Contents

same as in the real case...but not "$\mathbb{R}$eal"ly!!!

Let f be a function defined in a neighbourhood of z_0. If $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists. Then f is said to be *differentiable at* z_0 and the limit is denoted as $f'(z_0)$.

Eg: $f(z) = z^n$

$$\lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0}$$

$$z^{n-1} + z^{n-2} z_0 + \cdots + z_0^{n-1}$$

$$\lim_{z \to z_0} z^i z_0^j = z_0^{i+j}$$

$$= n z_0^{n-1}$$

Given $\epsilon > 0$ there exists a $\delta > 0$ such that $|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)| < \epsilon$ for all z such that $0 < |z - z_0| < \delta$.

Define $\eta(z)$ for $0 < |z - z_0| < \delta$ by

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0)$$

then $\eta(z)$ is defined for $z \neq z_0$; Set $\eta(z_0) := 0$. Then f is differentiable at z_0 implies that η is continuous also at z_0.

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Given $\varepsilon > 0 \ \exists\ \delta > 0 \ni |\eta(z) - \underbrace{\eta(z_0)}_{=0}| < \varepsilon \ \forall\ |z - z_0| < \delta$.

Proposition: If f is differentiable at z_0 then f is continuous at z_0.

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0)$$

continuous at z_0

f is differentiable on a domain D if it is differentiable at every $z \in D$.

Arithmetic of differentiability

Let f and g be two functions differentiable at z. Then

- $(f \pm g)'(z) = f'(z) \pm g'(z)$.
- $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.
- if $g(z) \neq 0$ then $(f/g)'(z) = f'(z)g(z) - f(z)g'(z)/g(z)^2$
- Chain rule hold for composition of differentiable functions.

1. $|z|$ at 0

$$\underset{z \to 0}{\mathrm{Lt}} \frac{|z|}{z} \text{ does not exist}$$

$$\underset{x \to 0}{\mathrm{Lt}} \frac{|x|}{x} \text{ does not exist}$$

$$f'(0) = \lim_{z \to 0} \frac{|z|^2}{z} = \lim_{z \to 0} \frac{z \cdot \bar{z}}{z} = \lim_{z \to 0} \bar{z} = 0$$

1. $|z|$ at 0
2. $\boxed{|z|^2 \text{ at } 0}$

$$\lim_{\substack{z \to z_0 \\ \neq 0}} \frac{|z|^2 - |z_0|^2}{z - z_0} = \lim_{h \to 0} \frac{|z_0+h|^2 - |z_0|^2}{h}$$

$$|z_0+h|^2 = (z_0+h)\left(\overline{z_0+h}\right) = z\bar{z}_0 + z_0\bar{h} + \bar{z}_0 h + h\bar{h}$$

$$-\ |z_0|^2 = z_0\bar{z}_0$$

$$= z_0\bar{h} + \bar{z}_0 h + h\bar{h}$$

$$\lim_{h \to 0}\left(\frac{z_0\bar{h}}{h} + \bar{z}_0 + \bar{h}\right) \text{ does not exist.}$$

1. $|z|$ at 0
2. $|z|^2$ at 0
3. $\boxed{\bar{z}}$

$$f'(0) = \lim_{h \to 0} \frac{\bar{h}}{h} \quad \text{does not exist}$$

$$\lim_{h_1 \to 0} \frac{\bar{h}_1}{h_1} = 1 \neq \lim_{ih_2 \to 0} \frac{-ih_2}{ih_2} = -1$$

$$f'(z) = \lim_{h \to 0} \frac{\overline{z+h} - \bar{z}}{h} = \lim_{h \to 0} \frac{\bar{h}}{h}$$

does not exist.

1. $|z|$ at 0
2. $|z|^2$ at 0
3. $\bar{z}$
4. $\bar{z}/z$ at 0 (Ex)

f is differentiable at $z = x + iy \quad \longleftrightarrow \quad u$ and v differentiable at (x, y)?

Turns out that the complex differentiability of f is stronger than real differentiability.

Complex differentiability of $f \Rightarrow$ Real differentiability of u, v.
Real differentiability of u, $v \not\Rightarrow$ Complex differentiability of f.

