Mixing and fast dynamo with random ABC flows

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Outline

1. The mixing problem
2. The kinematic dynamo
3. ABC flows
4. Random dynamics: main results

Passive transport in $\mathbb{T}^3$

Let $u(t, \boldsymbol{x}) \in \mathbb{R}^3$ be a vector field with $\boldsymbol{x} \in \mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$.
Assume that $\nabla \cdot u = 0$. The flow given by the vector field is defined by

$$\frac{d}{dt} X_t(\boldsymbol{x}) = u(t, X_t(\boldsymbol{x})), \quad X_0(\boldsymbol{x}) = \boldsymbol{x}.$$

1. Passive scalar $\rho(t, \boldsymbol{x}) \in \mathbb{R}$ advected by u.

$$\partial_t \rho + u \cdot \nabla \rho = 0, \quad \rho(0, \boldsymbol{x}) = \rho_0(\boldsymbol{x}).$$

   Solution along characteristics: $\rho(t, X_t(\boldsymbol{x})) = \rho_0(\boldsymbol{x})$.

2. Passive vector $v(t, \boldsymbol{x}) \in \mathbb{R}^3$ advected by u.

$$\partial_t v + (u \cdot \nabla) v - (v \cdot \nabla) u = 0, \quad \nabla \cdot v = 0, \quad v(0, \boldsymbol{x}) = v_0(\boldsymbol{x}).$$

   Solution along characteristics: $v(t, X_t(\boldsymbol{x})) = (D_{\boldsymbol{x}} X_t)^\top v_0(\boldsymbol{x})$.

The scalar problem: mixing

$$\partial_t \rho + u \cdot \nabla \rho = 0, \quad \rho(0, \boldsymbol{x}) = \rho_0(\boldsymbol{x}), \quad ⨍ \rho_0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0.$$

The transport equation presents some conserved quantities, e.g.

$$⨍ \rho(t, \boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = ⨍ \rho_0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \text{and} \quad \|\rho(t)\|_{L^p} = \|\rho_0\|_{L^p}.$$

- **Question:** Can we find div-free vector fields u that *mix* any ρ_0?

Figure: Action of an alternating shear flow in $\mathbb{T}^2$.

A measure of mixing

How can we measure the degree of mixedness?

- Mathew, Mezić, Petzold (2005); Thiffeault, Doering (2006)
  The $\dot{H}^{-s}$ norm of θ for some $s > 0$: e.g. in $\mathbb{T}^d$

$$\|\rho(t)\|_{\dot{H}^{-s}}^2 = \|\nabla^{-s}\rho(t)\|_{L^2}^2 = \sum_{k\in\mathbb{Z}^d : k\neq 0} \frac{|\hat{\rho}(t,k)|^2}{|k|^{2s}}$$

We say u divergence free **mixes** ρ_0 mean zero with rate $r(t)$ if

$$\|\rho(t)\|_{\dot{H}^{-1}} \lesssim r(t)\|\rho_0\|_{\dot{H}^1} \quad \text{where } r(t) \to 0 \text{ as } t \to \infty$$

- Crippa, De Lellis (2008); Seis (2013); Iyer, Kiselev, Xu (2014)...
  If u Lipschitz, mixing can be at most exponential: $r(t) \gtrsim e^{-\lambda t}$
- Shear and radial flows: *polynomial* mixers $r(t) \sim t^{-\lambda}$
- Yao, Zlatos (2017)...: *exponential* mixers $r(t) \sim e^{-\lambda t}$

Mixing with random vector fields

What about (div-free) vector fields that depend on a random parameter?

