MSO202A-Complex Analysis

18 September, 2018

PART-S

Max. marks: 58

- **This Question paper has two parts: Part O and Part S.**
- **The answers to Part O are to be written in the space provided.** **<u>The Part O answer sheets will be collected after 30 minutes</u>. Start answering Part S after having finished Part O.**
- **Please enter your NAME and ROLL NUMBER on the answer booklet and on the Part O answer sheet.**
- **Avoid graphical explanations or justifications. They will not be considered as proof.**
- $\mathbb{R}$ **stands for real numbers;** $\mathbb{N}$ **stands for the set** $\{1, 2, 3, \ldots\}$**;**

Answer all the questions. Any **statements or theorems** used without proof should be stated clearly and precisely. **They carry marks**. Marks alloted to each question is indicated in brackets.

1. (a) State Liouville's theorem. Using Liouville's theorem deduce that the image of a non-constant entire function $h : \mathbb{C} \to \mathbb{C}$ cannot be a finite set. (1+2)

   (b) Use De Moivre's formula to obtain all the distinct roots of the polynomial $z^3 - \lambda$, for a fixed $\lambda \in \mathbb{R}^{>0}$. (2)

   (c) Let $h : \mathbb{C} \to \mathbb{C}$ be a analytic function such that $h(z)^2 = \overline{h(z)}$ for all $z \in \mathbb{C}$. List all the distinct functions h satisfying the above condition. Give sufficient justification for your answer. (7)

LIOUVILLE'S THEOREM: An entire bounded function is constant 1

Let $h : \mathbb{C} \to \mathbb{C}$ be entire.

Suppose, $h(\mathbb{C})$ is finite, say $\{z_1, z_2, \ldots, z_n\}$ 1

Then $|h(z)| \leq \max\limits_{i=1,\ldots,n}\{|z_i|\} = M$ 1

$\Rightarrow$ h is bounded & entire $\Rightarrow$ h is constant $\Rightarrow\!\Leftarrow$

(1)(b) $z^3 = \lambda$. Let $z = re^{i\theta}$

$\Rightarrow r^3 e^{i3\theta} = \lambda$

$\therefore r = \lambda^{1/3}$, $3\theta = 2n\pi \Rightarrow \theta = \frac{2n\pi}{3}$

$\therefore$ Distinct roots are $\lambda^{1/3}, \lambda^{1/3} e^{\frac{i2\pi}{3}}, \lambda^{1/3} e^{i\pi/3}$.

2

(1)(c) $h(z)^2 = \overline{h(z)}$

$\Rightarrow h(z)^3 = |h(z)|^2$ 1

$h(z)^3$ is analytic and real-valued $\therefore h(z)^3 = \text{constant}$ (say λ) 1

$\Rightarrow h(z) \in \{\lambda^{1/3}, \lambda^{1/3} e^{i2\pi/3}, \lambda^{1/3} e^{i\pi/3}\}$ 1

Or arriving at $h(z) \in \{\lambda^{1/3}, \lambda^{1/3} e^{i2\pi/3}, \lambda^{1/3} e^{i\pi/3}\}$ by any other argument 3

By 1(a), if h is entire and its image is finite then h is constant. 1

$\Rightarrow h(z) = c$ (constant) 1

Given, $h(z)^2 = \overline{h(z)}$. Let $c = x + iy$ 1

then $(x+iy)^2 = x - iy$ 1

$\Rightarrow x^2 - y^2 = x$ & $2xy = -y$

$y = 0$ or $x = -1/2$

$\Downarrow$ $\Downarrow$

$x = 0$ or $x = 1$ $\quad$ $y = \frac{\pm\sqrt{3}}{2}$

$\therefore$ Any analytic fn $\ni h(z)^2 = \overline{h(z)}$

is $h(z) = c$, where $c \in \left\{0, 1, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\right\}$

$\forall z \in \mathbb{C}$

COMMENT: Using alternate methods

to arrive at $h(z) \in \left\{0, 1, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\right\}$

get a maximum of 5 marks, depending on the accuracy of the arguments used.

