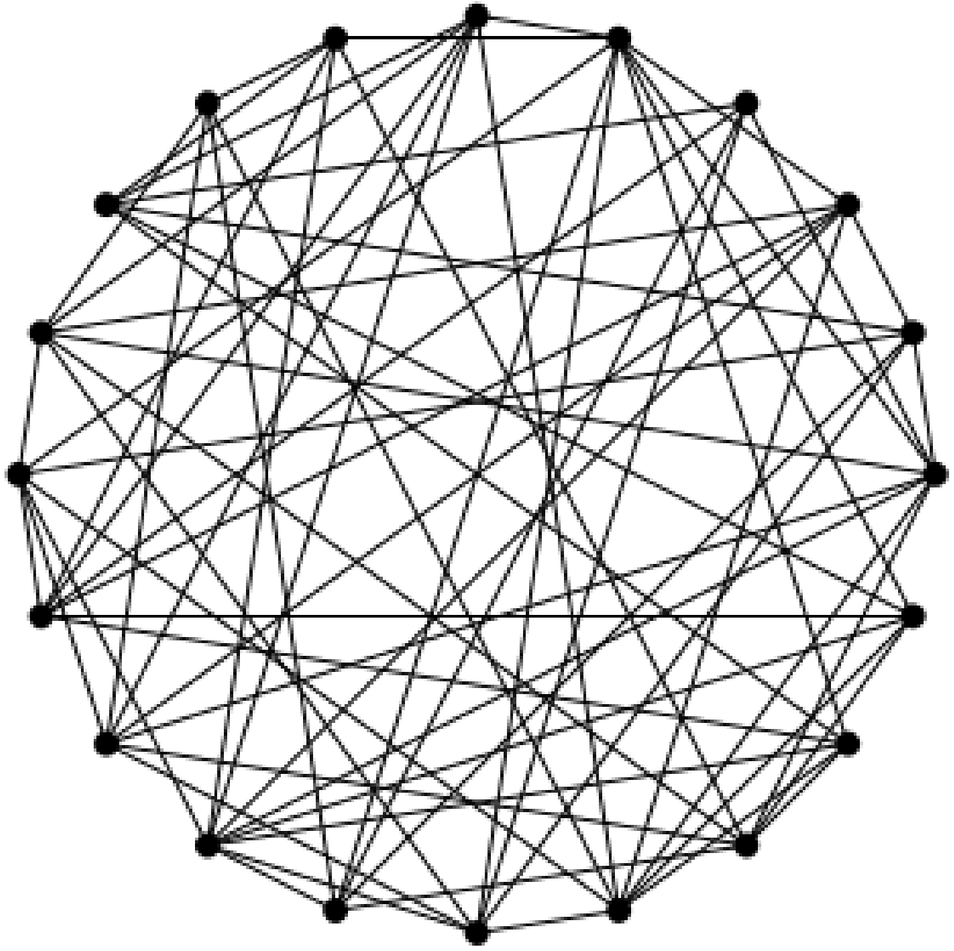
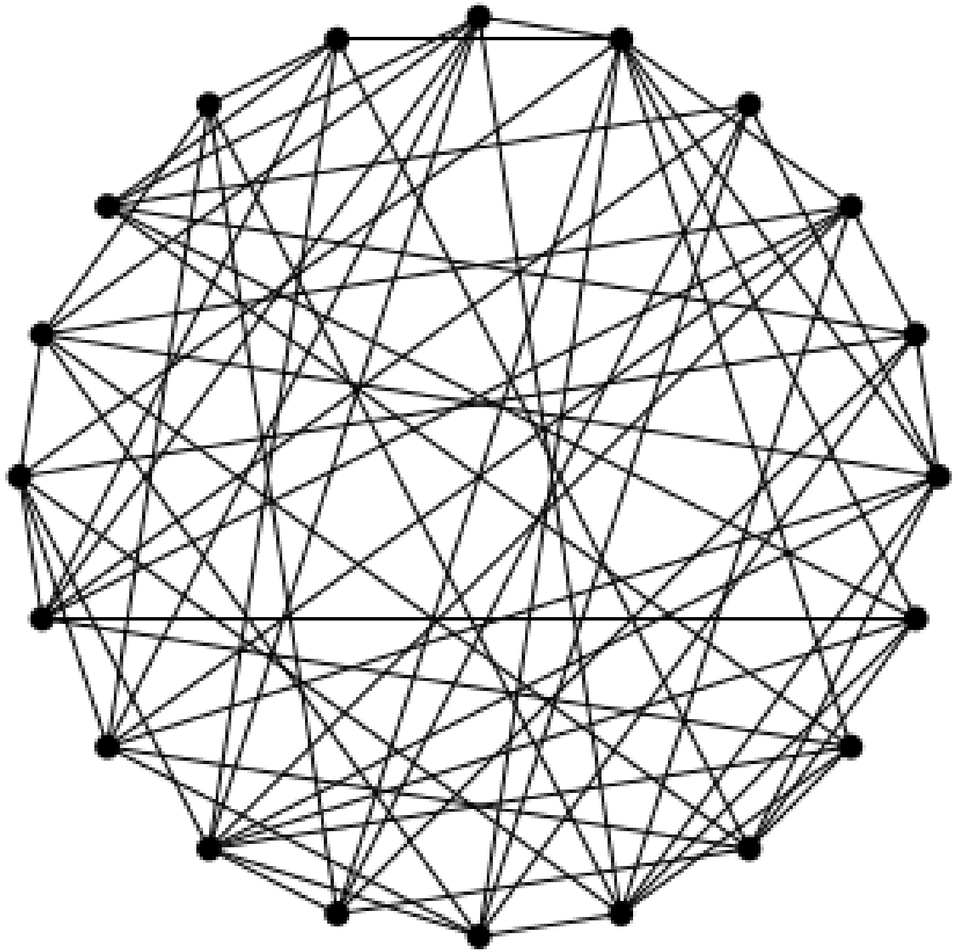
Where is the Triangle?

