LECTURE - 8

CAUCHY'S THEOREM.

Lecture 8: Cauchy's theorem

We saw in the last lecture that if f has an antiderivative in a domain D, then for any closed contour C in D we have

$$\int_C f(z)\,dz = 0 .$$

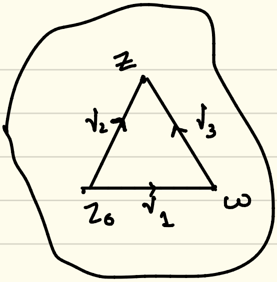
It turns out that $\int_C f(z)\,dz = 0$ is equivalent to f having an antiderivative.

(Indeed, if $\int_C f(z)\,dz = 0$ for any closed contour C in D then $\int_{z_1}^{z_2} f(z)\,dz$ is independent of the contour. Now, define for $z \in D$

$$F(z) := \int_{z_0}^{z} f(\zeta)\,d\zeta .$$

Consider $\dfrac{F(z)-F(\omega)}{z-\omega} - f(\omega)$

$$= \frac{\int_{z_0}^{z} f(\zeta)\,d\zeta - \int_{z_0}^{\omega} f(\zeta)\,d\zeta}{z-\omega} - f(\omega) \qquad (*)$$



$$\int_{\gamma_1+\gamma_3-\gamma_2} f(z)\,dz = 0 \quad (\text{given}).$$

$$\Rightarrow \int_{\gamma_3} f(z)\,dz = \int_{\gamma_2} f(z)\,dz - \int_{\gamma_1} f(z)\,dz$$

So, (*) becomes

$$\left(\frac{\int_{\omega}^{z} f(\zeta)\,d\zeta}{z-\omega}\right) - f(\omega)$$

$$\therefore \left| \frac{F(z)-F(\omega)}{z-\omega} - f(\omega) \right| = \frac{1}{|z-\omega|} \left| \int_{\omega}^{z} \big(f(\zeta)-f(\omega)\big)\,d\zeta \right|$$

$$\leq \frac{1}{|z-\omega|} \cdot |z-\omega| \sup_{\zeta \in \overline{z\omega}} |f(\zeta)-f(\omega)|$$

as $z \to \omega$, $|f(\zeta)-f(\omega)| \to 0$ $(\because |\zeta-\omega| \leq |z-\omega|)$

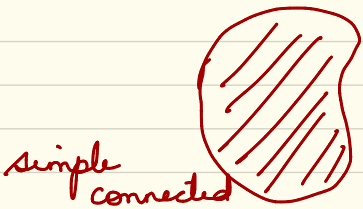
<u>Question</u>: Under what condition on f, do we have $\int_C f(z)dz = 0$ for any closed contour C.

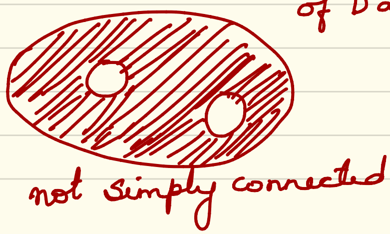
For simplicity of our discussion we always consider simple closed contours.

The answer to the above question is a centre-piece in complex analysis: CAUCHY'S THEOREM

<u>THEOREM</u>: (Cauchy's theorem): Let f be an analytic function on a simply connected domain D and C be a simple closed contour lying in D then $\int_C f(z)dz = 0$

Defn: every simple closed contour in D contains points of D alone.





(We make use of Green's theorem so we assume that f' is also continuous. The proof of the general statement involves topological arguments. So the proof will not be discussed).

Pf: Let $f(z) = u(x,y) + iv(x,y)$.

Let $\gamma(t) = x(t) + iy(t)$, $a \le t \le b$, be the contour C.

Then $\displaystyle\int_a^b f(\gamma(t))\,\gamma'(t)\,dt$

$$= \int_a^b \Big[u(x(t),y(t)) + iv(x(t),y(t))\Big]\Big[x'(t) + iy'(t)\Big]dt$$

$$= \int_a^b (ux' - vy')(t)\,dt + i\int_a^b (vx' + uy')(t)\,dt.$$

$$= \int_a^b (u\,dx - v\,dy) + i\int_a^b (v\,dx + u\,dy)$$

Green's theorem:

$$\int_{\partial D} M\,dx + N\,dy = \iint_D (N_x - M_y)\,dx\,dy.$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

$$= \iint_R (-v_x - u_y)\,dx\,dy + i\iint_R (u_x - v_y)\,dx\,dy$$

where R is the region enclosed in C.

$= 0$ (by CR-equations)

Consequences of Cauchy's theorem:

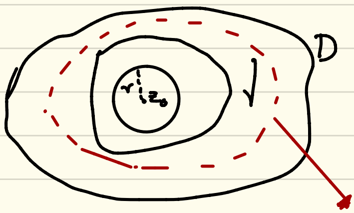
1. Existence of anti-derivative: (already seen)

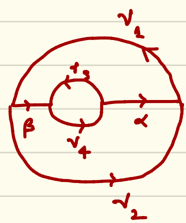
2. Independence of path: $\int_{z_1}^{z_2} f(z)\,dz$ is independent of path chosen from z_1 to z_2.

