Quantum Cosmology

Project report for PH 303: Supervised Learning Project

Navdha
(Roll number: 210260035)

Project supervisor: Prof. Vikram Rentala

Department of Physics
Indian Institute of Technology Bombay

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1 Causal Structure

1.1 Introduction

As we are aware from Special Relativity, associated with each event is a light cone, half of which represents the future of that point and the other half the past in the spacetime. In General Relativity, this notion maybe extended, but only locally. It is only locally that the causal structure of the spacetime remains the same. Differences occur globally because of non-trivial topologies, singularities, and twisting of light cones as one moves from point to point. In this chapter, an account of basic definitions and results concerning the causal structure of spacetime in General Relativity is provided. The discussion follows for an arbitrary four-dimensional spacetime manifold (M, g_{ab}). Other notation is explained as the chapter proceeds.

1.2 Basic Definitions and Results

Consider a manifold (M, g_{ab}). If $p \in M$, it is important to note that the light cone passing through p is a subset of V_p, the tangent space of p. In a non-simply connected manifold, it might not be possible to make a continuous designation of the future and the past. The way I think about it is, imagine standing on a M$\ddot{o}$bius strip, where walking forward means going forward in time. While M$\ddot{o}$bius strip is one very specific example of a non-simply connected region, it serves well to explain the image given in Wald. As we move forward, we ultimately reach the same point in time, only with our light cone inverted. Thus, the notion of past and future at that point are not very well-defined.

Definition 1.1. *If in a given spacetime, a continuous designation of future and past can be made, it is called* **time-orientable spacetime**.

A timelike/null-vector in the future of the light cone at a point is called future directed. Some results concerning time-orientable spaces are discussed below.

Lemma 1.1. *If (M, g_{ab}) is time orientable, $\exists$ a (highly non-unique) smooth non-vanishing timelike vector field t^a on M.*

Proof. Something that I did not clearly understand, have put it on hold. ☐

Theorem 1.2. *Let (M, g_{ab}) be an arvitrary spacetime and let $p \in M$. Then $\exists$ a convex normal neighbourhood. of p, i.e., an open set U with $p \in U$ such that $\forall q, r \in U, \exists$ a unique geodesic connecting q, r, staying within U.*

Proof of the above theorem is non-trivial, and has been skipped.

Definition 1.2 (Chronological future)**.** *The set of all points reached by future-directed timelike geodesics starting from p and contained within U (p and U being as defined above) is called the chronological future of the point p, represented as $I^+(p)$.*

Definition 1.3 (Causal curve)**.** *A curve that is not spacelike, i.e., a curve for which at every point there exists a timelike or null geodesic is called a causal curve.*

Definition 1.4 (Causal future)**.** *The set of all points reached by future-directed causal curves starting from p and contained within U is called the causal future of the point p, represented as $J^+(p)$.*

Some properties and results concerning causal and chronological futures are given below:

- $I^+(S) = \bigcup_{s \in S} I^+(s)$ where $S \subset M$
- $I^+[I^+(S)] = I^+(S) \subset J^+[J^+(S)] = J^+(S)$
- $x \in I^+(y)$ iff $y \in I^-(x)$
- $I^+(x)$ and $I^+(S)$ are open sets $\forall x \in M$ and $S \subset M$

While the first and the third property act as definitions, property two is trivially true. The last property can also be proved easily.

Corollary 1.2.1. *If q $\in J^+(p) - I^+(p)$, then any causal curve connecting p to q must be a null geodesic.*

Proof. Let q $\in J^+(p) - I^+(p)$ and let λ be causal curve between p and q . Using theorem 1.2, we can cover λ by convex normal neighbourhoods.
Since λ is a causal curve, it is the continuous image of a closed, bounded, and hence compact interval and so it is compact. Therefore, it can be covered by finitely many such neighbourhoods. Assume that λ is not a null geodesic in any such neighbourhood. Again, using theorem 1.2 and Lemma 1.1, we should be able to find a timelike deformation of λ in that neighbourhood. and extend it to other neighbourhoods continuously. $\implies \lambda \in I^+(p)$, which is a contradiction. □

We've already seen in the properties mentioned above that $I^+(p) \subset J^+(p)$, which implies that $\overline{I^+(p)} \subset \overline{J^+(p)}$. Using the similar arguments as written in the above proof, we show that $J^+(p) \subset \overline{I^+(p)}$. Let q be a point such that $q \in \overline{I^+(p)}$ but $q \notin J^+(p)$. If $q \in I^+(p)$, the proof is trivial. Hence we assume that $q \notin I^+(p)$. We have to now show that $\exists$ a null geodesic between p and q (it cannot be timelike). Note that this is also true since it must be that $q \in \dot{I}^+(p)$, for $\overline{I^+(p)} = I^+(p) \cup \dot{I}^+(p)$. Since by definition, $\dot{I}^+(p)$ is a collection of all points reached by future-directed null geodesics emanating from p, our proof is complete.
Some more results:

- $\overline{J^+(p)} = \overline{I^+(p)}$
- $I^+(p) = int(J^+(p))$
- $\dot{I}^+(S) = \dot{J}^+(p)$

Definition 1.5. *A subset $S \subset M$ is called* **achronal** *if $\exists\ p, q \in S$ such that $q \in J^+(p)$, i.e., $I^+(S) \cap S = \phi$.*

An important theorem about achronal spaces follows:

Theorem 1.3. *Let (M, g_{ab}) be a time-orientable spacetime and let $S \subset M$. Then if $\dot{I}^+(S) \neq \phi$, it is an achronal, three-dimensional, embedded, C^0-submanifold of M.*

Proof. Let $q \in \dot{I}^+(S)$. We show that $I^+(q) \subset I^+(S)$.
Consider $p \in I^+(q)$. Since $I^+(q)$ is an open set, $\exists$ an open neighbourhood U of q such that $q \in U \subset I^-(p)$. Since $q \in \dot{I}^+(S)$, $U \cap I^+(S) \neq \phi$. $\implies$ $I^+(U \cap I^+(S)) \subset I^+(S)$, and thus $I^+(q) \subset I^+(S)$. Similarly, $I^-(q) \subset M - I^+(S)$. (This is clear in the image below.) If $\dot{I}^+(S)$ is not achronal, $\exists q, r \in I^+(S)$ such that $r \in I^+(q) \subset I^+(S)$. This is impossible, since $I^+(S) \cap \dot{I}^+(S) = \phi$. Therefore, $\dot{I}^+(S)$ is achronal.

For the second part, we introduce Riemannian normal coordinates x^0, x^1, x^2, x^3 at $q \in \dot{I}^+(S)$ and consider a sufficiently neighbourhood of q where $\left(\frac{\partial}{\partial x^0}\right)^a$. Using achronality of $\dot{I}^+(S)$, value of x^0 at intersection point must be a continuous function of the coordinates (x^1, x^2, x^3) and so the above map from a neighbourhood of q in $\dot{I}^+(S) \longrightarrow R^3$ is a homeomorphism in the induced topology on $\dot{I}^+(S)$. Performing the same construction $\forall q \in \dot{I}^+(S)$ we get a C^0-compatible family of charts covering $\dot{I}^+(S)$ which makes $\dot{I}^+(S)$ an embedded submanifold. □

Definition 1.6. *A continuous curve λ is called future-directed timelike (resp. causal) if $\forall p \in \lambda \exists$ a convex normal neighbourhood U of p such that if $\lambda(t_1), \lambda(t_2) \in U$ with $t_1 < t_2$, then there exists a future-directed* **differentiable** *timelike (resp. causal) curve in U from $\lambda(t_1) to \lambda(t_2)$. The timelike nature of a continuous curve is unchanged by a continuous, one-to-one reparameterization.*

Extendibility of a continuous curve

Let $\lambda(t)$ be a future-directed causal curve. We say that $p \in M$ is a **future endpoint** of λ if for every neighbourhood O of p $\exists$ a t_0 such that $\lambda(t) \in O\ \forall\ t > t_0$. It is important to note that a curve can have only one future endpoint, for if $\lambda(t)$ had two, say x and y, and let the t_0 analogue for x and y be t_x and t_y respectively. Using the Hausdorff property of M, we can find two nbs U and V of x and y respectively such that $x \in U$, $y \in V$, and $U \cap V = \phi$. WLOG, assume that $t_x \geq t_y$. Then say $\lambda(t_0) \in U\ \forall\ t_0 > t_x \geq t_y$, and so $\lambda(t_0) \in V$ since $t_0 > t_y$. Hence $t_0 \in U \cap V$, which is a contradiction. Also, the endpoint need not lie on the curve.
λ is said to be **future inextendible** if it has no future endpoint. If λ is a differentiable causal curve with future endpoint p then it may not be possible to extend λ beyond p as a differentiable causal curve, but it can be extended as a continuous causal curve by adjoining a continuous causal curve to λ at p.

Lemma 1.4. *Let λ be a past inextendible causal curve passing through point p. Then through any $q \in I^+(p), \exists$ a past inextendible timelike curve γ such that $\gamma \in I^+(\lambda)$.*

Proof. WLOG, assume that the curve parameter $t \in [0, \infty)$. Choose an arbitrary Riemannian metric on M. Using Lemma 1.1, we can construct a timelike curve $\gamma(t)$ for $t \in [0, 1]$ which starts at q and satisfies the property that $\gamma \subset J^+(\lambda)$ and $d[\gamma(t), \lambda(t)] < \dfrac{c}{1+t}$ where c is a constant and d is the greatest lower bound of length (measured using the Riemannian metric) of all curves connecting a point $\gamma(t)$ to $\lambda(t')$ with $t' \in [0,1]$. Using induction, extend t to $[0,\infty)$. Resulting γ is past inextendible because any end point of γ is also an endpoint of λ. γ is the required curve. □

Convergence of causal curves

Definition 1.7 (Convergence point). *Let $\{\lambda_n\}$ be a sequence of causal curves. If given any open neighbourhood O of a point p, $\exists$ an N such that $\lambda_n \cap O \neq \phi \; \forall \; n > N$, p is called the* ***convergence point*** *of $\{\lambda_n\}$.*

Definition 1.8 (Limit point). *$p \in M$ is said to be a* ***limit point*** *of $\{\lambda_n\}$ if every open neighbourhood of p intersects infinitely many λ_n.*

Definition 1.9 (Convergence curve). *A curve λ is said to be a* ***convergence curve*** *of $\{\lambda_n\}$ if each $p \in \lambda$ is a convergence point.*

Definition 1.10 (Limit curve). *A curve λ is said to be a* ***limit curve*** *of $\{\lambda_n\}$ if $\exists$ a subseq $\{\lambda'_n\}$ for which λ is a convergence curve.*

Thus, if λ is a limit curve of λ_n, each $p \in \lambda$ is a limit point, but the converse is false.

Lemma 1.5. *Let λ_n be a seq of future inextendible causal curves which have a limit point p. Then $\exists$ a future inextendible causal curve λ passing through p which is a limit curve of $\{\lambda_n\}$.*

Proof. A brief sketch: □

Theorem 1.6. *Let C be a closed subset of the spacetime manifold M. Then every point $p \in I^+(C)$ with $p \notin C$ lies on a null geodesic λ which lies entirely in $I^+(C)$ and either is past inextendible or has a past endpoint on C.*

Proof. Choose a seq of $\{q_n\}$ in $I^+(C)$ which converges to p. Let λ_n be a past directed timelike curve connecting q_n to a point in C. □

1.3 Causality conditions

All spacetimes in GR locally have the same qualitative causal structure as in STR, but this is not true globally. For example, construct a flat spacetime with topology $S^1 \times \mathbb{R}^3$ by identifying $t = 0$ and $t = 1$ hyperplanes of Minkowski spacetime. In such a case, $\forall \; p \in M, \; I^+(p) = I^-(p) = M$, i.e., there exist closed timelike curves (CTCs from hereon) on this manifold. Even without constructing such "artificial" topologies, we can come up with examples of metrics that enable the existence of CTCs on M. Suppose we have a manifold M with topology[1] $\mathbb{R}^4$.
We take the example of the Gödel universe. Gödel metrics are the ones that can be written as $g_{\mu\nu} = b_{\mu\nu} - u_\mu u_\nu$ where the metric $g_{\mu\nu}$ is the difference between a background metric of an Einstein space of one-dimension lower than the entire spacetime and tensor product of two timelike vectors u_μ.
In our simple example, the line element for the metric $g_{\mu\nu}$ is given by

$$ds^2 = -dt^2 + dx^2 - \frac{1}{2}e^{2x}dy^2 + dz^2 - e^x(dt \cdot dy + dy \cdot dt) \tag{1}$$

which can be written in matrix form as

$$\begin{bmatrix} -1 & 0 & -e^x & 0 \\ 0 & 1 & 0 & 0 \\ -e^x & 0 & -\frac{1}{2}e^{2x} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies g^{\mu\nu} = \begin{bmatrix} 1 & 0 & -2e^{-x} & 0 \\ 0 & 1 & 0 & 0 \\ -2e^{-x} & 0 & 2e^{-2x} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let $u_\mu = (1 \; 0 \; e^x \; 0)$ The tensor product of u_μ with itself is then

$$u_\mu u_\nu = \begin{bmatrix} 1 & 0 & e^x & 0 \\ 0 & 1 & 0 & 0 \\ e^x & 0 & e^{2x} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and so

[1] The discussion that follows has been taken from this link.

$$b_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{e^{2x}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which has rank $= 3$, which is one less than the dimension of the spacetime. Also note that $u_\mu u^\mu = -1$, which makes u^μ a timelike unit vector.

