**MSO202A COMPLEX ANALYSIS**
**Assignment 5**

**Exercise Problems:**

1. Evaluate the integral $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{ze^{zt}}{(z+1)^3}\,dz$ where $C$ is a counter-clockwise oriented simple closed contour enclosing $z=-1$.

2. Write down the Taylor series centred at the given point for the following functions and find its disc of convergence:
$(i)$ $f(z)=\dfrac{1}{z^2}$ at $z_0\neq 0$ $\quad(ii)$ $f(z)=\dfrac{6z+8}{(2z+3)(4z+5)}$ at $z_0=1$
$(iii)$ $f(z)=\dfrac{e^z}{z+1}$ at $z_0=1$.

3. Let $f,g\colon\mathbb{C}\to\mathbb{C}$ be analytic functions such that $f(a_n)=g(a_n), n=1,2,\ldots$ for a bounded sequence of distinct complex numbers. Show that $f\equiv g$ on $\mathbb{C}$.

4. Derive the Taylor series representation of $\frac{1}{1-z}$ around $i$.
$$\frac{1}{1-z}=\sum_{n=1}^{\infty}\frac{(z-i)^n}{(1-i)^{n+1}} \text{ where } |z-i|<\sqrt{2}.$$

5. Let $f$ be analytic in a simply connected domain $D$ and $\gamma$ be a simple closed curve in $D$ oriented counterclockwise. Suppose $z_0$ is the only zero of $f$ in the region enclosed by $\gamma$. Show that $\int_\gamma \frac{f'(z)}{f(z)}dz=2\pi i m$ where $m$ is the order of zero of $f$ at $z_0$. (*If $f(z)=(z-z_0)^m g(z)$ where $g(z)$ is analytic at $z_0$ and $g(z_0)\neq 0$ then $f$ is said to have a zero of order $m$.*)

6. (Mean Value Theorem) Let $D$ be a simply connected domain and $f:D\to\mathbb{C}$ be an analytic function. Then prove that $f(z_0)=\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})d\theta$ for every $r>0$ such that $B(z_0,r)$ is contained in $D$.

7. Find the maximum of the function $|f|$ on $\overline{\mathbb{D}}$ if $(a)$ $f(z)=z^2-z$ $\;(b)$ $f(z)=\sin z$.

**Problem for Tutorial:**

8. Let $f$ be entire and $|f(z)|\le a+b|z|^n$ for some positive real numbers $a$ and $b$ and $n\in\mathbb{N}$. Show that $f$ is a polynomial of degree at most $n$.

9. Let $f\colon\mathbb{C}\to\mathbb{C}$ be a non-constant entire function. Let $z_0\in\mathbb{C}$ and $r>0$ be arbitrary. Show that the image of $f$ intersects the disc $B_r(z_0)=\{z:|z-z_0|<r\}$. (Hence image of a non-constant analytic function intersects every disc in $\mathbb{C}$. )

10. Let $f$ and $g$ be nonzero analytic functions defined on the disc $\mathbb{D}$ with $|f(z)|\le|g(z)|\forall z$. Assume that $z_0$ is a zero for $g(z)$ of order $n$. Show that $z_0$ is a zero for $f(z)$ of order at least $n$.