker Product please see [2]

2 Bounding sum of first singular value squared for Kronecker Sum of at least one traceless matrix

Let A, B be square complex matrices of size n and I be an identity matrix of size n. Let $\otimes$ mean Kronecker Product. Let σ_X mean biggest singular value of matrix X. Let $\|\cdot\|$ mean matrix 2-norm or vector euclidean norm.

Lemma 2.1 *If either A or B are traceless then* $\|A \otimes I + I \otimes B\|^2 \leq \sigma_A^2 + \sigma_B^2$

Proof. Largest singular value of matrix X can be defined as $\sigma_X = \max_{\|v\|=1} \|Xv\|$. Now let us consider vector v. Using basic law of cosines for vectors we get:

$$\begin{aligned}
&\|(A \otimes I + I \otimes B)v\|^2 = \\
&\|(A \otimes I)v + (I \otimes B)v\|^2 = \\
&\|(A \otimes I)v\|^2 + \|(I \otimes B)v\|^2 + 2\|(A \otimes I)v\|\|(I \otimes B)v\| \cos\left((A \otimes I)v, (I \otimes B)v\right)
\end{aligned} \tag{1}$$

First, let us consider upper bound on $\|(A \otimes I)v\|^2$. This is of course $\sigma_{A \otimes I}$. By properties of Kronecker Product we have

$$\|(A \otimes I)v\| \leq \sigma_{A \otimes I} = \sigma_A \sigma_I = \sigma_A \tag{2}$$

We can perform analogous reasoning for matrix $I \otimes B$.

Next let us consider $\cos\left((A \otimes I)v, (I \otimes B)v\right)$. Recall basic fact that $\cos(A, B) = \frac{\langle A, B\rangle}{\|A\|\|B\|}$. First, we will focus on numerator. We will use cyclic property of trace and use assumption that $\|v\|^2 = 1 = vv^H$

* *Title inspired by*: **https://mathoverflow.net/questions/89880/bounding-sum-of-first-singular-values-squared-for-kronecker-sum-of-traceless-mat**

$$
\begin{aligned}
&\langle (A \otimes I)v, (I \otimes B)v \rangle = \\
&Tr([(A \otimes I)v]^H (I \otimes B)v) = \\
&Tr(v^H (A^H \otimes I^H)(I \otimes B)v) = \\
&Tr(vv^H (A^H \otimes I^H)(I \otimes B)) = \\
&Tr((A^H \otimes I)(I \otimes B)) = \\
&Tr(A^H \otimes B) = Tr(A^H)Tr(B) = Tr(A)Tr(B)
\end{aligned}
\tag{3}
$$

Of course if either A or B are traceless then numerator is equal to 0 and whole $2\|(A \otimes I)v\|\|(I \otimes B)v\| \cos((A \otimes I)v, (I \otimes B)v)$ is equal to 0. Combining that fact and equation 2 we have

$$
\|A \otimes I + I \otimes B\|^2 \leq \sigma_A^2 + \sigma_B^2 \tag{4}
$$

Which completes the proof.

3 Solution to Problem 5

Let us restate the problem in its algebraic form:

Proposition 1 *Show that the sum of squares of two largest singular values is bounded by $\frac{1}{2}$ for any Kronecker (tensor) sum, $S = A \otimes I_4 + I_4 \otimes B$, where A and B denote traceless matrix of size four satisfying Tr$(A^\dagger A) +$ Tr$(B^\dagger B) = \frac{1}{4}$.*

First we will bound the square first singular value

Proposition 2 *First largest squared singular value of S is bounded by $\frac{1}{4}$*

Proof. Leveraging Lemma 2.1 we can conclude

$$
\|A \otimes I + I \otimes B\|^2 \leq \sigma_A^2 + \sigma_B^2 \leq \mathrm{Tr}(A^\dagger A) + \mathrm{Tr}(B^\dagger B) = \frac{1}{4} \tag{5}
$$

Now, we know that the second largest singular value cannot be greater than the first one. So the square of second one can be also at most $\frac{1}{4}$. Summing both of those we can conclude that the squares of first two largest singular values are bounded by $\frac{1}{2}$.

Which proves the proposition defined by Problem 5.

References

[1] Paweł Horodecki, Łukasz Rudnicki, and Karol Życzkowski. Five open problems in quantum information, 2020.

[2] Kathrin Schacke. On the kronecker product. *Master's thesis, University of Waterloo*, 2004.