Examining CTCs in the Gödel universe We first make a change of coordinates from (t, x, y, z) to (t, r, ϕ, z) (cylindrical transformation)

The line element in the new coordinates is then

$$ds^2 = -dt^2 + dr^2 + dz^2 - \sinh^2 r(\sinh^2 r - 1)d\phi^2 + \sqrt{2}\sinh^2 r(d\phi \cdot dt + dt \cdot d\phi) \tag{2}$$

where $t \in (-\infty, \infty)$, $r \in (0, \infty)$, $z \in (-\infty, \infty)$, and $\phi \in [0, 2\pi]$. Consider a curve $\gamma(s)$ with all coordinates constant ($r = r_C$) except for ϕ, s being the parameter. A tangent vector to the curve is then given by

$$V^\mu = \frac{d\gamma^\mu}{ds} = \left(0, 0, \frac{d\phi}{ds}, 0\right) \tag{3}$$

For γ to be timelike, vectors tangent to it should be timelike at all its points. Line element along the curve is

$$ds^2 = -\sinh^2 r_C(\sinh^2 r_C - 1)d\phi^2 \tag{4}$$

since $dz = dt = dr = 0$, r_C being the constant radial coordinate at which the curve is positioned. For V (the tangent vector at some arbitrarily chosen point) to be timelike, it must be that

$$V^\mu V_\mu < 0, \text{ i.e., } g_{\mu\nu}V^\mu V^\nu < 0$$

or $g_{\phi\phi}V^\phi V^\phi < 0$ where $V^\phi = \frac{d\phi}{ds}$. Using (2), we have

$$-\sinh^2 r_C(\sinh^2 r_C - 1)\left(\frac{d\phi}{ds}\right)^2 < 0$$

or $\sinh^2 r_C > 1$. Solving for r_C, we have,

$$r_C > \log\left(1 + \sqrt{2}\right) \tag{5}$$

i.e., for $r > r_C$ the continuous timelike curves of constant r, z, $\&$ t are closed. Even though the CTCs are constant in time coordinate, some proper time elapses during our travel and when we return to $\phi = 0$, we have travelled back in time. This can be thought of as the light cones tripping over as we move along the curve.

Even if a spacetime does not possess CTCs, it can be on the verge of violating causality. For example, on a cylindrical manifold with the metric as defined in (2). From such curves then, if we remove a point, we get a spacetime that is arbitrarily close to violating causality.

Definition 1.11 (Strong causality)**.** *A spacetime (M, g_{ab}) is said to be* **strongly causal** *if $\forall$ $p \in M$ and every neighbourhood O of p, there exists a neighbourhood V of p contained in O such that no causal curve intersects V more than once.*

Therefore, if a spacetime violates strong causality at p, then near p, there exist causal curves that come arbitrarily close to intersecting themselves. In the given figure, strong causality is violated at p. The following lemma is a consequence of strong causality:

Lemma 1.7. *Let (M, g_{ab}) be strongly causal and let $K \subset M$ be compact. Then every causal curve λ confined within K must have past and future endpoints in K.*

Proof. WLOG, take the curve parameter of the curve λ to be t. Let t run from $-\infty$ to ∞. Then $\{t_i\}$ is an increasing sequence of numbers that diverges to infinity and $p_i = \lambda(t_i)$. $\{p_i\}$ is a sequence in K and thus its infinite subset, which means that it has a limit point in K (since K is compact). Let this limit point be p. Assume that p is not the future endpoint of λ, i.e., $\exists$ an open neighbourhood O of p such that there does not exist a $t_0 \in \mathbb{R}$ for which $\lambda(t) \in O$ $\forall t > t_0$. This then holds for every $V \subset O$ also. $\implies$ λ intersects every such V more than once, since infinitely many points of $\{\lambda(t_i)\}$ enter V (since O is an open neighbourhood of p and p is the limit point of V). This contradicts the strong causality condition. Thus p is a future endpoint of λ. Similarly, we can find a past endpoint also. $\square$

It is important to note that strong causality is not a good enough condition - one can produce more complicated examples - for example, the cases where strong causality is satisfied but a modification of g_{ab} in an arbitrarily small neighbourhood of a point produces CTCs, and so does not fully express the condition that one is not on the verge of causality violation. We now define a stronger condition to avoid such cases.

Let t^a be a timelike vector at point $p \in M$ and define

$$\tilde{g}_{ab} = g_{ab} - t_a t_b \tag{6}$$

where g_{ab} is the spacetime metric at a point $p \in M$. It is easy to see that $\tilde{g}_{ab}$ is also a Lorentzian metric at p. Also, the light cone of $\tilde{g}_{ab}$ is strictly larger than that of g_{ab}, i.e., every timelike and null vector of g_{ab} is a timelike vector of $\tilde{g}_{ab}$. If V_a is a timelike/null vector of g_{ab}, $\tilde{g}_{ab}V^aV^b = g_{ab}V^aV^b - V^aV^bt_at_b$ (inner product of a future directed and past directed vector > 0), and so $\tilde{g}_{ab}V^aV^b < 0$, and thus $\tilde{g}_{ab}$ contains timelike vectors of g_{ab} and more.

Definition 1.12 (Stable causality)**.** *A spacetime is called stably causal if $\exists$ a cnts non-vanishing timelike vector field t^a such that the spacetime $(M, \tilde{g}_{ab})$ possesses no closed timelike curves, with $\tilde{g}_{ab}$ being as defined above.*

The following theorem shows the existence of a "global time function".

Theorem 1.8. *A spacetime (M, g_{ab}) is stably causal iff there exists a differentiable function f on M s.t. $\nabla^a f$ is a past directed timelike vector field.*

Proof. $\implies$ Since $\nabla^a f$ is a past directed timelike vector field, along every future directed timelike curve with tangent V^a, we have $g_{ab}V^a\nabla^a f > 0$ and so $V(f) > 0$, which means that f strictly increased along every future directed timelike curve. f cannot, therefore, return to its initial value along any timelike curve, which proves that there are no closed timelike curves in (M, g_{ab}). However, this does not prove stable causality. For stable causality, let $t^a = \nabla^a f$ and $\tilde{g}_{ab} = g_{ab} - t_a t_b$. Then the inverse metric $\tilde{g}^{ab}$ is

$$\tilde{g}^{ab} = g^{ab} + \frac{t^a t^b}{1 - t^c t_c}$$

Therefore,

$$\tilde{g}^{ab}\nabla_a f \nabla_b f = g^{ab}\nabla_a f \nabla_b f + \frac{t^a t^b}{1 - t^c t_c}\nabla_a f \nabla_b f = t^a t_a + \frac{(t^a t_a)^2}{1 - t^c t_c} = \frac{t^a t_a}{1 - t_c t^c} < 0$$

(since $t^a = \nabla^a f$). Hence $\tilde{g}^{ab}]\nabla_b f$ is a timelike vector in the metric $\tilde{g}_{ab}$, and thus no CTCs exist in $(M, \tilde{g}_{ab})$, by the arguments as given above, and thus (M, g_{ab}) is stably causal.
$\impliedby$ We know that (M, g_{ab}) is stably causal and want to construct an f s.t. $\nabla^a f$ is a past directed timelike vector field. We use the paracompactness (the property that every open cover has a locally finite refinement) of M and define a cnts volume measure μ on M s.t. the total volume of M is finite, i.e., $\mu[M] < \infty$. The proof that this can be done has been skipped for simplicity. Define $F(p) = \mu[I^+(p)]$. Then F strictly increases along all future directed timelike curves with non-vanishing tangents. But F need not be cnts. For stably causal spacetimes, though, we can obtain a cnts function by averaging F over nearby spacetimes with "opened" out light cones. To be more precise, let t^a be a timelike vector field (with $\tilde{g}_{ab}$ as deinfed in (6)) s.t. $\tilde{g}_{ab}$ has no closed timelike curves. For $0 \geq \alpha \geq 1$, define

$$(g_\alpha)_{ab} = g_{ab} - \alpha t_a t_b \tag{7}$$

and

$$F(p) = \mu[I_\alpha^-(p)] \tag{8}$$

where $I_\alpha^-(p)$ is the chronological past in the metric $(g_\alpha)_{ab}$. Once we average over α, we'll see that F increases strictly along causal curves, or $g_{ab}V^a\nabla^b f > 0$ for every V^a (defined above), which makes F the desired function. □

Corollary 1.8.1. *Stable causality implies strong causality.*

1.4 Foliation of a globally hyperbolic spacetime

Definition 1.13 (Domain of Dependence)**.** *Let S be a closed, achronal set (possibly with an edge). We define the future domain of dependence of S as*

$$D^+[S] = \{p \in M \mid \textit{every past inextendible causal curve through p intersects S}\}$$

By this definition, we have

$$S \subset D + [S] \subset J^+[S]$$

The set $D^+[S]$ is of interest because any signal sent to a point in $D^+[S]$ must have "registered" on S. Thus, with appropriate initial conditions, we must be able to determine what happens at any $p \in D^+[S]$. Conversely, if $p \in I^+[S]$, but $p \notin D^+[S]$ then it should be possible to send a signal to p without influencing S. In a similar manner as definition 1.9, one may define $D^-[S]$ and the full domain of dependence $D[S]$ is then given by

$$D[S] = D^+[S] \cup D^-[S]$$

$D[S]$ represents the complete set of events for which all conditions should be determined by knowledge of conditions on S.

Definition 1.14. *(Cauchy Surface/Slice) A closed achronal set Σ for which $D(\Sigma) = M$ is called a Cauchy surface.*

It follows that Σ has no edge, i.e., Σ is a Cauchy slice, and every Cauchy slice is an embedded $(n-1)$ dimensional C^0 submanifold of M. Since Σ is achronal, we can think of it representing an instant of "time".

Definition 1.15 (Globally Hyperbolic spacetime)**.** *If (M,g) possesses a Cauchy surface Σ, it is said to be globally hyperbolic.*

If we restrict ourselves to globally hyperbolic surfaces, knowing the initial conditions at some slice of spacetime is sufficient to predict the evolution of that spacetime. We expect, in general, that all physically realizable spacetimes are globally hyperbolic (for predictability to hold).

We now state the following theorem without proof -

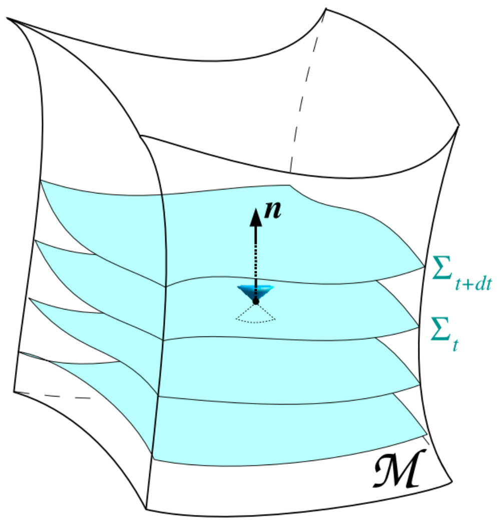


Figure 1: Foliation of the spacetime M by a family of spacelike hypersurfaces $(\Sigma_t), t \in \mathbb{R}$

Theorem 1.9. *Let (M, g) be a globally hyperbolic spacetime. Then M is stably causal. Furthermore, a global time function f can be chosen s.t. each surface of const f is a Cauchy Surface. Thus, M can be foliated by Cauchy surfaces and the topology of M is $\mathbb{R} \times \Sigma$, where Σ denotes any Cauchy slice. All Cauchy slices are homomorphic to one another.*

The stable causality leads to the existence of a global time function (theorem 1.6), and thus a global time parameter for Cauchy surfaces. We define Cauchy horizon of a set S as

$$H^+[S] = \overline{D^+[S]} - (\overline{D^+[S]}) \cap I^-[D^+[S]]$$

The Cauchy horizon $H^+[S]$ is closed, achronal, and $H[S] = \dot{D}[S]$. As a corrollary, if $M$is connected then a non-empty closed achronal set Σ is a Cauchy surface for (M, g) iff $H(\Sigma) = \phi$.

Definition 1.16 (Foliation)**.** *By foliation or slicing, it is meant that there exists a smooth scalar field $\hat{t}$ on M, which is regular (in the sense that its gradient never vanishes), such that each hypersurface is a level surface of this scalar field, or*

$$\forall t \in \mathbb{R},\ \Sigma_t := \{p \in M | \hat{t}(p) = t\} \tag{9}$$

Since $\hat{t}$ is regular, there does not exist a $p \in M$ such that for distinct t, t', $\hat{t}(p) = t$ and $\hat{t}(p) = t'$. Hence, it must be that

$$\Sigma_t \cap \Sigma_{t'} = \phi \text{ for } t \neq t'$$

In the following, we do no longer distinguish between t and $\hat{t}$, i.e. we skip the hat in the name of the scalar field. Each hypersurface Σ_t is called a leaf or a slice of the foliation. We assume that all Σ_t are spacelike and that the foliation covers M (fig. (1),

$$M = \bigcup_{t \in \mathbb{R}} \Sigma_t$$

2 Hartle-Hawking wavefunction

We have developed the mathematical background required to study the Initial Value Formulation of GR.

