Notes on the Zeta Function

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1 Introduction

Here are some notes taken during this advanced course on the Bernoulli Numbers and the Zeta Function at the Ross Mathematics Program 2022. Lectures were given by Dr. Stefan Patrikis.

Contents

2 Positive Zeta Values

Part of the motivation behind the Bernoulli Numbers involves sums of the form

$$\sum_{i=1}^{n} i^k$$

for $k \in \mathbb{Z}^+$. These numbers show up frequently in analysis, and are useful for deriving expressions for the Riemann Zeta function, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

Definition 1

$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0$. Our first recursive definition of these numbers is:

$$(m+1)B_m = -\sum_{i=0}^{m-1} \binom{m+1}{i} B_i$$

It can be shown that this formulation has an interchangeable definition.

Theorem 1

The Bernoulli Numbers can be modelled via a generating function.

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}$$

Corollary 2

Oddly indexed Bernoulli Numbers are 0.

We now consider a Fourier Series approach to computing the Zeta function with Bernoulli Numbers. Let's first build up some theory on Fourier Analysis. Let $f : [a, b] \to \mathbb{C}$ be Riemann integrable. Set $L = b - a$, where f is L-periodic.

Definition 2

$\forall n \in \mathbb{Z}$ let the n-th Fourier coefficient be

$$\hat{f}(n) = \frac{1}{L} \int_a^b f(x) e^{-2\pi i n x / L}\, dx$$

and the Fourier series of f:

$$f(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x / L}$$

Example 3. Compute $\hat{f}(n)$ for $\cos(\alpha x), x \in [-\pi, \pi], \alpha \in \mathbb{C} \backslash \mathbb{Z}$

Solution: We know $f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{i\pi n}$, where $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$. So

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\alpha x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\alpha x}+e^{-i\alpha x}}{2}e^{-inx}\,dx$$

$$= \frac{1}{4\pi}[\int_{-\pi}^{\pi}(e^{i\alpha x}+e^{-i\alpha x})e^{-inx}\,dx]$$

Solving both integrals yields

$$\hat{f}(n) = \frac{1}{4\pi}[\frac{e^{i\pi(\alpha-n)}-e^{-i\pi(\alpha-n)}}{i(\alpha-n)} + \frac{e^{-i\pi(\alpha+n)}-e^{\pi i(\alpha+n)}}{-i(\alpha+n)}]$$

$$= \frac{1}{4\pi}[\frac{2\sin(\pi(\alpha-n))}{\alpha-n} + \frac{2\sin(\pi(\alpha+n))}{\alpha+n}]$$

Expansion into sums and differences of sin and cos leads to much cancellation and finally,

$$\hat{f}(n) = \frac{\alpha\sin(\pi\alpha)(-1)^n}{\pi(\alpha^2-n^2)}$$

Theorem 4

Let $f : \mathbb{R} \to \mathbb{C}$ be riemann integrable on any closed interval, and periodic with period L. Then the Fourier coefficients of f don't depend on which L-interval is used to compute them.

Proof. The periodicity conditions here trivialize the solution. Formally, integrating by parts will yield that the result reduces to showing

$$f(t) - f(t+L) = f(t_0) - f(t_0+L)$$

which must be true, since they are both 0 by periodicity.

□

Theorem 5

When f is twice continuously differentiable on the circle $\mathbb{R}/2\pi\mathbb{Z}$, it satisfies

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$$

Proof. We'll prove this by finding a form for the fourier coefficients, and bounding the resultant by some constant. We have

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}\,d\theta = \frac{1}{2\pi}f(\theta)\frac{-e^{-in\theta}}{in}\Big|_0^{2\pi} + \frac{1}{2\pi in}\int_0^{2\pi} f'(\theta)e^{-in\theta}\,d\theta$$

$$= \frac{1}{2\pi in} \int_0^{2\pi} f'(\theta)e^{-in\theta}\, d\theta$$

Integrating by parts again, this is equal to

$$\frac{1}{2\pi in} f'(\theta) \frac{-e^{-in\theta}}{in}\Big|_0^{2\pi} + \frac{1}{2\pi(in)^2} \int_0^{2\pi} f''(\theta)e^{-in\theta}\, d\theta$$

$$-\frac{1}{n^2} \int_0^{2\pi} f''(\theta)e^{-in\theta}\, d\theta \leq -\frac{1}{n^2} \int_0^{2\pi} |f''(\theta)|\, d\theta$$

So

$$|\hat{f}(n)| \leq \frac{1}{n^2} \int_0^{2\pi} |f''(\theta)\, d\theta$$

Bound the integral by B, which must be constant by our assumptions of continuity and differentiability. Then

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)| \leq \sum_{n\in\mathbb{Z}} \frac{1}{n^2} \int_0^{2\pi} |f''(\theta)|\, d\theta \leq \sum_{n\in\mathbb{Z}} (\frac{1}{n^2} \cdot B) = B \sum_{n\in\mathbb{Z}} \frac{1}{n^2} < \infty$$

□

Now, lets consider these Bernoulli Numbers through the lense of Fourier Analysis. We begin with Bernoulli polynomials.

Definition 3

Define the Bernoulli polynomials $B_n(x)$, $n \in \mathbb{Z}_{\geq 0}$ by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}$$

We'll begin with a simple lemma about power series.

Lemma 6

Let $f(t) = \sum_{n=0}^{\infty} a_n t^n$ and $g(t) = \sum_{n=0}^{\infty} b_n t^n$ Then

$$f(t)g(t) = \sum_{n=0}^{\infty} [(\sum_{k=0}^{n} a_{n-k} b_k) t^n]$$

Proof. A simple expansion of fg and reordering & factoring of terms willl yield the result. □

Lemma 7

$B_k(x) = \sum_{i=0}^{k} \binom{k}{i} B_i x^{k-i}$

Now, we prove a series of lemmas that will help us derive neat properties for ζ and B_m.

