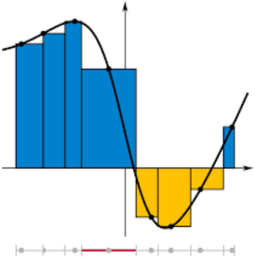
Contents



$$\sum_{i=0}^{n-1} f(c_i)(x_{i+1} - x_i)$$

$$c_i \in [x_i, x_{i+1}]$$

The Riemann–Stieltjes integral of a real-valued function f of a real variable with respect to a real function g is denoted by

$$\int_a^b f(x)\, dg(x)$$

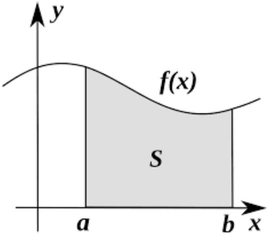
and defined to be the limit, as the norm of the partition

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

of the interval $[a, b]$ approaches zero, of the approximating sum

$$S(P, f, g) = \sum_{i=0}^{n-1} f(c_i)(g(x_{i+1}) - g(x_i))$$

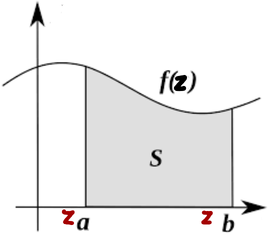
where c_i is in the i-th subinterval $[x_i, x_{i+1}]$. The two functions f and g are respectively called the integrand and the integrator.

