Complex Analysis - MSO202A

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Contents

A *complex number* is an ordered pair of real numbers (x, y).
x is called the *real* part.
y is called the *imaginary* part.

$(x, y) = x(1, 0) + y(0, 1)$. (Recall that $\mathbb{R}^2$ is a vector space over $\mathbb{R}$)
Denoting $(0, 1)$ as i we have the representation $(x, y) = x + iy$. This is the representation we use!!

The real and imaginary parts don't interact over $+$.

$x + iy = x' + iy'$ if and only if $x = x'$, $y = y'$.

Definition of a complex number

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The set of all complex numbers is denoted by $\mathbb{C}$.

- $\mathbb{C}$ is a $\mathbb{R}$-vector space.
- Consider the maps
  $Re : \mathbb{C} \to \mathbb{R}$ given by $x + iy \mapsto x$.
  $Im : \mathbb{C} \to \mathbb{R}$ given by $x + iy \mapsto y$.

  They are $\mathbb{R}$-linear.
- $\mathbb{C}$ is a field: $(x_1 + iy_1) + (x_2 + iy_2) := x_1 + x_2 + i(y_1 + y_2)$
  $(x_1 + iy_1).(x_2 + iy_2) := x_1x_2 - (y_1y_2) + i(x_1y_2 + x_2y_1)$

- $z = x + iy$ is the vector $(\mathbf{x}, \mathbf{y})$ on the $\mathbb{R}^2$-plane (or complex plane).
- length of the z is the length of the line segment from $(\mathbf{0}, \mathbf{0})$ to $(\mathbf{x}, \mathbf{y})$. Denoted as $|z|$.

Length of z

For $z = x + iy$, we have $|z| = \sqrt{x^2 + y^2}$.
It is referred also as *modulus of z*.

- Reflection of z about the x-axis is called the conjugate of z, denoted as $\bar{z}$.

Conjugate of z

For $z = x + iy$, we have $\bar{z} = x - iy$

- Note that $|z|^2 = z.\bar{z} = \bar{z}.z$.
  And, $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$.

Properties of $\bar{z}$ and $|z|$

- $Re(z) = \frac{z+\bar{z}}{2}$, $Im(z) = \frac{z-\bar{z}}{2i}$;

  $|Re(z)| \leq |z|$ and $|Im(z)| \leq |z|$.
- $|z| = |\bar{z}|$.
- $|z| = 0$ if and only if $z = 0$.
- $|z_1 z_2| = |z_1||z_2|$, $\quad |\frac{z_1}{z_2}| = \frac{|z_1}{|z_2|}$
- Triangle inequality: $\quad |z_1 + z_2| \leq |z_1| + |z_2|$.
- Parallelogram identity: $\quad |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

Recall that every point on the real plane (except $(0,0)$) can be written in the form $(r\cos\theta, r\sin\theta)$ for a suitable $r \in \mathbb{R}^{>0}$ and $\theta \in \mathbb{R}$. This gives rise to the polar form of a complex number.

The polar form of a complex number $z \neq 0$ is the representation of z in its polar co-ordinates.

Polar form of $z = x + iy$

If $(x, y) = (r\cos\theta, r\sin\theta)$ in polar co-ordinates then $r(\cos\theta + i\sin\theta)$ is the polar form of z.

Notation

$e^{i\theta} := \cos\theta + i\sin\theta$.

So, polar form of $z = re^{i\theta}$.

Properties of polar form

- $re^{i\theta} = re^{i(\theta+2\pi n)}$ for any integer n.
- $(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$.
- $\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$.

Polar form of z

Let $z = re^{i\theta}$.

$|z| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r$

θ is called the *argument of z*. Denoted as $\arg(z)$.

It is a multi-valued function from $\mathbb{C}^* \to \mathbb{R}$.

The *principal value* of $\arg(z)$ is the unique value of $\arg(z)$ satisfying $-\pi < \arg(z) \leq \pi$. It is denoted as $\mathrm{Arg}(z)$.

Properties of $\arg(z)$

- $re^{i\theta} = r'e^{i\theta'} \iff r = r'$ and $\theta = \theta' + 2n\pi$.
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) (mod\ 2\pi)$.
- $\arg(\frac{z_1}{z_2}) = \arg(z_1) - \arg(z_2) (mod\ 2\pi)$

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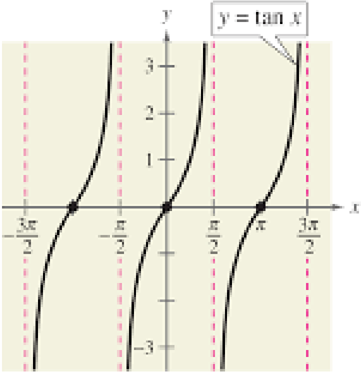


PERIOD: π

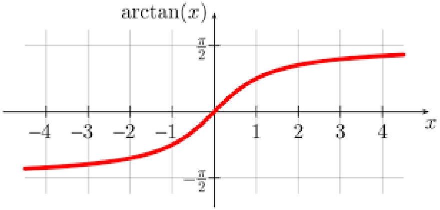
DOMAIN: ALL $x \neq \frac{\pi}{2} + n\pi$

RANGE: $(-\infty, \infty)$

VERTICAL ASYMPTOTES: $x = \frac{\pi}{2} + n\pi$

SYMMETRY: ORIGIN

If $x + iy = re^{i\theta}$ and $-\pi/2 < \theta < \pi/2$ then $\tan\theta = (y/x)$.



If $x + iy = re^{i\theta}$ and $-\pi/2 < \theta < \pi/2$ then $\theta = \tan^{-1}(y/x)$.
Since $\frac{y}{x} = \frac{-y}{-x}$, so,

Convention

$$\mathrm{Arg}(z) = \begin{cases} \tan^{-1}(y/x), & \text{if } x > 0 \\ \pi + \tan^{-1}(y/x), & \text{if } x < 0, y \geq 0 \\ -\pi + \tan^{-1}(y/x), & \text{if } x < 0, y < 0 \\ -\dfrac{\pi}{2}, & \text{if } x = 0, y < 0 \\ \dfrac{\pi}{2}, & \text{if } x = 0, y > 0 \end{cases} \tag{1}$$

de Moivre's Theorem

If m is any integer then

$$(e^{i\theta})^m = e^{im\theta},$$

$$i.e., \ (\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta.$$

The roots of $\omega^n = z = re^{i\theta}$ are given by $\sqrt[n]{r}e^{i(\theta+2k\pi)/n}$ where $k = 0, 1, \ldots, n-1$.