CHAPTER 2

Linear classifers

1 Classification

A binary *classifier* is a mapping from $\mathbb{R}^d \to \{-1, +1\}$. We'll often use the letter h (for hypothesis) to stand for a classifier, so the classification process looks like:

$$x \to \boxed{h} \to y \ .$$

Actually, general classifiers can have a range which is any discrete set, but we'll work with this specific case for a while.

Real life rarely gives us vectors of real numbers; the x we really want to classify is usually something like a song, image, or person. In that case, we'll have to define a function $\varphi(x)$, whose domain is $\mathbb{R}^d$, where φ represents *features* of x, like a person's height or the amount of bass in a song, and then let the $h : \varphi(x) \to \{-1, +1\}$. In much of the following, we'll omit explicit mention of φ and assume that the $x^{(i)}$ are in $\mathbb{R}^d$, but you should always have in mind that some additional process was almost surely required to go from the actual input examples to their feature representation.

In *supervised learning* we are given a training data set of the form

$$\mathcal{D}_n = \left\{ \left(x^{(1)}, y^{(1)} \right), \ldots, \left(x^{(n)}, y^{(n)} \right) \right\} \ .$$

We will assume that each $x^{(i)}$ is a $d \times 1$ *column vector*. The intended meaning of this data is that, when given an input $x^{(i)}$, the learned hypothesis should generate output $y^{(i)}$.

What makes a classifier useful? That it works well on *new* data; that is, that it makes good predictions on examples it hasn't seen. But we don't know exactly what data this classifier might be tested on when we use it in the real world. So, we have to *assume* a connection between the training data and testing data; typically, they are drawn independently from the same probability distribution.

My favorite analogy is to problem sets. We evaluate a student's ability to *generalize* by putting questions on the exam that were not on the homework (training set).

Given a training set $\mathcal{D}_n$ and a classifier h, we can define the *training error* of h to be

$$\mathcal{E}_n(h) = \frac{1}{n} \sum_{i=1}^{n} \begin{cases} 1 & h(x^{(i)}) \neq y^{(i)} \\ 0 & \text{otherwise} \end{cases} \ .$$

For now, we will try to find a classifier with small training error (later, with some added criteria) and hope it *generalizes well* to new data, and has a small *test error*

$$\mathcal{E}(h) = \frac{1}{n'} \sum_{i=n+1}^{n+n'} \begin{cases} 1 & h(x^{(i)}) \neq y^{(i)} \\ 0 & \text{otherwise} \end{cases}$$

on n' new examples that were not used in the process of finding the classifier.

2 Learning algorithm

A *hypothesis class* $\mathcal{H}$ is a set (finite or infinite) of possible classifiers, each of which represents a mapping from $\mathbb{R}^d \to \{-1, +1\}$.

A *learning algorithm* is a procedure that takes a data set $\mathcal{D}_n$ as input and returns an element h of $\mathcal{H}$; it looks like

$$\mathcal{D}_n \longrightarrow \boxed{\text{learning alg } (\mathcal{H})} \longrightarrow h$$

We will find that the choice of $\mathcal{H}$ can have a big impact on the test error of the h that results from this process. One way to get h that generalizes well is to restrict the size, or "expressiveness" of $\mathcal{H}$.

3 Linear classifiers

We'll start with the hypothesis class of *linear classifiers*. They are (relatively) easy to understand, simple in a mathematical sense, powerful on their own, and the basis for many other more sophisticated methods.

A linear classifier in d dimensions is defined by a vector of parameters $\theta \in \mathbb{R}^d$ and scalar $\theta_0 \in \mathbb{R}$. So, the hypothesis class $\mathcal{H}$ of linear classifiers in d dimensions is the *set* of all vectors in $\mathbb{R}^{d+1}$. We'll assume that θ is an $d \times 1$ column vector.

Given particular values for θ and θ_0, the classifier is defined by

$$h(x; \theta, \theta_0) = \text{sign}(\theta^T x + \theta_0) = \begin{cases} +1 & \text{if } \theta^T x + \theta_0 > 0 \\ -1 & \text{otherwise} \end{cases} .$$

Let's be careful about dimensions. We have assumed that x and θ are both $d \times 1$ column vectors. So $\theta^T x$ is 1×1, which in math (but not necessarily numpy) is the same as a scalar.

Remember that we can think of θ, θ_0 as specifying a hyperplane. It divides $\mathbb{R}^d$, the space our $x^{(i)}$ points live in, into two half-spaces. The one that is on the same side as the normal vector is the *positive* half-space, and we classify all points in that space as positive. The half-space on the other side is *negative* and all points in it are classified as negative.