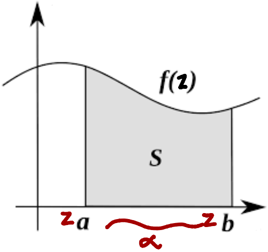


$$\int_a^b f(x)dx$$

$$S(P,f) = \sum f(c_i)(x_{i+1} - x_i)$$

$f : \mathbb{C} \to \mathbb{C}$



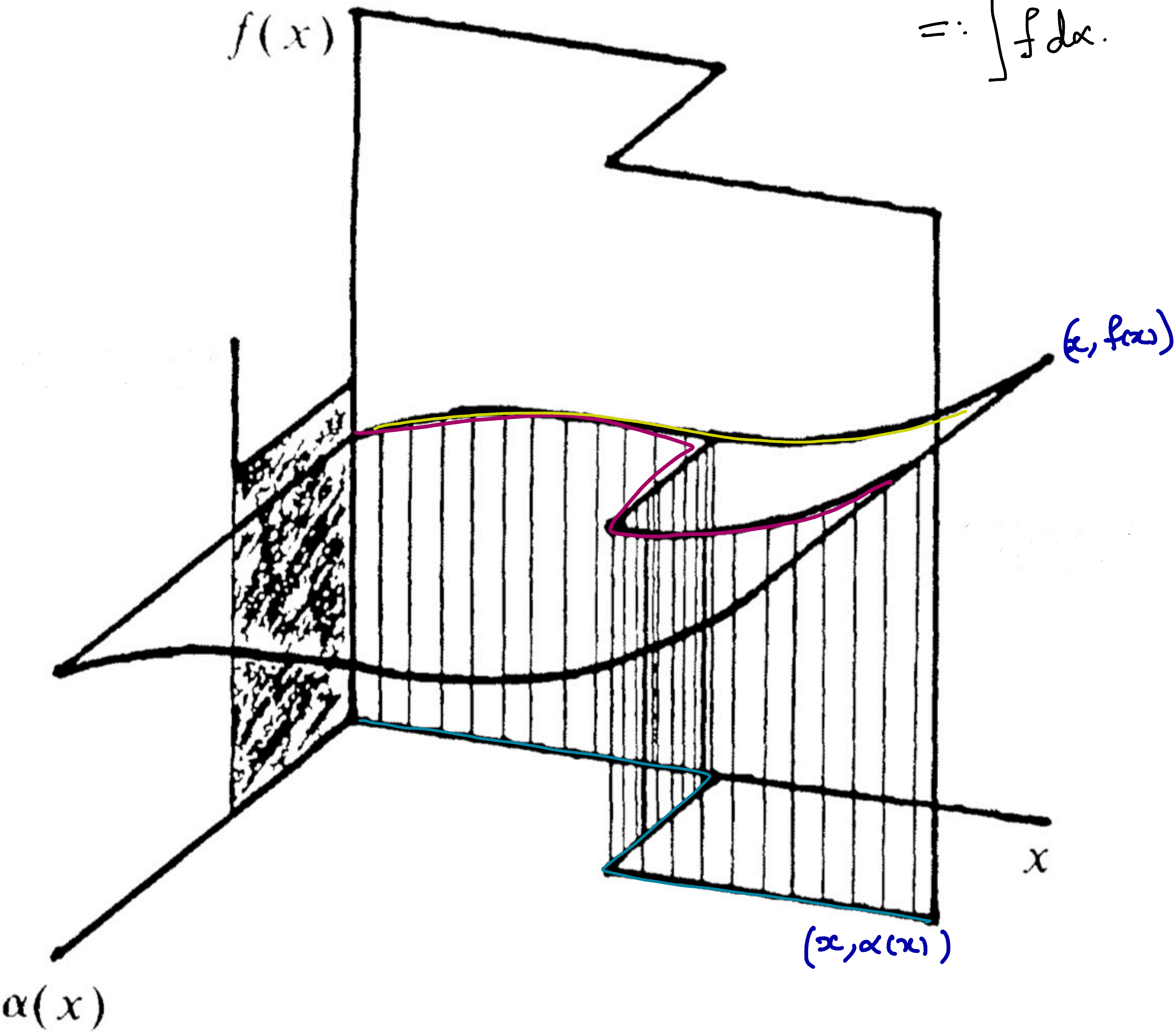


$f : \mathbb{C} \to \mathbb{C}$

$$\int_{z_a}^{z_b} f(z)dz := \int_{\alpha} f(z)dz$$

$f : \mathbb{R} \to \mathbb{R}$

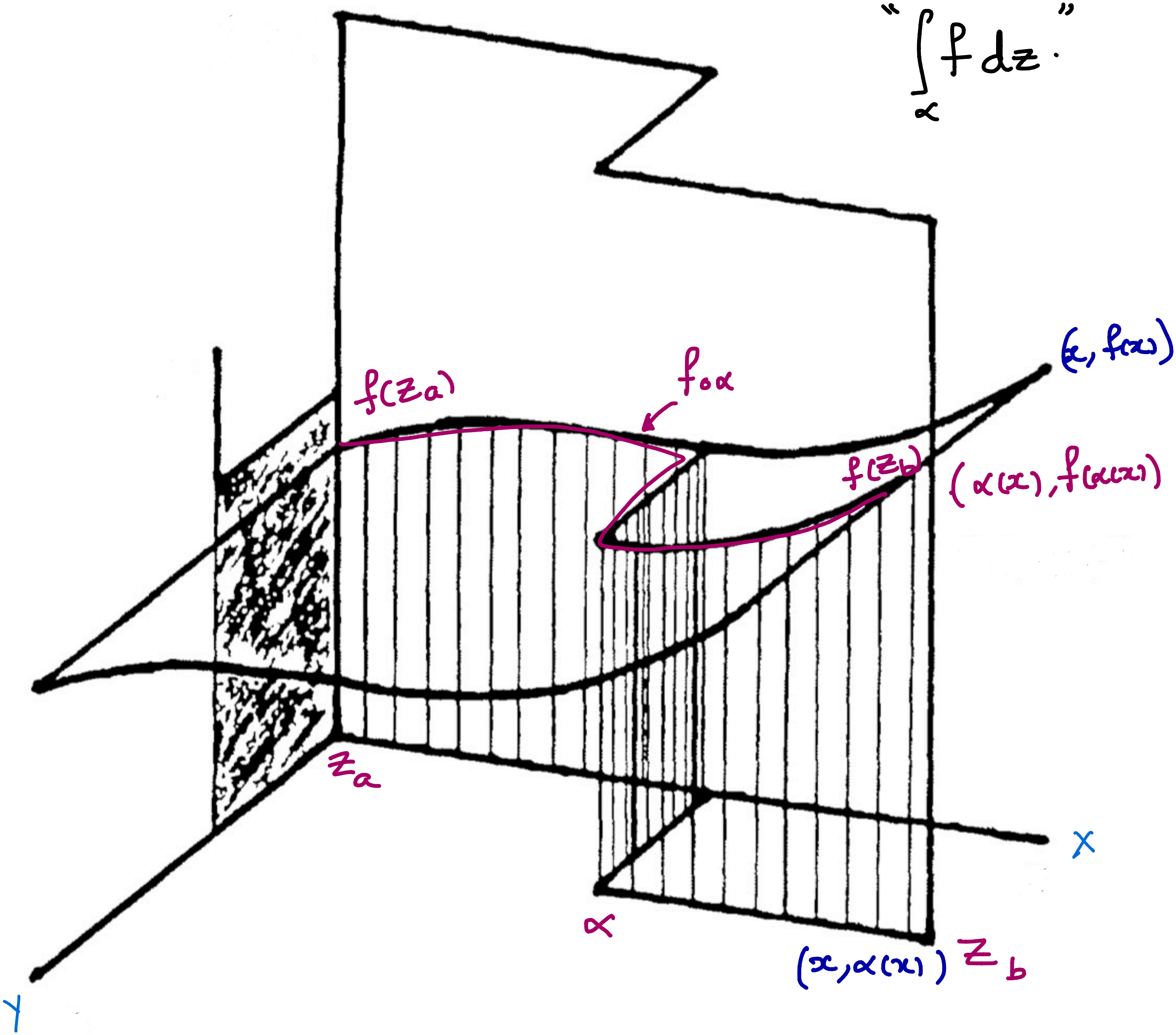
$$\sum_i f(\alpha(\xi_i))\,[\alpha(x_{i+1}) - \alpha(x_i)] =: \int f\, d\alpha.$$