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> Complex differentiability of $f \Rightarrow$ Real differentiability of u, v.
> Real differentiability of u, $v \not\Rightarrow$ Complex differentiability of f.

Eg:$f(z) = |z|^2$. Here u and v are (real) differentiable everywhere but f is not differentiable anywhere except at 0.

(Recall, $u : \mathbb{R}^2 \to \mathbb{R}$ is differentiable`if´u_x and u_y exist and are continuous.)

What conditions on u, v imply that f is differentiable?

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Let's see what happens when f is differentiable..*i*.e., what is necessary for f to be differentiable?

f is differentiable at z_0

$$\Rightarrow \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \text{ exists}$$

Taking $h = h_1 + i.0$, we note that the

$$\lim_{h_1 \to 0} \frac{f(z_0 + h_1) - f(z_0)}{h_1} = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Taking $h = 0 + ih_2$,

$$\lim_{h_2 \to 0} \frac{f(z_0 + ih_2) - f(z_0)}{ih_2} = \frac{1}{i} u_y(x_0, y_0) + iv_y(x_0, y_0) = v_y(x_0, y$$

$= v_y - iu_y$

Hence, if f is differentiable at z_0 then u_x, u_y, v_x, v_y exist and satisfy

$$u_x = v_y \qquad u_y = -v_x$$

There are the **Cauchy Riemann equations**.

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$=$

$f'(z_0)$

$=$

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$f(z) = |z|^2$ is not differentiable at $z_0 \neq 0$.

$u = x^2 + y^2$, $v = 0$

$u_x = 2x \qquad v_x = 0$

$u_y = 2y \qquad v_y = 0$

} CR-equations not satisfied.

u, v differentiable + CR- equation $\overset{?}{\Longrightarrow}$ f is differentiable?

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we skip the proof!!!!!

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Some consequences of the CR-equations:

Let f be differentiable $\underbrace{\text{at } z \text{ such that } f'(z) = 0}_{\forall z \in \underset{\text{open set}}{U}}$ then f is constant.

If $|f|$ or $Re(f)$ or $Im(f)$ is constant then f is constant or it is not differentiable.

Some consequences of the CR-equations:

Let f be differentiable at $\underbrace{z \text{ such that } f'(z)}_{\forall z \in B_r(z_0).} = 0$ then f is constant.

$f'(z) = 0 \quad \forall z \in B_r(z_0)$

$u_x(x,y) + i v_x(x,y) = 0 \quad \forall (x,y) \in B_r(x_0,y_0).$

$\Rightarrow u_x(x,y) = 0 = v_x(x,y) \quad \forall (x,y) \in B_r(x_0,y_0).$

$\Rightarrow u, v$ are independent of x

ie $u(x,y) = p(y)$, $v(x,y) = q(y)$

$f'(z) = v_y - i u_y \quad \therefore \quad v_y(x,y) = 0 = u_y(x,y) \quad \forall (x,y) \in B_r(x_0,y_0)$

$\Rightarrow p'(y) = 0 = q'(y)$

Polar form of the Cauchy-Riemann equations

$$x(r,\theta) = r\cos\theta; y(r,\theta) = r\sin\theta$$
$$\delta x/\delta r = \cos\theta; \delta x/\delta\theta = -r\sin\theta$$
$$\delta y/\delta r = \sin\theta; \delta y/\delta\theta = r\cos\theta$$
$$u = u(x,y); v = v(x,y)$$
$$u_r = u_x\delta x/\delta r + u_y\delta y/\delta r = u_x\cos\theta + u_y\sin\theta$$
$$v_r = v_x\delta x/\delta r + v_y\delta y/\delta r = v_x\cos\theta + v_y\sin\theta$$
$$u_\theta = u_x\delta x/\delta\theta + u_y\delta y/\delta\theta = -ru_x\sin\theta + ru_y\cos\theta$$
$$v_\theta = v_x\delta x/\delta\theta + v_y\delta y/\delta\theta = -rv_x\sin\theta + rv_y\cos\theta$$

Apply the CR equations and we get

$$u_r = \frac{1}{r}v_\theta; \qquad v_r = -\frac{1}{r}u_\theta$$