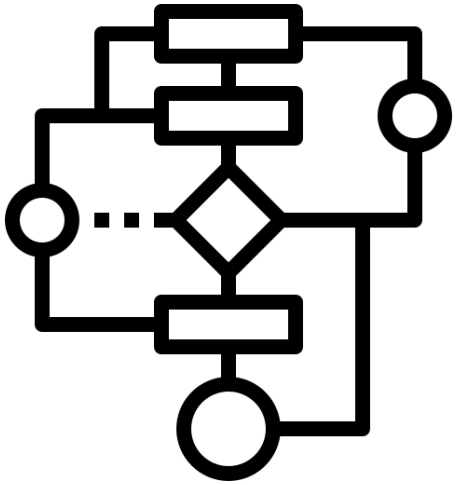

CS 310: Algorithms

Lecture 11

Instructor: Naveed Anwar Bhatti

Slides by Kevin Wayne (**heavily modified by Naveed Bhatti**).
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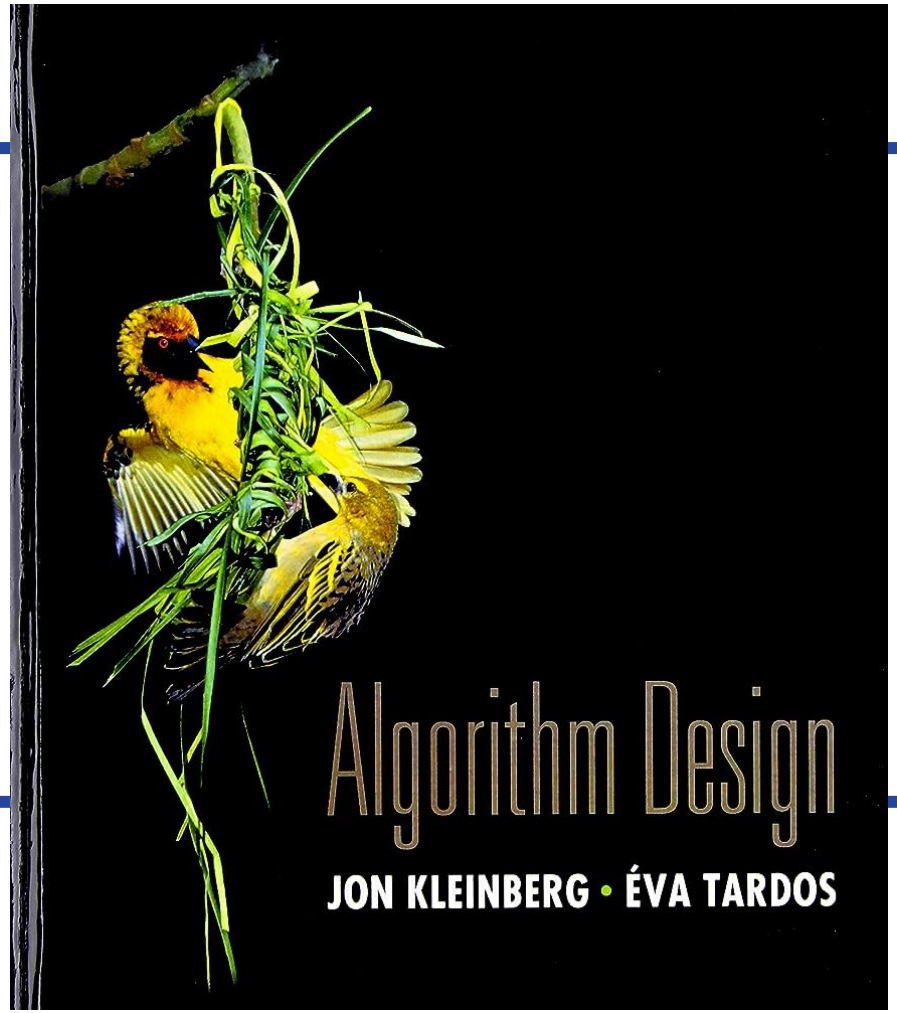

Chapter 5:
Divide and Conquer

Section 5.3:
Counting Inversions

Slides by Kevin Wayne (**heavily modified by Naveed Bhatti**).
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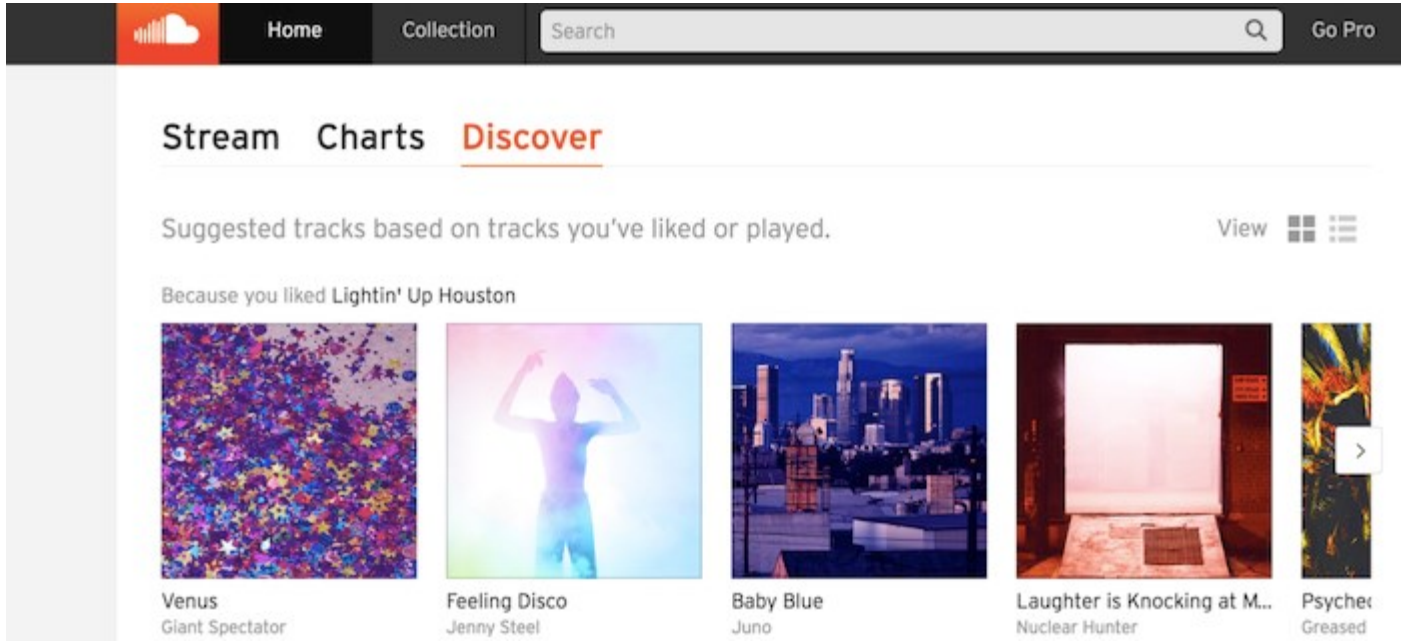
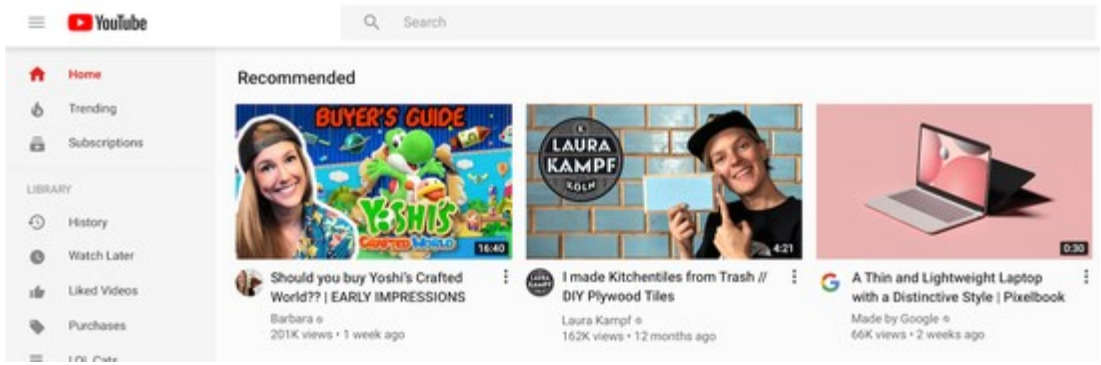
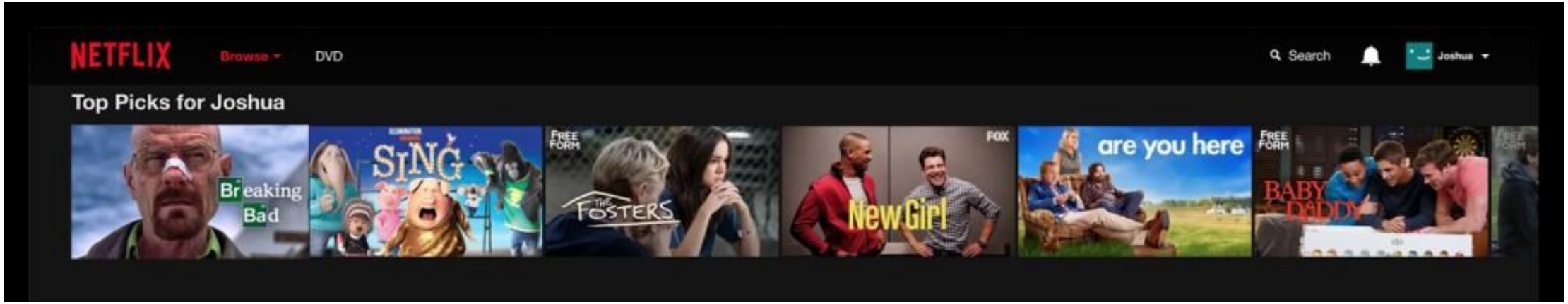
Motivation: Recommendation Systems







Motivation: Recommendation Systems

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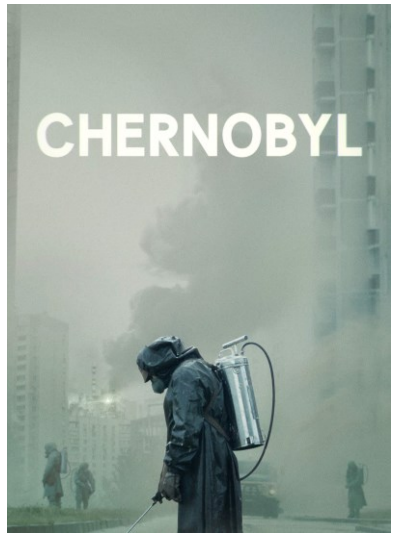
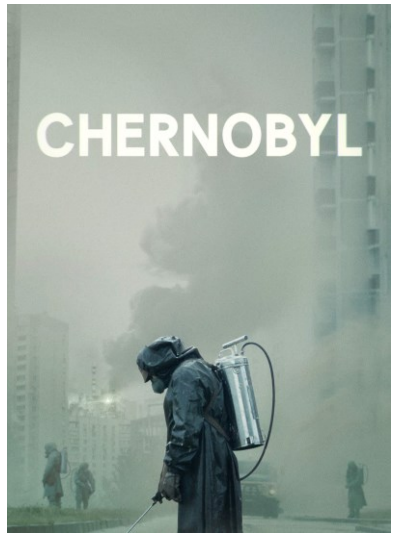
				
2	4	1	3	5

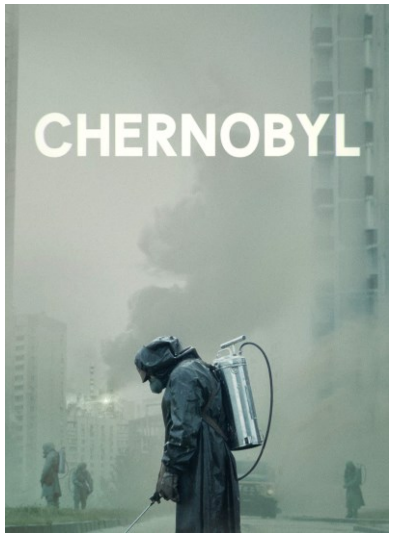
Counting inversions

1	2	3	4	5
				
2	4	1	3	5

(2,1)

Counting inversions

1 2 3 4 5

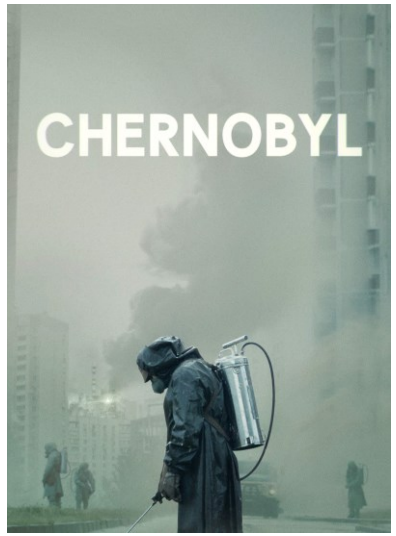


2 4 1 3 5

(2,1) (4,1)

Counting inversions

1 2 3 4 5



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(2,1) (4,1) (4,3)

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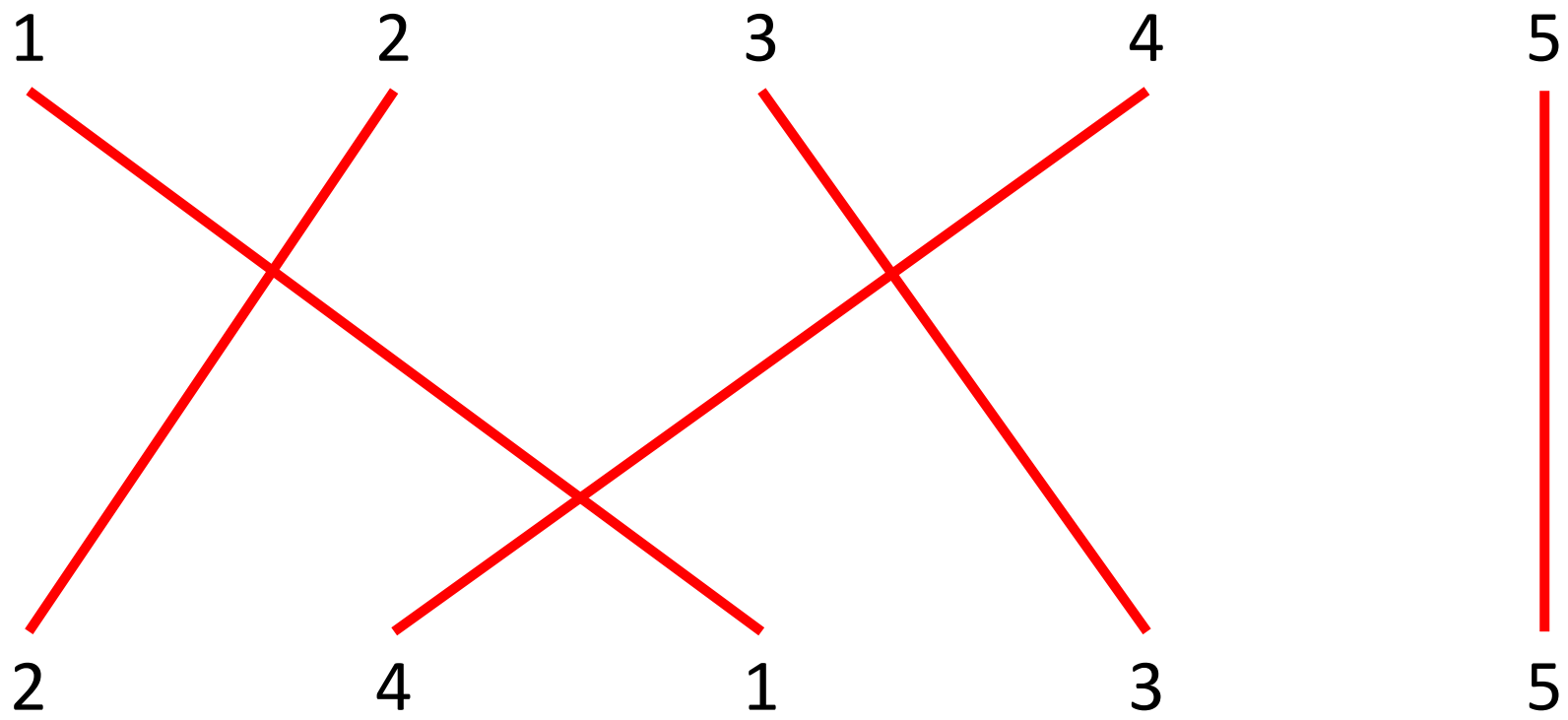

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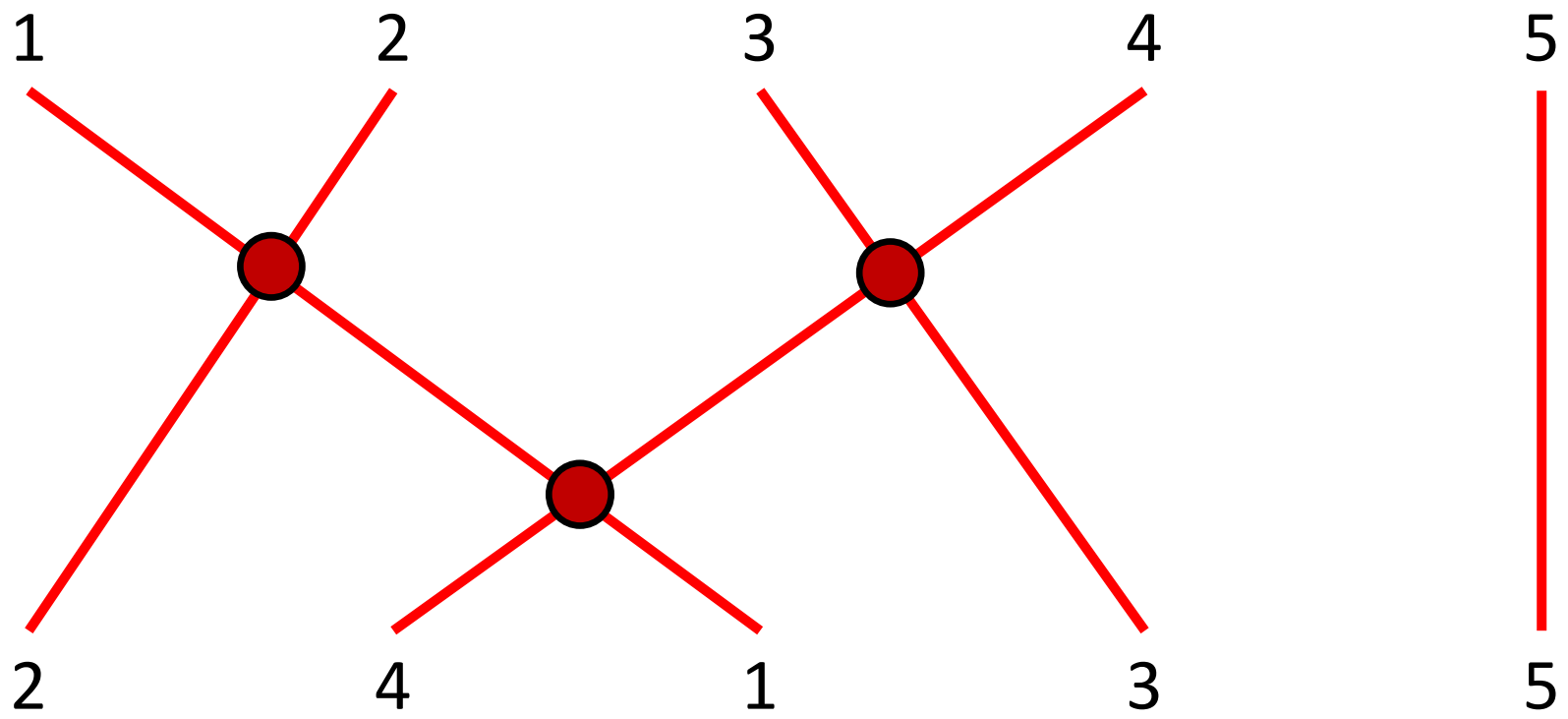

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			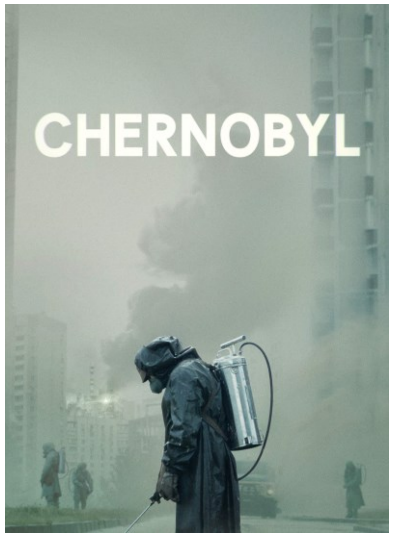	
2	4	1	3	5

Counting inversions: Divide and Conquer (Warm Up!)

- Divide: separate list into two halves A and B.
- Conquer: recursively count inversions in each list.
- Combine: count inversions (a, b) with $a \in A$ and $b \in B$.
- Return sum of three counts.

3	7	11	5	9	1	6	13	4	2	15	8	10	14	12	16

(3,1) (7,5) (7,1) (7,6) (11,5) (11,9) (11,1) (11,6) (5,1) (9,1) (9,6) = **11**

3	7	11	5	9	1	6	13

(4,2) (15,8) (15,10) (15,14) (15,12) (14,12) = **6**

4	2	15	8	10	14	12	16

(3,2) (7,4) (7,2) (11,4) (11,2) (11,8) (11,10) (5,4) (5,2) (9,4) (9,2) (9,8) (6,4) (6,2) (13,4) (13,2) (13,8) (13,10) (13,12) = **19**

3	7	11	5	9	1	6	13

4	2	15	8	10	14	12	16

Return = 11 + 6 +19 = 36

Counting inversions: how to combine two subproblems?

Q. How to count inversions (a, b) with **a ∈ A** and **b ∈ B** ***efficiently***?

A. Easy if A and B are sorted!

3	7	11	5	9	1	6	13

4	2	15	8	10	14	12	16

Counting inversions: how to combine two subproblems?

Q. How to count inversions (a, b) with **a ∈ A** and **b ∈ B**?

A. Easy if A and B are sorted!

Algorithm:

- Scan *A* and *B* from left to right.
- Compare a_i and b_j.
- If $a_i < b_j$, then a_i is not inverted with any element left in *B*.
- If $a_i > b_j$, then b_j is inverted with every element left in *A*.
- Append smaller element to sorted list *C*.

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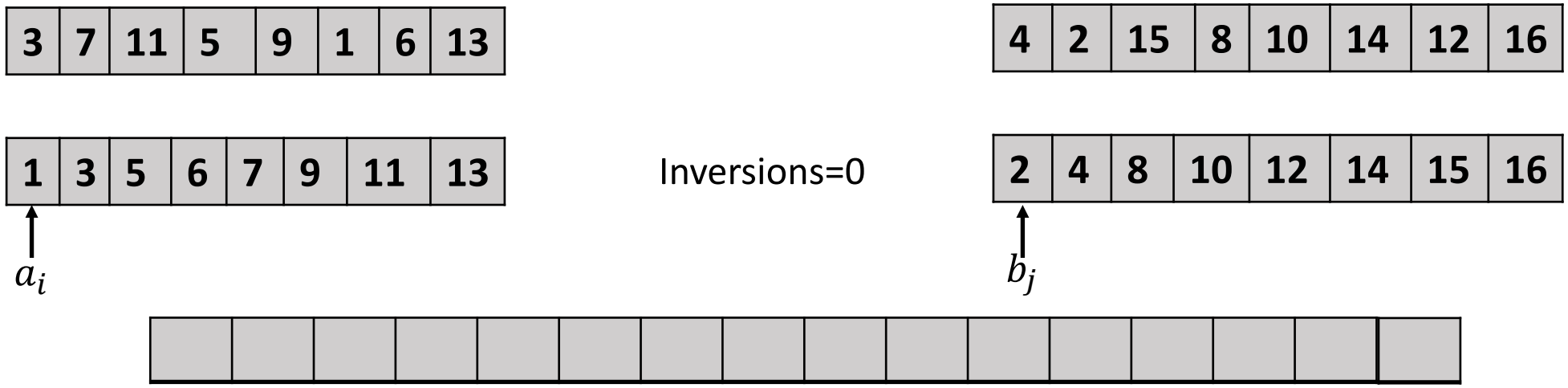

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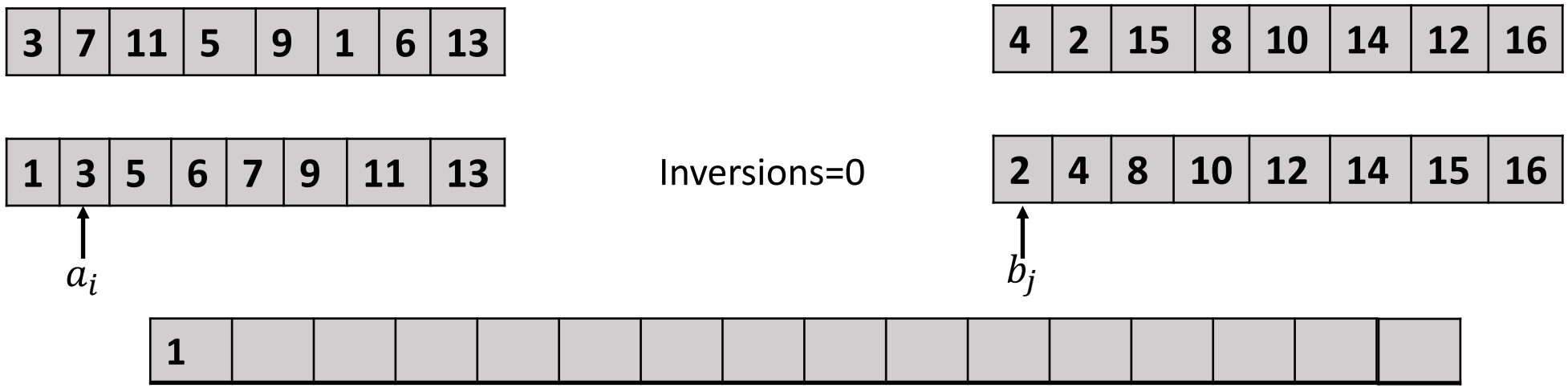

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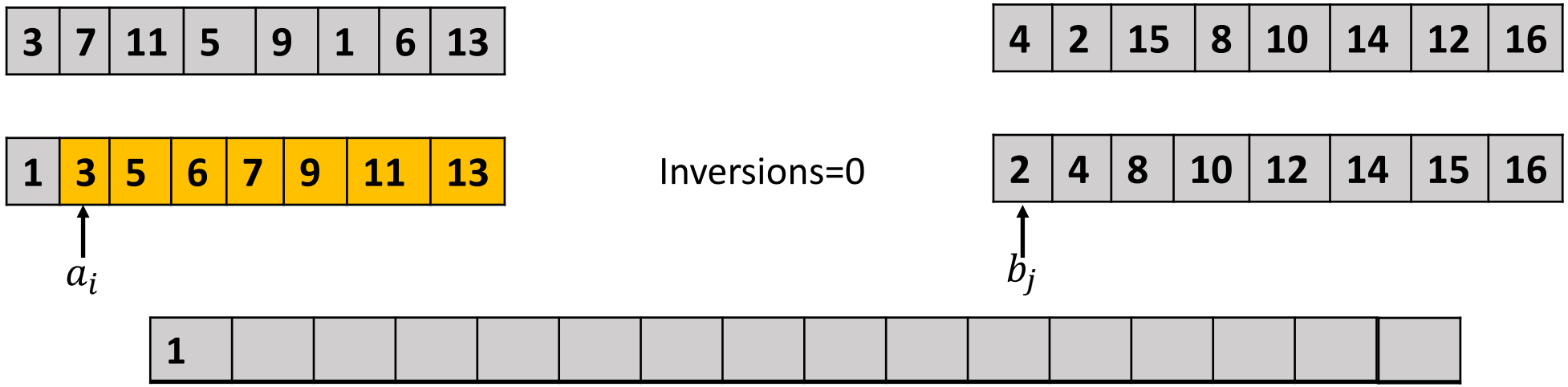

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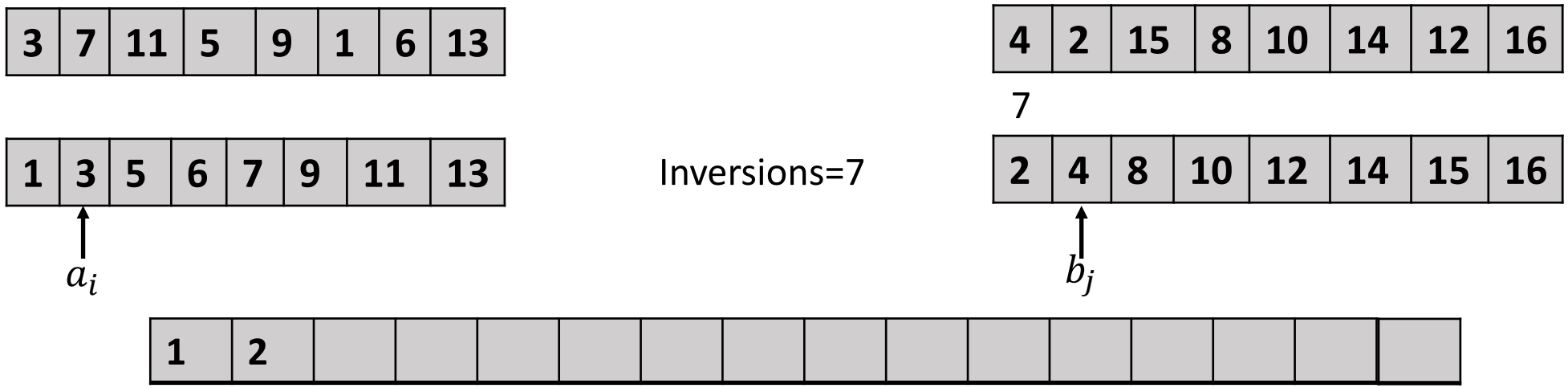

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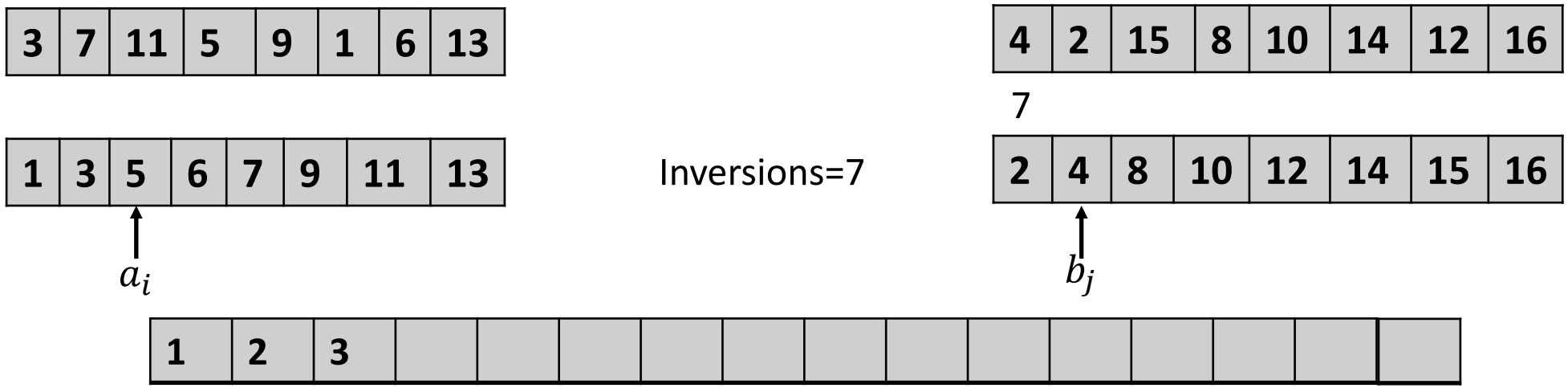