Proof. We have

$$\frac{te^{xt}}{e^t - 1} = \frac{t}{e^t - 1}e^{xt} = (\sum_{m=0}^{\infty} B_m \cdot \frac{t^m}{m!})(\sum_{m=0}^{\infty} \frac{(xt)^m}{m!})$$

$$= \sum_{n=0}^{\infty}(\sum_{m=0}^{n}(\frac{B_m}{m!} \cdot \frac{x^{m-n}}{(n-m)!}))t^n = \sum_{n=0}^{\infty}(\sum_{m=0}^{n} B_m \cdot x^{m-n}\binom{n}{m}\frac{1}{n!})t^n$$

$$= \sum_{n=0}^{\infty}(\sum_{m=0}^{n}\binom{n}{m}B_m x^{m-n})\frac{t^n}{n!}$$

Thus,

$$\sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!} = \sum_{n=0}^{\infty}(\sum_{m=0}^{n}\binom{n}{m}B_m x^{m-n})\frac{t^n}{n!}$$

$$\Longrightarrow B_k(x) = \sum_{i=0}^{k}\binom{k}{i}B_i x^{k-i}$$

□

Lemma 8

For $k \geq 2, B_k(0) = B_k(1)$.

Proof. When $x = 0$,

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k(0)\frac{t^k}{k!} = B_0(0) + tB_1(0) + \sum_{k=2}^{\infty} B_k(0)\frac{t^k}{k!}$$

$$1 - \frac{1}{2}t + \sum_{k=2}^{\infty} B_k(0)\frac{t^k}{k!}$$

Thus,

$$\sum_{k=2}^{\infty} B_k(0)\frac{t^k}{k!} = \frac{t}{e^t - 1} - 1 + \frac{1}{2}t = \frac{t + e^t(t-2) + 2}{2(e^t - 1)}$$

When $x = 1$,

$$\frac{te^t}{e^t - 1} = \sum_{k=0}^{\infty} B_k(1)\frac{t^k}{k!} = B_0(1) + tB_1(1) + \sum_{k=2}^{\infty} B_k(0)\frac{t^k}{k!}$$

$$1 + \frac{1}{2}t + \sum_{k=2}^{\infty} B_k(0)\frac{t^k}{k!}$$

Thus,

$$\sum_{k=2}^{\infty} B_k(0)\frac{t^k}{k!} = \frac{te^t}{e^t - 1} - 1 - \frac{1}{2}t = \frac{t + e^t(t-2) + 2}{2(e^t - 1)}$$

It follows that
$$\sum_{k=2}^{\infty} B_k(0)\frac{t^k}{k!} = \sum_{k=2}^{\infty} B_k(1)\frac{t^k}{k!} \implies B_k(0) = B_k(1)$$

□

Lemma 9
For $k \geq 1, B_k'(x) = kB_{k-1}(x)$, and $\int_0^1 B_k(x)\,dx = 0$

Proof. We have

$$B_k(x) = \sum_{i=0}^{k} \binom{k}{i} x^{k-i} B_i \implies B_k'(x) = \sum_{i=0}^{k-1} (k-i)\binom{k}{i} x^{k-i-1} B_i$$

$$= \sum_{i=0}^{k-1} (k-i)(\frac{k!}{(k-i)!i!}) x^{k-i-1} B_i = k\sum_{i=0}^{k-1} \frac{(k-1)!}{(k-i-1)!i!} x^{k-i-1} B_i$$

$$= k\sum_{i=0}^{k-1} \binom{k-1}{i} x^{k-i-1} B_i = kB_{k-1}(x)$$

Now we compute the integral:

$$\int_0^1 B_k(x)\,dx = \int_0^1 \sum_{i=0}^{k} \binom{k}{i} x^{k-i} B_i\,dx = \sum_{i=0}^{k} \binom{k}{i} B_i \int_0^1 x^{k-i}\,dx = \sum_{i=0}^{k} \binom{k}{i} B_i \frac{x^{k-i+1}}{k-i+1}\bigg|_0^1$$

Continued algebraic manipulations yield

$$= \frac{1}{k-i+1} \sum_{i=0}^{k} \binom{k}{i} B_i = \frac{1}{k-i+1} \sum_{i=0}^{k} \frac{k!}{(k-i)!i!} B_i$$

$$= \frac{1}{k+1}(\sum_{i=0}^{k-1} \binom{k+1}{i} B_i + (k+1)B_k) = \frac{1}{k+1}(\sum_{i=0}^{k-1} \binom{k+1}{i} B_i - \sum_{i=0}^{k-1} \binom{k+1}{i} B_i) = 0$$

□

We'll now consider $B_k(x)$ as a function on $[0,1]$, and compute its Fourier coefficients. We'll let $B_k(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}$. We also have the standard definition: $B_k(x) = \sum_{i=0}^{k} \binom{k}{i} x^{k-i} B_i$ Computing the fourier coefficients:

$$\hat{f}(n) = \int_0^1 B_k(x)e^{-2\pi inx}\,dx = -\frac{1}{2\pi in} B_k(x)e^{-2\pi inx}\bigg|_0^1 + \frac{1}{2\pi in}\int_0^1 B_k'(x)e^{-2\pi inx}\,dx$$

$$= -\frac{1}{2\pi in}\left(B_k(0)e^{-2\pi in} - B_k(1)\right) + \frac{1}{2\pi in}\int_0^1 kB_{k-1}(x)e^{-2\pi inx}\,dx = \frac{k}{2\pi in}\int_0^1 B_{k-1}(x)e^{-2\pi inx}\,dx$$

For the case of $k = 1$, we know $B_k(x) = x - \frac{1}{2}$. This suggests the following formula:

$$\hat{f}_k(n) = -(\frac{1}{2\pi in})^k \cdot (k!)$$

This is easy to check by induction.