2.1 The Wheeler-DeWitt Equation

2.1.1 Arnowitt-Deser-Misner (ADM) formulation of General Relativity

We turn our attention to Einstein's equation in vacuum, $G_{ab} = 0$. The first issue to consider is the nature of initial value formulation of this theory. In other theories of classical physics, we are given the soacetime background and our task is to determine the time evolution of the quantities in the baclground from their initial values and time derivatives. However, in general relativity, we are solving fir the spacetime itself. We must now view general relativity in terms of the evolution of some quantity. Let (M, g_{ab}) be a globally hyperbolic spacetime for the following discussion.

Geometry of hypersurfaces [3]

A hypersurface $\Sigma \subset M$ is the image of a manifold $\hat{\Sigma}$ (of one dimension lower) by an embedding $\phi : \hat{\Sigma} - \to M$, i.e., $\Sigma = \phi(\hat{\Sigma})$. The embedding defines the pushforward map ϕ_* from vectors on Σ to vectors on M and pullback map from dual vectors on M to dual vectors on Σ. The pullback is naturally extended to multilinear forms, including the metric g. We obtain the induced metric $h_{\mu\nu}$ as

$$h = \phi^* g \tag{10}$$

on Σ. A hypersurface is spacelike if induced metric is Riemannian, i.e., has $(+, +, +)$ signature., timelike if the induced metric is Lorentzian (signature $(-, +, +)$). We ignore the case of a null hypersurface for now.

Globally asymptotic spacetimes can be foliated into a family of spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$, meaning that there exists a smooth scalar field $\hat{t}$ on M with a non-vanishing gradient of which every hypersurface is a level curve i.e., $\Sigma_t := \left\{p \in M | \hat{t}(p) = t\right\}$ (definition 1.16). The hypersurfaces are called the "leaves" of the foliation. Once we have the hypersurface, on each hypersurface, introduce a coordinate system $(x^i) = (x^1, x^2, x^3)$ called the spatial coordinates. If these vary smoothly between each hypersurface, then $(x^\alpha) = (t, x^i)$ is a well behaved coordinate system on M. Let $\partial_\alpha = (\partial_t, \partial_i)$ be the natural basis on the tangent spaces of M associated with the coordinates (x^α). Define a unit vector

$$\mathbf{n} = -N\vec{\vec{\nabla}}t \tag{11}$$

where $N = (-\nabla t \cdot \nabla t)^{(-1/2)}$, so that $\mathbf{n} \cdot \mathbf{n} = -1$. Since the hypersurface is defined by the level curves of a scalar field t on M, the gradient $\nabla_\mu t$ is normal to Σ in the sense that $u^\mu \nabla_\mu t = 0$ for every vector $\mathbf{u} \in \Sigma$. This means that its vector dual, $\nabla^\mu t$ is also normal to Σ. Thus the vector $\mathbf{n}$ is a unit normal vector to Σ. The minus sign in (11) is chosen so that $\mathbf{n}$ is future oriented if the scalar field t is increasing towards the future. Call N the lapse function.

Also, define the normal evolution vector $\mathbf{m} = N\mathbf{n}$. Then we have

$$\langle \nabla t, m \rangle = m_\mu \nabla^\mu t = N n_\mu \nabla^\mu t = -N^2(\nabla t \cdot \nabla t) = 1 \tag{12}$$

Consider a point $p \in \Sigma_t$ and a neighbouring point $p' = p + \mathbf{m}\delta t$. Then

$$\hat{t}(p') = \hat{t}(p + \mathbf{m}\delta t) = \hat{t}(p) + t + \delta t \mathbf{m} \cdot \nabla t \tag{13}$$

which is equal to $t + \delta t$ (using (12)). Thus $\mathbf{m}$ carries a hypersurface Σ_t to a neighbouring hypersurface $\Sigma_{t+\delta t}$, justifying the name normal evolution vector. Physically, the hypersurface Σ_t is locally the set of simultaneous events. The proper time between nearby events p and p' described above is

$$d\tau = \sqrt{-g(\mathbf{m}\delta t, \mathbf{m}\delta t)} = \sqrt{-(-N^2)}\delta t = N\delta t \tag{14}$$

This justifies the name lapse function. In figure (2), the points p and p' refer to P_1 and P_2 respectively. The normal evolution vector in this case is n^μ.

Now let us come back to the coordinate system we have on the hypersurface. Let $(\partial_\alpha) = (\partial_\mathbf{t}, \partial_\mathbf{i})$ be the natural basis on the tangent spaces of M associated with the coordinates (x_α).The dual basis is given as $\mathbf{d}x^\alpha$. In particular, $\mathbf{d}t = \vec{\nabla}t$. Then $\langle \nabla t, \partial_\mathbf{t} \rangle = 1$, which is the same property exhibited by the normal evolution vector $\mathbf{m}$.

Then using Theorem 1.9, we may conclude that M can be foilated by Cauchy surfaces parameterized by a global time function t. Let n^μ be the normal unit vector to Σ_t. Then $g_{\mu\nu}$ induces a spatial (a 3D Riemannian) metric on each Σ_t, let's call this metric $h_{\mu\nu}$. Then we have

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \tag{15}$$

Let t^μ be a vector field on M such that $t^a \nabla_a t = 1$. We now decompose t^μ into its parts normal and tangential to Σ_t by defining the lapse function N and shift vector N^a with respect to t^a given by

$$N = -t^a n_a = (n^a \nabla_a t)^{-1} \tag{16}$$

$$N_a = h_{ab} t^b \tag{17}$$

Here t^a represents the 'flow of time' throughout spacetime (it can be thought to be doing that). The minus sign in (11) is chosen so that the vector N is future-oriented if the vector field t^μ is increasing towards the future. As we move forward in spcatime we go to Σ_t from Σ_0 following the integral curves of t^a. Thus moving forward in time means changing the spatial metric on Σ from $h_{ab}(0)$ to $h_{ab}(t)$, and so (M, g_{ab}) represents the time development of Riemannian metric on a fixed 3D manifold. We can therefore claim that h_{ab} is the dynamical variable in GR[2] Fig. (1) nicely explains the idea behind constructing the construction of the lapse function and the shift vector. The proper distance between P_1 and P_2 is given as

$$d\tau = N dt \tag{18}$$

The normal vector n^μ takes spatial coordinates x^i on Σ_t to some other spatial coordinates on Σ_{t+dt}. Let P_3 be the point on Σ_{t+dt} with same spatial coordinates as P_1 on Σ_t. The vector from P_2 to P_3 defines the shift vector N^i, which describes the distortion of Σ_t as it evolves in time.

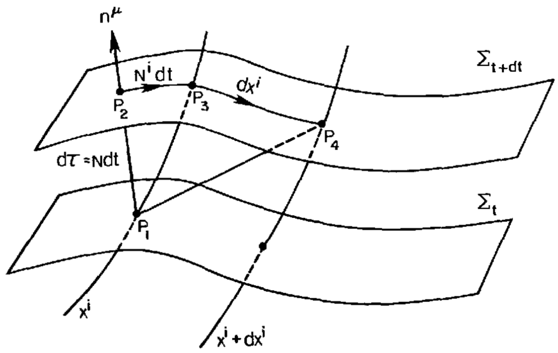


Figure 2: The lapse function N and the shift vector N^i

In S^3, it is easy to see why n^μ would have vanishing spatial components. In such a case, N^i would be zero, since n^μ takes us to the same spatial coordinates (i.e., point P_2 and P_3 would coincide for a sphere). The proper length from P_1 to P_4 is defined in terms of $g_{\mu\nu}$ as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{19}$$

In terms of the lapse shift,

$$ds^2 = (Ndt)^2 - h_{ij}(N^i dt + dx^i)(N^j dt + dx^j) \tag{20}$$

The above expression is not obvious from our discussion so far or even the argument presented in [4]. I follow the steps given in [3] to derive this equality. Let us go back to the discussion following Eq. (14). From there, we know that

$$\langle \nabla t, \partial_t \rangle = 1 \tag{21}$$

as well as

$$\langle \nabla t, \mathbf{m} \rangle = 1 \tag{22}$$

We also must define the shift vector in an intuitive manner. Define

$$\mathbf{N} = \partial_t - \mathbf{m} \tag{23}$$

where $\mathbf{N} = N^i \partial_i$ is defined as the shift vector. The vectors ∂_t and $\mathbf{m}$ coincide when the lines on which the spatial coordinates are constant are orthogonal to the hypersurfaces. In general, however, they differ by the shift vector. Also note that

$$\langle \nabla t, \mathbf{N} \rangle = \langle \nabla t, \partial_t \rangle - \langle \nabla t, \mathbf{m} \rangle = 1 - 1 = 0 \tag{24}$$

Hence, the shift vector is tangent to the hypersurfaces. Also, from the definition of $\mathbf{m}$, we have

$$\partial_t = N\mathbf{n} + \mathbf{N} \tag{25}$$

Turning the above equation around, [3] we get

$$\mathbf{n} = \frac{1}{N}(\partial_t - \mathbf{N}); \quad n^\alpha = \frac{1}{N}(1, -N^1, -N^2, -N^3) \tag{26}$$

[2] N, N_a are not considered dynamical since they merely prescribe how to move forward in time. We'll see later that they are essentially Lagarange multipliers.

[3] Here N and $\mathbf{N}$ are different quantites!

The vector dual, $n_\alpha = -N\partial_\mu t = -N\mathbf{d}t = (-N, 0, 0, 0)$. From here, it is now easy to derive the metric components. We begin by writing

$$g_{00} = g(\partial_t, \partial_t) = (\mathbf{m} + \mathbf{N}) \cdot (\mathbf{m} + \mathbf{N}) = -N^2 + N^i N_i \tag{27}$$

The remaining components can be derived as

$$g_{0i} = (\mathbf{m} + \mathbf{N}) \cdot \partial_i = 0 + \langle N_j \mathbf{d}x^j, \partial_i \rangle = N_i \tag{28}$$

and

$$g_{ij} = h_{ij} \tag{29}$$

Comparing eq. (13) and eq. (14), we have

$$g_{\mu\nu} = \begin{bmatrix} N^2 - N_i N_j h^{ij} & -N_j \\ -N_i & -h_{ij} \end{bmatrix} \tag{30}$$

Thus we also have the inverse metric,

$$g^{\mu\nu} = \begin{bmatrix} \dfrac{1}{N^2} & -\dfrac{N^j}{N^2} \\ -\dfrac{N^j}{N^2} & -\dfrac{N^i N^j}{N^2} - h^{ij} \end{bmatrix} \tag{31}$$

where $h^{ij} = (h_{ij})^{-1}$. It can also be noted that $\sqrt{-g} = N\sqrt{h}$. We now define the extrinsic curvature of Σ_i

$$K_{ij} := \frac{1}{2N}\left(\nabla_k N_i + \nabla_i N_k - \frac{\partial h_{ij}}{\partial t}\right) \tag{32}$$

which measures the curvature of Σ_t with respect to the enveloping 4-geometry. It is not clear why this quantity needs to be introduced. Again, I use [3] to provide a rigorous way of understanding the extrinsic curvature tensor. Let us first define an important map.

Definition 2.1 (Weingarten map or shape operator)**.** *The Weingarten map or shape operator* $\chi : T_p(\Sigma) \to T_p(\Sigma)$ *is defined by* $\chi(\mathbf{v}) = \nabla_{\mathbf{v}}\mathbf{n}$*. Note that it is self-adjoint in the sense that* $\mathbf{u} \cdot \chi(\mathbf{v}) = \chi(\mathbf{v}) \cdot \mathbf{u}$*.*

This means its eigenvalues, called the principal curvatures, are real-valued. This curvature is intrinsic.