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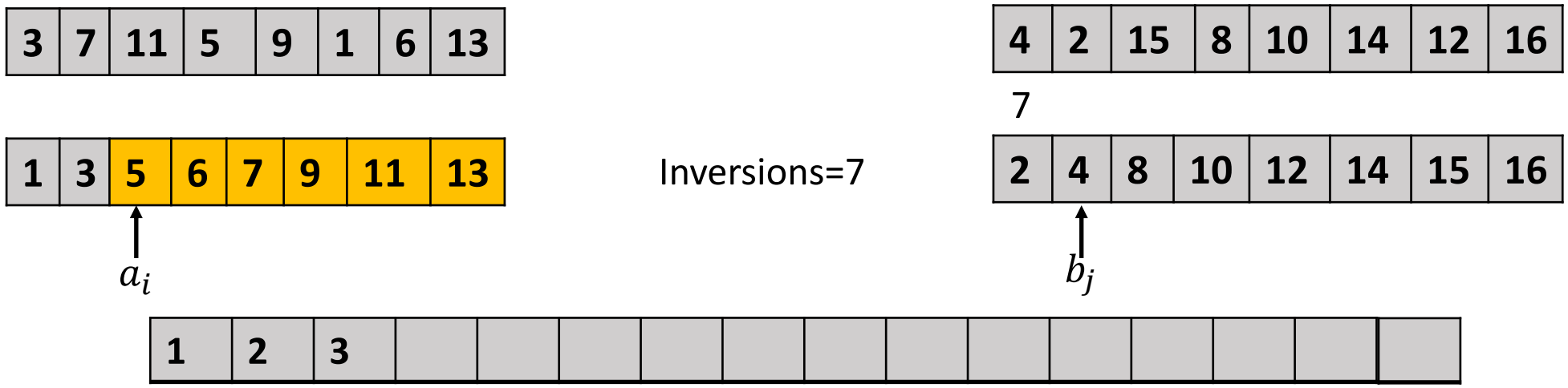

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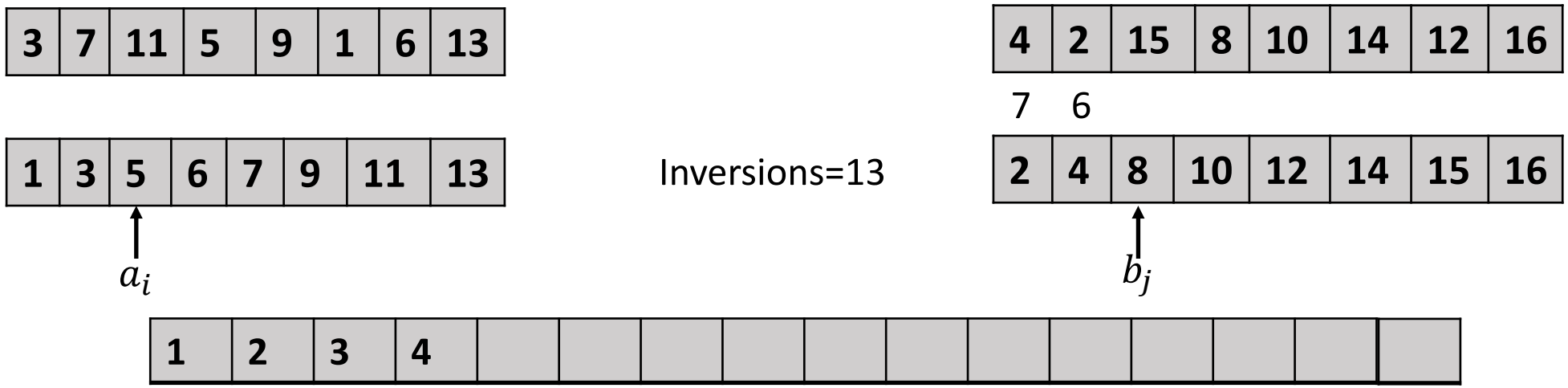

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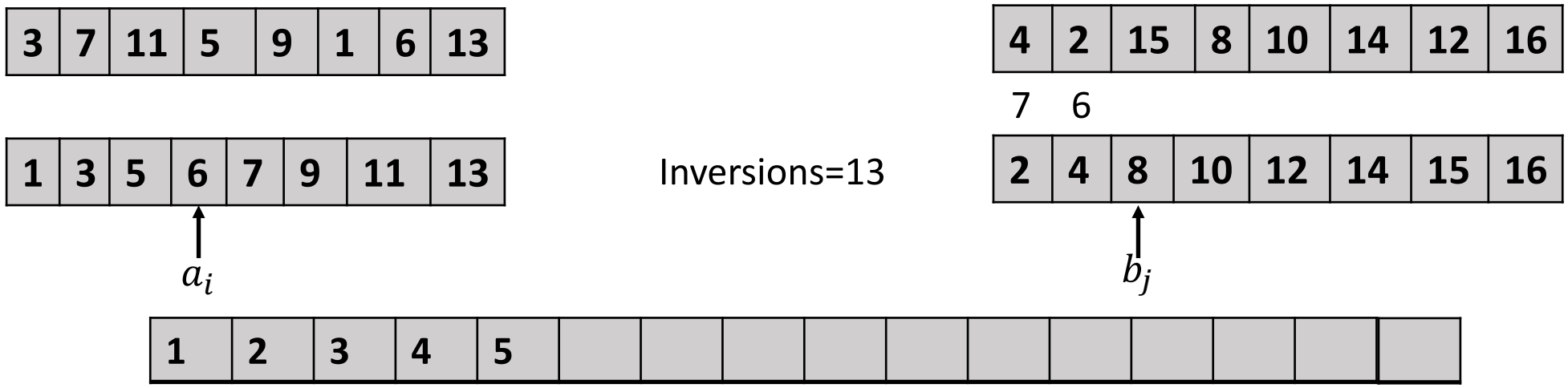

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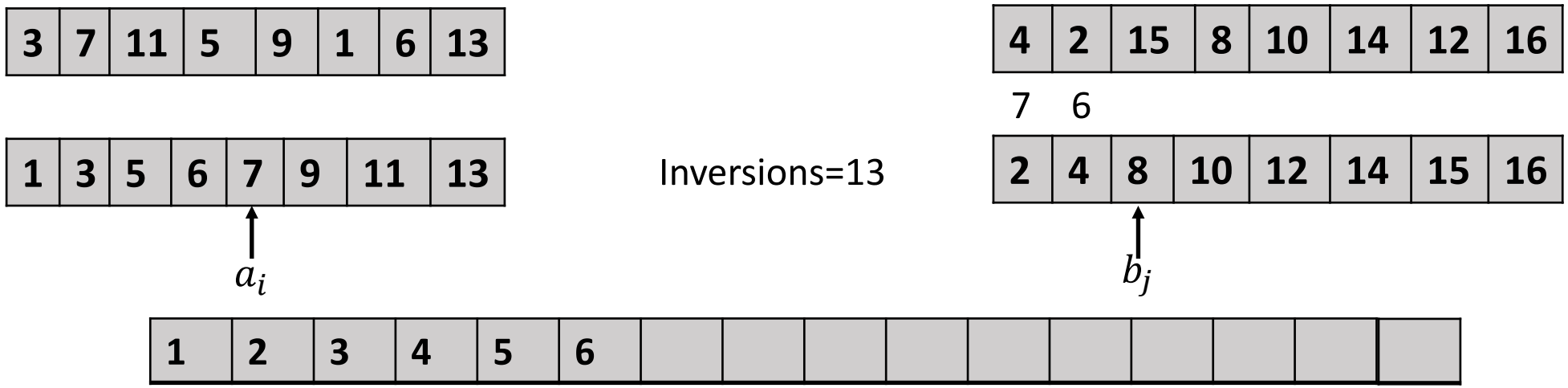

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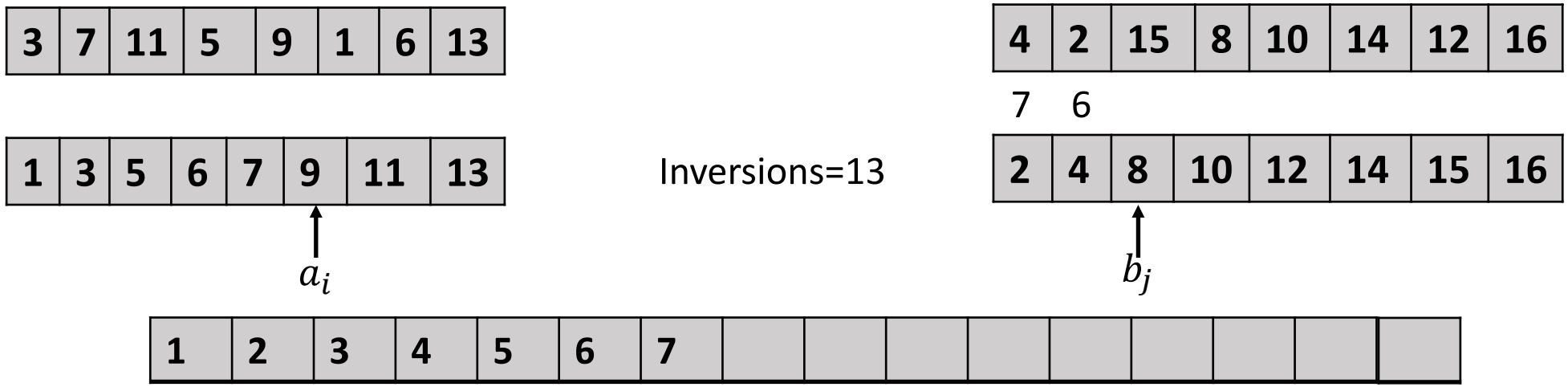

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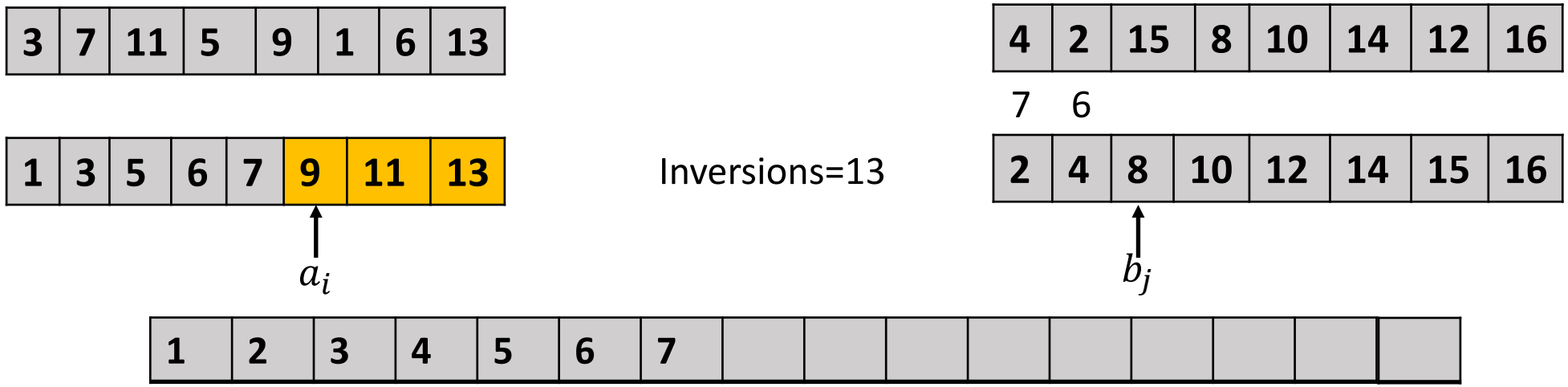

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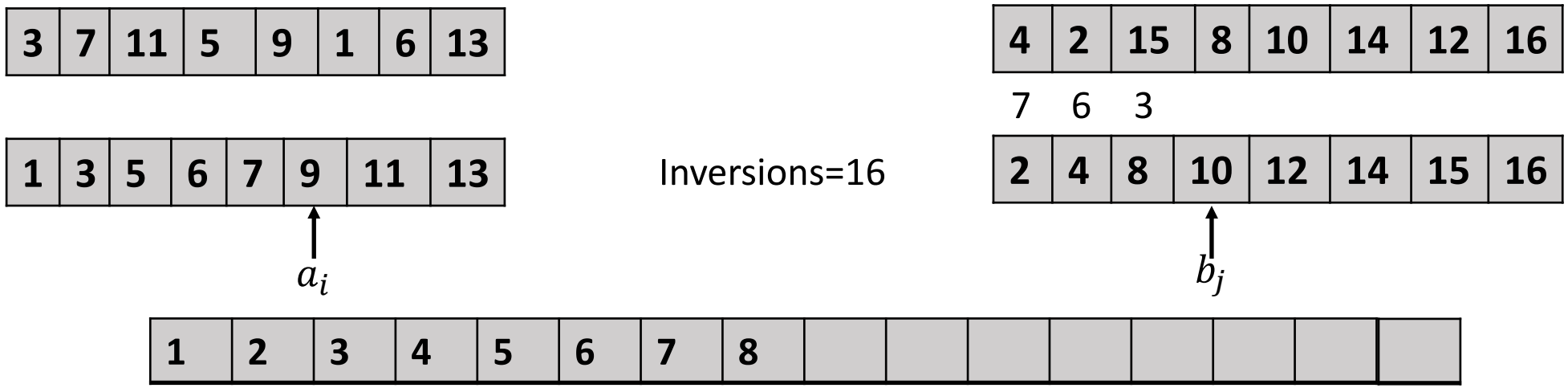

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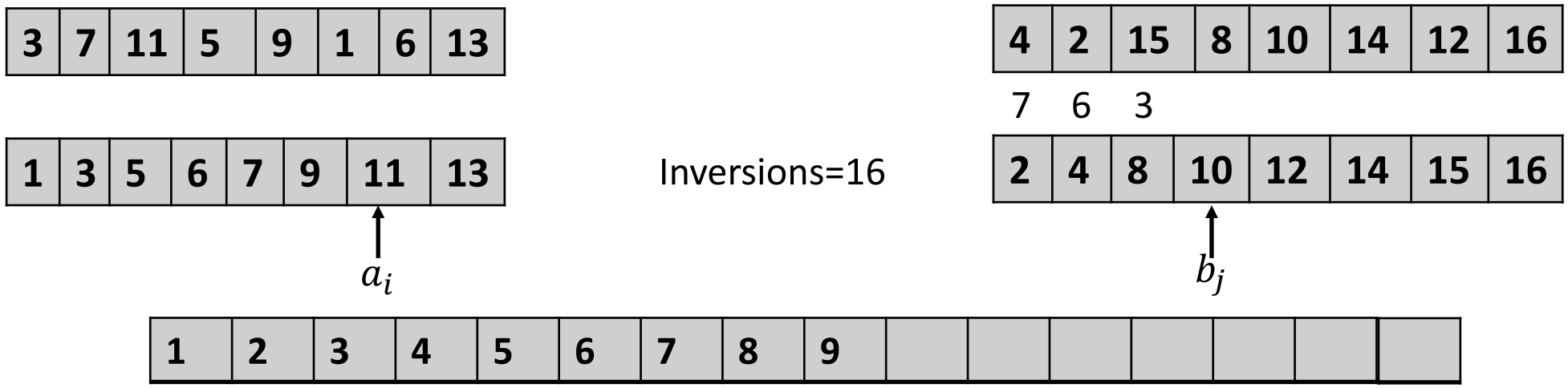

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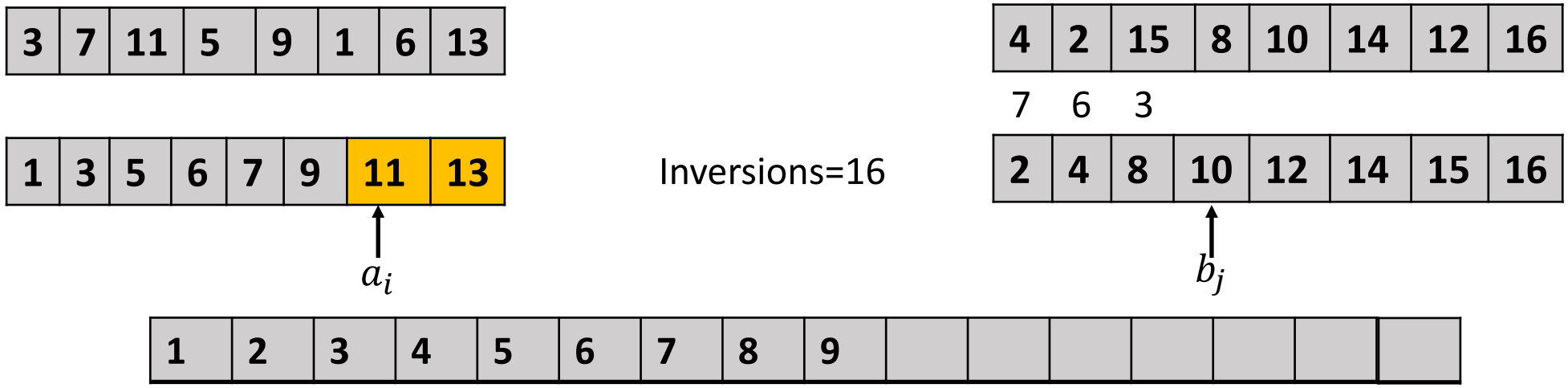

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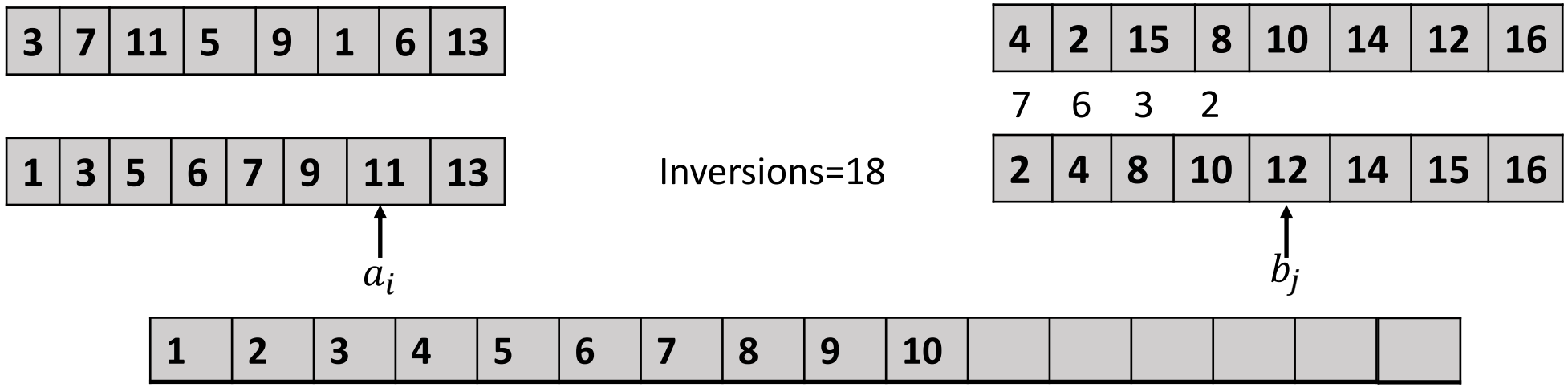

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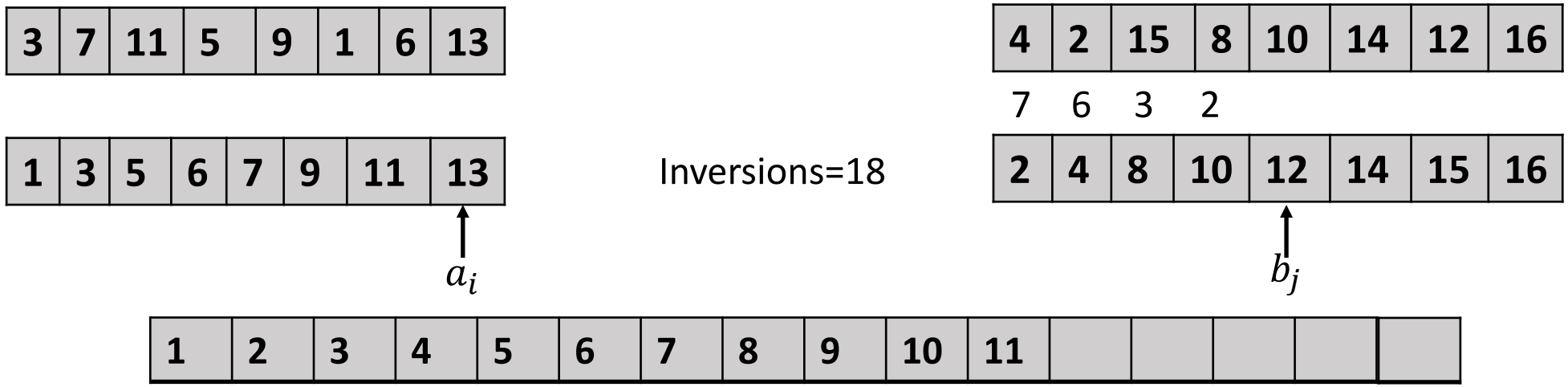

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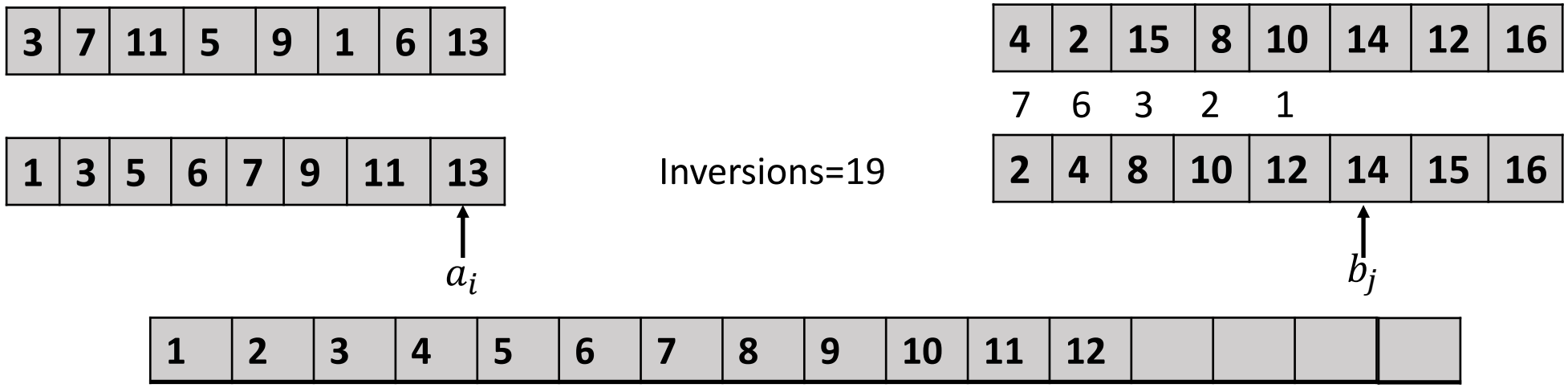

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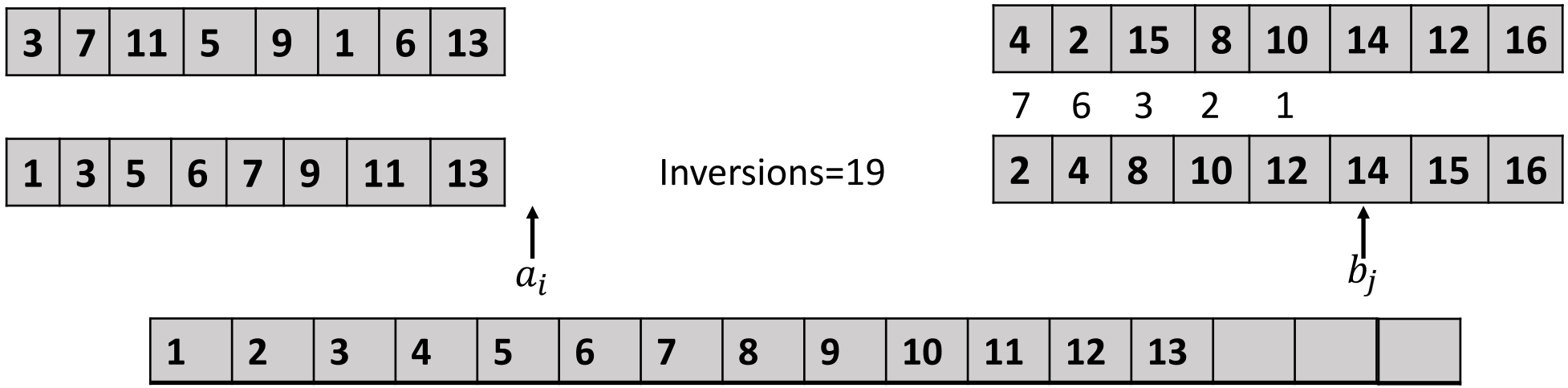

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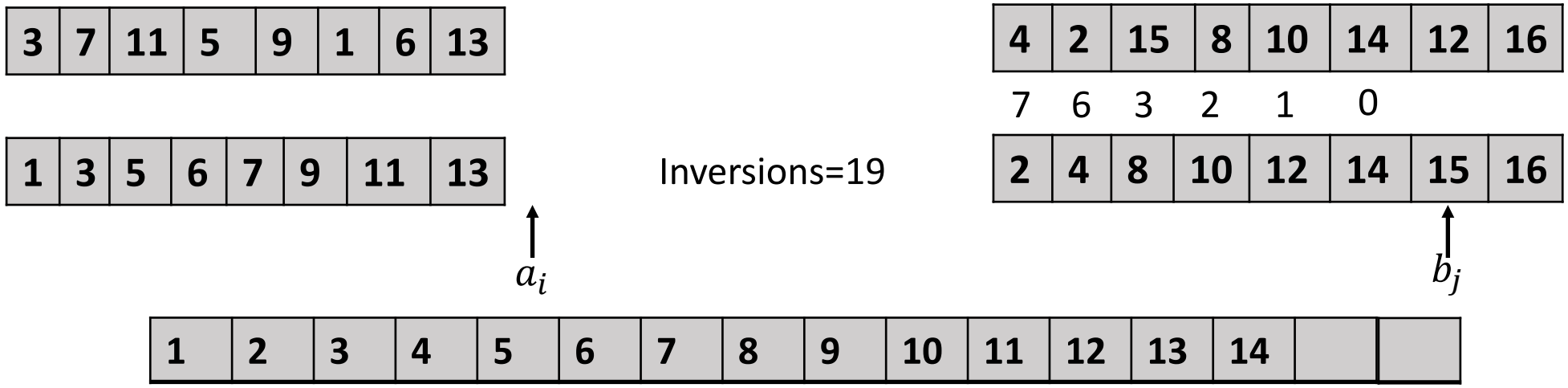

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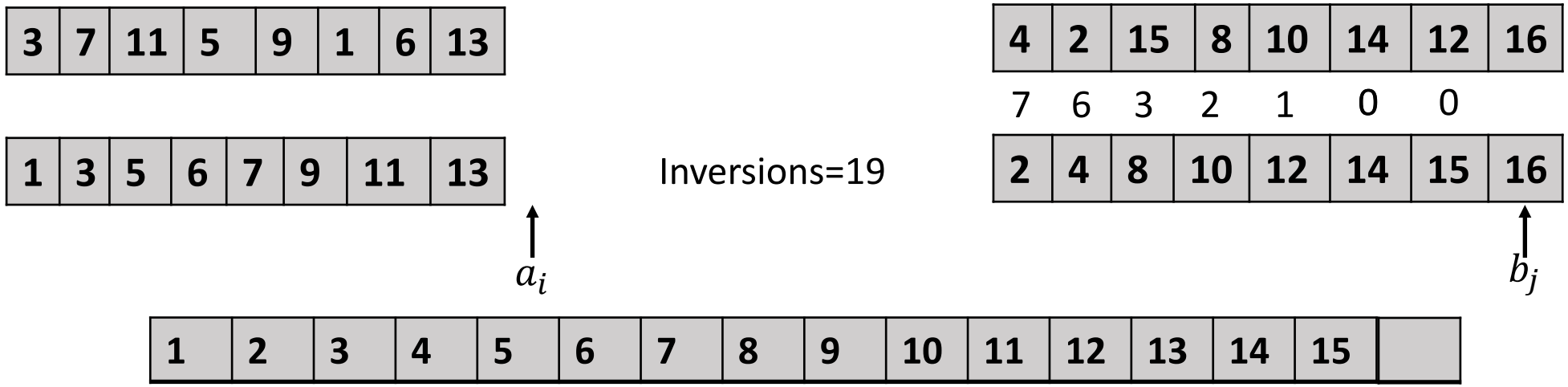


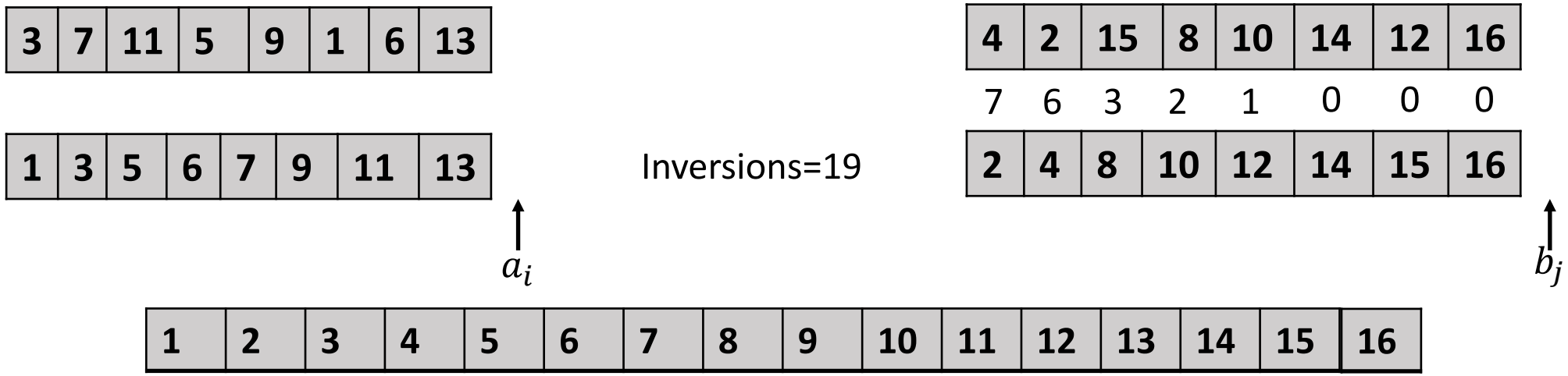
Counting inversions: how to combine two subproblems?

Q. How to count inversions (a, b) with **a ∈ A** and **b ∈ B**?

A. Easy if A and B are sorted!

Algorithm:

- Scan *A* and *B* from left to right.
- Compare a_i and b_j.
- If $a_i < b_j$, then a_i is not inverted with any element left in *B*.
- If $a_i > b_j$, then b_j is inverted with every element left in *A*.
- Append smaller element to sorted list *C*.



Counting inversions: how to combine two subproblems?