Now, we arrive at the grand result, a new expression for even zeta values:

Theorem 10

For all $m \in \mathbb{Z}^+$,

$$\zeta(2m) = (-1)^m \pi^{2m} \frac{2^{2m-1}}{(2m-1)!}(-\frac{B_{2m}}{2m})$$

Proof. We'll keep in mind the fourier results on $B_k(x)$. When k is even, we see $\hat{f}_k(n) = \hat{f}_k(-n)$. Thus, when k is even, we get

$$B_k(x) = \sum_{n\in\mathbb{Z}} \hat{f}_k(n)e^{2\pi inx} = \sum_{n\in\mathbb{Z}^+} (\hat{f}_k(n)e^{2\pi inx} + \hat{f}_k(-n)e^{-2\pi inx}) = \sum_{n\in\mathbb{Z}^+} (\hat{f}_k(n)e^{2\pi inx} + \hat{f}_k(-n)e^{-2\pi inx})$$

$$\sum_{n\in\mathbb{Z}^+} \hat{f}_k(n)(e^{2\pi inx} + e^{-2\pi inx}) = \sum_{n\in\mathbb{Z}^+} \hat{f}_k(n)(2\cos(2\pi inx))$$

$$= \sum_{n\in\mathbb{Z}^+} -(\frac{1}{2\pi in})^k \cdot k! \cdot 2\cos(2\pi nx) = -(\frac{1}{2\pi i})^k \cdot k! \sum_{n\in\mathbb{Z}^+} \frac{1}{n^k}\cos(2\pi nx)$$

We also have $B_k(0) = (-\frac{1}{2\pi i})^k \cdot k! \sum_{n=1}^{\infty} \frac{1}{n^k}$. So

$$\sum_{k=1}^{\infty} \frac{1}{n^k} = B_k(0) \cdot \frac{(-2\pi i)^k}{k!}$$

Recall k is even. So let $k = 2m$ for positive m. Then we can reformulate:

$$\zeta(2m) = B_{2m}(0) \cdot \frac{(-2\pi i)^{2m}}{(2m)!} = (-1)^m \pi^{2m} \frac{2^{2m-1}}{(2m-1)!}(-\frac{B_{2m}}{2m})$$

□

3 Analytic Continuation of Zeta

Now, we have an explicit formula for $\zeta(2m)$ in terms of Bernoulli Numbers. Let's extend $\zeta(s)$ to a Holomorphic Function on $\mathbb{C}\backslash\{1\}$ (as opposed to $\Re(s) > 1$), to find odd negative values entries of ζ.

Definition 4

We define the gamma Function

$$\Gamma(z) = \int_0^{\infty} x^{z-1}e^{-x}\,dx, \quad \Re(z) > 0$$

Remark 11. For $n \in \mathbb{Z}^+, \Gamma(n) = (n-1)!$

A similar integration by parts argument yields $\Gamma(z+1) = z\Gamma(z)$ for $\Re(z) > 0$. This suggests a way for us to analytically continue Γ over $\mathbb{C}$. Define

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}, \quad \Re(s) > -1$$

and iterate to extend over $\Re(s) > -2, -3, ...$, until Γ is defined over $\mathbb{C}\backslash\mathbb{Z}_{\leq 0}$

Claim 12. Γ is holomorphic over $\mathbb{C}\backslash\mathbb{Z}_{\leq 0}$

Definition 5

Let

$$\xi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$$

For $\Re(s) > 1$,

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})n^{-s} = \int_0^\infty e^{-t}t^{\frac{s}{2}}\frac{dt}{t}n^{-s}\pi^{-\frac{s}{2}} = \int_0^\infty e^{-t}(\frac{t}{\pi n^2})^{\frac{s}{2}}\frac{dt}{t}.$$

Let $y = \frac{t}{\pi n^2}$. Then

$$= \int_0^\infty e^{-\pi n^2 y}y^{\frac{s}{2}}\frac{dy}{y}$$

When we sum $\sum_{n=1}^\infty$, it follows that

$$\xi(s) = \sum_{n=1}^\infty \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})n^{-s} = \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 y}y^{\frac{s}{2}}\frac{dy}{y} = \int_0^\infty \sum_{n=1}^\infty e^{-\pi n^2 y}y^{\frac{s}{2}}\frac{dy}{y}$$

Definition 6

We define a new function

$$\Theta(y) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 y} = 1 + 2\sum_{n=1}^\infty e^{-\pi n^2 y}$$

Theorem 13

Let Θ be as defined above. Then

$$\Theta(y) = \frac{1}{\sqrt{y}}\Theta(1/y)$$

Proof. A corollary of Poisson's Summation gives $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$. Using this, we can use Fourier analysis to deduce the identity. Details are omitted, as the identity for the generalized Θ function is proved later. $\square$

This formulation will come incredibly handy for computing $\zeta(1-2m)$. Let's see these consequences. We can write

$$\xi(s)=\int_0^1 (\frac{\Theta(y)-1}{2})y^{\frac{s}{2}}\frac{dy}{y}+\int_1^\infty (\frac{\Theta(y)-1}{2})y^{\frac{s}{2}}\frac{dy}{y}.$$

We have

$$\int_0^1 (\frac{\Theta(y)-1}{2})y^{\frac{s}{2}}\frac{dy}{y}=-\frac{1}{2}\int_0^1 y^{\frac{s}{2}}\frac{dy}{y}+\frac{1}{2}\int_0^1 \Theta(y)y^{\frac{5}{2}}\frac{dy}{y}$$

$$=-\frac{1}{2}\frac{y^{\frac{s}{2}}}{s/2}\bigg|_0^1+\frac{1}{2}\int_1^\infty \Theta(1/x)x^{-\frac{s}{2}}\frac{dx}{x}=-\frac{1}{s}+\frac{1}{2}\int_1^\infty \Theta(x)x^{\frac{1-s}{2}}\frac{dx}{x}$$

$$=-\frac{1}{s}+\frac{1}{2}\int_1^\infty \Theta(x)x^{\frac{1-s}{2}}\frac{dx}{x}=-\frac{1}{s}+\int_1^\infty \frac{\Theta(x)-1}{2}x^{\frac{1-s}{2}}\frac{dx}{x}+\frac{1}{2}\int_1^\infty x^{\frac{1-s}{2}}\frac{dx}{x}$$

$$=-\frac{1}{s}-\frac{1}{1-s}+\int_1^\infty \frac{\Theta(x)-1}{2}x^{\frac{1-s}{2}}\frac{dx}{x}$$

Now, we have

$$\xi(s)=-\frac{1}{s}-\frac{1}{1-s}+\int_1^\infty \frac{\Theta(x)-1}{2}x^{\frac{1-s}{2}}\frac{dx}{x}+\int_1^\infty (\frac{\Theta(x)-1}{2})x^{\frac{s}{2}}\frac{dx}{x}.$$

These integrals converge for any $s\in\mathbb{C}$, to holomorphic functions. The rapid decrease of $\Theta(x)$ by exponentials brings down $x^{s/2}$. What's the punchline?