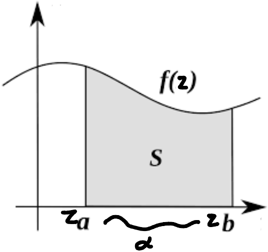


$f : \mathbb{C} \to \mathbb{C}$

$$\sum_i f(\alpha(\xi_i))\left[\alpha(x_{i+1}) - \alpha(x_i)\right]$$

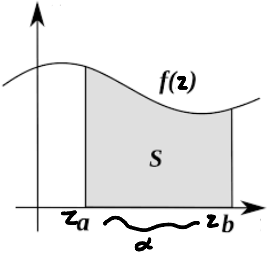
$$\text{“}\int_\alpha f\,dz\text{”}$$





$$\int_{z_a}^{z_b} f(z)dz := \int_{\alpha} f(z)dz = \int_a^b f(\alpha(t))\alpha'(t)dt$$

$$S(P, f, \alpha) = \sum f(\alpha(\zeta_i))(\alpha(x_{i+1}) - \alpha(x_i))$$



$$\int_{z_a}^{z_b} f(z)dz := \int_{\alpha} f(z)dz := \int_a^b f(\alpha(t))\alpha'(t)dt$$

definition

working definition

$$\cancel{S(P, f, \alpha) = \sum f(\alpha(\zeta_i))(\alpha(x_{i+1}) - \alpha(x_i))}$$

Contour Integral

$$\int_{\alpha} f(z)dz := \int_{a}^{b} f(\alpha(t))\alpha'(t)dt$$

$$\int_{\alpha} f(z)dz := \int_a^b f(\alpha(t))\alpha'(t)dt$$

What is a contour?

$$\int_{\alpha} f(z)dz := \int_a^b f(\alpha(t))\alpha'(t)dt$$

What is a contour?

Is the integral independent of the choice of the contour??

How does one evaluate this integral???

$$\int_{\alpha} f(z)dz := \int_a^b f(\alpha(t))\alpha'(t)dt$$

What is a contour?

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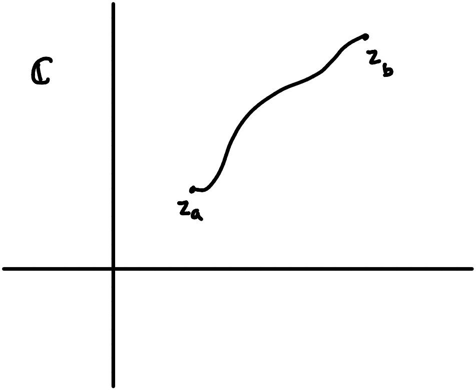
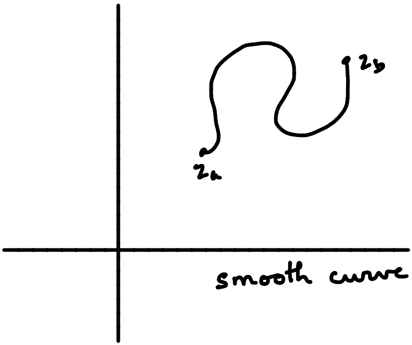
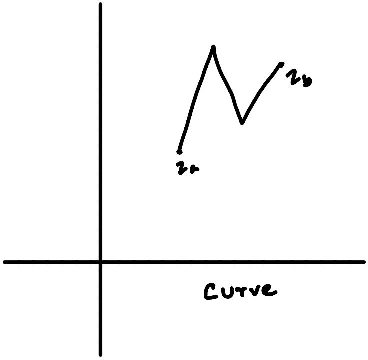
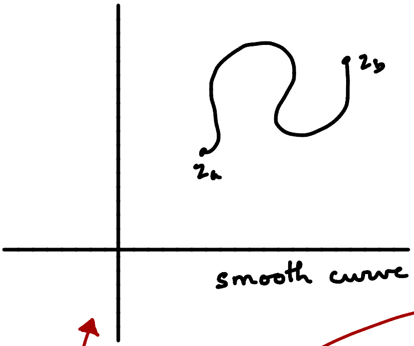


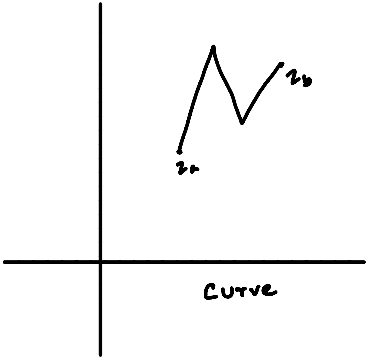
Contour ?