Definition 2.2 (Extrinsic curvature)**.** *The extrinsic curvature* $\mathbf{K} : T_p(\Sigma) \times T_p(\Sigma) \to \mathbb{R}$*, also called the second fundamental form, is defined through the shape operator by*

$$\mathbf{K}(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \chi(\mathbf{v}) = g(\mathbf{u}, \nabla_{\mathbf{v}}\mathbf{n}) \tag{33}$$

From here, it is now easy to find the tensor form of K, given the definition of the shift vector in terms of $\mathbf{n}$. For FRW geometry,

$$g_{\mu\nu}dx^\mu dx^\nu = dt^2 - R^2\left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{34}$$

Equating this to eq. (20), we get $N = 1$ and $N_i = 0$. Define the trace of the extrinsic curvature tensor, K as

$$K := K^i{}_i = h^{ij} K_{ij} \tag{35}$$

The extrinsic curvature tensor for FRW spacetime is therefore

$$K_{ij} = -\frac{1}{2N}\frac{\partial h_{ij}}{\partial t} \tag{36}$$

which comes out to be $-\dfrac{\dot{R}}{R}h_{ij}$. Therefore,

$$K = -\frac{3\dot{R}}{R} = -3H \tag{37}$$

We now write the Ricci scalar in terms of the lapse, the shift, and the induced metric -

$$\mathcal{R} = K^2 - K_{ij}K^{ij} - {}^3\mathcal{R} \tag{38}$$

This will be used for the construction of the Hamiltonian. The Ricci scalar as defined in eq. (38) is the same as the Ricci scalar we would calculate using the regular method, i.e., by evaluating the Christoffel symbols from the metric and finding the Riemann tensor. The gravitational Lagrangian density is given as

$$\mathcal{L}\left[g_{\mu\nu}\right] = -\frac{\sqrt{-g}\mathcal{R}}{16\pi G} \tag{39}$$

In terms of the lapse function, the shift vector, and the extrinsic curvature tensor this becomes

$$\mathcal{L}\left[N, N_i, h_{ij}\right] = -\frac{\sqrt{h}N}{16\pi G}(K^2 - K_{ij}K^{ij} - {}^3\mathcal{R}) \tag{40}$$

Note that here, K_{ij} and its trace involves time derivatives of h_{ij} and spatial derivatives of N_i, while ${}^3\mathcal{R}$ involves only spatial derivatives of h_{ij}. The RHS of equation (40) does not contain time derivatives of N or N_i, which means that N and N_i are not dynamical variables and the momenta conjugate to these quantities are zero, i.e.,

$$\pi = \frac{\delta\mathcal{L}}{\delta\dot{N}} = 0 \tag{41}$$

and

$$\pi^i = \frac{\delta\mathcal{L}}{\delta\dot{N}_i} = 0 \tag{42}$$

where π and π^i are the momenta conjugate to N and N_i respectively. Eq. (41) and (42) are soemtimes referred to as "primary constraints". Defining the momentum conjugate to h_{ij} as π^{ij}

$$\pi^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{ij}} \tag{43}$$

From eqs. (39) and (40) we have

$$\pi^{ij} = N\sqrt{h}\left[(h^{ik}h^{jl} - h^{ij}h^{kl})K_{kl} + (h^{ki}h^{lj} - h^{kl}h^{ij})K_{kl}\right]\left(\frac{-1}{2N}\right) \tag{44}$$

i.e.,

$$\pi^{ij} = \sqrt{h}\frac{K^{ij} - h^{ij}K}{16\pi G} \tag{45}$$

The Hamiltonian density by the Legendre transform is given as

$$\mathcal{H} = \pi^{ij}\dot{h}_{ij} - \mathcal{L} \tag{46}$$

(since the rest of the conjugate momenta are zero). The corresponding Hamiltonian is given as

$$H = \int_{\Sigma_t}\mathcal{H}d^3x \tag{47}$$

which is equal to

$$H = \int d^3x(N\mathcal{H}_G + N_i\mathcal{H}^i) \tag{48}$$

where

$$\mathcal{H}_G := \sqrt{h}\frac{K_{ij}K^{ij} - K^2 - {}^3\mathcal{R}}{\sqrt{16\pi G}} \text{ and } \mathcal{H}^i := \frac{-2\nabla_j\pi^{ij}}{16\pi G}$$

Using eqs. (46) and (47), we get

$$\mathcal{H}_G = \frac{16\pi G}{2\sqrt{h}}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl})\pi^{ij}\pi^{kl} \tag{49}$$

which we rewrite as

$$\mathcal{H}_G = \frac{16\pi G}{2\sqrt{h}}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl})\pi^{ij}\pi^{kl} := 16\pi G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{h}\frac{{}^3\mathcal{R}}{16\pi G} \tag{50}$$

Also note that $\pi = \pi^i = 0$ (Eqs. (41) and (42)) hold at all times, which means that $\dot{\pi} = \dot{\pi}^i = 0$. Writing the canonical equations of motion, we have

$$\dot{\pi} = \frac{\delta H}{\delta N} = 0 \tag{51}$$

and

$$\dot{\pi}^i = \frac{\delta H}{\delta N_i} = 0 \tag{52}$$

and so the Hamiltonian H is independent of N and N_i, which proves that they are not dynamical variables. This also gives us secondary constraints, $\mathcal{H}_G = \mathcal{H}^i = 0$. The true dynamical variable is h_{ij} whose evolution is described by the equation

$$\mathcal{H}_G = 16\pi G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{h}\frac{{}^3\mathcal{R}}{16\pi G} = 0 \tag{53}$$

2.1.2 Quantization

So far the discussion has been entirely classical. There are two steps involved in quantization, given below -

1. Identify Eq. (53) as the zero-energy Schrödinger equation, $\mathcal{H}_G(\pi_{ij}, h_{ij})\Psi\left[h_{ij}\right] = 0$ where the state vector Ψ is the wavefunction of the Universe.
2. Next is to follow the canonical quantization procedure, wherein we replace the observables by their corresponding operators in quantum mechanics. Thus, in eq. (53), we replace $\pi^i \to -i\dfrac{\partial}{\partial q_i}$ and $\pi^{ij} \to -i\left(\dfrac{1}{16\pi G}\right)^{3/2}\dfrac{\delta}{\delta h_{ij}}$.

These two steps lead to the **Wheeler-DeWitt equation**

$$\left[\frac{G_{ijkl}}{(16\pi G)^2}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} + \frac{\sqrt{h}\;{}^3\mathcal{R}}{16\pi G}\right]\Psi\left[h_{ij}\right] = 0 \tag{54}$$

Given this equation, we can easily generalize it to include cosmological constant and matter fields, which gives us the following equation:

$$\left[\frac{G_{ijkl}}{(16\pi G)^2}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} + \frac{\sqrt{h}\;{}^3\mathcal{R} - 2\Lambda}{16\pi G} - \mathcal{T}\right]\Psi\left[h_{ij}, \phi\right] = 0 \tag{55}$$

where $\mathcal{T} = T^0_{\ 0}\left(\phi, -i\dfrac{\partial}{\partial\phi}\right)$ and $T_{\mu\nu}$ is the stress-energy tensor of the matter fields. Note that we did not take care of ordering of the operators while writing either (54) or (55). This is a general problem that arises in canonical quantization. We can parameterize the effect of this factor ordering by a constant a and the corresponding Hamiltonian is obtained by substituting $\pi\pi \to -q^{-a}\left[\dfrac{\partial}{\partial q}q^a\dfrac{\partial}{\partial q}\right]$. In eq. (54), we have set $a = 0$. In the discussion that follows, factor ordering does not affect any of the semi-classical calculations, so the choice of a is for our convenience.

A striking feature of the wavefunction of the Universe, Ψ, is that it is independent of time - it depends only on h_{ij} and the matter-field content ϕ. I believe that here they are referring to explicit time dependence, for time-dependence is implicitly present due to h_{ij}. Likewise, the interpretation of the wavefunction Ψ has to be resolved. In any case, it must be that in the limit $\Psi \to \exp iS_{E-H}/\hbar$, we must approach classical limit. Also, there are many solutions to eq. (55), to reach a particular solution, we need information about the initial quantum state (just like we need initial state to completely get the evolution of the quantum state of an electron with time). I analyze the wavefunction of the Universe in more detail in the next section.

2.2 The wavefunction of the Universe

In our discussion in the previous section, $\Psi\left[h_{ij}, \phi\right]$ was defined on an infinite-dimensional space of all possible 3-geometries and matter-field configurations, known as superspace. Superspace is the dynamic arena of space much like spacetime (the totality of all points (x, t)) is the dynamic arena of a point. We assume a mini-superspacemodel in which the only degree of freedom is that of R, the scale factor of a closed homogeneous and isotropic universe and a homogeneous, massive scalar field ϕ. We write the cosmological constant as $\Lambda = 8\pi G\rho_{vac}$.

Classical evolution of scale factor for a closed FRW model is governed by Friedmann equation

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{1}{R^2} = \frac{\Lambda}{3} \tag{56}$$

to which we have the de-Sitter solution, $R(t) = R_0 \cosh\frac{t}{R_0}$ where $R_0 = \sqrt{\frac{3}{\Lambda}}$. R here describes a universe that in the past was infinitely large, contracts to a minimum sixe R_0 at $t = 0$ and then expands to infinity as $t \to \infty$. Note that since $\cosh x > 1$ for all x, the range $0 < R < R_0$ is forbidden for the scale factor. Let us rederive the Wheeler-DeWitt equation for FRW metric (we can obviously use (55), but it is more instructive to start with the action of FRW spacetime). The action is given as

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}(\mathcal{R} + 2\Lambda) \tag{57}$$

Integrating over the spatial part, we have

$$S = -\frac{3\pi}{4G}\int dt(R\dot{R} - R + \frac{R^3\Lambda}{3}) \tag{58}$$

The Lagrangian density is then defined as the time derivative of the action,

$$L := \frac{\delta S}{\delta t} = -\frac{3\pi}{4G}\left(\dot{R}^2 R - R + \frac{\Lambda R^3}{3}\right) \tag{59}$$

Therefore, the momentum conjugate to our only degree of freedom (R) is

$$\pi_R := \frac{\delta L}{\delta \dot{R}} = -\frac{3\pi \dot{R} R}{2G} \tag{60}$$

The Hamiltonian density, again obtained through Legendre transform is

$$H = \pi_R \dot{R} - L \tag{61}$$

which is equal to

$$H = -\frac{G}{3\pi R}\pi_R^2 + \frac{3\pi}{4G}R\left(1 - \frac{\Lambda}{3}R^3\right) \tag{62}$$

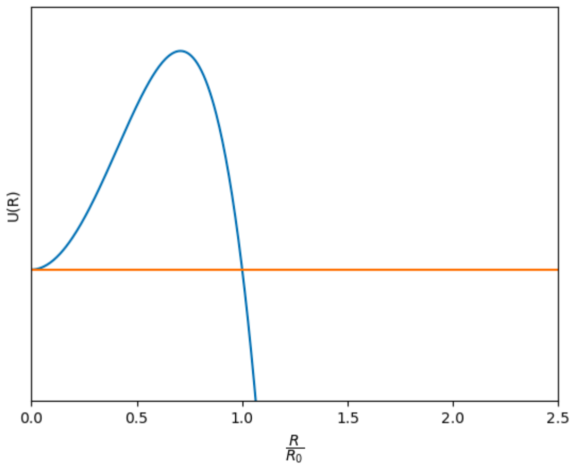


Figure 3: $U(R)$ as a function of $\frac{R}{R_0}$

To obtain the quantum mechanical equation of evolution, we follow the quantization procedure as listed in section 2.1.2. The equation is then $H\Psi[R] = 0$ with $\pi_R \to i\frac{\partial}{\partial R}$, which gives

$$\left[\frac{\partial^2}{\partial R^2} - \frac{9\pi^2}{4G^2}\left(R^2 - \frac{\Lambda}{3}R^4\right)\right]\Psi(R) = 0 \tag{63}$$

By choosing factor ordering corresponding to $a = 0$ (see the discussion following Eq. (55)), we observe that Eq. (63) is identical to TISE for a particle with zero energy and potential energy as a function of R given by

$$U(R) = \frac{9\pi^2 R_0^2}{4G^2}\left[\frac{R^2}{R_0^2} - \left(\frac{R}{R_0}\right)^4\right] \tag{64}$$

This is plotted in fig. 3. For $\frac{R}{R_0} < 1$, $U(R) > E = 0$, which makes it classically forbidden. This means that a zero-energy particle will be stuck at $R = 0$ if we only consider classical approach. But there is a quantum mechanical probability that the particle will tunnel through the barrier and emerge at the classical turning point $R = R_0$. To find the tunneling probability, we use the WKB approximation. We use Wick rotation, so that

$t_E = it$ where t_E is the Euclidean time. One can think of the tunneling rate as being associated with a classical motion in imaginary time as the decay rate is related to imaginary part of the energy, i.e., if

$$\Psi \propto (\exp -iEt/\hbar) \tag{65}$$

the imaginary part of E leads to a decay in the wavefunction. Thus, decay rate in WKB approximation $\propto \exp(-S_E)$ where S_E is the Euclidean action. In 4-dimensions, we visualize the deSitter space as a 3-sphere evolving in time, with curvature $= 4\Lambda$ where $\Lambda =$ cosmological constant. Note that in this entire discussion, we have set $\phi =$ constant, which has been absorbed in the cosmological constant. The Euclidean version of the deSitter space is therefore a 4-sphere with radius $R_0 = \sqrt{\frac{3}{\Lambda}}$, and volume $V_4 = \frac{8\pi^2 R_0^2}{3}$ which is equal to $8\pi^2 \cdot 3\Lambda^2$. As discussed above, the curvature/Ricci scalar is given as $\mathcal{R} = -4\Lambda$ where the negative sign comes from the fact that the Ricci scalar is derived from the Riemann tensor, which includes multiplication of first derivatives of the metric to the 2nd order, leading to an overall $\iota \cdot \iota = -1$ factor in $\mathcal{R}$.

Thus, $\mathcal{R} + 2\Lambda = -2\Lambda$, which is a constant. The Euclidean action is given by eq. (57), wherein we substitute the coordinates as $r, \theta, \phi, \iota t$. Since $\mathcal{R} + 2\Lambda$ is a constant, this is equivalent to multiplying by volume and removing the integral sign, which gives us

$$S_E = \frac{\mathcal{R} + 2\Lambda}{16\pi G} V_4 = -\frac{3\pi}{\Lambda G} < 0 \tag{66}$$

With this prescription, we actually have two possible wavefunctions at $R = R_0$, and thus two possible expressions for tunneling probability, which come from the WKB apprximation itself, for the factor in the exponent may be $\exp(\pm \int \iota p(x')dx'/\hbar)$. There has been much debate about what sign to choose, but [4] claims that it depends on the initial state or the boundary conditions, and so with this prescription, we write

$$P_T \propto \exp\left(-\left|\frac{3\pi}{G\Lambda}\right|\right) \tag{67}$$

We can also solve eq. (53) directly by first rewriting it with factor ordering corresponding to[4] $a = 1$ as

$$\left[\frac{\partial^2}{\partial R^2} - \frac{1}{R}\frac{\partial}{\partial R} - \frac{9\pi^2}{4G^2}\left(R^3 - \frac{\Lambda}{3}R^4\right)\right]\Psi(R) = 0 \tag{68}$$

Solution to the above equation is a linear combination of Airy functions $Ai[z(R)]$ and $Bi[z(R)]$ where $z(R) = \left(\frac{3\pi R_0^2}{4G}\right)^{2/3}\left(1 - \frac{R^2}{R_0^2}\right)$. The solutions discussed in all the papers match, but the disagreement occurs only at the boundary conditions.