We have now extended ξ from $\Re(s)>1$ to $\mathbb{C}\backslash\{0,1\}$. Moreover, $\forall s\in\mathbb{C}\backslash\{0,1\}$, our extension shows $\xi(s)=\xi(1-s)$. Finally, ξ is holomorphicon $\mathbb{C}\backslash\{0,1\}$.

4 Negative values of Zeta

Now we've built up some machinery to find negative entries for ζ. Consider $\zeta(s)=\frac{\xi(s)}{\pi^{-\frac{s}{2}}\Gamma(s/2)}$. Then

Theorem 14

$\zeta(s)$ extends to a holomorphic function on $\mathbb{C}\backslash\{1\}$ with $\zeta(-2m)=0,\forall m\in\mathbb{Z}^+$, and

$$\zeta(1-2m)=-\frac{B_{2m}}{2m},\forall m\in\mathbb{Z}^+$$

and $\zeta(0)=-\frac{1}{2}$.

Proof. We first outline some properties that we need for Γ.

(i) $\Gamma(s)$ is never 0. This follows from

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin(\pi-s)},\quad s\in\mathbb{C}\backslash\mathbb{Z}_{\leq 0}$$

Then $\frac{1}{\Gamma(s)} = \frac{\sin(\pi s)\cdot\Gamma(1-s)}{\pi}$ remains bounded.

(ii)

$$\zeta(1-s) = \frac{\zeta(1-s)}{\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})} = \frac{\zeta(s)}{\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})} \cdot \frac{\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})}{\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})} = \zeta(s)\cdot\pi^{-s+\frac{1}{2}}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}$$

(iii) $\Gamma(s)\cdot\Gamma(s+\frac{1}{2}) = \frac{2\sqrt{\pi}}{2^s}\Gamma(2s)$

These properties yield:

$$\Gamma(\frac{1-s}{2})\Gamma(1-\frac{1-s}{2}) = \frac{\pi}{\sin(\pi(\frac{1-s}{2}))} = \frac{\pi}{\frac{\cos(\pi s)}{2}} \tag{1}$$

and

$$\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}) = \frac{2\sqrt{\pi}}{2^{\frac{s}{2}}}\Gamma(s) \tag{2}$$

Combining these gives

$$\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})} = \frac{2\sqrt{\pi}}{2^{\frac{s}{2}}}\Gamma(s)\cdot\frac{\cos(\frac{\pi s}{2})}{\pi}$$

Thus, $\zeta(1-s) = \zeta(s)\pi^{-s}\cdot 2^{-\frac{s}{2}+1}\Gamma(s)\cos(\pi s/2)$ Finally,

$$\zeta(1-2m) = \zeta(2m)\pi^{-2m}\cdot 2^{-2m+1}\Gamma(2m)\cos(\pi m)$$

$$= (-1)^m\pi^{2m}\frac{2^{2m-1}}{(2m-1)!}(\frac{-B_{2m}}{2m})\pi^{-2m}\cdot 2^{-2m+1}(2m-1)!(-1)^m$$

Cancelling yields:

$$\zeta(1-2m) = \frac{-B_{2m}}{2m}$$

□

5 Analogues to L-functions

Now, we'll consider generalizations of the Zeta function, namely, L-functions. We define Dirichlet Characters:

Definition 7
Define a character

$$\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to S^1 = \{z\in\mathbb{C} \big| \, |z| = 1\}$$

where χ is a group homomorphism ($\chi(ab) = \chi(a)\chi(b) \;\; \forall a,b\in(\mathbb{Z}/m\mathbb{Z})^\times$).

We can extend χ to a function on $\mathbb{Z}$ by

$$\chi : \mathbb{Z}\to\mathbb{C}$$

$$a \mapsto \begin{cases} \chi(a \bmod m) & (a,m) = 1 \\ 0 & (a,m) > 1 \end{cases}$$

We'll note that we can see χ having period m. A primitive character mod m means that there is no proper divisor $d|m$ st there exists χ_d st $\chi = \chi_d \circ$(reduction $\pmod d$ on $(\mathbb{Z}/m\mathbb{Z})^\times$)

The Dirichlet L-function $L(\chi, s)$ is defined by $L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$

Our goal is to generalize our results on the Zeta Function to Dirichlet L-functions.

Remark 15. A similar Euler-Product form can be derived here: For $\Re(s) > 1$ and $\chi \pmod m$.

$$\prod_p (1 - \chi(p)p^{-s})^{-1}$$

Definition 8

Define the Gauss sum

$$G(\chi) = \sum_{k=0}^{m-1} \chi(k) e^{2\pi i k/m}$$

This sum is quite important for calculations going forward. We thus establish a crucial identity on the Gauss sum that will be useful later, namely that $|G(\chi, 1)|^2 = m$. We'll go through a line of lemmas before reaching this identity.

Lemma 16

Let χ be primitive $\pmod m$. Then for any divisor $m_1|m$, there exists c such that $(c, m) = 1, c \equiv 1 \pmod{m_1}$ with $\chi(c) \neq 1$.

In fact, this can be thought of as an equivalent formulation of χ being primitive. This helps us show:

Lemma 17

If χ is primitive, then $G(\chi, a) = \chi(a)G(\chi, 1)$ for every $a \in \mathbb{Z}/m\mathbb{Z}$

Proof. We separate into cases.