Curve:
Smooth Curve
Contour



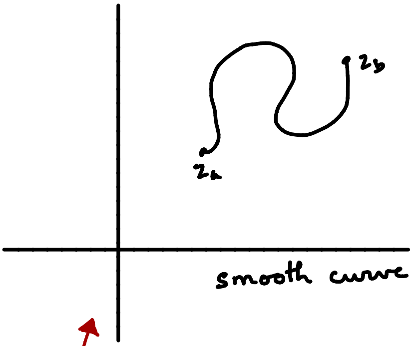


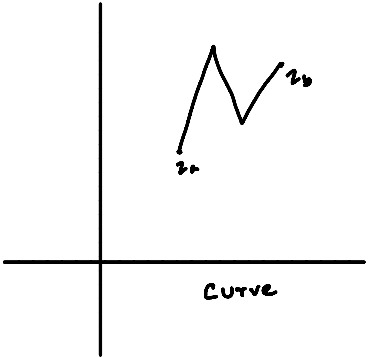
$\alpha : [a, b] \xrightarrow[\text{continuous}]{} \mathbb{C} \ni \alpha(a) = z_a \; ; \; \alpha(b) = z_b .$

Curve:

Smooth Curve

Contour



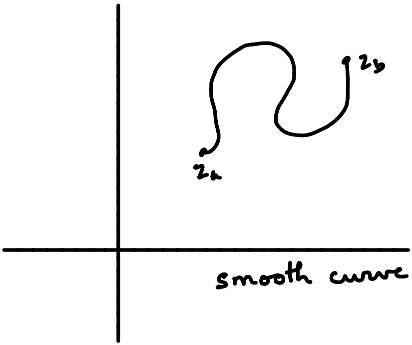


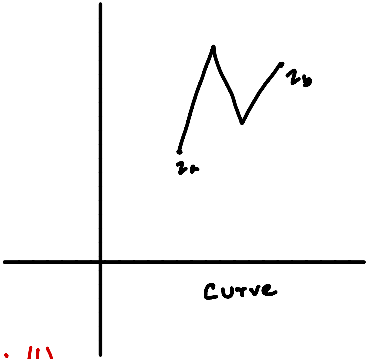
Curve:

Smooth Curve

Contour

$\alpha : [a,b] \to \mathbb{C} \ni \alpha(a) = z_a; \alpha(b) = z_b$. & α' exists and is continuous.





$\alpha(t) = x(t) + i y(t)$

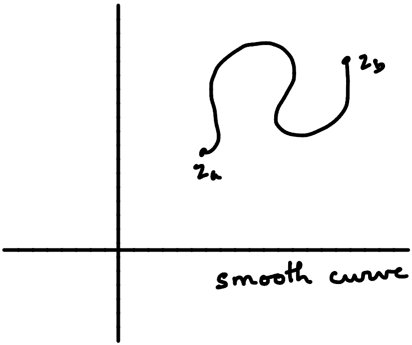
$\alpha'(t) = x'(t) + i y'(t)$

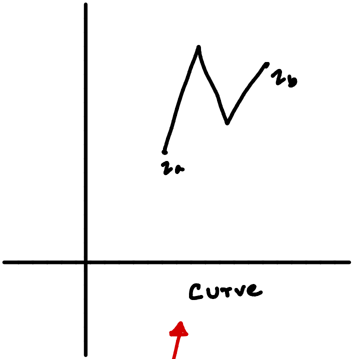
Curve:

Smooth Curve

Contour

$\alpha : [a,b] \to \mathbb{C} \ni \alpha(a) = z_a ; \alpha(b) = z_b$. & α' exists and is continuous.





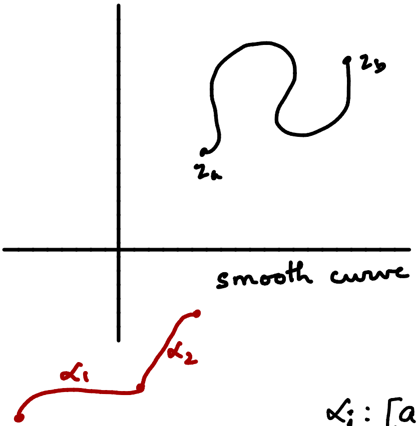
contour

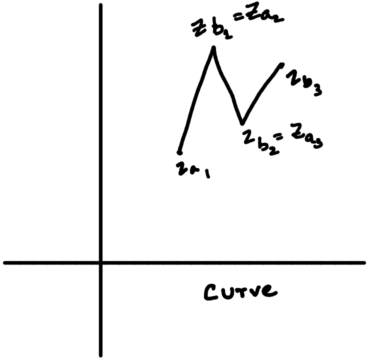
Curve:

Smooth Curve

Contour

Finitely many curves "joined".





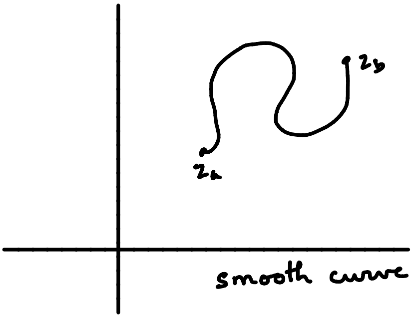
$$\alpha_i : [a_i, b_i] \to \mathbb{C} \quad , \; i = 1, 2$$

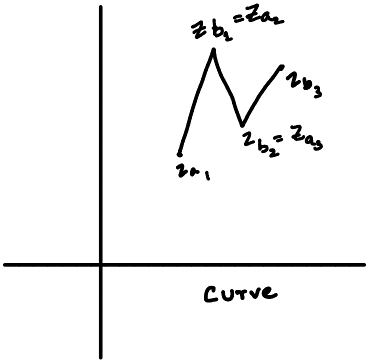
$$\ni \boxed{\alpha_1(b_1) = \alpha_2(a_2)} \quad , \; i = 1, 2$$