NOTE: $h(z) = c \quad \forall z \in \mathbb{C}$

$c \in \left\{0, 1, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\right\}$

is a "stronger" claim than

$h(z) \in \left\{0, 1, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\right\}$

2. Let $\mathbb{H} = \{z : Im(z) \geq 0\}$ denote the upper half plane. Let $\mathbb{D} = \{z : |z| \leq 1\}$ be the closed unit disk in $\mathbb{C}$.

   (a) Obtain a Möbius transformation $\phi : \mathbb{H} \to \mathbb{D}$ such that $0 \mapsto i$, $1 \mapsto -1, i \mapsto 0$ and such that ϕ^{-1} has a pole at $-i$. Also, obtain a formula for $\phi^{-1}(z)$. (4)

   (b) Let $f : \mathbb{H} \to \mathbb{D}$ be an analytic function such that $f(i) = 0$. Let $|f(2i)| = \frac{1}{3}$. Then compute $|f(3i)|$. (*Use ϕ^{-1} as obtained in 2(a), if required.*) (8)

(2)(a). $\phi^{-1} : \mathbb{D} \to \mathbb{H}$

$$\left.\begin{array}{l} \ni \quad i \longmapsto 0 \\ \quad -1 \longmapsto 1 \\ \quad -i \longmapsto \infty \end{array}\right\} \Rightarrow \phi^{-1}(z) = \left(\frac{z-i}{z+i}\right)\left(\frac{-1+i}{-1-i}\right)$$

1

$$= \frac{-i(z-i)}{z+i}$$

1

Let $\omega = -i\left(\frac{z-i}{z+i}\right)$. 1

Expressing z in terms of ω, $z = \dfrac{-1-i\omega}{\omega+i}$

1

$\therefore \phi : \mathbb{H} \to \mathbb{D}$

$\ni \quad 0 \longmapsto i$

$\quad 1 \longmapsto -1$

$\quad i \longmapsto 0$

is given by

$$\omega \longmapsto \frac{-1-i\omega}{\omega+i}$$

(2)(b) Let $f : \mathbb{H} \to \mathbb{D}$ be analytic $\ni f(i) = 0$.

Consider $f \circ \phi^{-1} : \mathbb{D} \to \mathbb{D}$ 1

$$(f \circ \phi^{-1})(0) = f(i) = 0 \quad 1$$

By Schwarz lemma, for an analytic function $f : \mathbb{D} \to \mathbb{D} \ni f(0) = 0$ we have $|f(z)| \le |z|$. Further, if equality holds for $0 \ne |z| < 1$ then $f(z) = \beta z$ for some $|\beta| = 1$. 1

1

$$\therefore \ |f \circ \phi^{-1}(z)| \le |z| \quad \forall z \in \mathbb{D}$$

1

$$|f(2i)| = |f \circ \phi^{-1}(\phi(2i))| \le |\phi(2i)| = |1/3i| = 1/3$$

Given $|f(2i)| = 1/3 \Rightarrow |f \circ \phi^{-1}(\phi(2i))| = |\phi(2i)|$

1

$\therefore$ By Schwarz lemma,

$$f \circ \phi^{-1}(z) = \beta \cdot z \text{ for some } \beta \ni |\beta| = 1$$

1

Now, $\left|f(3i)\right| = \left|f \circ \phi^{-1}(\phi(3i))\right|$

$$= |\beta| \, |\phi(3i)| = \left|\frac{2}{4i}\right| = \frac{1}{2}.$$

1

COMMENTS: No marks for assuming $f = \phi^{-1}$ or even "claiming" $f = \phi^{-1}$ (which is wrong).

