LECTURE - 20

Linear Fractional Transformations

Möbius transformation:

Let $a,b,c,d \in \mathbb{C} \ni ad-bc \neq 0$.

Then $f(z) = \dfrac{az+b}{cz+d}$ is called a Möbius transformation.

Properties: (1) f is diff'ble for $z \neq -d/c$

$$f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$$

② Composition of möbius transformation is again a Möbius transformation

$$g(z) = \frac{Az+B}{Cz+D}$$

$$g \circ f(z) = \frac{(Aa+Bc)z + Ab+Bd}{(Ca+Dc)z + Cb+Dd}.$$

$$(Aa+Bc)(Cb+Dd) - (Ab+Bd)(Ca+Dc)$$
$$= (AD-BC)(ad-bc) \neq 0.$$

3 Möbius transformation takes circles and straight lines to circles & straight lines

Equation of a line : $\dfrac{|z-p|}{|z-q|} = 1$

(gives the $\perp^r$ bisector joining a to b).

Equation of a circle : $\left|\dfrac{z-p}{z-q}\right| = k.$ $(k \neq 1)$

$$(x-a_1)^2 + (y-a_2)^2 = k^2\left[(x-b_1)^2 + (y-b_2)^2\right]$$

$$x^2(1-k^2) + y^2(1-k^2) - 2a_1x + k^2(2b_1x)$$
$$- 2a_2y + k^2(2b_2y) = k^2(b_1^2+b_2^2) - (a_1^2+a_2^2)$$

$$x^2 + y^2 - 2\frac{(a_1+k^2b_1)}{1-k^2}x - 2\frac{(a_2+k^2b_2)}{1-k^2}y = K$$

$$\left(x - \frac{a_1+k^2b_1}{1-k^2}\right)^2 + \left(y - \frac{a_2+k^2b_2}{1-k^2}\right)^2 = K + \frac{(a_1+k^2b_1)^2}{(1-k^2)^2} + \frac{(a_2+k^2b_2)^2}{(1-k^2)^2}$$

$$\left|\frac{z-p}{z-q}\right| = k \quad \& \quad f(z) = w$$

$$\Rightarrow \left|\frac{w-P}{w-Q}\right| = k$$

(on $\mathbb{C} \setminus \{-d/c\}$)

(4) Möbius transformation are invertible. The inverse is also a Möbius transformation.

$$\frac{az+b}{cz+d} = w \quad \Rightarrow \quad z = \left(\frac{dz-b}{-cz+d}\right)\frac{1}{ad-bc}.$$

(5) Every Möbius transformation is a composition of translation, Inversion, rotation and magnification

$$\tau_a(z) = z + a \quad \to \text{translation}$$

$$\rho_\theta(z) = e^{i\theta} z \quad - \text{rotation}$$

$$m_\alpha(z) = \alpha z \quad (\alpha > 0) \to \text{magnification}$$

$$j(z) = \frac{1}{z} \quad (z \neq 0) \to \text{inversion}.$$

(Note: all the above are Möbius transformations).

$$\frac{az+b}{cz+d} = t_2 \circ r \circ m \circ j \circ t_1(z)$$

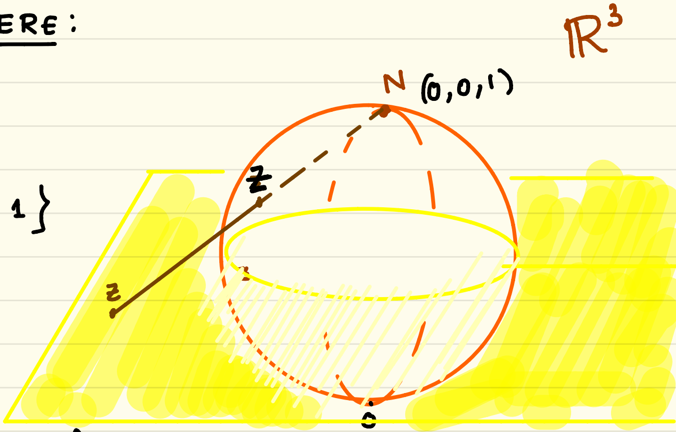
$c \neq 0$ $\quad t_1(z) = z + d/c, \quad m(z) = \left|\frac{ad-bc}{c^2}\right| \cdot z$

$$r(z) = \frac{ad-bc}{|ad-bc|} \, \frac{|c|^2}{c^2} \cdot z$$

$$t_2(z) = z + \frac{a}{c} .$$

$$c = 0 : \quad rmt_1(z) = \frac{az+b}{d} .$$

Riemann Sphere:



Eqn of sphere:
$\{(x,y,z) / x^2+y^2+z^2 = 1\}$

Eqn of line through $(0,0,1)$
and $(x,y,0)$: $tN + (1-t)Z \qquad 0 \leq t$

$$= \{((1-t)x, (1-t)y, t)\}$$

Intersection of the line with sphere.

$$(1-t)^2 x^2 + (1-t^2) y^2 + t^2 = 1$$

$$|z|^2 (1-t^2) + t^2 = 1 . \implies t = \frac{|z|^2 - 1}{|z|^2 + 1}$$

Thus, $(x, y, z) = Z$

is $\left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right)$.

