

$$\int_0^\infty f(x|h)f(h)dh = \int_0^\infty \prod_{j=1}^n \frac{1}{kh} \sum_{i=1}^k K\left(\frac{X_{2,j}-X_{1,i}}{h}\right) f(h) dh$$

Suppose K is the gaussian kernel, then:

$$\begin{aligned} &= \int_0^\infty \prod_{j=1}^n \frac{1}{kh} \sum_{i=1}^k \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X_{2,j}-X_{1,i}}{h}\right)^2} f(h) dh \\ &= \int_0^\infty \left(\frac{1}{kh}\right)^n \prod_{j=1}^n \sum_{i=1}^k \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X_{2,j}-X_{1,i}}{h}\right)^2} f(h) dh \\ &= \left(\frac{1}{k}\right)^n \int_0^\infty \left(\frac{1}{h}\right)^n \prod_{j=1}^n \sum_{i=1}^k \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X_{2,j}-X_{1,i}}{h}\right)^2} f(h) dh \\ &= \left(\frac{1}{k\sqrt{2\pi}}\right)^n \int_0^\infty \left(\frac{1}{h}\right)^n \prod_{j=1}^n \sum_{i=1}^k e^{-\frac{1}{2}\left(\frac{X_{2,j}-X_{1,i}}{h}\right)^2} f(h) dh \\ &= \left(\frac{1}{k\sqrt{2\pi}}\right)^n \int_0^\infty \left(\frac{1}{h}\right)^n \sum_{i_n=1}^k \sum_{i_{n-1}=1}^k \cdots \sum_{i_2=1}^k \sum_{i_1=1}^k e^{-\frac{1}{2h^2} \sum_{L=1}^n (X_{2,L}-X_{1,i_L})^2} f(h) dh \end{aligned}$$

Recall our choice of prior is:  $f(h) = \frac{2B}{\sqrt{\pi}} \frac{1}{h^2} e^{-\frac{B^2}{h^2}} I_{(0,\infty)}$

As B is fixed we treat it like a constant.

Plugging this in yields:

$$\begin{aligned} &= \left(\frac{1}{k\sqrt{2\pi}}\right)^n \int_0^\infty \left(\frac{1}{h}\right)^n \sum_{i_n=1}^k \sum_{i_{n-1}=1}^k \cdots \sum_{i_2=1}^k \sum_{i_1=1}^k e^{-\frac{1}{2h^2} \sum_{L=1}^n (X_{2,L}-X_{1,i_L})^2} \frac{2B}{\sqrt{\pi}} \frac{1}{h^2} e^{-\frac{B^2}{h^2}} I_{(0,\infty)} dh \\ &= \left(\frac{1}{k\sqrt{2\pi}}\right)^n \frac{2B}{\sqrt{\pi}} \int_0^\infty \sum_{i_n=1}^k \sum_{i_{n-1}=1}^k \cdots \sum_{i_2=1}^k \sum_{i_1=1}^k \left(\frac{1}{h}\right)^{n+2} e^{-\frac{1}{2h^2} (2B^2 + \sum_{L=1}^n (X_{2,L}-X_{1,i_L})^2)} dh \\ &= \left(\frac{1}{k\sqrt{2\pi}}\right)^n \frac{2B}{\sqrt{\pi}} \sum_{i_n=1}^k \sum_{i_{n-1}=1}^k \cdots \sum_{i_2=1}^k \sum_{i_1=1}^k \int_0^\infty \left(\frac{1}{h}\right)^{n+2} e^{-\frac{1}{2h^2} (2B^2 + \sum_{L=1}^n (X_{2,L}-X_{1,i_L})^2)} dh \end{aligned}$$

let  $u = h^2$   $du = 2h dh$

Integral bounds stay the same then:

$$= \left(\frac{1}{k\sqrt{2\pi}}\right)^n \frac{B}{\sqrt{\pi}} \sum_{i_n=1}^k \sum_{i_{n-1}=1}^k \cdots \sum_{i_2=1}^k \sum_{i_1=1}^k \int_0^\infty \left(\frac{1}{u}\right)^{\frac{n+1}{2}} e^{-\frac{1}{2u} (2B^2 + \sum_{L=1}^n (X_{2,L}-X_{1,i_L})^2)} du$$

The function inside the integral is  $IG\left(\frac{n+3}{2}, .5(2B^2 + \sum_{L=1}^n (X_{2,L}-X_{1,i_L})^2)\right)$

Thus, the integral evaluates to:  $\frac{\Gamma(\frac{n+3}{2})}{(.5(2B^2 + \sum_{L=1}^n (X_{2,L}-X_{1,i_L})^2))^{\frac{n+3}{2}}}$

So the marginal likelihood is:

$$= \left(\frac{1}{k\sqrt{2\pi}}\right)^n \frac{B}{\sqrt{\pi}} \sum_{i_n=1}^k \sum_{i_{n-1}=1}^k \cdots \sum_{i_2=1}^k \sum_{i_1=1}^k \frac{\Gamma(\frac{n+3}{2})}{(.5(2B^2 + \sum_{L=1}^n (X_{2,L}-X_{1,i_L})^2))^{\frac{n+3}{2}}}$$

The sum is iterated and very large, but it is still odd that the form is closed. I meant that the form is closed under the Gaussian Kernel, I may have unintentionally misled you earlier sorry. It is not entirely obvious how things change as k increases or as n increases.