Shock tube simulation with SPH

October 20, 2015

1 Introduction

In this assignment, you are asked to simulate the classical shock-tube problem using the Smoothed Particle Hydrodynamics (SPH) technique. For the Euler equations of gas dynamics, the one-dimensional Riemann problem is defined by two input states (left, right) abutting an interface, which results in a self-similar solution along lines with constant x . The solution is composed of different states, delineated by waves which emanate from the interface. t The waves are ordered by their speeds λ_i, so that $\lambda_1 < \lambda_2 < \lambda_3$. Of the three waves, the slowest and the fastest waves can be either a shockwave or an expansion wave, while the intermediate wave is called a contact discontinuity. All variables are smooth and discontinuous across an expansion wave and shockwave respectively. The contact discontinuity is special because the pressure and normal component of velocity are constant while the density and energy change discontinuously across it. The contact discontinuity is a linear wave which travels with the local fluid velocity.

2 Equations of Motion

The Euler equations governing the time rate of change of density, velocity and thermal energy can be written as

$$\frac{d\rho}{dt} = -\rho(\nabla \cdot \mathbf{v}) \tag{1a}$$

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}(\nabla p) \tag{1b}$$

$$\frac{de}{dt} = -\frac{p}{\rho}(\nabla \cdot \mathbf{v}) \tag{1c}$$

The equations are closed with an equation of state (EOS). In this work, we use the ideal gas EOS:

$$p = (\gamma - 1)\rho e \tag{2}$$

In the above equations, $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$ is the material derivative, e is the specific thermal energy and $\gamma = 1.4$, is the adiabatic index. To establish the notations used, the vector difference $\mathbf{A}_a - \mathbf{A}_b$ is denoted by $\mathbf{A}_{ab}$ and the symbol $\bar{B}_{ab}$ denotes the arithmetic mean $(B_a + B_b)/2$ of a scalar property B.

3 SPH formulation

You will use the standard SPH introduced by Monaghan, the discrete equations of which will be described next. The density for a target particle a is determined from the particle distribution as

$$\rho_a = \sum_{b \in \mathcal{N}(a)} m_b \tilde{W}_{ab} \tag{3}$$

where the kernel interaction between a pair of particles is summetrized as either $\tilde{W}_{ab} = W_{ab}(\frac{h_a+h_b}{2})$ or $\tilde{W}_{ab} = (W_{ab}(h_a) + W_{ab}(h_b))/2$.

For the momentum equation we have:

$$\frac{d\mathbf{v}_a}{dt} = -\sum_{b \in \mathcal{N}(a)} m_b \left(\frac{p_a}{\rho_a^2} + \frac{p_b}{\rho_b^2} + \Pi_{ab} \right) \nabla \tilde{W}_{ab} \tag{4}$$

The artificial viscosity Π_{ab} is introduced as a viscous pressure in this equation. This viscosity is constructed to approximate a shear and bulk viscosity in the continuum limit. It is given as:

$$\Pi_{ab} = \frac{-\alpha \tilde{c}_{ab} \mu_{ab} + \beta \mu_{ab}^2}{\tilde{\rho}_{ab}} \tag{5a}$$

$$\mu_{ab} = \frac{h_{ab} \boldsymbol{v}_{ab} \cdot \boldsymbol{x}_{ab}}{|x_{ab}|^2 + \eta^2} \tag{5b}$$

and is only activated for approaching particle pairs (that is, $\mathbf{v}_{ab} \cdot \mathbf{r}_{ab} \leq 0$). The viscosity depends on two numerical parameters α and β. The terms $\tilde{\rho}_{ab}$ and $\tilde{c}_{ab}$ denotes the arithmetic mean of the density and sound speed between two particles. The energy equation is discretized as

$$\frac{de_a}{dt} = \frac{1}{2} \sum_{b \in \mathcal{N}(a)} m_b \left(\frac{p_a}{\rho_a^2} + \frac{p_b}{\rho_b^2} + \Pi_{ab} \right) \mathbf{v}_{ab} \cdot \nabla \tilde{W}_{ab} \tag{6}$$

The artificial viscosity Π_{ab} is constructed to ensure that the viscous contribution to the thermal energy is always positive definite.

4 Problem Formulation

The initial conditions for the problem describe two states (left and right) of a quiescent gas separated by an imaginary diaphragm. The states are given as $(\rho_l, p_l, u_l) = (1.0, 1.0, 0.0)$ and $(\rho_r, p_r, u_r) = (0.25, 0.1, 0)$ on the left and right hand sides of the diaphragm which is placed at x = 0.

For the simulation, use 400 particles in the domain $x \in [0.5, 0.5]$, with 320 particles uniformly distributed in the interval $x \in [0.5, 0)$ giving $\Delta x_l = 0.0015625$. The remaining particles are placed in the interval $x \in (0, 0.5]$ with $\Delta x_r = 0.00625$. The mass of the particles are constant so that $= m/\Delta x$ gives the desired initial density profile.

- The particle smoothing lengths are set constant equal to $h = 2\Delta x_r$.
- Use the values $\alpha = \beta = 1$ for the artificial viscosity.

- For the time-integration, use a constant time step of $\Delta t < 1e^{-4}s$ and generate results (pressure, velocity, density and energy) at the final time $T = 0.2s$. You can use a one-step Euler integrator for the time integration.

- Use the Cubic-spline kernel for the computations ($q = |x|/h$):

$$W(q,h) = \frac{\sigma}{h^d} \begin{cases} 1 - \frac{3}{2}q^2 + \frac{3}{4}q^3 & 0 \leq q \leq 1 \\ \frac{1}{4}(2-q)^3 & 1 \leq q \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

  where d is the number of spatial dimensions and σ is a normalizing constant equal to $2/3$, $\frac{10}{7\pi}$ and $\frac{1}{\pi}$ in one, two and three dimensions respectively.

Compare your results with the exact solution.