$Merge - And - Count(A_1, A_2)$

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2 $x \leftarrow length(A_1)$

3 $inversions \leftarrow 0$

4 $i \leftarrow 1; j \leftarrow 1$

5 **for** $k = 1$ **to** m

6 **if** $A_1[i] \leq A_2[j]$

7 $A[k] \leftarrow A_1[i]$

8 $i \leftarrow i + 1$

9 $x \leftarrow x - 1$

10 **else** $A[k] \leftarrow A_2[j]$

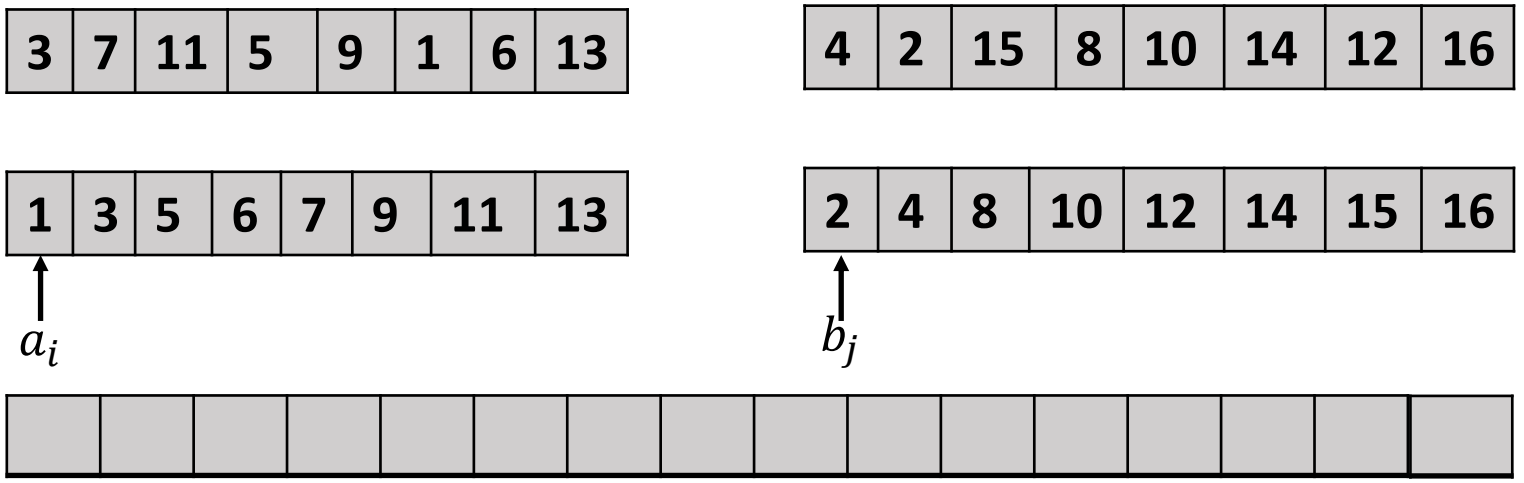
11 $j \leftarrow j + 1$

12 $inversions \leftarrow inversions + x$

13 **return** $(A, inversions)$

$inversions$=0

x=8



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9 $x \leftarrow x - 1$

10 **else** $A[k] \leftarrow A_2[j]$

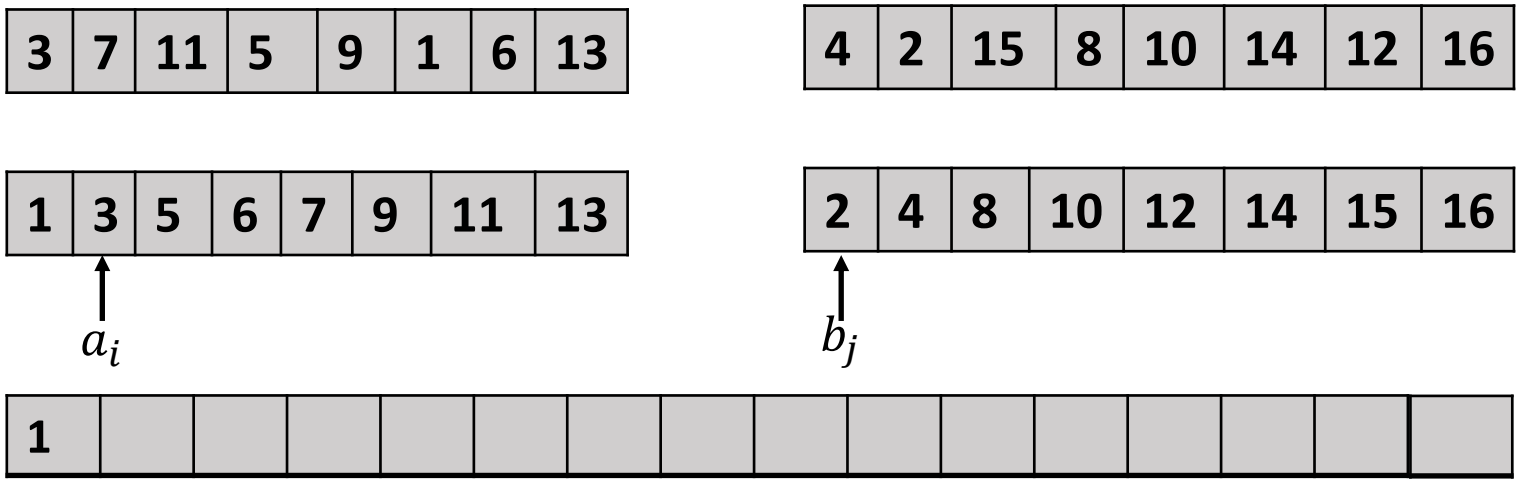
11 $j \leftarrow j + 1$

12 $inversions \leftarrow inversions + x$

13 **return** $(A, inversions)$

$inversions$=0

x=7



Counting inversions: how to combine two subproblems?

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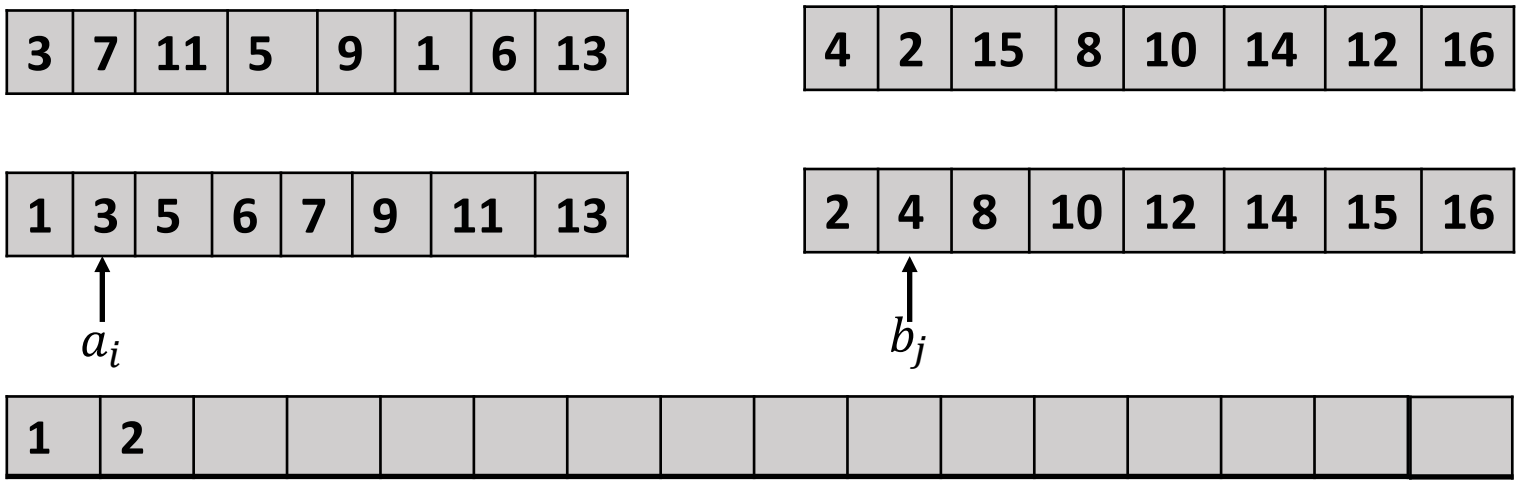

Counting inversions: how to combine two subproblems?

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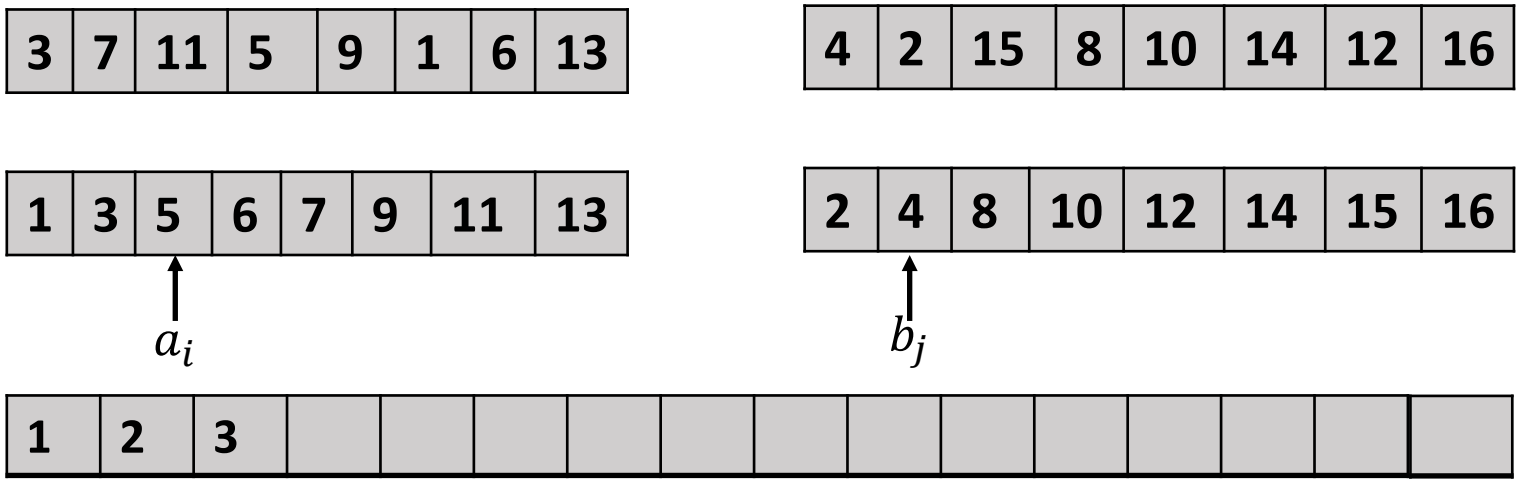
11 $j \leftarrow j + 1$

12 $inversions \leftarrow inversions + x$

13 **return** $(A, inversions)$

$inversions$=7

x=6



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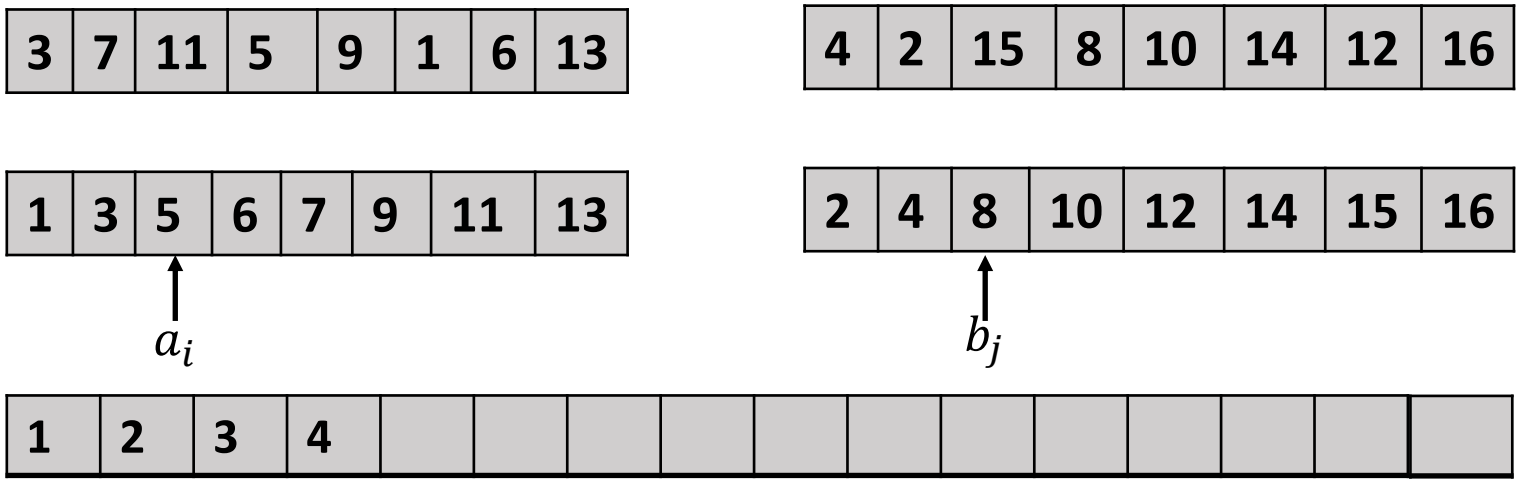
10 **else** $A[k] \leftarrow A_2[j]$
11 $j \leftarrow j + 1$
12 $inversions \leftarrow inversions + x$
13 **return** $(A, inversions)$

$inversions$=13

x=6



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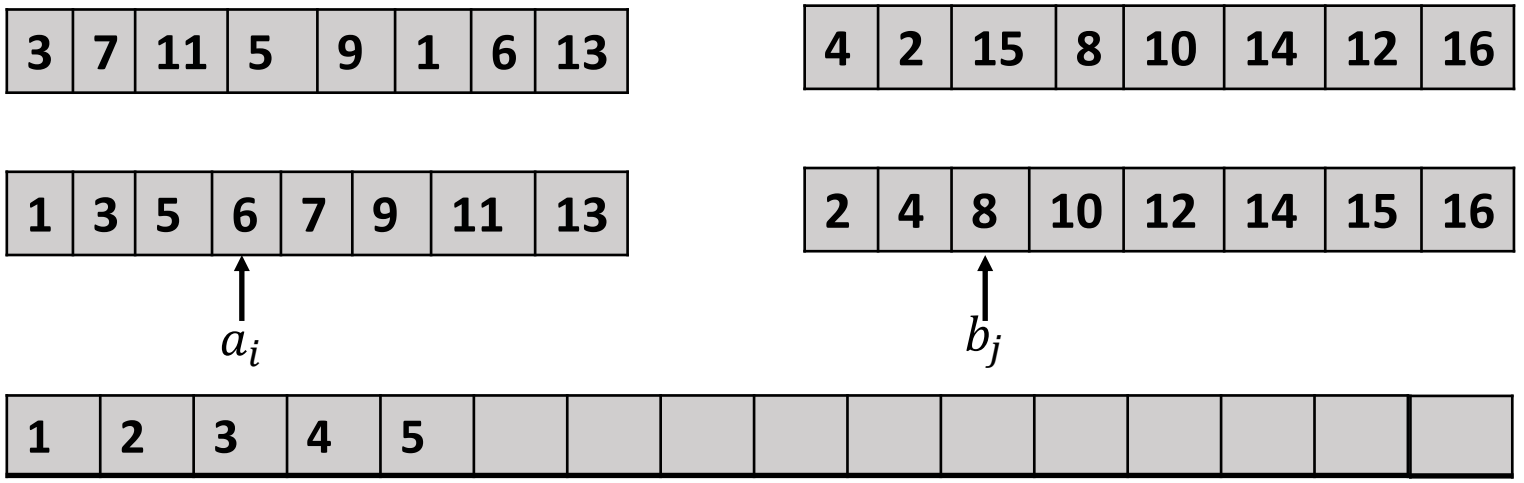
11 $j \leftarrow j + 1$

12 $inversions \leftarrow inversions + x$

13 **return** $(A, inversions)$

$inversions$=13

x=5



Counting inversions: how to combine two subproblems?

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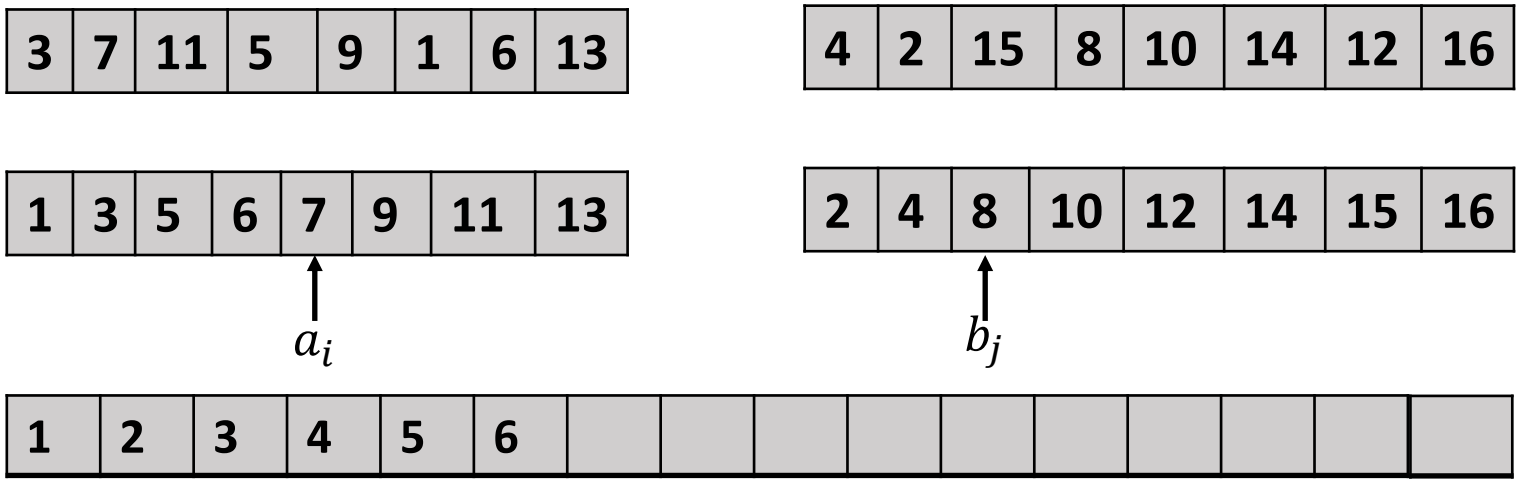
13 **return** $(A, inversions)$

$inversions$=13

x=4



Counting inversions: how to combine two subproblems?

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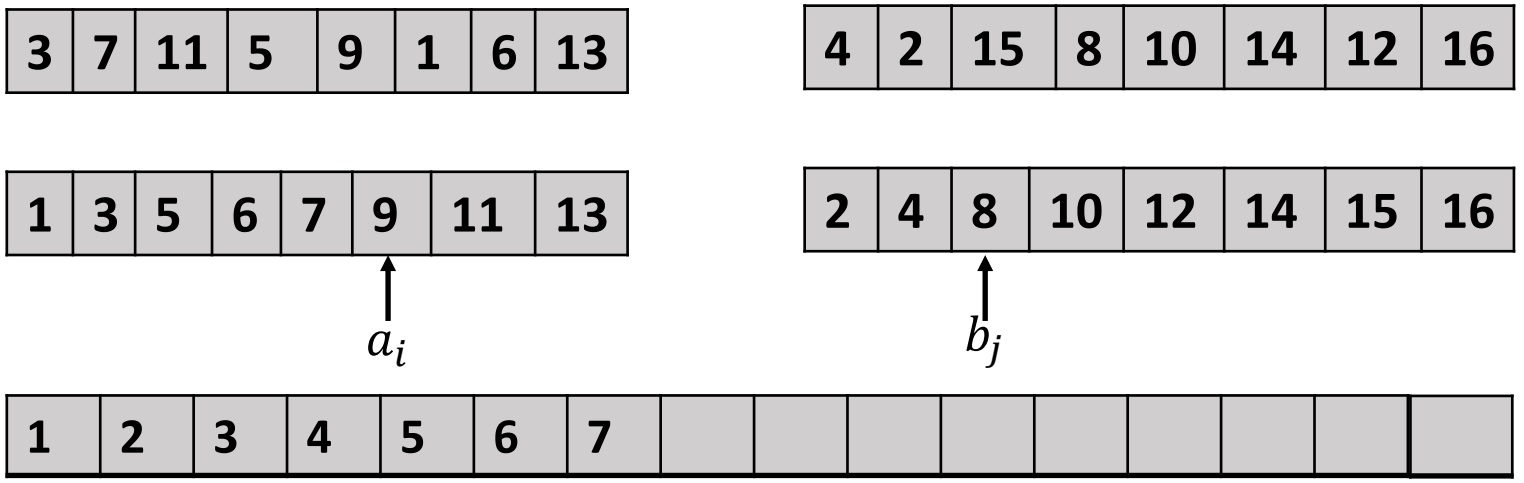
$inversions$=13

x=3



Counting inversions: how to combine two subproblems?

