

# Contents

- 1 Lecture 4
  - Differentiability
  - Cauchy-Riemann equations

same as in the real case...but not “ $\mathbb{R}$ Real”ly!!!

Let  $f$  be a function defined in a neighbourhood of  $z_0$ . If  $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$  exists. Then  $f$  is said to be *differentiable at  $z_0$*  and the limit is denoted as  $f'(z_0)$ .

Eg:  $f(z) = z^n$

$$\lim_{z \rightarrow z_0} \frac{z^n - z_0^n}{z - z_0} \leftarrow z^{n-1} + z^{n-2}z_0 + \dots + z_0^{n-1}$$

$$\lim_{z \rightarrow z_0} z^i z_0^j = z_0^{i+j}$$

$$= n z_0^{n-1}$$

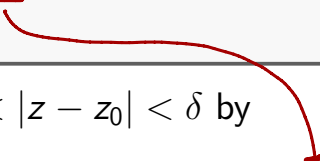
Given  $\epsilon > 0$  there exists a  $\delta > 0$  such that  
$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon \text{ for all } z \text{ such that } 0 < |z - z_0| < \delta.$$

Define  $\eta(z)$  for  $0 < |z - z_0| < \delta$  by

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0)$$

then  $\eta(z)$  is defined for  $z \neq z_0$ ; Set  $\eta(z_0) := 0$ . Then  $f$  is differentiable at  $z_0$  implies that  $\eta$  is continuous also at  $z_0$ .

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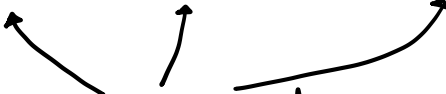
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$$\text{Given } \epsilon > 0 \exists \delta > 0 \ni \left| \eta(z) - \underbrace{\eta(z_0)}_0 \right| < \epsilon \quad \forall |z - z_0| < \delta.$$

Proposition: If  $f$  is differentiable at  $z_0$  then  $f$  is continuous at  $z_0$ .

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0)$$


continuous at  $z_0$

$f$  is differentiable on a domain  $D$  if it is differentiable at every  $z \in D$ .


## Arithmetic of differentiability

Let  $f$  and  $g$  be two functions differentiable at  $z$ . Then

- $(f \pm g)'(z) = f'(z) \pm g'(z)$ .
- $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$ .
- if  $g(z) \neq 0$  then
$$(f/g)'(z) = f'(z)g(z) - f(z)g'(z)/g(z)^2$$
- Chain rule hold for composition of differentiable functions.

1  $|z|$  at 0

$\lim_{z \rightarrow 0} \frac{|z|}{z}$  does not exist

  $\lim_{x \rightarrow 0} \frac{|x|}{x}$  does not exist

$$f'(0) = \lim_{z \rightarrow 0} \frac{|z|^2}{z} = \lim_{z \rightarrow 0} \frac{z \cdot \bar{z}}{z} = \lim_{z \rightarrow 0} \bar{z} = 0$$

1  $|z|$  at 0

2  $|z|^2$  at 0

$$\lim_{z \rightarrow z_0} \frac{|z|^2 - |z_0|^2}{z - z_0} = \lim_{h \rightarrow 0} \frac{|z_0 + h|^2 - |z_0|^2}{h}$$

$z_0 \neq 0$

$$\begin{aligned} |z_0 + h|^2 &= (z_0 + h)(\overline{z_0 + h}) = z_0 \bar{z}_0 + z_0 \bar{h} + \bar{z}_0 h + h \bar{h} \\ - |z_0|^2 &= z_0 \bar{z}_0 \\ \hline &= z_0 \bar{h} + \bar{z}_0 h + h \bar{h} \end{aligned}$$

$$\lim_{h \rightarrow 0} \left( \frac{z_0 \bar{h}}{h} + \bar{z}_0 + \bar{h} \right) \text{ does not exist.}$$



$$f'(0) = \lim_{h \rightarrow 0} \frac{\overline{h}}{h} \text{ does not exist}$$

1  $|z|$  at 0

2  $|z|^2$  at 0

3  $\boxed{\overline{z}}$

$$\lim_{h_1 \rightarrow 0} \frac{\overline{h_1}}{h_1} = 1 \neq \lim_{ih_2 \rightarrow 0} \frac{-ih_2}{ih_2} = -1$$

$$f'(z) = \lim_{h \rightarrow 0} \frac{\overline{z+h} - \overline{z}}{h} = \lim_{h \rightarrow 0} \frac{\overline{h}}{h} \text{ does not exist.}$$

1  $|z|$  at 0

2  $|z|^2$  at 0

3  $\bar{z}$

4  $\bar{z}/z$  at 0 (ex)

$f$  is differentiable at  $z = x + iy \iff u$  and  $v$   
differentiable at  $(x, y)$ ?

Turns out that the complex differentiability of  $f$  is stronger than real differentiability.

Complex differentiability of  $f \Rightarrow$  Real differentiability of  $u, v$ .

