Lecture -2

Sequences & Series

Lecture 2

Sequence: An infinite set of complex numbers indexed by $\mathbb{N}$.

Eg: $\{1/n\}$, $\{i^n\}$

Subsequence: A subset of a sequence indexed by a strictly increasing sequence of natural numbers is called a subsequence of the given sequence.

Eg: $\{1/2n\}$, $\{i^{4k}\}$

$\subseteq$ $\quad$ $\subseteq$

$\{1/n\}$ $\quad$ $\{i^n\}$

A sequence is said to converge to a limit "l" if given $\varepsilon > 0$ $\exists$ $N > 0$ $\ni$

$$|z_n - l| < \varepsilon \quad \forall \; n \geq N.$$

[Denoted as $\lim_{n \to \infty} z_n = l$

Eg: $\{1/n\}$ converges to 0.

REMARK: $\{z_n\}$ converges to z $\Leftrightarrow$ $\{\operatorname{Re}(z_n)\}$ converges to $\operatorname{Re} z$ and $\{\operatorname{Im} z_n\}$ converges to $\operatorname{Im} z$.

(Indeed, $z_n \to z \Rightarrow$ given $\varepsilon > 0$ $\exists N > 0 \ni$
$|z_n - z| < \varepsilon \ \forall n \geq N$

$$|\mathrm{Re}(z_n - z)| \leq |z_n - z| < \varepsilon$$

$$|\mathrm{Im}(z_n - z)| \leq |z_n - z| < \varepsilon$$

so $\mathrm{Re}\, z_n \to \mathrm{Re}\, z$ & $\mathrm{Im}\, z_n \to \mathrm{Im}\, z$

Conversely, if $\mathrm{Re}\, z_n \to \mathrm{Re}\, z$ & $\mathrm{Im}\, z_n \to \mathrm{Im}\, z$ then,

given $\varepsilon > 0$, choose $N \gg 0 \ni$

$$\left.\begin{array}{r} |\mathrm{Re}\, z_n - \mathrm{Re}\, z| < \varepsilon/2 \\ |\mathrm{Im}\, z_n - \mathrm{Im}\, z| < \varepsilon/2 \\ \forall n \geq N \end{array}\right\} \Rightarrow |z_n - z| = \sqrt{\varepsilon^2/4 + \varepsilon^2/4} = \varepsilon/\sqrt{2} < \varepsilon \quad \forall n \geq N.$$

as required.)

<u>Cauchy sequence</u>: A sequence $\{z_n\}$ is said to be <u>Cauchy</u> if given $\varepsilon > 0$ $\exists N > 0$

$$\ni |z_n - z_m| < \varepsilon \quad \forall n, m \geq N$$

REMARK: 1 $\{z_n\}$ is Cauchy $\Leftrightarrow$ $\{\mathrm{Re}\, z_n\}$, $\{\mathrm{Im}\, z_n\}$ are Cauchy sequences (of real numbers)
(Pf same as earlier)

2 Every Cauchy seq in $\mathbb{C}$ converges.

(<u>Pf</u>: $\{z_n\}$ is Cauchy $\Rightarrow$ $\{\mathrm{Re}\, z_n\}$, $\{\mathrm{Im}\, z_n\}$ are Cauchy sequences (of real nos)

$$\Rightarrow \mathrm{Re}\, z_n \to a, \quad \mathrm{Im}\, z_n \to b$$

Hence $z_n = \mathrm{Re}\, z_n + i\,\mathrm{Im}\, z_n \longrightarrow a + ib$.)

3 If the sequence $\{z_n\}$ converges to l then $\{|z_n|\}$ converges to $|l|$.

(Pf: Given $\varepsilon > 0$ $\exists N > 0$ $\ni$ $|z_n - l| < \varepsilon$ $\forall n \geq N$

Since $||z_n| - |l|| \leq |z_n - l|$

we have $|z_n| \to |l|$)

SERIES:

A sum of the form $\sum_{n=0}^{\infty} z_n$ is called a series.

Corresponding to a series, we associate its sequence of <u>partial sums</u>:

Let $\sum_{n=0}^{\infty} z_n$ be a series. Then the k-th partial sum $S_k := \sum_{n=0}^{k} z_n$.

Eg: $\sum_{n=0}^{\infty} z^n$, $S_k = 1 + z + \cdots + z^{k-1} = \dfrac{1 - z^k}{1 - z}$ $(z \neq 1)$

The series $\sum_{n=0}^{\infty} z_n$ is said to converge to l if the sequence of partial sums $\{S_k\}$ converges.