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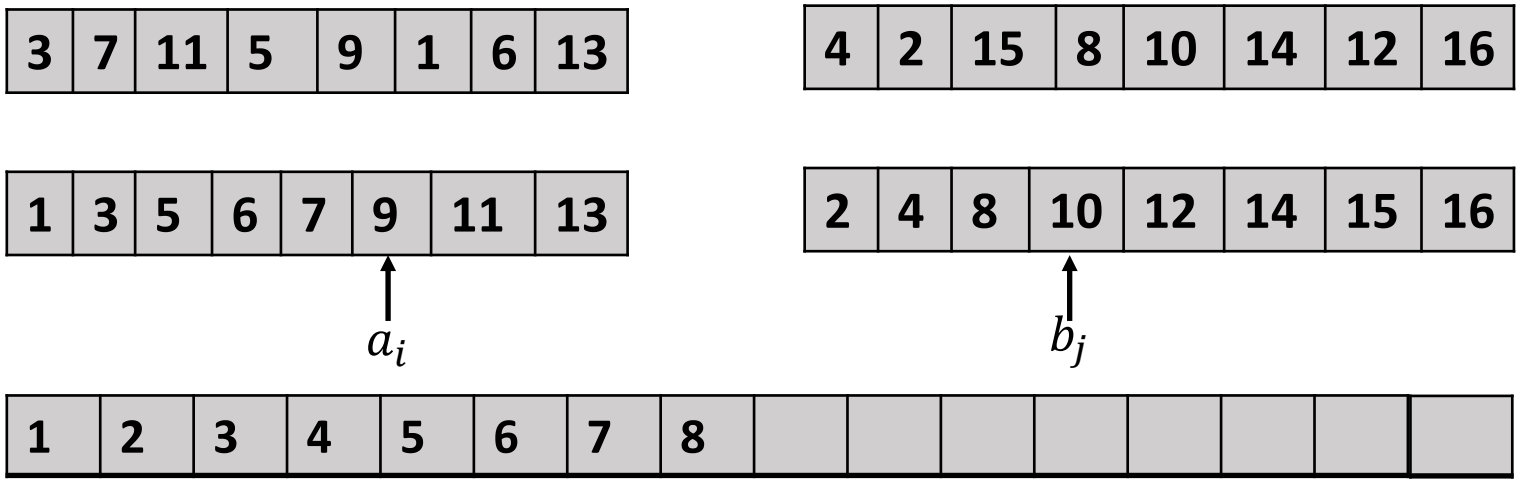
11           j ← j + 1
12           inversions ← inversions + x
13   return (A, inversions)
```

$inversions$=16

x=3



Counting inversions: how to combine two subproblems?

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11 $j \leftarrow j + 1$
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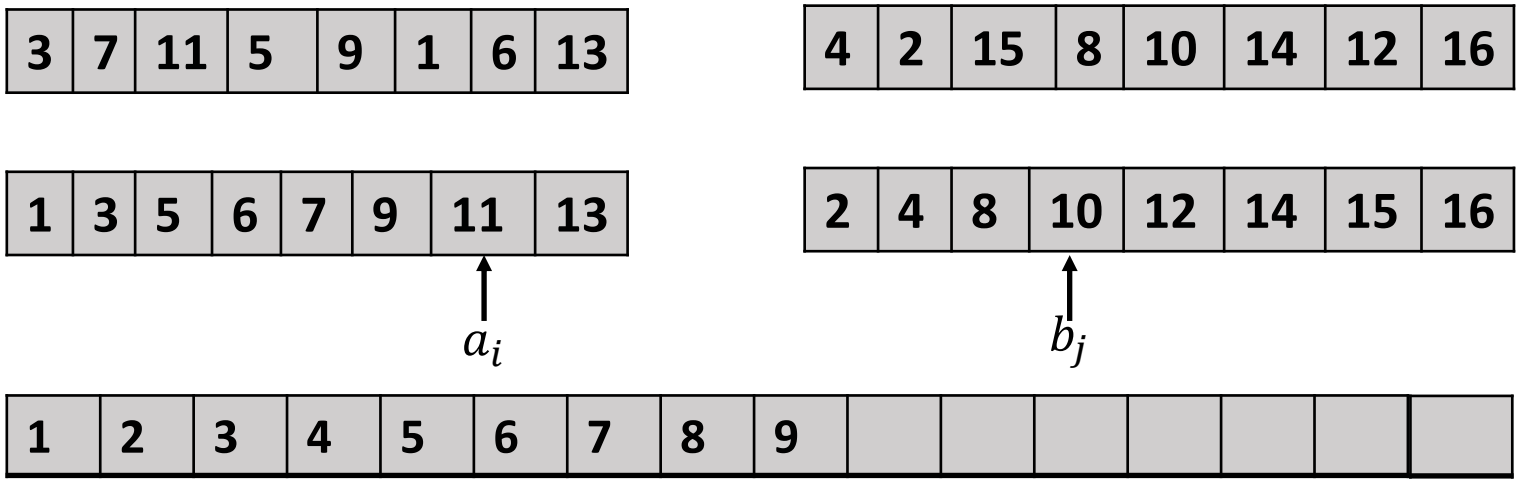
13 **return** $(A, inversions)$

$inversions$=16

x=2



Counting inversions: how to combine two subproblems?

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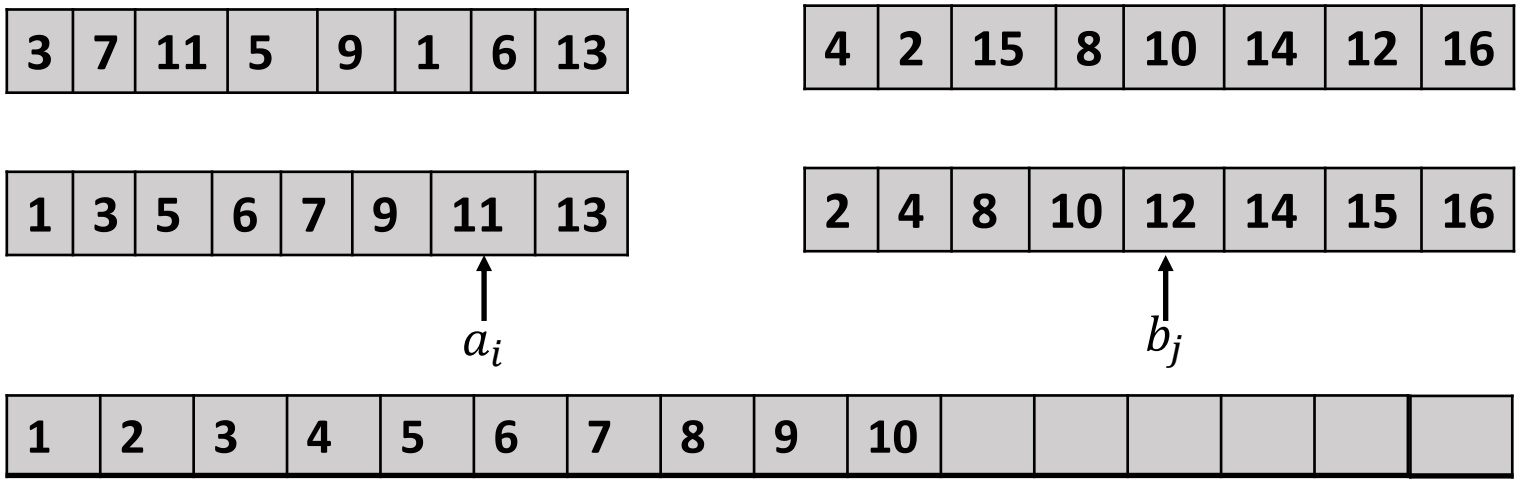
12 $inversions \leftarrow inversions + x$
13 **return** $(A, inversions)$

$inversions$=18

x=2



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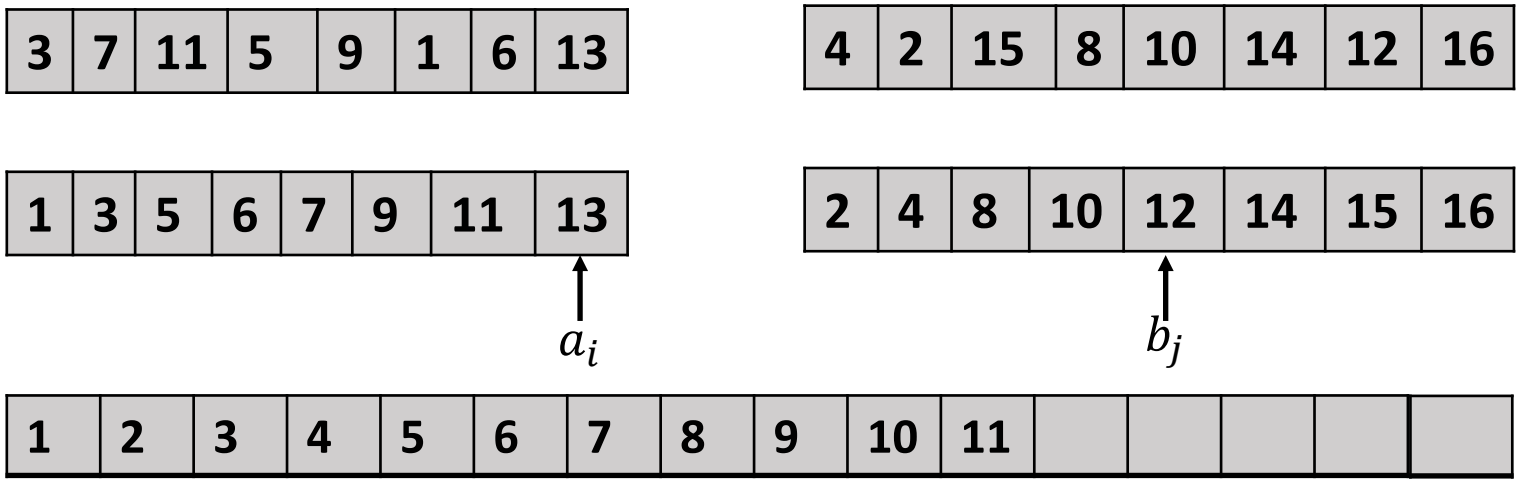
$inversions$=18

x=1



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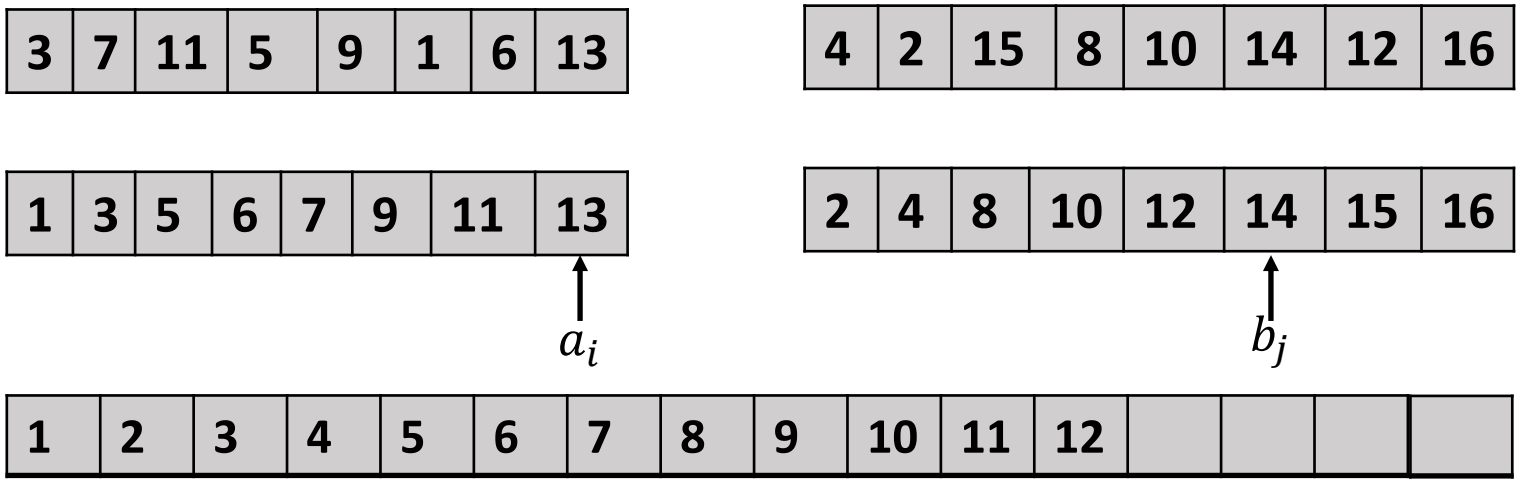
12 $inversions \leftarrow inversions + x$
13 **return** $(A, inversions)$

$inversions$=19

x=1



Counting inversions: how to combine two subproblems?

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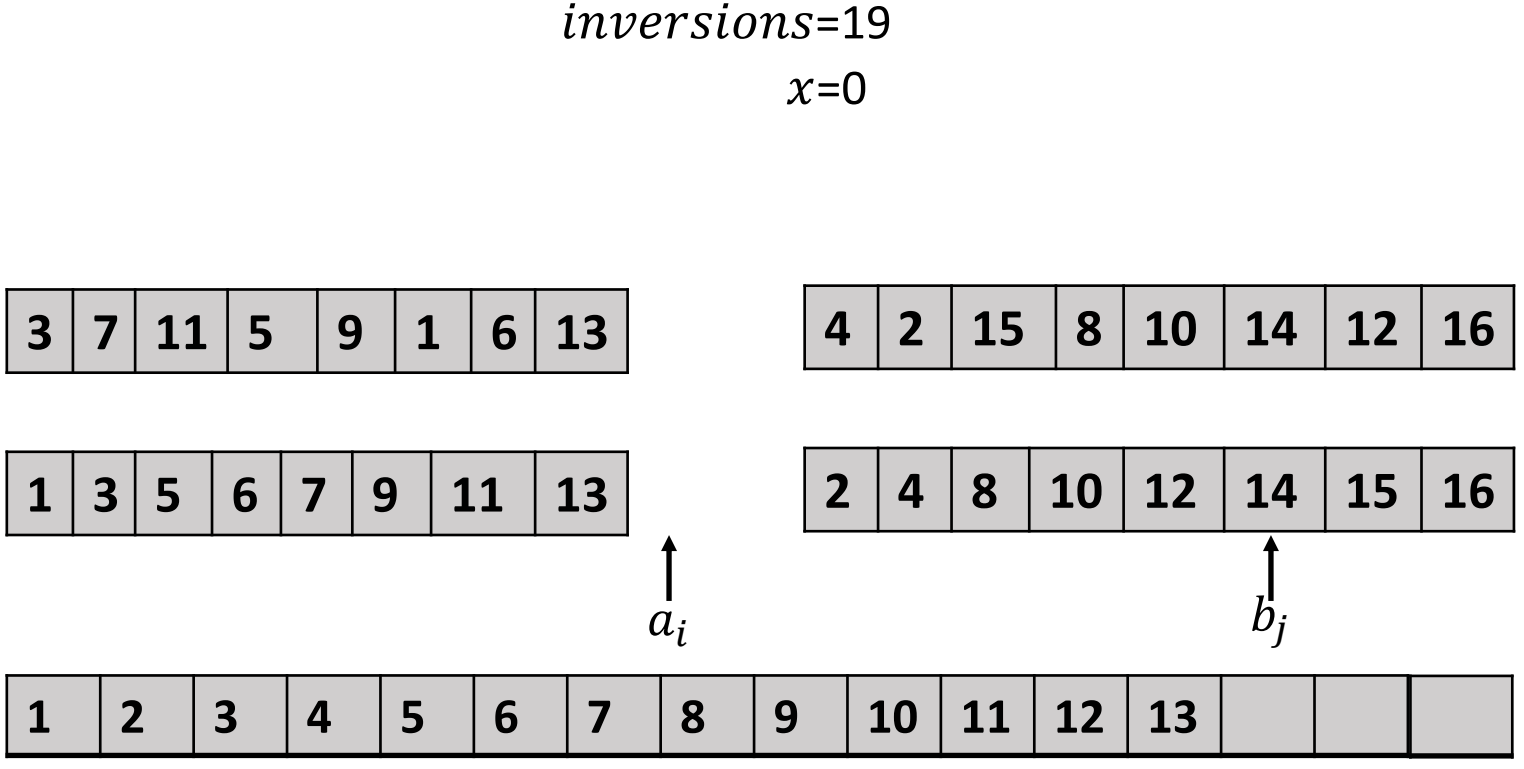

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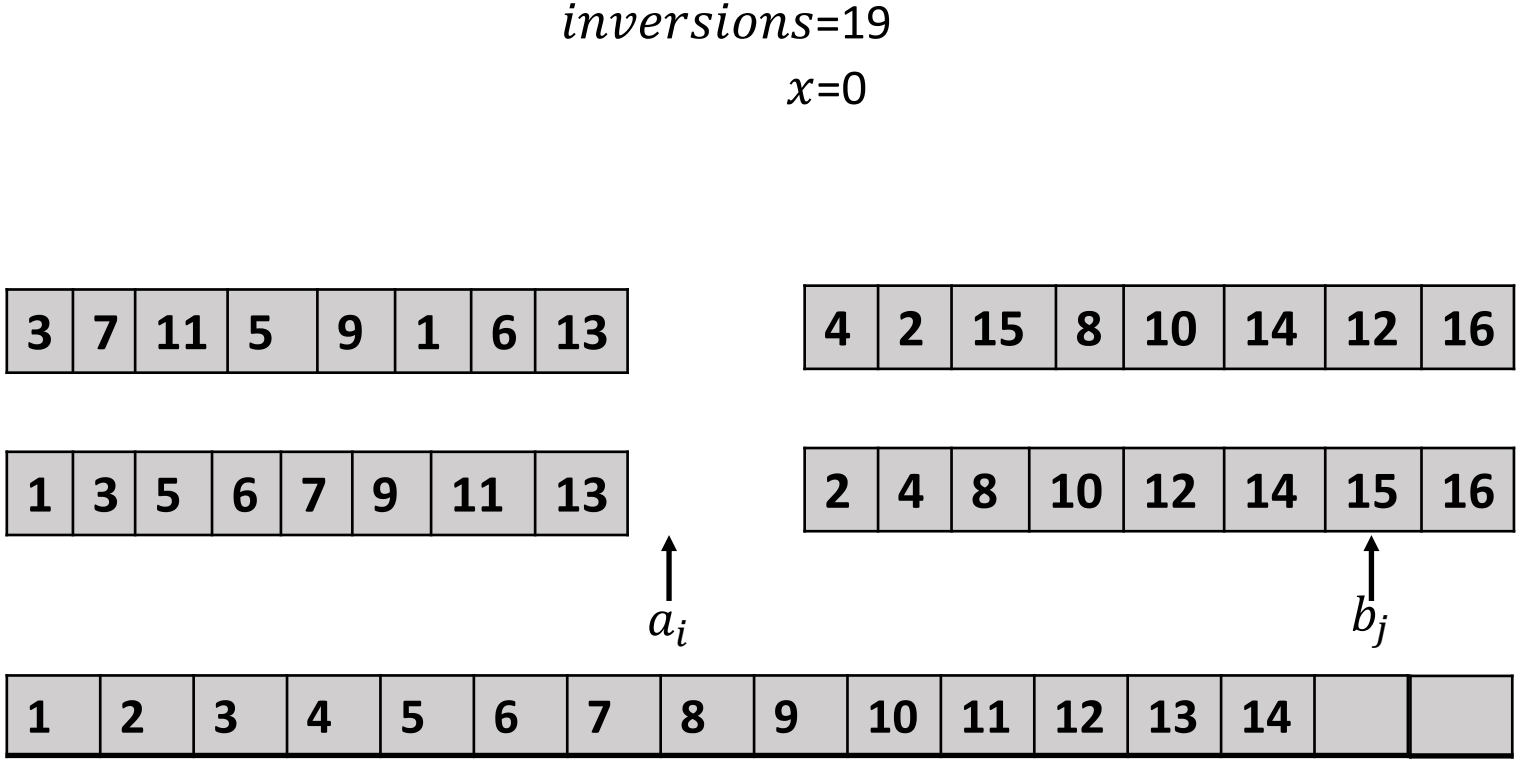

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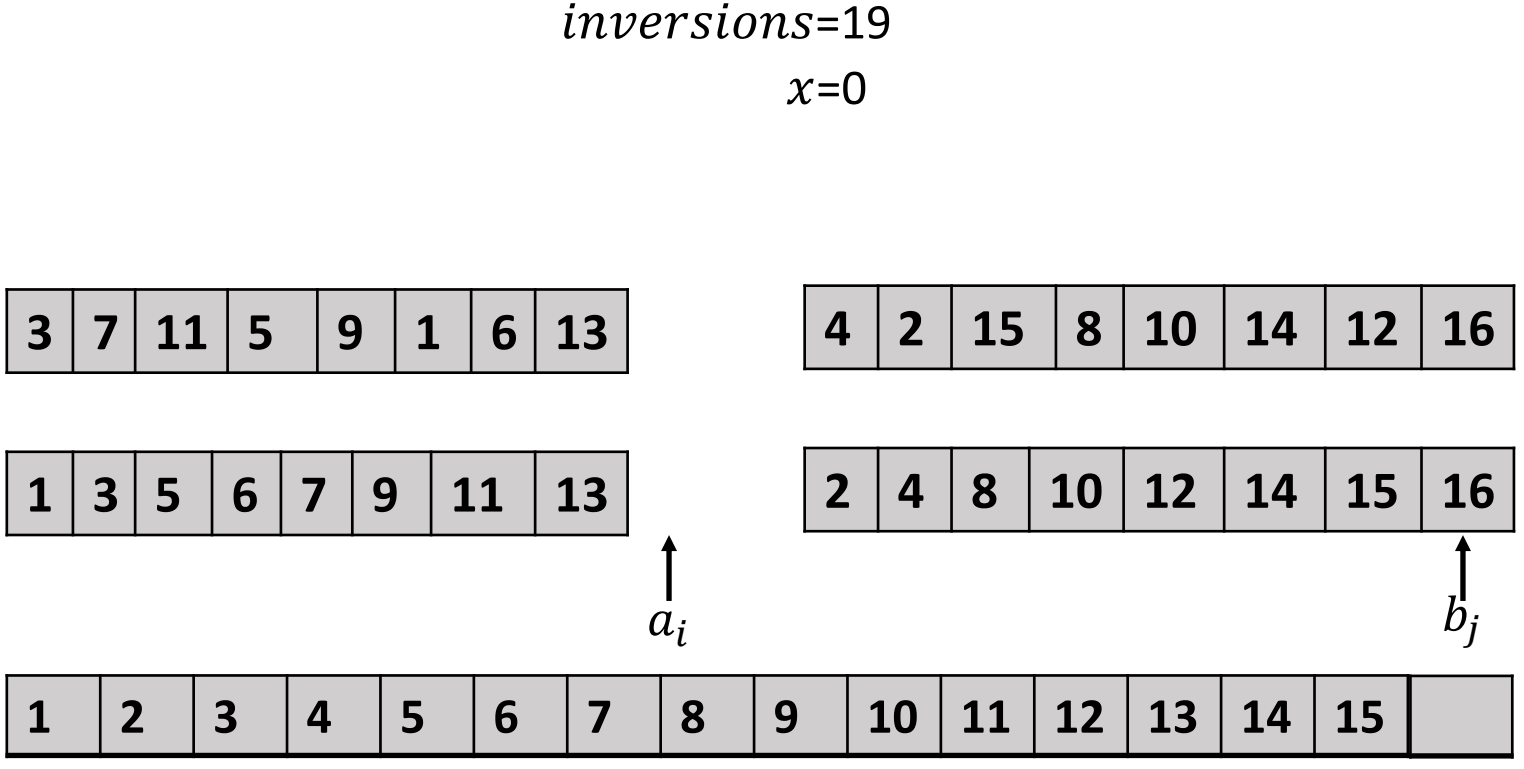

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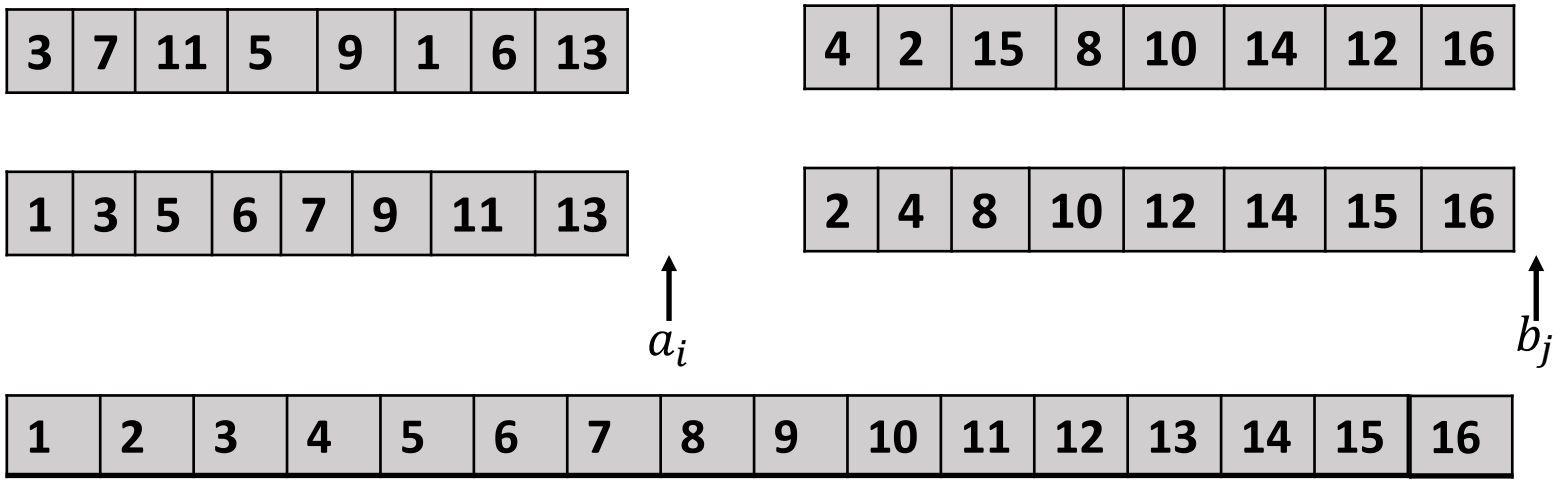
13 **return** $(A, inversions)$

$inversions$=19

x=0



Counting inversions: how to combine two subproblems?

$Sort - And - Count(A)$

1 **if** $(length(A) > 1)$

2 $A_1 \leftarrow A[1 \cdots \lfloor n/2 \rfloor]$ ← $\Theta(1)$

3 $A_2 \leftarrow A[\lfloor n/2 \rfloor + 1 \cdots n]$ ← $\Theta(1)$

4 $(A_1, A_{1\_inversions}) \leftarrow Sort - And - Count(A_1)$ ← $T(n/2)$

5 $(A_2, A_{2\_inversions}) \leftarrow Sort - And - Count(A_2)$ ← $T(n/2)$

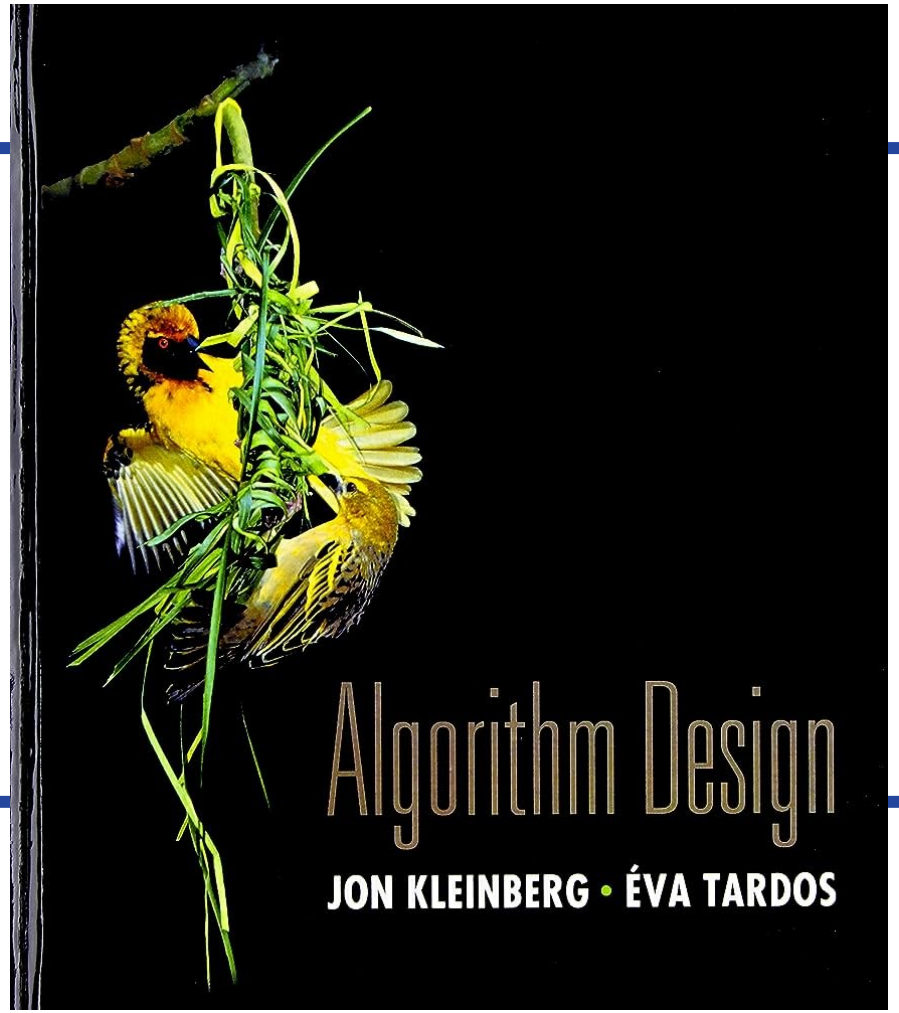
6 $(A, A_{inversions}) \leftarrow Merge - And - Count(A_1, A_2)$ ← $\Theta(n)$

7 **return** A

Counting inversions: divide-and-conquer algorithm analysis

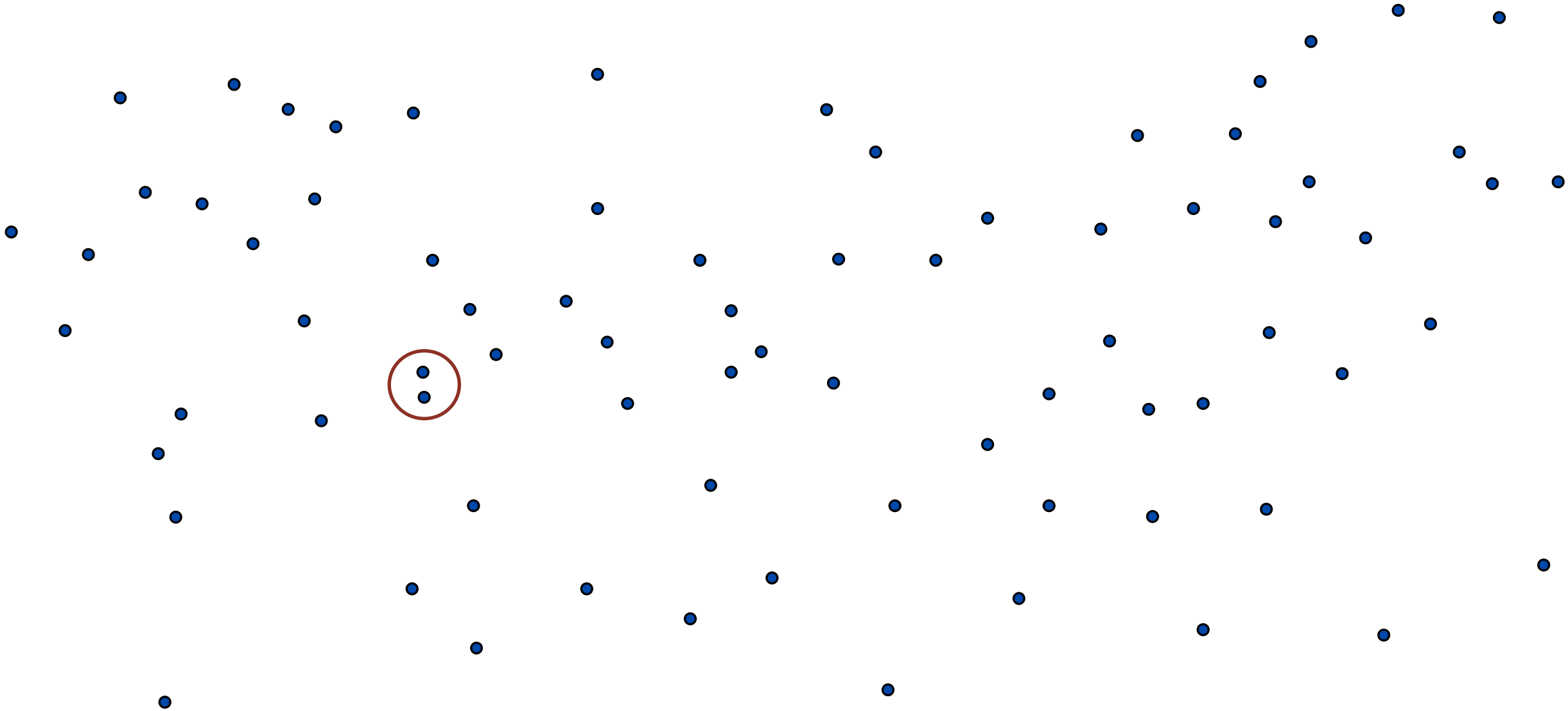
- Proposition. The sort-and-count algorithm counts the number of inversions in a permutation of size n in $O(n \log n)$ time.
- **Proof:** The worst-case running time $T(n)$ satisfies the recurrence:

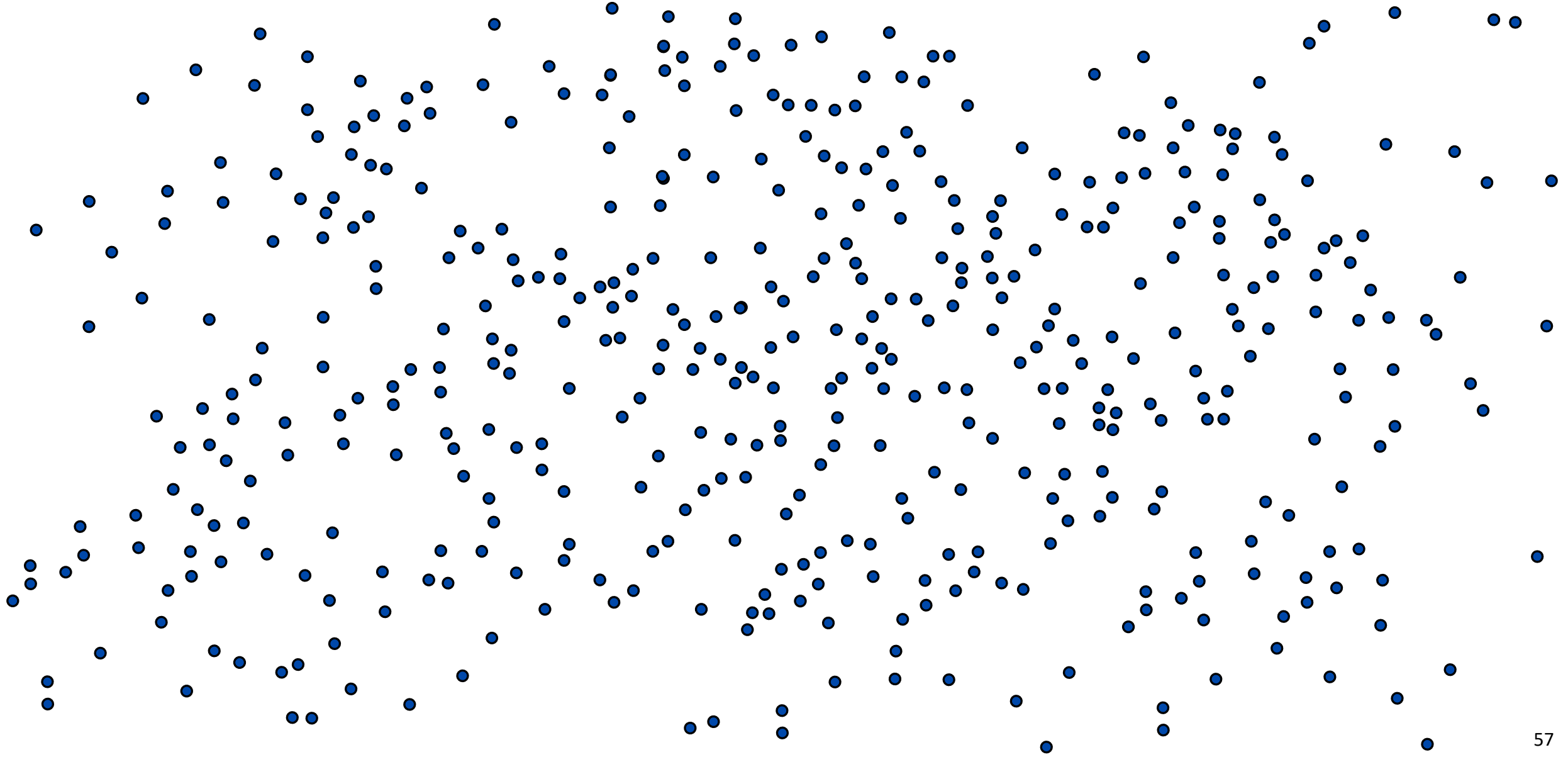
$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T\left(\frac{n}{2}\right) + \Theta(n) & \text{if } n > 1 \end{cases}$$



Chapter 5:
Divide and Conquer

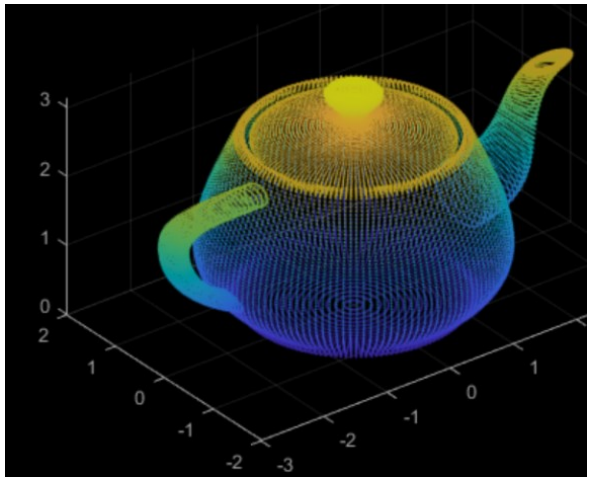
Section 5.4:
Closed Pair of Points

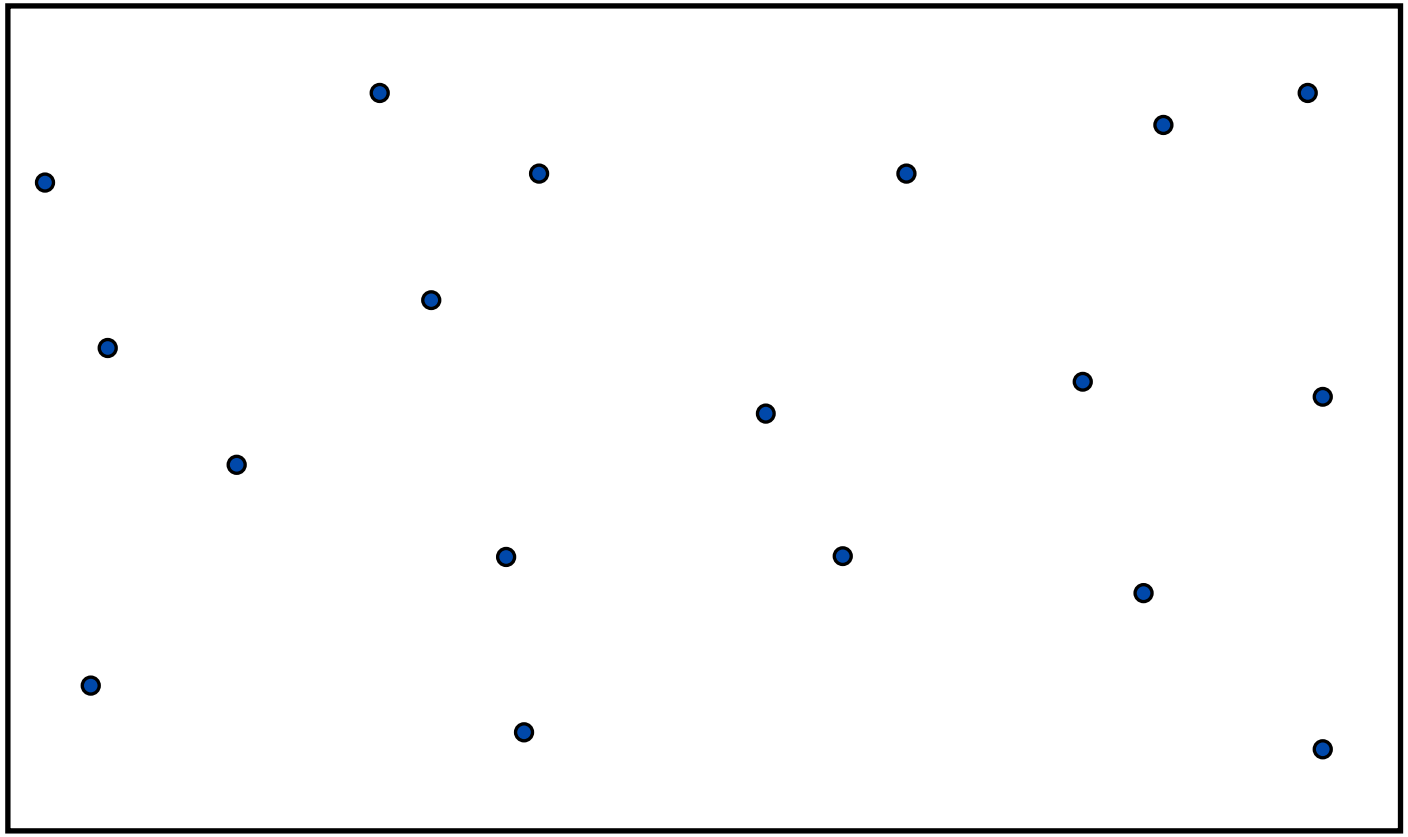


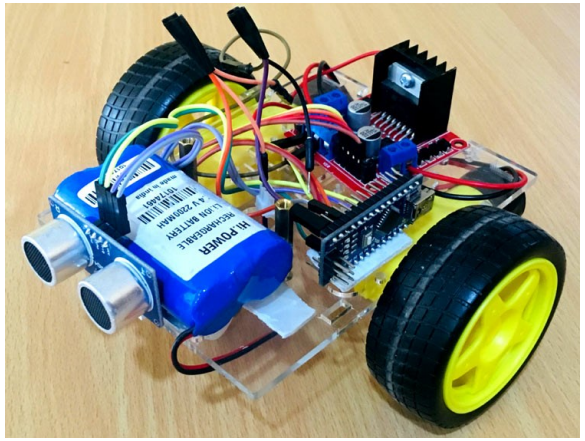


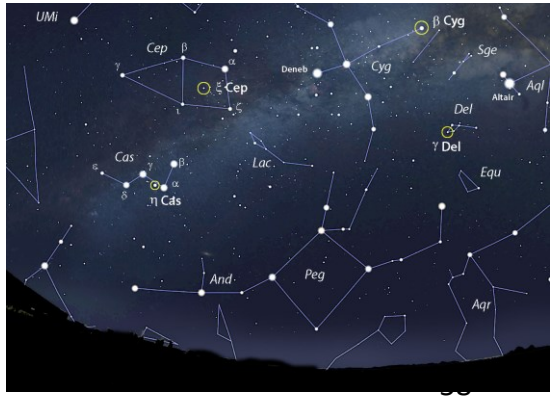
Closest pair of points