- Bedrossian, Blumenthal, Punshon-Smith (2018–2019)
  Stochastically forced Navier–Stokes in $\mathbb{T}^2$ (and hyper-viscous in $\mathbb{T}^3$)

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta u + Q\dot{W}_t \\ \nabla \cdot u &= 0 \end{cases}$$

- Blumenthal, Coti Zelati, Gvalani (2023); Cooperman (2023)
  Pierrehumbert model: Alternating shear flows in $\mathbb{T}^2$ with random phase or random duration

$$u_h(\boldsymbol{x}, \omega) = \begin{pmatrix} \sin(y + \omega_1) \\ 0 \end{pmatrix}, \quad u_v(\boldsymbol{x}, \omega) = \begin{pmatrix} 0 \\ \sin(x + \omega_2) \end{pmatrix}$$

The vector problem: kinematic dynamo

1. Maxwell equations for a homogeneous moving conductor

$$\nabla \times E = -\partial_t H, \qquad \nabla \cdot E = q,$$
$$\nabla \times H = j, \qquad \nabla \cdot H = 0.$$

2. Ohm's law: $j = \sigma(E + u \times H)$,

$$\partial_t H + (u \cdot \nabla)H - (H \cdot \nabla)u = \varepsilon \Delta H, \quad \nabla \cdot H = 0.$$

Coupled with the Euler equations for an inviscid fluid

$$\partial_t u + (u \cdot \nabla)u + \nabla p = F_{\text{ext}}, \quad \nabla \cdot u = 0.$$

The kinematic dynamo equations assume H to be so small compared to u that it does not play a role in the Euler equations: $F_{\text{ext}} = j \times H \sim 0$.

Kinematic fast dynamo problem

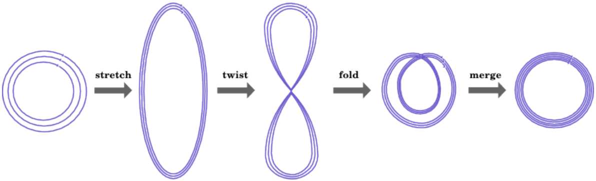
$F_{\text{ext}} \to 0$ and $\varepsilon \to 0$: *nondissipative kinematic dynamo* equations,

$$\partial_t H + (u \cdot \nabla) H - (H \cdot \nabla) u = 0, \quad \nabla \cdot H = 0.$$

We say that u is a (nondissipative) **kinematic fast dynamo** if

$$\|H(t)\|_{L^2} \geq C e^{\lambda t} \|H_0\|_{L^2}.$$

- Vainshtein, Zeldovich (1972): Stretch–Twist–Fold picture



Jennifer Schober (EPFL)

ABC flows

The ABC vector fields are smooth vector fields of the form

$$u(x, y, z) = \begin{pmatrix} \mathrm{A} \sin z + \mathrm{C} \cos y \\ \mathrm{B} \sin x + \mathrm{A} \cos z \\ \mathrm{C} \sin y + \mathrm{B} \cos x \end{pmatrix},$$

where $\mathrm{A}, \mathrm{B}, \mathrm{C} \in \mathbb{R}$, and $\boldsymbol{x} = (x, y, z) \in \mathbb{T}^3$.

- Beltrami (1889)
  The ABC vector fields have the Beltrami property: $\nabla \times u = \lambda u$.
- Arnold (1965)
  Interested in the topology of solutions to **3D Euler**: $\omega = \nabla \times u$,

$$\partial_t \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = 0, \quad \nabla \cdot u = 0.$$

- Childress (1970)
  Proposed as an example of dynamo in the context of MHD.

ABC flows

- Hénon (1966), Dombre et al. (1986), Zhao et al. (1993)...
  Evidence of chaotic streamlines in the ABC flow for different configurations of $A, B, C \neq 0$.

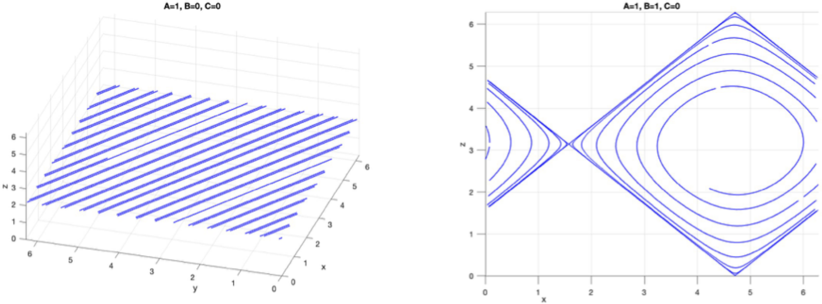


Figure: If either A, B or C are zero, the ABC flow is integrable.