3. (a) Let f be an entire function. Let $|f(z)| \leq a|z|^m$, where $a \in \mathbb{R}^{>0}$ and $m \in \mathbb{N}$. Show that f is a polynomial of degree at most m. (5)

(b) Let $B_1(0)$ denote the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. Determine all the distinct analytic functions $f : B_1(0) \to \mathbb{C}$ satisfying the condition $f(1/n) = n^2 [f(1/n)]^3$ for all $n \in \mathbb{N}$. Give proper justification. (7)

(3)(a) Given f is entire, so it has a Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \forall z.$ (1)

where $a_n = \frac{1}{2\pi i} \int_{C_r(0)} \frac{f(\omega)}{\omega^{n+1}} d\omega$. for any $r > 0$. (1)

By ML-inequality,

$$|a_n| \leq \frac{1}{2\pi} \sup_{\omega \in C_r(0)} \frac{|f(\omega)|}{r^{n+1}} \cdot 2\pi r$$ (1)

$$= \frac{1}{2\pi} \cdot \frac{a r^m}{r^n} \cdot 2\pi = \frac{a}{r^{n-m}}$$

For $n > m$, we get $\frac{a}{r^{n-m}} \to 0$ as $r \to \infty$. (1)

(1)

$\therefore |a_n| = 0 \quad \forall n > m$

$\Rightarrow f$ is a polynomial of degree atmost m.

3(b)

Let $f : B_1(0) \to \mathbb{C}$

$\Rightarrow f(1/n) = n^2 f(1/n)^3 \quad \forall n \in \mathbb{N}$

$\Rightarrow \frac{1}{n^2} f(1/n) - f(1/n)^3 = 0 \quad \forall n \in \mathbb{N}$. (1)

$\Rightarrow f(1/n)\left[1/n^2 - f(1/n)^2\right] = 0 \quad \forall n \in \mathbb{N}$

$\Rightarrow f(1/n) = 0$ or $1/n^2 - f(1/n)^2 = 0$ for each $n \in \mathbb{N}$

$\Rightarrow f(1/n) = 0$ for infinitely many n (1) — (a)

or $\frac{1}{n^2} - f(1/n)^2 = 0$ for infinitely many n — (b)

ie $\frac{1}{n} - f(1/n) = 0$ for infinitely many n (1) — (c)

or $\frac{1}{n} + f(1/n) = 0$ " " " "

By identity theorem, let f be analytic in a connected domain D, if $\{z_n\}$ is a sequence in D, of distinct terms $\ni z_n \to z_0 \in D$ and $f(z_n) = 0 \ \forall n$ then $f \equiv 0$ on D.

(1)

In case (a), $f(1/n) = 0$ for a subsequence of $\{1/n\} \to 0$

$\therefore$ By identity theorem $f \equiv 0$ (1)

In case (b), $\frac{1}{n} = f(1/n)$ for a subsequence of $\{1/n\} \to 0$

(1)

$\therefore$ By Uniqueness theorem $f(z) = z \quad \forall z \in B_1(0)$

In case (c), $f(1/n) = -1/n$ for a subsequence of $\{1/n\} \to 0$

$\therefore$ By Uniqueness theorem $f(z) = -z \quad \forall z \in B_1(0)$

(1)

COMMENTS: (1) For solutions in which

$$\left.\begin{aligned} f(z) &\equiv 0 \\ &\equiv z \\ &\equiv -z \end{aligned}\right\}$$

are shown to satisfy the relation

a maximum of 3 marks can be given.

(2) Alternate solutions may be marked appropriately: 4 marks for justifying $f \equiv 0$, $f = id$, $f = -id$ are the only analytic maps satisfying it.