- **Closest pair problem:** Given n points in the plane, find a pair of points with the smallest Euclidean distance between them.









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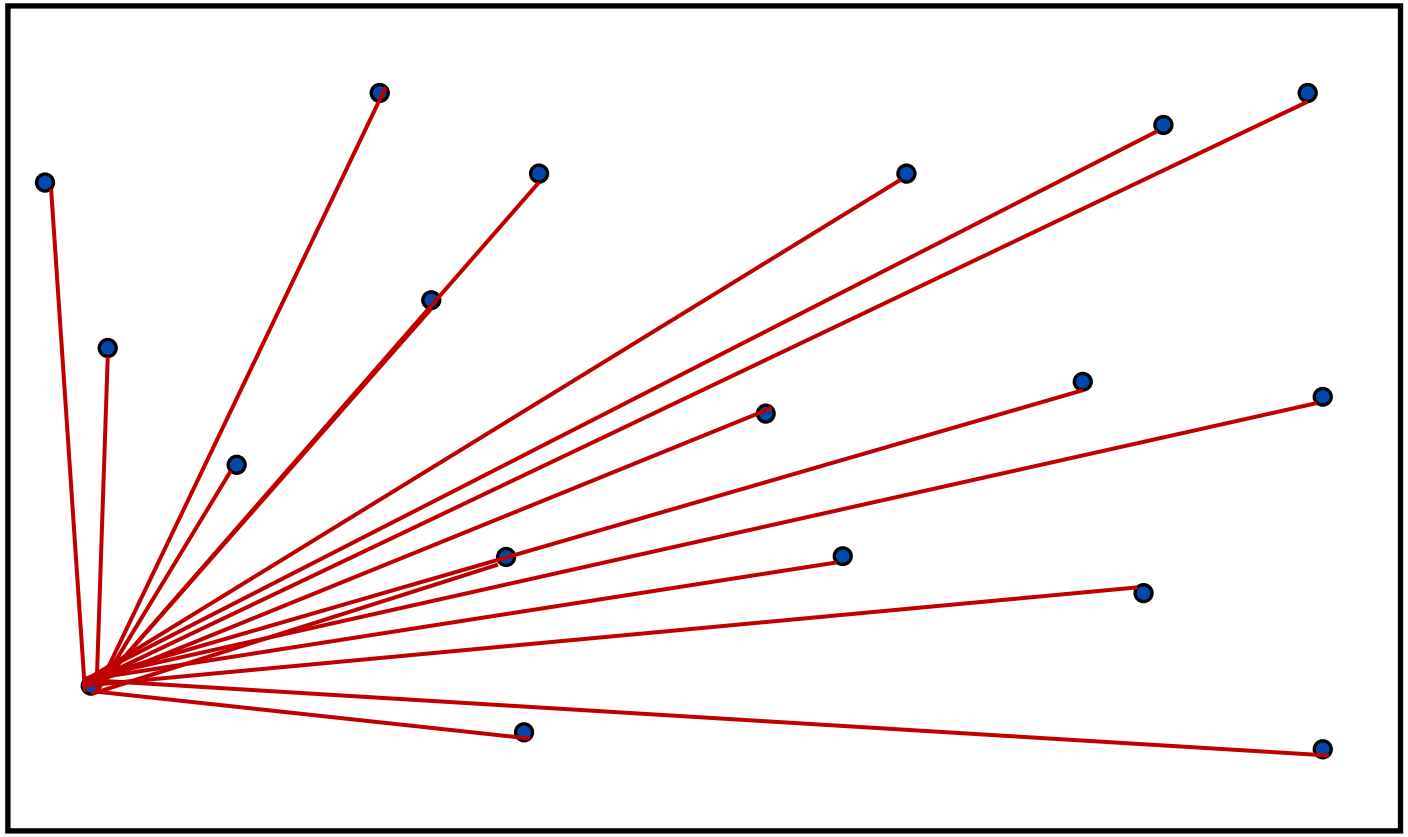

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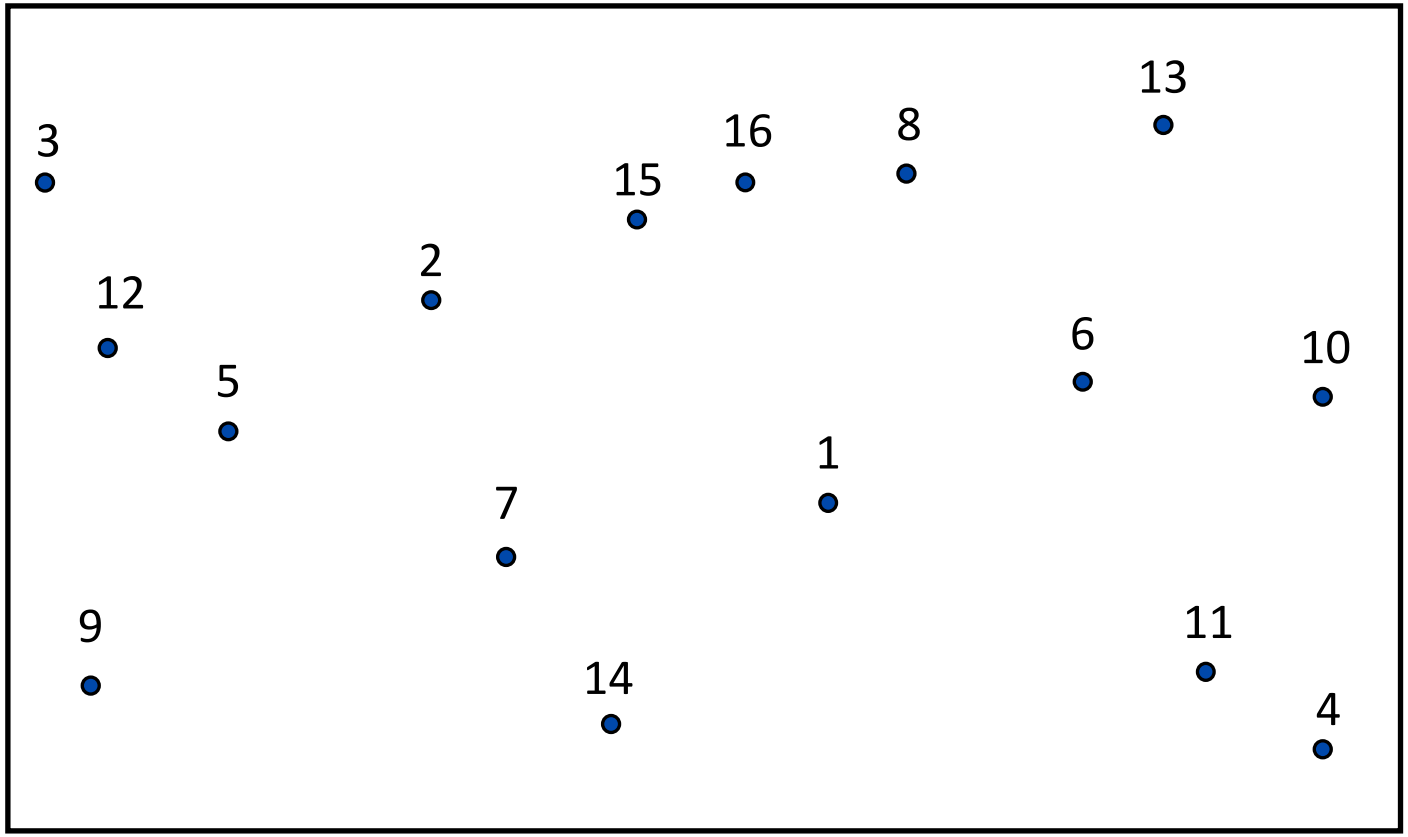
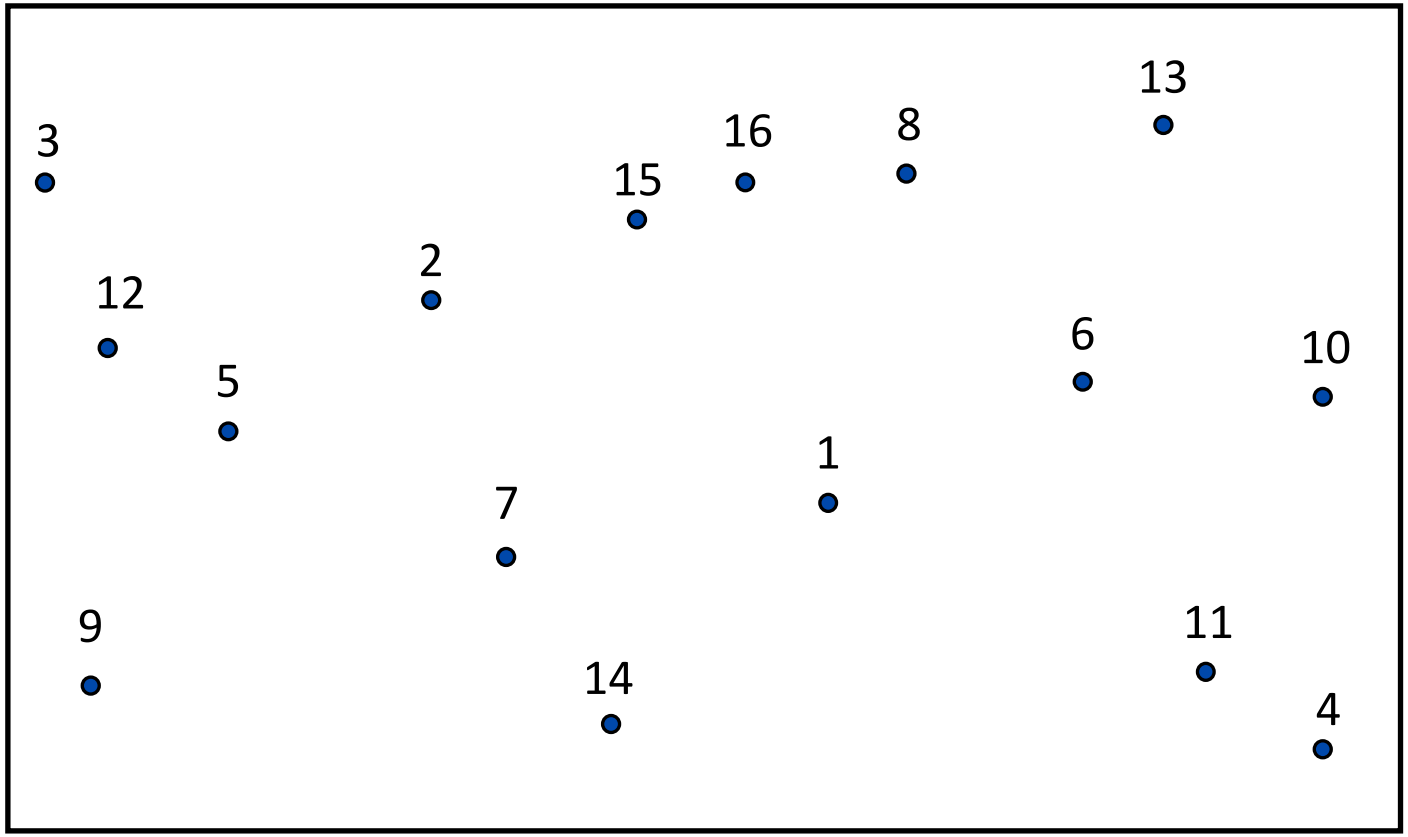


(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17)

Closest pair of points: Sorting Solution

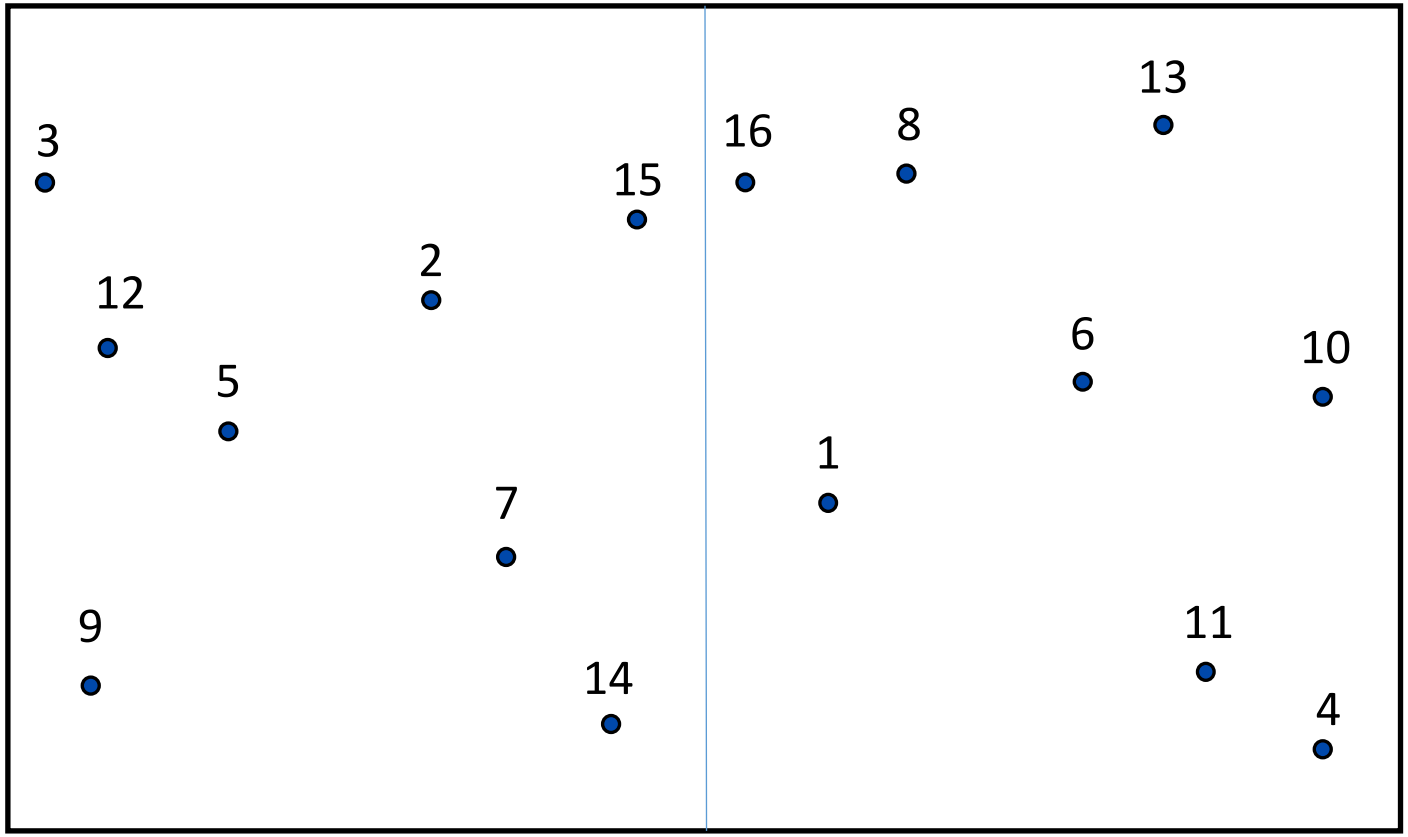
- Sort points by x-coordinate



(3, 9, 12, 5, 2, 7, 14, 15, 16, 1, 8, 6, 13, 11, 10, 4)

Closest pair of points

- Divide: draw vertical line L so that n / 2 points on each side.

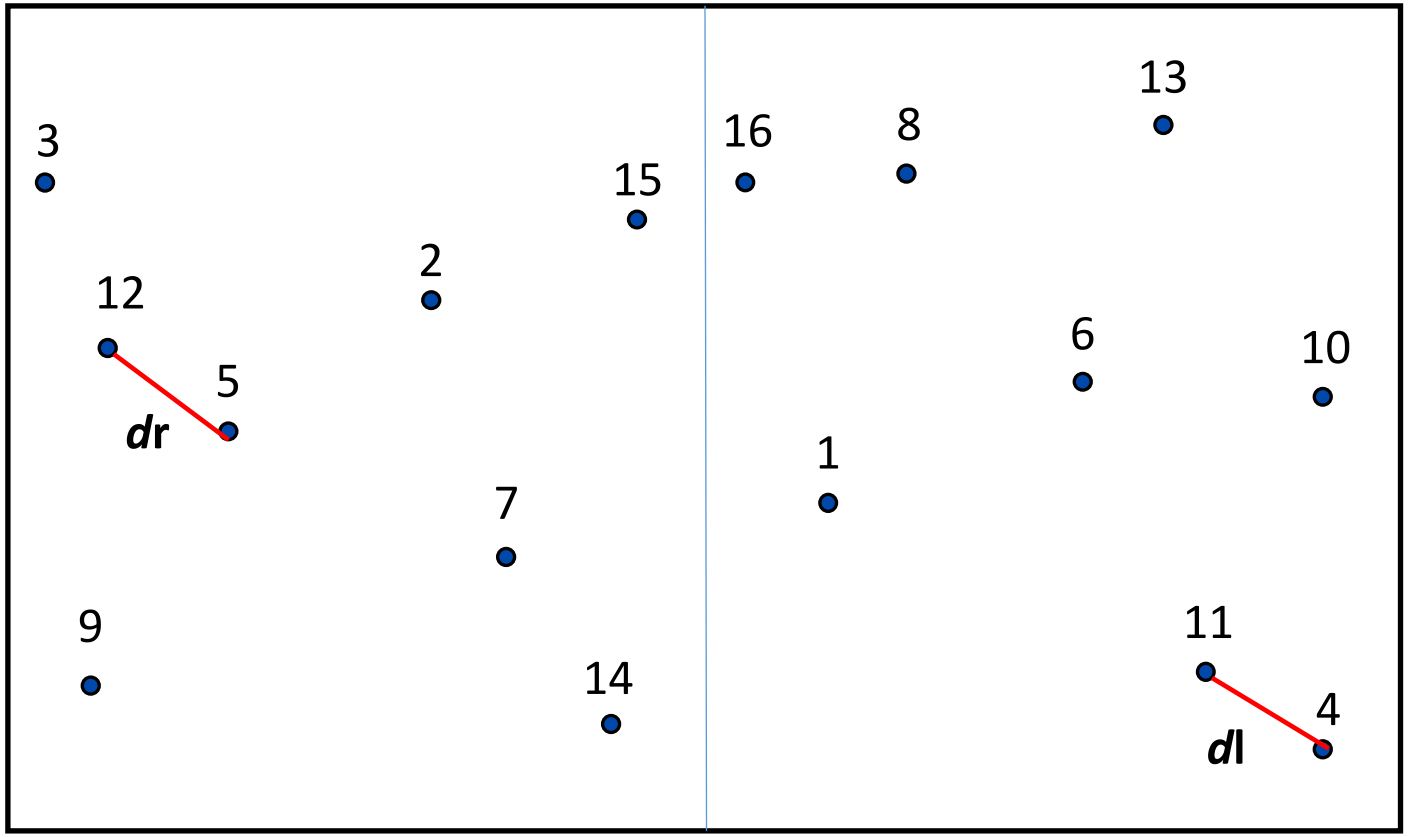


(3, 9, 12, 5, 2, 7, 14, 15)

(16, 1, 8, 6, 13, 11, 10, 4)

Closest pair of points

- Conquer: find closest pair in each side recursively.

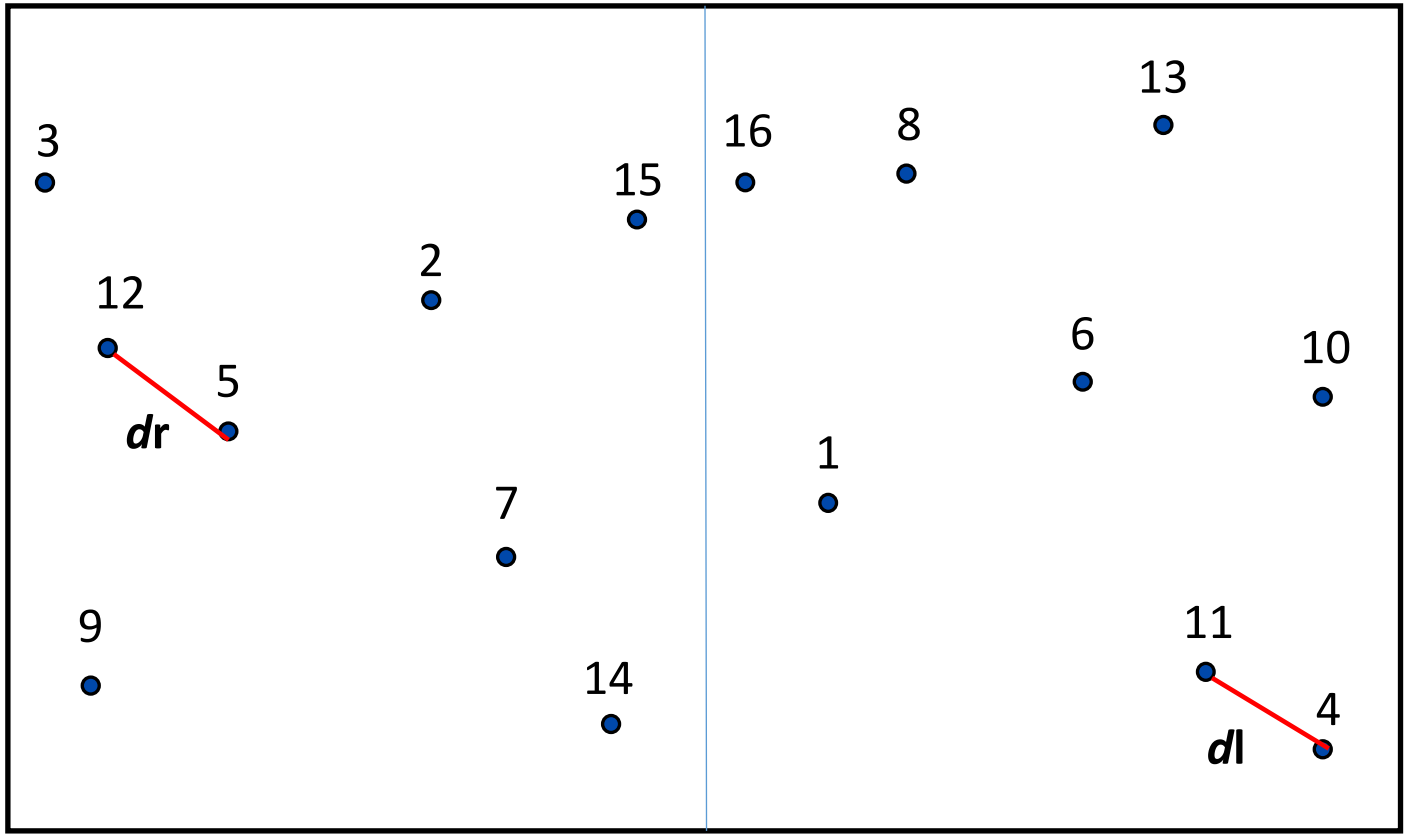


(3, 9, 12, 5, 2, 7, 14, 15)

(16, 1, 8, 6, 13, 11, 10, 4)

Closest pair of points

- Conquer: find closest pair in each side recursively.

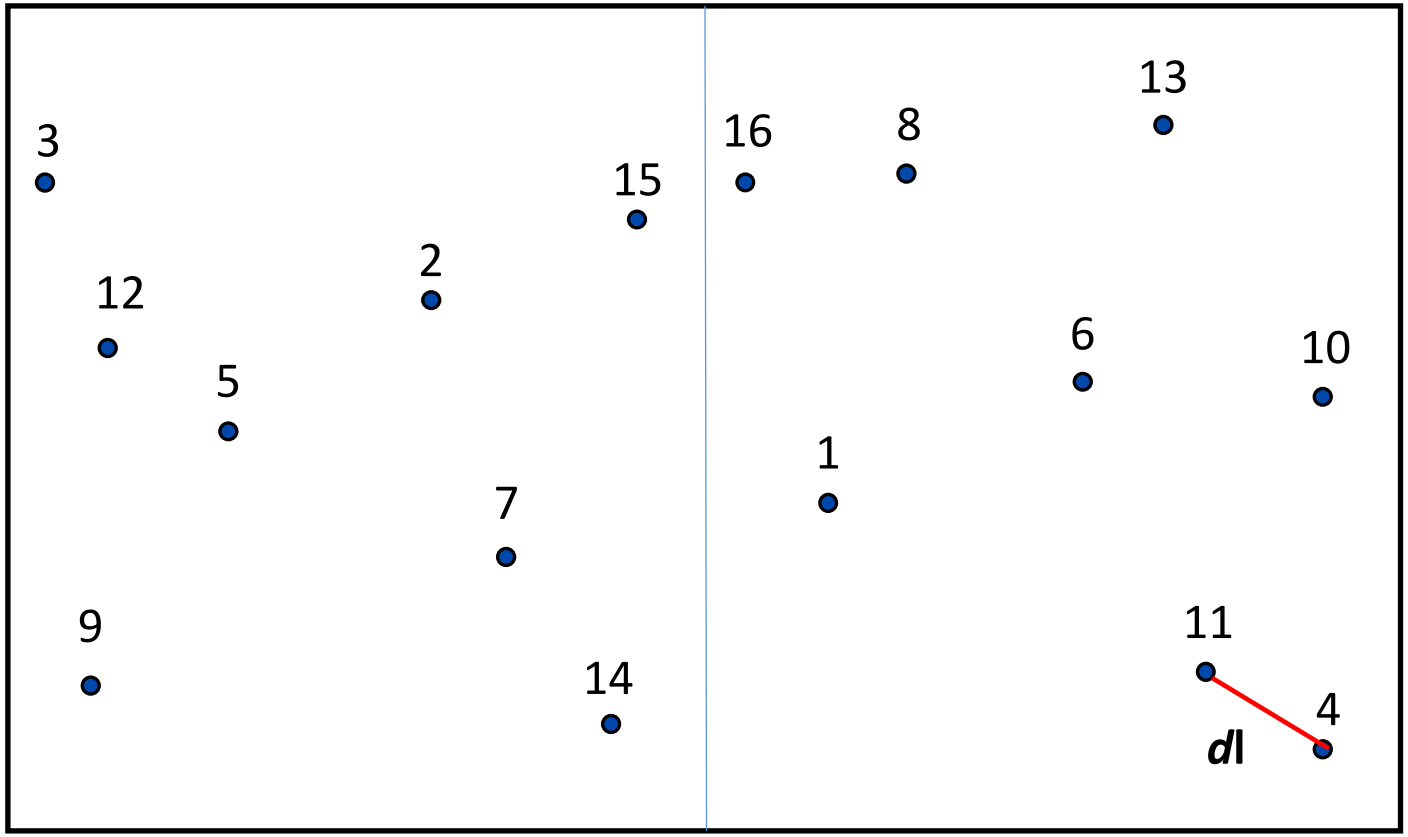


$d = \min(d_l, d_r)$

(3, 9, 12, 5, 2, 7, 14, 15)

(16, 1, 8, 6, 13, 11, 10, 4)

Closest pair of points



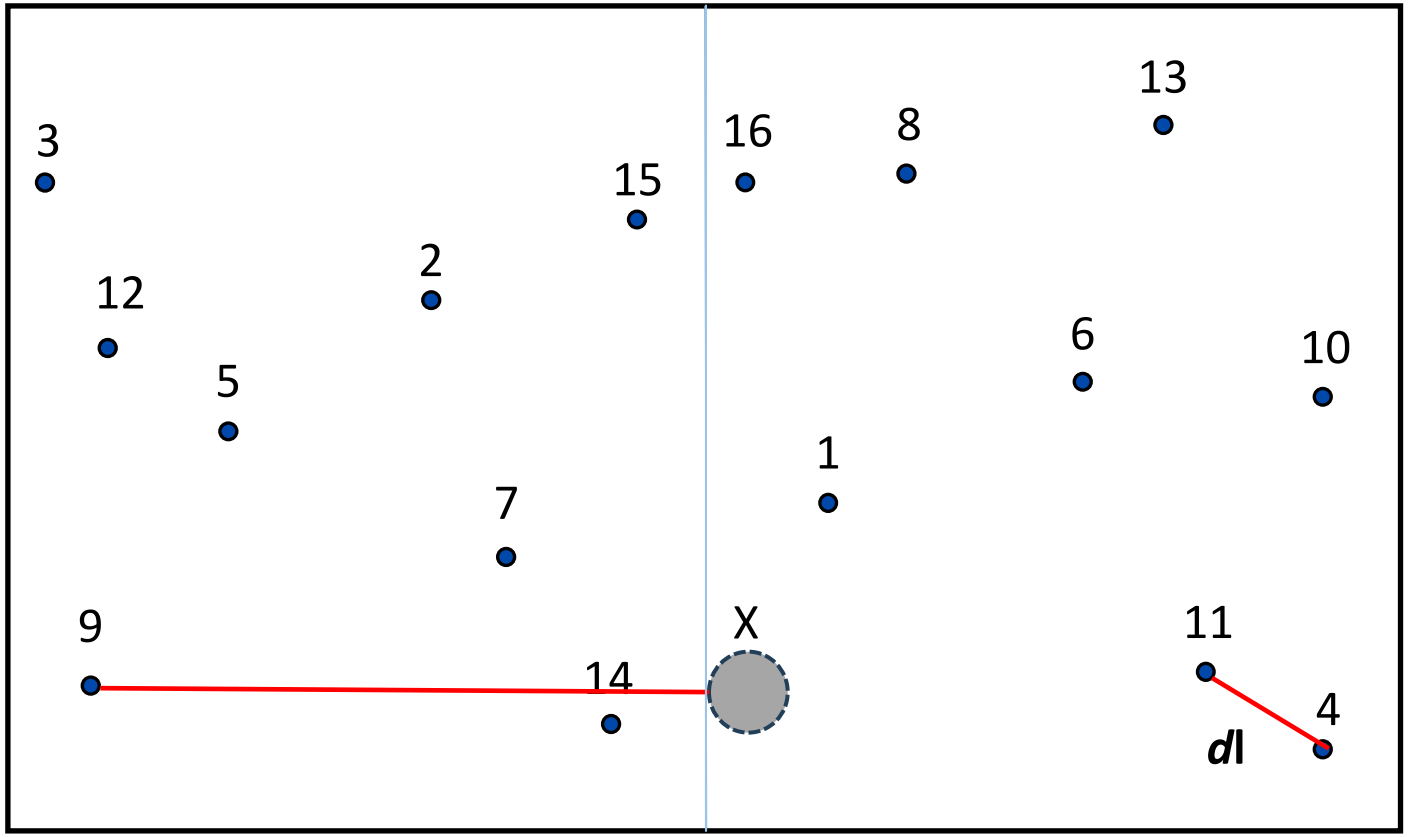
$d = \min(dl, dr)$

(3, 9, 12, 5, 2, 7, 14, 15)

(16, 1, 8, 6, 13, 11, 10, 4)

Closest pair of points

- Closest pair problem. Given n points in the plane, find a pair of points with the smallest Euclidean distance between them.



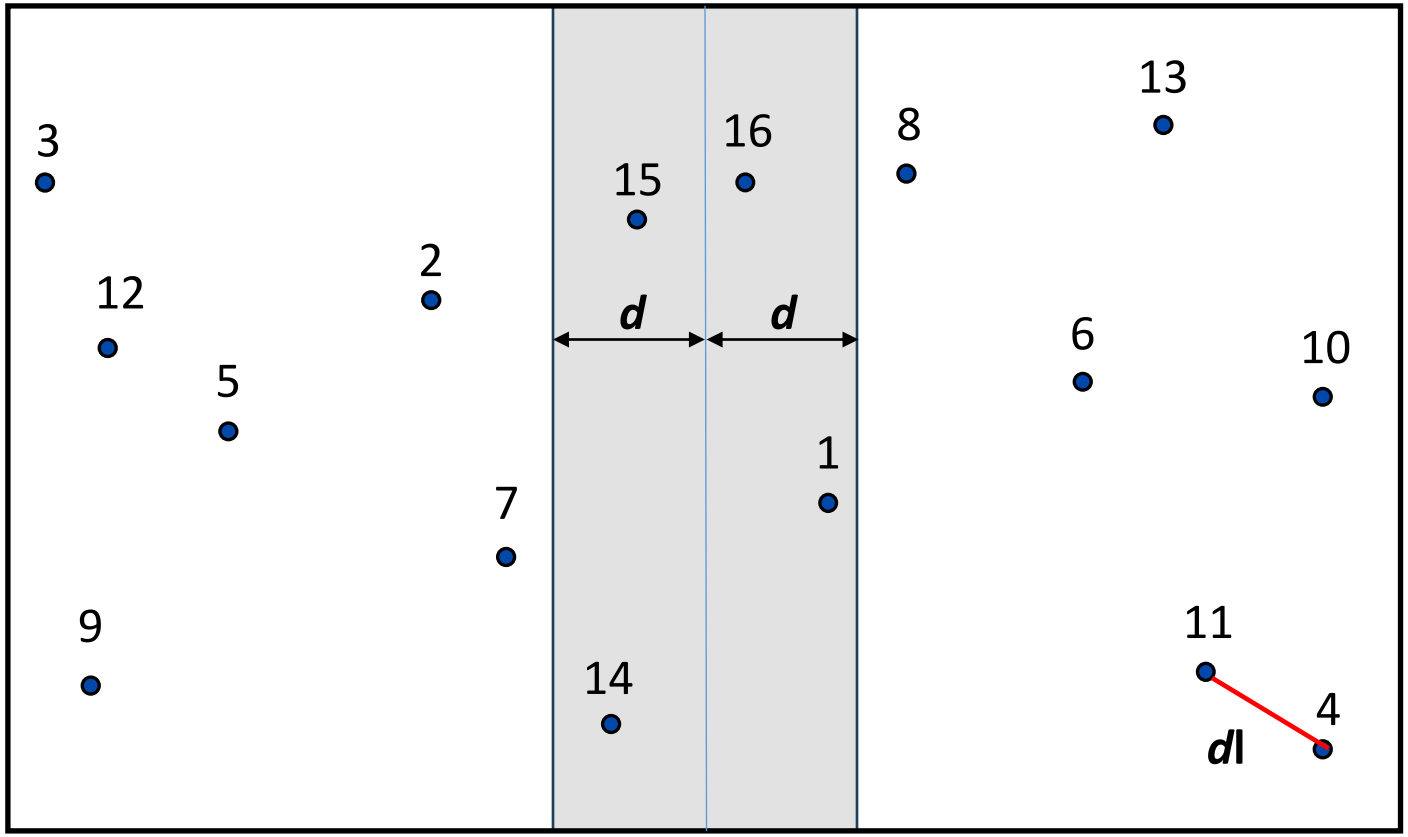
d= min(dl,dr)

(3, 9, 12, 5, 2, 7, 14, 15)

(16, 1, 8, 6, 13, 11, 10, 4)

Closest pair of points

- Observation: suffices to consider only those points within d of line L



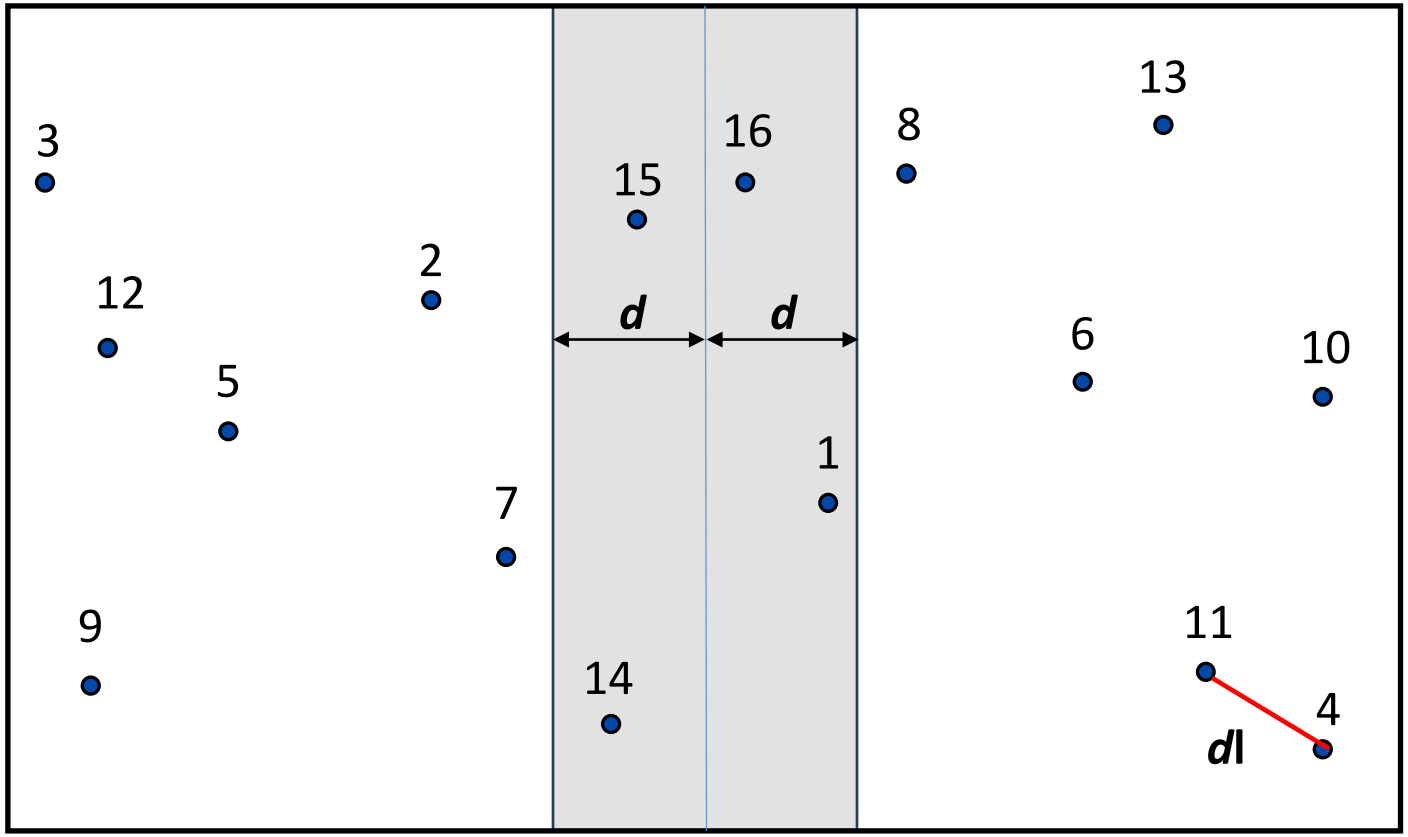
d= min(dl,dr)

(3, 9, 12, 5, 2, 7, 14, 15)

(16, 1, 8, 6, 13, 11, 10, 4)

Closest pair of points

- Observation: suffices to consider only those points within d of line L

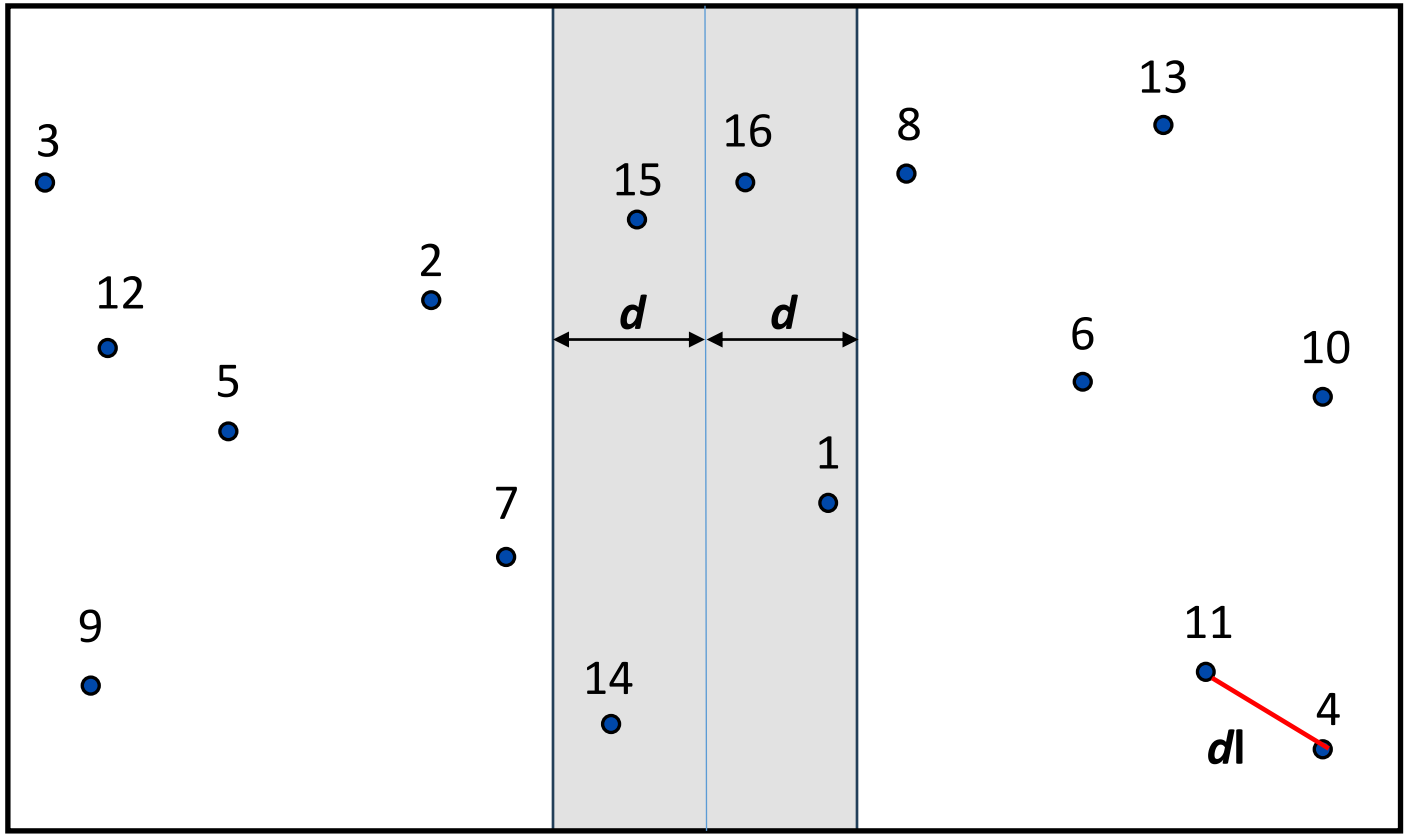


d= min(dl,dr)

(14,15,16,1)

Closest pair of points

- Reorder based on Y-axis

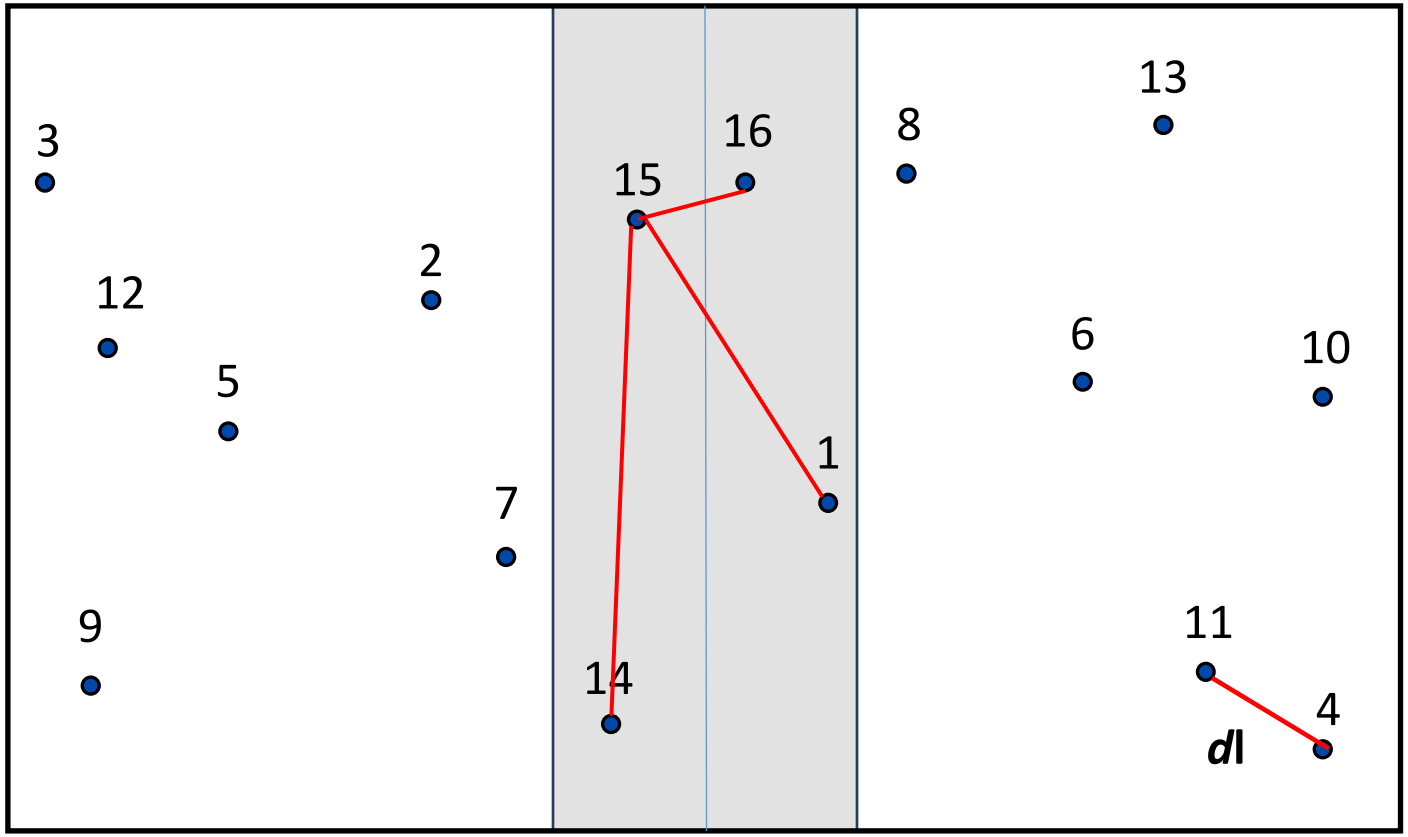


(16,15,1,14)

***d*= min(dl,dr)**

Closest pair of points

- Reorder based on Y-axis

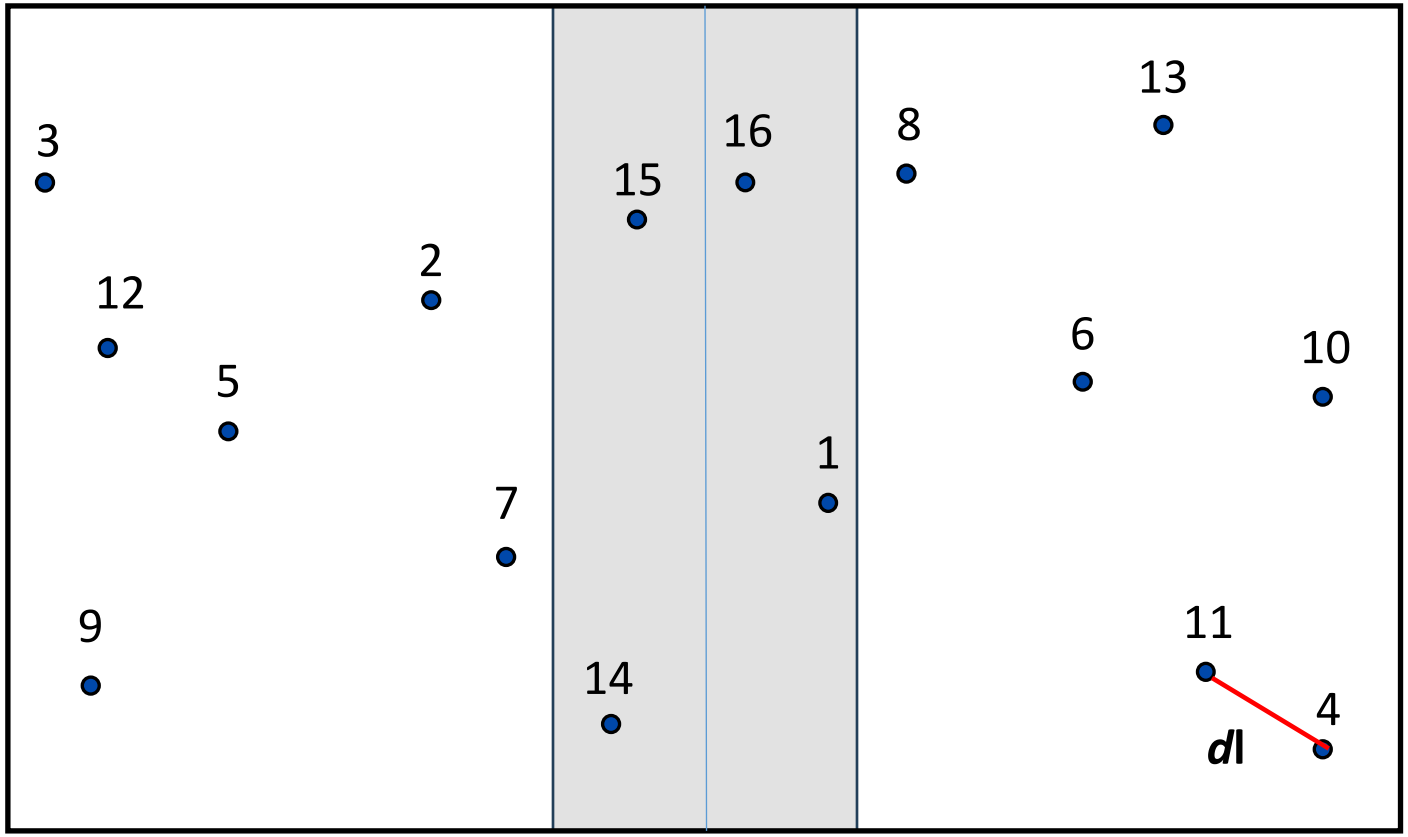


(16,15,1,14)

***d*= min(dl,dr)**

Closest pair of points

- Reorder based on Y-axis

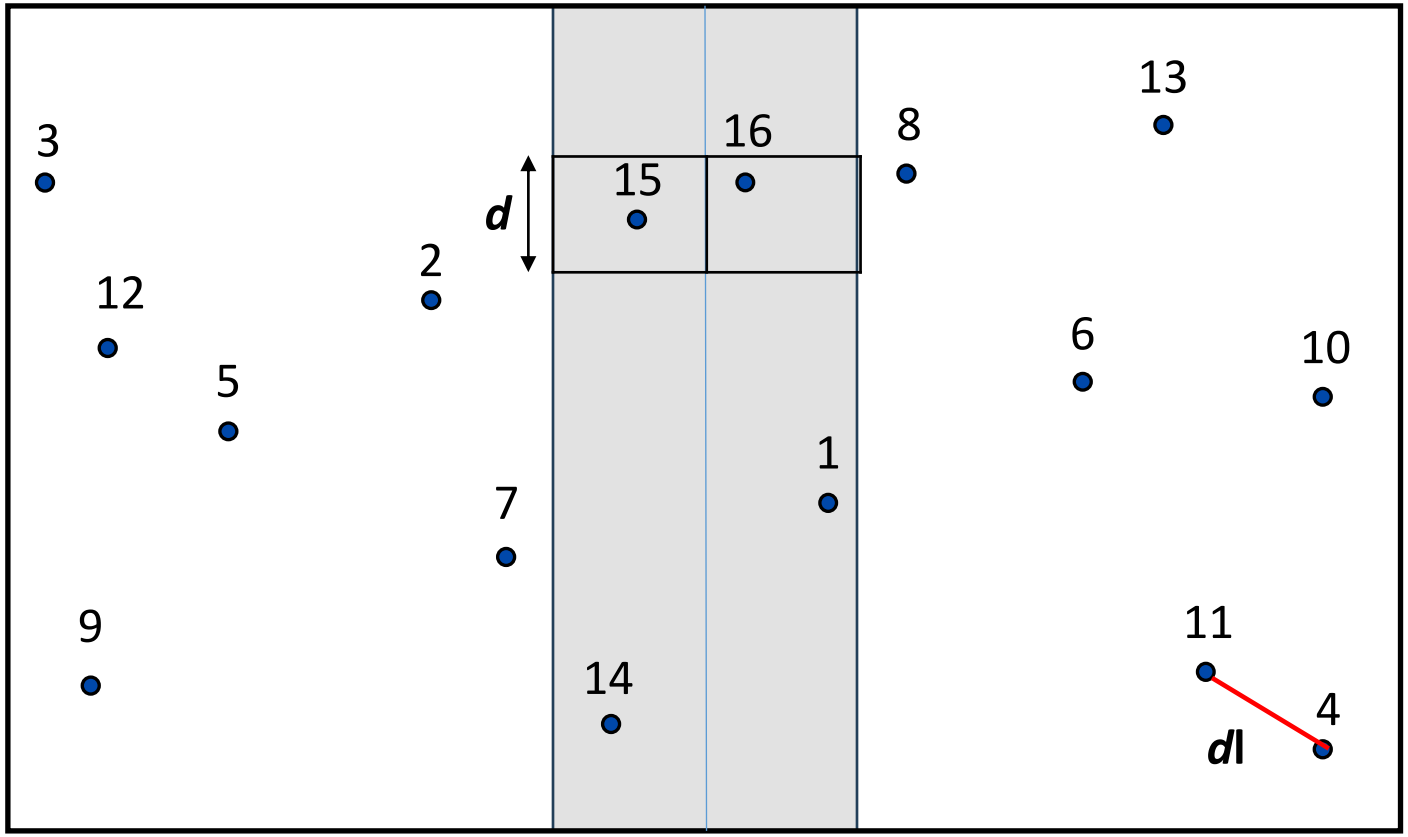


(16,15,1,14)

***d*= min(dl,dr)**

Closest pair of points

- Reorder based on Y-axis

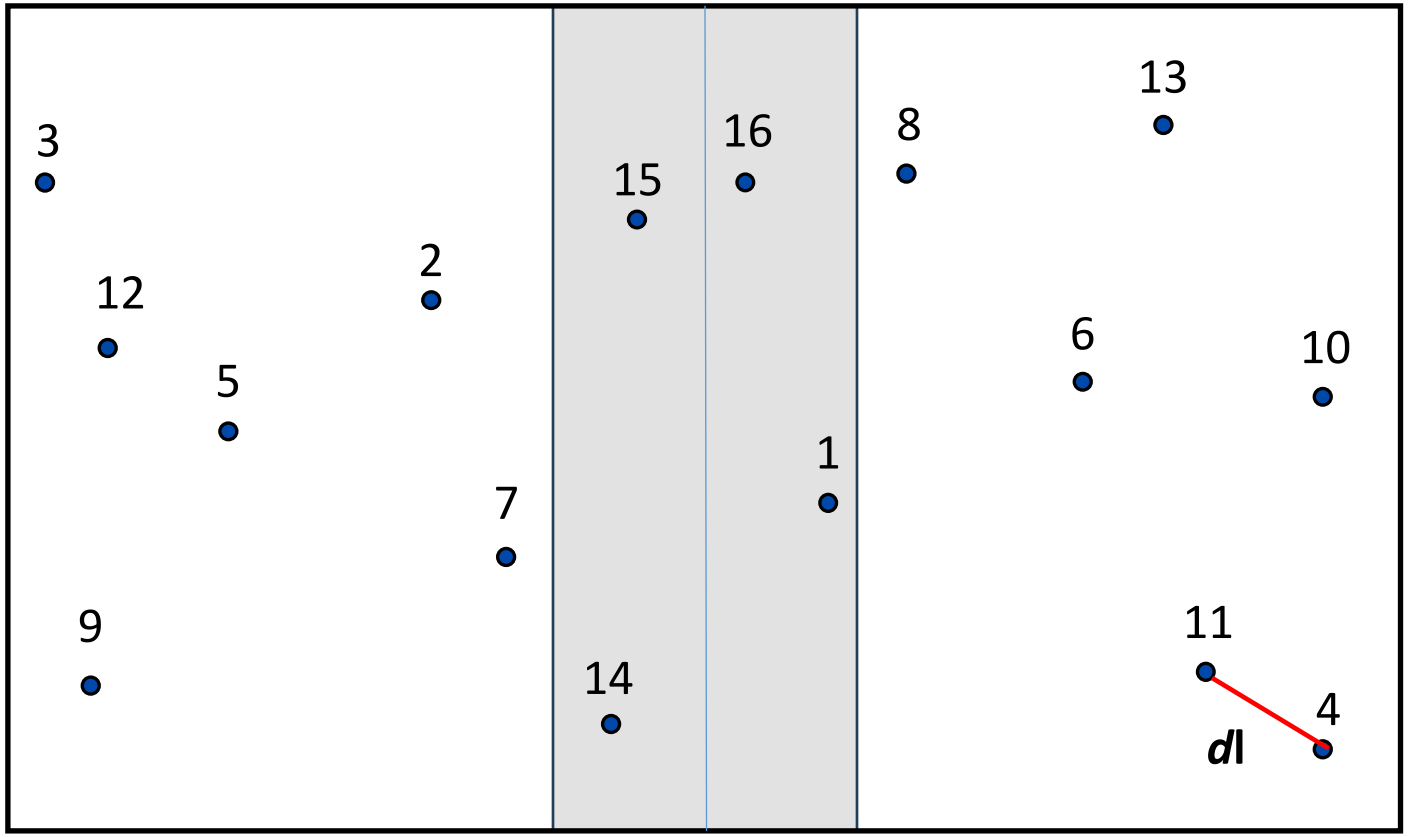


(16,15,1,14)

***d*= min(dl,dr)**

Closest pair of points

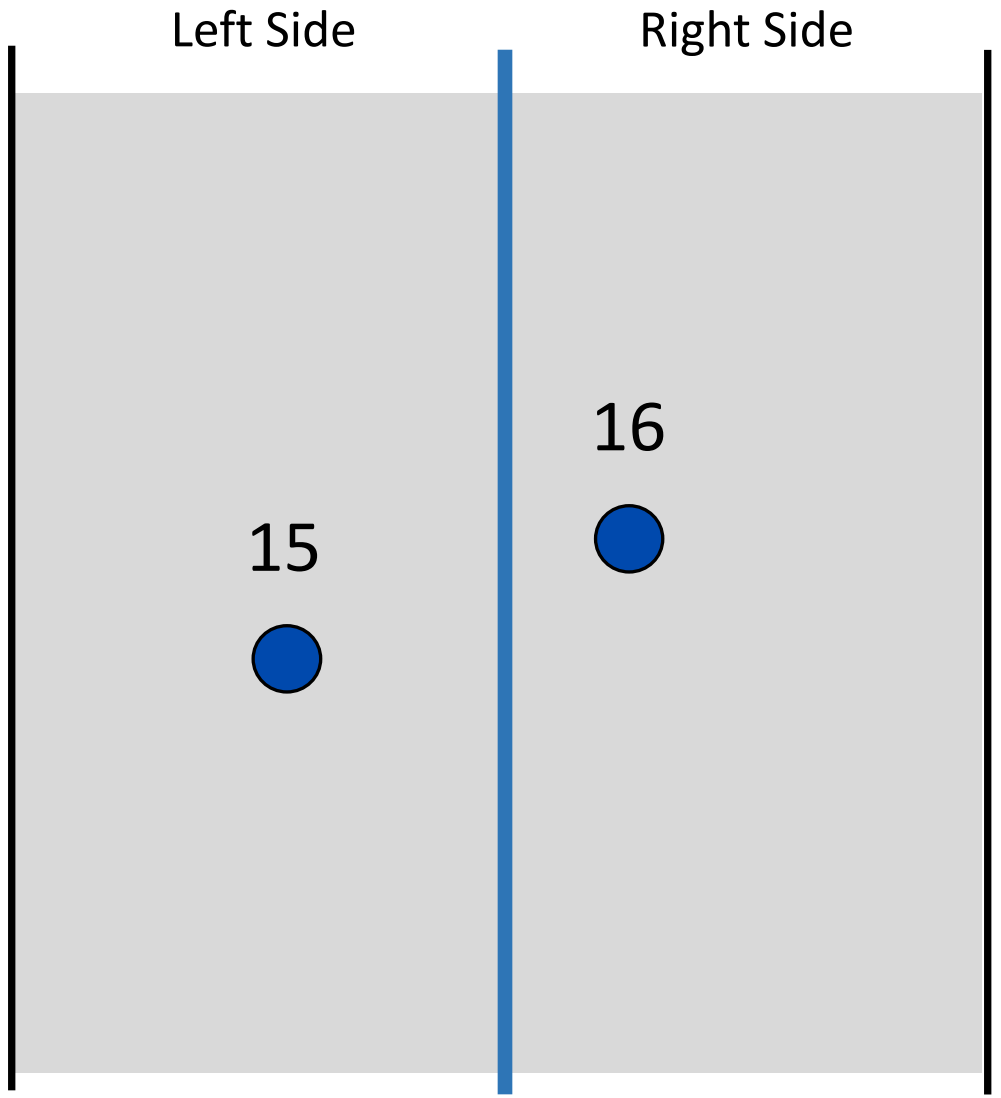
- Reorder based on Y-axis



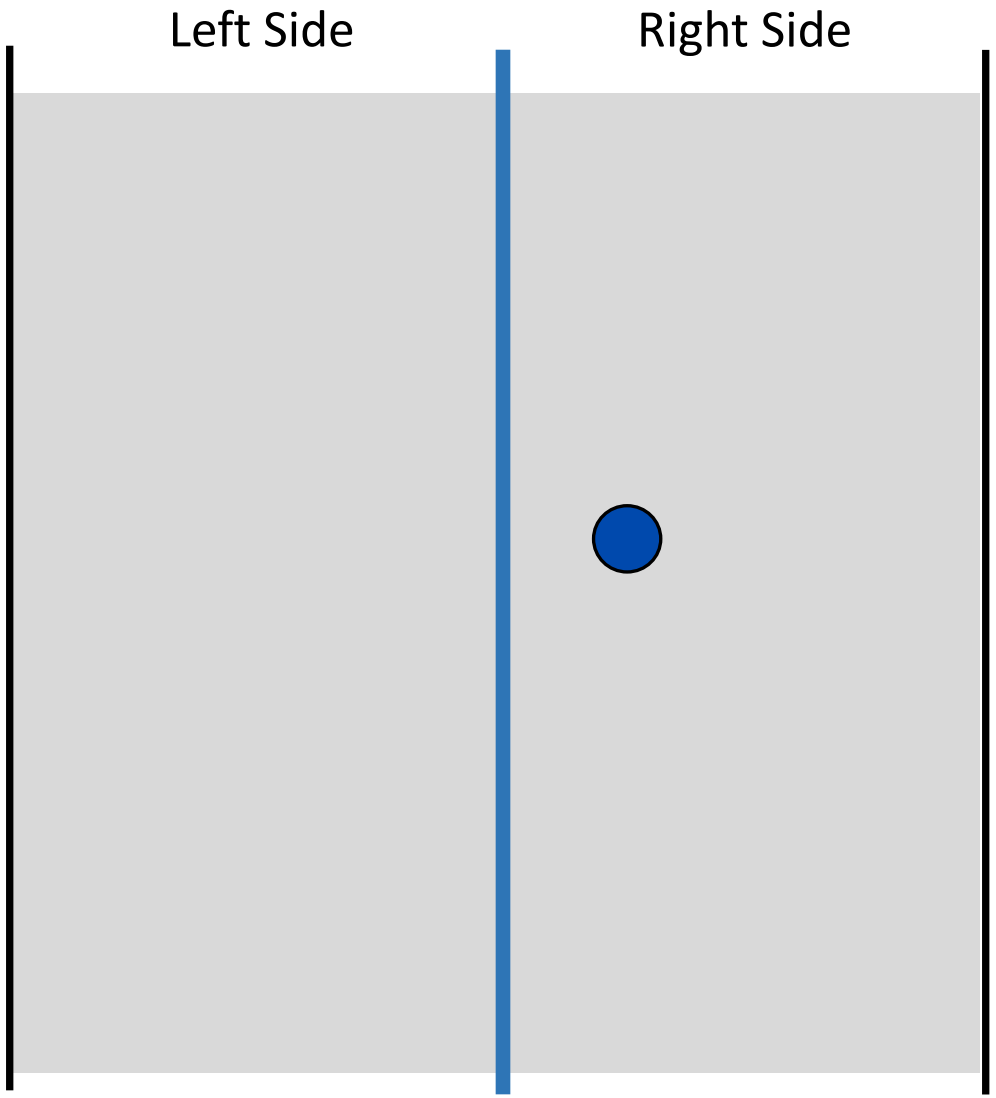
(16,15,1,14)

***d*= min(dl,dr)**

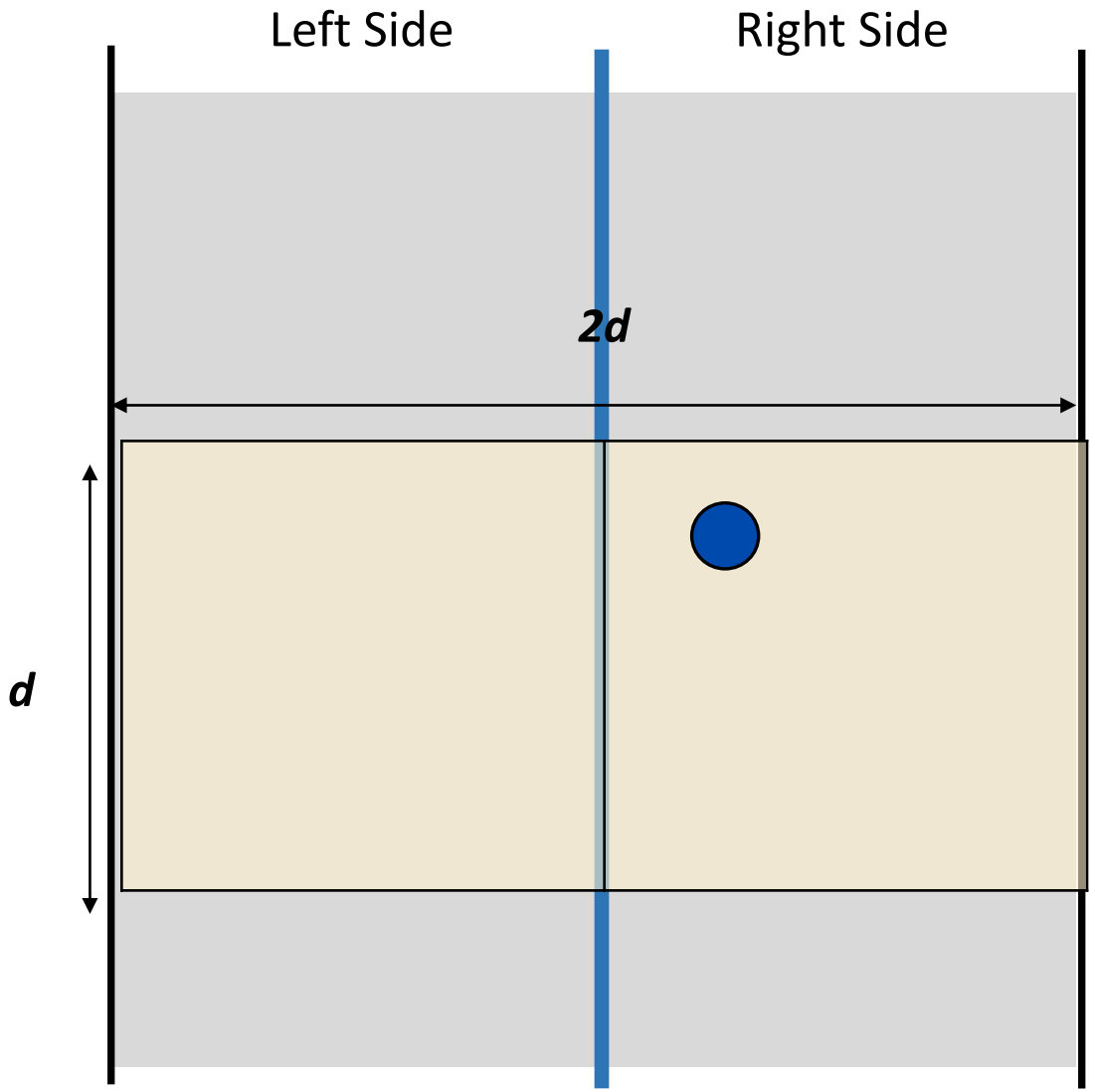
16
15
1
14



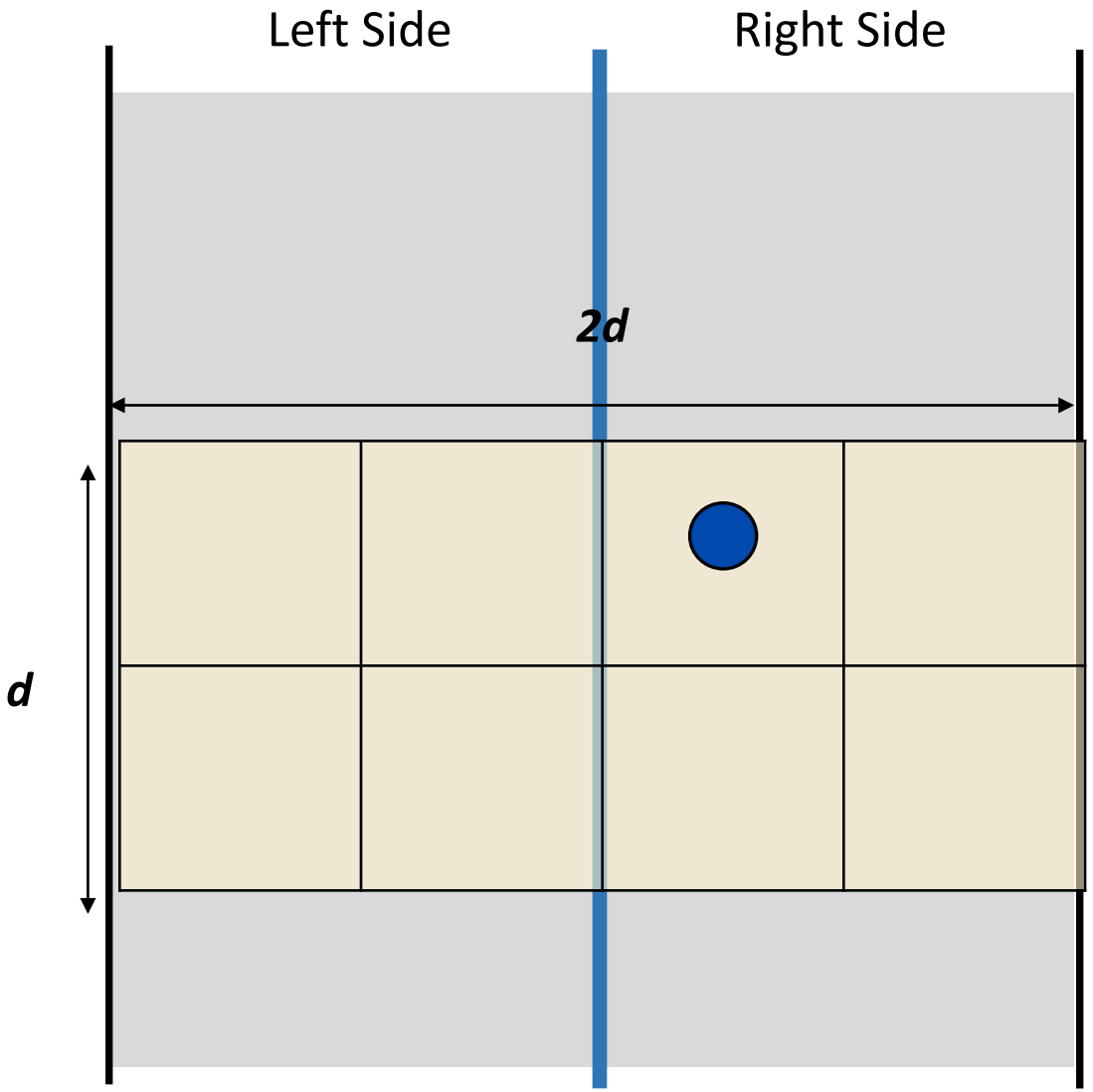
16
15
1
14
P_4
P_5
P_6
P_7
P_8
…
$P_{\sim n}$