Case 1: $(a, m) = 1$

We note that as k runs through $\mathbb{Z}/m\mathbb{Z}$, ak runs through $\mathbb{Z}/m\mathbb{Z}$. Let $y = ak$. Then $\chi(y) = \chi(a)\chi(k) \implies \chi(k) = \overline{\chi(a)}\chi(y)$ So

$$G(\chi, a) = \sum_{k \in \mathbb{Z}/m\mathbb{Z}} \chi(k) e^{2\pi i a k/m} = \sum_{k \in \mathbb{Z}/m\mathbb{Z}} \overline{\chi(a)}\chi(y) e^{2\pi i y/m}$$

$$= \overline{\chi(a)}G(\chi, 1)$$

Case 2: $(a, m) > 1$

It is enough to show $(\chi, a) = 0$. Suppose $m_1|m$. Let $d = (a, m)$, $a_1 = \frac{a}{d}$, $m_1 = \frac{m}{d}$. Choose c that satisfies the previous lemma. Then

$$\chi(c)G(\chi, a) = \sum_{k \in \mathbb{Z}/m\mathbb{Z}} \chi(ck)e^{2\pi iak/m}$$

Then these $y = ck$ hit all of $\mathbb{Z}/m\mathbb{Z}$. Recall $c \equiv 1 \pmod{m_1}$. Then

$$e^{2\pi iak/m} = e^{2\pi ika/m} = e^{2\pi ika_1/m_1} = e^{2\pi ikca_1/m_1} = e^{2\pi iya/m}$$

So $\chi(c)G(\chi, a) = \sum_{y \in \mathbb{Z}/m\mathbb{Z}} \chi(y)e^{2\pi iya/m} = G(\chi, a)$. But $\chi(c) \neq 1$. So

$$\chi(c)G(\chi, a) = G(\chi, a) \implies G(\chi, a) = 0$$

and we are done. □

Finally, we arrive at this well-known result:

Theorem 18

Let χ be a primitive character $\pmod{m}$, Then $|G(\chi, 1)|^2 = m$

Proof. This can be done with simple computations:

$$|G(\chi, 1)|^2 = \overline{G(\chi, 1)}G(\chi, 1) = \sum_{k \in \mathbb{Z}/m\mathbb{Z}} \overline{\chi(k)}e^{-2\pi ia/m}G(\chi, 1)$$

By the previous lemma,

$$= \sum_{k \in \mathbb{Z}/m\mathbb{Z}} e^{-2\pi ik/m}G(\chi, k) = \sum_{k \in \mathbb{Z}/m\mathbb{Z}} e^{-2\pi ik/m}\Big(\sum_{j \in \mathbb{Z}/m\mathbb{Z}} \chi(j)e^{2\pi ikj/m}\Big)$$

$$= \sum_{k \in \mathbb{Z}/m\mathbb{Z}} \Big(\sum_{j \in \mathbb{Z}/m\mathbb{Z}} e^{-2\pi ik/m}\chi(j)e^{2\pi ikj/m}\Big) = \sum_{k \in \mathbb{Z}/m\mathbb{Z}} \sum_{j \in \mathbb{Z}/m\mathbb{Z}} \chi(j)e^{2\pi ik(j-1)/m}$$

$$= \sum_{j \in \mathbb{Z}/m\mathbb{Z}} \chi(j) \sum_{k \in \mathbb{Z}/m\mathbb{Z}} e^{2\pi ik(j-1)/m}$$

Now, $\sum_{k \in \mathbb{Z}/m\mathbb{Z}} e^{2\pi ik(j-1)/m}$ is the sum over a geometric series. When $j = 1$, the sum goes to m; otherwise, the sum goes to 0. Thus we have

$$|G(\chi, 1)|^2 = \sum_{j \in \mathbb{Z}/m\mathbb{Z}} \chi(j) \sum_{k \in \mathbb{Z}/m\mathbb{Z}} e^{2\pi ik(j-1)/m} = \chi(1)m = m$$

□

Definition 9

We define an analogue for $\xi(s)$:

$$\Lambda(\chi, s) = (\frac{m}{\pi})^{\frac{s}{2}}\Gamma(\frac{s+p}{2})L(\chi, s)$$

where $p \in \{0, 1\}$ and $\chi(-1) = (-1)^p$.

We want to extend some of the complex properties to our analogue. Let's build up some theory before we arrive at a holomorphicity theorem for Λ

We have

$$\Gamma(\frac{s+p}{2}) = \int_0^\infty e^{-y}y^{\frac{s+p}{2}}\frac{dy}{y} = \int_0^\infty e^{-\pi n^2 t/m}(\frac{\pi n^2 t}{m})^{\frac{s+p}{2}}\frac{dt}{t}$$

Thus,

$$(\frac{m}{\pi})^{\frac{s+p}{2}}\Gamma(\frac{s+p}{2}\chi(n)n^{-s}) = \chi(n)\int_0^\infty e^{-\pi n^2 t/m}n^p t^{\frac{s+p}{2}}\frac{dt}{t}$$

As before, we can take the sum for $n = 1, \ldots, \infty$ to obtain:

$$(\frac{m}{\pi})^{p/2} \cdot \Lambda(\chi, s) = \sum_{n=1}^{\infty}\chi(n)\int_0^\infty e^{-\pi n^2 t/m}n^p t^{\frac{s+p}{2}}\frac{dt}{t}$$

$$= \int_0^\infty (\sum_{n=1}^{\infty}\chi(n)n^p e^{-\pi n^2 t/m})t^{\frac{s+p}{2}}\frac{dt}{t}$$

Now we define some generalization of the Θ function, which will be crucial as we move forward

Definition 10

Set

$$\Theta(\chi, y) = \sum_{n\in\mathbb{Z}}\chi(n)n^p e^{-\pi n^2 y/m}$$

Thus,

$$(\frac{m}{\pi})^{p/2} \cdot \Lambda(\chi, s) = \int_0^\infty \frac{\Theta(\chi, t)}{2}t^{\frac{s+p}{2}}\frac{dt}{t}$$