3. **Deformation theorem:**

Let f be holomorphic on a simply connected domain $D\setminus\{z_0\}$. Let γ be a closed contour in D containing z_0 be a point in the region enclosed by γ.

$$\text{Then } \int_{\gamma} f(z)\,dz = \int_{C_{z_0,r}} f(z)\,dz$$





$C_1 = \alpha + \gamma_1 + \beta - \gamma_3$ is a closed contour

$C_2 = \alpha - \gamma_2 + \beta + \gamma_4$, , , . . .

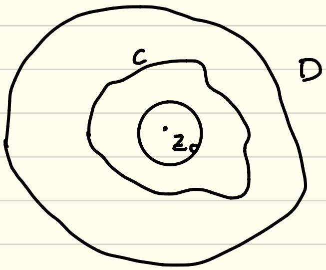
$$\int_{C_1} f(z)\,dz = 0 = \int_{C_2} f(z)\,dz \Rightarrow \int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_3+\gamma_4} f(z)\,dz$$

Cauchy Integral Formula:

Let f be an analytic fn on a simply connected domain D. Suppose $z_0 \in D$ and C be a simple closed curve in D enclosing z_0.

Then $$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0).$$

C (oriented anti-clockwise)



Pf: By deformation theorem $$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_{z_0,r}} \frac{f(z)}{z-z_0}\,dz$$

$$= \int_0^{2\pi} \frac{f(z_0+re^{it})}{re^{it}}\, rie^{it}\,dt$$

$$= i\int_0^{2\pi} f(z_0+re^{it})\,dt$$

Hence, $\left| \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz - f(z_0) \right|$

$$= \frac{1}{2\pi} \left| \int_0^{2\pi} \left(f(z_0 + re^{it}) - f(z_0) \right) dt \right|$$

$$\leq \frac{1}{2\pi} \times 2\pi \times \sup_{t \in [0, 2\pi]} \left| f(z_0 + re^{it}) - f(z_0) \right| \quad - (*)$$

Since f is continuous on D, in particular at z_0,

given $\varepsilon > 0$ $\exists$ $\delta > 0 \ni$ $|f(\zeta) - f(z_0)| < \varepsilon$
$\forall$ $|\zeta - z_0| < \delta$

if $r < \delta$, $(*)$ above becomes $\leq \varepsilon$.

Thus, $\boxed{\frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz = f(z_0)}$.

<u>Example:</u> $\int_{C_{0,5}} \frac{\cos z}{z} dz = 2\pi i (\cos 0) = 2\pi i$

Cauchy Integral Formula II:

Theorem 3: If f is analytic on a simply connected domain D then f has derivatives of all orders in D; for any $z_0 \in D$,

$$f^{n}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

where C is a simple closed contour (oriented counterclockwise) around z_0 in D.

Proof: Using CIF,

$$f'(z_0) = \operatorname*{Lt}_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

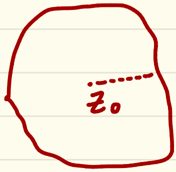
$$= \lim_{h\to 0}\left[\frac{\displaystyle\int_C \frac{f(z)}{z-(z_0+h)}\,dz - \int_C \frac{f(z)}{z-z_0}\,dz}{2\pi i h}\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{2\pi i h}\int_C \frac{f(z)\left[(z-z_0) - [z-(z_0+h)]\right]}{(z-z_0)(z-(z_0+h))}\right]$$

$$= \lim_{h \to 0} \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)(z-(z_0+h))} \, dz$$

Let $\alpha = \min\limits_{z \in C} |z - z_0| > 0$

$(\because z_0 \notin C$ & C is closed) and bounded



Then $\alpha \le |z - z_0| = |z - z_0 - h + h|$

$$\le |z-(z_0+h)| + |h|$$

For $|h| < \frac{\alpha}{2}$, $\quad |z - (z_0+h)| > \alpha/2$

We wish to show that $f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} dz$

Consider $\left| \int_C \left[\frac{f(z)}{(z-z_0)(z-z_0+h)} - \frac{f(z)}{(z-z_0)^2} \right] dz \right|$

$$\left| \int_C \frac{f(z)h}{(z-z_0)^2(z-z_0+h)} \, dz \right|$$

$$\left|\frac{f(z)}{(z-z_0)^2(z-(z_0+h))}\right| \leq \frac{|f(z)|}{\alpha^2 \cdot \alpha/2}$$

$$\therefore \sup_{z \in C} \frac{2|f(z)|}{\alpha^3} = M$$

By ML-inequality, we get

$$\left| \frac{f(z_0+h) - f(z_0)}{h} - \frac{1}{2\pi i}\int \frac{f(z)}{(z-z_0)^2}\,dz \right|$$

$$= \left| \frac{1}{2\pi i}\int_C \frac{f(z)h}{(z-z_0)^2(z-(z_0+h))}\,dz \right| \leq \frac{M \cdot h \cdot l}{2\pi}$$

where l = length of C.

The RHS $\to 0$ as $h \to 0$

$$\therefore f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$$

A similar argument as above for arbitrary $n > 0$, gives

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = 2\pi i\, f(z_0) \quad \text{if } n=0$$
$$= 2\pi i\, f^{(n)}(z_0) \quad \text{if } n \geq 1.$$

<u>REMARK</u>: If z_0 is not contained in the region enclosed by C then the above integral is 0 (by Cauchy's theorem)