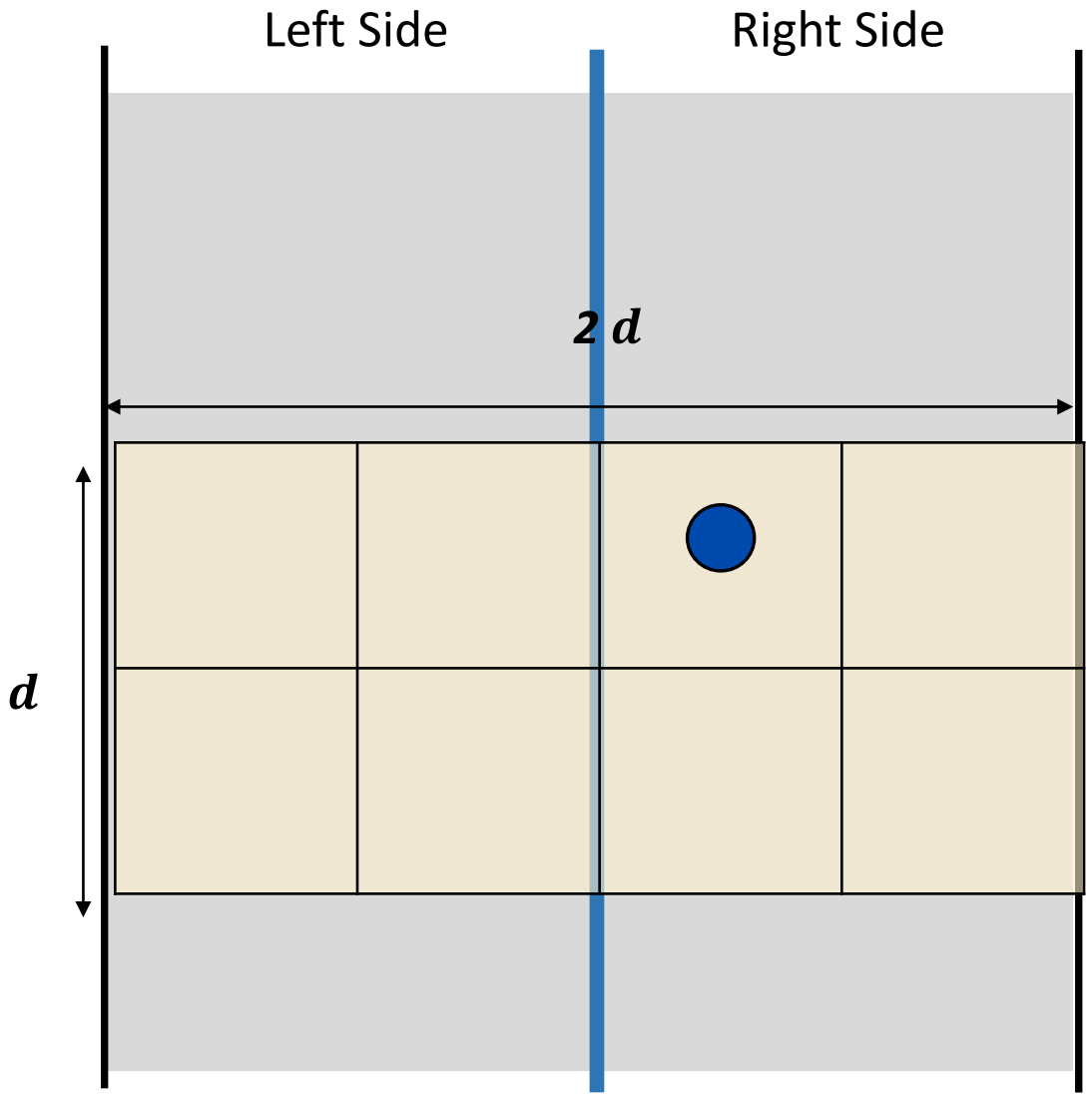
16
15
1
14
P_4
P_5
P_6
P_7
P_8
…
$P_{\sim n}$

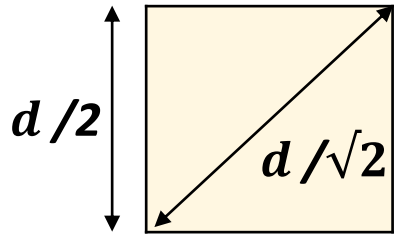


16
15
1
14
P_4
P_5
P_6
P_7
P_8
…
$P_{\sim n}$



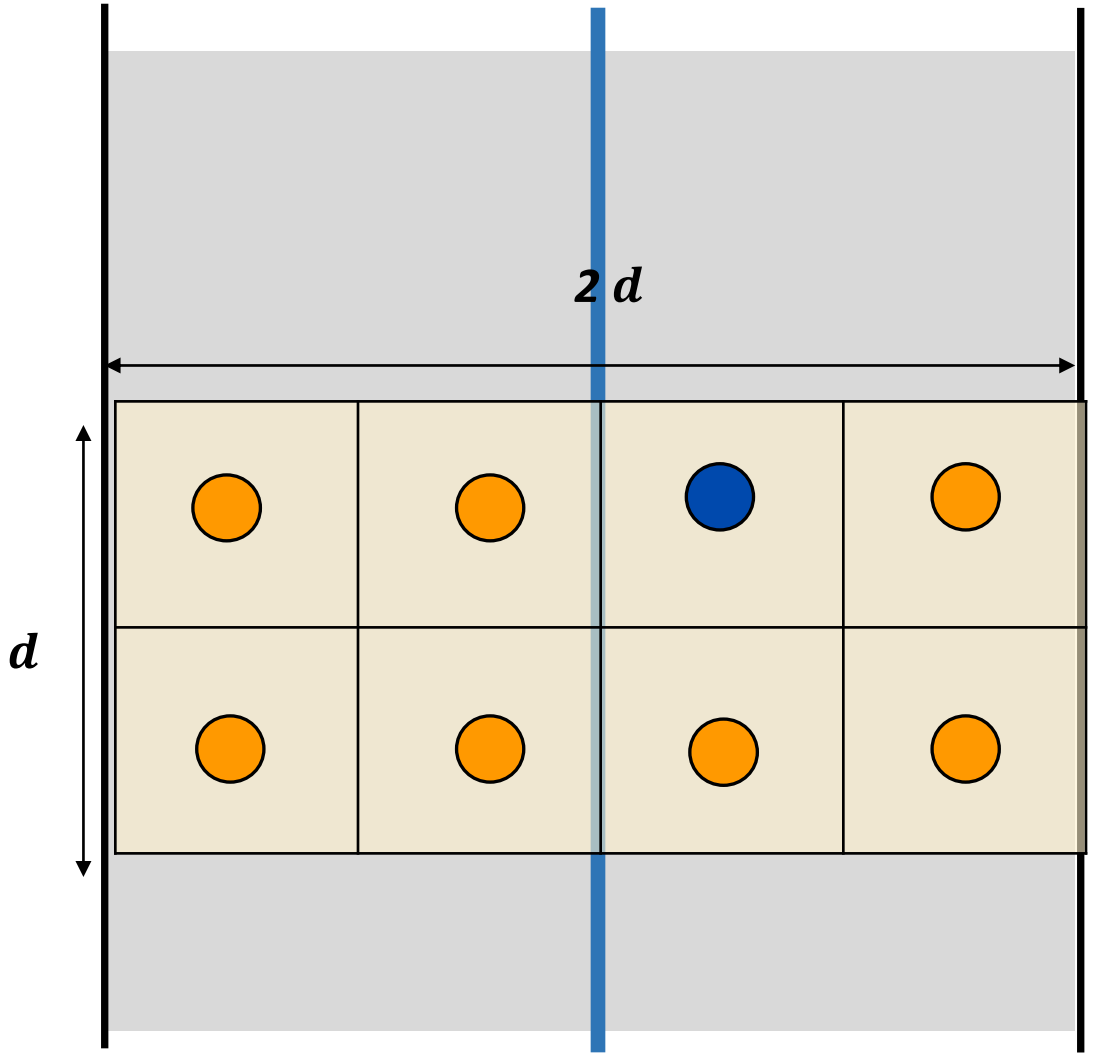
16
15
1
14
P_4
P_5
P_6
P_7
P_8
…
$P_{\sim n}$

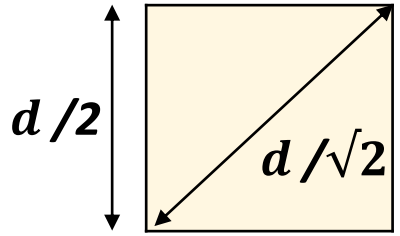




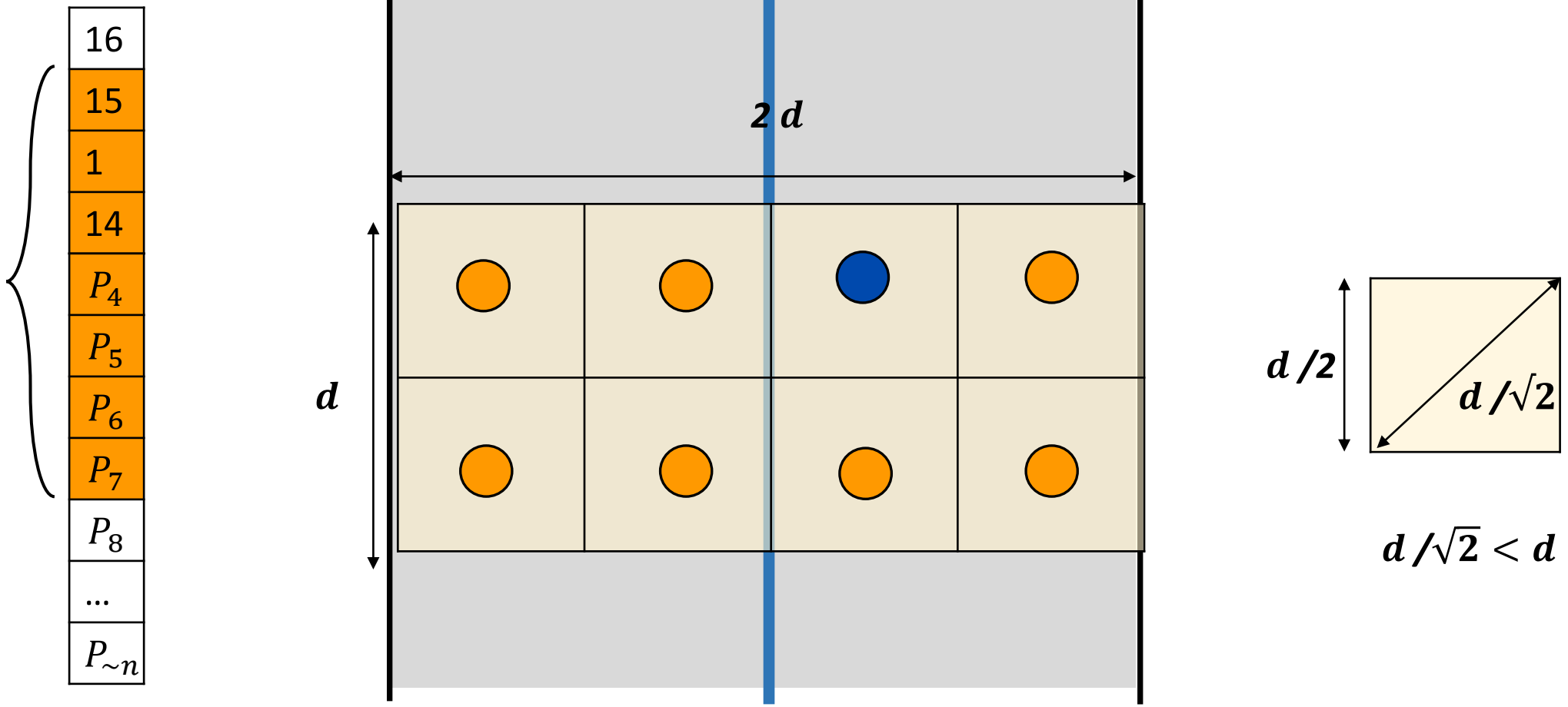
$$d/\sqrt{2} < d$$

16
15
1
14
P_4
P_5
P_6
P_7
P_8
…
$P_{\sim n}$

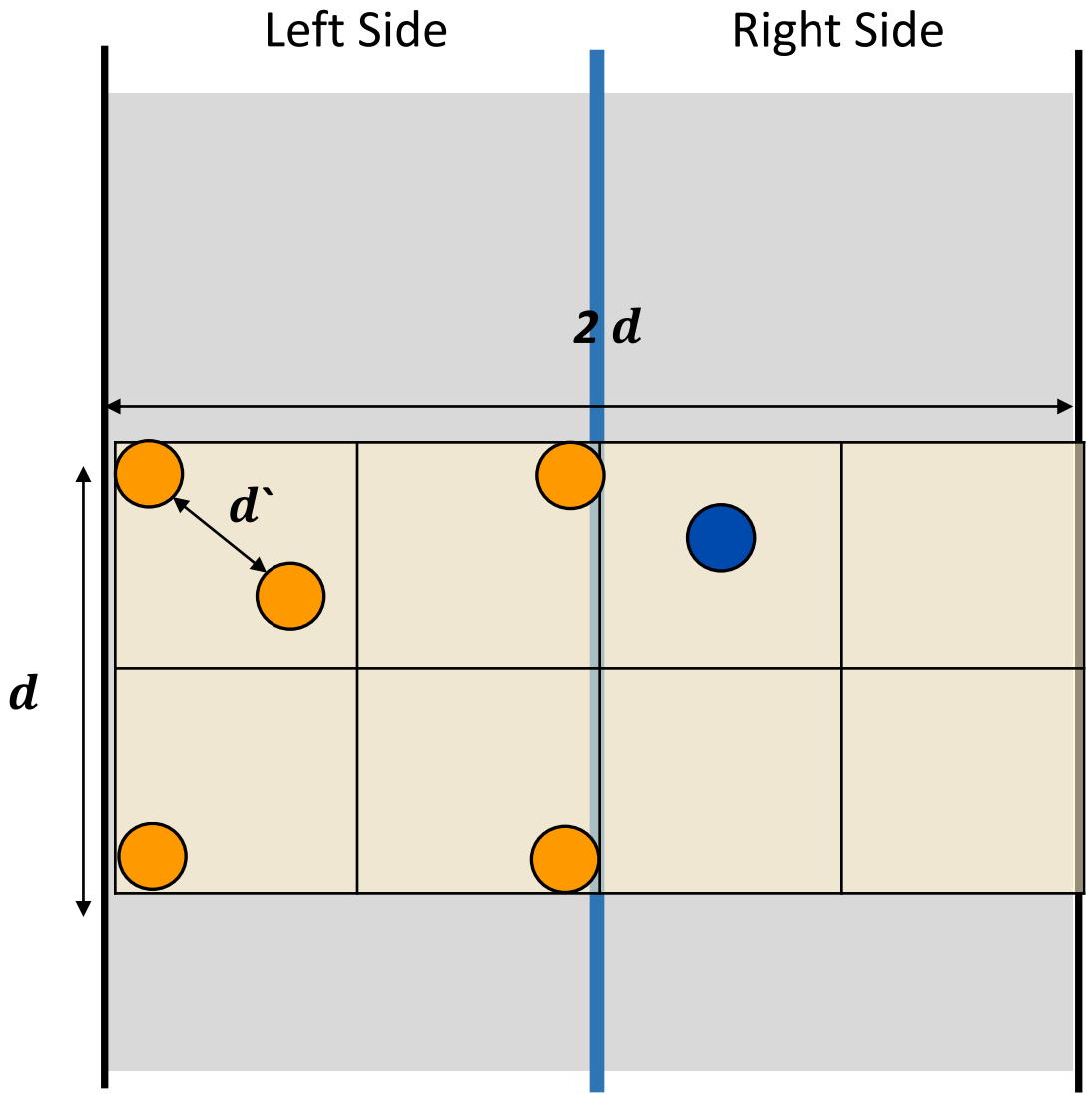




$$d/\sqrt{2} < d$$

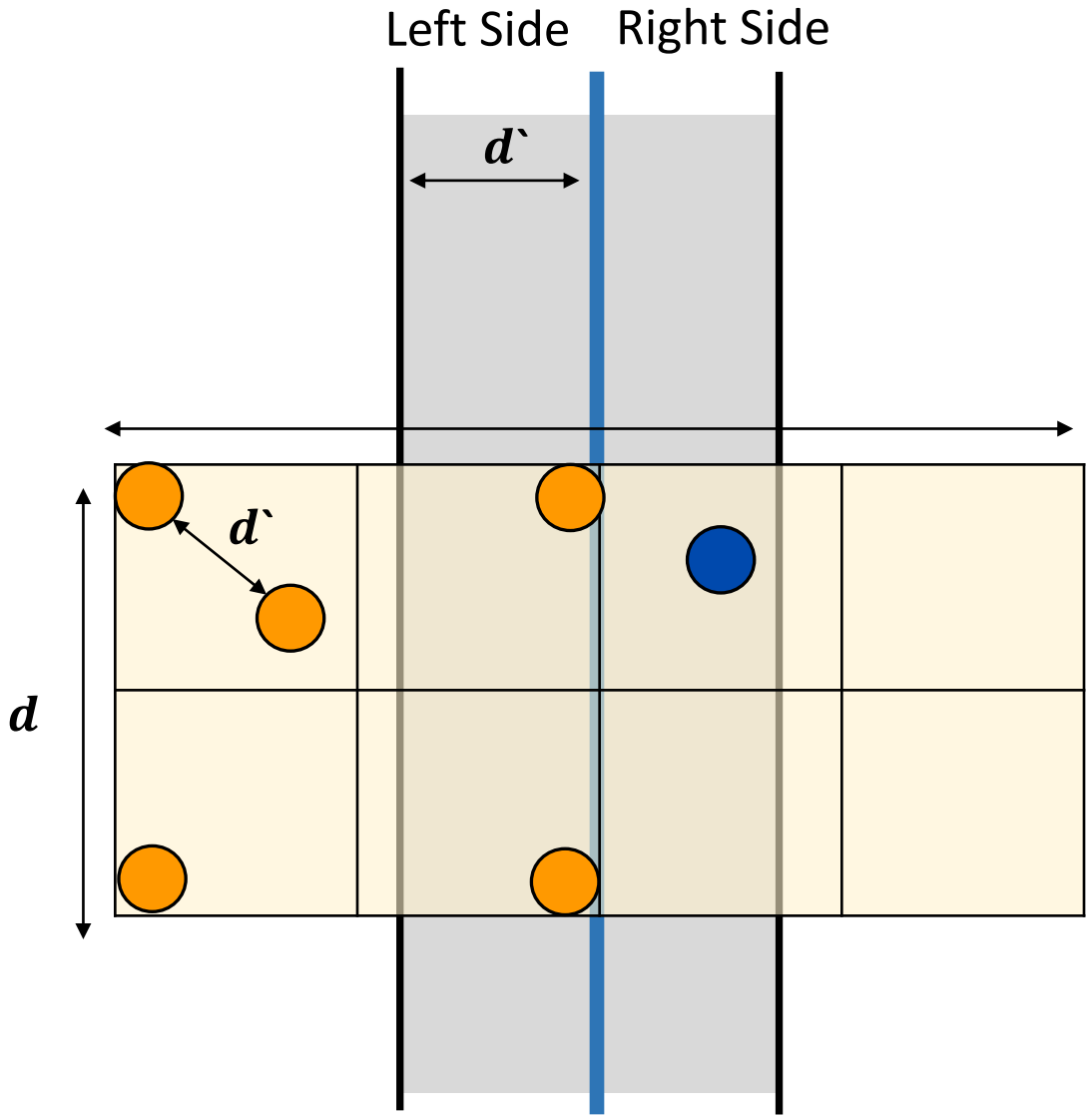

16
15
1
14
P_4
P_5
P_6
P_7
P_8
...
$P_{\sim n}$
Left Side
Right Side
$2\,d$
d
$d/2$
$d/\sqrt{2}$
$d/\sqrt{2} < d$

16
15
1
14
P_4
P_5
P_6
P_7
P_8
…
$P_{\sim n}$

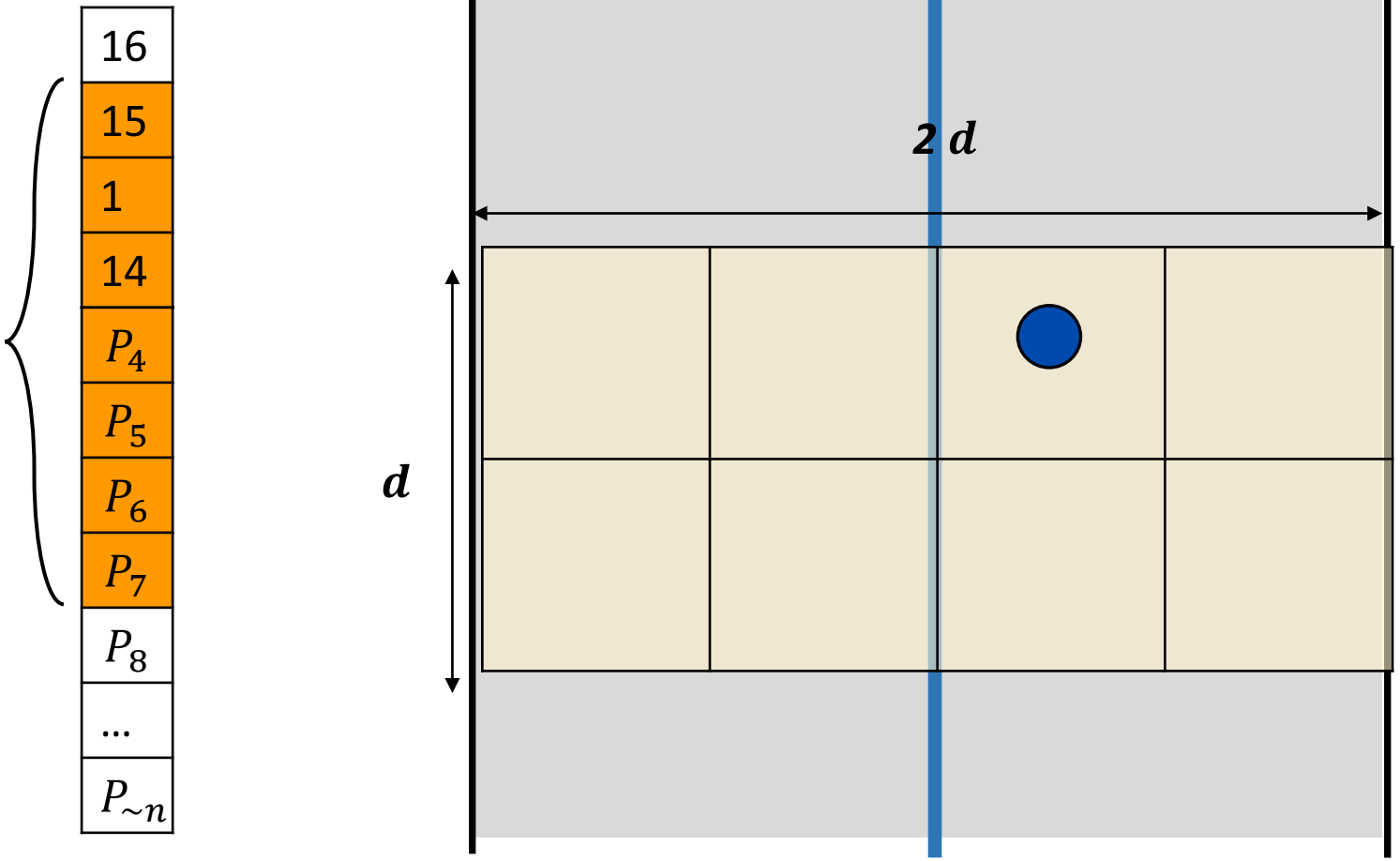


$$d` < d$$

16
15
1
14
P_4
P_5
P_6
P_7
P_8
…
$P_{\sim n}$



$$d` < d$$


16
15
1
14
P_4
P_5
P_6
P_7
P_8
...
$P_{\sim n}$
Left Side
Right Side
2 d
d

Closest Pair: Live Poll 1

What is the complexity of this algo?

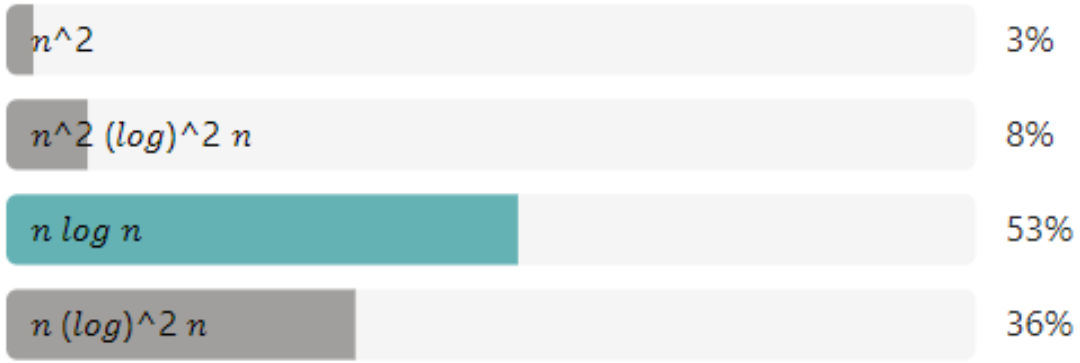
$A.\ n^2$
$B.\ n^2\ log^2 n$
$C.\ n\ log\ n$
$D.\ n\ log^2\ n$

Scan the QR code to vote or go to https://forms.office.com/r/zKJjPb1tCK

Closet Pair: Live Poll 1

Only people in my organization can respond, Record name

1. What is the complexity of this algo?



36 responses

1/1

Scan the QR code to vote or go to https://forms.office.com/r/zKJjPb1tCK

Finding Closest Point Algorithm

Sort points based on X-coordinates → $O(n \log n)$

CLOSEST-PAIR($p_1, p_2, \ldots, p_n$)

Compute vertical line L such that half the points are on each side of the line. → $O(n)$

$\delta_1 \leftarrow$ CLOSEST-PAIR(points in left half). → $T(n/2)$

$\delta_2 \leftarrow$ CLOSEST-PAIR(points in right half). → $T(n/2)$

$\delta \leftarrow \min \{ \delta_1 , \delta_2 \}$. → $O(1)$

$A \leftarrow$ list of all points closer than δ to line L. → $O(n)$

Sort points in A by y-coordinate. → $O(n \log n)$

Scan points in A in y-order and compare distance between each point and next 7 neighbors. → $O(n)$

If any of these distances is less than δ, update δ. → $O(1)$

RETURN δ.

Finding Closest Point Algorithm

$$O(n \log n) + O(log\ n).(O(n) + O(1) + O(n) + O(n\ log\ n) + O(n) + O(1))$$

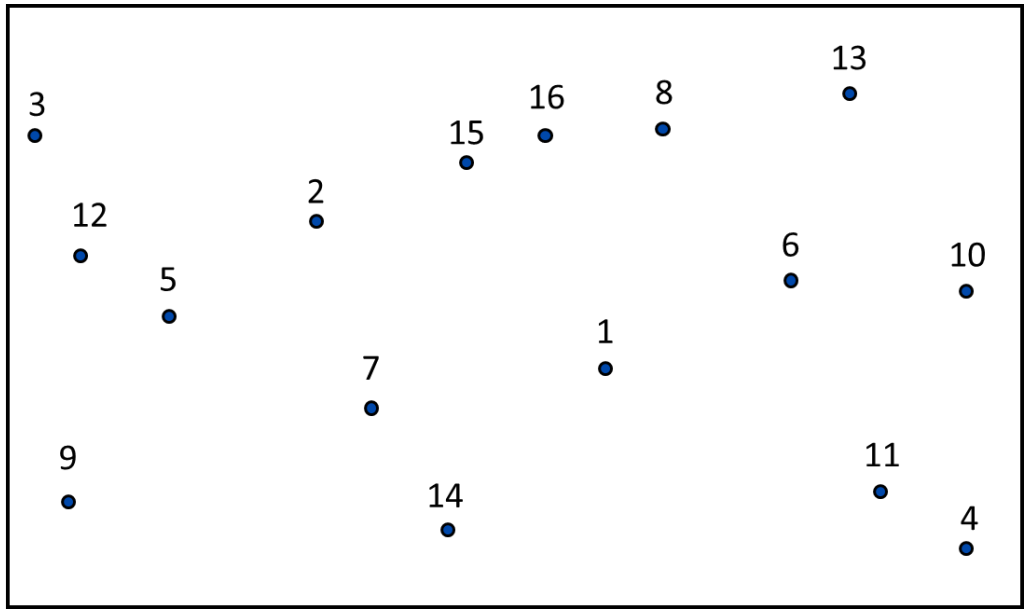
$$O\left(n\ log^2 n\right)$$

One Last Thing….

Can we bring it down to $n \log n$?

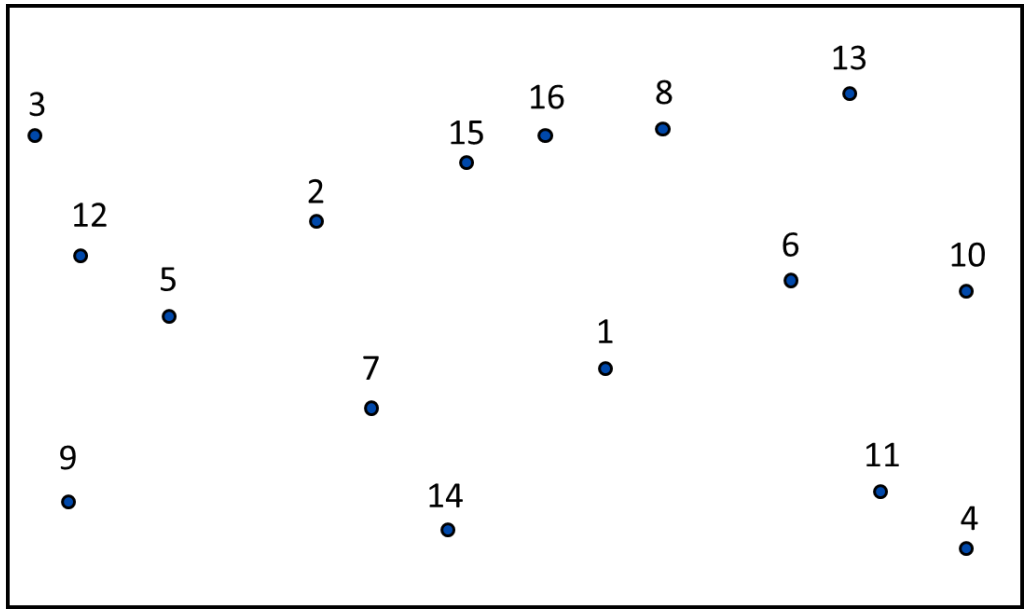
One Last Thing… Yup

Sorted X-axis



(3, 9, 12, 5, 2, 7, 15, 14, 16, 1, 8, 6, 13, 11, 10, 4)

Sorted Y-axis



(13, 8, 16, 3, 15, 2, 12, 6, 10, 5, 1, 7, 9, 11, 14, 4)

Sort points based on X-coordinates → $O(n \log n)$

CLOSEST-PAIR($p_1, p_2, \ldots, p_n$)

Compute vertical line L such that half the points are on each side of the line. → $O(n)$

$\delta_1 \leftarrow$ CLOSEST-PAIR(points in left half). → $T(n/2)$

$\delta_2 \leftarrow$ CLOSEST-PAIR(points in right half). → $T(n/2)$