Hartle and Hawking's boundary condition is explained below, but let us first take note of the wavefunctions obtained:

$$\Psi^V(R) \propto A_i\left[z(R)\right] A_i\left[z(R_0)\right] + B_i\left[z(R)\right] B_i\left[z(R_0)\right] \tag{69}$$

and

$$\Psi^{HH} \propto A_i\left[z(R)\right] \tag{70}$$

These solutions are shown in Fig. (4).

More specifically, Hartle-Hawking proposal for initial state $\Psi_0\left[h_{ij}\right] \propto \int \mathcal{D}g \exp(-S_E(g))$ where the path integral over geometries extends over all compact manifolds whose induced metric is h_{ij} and $S_E(g)$ is the Euclidean action associated with the manifold. The Hartle-Hawking wave function is expressed as a path integral over compact Euclidean geometries bounded by a given 3-geometry g,

$$\psi_{HH}(g, \phi) = \int^{(g,\phi)} e^{-S_E}$$

(In practice, one assumes that the dominant contribution to the path integral is given by the stationary points of the action and evaluates ψ_{HH} simply as $\psi_{HH} \propto e^{-S_E}$. The wave function $\psi_{HH}(R)$ for this model has only the growing solution under the barrier and a superposition of ingoing and outgoing waves with equal amplitudes in the classically allowed region. This wave function appears to describe a contracting and re-expanding universe). Since a compact manifold has no boundaries, the proposal is referred to as "no-boundary" boundary condition. We now extend the model by not assuming that ϕ is frozen at a value corresponding to the vacuum energy

[4] This does not change any of our semi-classical analysis, but leads to an easier-to-solve equation whose solutions are known functions.

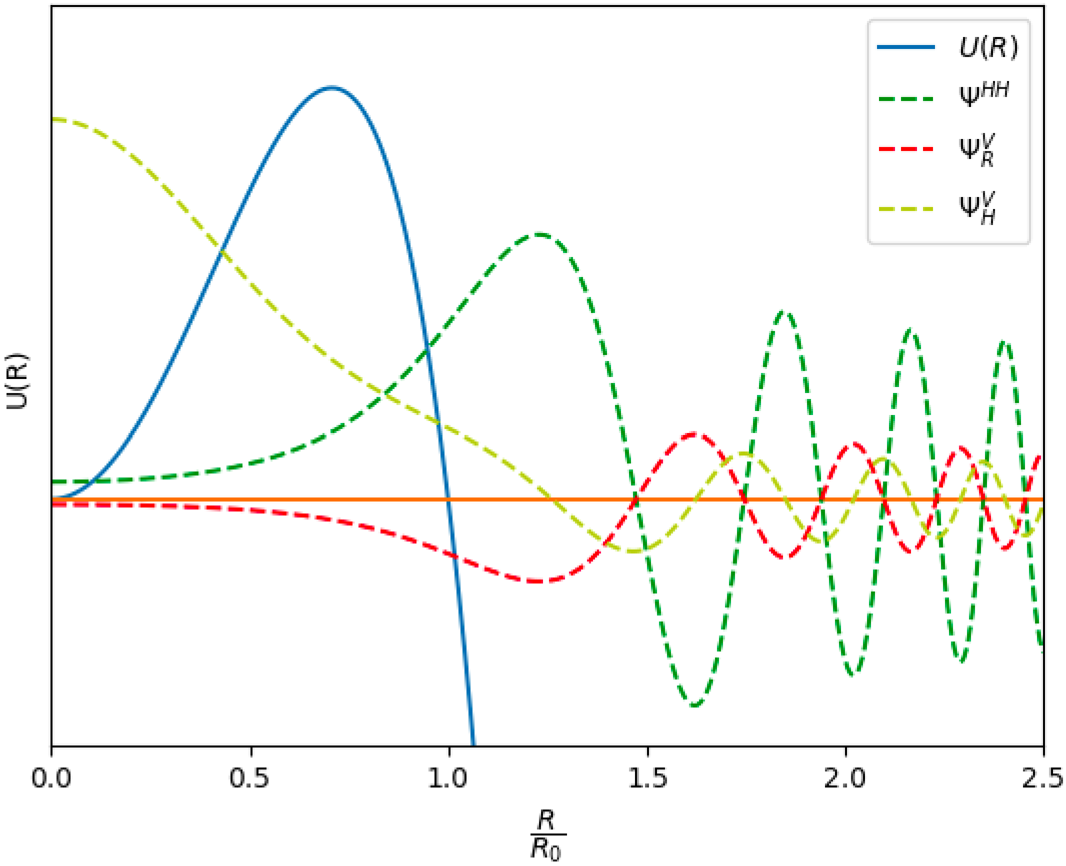


Figure 4: Mini-superspace wavefunctions for Hartle-Hawking (green) and Vilenkin (red and yellow) boundary conditions

$\rho_{VAC} = \dfrac{\Lambda}{8\pi G}$. Instead of assuming ϕ fixed now, assume that it is spatially homogeneous. The action is then given by

$$S = \int d^4x\sqrt{-g}\left[-\frac{\mathcal{R}}{16\pi G} + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)\right] \tag{71}$$

which in terms of the scale factor becomes

$$2\pi^2\int dt\left[-\frac{6R\dot{R}^2 - 6R}{16\pi G} + \frac{1}{2}R^3\dot{\phi}^2 - V(\phi)\right] \tag{72}$$

Now we have two degrees of freedom, ϕ and R, and the conjugate momentum corresponding to ϕ is given as

$$\pi_\phi := \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = 2\pi^2R^3\dot{\phi} \tag{73}$$

The hamiltonian density is therefore

$$H = \pi_R\dot{R} + \pi_\phi\dot{\phi} - \mathcal{L} \tag{74}$$

After pluggiing in the values of π_R, π_ϕ, the Wheeler-DeWitt equation $H\Psi(R,\phi) = 0$ becomes

$$\left[R^{-a}\frac{\partial}{\partial R}R^a\frac{\partial}{\partial R} - \frac{1}{R^2}\frac{3}{4\pi G}\frac{\partial^2}{\partial\phi^2} - U(R,\phi)\right]\Psi(R,\phi) = 0$$

where $U(R,\phi) = \dfrac{9\pi^2}{4G^2}\left[R^2 - R^4\dfrac{8\pi G}{3}V(\phi)\right]$. Take now the loose shoe approximation, $\phi = \phi_0 = const$. The $V(\phi_0)$ is then equivalent to the cosmological constant $\Lambda = 8\pi GV(\phi_0)$. If the potential has several extrema say at $\phi = \phi_i$, then using $P_T \propto \exp -S_E$, we observe that quantum tunneling exponentially favours the maximum with the largest value of $V(\phi_i)$. This result is compatible with one of the requirements of inflation, i.e., there is a scalar field displaced from the minimum of its potential, as we shall see shortly.

3 A review of scalar dynamics of Inflation

The simplest underlying field that can cause the exponential expansion of the universe within a fraction of a second is a single scalar field: The Inflaton Field ϕ. We don't specify the physical nature of the field ϕ, but simply use it as an order parameter to parameterize the time-evolution of the inflationary energy density. This itself provides several insights into the evolution of the universe. The dynamics of a field minimally coupled to gravity is governed by minimization of the following action:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] = S_{EH} + S_\phi \tag{75}$$

The action is the sum of the gravitational Einstein-Hilbert action, S_{EH}, and the action of a scalar field with canonical kinetic term, S_ϕ. The potential $V(\phi)$ describes the self interactions of the scalar field. The stress-energy tensor for scalar field is given by

$$T_{\mu\nu}(\phi) := -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\sigma\phi\partial_\sigma\phi + V(\phi)\right) \tag{76}$$

Equation of motion is given as

$$\frac{\delta S_\phi}{\delta\phi} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) + \frac{dV}{d\phi} \tag{77}$$

3.1 Slow-roll Inflation

Take FRW metric for $g_{\mu\nu}$ and restrict to a case of a homogeneous field $\phi(x,t) := \phi(x)$; the stress energy tensor reduces to that of a perfect fluid with

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{78}$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{79}$$

and so the ω corresponding to ϕ is

$$\omega_\phi := \frac{(1/2)\dot{\phi}^2 - V(\phi)}{(1/2)\dot{\phi}^2 + V(\phi)} \tag{80}$$

Thus, a scalar field can lead to accelerated expansion due to negative pressure ($\omega_\phi < -1/3$) if V dominates over the kinetic energy. We can substitute these expressions for the energy density and pressure in the Friedmann equations and the continuity equation for a flat FRW geometry. The Friedmann equations are given by:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} - \frac{k}{a^2} \tag{81}$$

and

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3P) \tag{82}$$

Multiplying equation (81) by a^2 and differentiating gives what is known as the continuity equation -

$$\dot{\rho} + 2\frac{\dot{a}}{a} - 6\frac{\dot{a}\ddot{a}}{a^2} = 0 \tag{83}$$

or

$$\frac{d\rho}{dt} + 3H(\rho + P) = 0 \tag{84}$$

Let us now substitute eq. (78) and (79) in eq. (84). From here, we get that

$$\frac{d\rho}{dt} + 3H(\rho_\phi + P_\phi) = \dot{\phi}\ddot{\phi} + \dot{V} + 3H\dot{\phi}^2 = 0 \tag{85}$$

Note that $\dot{V}$ is the derivative of $V(\phi)$ w.r.t ϕ. For a flat spacetime, $k = 0$, so we also have that

$$H^2 = \frac{\rho}{3} = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V\right) \tag{86}$$

Using equation (82),

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\left(2\dot{\phi}^2 - 2V(\phi)\right) \tag{87}$$

Rewriting this in terms of w_ϕ as defined in (80), we get

$$\frac{\ddot{a}}{a} = H^2\left(1 - \frac{3}{2}(1 + \omega_\phi)\right) \tag{88}$$

Here, using the equations derived above we define the first Hubble slow-roll parameter ϵ as:

$$\epsilon := \frac{3}{2}(1 + \omega_\phi) = \frac{\dot{\phi}^2}{2H^2} = -\frac{\dot{H}}{H^2} \tag{89}$$

so that

$$\frac{\ddot{a}}{a} = H^2(1 - \epsilon) \tag{90}$$

Clearly, accelerated expansion occurs if $\epsilon < 1$ (so that $\ddot{a} > 0$ if $a > 0$). The limit $P_\phi \to -\rho_\phi$ corresponds to $\epsilon \to 0$. In this limit the potential energy dominates over the kinetic energy, i.e.,

$$\dot{\phi}^2 << V(\phi) \tag{91}$$

This is the so called slow-roll limit for which the universe undergoes accelerated expansion in the presence of negative pressure (with $\omega_\phi - 1$). Accelerated expansion persists 'long enough' only if the second time derivative of ϕ in (85) is small enough. Thus, during inflation (slow-roll), we have

$$|\ddot{\phi}| << |3H\dot{\phi}|, |\dot{V}| \tag{92}$$

This enforces the smallness of the second Hubble slow-roll parameter η defined as

$$\eta := -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{93}$$

In the slow-roll limit ($\dot{\phi}^2 << V(\phi)$ the slow-roll conditions, $\epsilon, |\eta| < 1$ can be equivalently expressed in terms of the shape of the potential $V(\phi)$ as

$$\epsilon_V(\phi) := \frac{1}{2}\left(\frac{\dot{V}}{V}\right)^2 \tag{94}$$

and

$$\eta_V(\phi) = \frac{\ddot{V}}{V} \tag{95}$$

In the slow-roll regime the potential slow-roll parameters satisfy $\epsilon_V, |\eta_V| << 1$ The background evolution is thus given by

$$H^2 \approx \frac{1}{3}V(\phi) \approx const \tag{96}$$

and

$$\dot{\phi} \approx -\frac{\dot{V}}{3H} \tag{97}$$

The spacetime is approximately de Sitter with $a(t) \approx e^{Ht}$. Using equation (97) in the slow-roll limit, we get

$$\epsilon_V = \frac{1}{2}\left(\frac{\dot{V}}{V}\right)^2 \approx \frac{1}{2}\left(\frac{3H\dot{\phi}}{V}\right)^2 \approx \frac{3}{2}\left(\frac{\dot{\phi}^2}{V^2}\right) \approx \frac{\dot{\phi}^2}{2H^2} = \epsilon \tag{98}$$

Differentiating LHS and RHS of (97) w.r.t ϕ, we have

$$\ddot{V} \approx -3\dot{H}\dot{\phi} - 3H\ddot{\phi} \tag{99}$$

$$\implies \eta = -\frac{\ddot{\phi}}{H\dot{\phi}} = -\frac{\dot{H}\dot{\phi}}{H^2} + \frac{\ddot{V}}{3H^2} = \frac{\ddot{V}}{V} - \frac{\dot{V}^2}{V^2} = \eta_V - \epsilon_V \tag{100}$$

Thus, as long as the slow-roll approximation holds, the Hubble slow-roll parameters and the potential slow-roll parameters are related as follows

$$\epsilon_V \approx \epsilon \tag{101}$$

and

$$\eta \approx \eta_V - \epsilon_V \tag{102}$$

Inflation ends when the slow-roll conditions are violated, i.e, $\epsilon(\phi_{end}) = 1$ and $\epsilon_V(\phi_{end}) \approx 1$. To solve the flatness and horizon problem the number of e-folds before inflation ends must be ≥ 60. This has been discussed briefly in the next section.