Miniature - 10

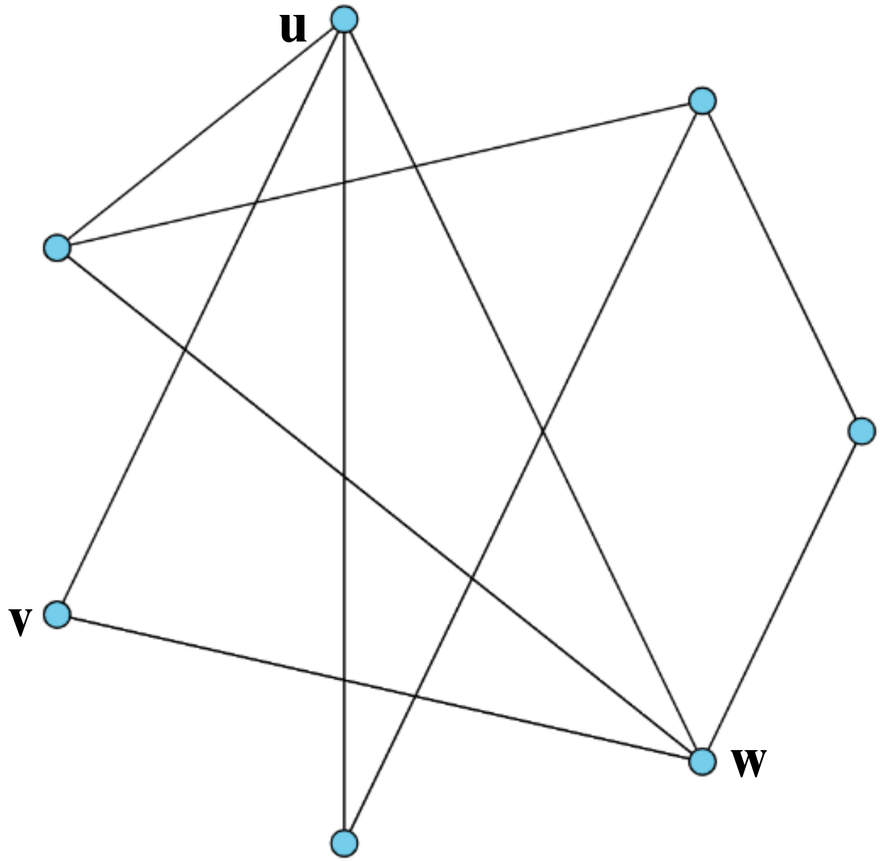


Where is the Triangle?