Curve:

Smooth Curve

Contour





$$\alpha_i : [a_i, b_i] \to \mathbb{C}, \quad i = 1, 2$$

$$\ni \alpha_1(b_1) = \alpha_2(a_2), \quad i = 1, 2$$

Join of α_1 & α_2

Then $\alpha_1 \sqcup \alpha_2 = \alpha : [a_1, b_1 + (b_2 - a_2)] \to \mathbb{C}$

$$\alpha(t) \begin{cases} = \alpha_1(t) \ \forall\, a_1 \le t \le b_1 \\ = \alpha_2(a_2 - b_1 + t) \ \forall\, b_1 \le t \le b_1 + (b_2 - a_2). \end{cases}$$

Curve:

Smooth Curve

Contour

$$\int_\alpha f(z)dz := \int_a^b \underbrace{f(\alpha(t))\alpha'(t)}_{?}dt$$

$g : [a,b] \to \mathbb{C}$

$g(t) = g_1(t) + i g_2(t)$

Then $\int_a^b g(t)\,dt := \int_a^b g_1(t)\,dt + i\int_a^b g_2(t)\,dt$

$$\int_\alpha f(z)dz := \int_a^b \underbrace{f(\alpha(t))\alpha'(t)}_{?}dt$$

$\alpha : [a,b] \longrightarrow \mathbb{C}$

$\alpha(t) = x(t) + i\,y(t)$

$\alpha'(t) = x'(t) + i\,y'(t)$

$f(\alpha(t)) = u(t) + i\,v(t)$

$$\int_\alpha f(z)dz := \int_a^b \underbrace{f(\alpha(t))\alpha'(t)dt}_{?}$$

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$$f(\alpha(t))\,\alpha'(t) = u(t)x'(t) - v(t)y'(t) + i\,\big(u(t)\,y'(t) + v(t)\,x'(t)\big)$$

$$= g_1(t) + i\,g_2(t).$$

$$\int_\alpha f(z)dz := \int_a^b f(\alpha(t))\alpha'(t)dt$$

$$\alpha : [a,b] \longrightarrow \mathbb{C}$$

$$f(\alpha(t)) = u(t) + i\,v(t)$$

$$\alpha(t) = x(t) + i\,y(t)$$

$$\alpha'(t) = x'(t) + i\,y'(t)$$

$$\int_a^b f(\alpha(t))\,\alpha'(t) = u(t)x'(t) - v(t)y'(t) + i\big(u(t)\,y'(t) + v(t)\,x'(t)\big)$$

$$\boxed{= \int_a^b \big(g_1(t) + i\,g_2(t)\big)dt}$$

How does one evaluate this integral???

$$\boxed{\int_\alpha f(z)dz := \int_a^b f(\alpha(t))\alpha'(t)dt}$$

What is a contour?

Is the integral independent of the choice of the contour??

Properties of contour integral

1. Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth curve.

Then $-\gamma : [a,b] \to \mathbb{C}$ defined as

$$-\gamma(t) := \gamma(a+b-t)$$ is the "opposite curve".

Properties of contour integral

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Then $-\gamma : [a,b] \to \mathbb{C}$ defined as

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"opposite curve".

$$\boxed{\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz}$$

Properties of contour integral

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$$\|$$

$$\int_a^b f(-\gamma(t))(-\gamma)'(t)\,dt$$

Properties of contour integral

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$$\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz$$

$$\parallel$$

$$\int_a^b f(-\gamma(t))(-\gamma)'(t)\,dt = \int_a^b f(\gamma(a+b-t))(-\gamma'(a+b-t))\,dt$$

$$= -\int_a^b f(\gamma(a+b-t))\,\gamma'(a+b-t)\,dt$$

$$s = a+b-t \quad , \quad ds = -dt$$

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$$\parallel$$

$$\int_a^b f(-\gamma(t))(-\gamma)'(t)\,dt = \int_a^b f(\gamma(a+b-t))(-\gamma'(a+b-t))\,dt$$

$$= -\int_a^b f(\gamma(a+b-t))\,\gamma'(a+b-t)\,dt$$

$$s = a+b-t \quad , \quad ds = -dt$$

$$= -\int_b^a f(\gamma(s))\,\gamma'(s)\,(-ds)$$

$$= -\int_a^b f(\gamma(t))\,\gamma'(t)\,dt = -\int_{\gamma} f(z)\,dz$$

Properties of contour integral

(1) Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth curve.

Then $-\gamma : [a,b] \to \mathbb{C}$ defined as

$-\gamma(t) := \gamma(a+b-t)$ is the opposite curve.

$$\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz$$

(2) $$\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$

Reparametrization.

(3) Let $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ be another parametrization of "γ" such that there exists $\alpha : [a,b] \to [\tilde{a}, \tilde{b}]$, smooth $\ni$ $\tilde{\gamma} \circ \alpha = \gamma$.

