LECTURE - 10. (L-13, L-14)

Application of Taylor's theorem.

→ Zeros of analytic functions

→ Identity theorem

→ Maximum Modulus principle.

§ Zeros of analytic functions:

Let f be an analytic function in a domain $D \supset B_R(z_0)$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

where $a_n = \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$.

Suppose $f(z_0)=0 \Rightarrow a_0 = 0$

($f \not\equiv 0$ on $B_R(z_0)$) Let $a_m \neq 0 \ni a_i = 0 \ \forall \ i < m$

Then $f(z) = \sum_{n=m}^{\infty} a_n (z-z_0)^n$

$$= (z-z_0)^m \underbrace{\sum_{n=0}^{\infty} a_{n+m} (z-z_0)^n}_{g(z)}.$$

$$f(z) = (z-z_0)^m g(z), \quad g(z_0) = a_m \neq 0$$

Now, $g(z)$ is analytic, hence continuous

$$\Rightarrow \quad g(z_0) \neq 0 \quad \Rightarrow g(z) \neq 0 \;\; \forall \; z \in B_\varepsilon(z_0)$$

$$\therefore \; f(z) \neq 0 \quad \forall \, z \neq z_0, \; z \in B_\varepsilon(z_0)$$

ie zeros of f are isolated.

§ Identity theorem

Suppose f is analytic on D (domain). If

$$\{z_n\} \subset D \;\ni\; z_n \longrightarrow z_0 \in D \;\ni\; f(z_n) = 0 \;\forall n$$

Then $f \equiv 0$ on D.

Pf: $\quad z_n \to z_0 \Rightarrow f(z_n) \to f(z_0)$

$$\therefore f(z_0) = 0.$$

Suppose $f \not\equiv 0$ on $B_R(z_0)$

By previous result, $\exists \; \varepsilon > 0 \ni f(z) \neq 0$

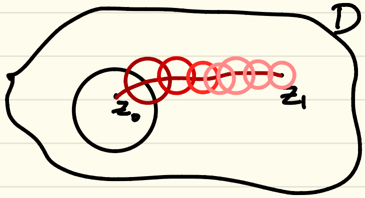
$\forall \, z \neq z_0, \, z \in B_\varepsilon(z_0)$

but $z_n \to z_0 \Rightarrow \exists \, N > 0 \ni z_n \in B_\varepsilon(z_0)$

$\forall \, n > N$

$\Rightarrow\!\Leftarrow$

Thus, $f \equiv 0$ on $B_R(z_0)$



Idea of proof.

Take $|z| = \varepsilon$, $\exists \{\zeta_n\} \to z \ni f(\zeta_n) = 0 \ \forall n$

by similar argument as above

$f \equiv 0$ on $B_{\varepsilon_2}(z)$. Proceeding thus

we get $f(z_1) = 0$ ▨

<u>Cor</u>: (Uniqueness theorem) Let f, g be analytic on $B_R(z_0) \ni f(z_n) = g(z_n)$ for $\{z_n\} \to z$. ${}^{\not z}$

Then $f = g$ on $B_R(z_0)$. (Pf: apply above to $f-g$)

§ <u>Maximum - Modulus principle</u>.

Let f be a non-constant analytic fn. on a domain G. Then $|f|$ does not attain a local maximum "in" G.

<u>Pf:</u> Suppose $\exists z_0 \in G \ni |f(z)| \leq |f(z_0)| \ \forall z \in B_R(z_0)$

Now $f(z_0) = \frac{1}{2\pi i} \int_{S_r(z_0)} \frac{f(w)\,dw}{w - z_0}$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}}\, rie^{it}\,dt$$

$$\boxed{f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\,dt}$$ "mean value property".

$B_r(z_0) \subset B_R(z_0)$

$$\Rightarrow \quad |f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})|\,dt$$

$$\leq |f(z_0)|$$

$$\therefore \quad \frac{1}{2\pi} \int_0^{2\pi} \Big| |f(z_0)| - |f(z_0 + re^{it})| \Big|\,dt = 0$$

$\Rightarrow \quad |f(z_0)| = |f(z_0 + re^{it})|$

$\Rightarrow$ $|f|$ is constant on $B_R(z_0)$

$\Rightarrow$ f is constant on $B_R(z_0)$

$\Rightarrow$ f = constant on G (by Uniqueness thm).

<u>Cor</u>: If f is analytic inside and on a simple closed curve C, then $|f|$ attains its maximum on the boundary.