MSO202A COMPLEX ANALYSIS
Assignment 6

Exercise Problems:

1. If $0 < |z| < 4$, show that $\dfrac{1}{4z - z^2} = \dfrac{1}{4z} + \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$.

   Proof: We have $0 < |z| < 4 \Rightarrow \frac{|z|}{4} < 1$.

$$\frac{1}{4z - z^2} = \frac{1}{z(4-z)} = \frac{1}{4z(1-\frac{z}{4})} = \frac{1}{4z}\sum_{n=0}^{\infty}(\frac{z}{4})^n = \frac{1}{4z} + \sum_{n=0}^{\infty}\frac{z^n}{4^{n+2}}$$

2. Write the two Laurent series in powers of z that represent the function $f(z) = \frac{1}{z(1+z^2)}$ in different domains.

   Proof: Let $0 < |z| < 1$. Then $\dfrac{1}{z(1+z^2)} = \dfrac{1}{z(1-(-z^2))} = \dfrac{1}{z} + \displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1}$.

   Let $|z| > 1$. Then $\dfrac{1}{z(1+z^2)} = \dfrac{1}{z^3(1-(-\frac{1}{z^2}))} = \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}z^{-2n-1}$.

3. Which of the following singularities are removable/pole:
   $(a)\ \dfrac{\sin z}{z^2 - \pi^2}$ at $z = \pi$, $(b)\ \dfrac{\sin z}{(z-\pi)^2}$ at $z = \pi$ $(c)\ \dfrac{z\cos z}{1 - \sin z}$ at $z = \pi/2$.

   Proof: (a) Since $z = \pi$ is a simple zero of $\sin z$, and $z^2 - \pi^2$, so $z = \pi$ is a removable singularity.
   (b) Since $z = \pi$ is a simple zero of $\sin z$, and a double zero of $(z-\pi)^2$ so $z = \pi$ is a simple pole of $\dfrac{\sin z}{(z-\pi)^2}$.
   (c) $z = \pi/2$ is a simple zero of $z\cos z$ and a double zero of $1 - \sin z$, so $z = \pi/2$ is a simple pole.

4. Find the residue at $z = 0$ of the following functions and indicate the type of singularity they have at 0. $(a)\ \dfrac{1}{z+z^2}$ $(b)\ z\cos\dfrac{1}{z}$ $(c)\ \dfrac{z - \sin z}{z}$ $(d)\ \dfrac{\cot z}{z^4}$.

   Proof: (a) 0 is a simple zero of $z + z^2$ so it is a simple pole of $\dfrac{1}{z+z^2}$.
   (b) The Laurent series of $z\cos\frac{1}{z}$ is $z\sum_{n=0}^{\infty}\frac{(-1)^{2n}}{n!}(1/z)^{2n}$ for $|z| > 0$. Hence f has an essential sigularity at $z = 0$.
   (c) Since $z = 0$ is a simple zero of z and a double 0 of $z - \sin z$, so $z = 0$ is a removable

singularity.
(d) $\frac{\cot z}{z^4}$ has pole of order 5 at $z = 0$ since $z^4 \sin z$ has a zero of order 5 at $z = 0$ and $\cos 0 = 1$.

5. Use Cauchy's residue theorem to evaluate the integral of each of the following functions around the circle $|z| = 3$. (a) $\dfrac{e^{-z}}{z^2}$, (b) $\dfrac{e^{-z}}{(z-1)^2}$, (c) $z^2 e^{\frac{1}{z}}$ and $(d)\dfrac{z+1}{z^2-2z}$.

   Proof: (a) $2\pi i \mathrm{Res}(f;0) = -2\pi i$; (b) $-2\pi i \mathrm{Res}(f;1) = 2\pi i e^{-1}$ (c) $2\pi i \mathrm{Res}(f;0) = \pi i/3$; (d) $2\pi i(\mathrm{Res}(f;0) + \mathrm{Res}(f;2)) = 2\pi i$.