Random ABC flows

Random vector $\omega = (\mathrm{A}, \alpha, \mathrm{B}, \beta, \mathrm{C}, \gamma) \in \Omega_0 = ([-U, U] \times [0, 2\pi))^3$.
Probability space $(\Omega_0, \mathcal{B}_0, \mathbb{P}_0)$. $\mathbb{P}_0$ uniform probability in Ω_0.

$$
\begin{aligned}
f_a(x,y,z) &= \begin{pmatrix} x + \mathrm{A}\sin(z+\alpha) \\ y + \mathrm{A}\cos(z+\alpha) \\ z \end{pmatrix}, \\
f_b(x,y,z) &= \begin{pmatrix} x \\ y + \mathrm{B}\sin(x+\beta) \\ z + \mathrm{B}\cos(x+\beta) \end{pmatrix}, \quad f_\omega(\boldsymbol{x}) = (f_c \circ f_b \circ f_a)(\boldsymbol{x}). \\
f_c(x,y,z) &= \begin{pmatrix} x + \mathrm{C}\cos(y+\gamma) \\ y \\ z + \mathrm{C}\sin(y+\gamma) \end{pmatrix},
\end{aligned}
$$

Iterative scheme: $\underline{\omega}^N = (\omega_1, \ldots, \omega_N) = \Omega_0^N$, $\underline{\omega} = (\omega_1, \ldots) \in \Omega = \Omega_0^{\mathbb{N}}$,

$$X_N(\boldsymbol{x}) = f_{\underline{\omega}^N}(\boldsymbol{x}) = (f_{\omega_N} \circ \ldots \circ f_{\omega_1})(\boldsymbol{x}).$$

Mixing result: decay of correlations

(H_1) Absolutely continuous noise: $\mathbb{P}_0 = \rho_0 \, \mathrm{d}Leb$, and $(\omega, \boldsymbol{x}) \mapsto f_\omega(\boldsymbol{x})$ is C^2.

(H_2) f_ω preserves the Lebesgue measure ($\Leftarrow \nabla \cdot u = 0$).

(H_3) There exists $L > 0$ such that $|D_{\boldsymbol{x}} f_\omega|, |(D_{\boldsymbol{x}} f_\omega)^{-1}| \leq L$, $\mathbb{P}_0-$a.s.

Theorem [Coti Zelati, NF (2024)]

For all $s > 0$, there exists a deterministic constant $\delta > 0$, and a random constant $D_{\underline{\omega}}$ almost surely positive such that for all mean free $g, h \in \dot{H}^s$,

$$\left| \int_{\mathbb{T}^3} g(\boldsymbol{x}) h(f_{\underline{\omega}^n}(\boldsymbol{x})) \, \mathrm{d}\boldsymbol{x} \right| \leq D_{\underline{\omega}} \|g\|_{\dot{H}^s} \|h\|_{\dot{H}^s} e^{-\delta n}$$

for all $n \in \mathbb{N}$. Moreover $\mathbb{E}|D|^q < \infty$ for all $q > 0$.

- Decay of *correlations* $\Rightarrow$ decay of $\dot{H}^{-s}$ norm (by duality):
  Let $g = \rho_0 \in H^1$, then $\|\rho(t)\|_{\dot{H}^{-1}} \leq D_{\underline{\omega}} \|\rho_0\|_{\dot{H}^1} e^{-\delta' t}$.

Random dynamics

Key idea: $(f_{\underline{\omega}^n})_{n\in\mathbb{N}}$ defines a random dynamical system.