Then $\displaystyle\int_{\gamma} f(z)\,dz = \int_{\tilde{\gamma}} f(z)\,dz$.

Reparametrization.

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Then $\displaystyle\int_{\gamma} f(z)\,dz = \int_{\tilde{\gamma}} f(z)\,dz$.

$$\int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \int_a^b f(\tilde{\gamma}\circ\alpha(t))\underbrace{(\tilde{\gamma}\circ\alpha)'(t)}_{\tilde{\gamma}'(\alpha(t))\,\alpha'(t)}\,dt$$

$$= \int_{\alpha(a)}^{\alpha(b)} f(\tilde{\gamma}(s))\,\tilde{\gamma}'(s)\,ds$$

$s = \alpha(t)$

$ds = \alpha'(t)\,dt$

Reparametrization.

(3) Let $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ be another parametrization of "γ" such that there exists $\alpha : [a,b] \to [\tilde{a}, \tilde{b}]$, smooth $\ni \tilde{\gamma} \circ \alpha = \gamma$.

Then $\displaystyle\int_{\gamma} f(z)\,dz = \int_{\tilde{\gamma}} f(z)\,dz$.

$$\int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \int_a^b f(\tilde{\gamma}\circ\alpha(t))\,\underbrace{(\tilde{\gamma}\circ\alpha)'(t)}_{\tilde{\gamma}'(\alpha(t))\,\alpha'(t)}\,dt$$

$$= \int_{\alpha(a)}^{\alpha(b)} f(\tilde{\gamma}(s))\,\tilde{\gamma}'(s)\,ds$$

$$s = \alpha(t)$$
$$ds = \alpha'(t)\,dt$$

$$= \int_{\tilde{a}}^{\tilde{b}} f(\tilde{\gamma}(s))\,\tilde{\gamma}'(s)\,ds$$

$$= \int_{\tilde{\gamma}} f(z)\,dz$$

(4) $$\left| \int_{\gamma} f(z)\,dz \right| \leq \int_{a}^{b} |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

④ $$\left|\int_\gamma f(z)\,dz\right| \le \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

If $\int_\gamma f(z)\,dz = 0$ TRUE!

Let $u = e^{-i\theta}$, $\theta = \arg\left(\int_\gamma f(z)\,dz\right)$.

$$= u\int_\gamma f(z)\,dz = \mathrm{Re}\left(\int_\gamma u f(z)\,dz\right)$$

$$= \mathrm{Re}\int_a^b u f(\gamma(t))\,\gamma'(t)\,dt$$

$$\underset{\text{(by defn)}}{=} \int_a^b \mathrm{Re}\left(f(\gamma(t))\,\gamma'(t)\right)dt$$

$$\underset{\text{(R.A)}}{\le} \int_a^b |f(\gamma(t))\,\gamma'(t)|\,dt$$

∎

ML-inequality.

Let $f : D \to \mathbb{C}$ be a continuous function on an open set D.

Let C be a contour in D given by $\gamma : [a,b] \to \mathbb{C}$.

Then $$\boxed{\left| \int_C f(z)\,dz \right| \leq M\ell}$$

where $M = \sup_{t \in [a,b]} \{ |f(\gamma(t))| \}$ (exists!)

$$\underbrace{\ell = \int_a^b |\gamma'(t)|\,dt}_{\text{length of } \gamma}.$$

ML-inequality.

Let $f : D \to \mathbb{C}$ be a continuous function on a domain D.

Let C be a contour in D given by $\gamma : [a,b] \to \mathbb{C}$.

Then $$\left| \int_C f(z)\,dz \right| \leq M\ell$$

where $M = \sup_{t \in [a,b]} \{ |f(\gamma(t))| \}$ (exists!)

$$\underbrace{\ell = \int_a^b |\gamma'(t)|\,dt}_{\text{length of } \gamma}.$$

Pf: $$\left| \int_a^b f(\gamma(t))\gamma'(t)\,dt \right| \leq \underbrace{\int_a^b |f(\gamma(t))||\gamma'(t)|\,dt}_{\text{Riemann integral}}$$

$$\leq M \cdot \ell. \qquad \blacksquare$$

Evaluation $\int_\alpha f(z)dz$

$$\int_{t=a}^{b} f(\alpha(t))\alpha'(t)\,dt$$

"first principles"

Evaluation $\int_\alpha f(z)dz$

$$\int_{t=a}^{b} f(\alpha(t))\alpha'(t)\,dt$$

"first principles"

Anti-derivative

Eg: $\int_\alpha e^z dz$

$e^z = (e^z)'$

Evaluation $\int_\alpha f(z)dz$

$$\int_{t=a}^{b} f(\alpha(t))\alpha'(t)\,dt$$

"first principles"

Anti-derivative

Eg: $\int_\alpha e^z dz$

$e^z = (e^z)'$

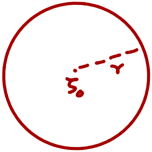
CAUCHY'S THEOREM

$\int_c f(z)dz = 0$

f- analytic

FIRST PRINCIPLES:-

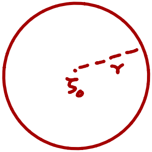
$$\int_{C_{\zeta_0,r}} (z-\zeta_0)^n \, dz$$