Eg: $\sum_{n=0}^{\infty} z^n$, $\quad S_k = \dfrac{1-z^k}{1-z}$ (for $z \neq 1$)

if $|z|<1$ then $|z|^k \to 0$; hence so does the sequence z^k.

Thus, $\lim_{k\to\infty} \dfrac{1-z^k}{1-z} = \dfrac{1}{1-z}$

Remark: 1 If $\sum_{n=0}^{\infty} z_n$ converges then $\sum_{n=k}^{\infty} z_n$ is a null sequence; in particular, $\{z_n\}$ is a null sequence

(Indeed, $\sum_{n=0}^{\infty} z_n = l \iff$ given $\varepsilon > 0$ $\exists$ $N > 0$ $\ni$

$$|l - S_k| < \varepsilon \quad \forall k > N$$

$$\iff \left|\sum_{n=k+1}^{\infty} z_n\right| < \varepsilon \quad \forall\ k > N.$$)

since m^{th} partial sum is $S_m - S_k$ (& $\lim_{m\to\infty} S_m - S_k = l - S_k$)

In above Eg: $\sum_{n=0}^{\infty} z^n$ does not converge for $|z| \geq 1$ since $\{z^n\}$ is not a null sequence.

Remark 2: "The series $\sum_{n=0}^{\infty} z_n$ cgs $\iff$ $\{S_k\}$ is Cauchy;

ie, $\sum_{n=0}^{\infty} z_n$ cgs iff given $\varepsilon > 0$ $\exists$ $N > 0$ $\ni \left|\sum_{k=n+1}^{m} z_k\right| < \varepsilon$ $\forall\ m > n > N$.

(Easy)

Definition: <u>Absolutely convergent series.</u>

A series $\sum_{n=0}^{\infty} z_n$ is said to be absolutely convergent if $\sum_{n=0}^{\infty} |z_n|$ is convergent.

Eg: $\sum_{n=0}^{\infty} \frac{i^n}{n^2}$.

<u>Remark</u>(3): A series which is absolutely convergent is also convergent (ie $\boxed{\text{absolutely cgt} \Rightarrow \text{cgt}}$)

(Warning: For sequences: cgt $\not\Rightarrow$ abs cgt).

Indeed, if $\sum_{n=0}^{\infty} |z_n|$ is cgt, then by earlier Remark 2 $\sum_{n=m+1}^{k} |z_n| < \varepsilon \quad \forall\, k > m > N$.

$$\therefore \left| \sum_{n=m+1}^{k} z_n \right| \leq \sum_{n=m+1}^{k} |z_n| < \varepsilon \quad \forall\, k > m > N.$$

Hence, $\sum_{n=0}^{\infty} z_n$ is cgt.

<u>Comparison test</u>: Let $\sum_{n=0}^{\infty} |a_n|$ be a convergent series (of real numbers).

If $|z_n| \le |a_n| \;\forall\; n > N_1$ then

$$\sum_{n=0}^{\infty} z_n \text{ also converges.}$$

Pf: $\sum_{n=0}^{\infty} |a_n|$ converges, so given $\varepsilon > 0 \;\exists\; N_2 > 0 \ni$
$\sum_{k=n+1}^{m} |a_n| < \varepsilon \;\forall\; m \geq n > N_2$

Consider $|S_n - S_m|$ where S_k is the k^{th} partial sum.

$$|S_n - S_m| = \left|\sum_{k=m+1}^{n} z_k\right| \le \sum_{k=m+1}^{n} |z_k| \le \sum_{k=m+1}^{n} |a_n| < \varepsilon$$

$\forall\; n, m \geq \max\{N_1, N_2\}$

$\Rightarrow \{S_n\}$ is Cauchy hence convergent. ☒

Infact, $\sum_{n=0}^{\infty} z_n$ is absolutely convergent.

<u>Ratio test</u>: Let $\sum_{n=0}^{\infty} z_n$ be a series such that $\lim_{n\to\infty} \frac{|z_{n+1}|}{|z_n|}$ exists and is equal to 'l'. Then

(i) $\sum_{n=0}^{\infty} z_n$ converges if $l < 1$

(ii) $\sum_{n=0}^{\infty} z_n$ diverges if $l > 1$

(iii) when $l = 1$, the series may or may not converge.