4 Eternal Inflation and its implications

4.1 Inflation and the Cauchy problem of the universe

To specify the initial condition of the universe we consider a spatial slice of constant time Σ. On the 3-surface Σ we define the positions and velocities of all matter particles. The laws of gravity and fluid dynamics are then used to evolve the system forward in time. There are several puzzles that arise in the theory of the big bang. The theory of inflation helps resolve most of them. These puzzles have been explained briefly below. Much of this section has been taken from [5].

4.1.1 Size of the universe

In "standard" FRW cosmology, without inflation, one simply postulates that about 10^{90} or more particles were here from the start. However, in the context of present-day cosmology, many of us hope that even the creation of the universe can be described in scientific terms. Thus, we are led to at least think about a theory that might explain how the universe got to be so big. The easiest way by far to get a huge number, with only modest numbers as input, is for the calculation to involve an exponential. The exponential expansion of inflation reduces the problem of explaining 1090 particles to the problem of explaining 60 or 70 e-foldings of inflation. Inflation is therefore losely understood as a rapid expansion of the universe shortly after it was formed.

4.1.2 Initial Homogeneity or the Horizon problem

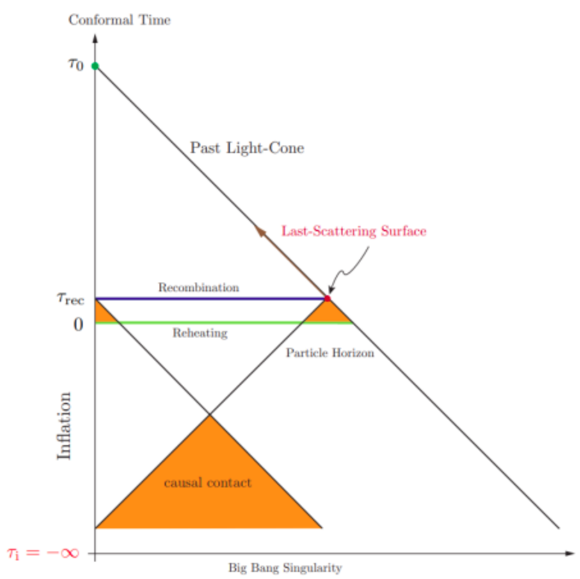


Figure 5: Conformal diagram for Inflation, the Big Bang singularity is pushed to conformal time $-\infty$

In FRW cosmology, we often take the universe to be homogeneous and isotropic. This is done because inhomogeneities are gravitationally unstable and therefore grow with time. CMB (Cosmic Microwave Background) observations reveal that inhomogeneities were small in the past, which means that they should have been even smaller earlier (since the CMB is so homogeneous). However, in the conventional Big Bang picture, we see that the early universe (CMB at last scattering) consisted of causally disconnected regions of space. There is no dynamical reason to explain the inhomogeneity of the universe in this picture. Mathematically describing the problem, we define the comoving horizon or the causal horizon as the maximum distance a light ray can travel between $t = 0$ and some time t. This can be written as

$$\tau = \int_0^t \frac{dt}{a} \tag{103}$$

This means

$$\tau = \int_0^a \frac{da}{a^2 H} = \int_0^a (d\ln a)\frac{1}{aH}) = \tag{104}$$

where $(aH)^{-1}$ is defined as the comoving Hubble radius, which is important in Inflation. In FRW cosmology, we know that the Hubble parameter H varies with time as $H \propto a^{-\frac{3}{2}(1+\omega)}$. From here, we get

$$\tau \propto a^{\frac{1}{2}(1+3\omega)} \tag{105}$$

where $\omega = \frac{1}{3}$ for radiation dominated (RD) universe and $\omega = 0$ for Matter-dominated universe. From here we get that for RD universe $\tau \propto a$ and for MD universe, $\tau \propto \sqrt{a}$. This means that the comoving horizon grows monotonically with time, which implies that comoving scales entering the horizon today have been far outside the horizon at CMB decoupling. This problem is resolved in Inflation as we require that the comoving Hubble radius $(aH)^{-1}$ shrinks during inflation, which extends conformal time to negative values. Therefore, any two patches of the universe are causally connected for negative conoformal times.

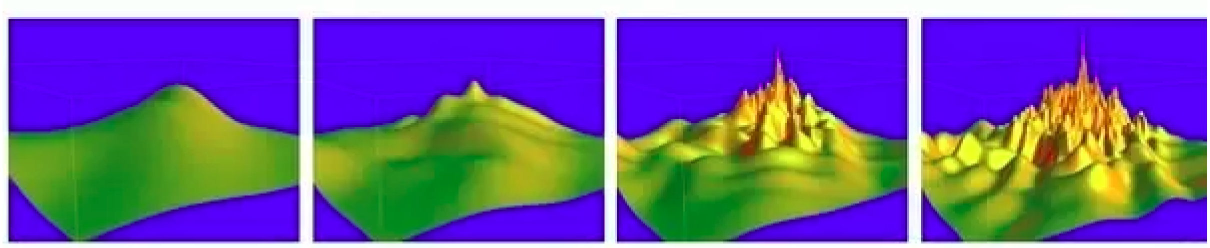


Figure 6: In this computer generated sequence, the universe evolves, inflating and expanding its terrain. The gentle valleys represent quiescent cosmic zones where all is stable (the bubble universes). The peaks symbolize the inflationary engine of universe creation.

4.1.3 Flatness problem

The flatness problem concerns the quantity $\Omega_k = \Omega - 1$ (assuming no contribution from the cosmological constant) $= \dfrac{\rho - \rho_{crit}}{\rho_{crit}}$ where $\rho_{crit} = 3H^2$. From the second Friedmann equation we know that

$$H^2 = \frac{\rho(a)}{3} - \frac{k}{a^2} \tag{106}$$

and let us define $\Omega(a)$ (which is different from Ω_k or Ω which are written as functions of time) as

$$\Omega(a) = \frac{\rho}{\rho_{crit}} \tag{107}$$

Also, by definition, $\rho = 3H^2\Omega$. We therefore get that

$$1 - \Omega(a) = \frac{-k}{a^2H^2} \tag{108}$$

In standard FRW cosmology, the Hubble radius grows with time, and so for $\Omega(a_0) \approx 1$ we need for Ω to have been extremely fine-tuned in the early universe as $|1 - \Omega|$ diverges with time. This problem is again solved in inflation as the Hubble radius decreases during Inflation. Added to these, the theory of Inflation solves several other problems, including the absence of magnetic monopoles [6].

4.2 Eternal Inflation

4.2.1 Eternal New Inflation

In quantum field theory, a false vacuum is a hypothetical vacuum that is relatively stable, but not in the most stable state possible. In this condition it is called metastable. It may last for a very long time in this state, but could eventually decay to the more stable one, an event known as false vacuum decay. A false vacuum exists at a local minimum of energy and is therefore not completely stable, in contrast to a true vacuum, which exists at a global minimum and is stable. Classically, there would be no way for a scalar field in a local minima to cross the potential barrier and reach the global minima.

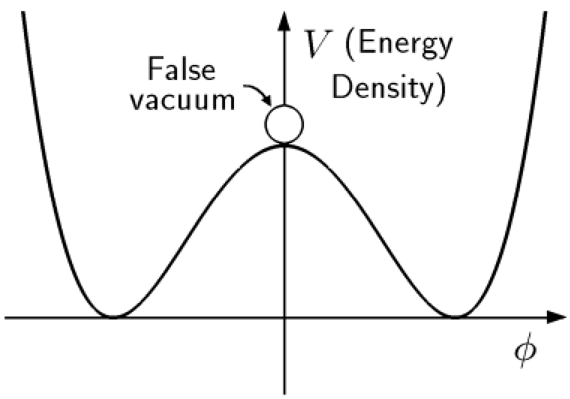
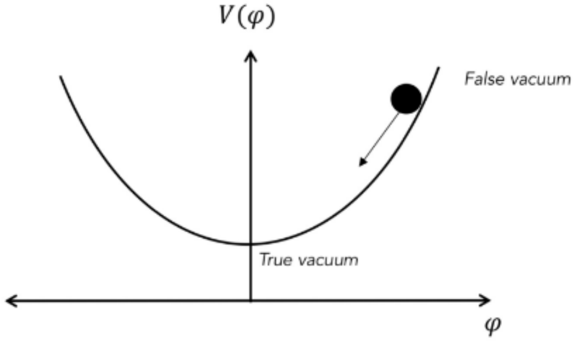
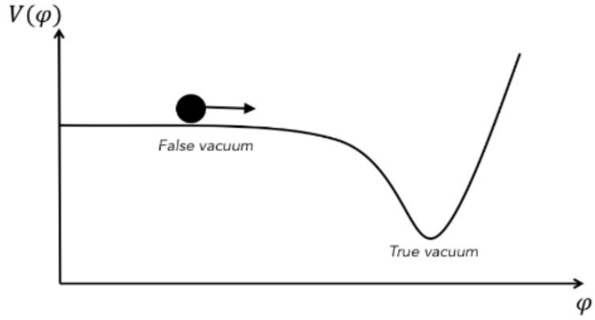


Figure 7: Generic form of the potential for the new inflationary scenario.

In the case of new inflation, the exponential expansion occurs as the scalar field rolls from the false vacuum state at the peak of the potential energy diagram (see Fig. 5) toward the trough. The eternal aspect occurs while the scalar field is hovering around the peak. This occurs because even though classically the field would roll off the hill, quantum-mechanically there is always an amplitude for it to remain at the top. Just like the decay of most metastable states, the false vacuum state decays exponentially with time in most models of new inflation. This means that the probability of finding the inflaton field in a metastable state, for example, as shown in fig. 5, does not fall sharply to zero but instead decays exponentially with



(a) Prototypical large-field model or chaotic inflation where a single monomial term dominates the potential



(b) Single-field slow roll inflation

time. The time scale for the decay of the false vacuum is controlled by

$$m^2 := -\frac{\partial^2 V}{\partial \phi^2}|_{\phi=0} \tag{109}$$

or wherever the inflaton field is at the top of the potential hill (this is $\phi = 0$ for fig. 5 and fig. 6(a)). This is an adjustable parameter as far as our use of the model is concerned, but m has to be small compared to the Hubble constant or else the model does not lead to enough inflation [7]. Since Hubble constant is a measure of the expansion of the false vacuum, we can conclude that the false vacuum state expands (or spreads out) faster than it decays. This means that Inflation does not end everywhere at once, but in localized patches. These localized patches are known as 'bubble universes' or 'pocket universes'.

4.2.2 Eternal chaotic Inflation

Chaotic inflation means chaotic initial conditions, i.e., different initial inflaton field values in different patches of space. The initial inflationary models had potentials with features such as local minima and plateaus, which were required to make the model work. But if we assume chaotic initial conditions, then inflation can work even with a very simple, polynomial potential. Therefore, chaotic inflation is often interchangeably used with polynomial potential, but we can make the potential more complex if we want to. As the field is classically rolling down the potential (Fig. 6(a)) the change in the field $\Delta\phi$ during time Δt will be modified by quantum fluctuations. Call this change in ϕ due to quantum fluctuations $\Delta\phi_{qu}$, which can be positive or negative, depending on the trajectory. During one time interval of duration $\Delta t = H^{-1}$ (one Hubble time) let us see what happens to one Hubble volume, H^{-3}. By definition, we know that in Hubble time, the universe expands by a factor of e. Therefore, the Hubble volume grows by a factor of $e^3 \approx 20$. Since correlations typically extend over about a Hubble length, by the end of one Hubble time, the initial Hubble-sized region grows and breaks up into 20 independent Hubble-sized regions.