Parametrize $C_{\zeta_0,r}$

$$\gamma : [0,2\pi] \longrightarrow \mathbb{C}$$
$$t \longmapsto \zeta_0 + re^{it}$$

$$\int_{C_{\zeta_0,r}} (z-\zeta_0)^n dz$$



Parametrize $C_{\zeta_0,r}$

$\gamma : [0,2\pi] \longrightarrow \mathbb{C}$

$t \longmapsto \zeta_0 + re^{it}$

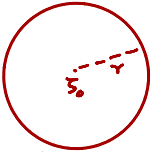
$$\int_\gamma (z-\zeta_0)^n dz = \int_0^{2\pi} (re^{it})^n \, rie^{it} \cdot dt$$

$$= i\int_0^{2\pi} r^{n+1} \cdot e^{i(n+1)t} dt$$

(De Moivre's theorem)

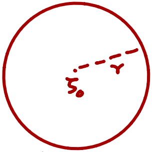
$$= ir^{n+1}\int_0^{2\pi} (\cos(n+1)t + i\sin(n+1)t)\, dt$$

$$\boxed{\int_{C_{\zeta_0,r}} (z-\zeta_0)^n dz}$$



$$\therefore \int_\gamma (z-\zeta_0)^n dz = i r^{n+1} \int_0^{2\pi} \left(\cos(n+1)t + i\sin(n+1)t\right) dt$$

$$\boxed{\int_{C_{\zeta_0,r}} (z-\zeta_0)^n dz}$$



$$\therefore \int_\gamma (z-\zeta_0)^n dz = i r^{n+1}\int_0^{2\pi} \left(\cos(n+1)t + i\sin(n+1)t\right)dt$$

$$(\text{if } n\neq -1) \quad = i r^{n+1}\int_0^{2\pi}\cos(n+1)t\,dt - r^{n+1}\int_0^{2\pi}\sin(n+1)t\,dt$$

$$= i r^{n+1}\frac{\sin(n+1)t}{n+1}\Big|_0^{2\pi} - r^{n+1}\frac{\cos(n+1)t}{n+1}\Big|_0^{2\pi}$$

$$= 0$$

$$\int_{C_{\zeta_0,r}} (z-\zeta_0)^n dz \quad = 0 \text{ if } n+1 \neq 0$$

$$\int_{C_{\zeta_0,r}} (z-\zeta_0)^n dz = 0 \text{ if } n+1 \neq 0$$

$$n=-1: \quad \int_\gamma (z-\zeta_0)^{-1} dz = \int_0^{2\pi} \frac{1}{re^{it}} r i e^{it} dt$$

$$= i \int_0^{2\pi} dt = 2\pi i$$

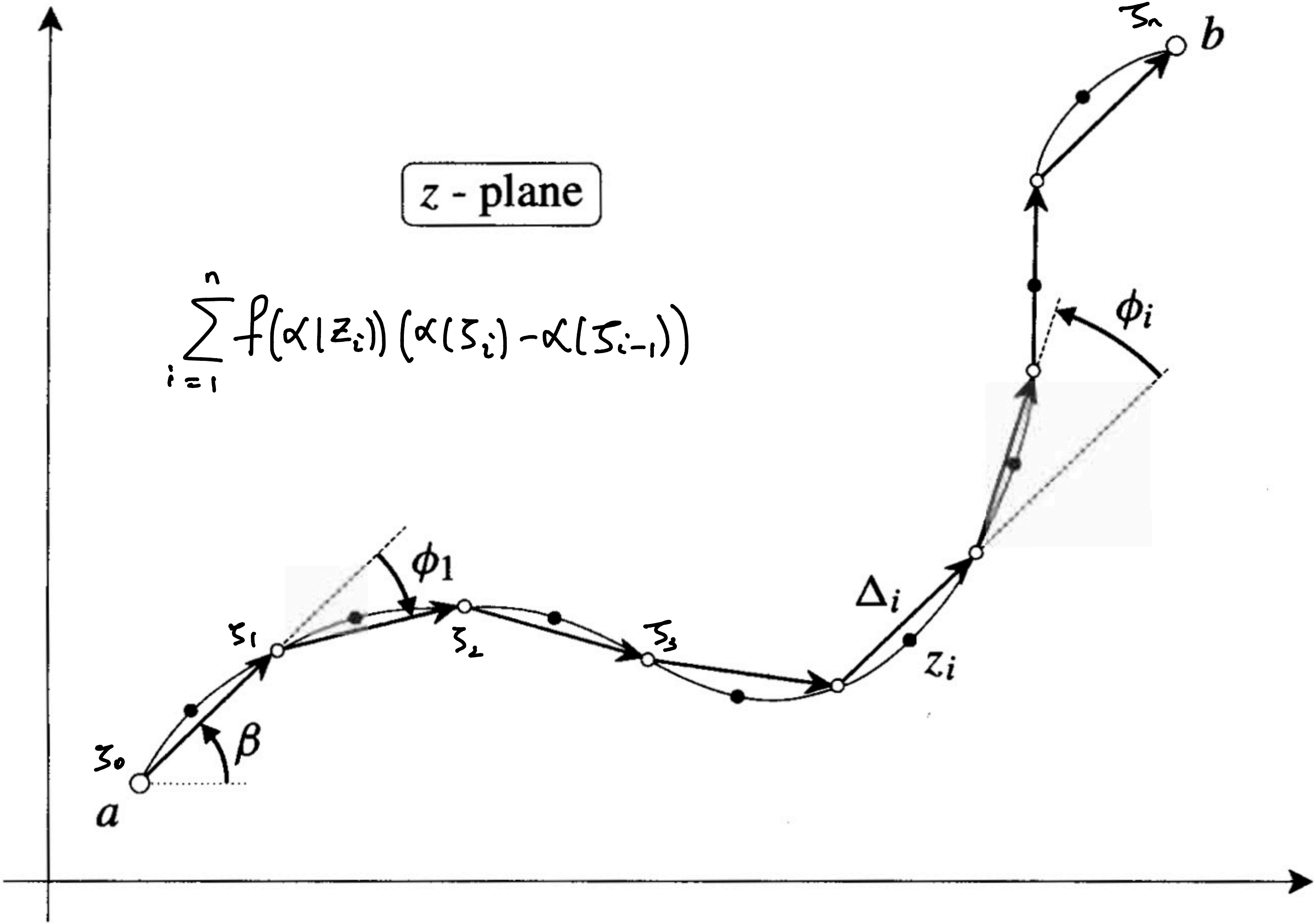
$$\therefore \int_{C_{\zeta_0,r}} (z-\zeta_0)^{-1} dz = 2\pi i$$

