Contents

same as in the real case...

A function is a map $f : \mathbb{C} \to \mathbb{C}$.

Corresponding to such an f there are two real-valued functions on $\mathbb{C}$:

$$Ref(f) = Re \circ f$$

$$Im(f) = Im \circ f$$

So, $f = Re(f) + i\ Im(f)$.

Now thinking of $\mathbb{C}$ as $\mathbb{R}^2$ what we have is two real-valued functions on $\mathbb{R}^2$:

$$u : \mathbb{R}^2 \to \mathbb{R} \quad \text{given by } u(x, y) = Re(f)(x + iy)$$

$$v : \mathbb{R}^2 \to \mathbb{R} \quad \text{given by } v(x, y) = Im(f)(x + iy)$$

So, $f(x + iy) = u(x, y) + iv(x, y)$.

Let U be a neighbourhood of z_0.
A function $f : U \subset \mathbb{C} \to \mathbb{C}$ (except possibly at z_0) is said to have a limit l at z_0 if, $f(z) \to l$ as $z \to z_0$.

Given an $\epsilon > 0$ there is a $\delta > 0$ such that $|f(z) - l| < \epsilon$ for all z such that $0 < |z - z_0| < \delta$.

Notation

$\lim_{z \to z_0} f(z) = l$

$f(z) \to l$ as $z \to z_0$ if and only if $Re\ f(z) \to Re(l)$ and $Im\ f(z) \to Im(l)$ as $z \to z_0$.

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Arithmetic of limits

Let g and f be two functions defined in a neighbourhood around z_0. Let

$$\lim_{z \to z_0} f(z) = l$$

and

$$\lim_{z \to z_0} g(z) = l'.$$

Then

1. $\lim_{z \to z_0} f(z) + g(z) = l + l'$
2. $\lim_{z \to z_0} f(z)g(z) = ll'$
3. further, if $l' \neq 0$, then $\lim_{z \to z_0} f(z)/g(z) = l/l'$
4. $\lim_{z \to z_0} cf(z) = cl$

Let $f : D \to \mathbb{C}$ be a function defined around z_0. If $\lim_{z \to z_0} f(z)$ exists and is equal to $f(z_0)$ then we say f is continuous at z_0.

$f(z) = u(x, y) + iv(x, y)$ is continuous if and only if u and v are continuous.

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$$
\begin{aligned}
& f(z) \to f(z_0) \text{ as } z \to z_0 = x_0 + iy_0 \\
\iff & u(x, y) \to Re\ f(z_0) = u(x_0, y_0) \\
& \text{and } v(x, y) \to Im\ f(z_0) = v(x_0, y_0) \\
& \text{as } (x, y) \to (x_0, y_0).
\end{aligned}
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(a) f is continuous $\iff$ u & v are continuous

(b) $f(z_n) \to f(z) \iff$
$u(x_n, y_n) \to u(x, y)$ & $v(x_n, y_n) \to v(x, y)$

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