   Problem for Tutorial:

6. Prove Jordan's inequality: Given any $R > 0$, $\int_0^\pi e^{-R\sin\theta} d\theta < \frac{\pi}{R}$.

   Proof: First of all, observe that we have the inequality: $\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1$ for $0 \le \theta \le \frac{\pi}{2}$. This can be immediately seen by noting that $\frac{\sin\theta}{\theta}$ is decreasing in $(0, \frac{\pi}{2}]$ (See footnote for a short proof*). Hence $\sin\theta \ge \frac{2\theta}{\pi}$. We thus get $e^{-R\sin\theta} \le e^{-\frac{2R\theta}{\pi}} \Rightarrow \int_0^{\pi/2} e^{-R\sin\theta} d\theta \le \int_0^{\pi/2} e^{-\frac{2R\theta}{\pi}} d\theta = \frac{\pi}{2R}(1 - e^{-R}) < \frac{\pi}{2R}$. Therefore, $\int_0^\pi e^{-R\sin\theta} d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta} d\theta < \frac{\pi}{R}$.

7. Find the Laurent series of the function $f(z) = \frac{6z+8}{(2z+3)(4z+5)}$ in the regions $\{z : \frac{5}{4} < |z| < \frac{3}{2}\}$, $\{z \in \mathbb{C} : |z| > \frac{3}{2}\}$, $\{z : |z| < \frac{5}{4}\}$.

   Proof: For $\frac{5}{4} < |z| < \frac{3}{2}$, $f(z) = \dfrac{6z+8}{(2z+3)(4z+5)} = \dfrac{1}{2z+3} + \dfrac{1}{4z+5} = \dfrac{1}{3(1+\frac{2z}{3})} + \dfrac{1}{4z(1+\frac{5}{4z})} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^{n+1}} z^n + \sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{4^{n+1}} \frac{1}{z^{n+1}}$.
   For $|z| < \frac{5}{4}$, $f(z) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{2^n}{3^{n+1}} + \frac{5^n}{4^{n+1}}\right) z^n$.
   For$|z| > \frac{3}{2}$, $f(z) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{3^n}{2^{n+1}} + \frac{5^n}{4^{n+1}}\right) z^{-(n+1)}$.

8. Find the isolated singularities and compute the residue of f: (a) $\frac{e^z}{z^2-1}$ (b) $\frac{3z}{z^2+iz+2}$ (c) $\cot\pi z$.

   Proof: (a) Singularities are $z = \pm 1$. As both are simple poles,
   $\mathrm{Res}(f;1) = \lim_{z\to 1}(z-1)\frac{e^z}{z^2-1} = \frac{e}{2}$;
   $\mathrm{Res}(f;-1) = \lim_{z\to -1}(z+1)\frac{e^z}{z^2-1} = \frac{-1}{2e}$ (b) Since $z^2 + iz + z = (z-i)(z+2i)$, the singularities are i, $-2i$. Both the singularities are simple poles so $\mathrm{Res}(f;i) = \lim_{z\to i}(z-i)\frac{3z}{z^2+iz+z} = 1$; $\mathrm{Res}(f;-2i) = 2$.

* $\frac{d}{d\theta}(\frac{\sin\theta}{\theta}) = \frac{\theta\cos\theta - \sin\theta}{\theta^2}$ which takes value 0 at $\theta = 0$; further, the derivative of $\frac{\theta\cos\theta - \sin\theta}{\theta^2}$ is $-\theta\sin\theta$ which is < 0 for $\theta \in [0, \pi/2]$

(c) Poles are at $z = \pm n$, $n \in \mathbb{N}$ each being simple. $\text{Res}(\cot \pi z; n) = \lim_{z\to n}(z - n)\frac{\cos \pi z}{\sin \pi z} = \lim_{z\to n}(z - n)\frac{(-1)^n \cos \pi z}{\sin \pi(z-n)} = \frac{1}{\pi}$.

9. Let $f(z) = \frac{\pi \cot \pi z}{(z+\frac{1}{2})^2}$. Compute the residue of f at the isolated singularities.

Proof: As computed above, we get $\lim_{z\to n}(z - n)\frac{\pi \cot \pi z}{(z+1/2)^2} = \frac{1}{(n+\frac{1}{2})^2}$. For $z = \frac{-1}{2}$, note that $\frac{-1}{2}$ is a simple zero of $\cos \pi z$ and a double zero of $(z + 1/2)^2$ so its a simple pole of $\frac{\pi \cot \pi z}{(z+1/2)^2}$. Hence $\text{Res}(\frac{\pi \cot \pi z}{(z+\frac{1}{2})^2}; \frac{-1}{2}) = \lim_{z\to \frac{-1}{2}} \frac{\pi \cot \pi z - 0}{z+\frac{1}{2}} = -\pi^2 \csc^2 \pi z|_{z=\frac{-1}{2}} = -\pi^2$.