4. (a) Find the Laurent series of $\frac{1}{2-3z+z^2}$ in the annulus $\sqrt{2} < |z+i| < \sqrt{5}$. (4)

(b) Determine the radius of convergence R of $\sum_{n=0}^{\infty} 2^n z^{n^2}$. (3)

(c) Let R be as in 4(b). Let $f(z) = \sum_{n=0}^{\infty} 2^n z^{n^2}$ for all $z \in B_R(0) = \{z \in \mathbb{C} : |z| < R\}$. Obtain the Taylor series around 0 of an antiderivative of $f(z)$. (3)

4(a) $$\frac{1}{2-3z+z^2} = \frac{1}{1-z} - \frac{1}{2-z}$$

$$= \frac{-1}{z+i\left[1-\frac{1+i}{z+i}\right]} - \frac{1}{2+i\left[1-\frac{z+i}{2+i}\right]} \quad 1$$

$$= \frac{-1}{z+i}\sum_{n=0}^{\infty}\left(\frac{1+i}{z+i}\right)^n - \frac{1}{2+i}\sum_{n=0}^{\infty}\left(\frac{z+i}{2+i}\right)^n \quad 1$$

1

for $\left|\frac{1+i}{z+i}\right| < 1 \Rightarrow |z+i| > \sqrt{2}$

and $\left|\frac{z+i}{2+i}\right| < 1 \Rightarrow |z+i| < \sqrt{5}$

1

4(b) Radius of convergence $= \dfrac{1}{\limsup \sqrt[n]{|a_n|}}$ 1

For the given series

$$\left.\begin{aligned} a_n &= 0 && \text{if } n \neq m^2 \\ &= 2^m && \text{if } n = m^2 \end{aligned}\right\} 1$$

$$\therefore \frac{1}{\limsup \sqrt[n]{|a_n|}} = \frac{1}{\lim \sqrt[n^2]{2^n}}$$

$$= \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{2}} = 1. \quad 1$$

$\therefore$ Radius of convergence $= 1$.

(4)(c) Let $g(z)$ be an antiderivative of $f(z)$ in $B_1(0)$.

Then $g(z) = \sum\limits_{n=0}^{\infty} a_n z^n$

$$g'(z) = f(z) \quad 1$$

$$\Rightarrow n a_n = \text{coeff of } z^{n-1} \text{ in } \sum_{n=0}^{\infty} 2^n z^{n^2}$$

$$\left.\begin{aligned} &= 0 \quad \text{if } n-1 \neq m^2 \\ &= 2^m \quad \text{if } n-1 = m^2 \end{aligned}\right\} \quad 1$$

$$\left.\therefore g(z) = \sum_{m=0}^{\infty} \frac{2^m}{m^2+1} \cdot z^{m^2+1}.\right\} \quad 1$$

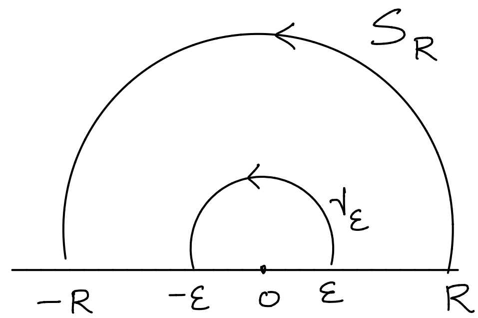
4. (a) Find the Laurent series of $\frac{1}{2-3z+z^2}$ in the annulus $\sqrt{2} < |z+i| < \sqrt{5}$. (4)

(b) Determine the radius of convergence R of $\sum_{n=0}^{\infty} 2^n z^{n^2}$. (3)

(c) Let R be as in 4(b). Let $f(z) = \sum_{n=0}^{\infty} 2^n z^{n^2}$ for all $z \in B_R(0) = \{z \in \mathbb{C} : |z| < R\}$. Obtain the Taylor series around 0 of an antiderivative of $f(z)$. (3)

5. Using a contour containing the arc $S_R : Re^{i\theta}$, $0 \leq \theta \leq \pi$ $(R > 0)$, evaluate $\int_0^{\infty} \frac{\cos ax - \cos bx}{x^2} dx$ where $0 < a < b$ are fixed real numbers. (12)