Now the change in ϕ as this happens is

$$\Delta\phi = \Delta\phi_{cl} + \Delta\phi_{qu} \tag{110}$$

We calculate $\Delta\phi_{qu}$ using free quantum field. This derivation has not been presented here, but from [8] and [9], we get that in the first order perturbation theory, $\langle\Delta\phi_{qu}\rangle$, the quantum fluctuation averaged over one of the 20 Hubble volumes at the end of time interval Δt will have a Gaussian probability distribution of order $\frac{H}{2\pi}$. Explicitly ([8] & [10]), let us first take the definition of the squared mass of the potential as given in eq. (109), assuming that the field is at a local minimum at $\phi_0 = 0$. Then for small values of ϕ,

$$V(\phi) \approx -\frac{1}{2}m^2\phi^2 \tag{111}$$

Now, this field also satisfies the classical field equation,

$$(\Box - m^2)\phi = 0 \tag{112}$$

We describe the growth of quantum fluctuations by $\langle \phi^2 \rangle$. Technically, this is a divergent quantity and should be renormalized, but we can subtract this constant infinite part so that

$$\langle \phi^2 \rangle = 0 \tag{113}$$

at $t = 0$. For $m << H$, we get (not derived here)

$$\langle \phi^2 \rangle = \frac{3H^4}{8\pi^2 m^2} \left[\exp\left(\frac{2m^2 t}{3H} \right) - 1 \right] \tag{114}$$

For $t << \dfrac{H}{m^2}$ this is equal to

$$\langle \phi^2 \rangle \approx \frac{H^3 t}{4\pi^2} \tag{115}$$

Therefore, under the approximations as stated above, $\langle \phi^2 \rangle \propto t$. This can be pictured as a Brownian motion of the field ϕ. As a result of the quantum fluctuations, the magnitude of ϕ on the horizon scale changes by $\pm \left(\dfrac{H}{2\pi} \right)$ per expansion time (H^{-1}). This has been derived formally in [11], where they derive $\Delta\phi$ to have a Gaussian probability distribution, with a width of order $\dfrac{H}{2\pi}$. Now, there is some probability that the RHS of eq. (110) is positive, i.e., the scalar field fluctuates up and not down. If this probability $> \dfrac{1}{20}$, then the number of inflating regions with $\phi > \phi_0$ would have increased after Δt. Now for each of these inflating regions, we start at the beginning, and the process goes on forever, and so the inflation is eternal. Therefore, the criterion for eternal inflation is that the probability for the scalar field $\Delta\phi$ to go up must be bigger than $1/e^3 \approx 1/20$. For a Gaussian probability distribution $\Delta\phi_{qu}$, this condition will be met provided that the standard deviation should be bigger than $0.61\Delta\phi_{cl}$. Now we can approximate $\Delta\phi_{cl}$ as

$$\Delta\phi_{cl} \approx \dot{\phi}_{cl} H^{-1} \tag{116}$$

We get from here

$$\Delta\phi_{qu} \approx \frac{H}{2\pi} > 0.61 |\dot{\phi}_{cl}| H^{-1} \tag{117}$$

$$\implies \frac{H^2}{|\dot{\phi}_{cl}|} > 3.8 \tag{118}$$

The probability that $\Delta\phi$ is positive tends to increase as one considers larger and larger values of ϕ, so sooner or later one reaches the point at which inflation becomes eternal. Let us now consider the example of the case where the scalar field has a potential $V(\phi) = \dfrac{1}{4}\lambda\phi^4$. The equation of motion for the field evolution is therefore

$$\ddot{\phi} + 3H\dot{\phi} + \lambda\phi^3 = 0 \tag{119}$$

In the slow-roll approximation, $\ddot{\phi} \approx 0$ and so we get

$$\dot{\phi} \approx -\frac{\lambda\phi^3}{3H} \tag{120}$$

where the Hubble constant is given as

$$H^2 = \frac{8\pi G}{3}\rho = \frac{2\pi\lambda}{3M_{pl}^2}\phi^4 \tag{121}$$

Condition for eternal inflation then becomes

$$\phi > 0.75\lambda^{-1/6} M_{pl} \tag{122}$$

Tha magnitude of λ is estimated based on the magnitude of density perturbations. This comes out to be $\approx 10^{-12}$. From here it is easy to see that ϕ comes out to be well above the Planck scale for this value of λ ($\approx 75M_{pl}$). The corresponding energy density however comes out to be well below the Planck scale, which has led to the belief that inflation is almost always eternal.

4.3 Difficulties in calculating probabilities

As soon as we attempt to define probabilities in an eternally inflating spacetime, we discover ambiguities. The problem is that the sample space is infinite, i.e., an eternally inflating universe produces an infinite number of pocket universes. The fraction of universes with any particular property is therefore equal to infinity divided by infinity, which is a meaningless ratio. To obtain a well-defined answer, we need to invoke some method of regularization.

To understand the nature of the problem, let us think about the integers as a model system with an infinite number of entities. We can then ask what fraction of the integers are odd. The most logical answer would be $1/2$. That is, if the string of integers is truncated after the N, then the fraction of odd integers in the string is exactly $1/2$ if N is even, and is $(N+1)/2N$ if N is odd. In any case, the fraction approaches $1/2$ as N approaches infinity. However, the ambiguity of the answer can be seen if one imagines other orderings for the integers. We may order integers as

$$1, 3, 2, 5, 7, 4, 9, 11, 6, ...$$

always writing two odd integers followed by one even integer. This series includes each integer exactly once, the integers are just arranged in an unusual order. However, if we truncate the sequence as shown above after the Nth entry, and then take the limit $N \to \infty$, we would conclude that $2/3$ of the integers are odd. Thus, we find that the definition of probability on an infinite set requires some method of truncation, and that the answer can depend nontrivially on the method that is used.

In the case of eternally inflating spacetimes, the natural choice of truncation might be to order the pocket universes in the sequence in which they form. However, since any two pocket universes are spacelike separated, this order also would vary from observer to observer.

5 Tunneling wave function

5.1 Using analytic continuation

To introduce the tunneling wavefunction, we assume the case of a closed FRW universe filled with a vacuum of constant energy density ρ_v. Let the parameter characterizing the amount of radiation be ϵ. Therefore, the total energy density of the universe is given by

$$\rho = \rho_v + \frac{\epsilon}{a^4} \tag{123}$$

The evolution equation, given by the second Friedmann equation is written as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{124}$$

where $k = 1$ in this case as the universe is taken to be closed. Substituting the energy density as derived in eq. (123), we get

$$p^2 + a^2 - \frac{a^4}{a_0^2} = \epsilon \tag{125}$$

where $p = -a\dot{a}$ and $a_0 = \dfrac{3}{4\sqrt{\rho_v}}$ (this is derived after taking Gaussian units). If we think of p as the momentum conjugate to a, we can think of eq. (125) as an equation for a 'particle' of energy ϵ moving in a potential which varies as a function of a as $U(a) = a^2 - \dfrac{a^4}{a_0^2}$. Note that this potential is the same as the one shown in Fig. (3). Since this potential has two classical turning points, there are two possible classical trajectories for a sufficiently small ϵ (so that it intersects the potential curve at two distinct points). Let the two distinct points of this intersection be a_1 and a_2. The universe can then start at $a = 0$, expand to a maximum radius a_1, at which $\dot{a}$ becomes 0. It then recollapses. Another possibility is for it to start with an infinite size, reach a minimum radius a_2, and then re-expand. We can extend the first possibility in the quantum cosmology picture. If we think of the universe as a 'particle' bound in potential $U(a)$, it can now quantum mechanically tunnel through the barrier instead of just recollapsing once it reaches $a = a_1$.

We treat this case semi-classically, and apply WKB approximation to find the semiclassical tunneling probability. This is given as

$$\mathcal{P} \approx e^{-2\gamma} \tag{126}$$

where

$$\gamma = \int_{a_1}^{a_2} |p(a)| da \tag{127}$$

$p(a)$ can be derived as a function of a using eq. (125). Note that when $\epsilon \to 0$, $|p(a)| = \left|\sqrt{\dfrac{a^4}{a_0^2} - a^2}\right|$ and so the corresponding probability is given by

$$\mathcal{P} \approx e^{-2\int_0^{a_0} |p(a)| da} = e^{-\dfrac{3}{8\rho_v}} \tag{128}$$

(the classical turning points are now 0 and ϵ). Thus, in this limit, the probability of tunneling is still non-zero. This means that in this case of absence of radiation of initially, tunneling takes place from nothing to a closed universe of radius a_0. Just as we did in Sec. 2.1.2, we now perform canonical quantization for eq. (125) by replacing the conjugate momentum p with the corresponding quantum mechanical operator $-i\dfrac{d}{da}$ to obtain the Wheeler-DeWitt equation for this simple model:

$$\left(\frac{d^2}{da^2} - a^2 + \frac{a^4}{a_0^2}\right)\psi(a) = 0 \tag{129}$$

(where $\epsilon = 0$)[5]. In the classically allowed region, this equation has outgoing and ingoing solutions. This similar to the two directions of propagation for the case of a potential barrier, except now the propagation is in the variable a, so the outgoing and ingoing solutions correspond to expanding and contracting solutions respectively. In the classically forbidden region ($0 < a < a_0$), the wavefunction has exponentially decaying and growing

[5] Again disregarding the ordering of operators

solutions. Near the classical turning point, these have similar magnitudes at $a = a_0$, but from our knowledge of Quantum Mechanics, the exponentially decaying solution dominates in the classically forbidden region, since the wavefunction has no reason to grow exponentially in the barrier.
The tunneling probability can also be expressed in the language of instantons.

Let us first understand what instantons are. An instanton is a classical solution to equations of motion with a finite, non-zero action, either in quantum mechanics or in quantum field theory. More precisely, it is a solution to the equations of motion of the classical field theory on a Euclidean spacetime. In such quantum theories, solutions to the equations of motion may be thought of as critical points of the action. The critical points of the action may be local maxima of the action, local minima, or saddle points. Instantons are important in quantum field theory because they appear in the path integral as the leading quantum corrections to the classical behavior of a system. The nucleated universe after tunneling is described by de Sitter space, and the under-barrier evolution can be semiclassically represented by the Euclideanized de Sitter space. This de Sitter instanton has the geometry of a four-sphere. By matching it with the Lorentzian de Sitter at $a = a_0$ we can symbolically represent the origin of the universe as shown in fig. (9). For 'normal' quantum tunneling (without gravity), the tunneling probability P is proportional to $exp(-S_E)$, where S_E is the instanton action. In our case,

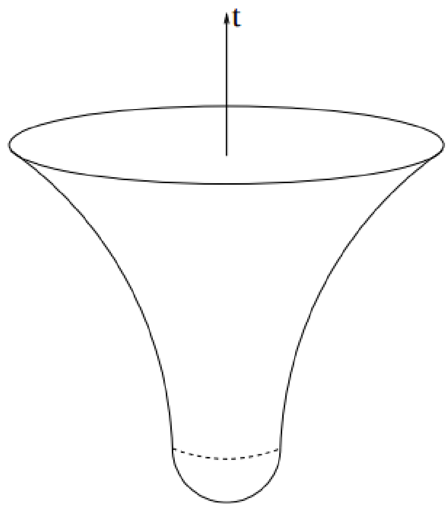


Figure 9: Birth of an inflationary universe: a schematic representation

$$S_E = \int d^4x\sqrt{-g}\left(-\frac{R}{16\pi G} + \rho_V\right) = -2\rho_V\Omega_4 = -\frac{3}{8}G^2\rho_V. \tag{130}$$

where $R = 32\pi G\rho_V$ is the scalar curvature, and $\Omega_4 = \dfrac{4\pi^2}{3}a_0^4$ is the volume of the four sphere.Hence, it is concluded in [12] that the probability of transition, $P \propto \exp\left(3/8G^2\rho_V\right)$.

5.2 Using a path integral

The tunneling wave function can alternatively be defined as a path integral

$$\psi_T(g,\phi) = \textstyle\int_{\emptyset}^{(g,\phi)} e^{iS}$$

where the integration is over paths interpolating between a vanishing 3-geometry $\emptyset$ ('nothing') and (g,ϕ). This definition is considered equivalent to the tunneling boundary condition in a wide class of models. Note that according to the definition given above, ψ_T is strictly not a wave function, but rather a propagator

$$\psi_T(g,\phi) = K(g,\phi|\emptyset)$$

where $K(g,\phi|g',\phi')$ is given by the path integral taken over Lorentzian histories interpolating between (g',ϕ') and (g,ϕ). One expects, therefore, that ψ_T should generally be singular at $g = \emptyset$. At present, the general definitions of the tunneling wave function remain largely formal, since it is not known how to solve the Wheeler-DeWitt equation, assuming that the topology of the universe[6] is fixed (possible modification of this equation accounting for topology change have been discussed in [13]) or how to calculate the path integral, except for simple models and small perturbations about them, or in the semiclassical limit.

5.3 An example of transition to TV bubble using analytic continuation

Let us begin by recalling tunneling in QM. If a particle faces a potential barrier of say height V_0 and width d, the probability of transmission/transmission coefficient is given by

$$|T|^2 = \frac{1}{1 + \dfrac{V_0^2\sinh^2\Omega d}{4E(V_0 - E)}} \tag{131}$$

[6]The "creation of the universe from nothing" is an example of a topology changing event.

where E is the energy of the particle and

$$\Omega d = \frac{1}{\hbar}\int_0^d \sqrt{2m(V_0 - E)} \tag{132}$$

Let us now consider a classical particle moving in potential well (much like a round bowl) of height ΔV. Then if $\dot{x}$ is the maximum velocity of this particle, it must be that

$$\Delta V = \frac{1}{2}\dot{x}^2 \tag{133}$$

$$\implies \int \sqrt{2\Delta V} dx = \int 2\Delta V dt = \int \left(\Delta V + \frac{1}{2}\dot{x}^2\right) dt \tag{134}$$

Now let us relate the classical and the quantum case. The Lagrangian of the quantum mechanical particle in a potential barrier is given as

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 - V \tag{135}$$

Let us now make a coordinate transformation $t \to i\tau$ (this is usually called a Wick rotation). The lagrangian transforms to

$$\mathcal{L}' = -\frac{1}{2}\dot{x}^2 - V = -\left(\frac{1}{2}\dot{x}^2 - (-V)\right) \tag{136}$$

which represents a particle moving in an inverted well. Thus, in this way, we can capture the main aspect of tunneling probability by classical Euclidean motion, which is where the role of instantons comes in. Tunneling amplitude according to WKB approximation is given by $e^{-2\gamma}$ where $\gamma = \frac{1}{\hbar}\int p dx$. We see that this is exactly the same as $e^{-S_E/\hbar}$ where S_E is the corresponding Euclidean action. Let us now perform this treatment for false vacuum decay, as presented in [14]. As mentioned in the last section, false vacuum is a metastable state, thus the final Euclidean action that we'll be considering will be the Euclidean action of the field from the metastable point to emergence and back. The Lagrangian is given as

$$\mathcal{L}_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{137}$$

Let us take

$$V = \frac{\lambda}{2}(\phi^2 - \eta^2)^2 - \frac{\epsilon}{2\eta}(\phi - \eta) \tag{138}$$

Note that this potential has two local minima in ϕ, $\phi = \pm\eta$. However, V for the case pf $\phi = -\eta$ is shifted up by ϵ while it is 0 for $\phi = \eta$. This means that $\phi = -\eta$ represents the false vacuum or metastable state while $\phi = \eta$ is the true vacuum (TV from hereon). Let us first Euclideanize the action and the lagrangian ($t \to i\tau$). The equation of motion thus obtained is

$$\frac{d^2\phi}{d\tau^2} + \nabla^2\phi = \frac{\partial V}{\partial \phi} = 2\lambda\phi(\phi^2 - \eta^2) - \frac{\epsilon}{2\eta} \tag{139}$$

The transition from FV to TV is just like a phase transition, not the entirety of the FV decays into TV because of the potential barrier, but we get a bubble of TV. We expect these bubbles of TV to possess hyper-spherical symmetry in the Euclidean space (since nucleated universe after tunneling is described by de Sitter space). So if the radius of a bubble is ρ_0, we get that $\rho_0^2 = \tau^2 + x^2$. Assume now that ϵ is a small number. Secondly, we assume that since ϵ is small, we should expect a large bubble, or $\rho_0 >> \frac{1}{\sqrt{\pi}\eta}$. Using these two approximations, we find ϕ to be

$$\phi \approx \eta \tanh\left(\sqrt{\pi}\eta(\rho - \rho_0)\right) \tag{140}$$

We've now got a Euclidean solution, which is called the instanton, which interpolates from a FV to a bubble of TV at $\epsilon = 0$. If we now analytically continue this solution to Lorentzian spacetime, we find that bubble of TV forms and expands. The approximation is self-consistent as $\rho_0 = \frac{3\sigma}{\epsilon}$, σ being the tension of the domain wall, and since ϵ is small, $\rho_0 >> 1$.