Theorem 19

Again, we have a neat identity for Θ:

$$\Theta(\chi, 1/y) = \frac{G(\chi)}{i^p\sqrt{m}}y^{p+\frac{1}{2}}\Theta(\overline{\chi}, y)$$

We now present a holomorphicity theorem for Λ

Theorem 20

Let χ be a non-trivial primitive Dirichlet Character. Then $\Lambda(\chi, s)$ extends to a holomorphic function on all of $\mathbb{C}$ This extension satisfies the functional equation

$$\Lambda(\chi, s) = \frac{G(\chi)}{i^p\sqrt{m}}\Lambda(\overline{\chi}, 1-s)$$

Proof. We have

$$(\frac{m}{\pi})^{p/2} \cdot \Lambda(\chi, s) = \int_0^1 + \int_1^\infty = \int_1^\infty \frac{\Theta(\chi, 1/y)}{2} y^{-(\frac{s+p}{2})}\frac{dy}{y} + \int_1^\infty \frac{\Theta(\chi, t)}{2} t^{\frac{s+p}{2}}\frac{dt}{t}$$
$$= \frac{G(\chi)}{i^p\sqrt{m}}\int_1^\infty \frac{\Theta(\overline{\chi}, y)}{2} y^{p+\frac{1}{2}-(\frac{s+p}{2})}\frac{dy}{y} + \int_1^\infty \frac{\Theta(\chi, t)}{2} t^{\frac{s+p}{2}}\frac{dt}{t}$$

which is holomorphic on all of $\mathbb{C}$. Multiplying by $\frac{i^p\sqrt{m}}{G(\chi)}$, we get

$$(\frac{m}{\pi})^{p/2}\frac{i^p\sqrt{m}}{G(\chi)} \cdot \Lambda(\chi, s) = \int_1^\infty \frac{\Theta(\overline{\chi}, y)}{2} y^{\frac{1-s+p}{2}}\frac{dy}{y} + \frac{i^p\sqrt{m}}{G(\chi)}\int_1^\infty \frac{\Theta(\chi, t)}{2} t^{\frac{s+p}{2}}\frac{dt}{t}$$

On the other hand, we can take

$$(\frac{m}{\pi})^{p/2}\Lambda(\overline{\chi}, 1-s) = \int_0^\infty \frac{\Theta(\overline{\chi}, t)}{2} t^{\frac{1-s+p}{2}}\frac{dt}{t} = \int_1^\infty \frac{\Theta(\overline{\chi}, t)}{2} t^{\frac{1-s+p}{2}}\frac{dt}{t} + \int_1^\infty \frac{\Theta(\overline{\chi}, \frac{1}{y})}{2} y^{-\frac{1-s+p}{2}}\frac{dy}{y}$$
$$= \int_1^\infty \frac{\Theta(\overline{\chi}, t)}{2} t^{\frac{1-s+p}{2}}\frac{dt}{t} + \frac{G(\overline{\chi})}{i^p\sqrt{m}}\int_1^\infty \frac{\Theta(\chi, y)}{2} y^{p+\frac{1}{2}-(\frac{1-s+p}{2})}\frac{dy}{y}$$

We notice that the $(\frac{m}{\pi})^{p/2}$ cancel; equating terms, we get $\frac{i^p\sqrt{n}}{G(\chi)}\Lambda(\chi, s)$ agrees with $\Lambda(\overline{\chi}, 1-s)$ as long as $\frac{i^p\sqrt{m}}{G(\chi)} = \frac{G(\overline{\chi})}{i^p\sqrt{m}}$. If we can show this identity, we are done.

Recall $|G(\chi)| = \sqrt{m} \implies G(\chi)\overline{G(\chi)} = m$. Using this, note

$$\overline{G(\chi)} = \sum_{k=0}^{m-1}\overline{\chi}(k)e^{-2\pi ik/m} = \chi(-1)\sum_{k=0}^{m-1}\overline{\chi}(k)e^{2\pi ik/m}$$
$$= \chi(-1)G(\overline{\chi}) = (-1)^p G(\overline{\chi})$$

Thus

$$\frac{i^p\sqrt{m}}{G(\chi)} = \frac{i^p\sqrt{m}\cdot\overline{G(\chi)}}{G(\chi)\cdot\overline{G(\chi)}} = \frac{G(\overline{\chi})}{i^p\sqrt{m}}$$

and we're done. □

6 Special Values of L-functions

We'll use these established holomorphicity theorems for our functional equation to precisely state some results for L-functions.

Definition 11

The generalized Bernoulli Numbers $B_{n,\chi}, n \in \mathbb{Z}_{\geq 0}$ are defined by

$$\sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!} = \sum_{a=1}^{n} \chi(a) \frac{te^{at}}{e^{mt}-1} = F_x(t)$$

Let $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet Character extended by 0 to $\chi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{C}$. Then χ has a Fourier transform

$$\hat{\chi}(a) = \frac{1}{m} \sum_{k=0}^{m-1} \chi(k) e^{-2\pi iak/m}$$

and we have Fourier Inversion Formula:

$$\chi(k) = \sum_{a\in\mathbb{Z}/m\mathbb{Z}} \hat{\chi(k)} e^{2\pi iak/m} = \frac{1}{m}\overline{\chi}(-a)G(\chi)$$

Again, we apply Fourier Inversion on $\chi(k)$ for primitive χ

$$\chi(k) = \sum_{a\in\mathbb{Z}/m\mathbb{Z}} \frac{1}{m}\overline{\chi}(-a)G(\chi)e^{2\pi iak/m} = \frac{G(\chi)\chi(-1)}{m} \sum_{a\in\mathbb{Z}/m} \hat{\chi}(a)e^{2\pi iak/m}$$

$$= \sum_{a\in\mathbb{Z}/m} \frac{1}{m}\overline{\chi}(-a)G(\chi)e^{2\pi iak/m} = \frac{G(\chi)\chi(-1)}{m} \sum_{a\in\mathbb{Z}/m} \overline{\chi}(a)e^{2\pi iak/m}$$

