Assignment - 1

From Gauss Divergence we have,

$$\int_{\Omega} \operatorname{div} F \, dx = \int_{\partial\Omega} F \cdot \eta \, dS.$$

Choose, $F(x) = (u(x), 0, \dots 0)$ – $(n-1)$ zero component.

$$\int_{\Omega} \operatorname{div} F(x) = \int_{\Omega} u_{x_1} \, dx$$

$$\overset{G \cdot D}{=} \int_{\partial\Omega} u(x) \cdot \eta_1 \, dS \quad (\eta_1 \text{ is the } 1^{st} \text{ component of unit outward normal } \eta).$$

Hence we have,

$$\int_{\Omega} u_{x_i} \, dx = \int_{\partial\Omega} u \eta_i \, dS \quad \text{for any } i = 1, 2, \dots, n. \quad \text{—— } (*)$$

Now for $u, v \in C^2(\Omega)$ we have $uv \in C^2(\Omega)$ and applying $(*)$ we have,

$$\int_{\Omega} (uv)_{x_i} \, dx = \int_{\partial\Omega} uv\eta_i \, dS$$

$$\Rightarrow \int_{\Omega} u_{x_i} v \, dx = - \int_{\Omega} u v_{x_i} \, dx + \int_{\partial\Omega} uv\eta_i \, dS. \quad \forall i = 1, 2, \dots, n. \quad \text{—— } (i)$$

(ii) replacing u_{x_i} in place of u and $v(x) \equiv 1$, in (i) we have,

$$\int_{\Omega} u_{x_i x_i} \, dx = \int_{\partial\Omega} u_{x_i} \eta_i \, dS$$

Summing $i = 1, 2, \dots, n$ we have,

$$\int_{\Omega} \Delta u \, dx = \int_{\partial\Omega} \nabla u \cdot \eta \, dS := \int_{\partial\Omega} \frac{\partial u}{\partial \eta} \, dS.$$

(iii) Replace v by v_{x_i} in (i) we have,

$$\int_{\Omega} u_{x_i} \cdot v_{x_i} = - \int_{\Omega} u v_{x_i x_i} \, dx + \int_{\partial\Omega} u v_{x_i} \eta_i \, dS$$

Summing $i = 1, 2, \dots, n$ we have the desired result.

(2) Classify the PDE :

(i) $\Delta u = u^2$ (Semilinear) (ii) $u_{tt} - u_{xx} = \sin x$ (Linear)

(iii) $u_t - u_{xx} = \cos(u)$ (semilinear) (iv) $|\nabla u| = 1$ (Nonlinear)

(v) $\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$ (quasilinear)

when $p = 2$; $\operatorname{div}(\nabla u) = 0$ or $\Delta u = 0$ (linear).

All are 2nd order PDE except (d) which is 1st order.

(3) $y'' + a(x) y' + b(x) y = f(x)$

$y(x_0) = y_0$; $y(x_1) = y_1$.

Consider $y'' + y = 0$ with $y(0) = 1$ & $y(\pi) = 1$ has no soln.

$\because$ any soln will look like $y(x) = A\cos x + B\sin x$

$1 = y(0) = A + B \cdot 0 \Rightarrow A = 1$

again, $1 = y(\pi) = A \cdot \pi + B \cdot 0 \Rightarrow A = -1/\pi$ } Not possible

Again, $y'' + y = 0$ with $y(0) = 1$ and $y(2\pi) = 1$ has infinitely many soln. $y(x) = A\cos x + B\sin x$ is a soln

Now, $1 = y(0) = A$ & $1 = y(2\pi) = +A$

$\therefore y(x) = \cos x + B\sin x$ ($\because$ B is arbitrary)

we have infinitely many soln.

(4) we know that $a_2(x) y'' + a_1(x) y' + a_0(x) y = f(x)$ can be put in S-L form

$(p(x) y')' + q(x) y = F(x)$

where $p(x) = \exp\left(\int \frac{a_1(x)}{a_2(x)} dx\right)$; $q(x) = \exp\left(\int \frac{a_1(x)}{a_2(x)} dx\right) \frac{a_0(x)}{a_2(x)}$.

and $F(x) = p(x) \frac{f(x)}{a_2(x)}$.

$\therefore x^2y'' + xy' + 2y = 0$ can be written as $(xy')' + \frac{2}{x}y = 0$

by multiplying the eqn by $\frac{1}{x^2}\exp\left(\int \frac{dx}{x}\right) = \frac{1}{x}$.

