Vector Algebra

1. **Problem statement :** The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one, Find the value of λ.
Solution: Let

$$\mathbf{a}=\begin{pmatrix}1\\1\\1\end{pmatrix},\mathbf{b}=\begin{pmatrix}2\\4\\-5\end{pmatrix},\mathbf{c}=\begin{pmatrix}\lambda\\2\\3\end{pmatrix} \tag{1}$$

$$(\mathbf{b}+\mathbf{c})=\begin{pmatrix}2\\4\\-5\end{pmatrix}+\begin{pmatrix}\lambda\\2\\3\end{pmatrix}=\begin{pmatrix}2+\lambda\\6\\-2\end{pmatrix} \tag{2}$$

Let $\mathbf{r}$ be the unit vector along with $(\mathbf{b}+\mathbf{c})$

$$\hat{\mathbf{r}}=\frac{(\mathbf{b}+\mathbf{c})}{\|\mathbf{b}+\mathbf{c}\|}=\frac{(\mathbf{b}+\mathbf{c})}{\sqrt{(2+\lambda)^2+(6)^2+(-2)^2}} \tag{3}$$

$$\implies \hat{\mathbf{r}}=\frac{1}{\sqrt{\lambda^2+4\lambda+4}}\begin{pmatrix}2+\lambda\\6\\-2\end{pmatrix} \tag{4}$$

Given $\mathbf{a}^\top(\hat{\mathbf{r}})=1$

$$\begin{pmatrix}1&1&1\end{pmatrix}\left(\frac{1}{\sqrt{\lambda^2+4\lambda+4}}\begin{pmatrix}2+\lambda\\6\\-2\end{pmatrix}\right)=1 \tag{5}$$

$$\implies \begin{pmatrix}1&1&1\end{pmatrix}\begin{pmatrix}2+\lambda\\6\\-2\end{pmatrix}=\sqrt{\lambda^2+4\lambda+4} \tag{6}$$

$$\implies 2+\lambda+6-2=\sqrt{\lambda^2+4\lambda+4} \tag{7}$$

$$\implies \lambda+6=\sqrt{\lambda^2+4\lambda+4} \tag{8}$$

$$\tag{9}$$

Squaring on both sides

$$(\lambda+6)^2=\left(\sqrt{\lambda^2+4\lambda+4}\right)^2 \tag{10}$$

$$\lambda^2+12(\lambda)+4=\lambda^2+4\lambda+4 \tag{11}$$

$$8(\lambda)=8 \tag{12}$$

$$\lambda=1 \tag{13}$$