$\delta \leftarrow \min \{ \delta_1 , \delta_2 \}$. → $O(1)$

$A \leftarrow$ list of all points closer than δ to line L. → $O(n)$

Sort points in A by y-coordinate. → $O(n \log n)$

Scan points in A in y-order and compare distance between each point and next 7 neighbors. → $O(n)$

If any of these distances is less than δ, update δ. → $O(1)$

RETURN δ.

Sort points based on X-coordinates → $O(n \log n)$

Sort points based on Y-coordinates → $O(n \log n)$

CLOSEST-PAIR($p_1, p_2, \ldots, p_n$)

Compute vertical line L such that half the points are on each side of the line. → $O(n)$

$\delta_1 \leftarrow$ CLOSEST-PAIR(points in left half). → $T(n/2)$

$\delta_2 \leftarrow$ CLOSEST-PAIR(points in right half). → $T(n/2)$

$\delta \leftarrow \min \{ \delta_1 , \delta_2 \}$. → $O(1)$

$A \leftarrow$ list of all points closer than δ to line L. → $O(n)$

Sort points in A by y-coordinate. → $O(n \log n)$

Scan points in A in y-order and compare distance between each point and next 7 neighbors. → $O(n)$

If any of these distances is less than δ, update δ. → $O(1)$

RETURN δ.

Sort points based on X-coordinates → $O(n \log n)$

Sort points based on Y-coordinates → $O(n \log n)$

CLOSEST-PAIR($p_1, p_2, \ldots, p_n$)

Compute vertical line L such that half the points are on each side of the line. → $O(n)$

$\delta_1 \leftarrow$ CLOSEST-PAIR(points in left half). → $T(n/2)$

$\delta_2 \leftarrow$ CLOSEST-PAIR(points in right half). → $T(n/2)$

$\delta \leftarrow \min \{ \delta_1 , \delta_2 \}$. → $O(1)$

$A \leftarrow$ list of all points closer than δ to line L. → $O(n)$

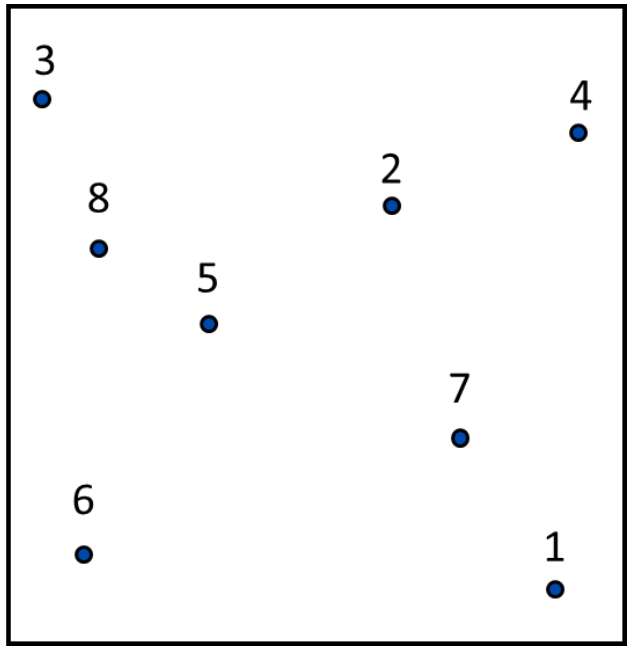
Sort points in A by y-coordinate. → $O(n)$

Scan points in A in y-order and compare distance between each point and next 7 neighbors. → $O(n)$

If any of these distances is less than δ, update δ. → $O(1)$

RETURN δ.

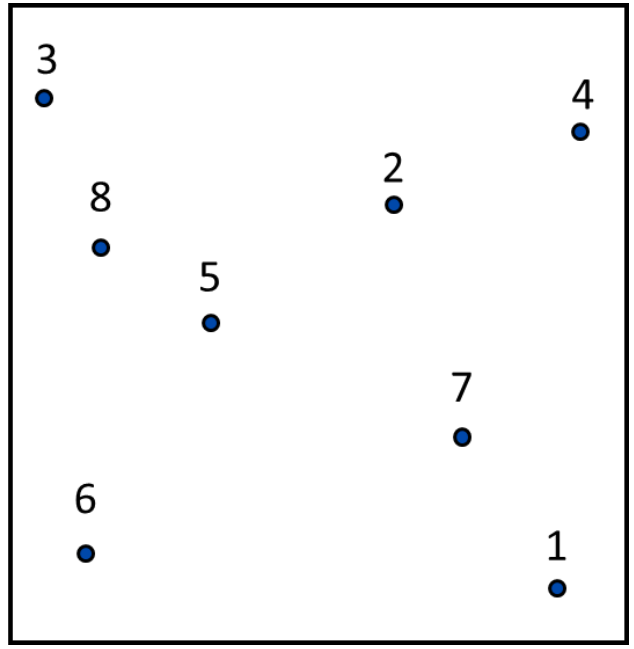
Sorted X-axis



(3 , 6 , 8 , 5 , 2 , 7 , 1 , 4)

{(**1**,8),(**2**,2),(**3**,4),(**4**,5),(**5**,6),(**7**,4),(**8**,1),(**8**,7)}

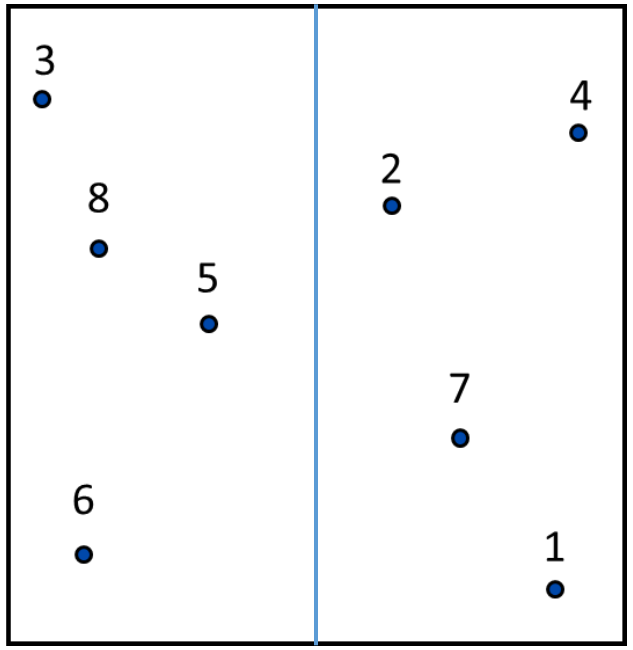
Sorted Y-axis



(1 , 6 , 7 , 5 , 8 , 2 , 4 , 3)

{(8,**1**),(2,**2**),(7,**4**),(4,**5**),(3,**6**),(5,**6**),(8,**7**),(1,**8**)}

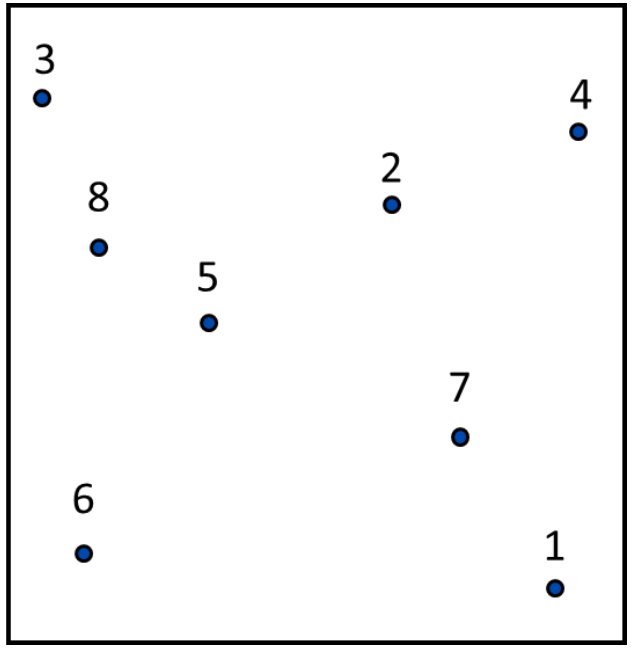
Sorted X-axis



(3 , 6 , 8 , 5 , 2 , 7 , 1 , 4)

{(**1**,8),(**2**,2),(**3**,4),(**4**,5),(**5**,6),(**7**,4),(**8**,1),(**8**,7)}

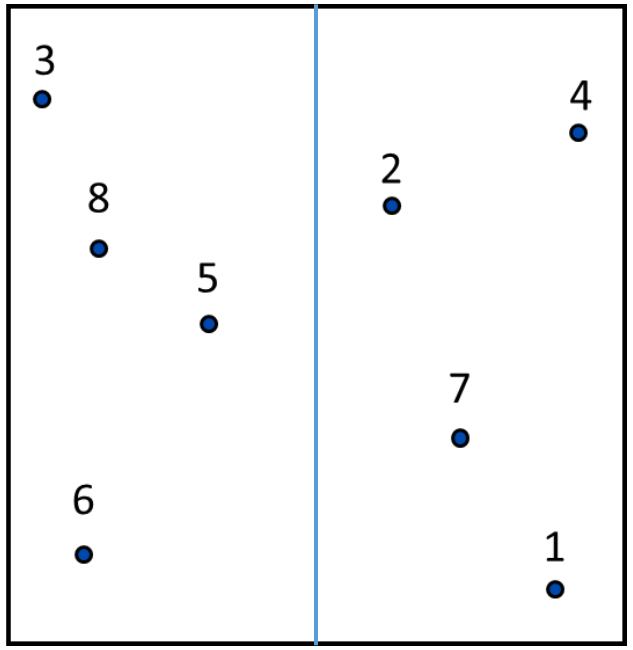
Sorted Y-axis



(1 , 6 , 7 , 5 , 8 , 2 , 4 , 3)

{(8,**1**),(2,**2**),(7,**4**),(4,**5**),(3,**6**),(5,**6**),(8,**7**),(1,**8**)}

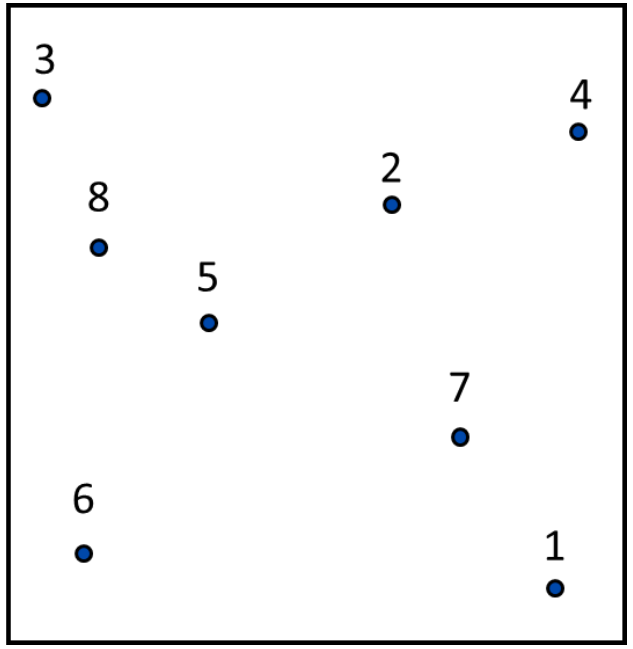
Sorted X-axis



(3 , 6 , 8 , 5) (2 , 7 , 1 , 4)

{(**1**,8),(**2**,2),(**3**,4),(**4**,5)} {(**5**,6),(**7**,4),(**8**,1),(**8**,7)}

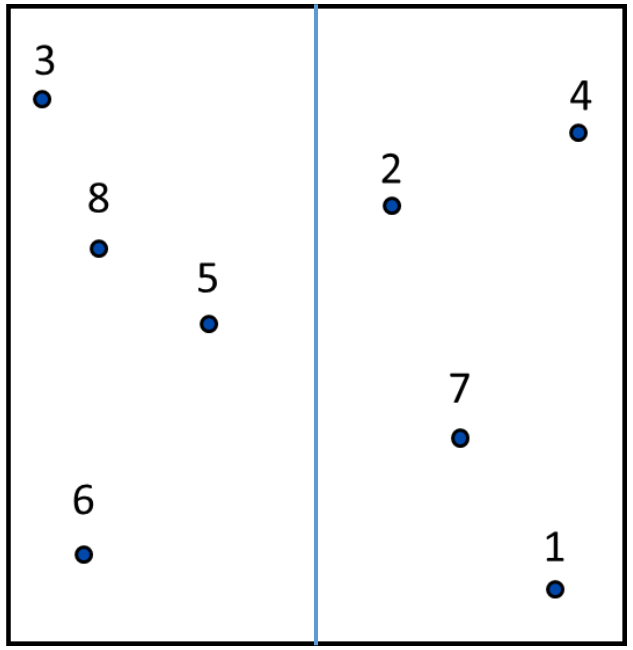
Sorted Y-axis



(1 , 6 , 7 , 5 , 8 , 2 , 4 , 3)