Let $f(z) = \dfrac{e^{iaz} - e^{ibz}}{z^2}$ 1



Let γ_R be given by

$S_R + \overrightarrow{-R,-\varepsilon} - \gamma_\varepsilon + \overrightarrow{\varepsilon,R}$ 1

Then $\int_{\gamma_R} f(z)\,dz = 0$ (By Cauchy's theorem) 1

$$\int_{\gamma_R} f(z)\,dz = \underbrace{\int_{S_R} f(z)\,dz}_{I} + \underbrace{\int_{-R}^{-\varepsilon} f(x)\,dx}_{(A)} - \underbrace{\int_{\gamma_\varepsilon} f(z)\,dz}_{II} + \underbrace{\int_{\varepsilon}^{R} f(x)\,dx}_{(B)} = 0$$

COMMENT: No marks for using contours not containing S_R.

Term I: $\left| \int_{S_R} f(z)\,dz \right| = \left| \int_0^{\pi} \frac{e^{ia(Re^{i\theta})} - e^{ib(Re^{i\theta})}}{R^2 e^{i2\theta}} \cdot Rie^{i\theta} d\theta \right|$

$$\leq \frac{e^{-aR\sin\theta} + e^{-bR\sin\theta}}{R} \cdot \pi \qquad (3)$$

$$\leq \frac{2\pi}{R} \quad \left(\because \begin{array}{l} e^{-aR\sin\theta} \leq 1 \\ e^{-bR\sin\theta} \leq 1 \\ \forall R >> 0 \end{array} \right)$$

$$\therefore \lim_{R\to\infty} \int_{S_R} f(z)\,dz = 0$$

Term III: By indentation lemma, (1)

$$\lim_{\varepsilon\to 0} \int_{r_\varepsilon} f(z)\,dz = i(\pi - 0)\,\mathrm{Res}(f;0)$$

(Since f has a simple pole at '0').

$$\mathrm{Res}(f,0) = \lim_{z\to 0} z \frac{(e^{iaz} - e^{ibz})}{z^2} = \lim_{z\to 0} iae^{iaz} - ibe^{ibz}$$

$$= i(a-b) \qquad (1)$$

$$\therefore \lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} f(z)\,dz = \pi(b-a) \qquad (1)$$

Term (A) + Term (B)

(Alternate : comparing real part instead is also correct)

$$= \int_{-R}^{-\varepsilon} \frac{e^{iax} - e^{ibx}}{x^2}\,dx + \int_{\varepsilon}^{R} \frac{e^{iax} - e^{ibx}}{x^2}\,dx$$

$$= \int_{\varepsilon}^{R} \frac{e^{-iax} - e^{-ibx}}{x^2}\,dx + \int_{\varepsilon}^{R} \frac{e^{iax} - e^{ibx}}{x^2}\,dx$$

(By change of parameter x to $-x$ and reversing orientation)

$$= \int_{\varepsilon}^{R} \frac{2\cos ax - 2\cos bx}{x^2}\,dx \qquad (3)$$

$$\therefore \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_{\varepsilon}^{R} \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{\pi(b-a)}{2}.$$

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PART-O

Name: ______________________ **Set A**

Roll Number: ____________ **Max. time: 30 minutes**

- **Please enter your NAME and ROLL NUMBER in the space provide.**
- **The answers are to be written in the space provided and returned. No justification needs to be provided for Part O.**
  The Part O answer sheets will be collected after 30 minutes.