Thus, for $\Re(s) > 1$,

$$\sum_{k=1}^{\infty} \chi(k)k^{-s} = \frac{G(\chi)\chi(-1)}{m} \sum_{k=1}^{\infty} \sum_{a\in\mathbb{Z}/m} \overline{\chi}(a)e^{2\pi iak/m}k^{-s}$$

Now, set $A_k(x) = \sum_{n\in\mathbb{Z}\backslash 0} \frac{e^{2\pi inx}}{n^k}$

Claim 21. For $k \geq 2$, $\forall x \in [0,1]$, $A_k(x) = \frac{-(2\pi i)^k}{k!} B_k(x)$

Then, plugging in $B_n(x)$, we get $L(\chi, k) = \frac{\chi(-1)G(\chi)}{2m} \sum_{a=1}^{m} \overline{\chi}(a) \frac{-(2\pi i)^k}{k!} B_k(\frac{a}{m})$. Applying the functional equation with Λ, we get

$$L(\chi, 1-k) = \frac{\Lambda(\chi, 1-k)}{(\frac{m}{\pi})^{\frac{1-k}{2}}\Gamma(\frac{1-k+p}{2})} = \Lambda(\overline{\chi}, k) \cdot \frac{G(\chi)}{i^p\sqrt{m}} \cdot \frac{1}{(\frac{m}{\pi})^{\frac{1-k}{2}}\Gamma(\frac{1-k+p}{2})}$$

$$= L(\overline{\chi}, k) \cdot \frac{G(\chi)}{i^p\sqrt{m}} \cdot \frac{(\frac{m}{\pi})^{\frac{k}{2}}\Gamma(\frac{k+p}{2})}{(\frac{m}{\pi})^{\frac{1-k}{2}}\Gamma(\frac{1-k+p}{2})}$$

$$= (\frac{\chi(-1)G(\overline{\chi})(2\pi i)^k}{2mk!} \sum_{i=1}^{m} \chi(j)B_k(\frac{j}{m}))\frac{G(\chi)}{i^p\sqrt{m}} \cdot (\frac{m}{\pi})^{k-\frac{1}{2}} \cdot \frac{\Gamma(\frac{k+p}{2})}{\Gamma(\frac{1-k+p}{2})}$$

Claim 22. If $p = 0$, $\frac{\Gamma(\frac{k+p}{2})}{\Gamma(\frac{1-k+p}{2})} = \frac{2(k-1)!}{\sqrt{\pi}2^k}(-1)^{\frac{k}{2}}$. If $p = 1$, $\frac{\Gamma(\frac{k+p}{2})}{\Gamma(\frac{1-k+p}{2})} = \frac{2\Gamma(k-1)}{\sqrt{\pi}2^{k-1}}(-1)^{\frac{k-1}{2}}$

We make the following observations of $L(\chi, 1-k)$: all powers of π and 2 cancel. After this simplification, we're left with

$$L(\chi, 1-k) = (-(-1)^p i^{k-p}(-1)^{\frac{k-p}{2}}(-1)^p m) \cdot \frac{m^{k-\frac{1}{2}}}{m\sqrt{m}} \cdot \frac{(k-1)!}{k!} \cdot \sum_{j=1}^{m} \chi(j) B_k(\frac{j}{m})$$
$$= -\frac{1}{k} \cdot m^{k-1} \sum_{j=1}^{m} \chi(j) B_k(\frac{j}{m})$$

Claim 23. $B_{k,\chi} = m^{k-1} \sum_{j=1}^{m} \chi(j) B_k(\frac{j}{m})$

Proof. This can be proven easily via simple expansion of the generating function definition. We have

$$\sum_{n=0}^{\infty}[m^{n-1} \sum_{j=1}^{m} \chi(j) B_n(\frac{j}{m})] \cdot \frac{t^n}{n!} = \sum_{j=1}^{m} \frac{\chi(j)}{m} \sum_{n=0}^{\infty} B_n(\frac{j}{m}) \frac{(mt)^n}{n!}$$
$$= \sum_{j=1}^{m} \frac{\chi(j)}{m} \frac{mt}{e^{mt}-1} e^{j/m \cdot mt} = \sum_{j=1}^{m} \chi(j) \frac{te^{it}}{e^{mt}-1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}$$

∎

We can conclude that

$$L(\chi, 1-k) = -\frac{B_{n,\chi}}{k}$$

7 P-adic Properties & Interpolation of L-values

Recall:

$$\zeta(1-n) = -\frac{B_n}{n}, \ \ \forall n \in \mathbb{Z}_{>1}$$

For χ primitive Dirichlet Character

$$L(\chi, 1-n) = -\frac{B_{n,\chi}}{n}, \ \ \forall n \in \mathbb{Z}_{\geq 1}$$

Now we turn to the following question: What can we say about divisibility properties of these (almost) rational numbers?

- What p divides the numerator
- What p divides the denominator
- Is there any congruence relation between these special values.

Theorem 24

Let p be prime, $n \in \mathbb{Z}_{\geq 1}$.

1. If $n \not\equiv 0 \pmod{p-1}$, then $-\frac{B_n}{n} = \zeta(1-n)$
$$\in \mathbb{Z}_{(p)} = \{a \in \mathbb{Q} |\ a = \frac{b}{c};\ b, c \in \mathbb{Z}\ \&\ (b,c) = 1,\ \text{with } p \nmid c\}$$
2. If $n \equiv 0 \pmod{p-1}$, then $p \cdot B_n \equiv -1 \pmod{p}$. ie (1)+(2) $\implies$ $B_n \in \sum_{(p-1)|n} -\frac{1}{p} + \mathbb{Z}$
3. Kummer congruences: Let $n \equiv n' \pmod{p-1}$ and are $\not\equiv 0 \pmod{p-1}$. Then
$$-\frac{B_n}{n} = \zeta(1-n) \equiv \zeta(1-n') = -\frac{B_{n'}}{n'} \pmod{p\mathbb{Z}_{(p)}}$$
More generally, for $n \equiv n' \pmod{(p-1)p^n}$ & $\not\equiv 0 \pmod{p-1}$,
$$(1-p^{n-1})\zeta(1-n) \equiv (1-p^{n'-1})\zeta(1-n') \pmod{p^{n+1}\mathbb{Z}_{(p)}}$$
4. Kummer criterion: The following are equivalent:
   - $\forall$ even m, $2 \leq m \leq p-3$, $p \nmid \zeta(1-m)$ in $\mathbb{Z}_{(p)}$
   - $p \nmid \#C\ell(\mathbb{Z}[\zeta_p])$ where $\mathbb{Z}[\zeta_p]$ is the ring of algebraic integers in $\mathbb{Q}[\zeta_p]$ and $C\ell(\cdot)$ is the class group