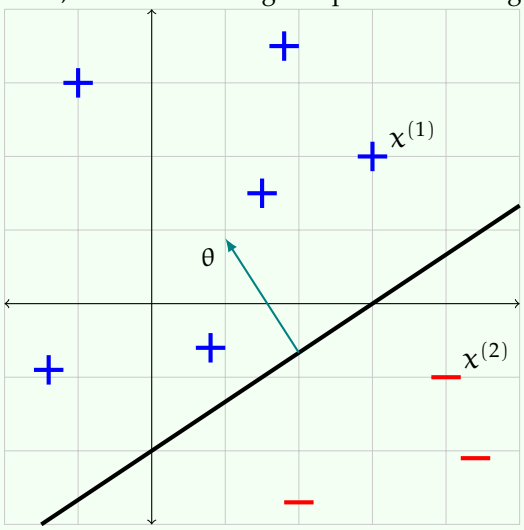
Example: Let h be the linear classifier defined by $\theta = \begin{bmatrix} -1 \\ 1.5 \end{bmatrix}, \theta_0 = 3$.

The diagram below shows several points classified by h. In particular, let $x^{(1)} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $x^{(2)} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

$$h(x^{(1)}; \theta, \theta_0) = \text{sign}\left(\begin{bmatrix} -1 & 1.5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 3\right) = \text{sign}(3) = +1$$

$$h(x^{(2)}; \theta, \theta_0) = \text{sign}\left(\begin{bmatrix} -1 & 1.5 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} + 3\right) = \text{sign}(-2.5) = -1$$

Thus, $x^{(1)}$ and $x^{(2)}$ are given positive and negative classfications, respectively.



Study Question: What is green vector normal to the hyperplane? Specify it as a column vector.

Study Question: What change would you have to make to θ, θ_0 if you wanted to have the separating hyperplane in the same place, but to classify all the points labeled '+' in the diagram as negative and all the points labeled '-' in the diagram as positive?

4 Learning linear classifiers

Now, given a data set and the hypothesis class of linear classifiers, our objective will be to find the linear classifier with the smallest possible training error.

This is a well-formed optimization problem. But it's not computationally easy!

We'll start by considering a very simple learning algorithm. The idea is to generate k possible hypotheses by generating their parameter vectors at random. Then, we can evaluate the training-set error on each of the hypotheses and return the hypothesis that has the lowest training error (breaking ties arbitrarily).

It's a good idea to think of the "stupidest possible" solution to a problem, before trying to get clever. Here's a fairly (but not completely) stupid algorithm.

RANDOM-LINEAR-CLASSIFIER($\mathcal{D}_n$, k, d)

for $j = 1$ **to** k
    randomly sample $\left(\theta^{(j)}, \theta_0^{(j)}\right)$ from $(\mathbb{R}^d, \mathbb{R})$
$j^* = \arg\min_{j \in \{1,\ldots,k\}} \mathcal{E}_n\left(\theta^{(j)}, \theta_0^{(j)}\right)$
return $\left(\theta^{(j^*)}, \theta_0^{(j^*)}\right)$

A note about notation.

This might be new notation: $\arg\min_x f(x)$ means the value of x for which $f(x)$ is the smallest. Sometimes we write $\arg\min_{x \in \mathcal{X}} f(x)$ when we want to explicitly specify the set $\mathcal{X}$ of values of x over which we want to minimize.

Study Question: What do you think happens to $\mathcal{E}_n(h)$, where h is the hypothesis returned by RANDOM-LINEAR-CLASSIFIER, as k is increased?

Study Question: What properties of $\mathcal{D}_n$ do you think will have an effect on $\mathcal{E}_n(h)$?

5 Evaluating a learning algorithm

How should we evaluate the performance of a *classifier* h? The best method is to measure *test error* on data that was not used to train it.

How should we evaluate the performance of a *learning algorithm*? This is trickier. There are many potential sources of variability in the possible result of computing test error on a learned hypothesis h:

- Which particular *training examples* occurred in $\mathcal{D}_n$
- Which particular *testing examples* occurred in $\mathcal{D}_n$,
- Randomization inside the learning *algorithm* itself

Generally, we would like to execute the following process multiple times:

- Train on a new training set
- Evaluate resulting h on a testing set *that does not overlap the training set*

Doing this multiple times controls for possible poor choices of training set or unfortunate randomization inside the algorithm itself.

One concern is that we might need a lot of data to do this, and in many applications data is expensive or difficult to acquire. We can re-use data with *cross validation* (but it's harder to do theoretical analysis).

CROSS-VALIDATE($\mathcal{D}$, k)

divide $\mathcal{D}$ into k chunks $\mathcal{D}_1, \mathcal{D}_2, \ldots \mathcal{D}_k$ (of roughly equal size)
for $i = 1$ **to** k
    train h_i on $\mathcal{D} \setminus \mathcal{D}_i$ (withholding chunk $\mathcal{D}_i$)
    compute "test" error $\mathcal{E}_i(h_i)$ on withheld data $\mathcal{D}_i$
return $\frac{1}{k}\sum_{i=1}^{k} \mathcal{E}_i(h_i)$

It's very important to understand that cross-validation neither delivers nor evaluates a single particular hypothesis h. It evaluates the *algorithm* that produces hypotheses.