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1. Let $x + iy = (1 + i\sqrt{3})^{60}$. Then $x =$ 2^{60} and $y =$ 0.
2. TRUE/FALSE: (**Negative marks -2 for wrong answer.**) The function $\frac{1}{z-1}$ has an anti-derivative in $\mathbb{C} \setminus \{1\}$. FALSE
3. The function given by $\frac{e^z - z - 1}{z^2}$ has an isolated singularity of type removable
4. Let $f(z) = z\cos\frac{1}{z-1}$. Then $\mathrm{Res}(f; 1) =$ $-1/2$.
5. Let σ denote the unit circle around 0 (oriented anti-clockwise). Then $\int_\sigma \frac{z^2+2z}{\sin^2 z} dz =$ $4\pi i$.
6. Let γ be a simple closed contour around 0. Let f be analytic on $\mathbb{C}$ such that $f(0) = 0$ and $f(z) \neq 0$ for $z \neq 0$. Let $\int_\gamma \frac{f'(z)}{f(z)} dz = 4\pi i$. Let g be analytic on $\mathbb{C}$ with a zero of order 5 at 0. Then f/g has a pole of order 3 at $z = 0$.
7. Let f be an entire function such that $\lim_{|z|\to\infty} \frac{f(z)}{z} = 2$ and $f(2) = 2$. Then $f(3) =$ 4.
8. Let $$\int_\gamma \frac{z^2 + z + 1}{z^2(z+1)dz} = x + iy$$ where γ is the contour given in the picture below:

   or $\mathrm{Log}\, 2 - \mathrm{Log}\sqrt{2} - 1$

   Then $x =$ $\mathrm{Log}\sqrt{2} - 1$, $y =$ $-1 - \pi/4$.
9. **Fill the blanks with no more than 3 words each (No partial marks):** The following is an equivalent formulation of Morera's theorem:
   Let D be simply connected domain. If f has an antiderivative on D then f is analytic on D.
   (or has a power series)

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PART-O

Name: ______________________ **Set B**

Roll Number: ____________ **Max. time: 30 minutes**

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1. Let γ be a simple closed contour around 0 (oriented anti-clockwise). Let f be analytic on $\mathbb{C}$ such that $f(0) = 0$ and $f(z) \neq 0$ for $z \neq 0$. Let $\int_\gamma \frac{f'(z)}{f(z)} dz = 4\pi i$. Let g be analytic on $\mathbb{C}$ with a zero of order 4 at 0. Then f/g has a pole of order 2 at $z = 0$.

2. Let f be an entire function such that $\lim_{|z|\to\infty} \frac{f(z)}{z} = 2$ and $f(2) = 2$. Then $f(4) =$ 6.

3. Let $\int_\gamma \frac{z^2 + z + \sqrt{3}}{z^2(z + \sqrt{3})dz} = x + iy$ where γ is the contour given in the picture below:

   Then $x = -\mathrm{Log}\ 2 + \mathrm{Log}(1+\sqrt{3})^{-1}$, $y = -\pi/6 - 1$.

4. **Fill the blanks with no more than 3 words each (No partial marks):** The following is an equivalent formulation of the converse of Morera's theorem:
   Let D be simply connected domain. If f is analytic on D then f has an antiderivative on D.
   (or has a power series) (or has a primitive)

5. Let $f(z) = z \cos \frac{1}{z-2}$. Then $\mathrm{Res}(f; 2) =$ $-1/2$.

6. Let σ denote the unit circle around 0 (oriented anti-clockwise). Then $\int_\sigma \frac{z^2+z}{\sin^2 z} dz =$ $2\pi i$.

7. Let $x + iy = (1 + i)^{60}$. Then $x = (\sqrt{2})^{60}$ and $y =$ 0.

8. TRUE/FALSE: (**Negative marks -2 for wrong answer.**) The function $\frac{1}{z-1}$ has an anti-derivative in $\mathbb{C} \setminus \{1\}$. FALSE

9. The function given by $\frac{e^z-1}{z^2}$ has an isolated singularity of type simple pole

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PART-O

Name: ______________________ **Set C**

Roll Number: __________ **Max. time: 30 minutes**

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1. Let γ be a simple closed contour around 0 (oriented anti-clockwise). Let f be analytic on $\mathbb{C}$ such that $f(0) = 0$ and $f(z) \neq 0$ for $z \neq 0$. Let $\int_\gamma \frac{f'(z)}{f(z)} dz = 4\pi i$. Let g be analytic on $\mathbb{C}$ with a zero of order 3 at 0. Then f/g has a pole of order 1 at $z = 0$.