5.4 Alternative proposals to the wavefunction

We have already discussed in detail the Hawking Hartle wavefunction in section 2. Let us look at some other proposals for wavefunction of the universe.

5.4.1 The DeWitt wave function

DeWitt suggested that the wave function should vanish for the vanishing scale factor,

$$\psi(a = 0) = 0 \tag{141}$$

The motivation for this is that $a = 0$ corresponds to the cosmological singularity, so the above equation implies that the probability for the singularity to occur is zero. In most general models it tends to give an identically vanishing wave function. No generalizations of the DeWitt boundary condition have yet been proposed.

5.4.2 The Linde wave function [1], [2]

Linde suggested that the wave function of the universe is given by a Euclidean path integral like ψ_{LL} but with the Euclidean time rotation performed in the opposite sense, $t \to +i\tau$, giving us

$$\psi_L \approx \int^{(g,\phi)} e^{S_E} \tag{142}$$

The problem with this proposal is that the Euclidean action is unbounded from above and the path integral is divergent. If one regards this as a general definition that applies beyond the simple model, integrations over matter fields and over inhomogeneous modes of the metric are divergent. The Linde wave function includes only the decaying solution under the barrier and a superposition of ingoing and outgoing modes with equal amplitudes outside the barrier.

5.5 Semiclassical probabilities

Quantum cosmology has some problems which arise when one tries to quantize a closed universe. The first problem stems from the fact that ψ, the wavfunction of the universe, is independent of time. This can be understood in the sense that the wave function of the universe should describe everything, including the clocks which show time. In other words, time should be defined intrinsically in terms of the geometric or matter variables. However, we have not yet found a general prescription that would give a function $t(g_{ij}, \phi)$ that would be "monotonic".
Another consideration is the way we calculate probabilities. Currently, the accepted definition is in terms of the conserved current of the Wheeler-DeWitt equation. We can rewrite the Wheeler-DeWitt equation symbolically as

$$(\nabla^2 - U)\psi = 0 \tag{143}$$

where U is a functional of g_{ij} and ϕ. This equation looks similar to Klein Gordon equation and has the conserved current

$$J = i(\psi * \nabla\psi - \psi\nabla\psi*), \nabla \cdot J = 0 \tag{144}$$

The conservation is a useful property, since we want probability to be conserved. But one runs into the same problem as with the Klein-Gordon equation: the probability defined above is not positive-definite. Although we do not know how to solve these problems in general, they can both be solved in the semiclassical domain. Let us consider the situation when some of the variables c describing the universe behave classically, while the rest of the variables q must be treated quantum-mechanically. The wavefunction of the universe can then be written as

$$\psi = \Sigma_k A_k(c) e^{iS_k(c)} \chi_k(c, q) := \Sigma_k \psi_k^{(c)} \chi_k \tag{145}$$

where the classical variables are described by the WKB wave functions $\psi_k^{(c)} = A_k(c)e^{iS_k(c)}$. Substituting this form of the wavefunction into the original WheelerDeWitt equation, we get

$$\nabla S \cdot \nabla S + U = 0 \tag{146}$$

The different solutions to this equation give us the wavefunctions to take summation over in the definition of the wavefunction of the universe. Hence, a semiclassical wave function $\psi^{(c)} = Ae^{iS}$ describes an ensemble of classical universes evolving along the trajectories of $S(c)$. A probability distribution for these trajectories can be obtained using the conserved current. Since the variables c behave classically, these probabilities do not change in the course of evolution and can be thought of as probabilities for various initial conditions. The time variable t can be defined as any monotonic parameter along the trajectories, and it can be shown that in this case the corresponding component of the current J is non-negative, $J_t \geq 0$. Moreover, one finds that the 'quantum' wave function χ satisfies the usual Schrodinger equation [2]

$$i\frac{d\chi}{dt} = H_\chi \chi \tag{147}$$

with an appropriate Hamiltonian H_χ. This semiclassical interpretation of the wave function ψ is valid to the extent that the WKB approximation for ψ_c is justified and the interference between different terms in the expansion of ψ can be neglected. Otherwise, time and probability cannot be defined, suggesting that the wave function has no meaningful interpretation. This makes sense, since in a universe where no object behaves classically (that is, predictably), no clocks can be constructed, no measurements can be made, and there is nothing to interpret.

6 Interplay between Eternal Inflation and the tunneling wavefunction

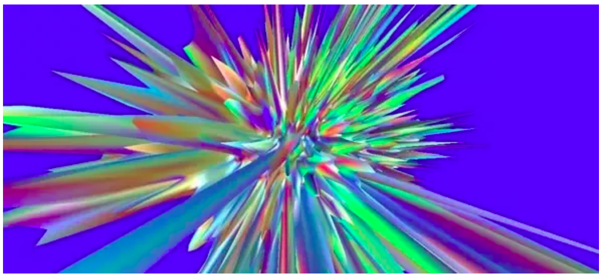


Figure 10: Fractal landscape resulting from eternal inflation

In section 4.2 we have seen that a generic feature of inflation is that it is of eternal character (see, for example, sec. 4.2.2). The evolution of the inflaton field ϕ is influenced by quantum fluctuations, and as a result thermalization (can be thought of as creation of bubbles of true vacuum) does not occur simultaneously in different parts of the universe. At any time, the universe consists of post-inflationary, thermalized regions embedded in an inflating background. Thermalized regions grow by annexing adjacent inflating regions, and new thermalized regions are constantly formed in the midst of the inflating background. At the same time, the inflating regions expand and their combined volume grows exponentially with time (sec. 4.2). It has also been shown that inflating regions form a self-similar fractal of dimension < 3 (see fig. 8) [15]. A natural question would then be if the universe could also be past eternal. . If it could, we would have a model of an infinite, eternally inflating universe without a beginning. We would then need no initial or boundary conditions for the universe, and quantum cosmology would arguably be unnecessary. This possibility was discussed in the early days of inflation, but it was later realized that the idea could not be implemented in the simplest model in which the inflating universe is described by an exact de Sitter space. The reason is that in the full de Sitter space, exponential expansion is preceded by an exponential contraction. If thermalized regions were allowed to form all the way to the past infinity, they would rapidly fill the space, and the whole universe would be thermalized even before the inflationary expansion could begin. It has also recently been shown that inflationary spacetimes are geodesically incomplete in the past[7]. This means that such spacetimes admit some incomplete geodesic, namely a geodesic which does not exist for all values of affine parameter. However, it is now believed that one of the key assumptions made in these theorems, the weak energy condition, is likely to be violated by quantum fluctuations in the inflating parts of the universe.

There is a theorem by Arvind Borde and Alan Guth which eliminates the possibility of not needing initial conditions. The theorem assumes that (i) the spacetime is globally expanding and (ii) that the expansion rate is bounded below by a positive constant, i.e.,

$$H > H_{min} \geq 0 \tag{148}$$

where the Hubble expansion rate H is defined as usual, as the relative velocity divided by the distance, with all quantities measured in the local comoving frame. Since we do not assume any symmetries of spacetime, the expansion rates are generally different at different places and in different directions. The bound is assumed to be satisfied at all points and in all directions. This is a very reasonable requirement in the inflating region of spacetime. The theorem then states that any spacetime with these properties is past geodesically incomplete.

It follows from the theorem that the inflating region has a boundary in the past, and some new physics (other than inflation) is necessary to determine the conditions at that boundary [2]. Quantum cosmology is the prime candidate for this role. The picture suggested by quantum cosmology is that the universe starts as a small, closed 3-geometry and immediately enters (or tunnels into) the regime of eternal inflation, with new thermalized regions being constantly formed.

[7] If such an incomplete geodesic exists, the spacetime is singular; an observer on such a timelike geodesic trajectory experiences a catastrophic halt to their proper time, as their worldline cannot be extended past a particular spacetime event. In the case of a curvature singularity an observer's clock stops because a spatial defect is encountered [16].

It is difficult to apply this theorem to general inflationary models, since there is no accepted definition of what exactly defines this class. However, in standard eternally inflating models, the future of any point in the inflating region can be described by a stochastic model for inflaton evolution, valid until the end of inflation, except for extremely rare large quantum fluctuations. The past for an arbitrary model is less certain, but we consider eternal models for which the past is like the future [7].

6.1 Testability of quantum cosmology in eternal inflation picture

In the picture of eternal inflation, the universe quickly forgets its initial conditions; since the number of thermalized regions to be formed in an eternally inflating universe is unbounded, a typical observer is removed arbitrarily far from the beginning, and the memory of the initial state is completely erased. This implies that any predictions that quantum cosmology could make about the initial state of the universe cannot be tested observationally. All three proposals for the wave function of the universe are therefore in equally good agreement with observations, as well as a wide class of other wave functions – as long as they give a non-vanishing probability for eternal inflation to start. There are some other wavefunctions as well, but they do not display semiclassical behaviour, and therefore cannot allow the existence of observers. The only case that requires special consideration is when there are some constants of nature, α_j, which are constant within individual universes, but can take different values in different universes of the ensemble (for example, the cosmological constant). In such a case, the memory of the initial state is never erased completely, since the values of α_j are always equal to their initial values. Following some of the discussion above, we can find the probability distribution of a universe to nucleate, given some value of α_j. Therefore, in an eternally inflating universe, the probability of observing some constant of nature α_j is given by

$$P_{obs}(\alpha) \propto P_{nucl}(\alpha)N(\alpha) \tag{149}$$

where $N(\alpha)$ is the number of observers in a particular bubble universe, the dependence of which on α is given by

$$N(\alpha;t) = B(\alpha)\exp\left(\chi(\alpha)t\right) \tag{150}$$

where $\chi(\alpha)$ is the rate of growth for that particular universe. This suggests that the most probable values of α_j should be the ones maximizing the expansion rate $\chi(\alpha)$. As time goes on, the number of observers in universes with this preferred set of α_j gets larger than the rest by an arbitrarily large factor. In fact, in the limit $t \to \infty$, this set has a 100% probability, while the probability of any other values is zero, i.e.,

$$P_{obs}(\alpha) \propto \delta(\alpha - \alpha_*) \tag{151}$$

where α_* is such that $\chi(\alpha_*)$ is maximum. We thus see that the probability of observing the constants α_j is determined entirely by the physics of eternal inflation and is independent of the nucleation probability $P_{nucl}(\alpha)$, unless $P_{nucl}(\alpha_*) = 0$. However, the problem is that the expansion rate $\chi(\alpha)$ and the values of α_j maximizing this rate depend on one's choice of the time coordinate t. There are two resolutions to this as proposed in [2]:

- There may be some preferred, on physical grounds, choice of the time variable t, which should be used in this case for the calculation of probabilities. For example, one could choose the proper time along the worldlines of comoving observers.
- One can take the point of view that no meaningful definition of probabilities is possible for observations in disconnected, eternally inflating universes.

7 Conclusion

In this report, I have presented an introduction to some of the fundamental ideas in quantum cosmology including eternal inflation and the wavefunction of the universe. Eternal inflation predicts the existence of multiple bubble universes in the sea of a single inflating universe. Inflation ends at different times in different regions of the universe, leading to multiple separated bubbles. The tunneling wavefunction of the universe has then been described, with an emphasis on Hawking and Hartle's no-boundary boundary condition wavefunction. The relation between quantum cosmology and eternal inflation has then been discussed, particularly in the context of the possibility of an eternal past of the universe. As the next step, further analysis on several other areas in quantum cosmology including the the beginning of inflation, the bubble of nothing, and cigar geometry may be carried out.

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