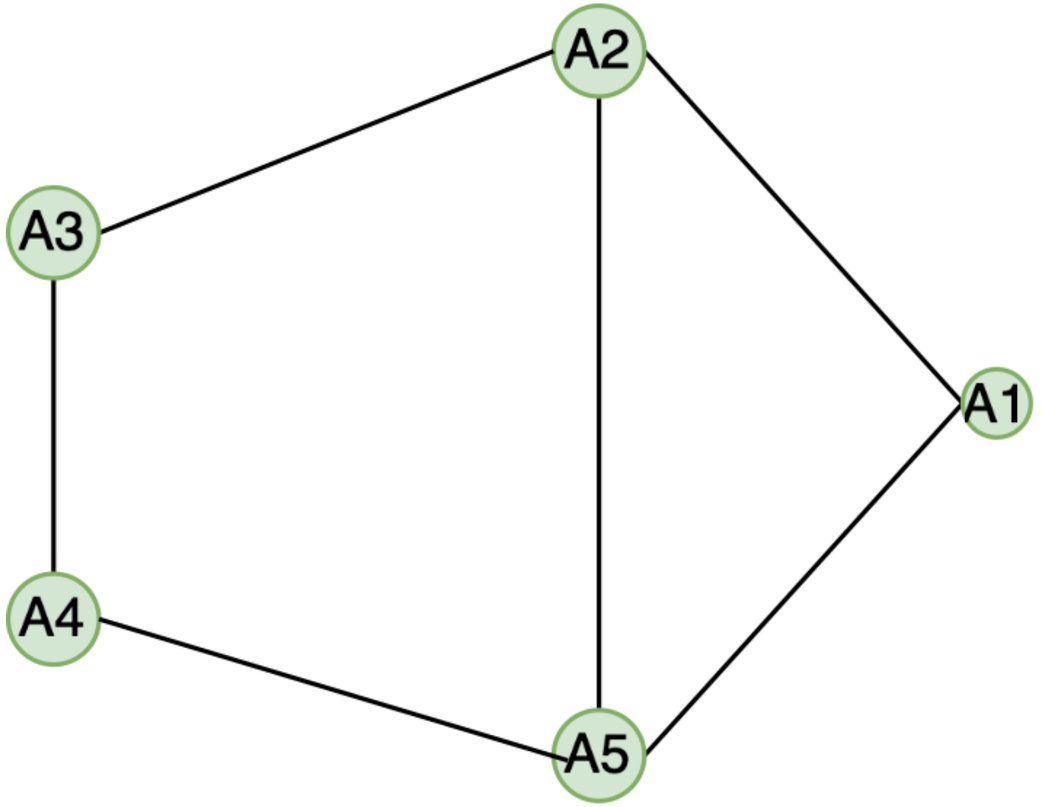
Does a given graph contain a triangle?



A triangle in a graph is defined as a set of three vertices where each vertex is connected to the other two by an edge i.e. three vertices **u, v, w**, every two of them connected by an edge.



Consider this graph with 5 vertices



Adjacency matrix of the graph :

$$a_{ij} = \begin{cases} 1 & \text{if } i \neq j \text{ and } \{i, j\} \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

