Desenvolvimento da fórmula de Gauss-Legendre com 4 pontos

$$I=\int_{x_i}^{x_f} f(x)\,dx \approx \frac{x_f-x_i}{2}\left[\sum_{k=1}^{4} f(x(a_k))\,w_k\right] =$$

$$= \frac{x_f-x_i}{2}\left[f(x(a_1))w_1+f(x(a_2))w_2+f(x(a_3))w_3+f(x(a_4))w_4\right]$$

1) Quem são a_1, a_2, a_3 e a_4?

Os valores $a_1, \ldots, a_4$ são as raízes do polinômio de Legendre de grau 4, $P_4(\alpha)$

$$P_4(\alpha)=\frac{1}{2^4 4!}\frac{d^4}{d\alpha^4}\left[(\alpha^2-1)^4\right]=\frac{1}{8}(35x^4-30x^2+3)$$

Resolvendo $P_4(\alpha)=\frac{1}{8}(35x^4-30x^2+3)=0$ temos

$a_1=-0.86114$, $a_2=-0.33998$, $a_3=+0.33998$,
$a_4=+0.86114$

Aplicando esses valores em $x(a_k)=\frac{x_i+x_f}{2}+\frac{x_f-x_i}{2}a_k$ obtemos

$$x(a_1)=x(-0,86114)=\frac{x_i+x_f}{2}-\frac{x_f-x_i}{2}0.86114$$

$$x(a_2)=x(-0,33998)=\frac{x_i+x_f}{2}-\frac{x_f-x_i}{2}0.33998$$

$$x(a_3)=x(0,33998)=\frac{x_i+x_f}{2}+\frac{x_f-x_i}{2}0,33998$$

$$x(a_4)=x(0,86114)=\frac{x_i+x_f}{2}+\frac{x_f-x_i}{2}0.86114$$

Por fim, precisamos calcular os valores dos pesos w_1, w_2, w_3 e w_4

(1) $w_1 = \int_{-1}^{1} L_1(\alpha)\, d\alpha$, onde $L_1(\alpha)$ é um polinômio interpolador de Lagrange

(2) $$L_1(\alpha) = \frac{(\alpha-\alpha_2)(\alpha-\alpha_3)(\alpha-\alpha_4)}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)} = \frac{(\alpha+0.33998)(\alpha-0.33998)(\alpha-0.86114)}{-0.751872}$$

$$= \frac{\alpha^3 - 0.86114\alpha^2 - 0.115586\alpha + 0.0995361}{-0.751871}$$

(3) $$L_2(\alpha) = \frac{(\alpha-\alpha_1)(\alpha-\alpha_3)(\alpha-\alpha_4)}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)} = \frac{(\alpha+0.86114)(\alpha-0.33998)(\alpha-0.86114)}{0.425638}$$
$$= \frac{\alpha^3 - 0.339981\alpha^2 - 0.741562\alpha + 0.252116}{0.425638}$$

Com isso temos

$$w_1 = \int_{-1}^{1} L_1(\alpha)\, d\alpha = \int_{-1}^{1} \frac{\alpha^3 - 0.86114\alpha^2 - 0.115586\alpha + 0.0995361}{-0.751871}\, d\alpha$$
$$= 0.498738 = w_4$$

$$w_2 = \int_{-1}^{1} L_2(\alpha)\, d\alpha = \int_{-1}^{1} \frac{\alpha^3 - 0.339981\alpha^2 - 0.741562\alpha + 0.252116}{0.425638}\, d\alpha$$
$$= 0.652147 = w_3$$

E com isso, temos todos os ingredientes necessários para implementar a fórmula de Gauss-Legendre com 4 pontos