The corresponding eigenvalue problem is

$$x^2y'' + xy' + (\lambda+2)y = 0 \quad (\text{Assuming } \lambda \text{ negative})$$

$\therefore$ Char eqn is $r^2 + \lambda + 2 = 0$, assuming $\lambda + 2 > 0$ we have

$$y(x) = c_1 \cos(\sqrt{\lambda+2}\ \ln|x|) + c_2 \sin(\sqrt{\lambda+2}\ \ln|x|)$$

Now from boundary condition: $y'(1) = 0 \Rightarrow c_2 = 0$.

$$\therefore y(x) = c_1 \cos(\sqrt{\lambda+2}\ \ln|x|)$$

Again, $y'(2) = 0 \Rightarrow \sqrt{\lambda+2}\ \ln 2 = n\pi \quad (n = 0, 1, 2 \ldots)$

$$\therefore y_n(x) = \cos\left(\frac{n\pi}{\ln 2} \ln x\right);\ 1 \le x \le 2.$$

⑤ $y'' + \lambda y = 0$ — (1)

$y(0) = y(\pi)$ & $y'(0) = y'(\pi)$ — Periodic BVP.

Case 1:- Let $-\lambda < 0$ then $\lambda = +\mu^2$ $(\mu \neq 0)$

we have, $y(x) = A\cos(\mu x) + B\sin(\mu x)$.

Hence from (1)

Now, $y(0) = y(\pi) \Rightarrow A \cdot 1 + B \cdot 0 = A\cos(\mu\pi) + B\sin(\mu\pi)$.

$$\Rightarrow A(1 - \cos\mu\pi) - B\sin\mu\pi = 0 \quad \text{— (i)}$$

Again $y'(0) = y'(\pi) \Rightarrow -A\mu\sin(\mu\cdot 0) + B\mu\cos(\mu\cdot 0) = -A\mu\sin(\mu\pi) + B\mu\cos(\mu\pi)$

$$\Rightarrow B\mu(1 - \cos\mu\pi) + A\mu\sin\mu\pi = 0 \quad \text{— (ii)}$$

For a non trivial soln

$$\begin{vmatrix} 1 - \cos\mu\pi & -\sin\mu\pi \\ \mu\sin\mu\pi & \mu(1-\cos\mu\pi) \end{vmatrix} = 0 \Rightarrow (1 - \cos\mu\pi)^2 + \sin^2\mu\pi = 0.$$

$$\Rightarrow 2 - 2\cos\mu\pi = 0.$$

$$\Rightarrow \cos\mu\pi = 1 \Rightarrow \mu = \pm 2n \quad (n \in \mathbb{N}).$$

$\therefore \lambda_n = 4n^2$ and the eigenfuncs are $y_n^1(x) = \cos(2nx)$ and $y_n^2(x) = \sin(2nx)$ which are linearly independent.

6. $y'' + \lambda y = 0$ —— ①

$y(0) = y(\pi) = 0$ —— ②

Find the eigenvalue & eigenfxn for the above problem.

Case 1 :- If $\lambda = 0$ then ① reduces to $y'' = 0$ whose soln is $y(x) = c_1 + c_2 x$ where c_1 and c_2 are arbitrary constants.

'y' will satisfy ② hence that is only possible if $c_1 = c_2 = 0$.

ie $y(x) = 0$ is the only soln of ① & ②

$\therefore \lambda = 0$ is not an eigenvalue

Case 2 :- If $\lambda \neq 0$ then let $\lambda = \mu^2$ $(\mu \in \mathbb{C})$.

$\therefore$ ① reduces to $y'' + \mu^2 y = 0$ whose soln is given by

$$y(x) = c_1 e^{i\mu x} + c_2 e^{-i\mu x}$$

For 'y' to satisfy ① and ②,

$$c_1 + c_2 = 0$$

and, $c_1 e^{i\mu\pi} + c_2 e^{-i\mu\pi} = 0$

this has a nontrivial soln iff $e^{-i\mu\pi} - e^{i\mu\pi} = 0$ —— ⊛

If $\mu = a + ib$; $a, b \in \mathbb{R}$ then ⊛ reduces to

$$e^{b\pi}(\cos a\pi - i \sin a\pi) - e^{-b\pi}(\cos a\pi + i \sin a\pi)$$

$$= (e^{b\pi} - e^{-b\pi}) \cos a\pi - i(e^{b\pi} + e^{-b\pi}) \sin a\pi$$

$$= 2 \sinh b\pi \cos a\pi - 2i \cosh b\pi \sin a\pi = 0.$$

$\therefore \sinh b\pi \cos a\pi = 0$ ① and $\cosh b\pi \sin a\pi = 0$ —— (ii)

$\because \cosh b\pi > 0$ $\forall b$, eq (ii) implies $a = n$; $n \in \mathbb{Z}$. further for this choice of a, $\cos a\pi \neq 0$.

So, $\sinh b\pi = 0 \Rightarrow b = 0$

But if $b = 0$ and $a = 0$ then $\mu = 0$ (Not an eigenvalue)

$\therefore \mu = n$; $n \in \mathbb{Z} \setminus \{0\}$.

So the eigenvalues are $\lambda_n = \mu^2 = n^2$ $(n = 1, 2, \ldots)$ and the eigenfxns $\phi_n(x) = \sin nx$. $(\because c_1 = -c_2)$.