Example 25. $p = 691| - \frac{B_{12}}{12} = \zeta(-11)$, so by Kummer, $691|\#C\ell(\mathbb{Z}[\zeta_{691}])$

Kummer criterion is proven using analytic class cumber formula for the number field $K = \mathbb{Z}[\zeta_p]$. For any number field K, the Dedekind zeta function of K is

$$\zeta_K(s) = \prod_{\substack{\text{prime ideals p} \\ \text{of \{alg. int of K\}}}} (1 - N(p)^{-s})^{-1}.$$

Then ζ_k extends to a holomorphic function on $\mathbb{C}\backslash\{1\}$ with $(s-1)\cdot\zeta_k(s)$ holomorphic on $\mathbb{C}$; its value at $s = 1$ is an interesting number: product of various interesting numbers.

A much deeper theorem (Herbrand-Ribot Theorem) produces elements of the class group.

We want to move towards a p-adic version of the L-functions. We'll first define and build some theory for the p-adics.

Fix a prime p. Say $p = 3$. Put integers into mod 3 boxes. Within these boxes, mod out by 9, then by 27, etc. The 3-adics will be a number system containing the integers in which two integers are close if they are in many common boxes. More precisely,

Definition 12

Fix prime p. Let $a \in \mathbb{Q}$ Write $a = p^r \cdot \frac{u}{v}, r \in \mathbb{Z}, u, v \in \mathbb{Z} : p \nmid (u \cdot v)$. Define the p-adic valuation of a to be $v_p(a) = r$. So $v_p : \mathbb{Q} \backslash \{0\} \to \mathbb{Z}$ and we set $v_p(0) = \infty$.

We note that a has a unique p-adic representation in this form.

Lemma 26

1. $\forall a, b \in \mathbb{Q}$, $v_p(ab) = v_p(a) + v_p(b)$
2. $v_p(a+b) \geq \min\left(v_p(a), v_p(b)\right)$

Proof. Let $a = \frac{a_1}{a_2}$ and $b = \frac{b_1}{b_2}$. We first prove (1).

We can represent

$$ab = \frac{a_1}{a_2} \cdot \frac{b_1}{b_2} = p^{r_1} \cdot \frac{u_1}{v_1}$$

Then $v_p(ab) = r_1$. Similarly, write $a = p^{r_2} \cdot \frac{u_2}{v_2}$ and $b = p^{r_3} \cdot \frac{u_3}{v_3}$. Then

$$p^{r_1} \cdot \frac{u_1}{v_1} = p^{r_2 + r_3} \frac{u_2}{v_2} \cdot \frac{u_3}{v_3}$$

So $v_p(ab) = v_p(a) + v_p(b) \implies r_1 = r_2 + r_3$ is only satisfied if $\frac{u_1}{v_1} = \frac{u_2}{v_2} \cdot \frac{u_3}{v_3}$. But this must be true by our assumptions on p and uniqueness of this p-adic representation.

Now we prove (2). □

Example: Consider the sequence $(a_n = 1 + p + p^2 + \cdots + p^{n-1} \in \mathbb{Z})_{n \geq 1}$.

$$\frac{1}{1-p} - a_n = \frac{1}{1-p} - \left(\frac{1-p^n}{1-p}\right) = \frac{p^n}{1-p} \text{ has } v_p\left(\frac{1}{1-p} - a_n\right) = n \to \infty \text{ as } n \to \infty$$

The p-adics will be a metric space where the sequence $(a_n)_{n \geq 1}$ converges to $\frac{1}{1-p}$

Definition 13

For $a \in \mathbb{Q}$, define the p-adic absolute value of a by $|a|_p = p^{-v_p(a)}$. The p-adic distance between $a, b \in \mathbb{Q}$ is $d_p(a, b) = |a - b|_p$, so a and b are close when $a - b \equiv 0 \pmod{(p^{\text{large}}) \cdot \mathbb{Z}_{(p)}}$

Lemma 27

1. $\forall a, b \in \mathbb{Q}$ $|a \cdot b|_p = |a|_p \cdot |b|_p$
2. $|a + b|_p \leq \max |a|_p, |b|_p$ with equality if $|a| \neq |b|_p$.
3. $|a|_p = 0 \iff a = 0$

We'll present a p-adic generalization of the Dirichlet L-function.

Theorem 28

For primitive χ, there exists a unique analytic function $L_p(\chi, s)$ st $\forall n \in \mathbb{Z}{\geq 1}$

$$L_p(\chi, 1-n) = (1 - (\chi\omega^{-n})(p)p^{n-1})L(\chi\omega^{-n}, 1-n) = (1 - (\chi\omega^{-n})(p)p^{n-1}) \cdot \frac{-B_{n,\chi\omega^{-n}}}{n}$$

Example 29. Let $\chi = \omega^n$. If $n \not\equiv 0 \pmod{p-1}$, then ω^n and χ are primitive $\pmod{()}p$, but for $(\chi\omega^{-n})$, we take the trivial character mod 1, not the trivial character mod p. Thus

$$L_p(\omega^n, 1-n) = [1 - (\chi\omega^{-n})(p)p^{n-1}]L(\chi\omega^{-n}, 1) = (1 - p^{n-1}) \cdot \zeta(1-n)$$