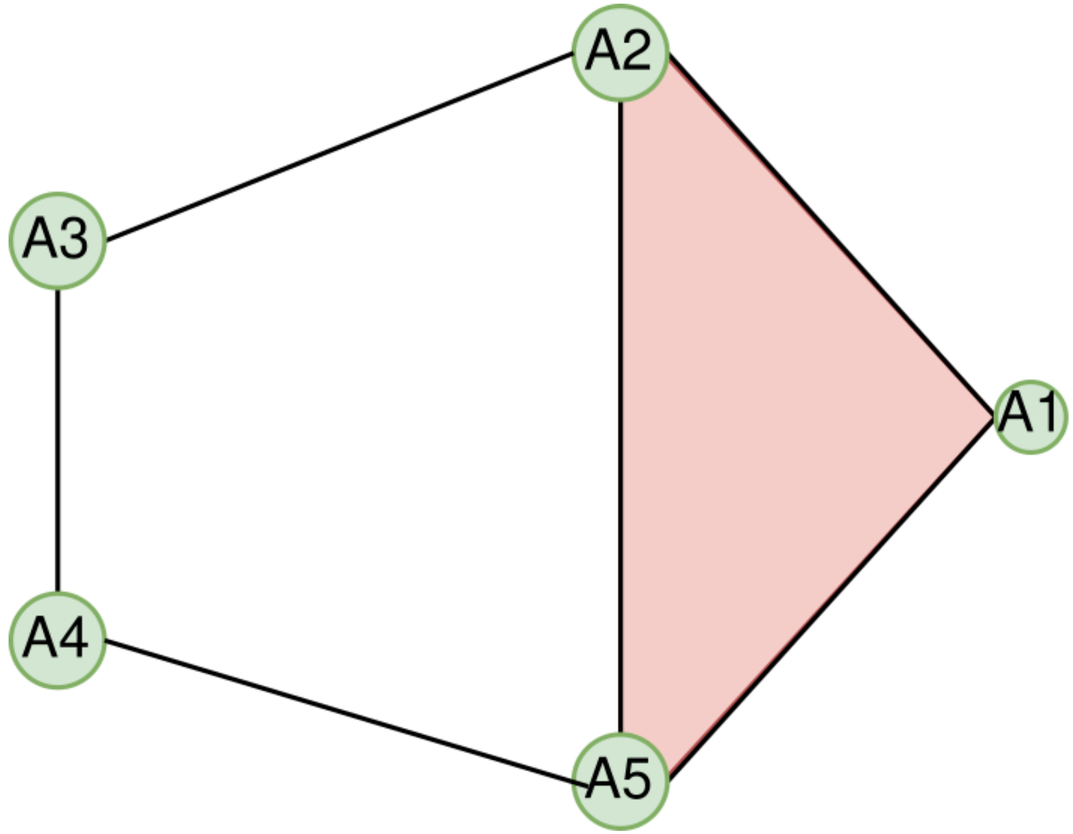
Matrix A

	A1	A2	A3	A4	A5
A1	0	1	0	0	1
A2	1	0	1	0	1
A3	0	1	0	1	0
A4	0	0	1	0	1
A5	1	1	0	1	0

A naive algorithm for finding a triangle inspects every triple of vertices, and thus it needs roughly **n^3** operations for a n-vertex graph.

Matrix A

	A1	A2	A3	A4	A5
A1	0	1	0	0	1
A2	1	0	1	0	1
A3	0	1	0	1	0
A4	0	0	1	0	1
A5	1	1	0	1	0



The known approach for breaking the n^3 barrier is algebraic and it is based on fast matrix multiplication.

Matrix A

	A1	A2	A3	A4	A5
A1	0	1	0	0	1
A2	1	0	1	0	1
A3	0	1	0	1	0
A4	0	0	1	0	1
A5	1	1	0	1	0

Matrix A^2

	A1	A2	A3	A4	A5
A1	2	1	1	1	1
A2	1	3	0	2	1
A3	1	0	2	0	2
A4	1	2	0	2	0
A5	1	1	2	0	3

Finding a triangle is equivalent to finding two adjacent vertices i, j with a common neighbor k. So we look for two indices i, j such that both $a_{ij} \neq 0$ and $b_{ij} \neq 0$.

Matrix A

	A1	A2	A3	A4	A5
A1	0	1	0	0	1
A2	1	0	1	0	1
A3	0	1	0	1	0
A4	0	0	1	0	1
A5	1	1	0	1	0

Matrix A^2

	A1	A2	A3	A4	A5
A1	2	1	1	1	1
A2	1	3	0	2	1
A3	1	0	2	0	2
A4	1	2	0	2	0
A5	1	1	2	0	3

Indices which are non-zero in both A and A^2 : {(1,2), (2,1), (1,5), (5,1), (2,5), (5,2)}

⟶ **Graph contains a triangle**

Simple matrix multiplication needs **n^3** arithmetic operations .

Known fast-matrix multiplication algorithms can do this in roughly **n^2.3** arithmetic operations

Timeline of matrix multiplication exponent

Year	Bound on omega	Authors
1969	2.8074	Strassen[1]
1978	2.796	Pan[10]
1979	2.780	Bini, Capovani [it], Romani[11]
1981	2.522	Schönhage[12]
1981	2.517	Romani[13]
1981	2.496	Coppersmith, Winograd[14]
1986	2.479	Strassen[15]
1990	2.3755	Coppersmith, Winograd[16]
2010	2.3737	Stothers[17]
2012	2.3729	Williams[18][19]
2014	2.3728639	Le Gall[20]
2020	2.3728596	Alman, Williams[21][22]
2022	2.371866	Duan, Wu, Zhou[23]
2024	2.371552	Williams, Xu, Xu, and Zhou[2]