- One-point chain: $\boldsymbol{x} \in \mathbb{T}^3$

$$P(\boldsymbol{x}, A) = \mathbb{P}_0[f_\omega(\boldsymbol{x}) \in A].$$

- Projective chain: $(\boldsymbol{x}, \boldsymbol{v}) \in \mathbb{T}^3 \times \mathbb{S}^2$

$$\hat{P}((\boldsymbol{x}, \boldsymbol{v}), \hat{A}) = \mathbb{P}_0\left[\left(f_\omega(\boldsymbol{x}), \frac{D_{\boldsymbol{x}} f_\omega \boldsymbol{v}}{|D_{\boldsymbol{x}} f_\omega \boldsymbol{v}|}\right) \in \hat{A}\right].$$

- Two-point chain: $(\boldsymbol{x}^1, \boldsymbol{x}^2) \in \mathbb{T}^3 \times \mathbb{T}^3 \setminus \Delta$

$$P^{(2)}((\boldsymbol{x}^1, \boldsymbol{x}^2), A^{(2)}) = \mathbb{P}_0\left[(f_\omega(\boldsymbol{x}^1), f_\omega(\boldsymbol{x}^2)) \in A^{(2)}\right].$$

Chapman-Kolmogorov:

$$P_{n+1}(\boldsymbol{x}, A) = \int P_n(\boldsymbol{x}', A) P(\boldsymbol{x}, \mathrm{d}\boldsymbol{x}').$$

Exponential ergodicity and mixing

Exponential ergodicity of the two-point chain $\Rightarrow$ Exponential mixing.

- Let X be a complete metric space.

Harris Theorem

Let P be a Markov-Feller chain and assume that

1. it is topologically irreducible, aperiodic, and admits a small set;
2. there exist $V : X \to [1, \infty)$, constants $\delta \in (0, 1)$, $b > 0$, and a compact set K such that $PV(\boldsymbol{x}) \le \delta V(\boldsymbol{x}) + b\chi_K(\boldsymbol{x})$.

The, P is $V-$geometrically ergodic, i.e. there exists an invariant measure μ in X and constants $k > 0$, $\sigma \in (0, 1)$ such that for all $\varphi \in L^\infty_V$,

$$\left| P_n\varphi(\boldsymbol{x}) - \int_X \varphi \, \mathrm{d}\mu \right| \le kV(\boldsymbol{x})\sigma^n.$$

- If X compact, then **2** is not needed: *uniform exponential ergodicity*.

Decay of correlations: heuristics

Exponential ergodicity of the two-point chain implies decay of correlations:

$$\mathbb{P}\left[\left|\int_{\mathbb{T}^3} g(\boldsymbol{x})h(f_{\underline{\omega}^n}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}\right| > \varepsilon^n\right] \leq \frac{1}{\varepsilon^{2n}}\mathbb{E}\left|\int_{\mathbb{T}^3} g(\boldsymbol{x})h(f_{\underline{\omega}^n}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}\right|^2$$
$$= \frac{1}{\varepsilon^{2n}}\int_{\mathbb{T}^3\times\mathbb{T}^3} h^{(2)}P_n^{(2)}g^{(2)}\,\mathrm{d}\boldsymbol{x}^1\,\mathrm{d}\boldsymbol{x}^2,$$

where $h^{(2)}(\boldsymbol{x}^1,\boldsymbol{x}^2) = h(\boldsymbol{x}^1)h(\boldsymbol{x}^2)$, $g^{(2)}(\boldsymbol{x}^1,\boldsymbol{x}^2) = g(\boldsymbol{x}^1)g(\boldsymbol{x}^2)$.

- If $P^{(2)}$ is exponentially ergodic then for some $\sigma \in (0,1)$

$$\int_{\mathbb{T}^3\times\mathbb{T}^3} h^{(2)}P_n^{(2)}g^{(2)}\,\mathrm{d}\boldsymbol{x}^1\,\mathrm{d}\boldsymbol{x}^2 \lesssim \|P_n^{(2)}g^{(2)}\|_{L^\infty} \lesssim \sigma^n.$$

- **Borel-Cantelli argument**: The probability of the $\limsup_{n\to\infty}$ of the correlations at time n being larger than ε^n is zero provided that

$$\sum_{n=1}^{\infty}\mathbb{P}\left[\left|\int_{\mathbb{T}^3} g(\boldsymbol{x})h(f_{\underline{\omega}^n}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}\right| > \varepsilon^n\right] < \infty.$$

How to deal with the diagonal

Main challenge: $X = \mathbb{T}^3 \times \mathbb{T}^3 \setminus \Delta$ is not compact, the Lyapunov-drift condition **2** for $P^{(2)}$ is difficult to verify.