{(8,**1**),(2,**2**),(7,**4**),(4,**5**),(3,**6**),(5,**6**),(8,**7**),(1,**8**)}

Sorted X-axis



(3 , 6 , 8 , 5) (2 , 7 , 1 , 4)

{(**1**,8),(**2**,2),(**3**,4),(**4**,5)} {(**5**,6),(**7**,4),(**8**,1),(**8**,7)}

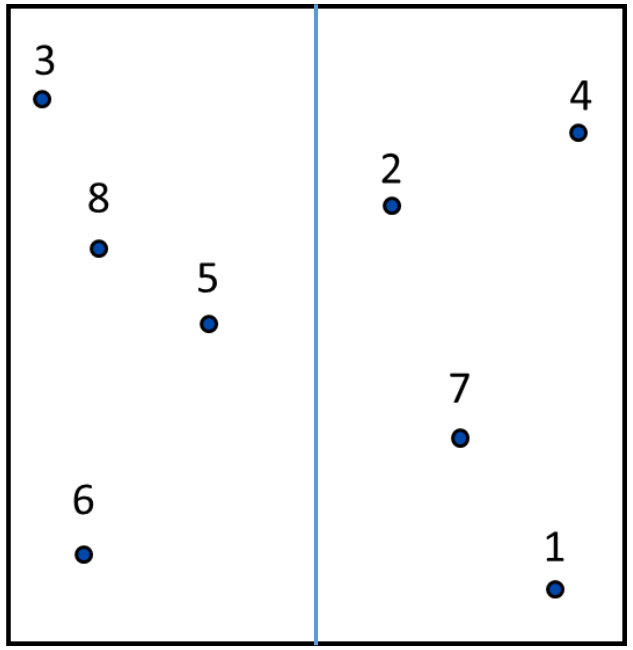
(1 , 6 , 7 , 5 , 8 , 2 , 4 , 3)

{(8,**1**),(2,**2**),(7,**4**),(4,**5**),(3,**6**),(5,**6**),(8,**7**),(1,**8**)}

If (***X_coordinate =***< 4)

else ()

Sorted X-axis



(3 , 6 , 8 , 5) (2 , 7 , 1 , 4)

{(**1**,8),(**2**,2),(**3**,4),(**4**,5)} {(**5**,6),(**7**,4),(**8**,1),(**8**,7)}

(1 , 6 , 7 , 5 , 8 , 2 , 4 , 3)

{(8,**1**),(2,**2**),(7,**4**),(4,**5**),(3,**6**),(5,**6**),(8,**7**),(1,**8**)}

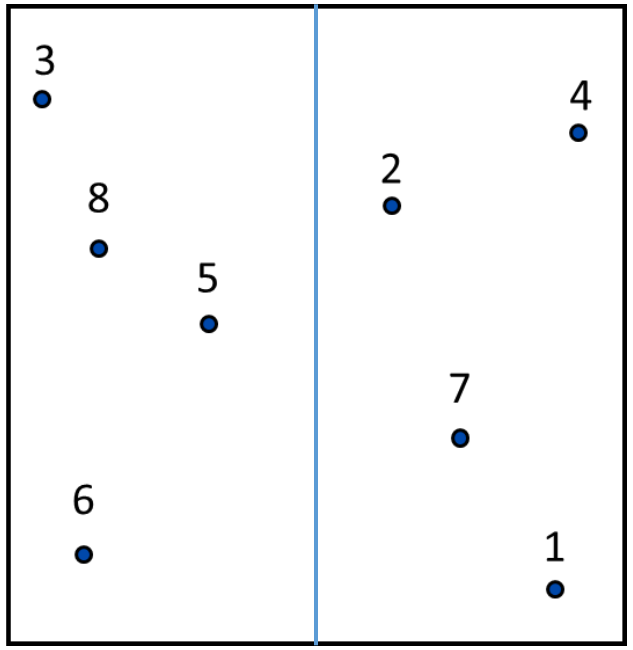
If (***X_coordinate =***< 4)

else ()

1

(8,**1**)

Sorted X-axis



(3 , 6 , 8 , 5) (2 , 7 , 1 , 4)

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(1 , 6 , 7 , 5 , 8 , 2 , 4 , 3)

{(8,**1**),(2,**2**),(7,**4**),(4,**5**),(3,**6**),(5,**6**),(8,**7**),(1,**8**)}

If (***X_coordinate*** =< 4)

6

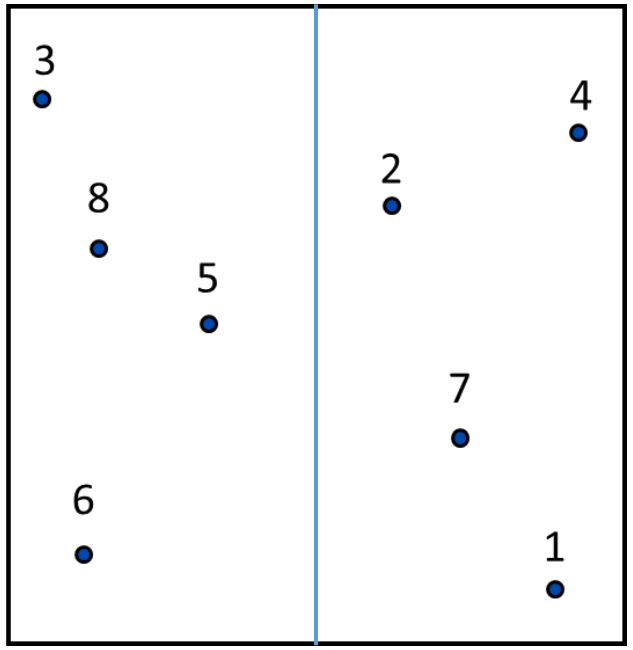
(2,**2**)

else ()

1

(8,**1**)

Sorted X-axis



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If (***X_coordinate*** =< 4)

6

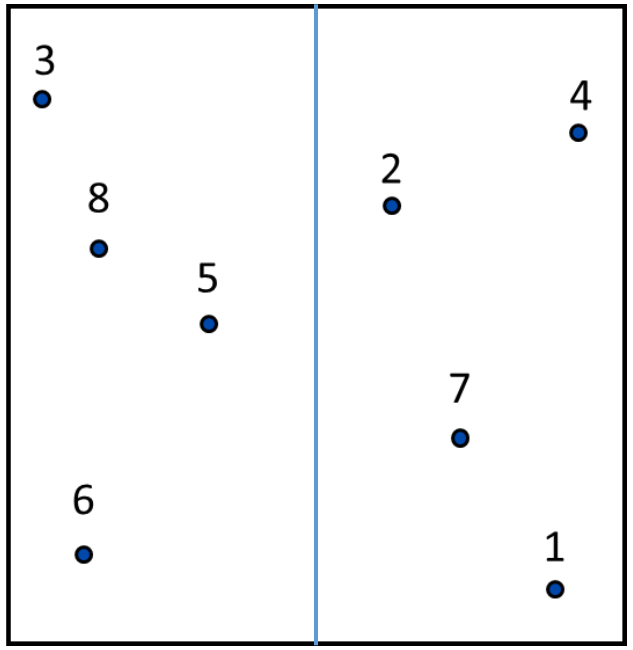
(2,**2**)

else ()

1 7

(8,**1**) (7,**4**)

Sorted X-axis



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If (***X_coordinate*** =< 4)

6 5

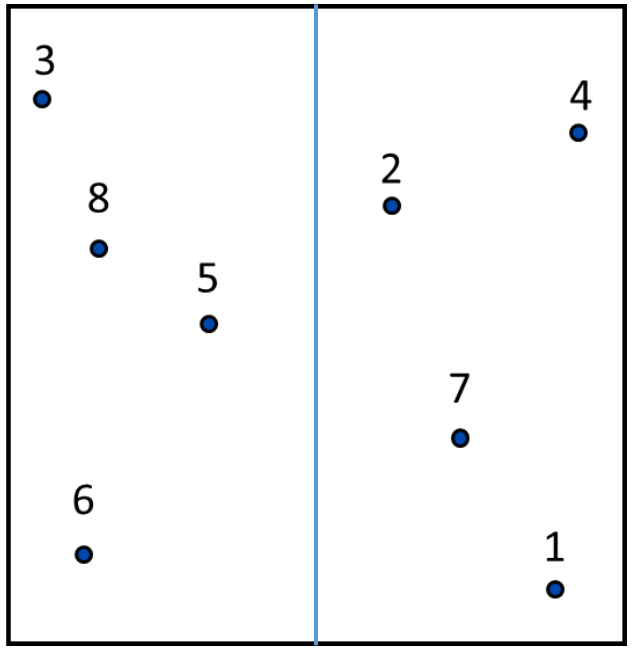
(2,**2**) (4,**5**)

else ()

1 7

(8,**1**) (7,**4**)

Sorted X-axis



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{(8,**1**),(2,**2**),(7,**4**),(4,**5**),(3,**6**),(5,**6**),(8,**7**),(1,**8**)}

If (***X_coordinate*** =< 4)

6 5 8

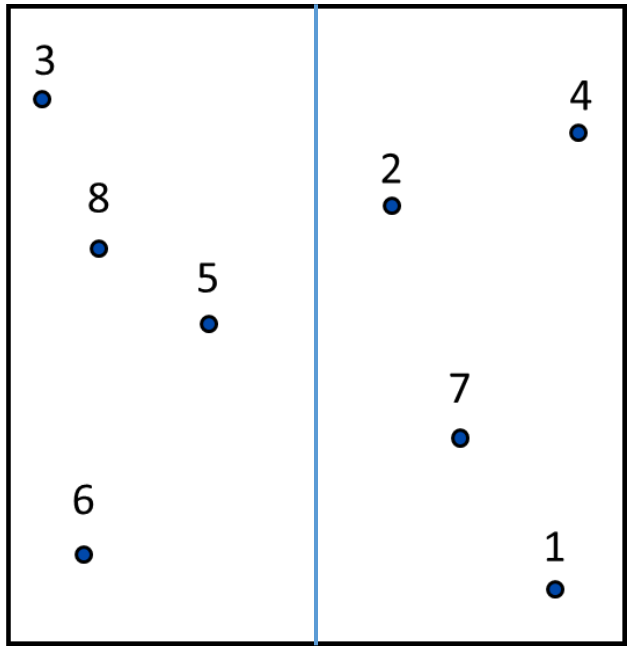
(2,**2**) (4,**5**) (3,**6**)

else ()

1 7

(8,**1**) (7,**4**)

Sorted X-axis



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{(8,**1**),(2,**2**),(7,**4**),(4,**5**),(3,**6**),(5,**6**),(8,**7**),(1,**8**)}

If (***X_coordinate =<*** 4)

6 5 8

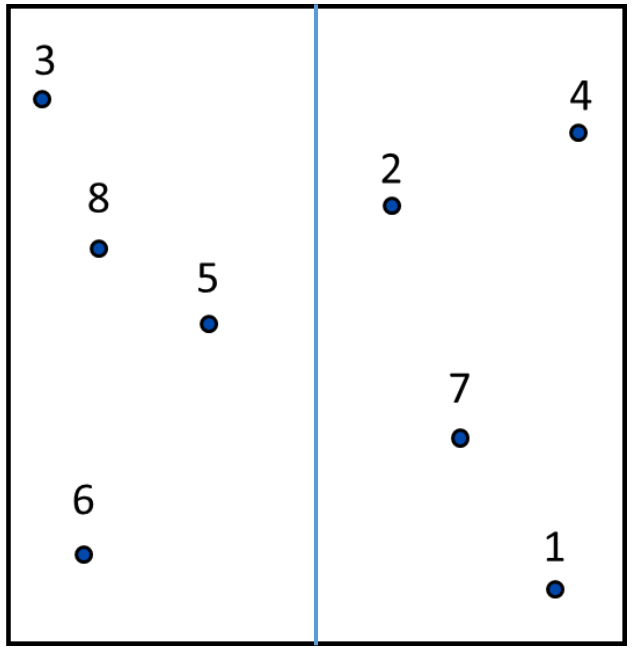
(2,**2**) (4,**5**) (3,**6**)

else ()

1 7 2

(8,**1**) (7,**4**) (5,**6**)

Sorted X-axis



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{(8,**1**),(2,**2**),(7,**4**),(4,**5**),(3,**6**),(5,**6**),(8,**7**),(1,**8**)}

If (***X_coordinate*** =< 4)

6 5 8

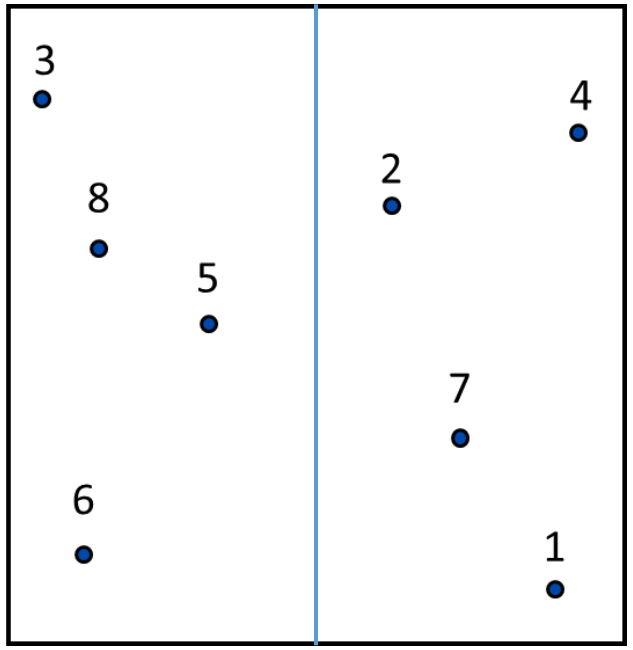
(2,**2**) (4,**5**) (3,**6**)

else ()

1 7 2 4

(8,**1**) (7,**4**) (5,**6**) (8,**7**)

Sorted X-axis



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{(8,**1**),(2,**2**),(7,**4**),(4,**5**),(3,**6**),(5,**6**),(8,**7**),(1,**8**)}

If (***X_coordinate*** =< 4)

6 5 8 3

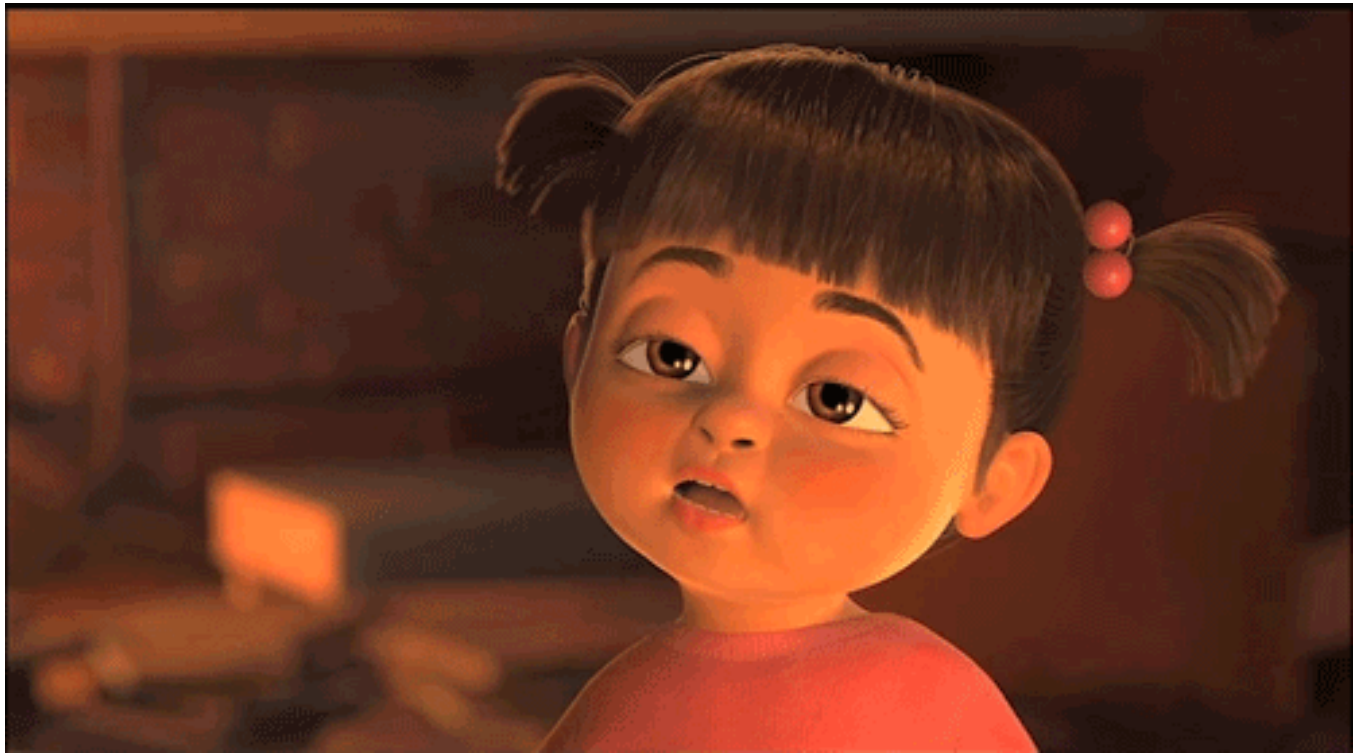
(2,**2**) (4,**5**) (3,**6**) (1,**8**)

else ()

1 7 2 4

(8,**1**) (7,**4**) (5,**6**) (8,**7**)

Thanks a lot



If you are taking a Nap, **wake up**.....Lecture Over