Every line on $\mathbb{C}$ corresponds to a circle on the sphere through N. under stereographic projection.

North pole = point at ∞.

Line : P, Q, ∞. $\qquad 0, 1, \infty$ real

$0, i, \infty$ im.

Circle : P, Q, R. $\qquad i, -i, 1$ - unit circle.

Convention: $\frac{az+b}{cz+d} = \infty$ if $\frac{cz+d}{az+b} = 0$, $\frac{a\infty + b}{c\infty + d} = \frac{a + b\cdot 0}{c + d\cdot 0}$

THEOREM: There is a unique Möbius transformation taking a triplet z_1, z_2, z_3 to $0, 1, \infty$.
Hence, there is a unique Möbius transf taking 1 triplet to another.

Pf: $\left(\frac{z - z_1}{z - z_3}\right)\left(\frac{z_2 - z_3}{z_2 - z_1}\right)$.

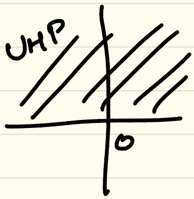
$f(z) = \infty \Leftrightarrow \frac{1}{f(z)} = 0$

$f(\infty) = \omega \Leftrightarrow f(1/z) = \omega$ for $z = 0$

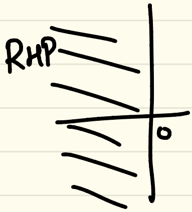
CONVENTION: if one of z_i's is ∞ the the ratio of terms containing it is 1.

The above theorem enables us to also map regions under Möbius transformation to other regions

Eg1: A Möbius transformation taking



to



real axis $\qquad\qquad$ imaginary axis

$0, 1, \infty \qquad\qquad 0, i, -i$

$$\left(\frac{z-0}{z+i}\right)\left(\frac{i+i}{i}\right) \longleftarrow\!\!\!\mid$$

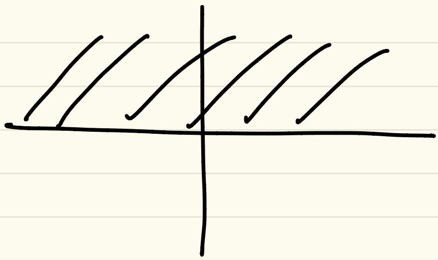
$$\parallel$$

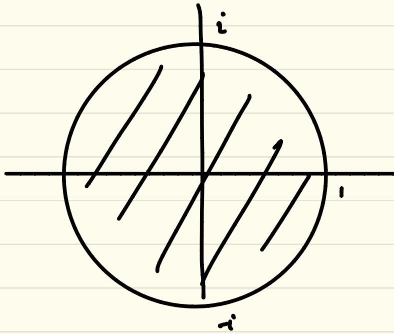
$$\frac{2z}{z+i} \qquad \frac{-2}{-1+i} \longleftarrow\!\!\!\mid -1$$

$$= \frac{-2(-1-i)}{2}$$

$$= 1+i \in \text{UHP}$$

Eg 2:





$$0,\ 1,\ \infty \qquad\qquad i,\ 1,\ -i$$

$$\underbrace{\left(\frac{z-i}{z+i}\right)\left(\frac{1+i}{1-i}\right)}_{i\left(\frac{z-i}{z+i}\right)}$$

$$i\left(\frac{z-i}{z+i}\right)$$

$$1 \longleftarrow 0$$

Exercises:

① Consider the inversion map $z \longmapsto 1/z \quad (z \neq 0)$

Determine the image of the line

$$|z| = |z-2| \text{ under the inversion.}$$

Soln: Let $\omega = \frac{1}{z}$, ie $z = \frac{1}{\omega}$

$$\therefore \left|\frac{1}{\omega}\right| = \left|\frac{1}{\omega} - 2\right|$$

$$\Rightarrow \underbrace{|2\omega - 1| = 1}_{} \quad (\forall |\omega| \neq 0)$$

denotes a circle with radius 1 and center at $1/2$.

② Show the $\left|\frac{z-7}{z-4}\right| = 2$ represents the circle

$|z-3| = 2$.

(Check).

③ Find Möbius transformation which take

(i) $-1, i, 1$ to $-1, -i, i$ (Determine its poles)

(ii) $-1-i, 0, 1+i$ to $0, 1, \infty$

(4)(a) Obtain a Möbius transformation taking $0, 1, \infty$ to, $1, 1+i, i$.

(b) Under this determine the image of

(i) the arc through -1 and $-i$

(ii) real axis

(iii) imaginary axis

(5) Find the inverse of the following Möbius transformations and find the fixed points of each (ie find the pts z $\ni$ $f(z) = z$)

(a) $\dfrac{z-1}{z+1}$ (b) $\dfrac{3z-4}{z-1}$ (c) iz (d) $\dfrac{2iz}{z+i}$

Soln (a) $w = \dfrac{z-1}{z+1}$

$$w(z+1) = z - 1 \Rightarrow z(w-1) = -w-1$$

$$z = \frac{1+w}{1-w}, \quad \text{inverse: } w \mapsto \frac{1+w}{1-w}$$

Fixed pts: $\dfrac{z-1}{z+1} = z$

$$\Rightarrow z - 1 = z^2 + z \Rightarrow z^2 = 1, \quad z = \pm i.$$