Pf: (i) Let $l < 1$. Then given $\varepsilon > 0$ $\exists$ $N > 0$

$$\ni \left| \frac{|z_{n+1}|}{|z_n|} - l \right| < \varepsilon \quad \forall \, n \geq N$$

In particular, $\dfrac{|z_{n+1}|}{|z_n|} < l + \varepsilon \quad \forall \, n \geq N$

Since $l < 1$, we can choose $\varepsilon > 0$ small enough such that $l + \varepsilon < 1$

ie $$\frac{|z_{n+1}|}{|z_n|} < \underbrace{l + \varepsilon}_{\rho} < 1 \quad \forall \, n \geq N$$

$$\therefore \; |z_{n+1}| < \rho |z_n| \quad \forall \, n \geq N$$

Thus, $|z_n| < \rho |z_{n-1}| < \rho^2 |z_{n-2}| < \cdots < \rho^{n-N} |z_N|$

$\forall \, n \geq N$

If $\rho < 1$ then $\sum \rho^n$ is cgt; so by comparison test $\sum_{n=0}^{\infty} z_n$ cgs.

(ii) $l > 1$ then $\dfrac{|z_{n+1}|}{|z_n|} > l - \varepsilon \quad \forall \, n \geq N$

Choose $\varepsilon > 0$, small enough $\ni$ $l > l - \varepsilon > 1$

So, $\dfrac{|z_{n+1}|}{|z_n|} > l - \varepsilon > 1 \quad \forall\, n \geq N$

$\therefore\ |z_{n+1}| > |z_n| \ \forall\, n \geq N$

In particular $|z_n|$ is not a null sequence.
Hence $\sum_{n=0}^{\infty} z_n$ does not converge. $\blacksquare$

(iii) same as in real case.

<u>Root test:</u> Let $\sum_{n=0}^{\infty} z_n$ be a series such that $\limsup \sqrt[n]{|z_n|} = l$. Then

(i) if $l < 1$, $\sum_{n=0}^{\infty} z_n$ converges

(ii) if $l > 1$ $\sum_{n=0}^{\infty} z_n$ diverges

(iii) if $l = 1$ the series may or may not converge.

<u>Pf:</u> $\limsup \sqrt[n]{|z_n|} = l$

(by defn) $\Rightarrow$ every convergent subsequence of $\{\sqrt[n]{|z_n|}\}$ has limit $\leq l$.

(i) if $l < 1$ then there are almost finitely many terms of the sequence which are greater than l (<1)

$$\therefore \sqrt[n]{|z_n|} \leq l < r < 1 \quad \forall\, n \geq N$$

$$\therefore |z_n| < r^n \quad \forall\, n \geq N$$

$\Rightarrow$ by comparison $\sum_{n=0}^{\infty} |z_n|$ is cgt

$\Rightarrow \sum_{n=0}^{\infty} z_n$ is cgt.

(ii) $l > 1$, there exists a subsequence of $\{\sqrt[n]{|z_n|}\}$ such that it converges to $l > 1$

$\Rightarrow$ there are infinitely many terms of the sequence $\{\sqrt[n]{|z_n|}\}$ which are > 1.

$\Rightarrow \{z_n\}$ is not a null sequence.

(iii) $l = 1$ (as in real case; look for examples).

Appendix: lim sup of a real sequence.

Let $\{a_n\}$ be a sequence of real numbers.

Then $\limsup \{a_n\} = \sup\{$ limits of convergent subsequences $\}$

If limits and supremum can take value in $\mathbb{R} \cup \{\infty\}$, the above always exists.

Example 1: $\left\{ \sin \frac{n\pi}{2} \right\}$

$$= \{\underline{\underline{0}}, \underline{\underline{1}}, \underline{\underline{0}}, \underline{\underline{-1}}, \underline{\underline{0}}, \underline{\underline{1}}, \underline{\underline{0}}, \underline{\underline{-1}}, \dots \}$$

convergent subsequences are eventually $\begin{cases} \{0, 0, \dots\} \\ \{1, 1, \dots\} \\ \{-1, -1, \dots\} \end{cases}$

$\therefore \{$ limits of cgt subsequences $\} = \{0, -1, 1\}$

$$\limsup \left\{ \sin \frac{n\pi}{2} \right\} = 1$$

Example 2: $\left\{ 1, \frac{1}{2}, 3 + \frac{1}{3}, \frac{1}{4}, 5 + \frac{1}{5}, \dots \right\}$

odd term $a_{2n+1} = (2n+1) + \frac{1}{2n+1}$

even term $a_{2n} = \frac{1}{2n}$

$\{a_2, a_4, \dots\}$ converges to 0

$\{a_1, a_3, \dots\}$ diverges to ∞

$\limsup a_n = \infty$

Properties:

Let $\limsup\{a_n\} = l$, $\limsup\{b_n\} = l'$

1. $\limsup\{a_n + b_n\} \le l + l'$
2. if $l, l' > 0$ then
$$\limsup_{n\to\infty} a_n b_n \le l\,l'$$
3. $\limsup c\{a_n\} = c\,l$. $(c \geq 0)$