Real differentiability of  $u, v \not\Rightarrow$  Complex differentiability of  $f$ .

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Real differentiability of  $u, v \not\Rightarrow$  Complex differentiability of  $f$ .

Eg:  $f(z) = |z|^2$ . Here  $u$  and  $v$  are (real) differentiable everywhere but  $f$  is not differentiable anywhere except at 0.

(Recall,  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  is differentiable if  $u_x$  and  $u_y$  exist and are continuous.)

What conditions on  $u, v$  imply that  $f$  is differentiable?

What conditions on  $u$ ,  $v$  imply that  $f$  is differentiable?

Let's see what happens when  $f$  is differentiable..i.e., what is necessary for  $f$  to be differentiable?

$$f \text{ is differentiable at } z_0 \\ \Rightarrow \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \text{ exists}$$

Taking  $h = h_1 + i \cdot 0$ , we note that the

$$\lim_{h_1 \rightarrow 0} \frac{f(z_0 + h_1) - f(z_0)}{h_1} = u_x(x_0, y_0) + i v_x(x_0, y_0).$$

Taking  $h = 0 + i h_2$ ,

$$\lim_{h_2 \rightarrow 0} \frac{f(z_0 + i h_2) - f(z_0)}{i h_2} = \frac{1}{i} u_y(x_0, y_0) + i v_y(x_0, y_0) = v_y(x_0, y_0) - i u_y(x_0, y_0)$$

Hence, if  $f$  is differentiable at  $z_0$  then  $u_x$ ,  $u_y$ ,  $v_x$ ,  $v_y$  exist and satisfy

$$u_x = v_y \quad u_y = -v_x$$

There are the **Cauchy Riemann equations**.

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$f'(z_0)$

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Going back to ... What conditions on  $u, v$  imply that  $f$  is differentiable?

$f(z) = |z|^2$  is not differentiable  
at  $z_0 \neq 0$ .

$$u = x^2 + y^2, \quad v = 0$$

$$u_x = 2x \quad v_x = 0$$

$$u_y = 2y \quad v_y = 0$$

} CR-equations  
not satisfied.

$u, v$  differentiable + CR-equation  
 $\stackrel{?}{\Rightarrow} f$  is differentiable?

Going back to ... What conditions on  $u$ ,  $v$  imply that  $f$  is differentiable?

$f$  is differentiable at  $z_0$  if  $u_x$ ,  $u_y$ ,  $v_x$ ,  $v_y$  exist, are continuous and satisfy

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we skip the proof!!!!

Going back to ... What conditions on  $u$ ,  $v$  imply that  $f$  is differentiable?

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we skip the proof!!!!

Some consequences of the CR-equations:

Let  $f$  be differentiable at  $z$  such that  $f'(z) = 0$  then  
 $f$  is constant.

$\forall z \in U$   
open set

If  $|f|$  or  $\operatorname{Re}(f)$  or  $\operatorname{Im}(f)$  is constant then  $f$  is constant or it is not differentiable.

Some consequences of the CR-equations:

Let  $f$  be differentiable at  $z$  such that  $f'(z) = 0$  then  $f$  is constant.  
 $\forall z \in B_r(z_0)$ .

$$f'(z) = 0 \quad \forall z \in B_r(z_0)$$

$$u_x(x, y) + i v_x(x, y) = 0 \quad \forall (x, y) \in B_r(x_0, y_0).$$

$$\Rightarrow u_x(x, y) = 0 = v_x(x, y) \quad \forall (x, y) \in B_r(x_0, y_0).$$

$\Rightarrow u, v$  are independent of  $x$

$$\text{i.e. } u(x, y) = p(y), \quad v(x, y) = q(y)$$

$$f'(z) = v_y - i u_y \quad \therefore \quad v_y(x, y) = 0 = u_y(x, y) \quad \forall (x, y) \in B_r(x_0, y_0)$$

$$\Rightarrow p'(y) = 0 = q'(y)$$

# Polar form of the Cauchy-Riemann equations

$$x(r, \theta) = r \cos \theta; y(r, \theta) = r \sin \theta$$

$$\delta x / \delta r = \cos \theta; \delta x / \delta \theta = -r \sin \theta$$

$$\delta y / \delta r = \sin \theta; \delta y / \delta \theta = r \cos \theta$$

$$u = u(x, y); v = v(x, y)$$

$$u_r = u_x \delta x / \delta r + u_y \delta y / \delta r = u_x \cos \theta + u_y \sin \theta$$

$$v_r = v_x \delta x / \delta r + v_y \delta y / \delta r = v_x \cos \theta + v_y \sin \theta$$

$$u_\theta = u_x \delta x / \delta \theta + u_y \delta y / \delta \theta = -r u_x \sin \theta + r u_y \cos \theta$$

$$v_\theta = v_x \delta x / \delta \theta + v_y \delta y / \delta \theta = -r v_x \sin \theta + r v_y \cos \theta$$

Apply the CR equations and we get

$$u_r = \frac{1}{r} v_\theta; \quad v_r = -\frac{1}{r} u_\theta$$