2. Let f be an entire function such that $\lim_{|z|\to\infty} \frac{f(z)}{z} = 2$ and $f(2) = 2$. Then $f(5) =$ 8.

3. Let $\int_\gamma \frac{z^2+z+1}{z^2(z+1)dz} = x + iy$ where γ is the contour given in the picture below:

   Then $x =$ $\log\sqrt{2} - 1$, $y =$ $-1 - \pi/4$.

4. **Fill the blanks with no more than 3 words each (No partial marks):** The following is an equivalent formulation of Morera's theorem:
   Let D be simply connected domain. If f has an antiderivative on D then f is analytic on D.
   or (has a power series)

5. Let $f(z) = z\cos\frac{1}{z+1}$. Then $\text{Res}(f; -1) =$ $-1/2$.

6. Let σ denote the unit circle around 0 (oriented anti-clockwise). Then $\int_\sigma \frac{z^2+3z}{\sin^2 z} dz =$ $6\pi i$.

7. Let $x + iy = (1 - \sqrt{3}i)^{60}$. Then $x =$ 2^{60} and $y =$ 0.

8. TRUE/FALSE: (**Negative marks -2 for wrong answer.**) The function $\frac{1}{z-1}$ has an anti-derivative in $\mathbb{C} \setminus \{1\}$. FALSE

9. The function given by $\frac{e^z - z}{z^2}$ has an isolated singularity of type pole of order 2.

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PART-O

Name: ____________________________ **Set D**

Roll Number: ______________ **Max. time: 30 minutes**

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1. Let $x + iy = (1-i)^{60}$. Then $x =$ $-(\sqrt{2})^{60}$ and $y =$ 0.
2. TRUE/FALSE: (**Negative marks -2 for wrong answer.**) The function $\frac{1}{z-1}$ has an anti-derivative in $\mathbb{C} \setminus \{1\}$. FALSE
3. The function given by $\frac{e^z - 1}{z^2}$ has an isolated singularity of type pole. of order 1
4. Let $f(z) = z \cos \frac{1}{z-2}$. Then $\mathrm{Res}(f; 2) =$ $-1/2$.
5. Let σ denote the unit circle around 0 (oriented anti-clockwise). Then $\int_\sigma \frac{z^2+4z}{\sin^2 z} dz =$ $8\pi i$.
6. Let γ be a simple closed contour around 0. Let f be analytic on $\mathbb{C}$ such that $f(0) = 0$ and $f(z) \neq 0$ for $z \neq 0$. Let $\int_\gamma \frac{f'(z)}{f(z)} dz = 4\pi i$. Let g be analytic on $\mathbb{C}$ with a zero of order 4 at 0. Then f/g has a pole of order 2 at $z = 0$.
7. Let f be an entire function such that $\lim_{|z|\to\infty} \frac{f(z)}{z} = 2$ and $f(2) = 2$. Then $f(1) =$ 0.
8. Let $$\int_\gamma \frac{z^2 + z + \sqrt{3}}{z^2(z+\sqrt{3})dz} = x + iy$$ where γ is the contour given in the picture below:

   Then $x =$ $(\mathrm{Log}\ 2 - \mathrm{Log}(1+\sqrt{3}) + 1)$, $y =$ $(\pi/6 + 1)$.
9. **Fill the blanks with no more than 3 words each (No partial marks):** The following is an equivalent formulation of the converse of Morera's theorem:
   Let D be simply connected domain. If f is analytic on D then f has an anti derivative on D.
   (or has a power series)