- There is an invariant measure supported at $\Delta = \{\boldsymbol{x}^1 = \boldsymbol{x}^2\}$!

Steps to overcome this problem.

1. $\mathbb{T}^3$ and $\mathbb{T}^3 \times \mathbb{S}^2$ compact:
   Prove *uniform* exponential ergodicity for P and $\hat{P}$.
2. Positivity of the Top Lyapunov exponent:

$$\lambda_1(\underline{\omega}, \boldsymbol{x}) = \lim_{n\to\infty} \frac{1}{n} \log |D_{\boldsymbol{x}} f_{\underline{\omega}^n}|.$$

MET: If P ergodic, the limit exists and it is a.s. constant over $(\underline{\omega}, \boldsymbol{x})$.
Furstenberg criterion: Sufficient conditions for $\lambda_1 > 0$.
Growth of $|D_{\boldsymbol{x}} f_\omega|$ implies some repulsion from the diagonal,

$$f_\omega(\boldsymbol{x}^1) \sim f_\omega(\boldsymbol{x}^2) + D_{\boldsymbol{x}^2} f_\omega(\boldsymbol{x}^1 - \boldsymbol{x}^2).$$

The nondissipative kinematic dynamo

Going back to the kinematic dynamo equations in $\mathbb{T}^3$,

$$\partial_t H + (u \cdot \nabla) H - (H \cdot \nabla) u = 0, \quad \nabla \cdot H = 0, \tag{KD}$$

with $u = u_{\mathrm{abc}}$ the random ABC vector field

$$f_{\underline{\omega}^n}(\boldsymbol{x}) = f_{\underline{\omega}^{n-1}}(\boldsymbol{x}) + u_{\mathrm{abc}}(\omega_n, f_{\underline{\omega}^{n-1}}(\boldsymbol{x})), \quad t \in [n-1, n).$$

Theorem [Coti Zelati, NF (2024)]

For all $p \in [1, \infty]$, there exist deterministic constants $c, \lambda > 0$ such that the solution to (KD) advected by the random ABC vector field u_{abc} satisfies

$$\|H(t)\|_{L^p} \geq c \|H_0\|_{L^p} e^{\lambda t},$$

for all $t > 0$. In particular u_{abc} is a kinematic fast dynamo.

The nondissipative kinematic dynamo

This result come as a byproduct of our study of the decay of correlations.

- *Nonrandom* multiplicative ergodic theorem:
  If $\hat{P}$ ergodic, then for a.a. $(\underline{\omega}, \boldsymbol{x})$ and all $\boldsymbol{v} \in \mathbb{S}^2$

$$\lambda_1 = \lim_{n\to\infty} \frac{1}{n} \log |D_{\boldsymbol{x}} f_{\underline{\omega}^n} \boldsymbol{v}|.$$

- If the top Lyapunov exponent is positive $\lambda_1 > 0$:
  For all $\varepsilon \in (0, \lambda_1)$, $\exists c > 0$ such that for a.a. $(\underline{\omega}, \boldsymbol{x})$ and all $\boldsymbol{v} \in \mathbb{S}^2$

$$|D_{\boldsymbol{x}} f_{\underline{\omega}^n} \boldsymbol{v}| \geq c e^{(\lambda_1 - \varepsilon)n}.$$

- Since the flow map generated by u_{abc} is measure preserving,

$$\int_{\mathbb{T}^3} |H(t, \boldsymbol{x})| \, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{T}^3} |H(t, X_t(\boldsymbol{x}))| \, \mathrm{d}(X_t)_{\#}\boldsymbol{x} = \int_{\mathbb{T}^3} |H(t, X_t(\boldsymbol{x}))| \, \mathrm{d}\boldsymbol{x}.$$

  H passive vector advected by u_{abc}: $H(t, X_t(\boldsymbol{x})) = (D_{\boldsymbol{x}} X_t)^\top H_0(\boldsymbol{x})$.

Some Key References

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