$$\int_{C_{\zeta_0,r}} (z-\zeta_0)^n dz = 0 \text{ if } n+1 \neq 0$$

$n = -1$: $\displaystyle\int_\gamma (z-\zeta_0)^{-1} dz = \int_0^{2\pi} \frac{1}{re^{it}} \, r i e^{it} dt$

$$= i\int_0^{2\pi} dt = 2\pi i$$

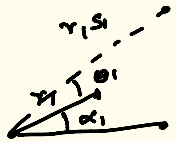
$$\therefore \int_{C_{\zeta_0,r}} (z-\zeta_0)^{-1} dz = 2\pi i \text{ if } n=-1$$



z - plane
$\sum_{i=1}^{n} f(\alpha(z_i))(\alpha(\zeta_i)-\alpha(\zeta_{i-1}))$
ζ_n
b
ϕ_i
ϕ_1
ζ_1
ζ_2
ζ_3
Δ_i
z_i
β
ζ_0
a

$$\left.\begin{aligned} \alpha(\zeta_1) - \alpha(\zeta_0) &= r_1 e^{i\alpha_1} \\ f(\alpha(z_1)) &= s_1 e^{i\theta_1} \end{aligned}\right\} \quad f(\alpha(z_1))\big(\alpha(\zeta_1) - \alpha(\zeta_0)\big) \quad T_1$$

$$= r_1 s_1 e^{i(\alpha_1 + \theta_1)}$$

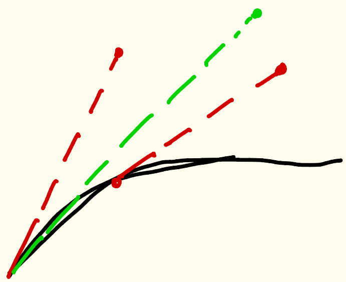


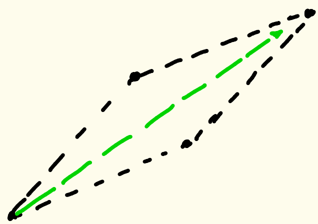
T_1

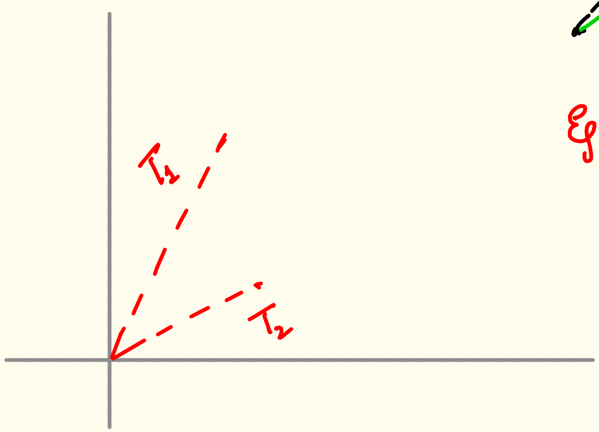
$$f(\alpha(z_1))\big(\alpha(\zeta_1) - \alpha(\zeta_0)\big)$$

$$+ f(\alpha(z_2))\big(\alpha(\zeta_2) - \alpha(\zeta_1)\big)$$

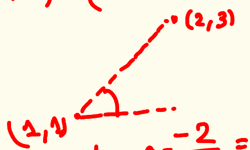
T_2



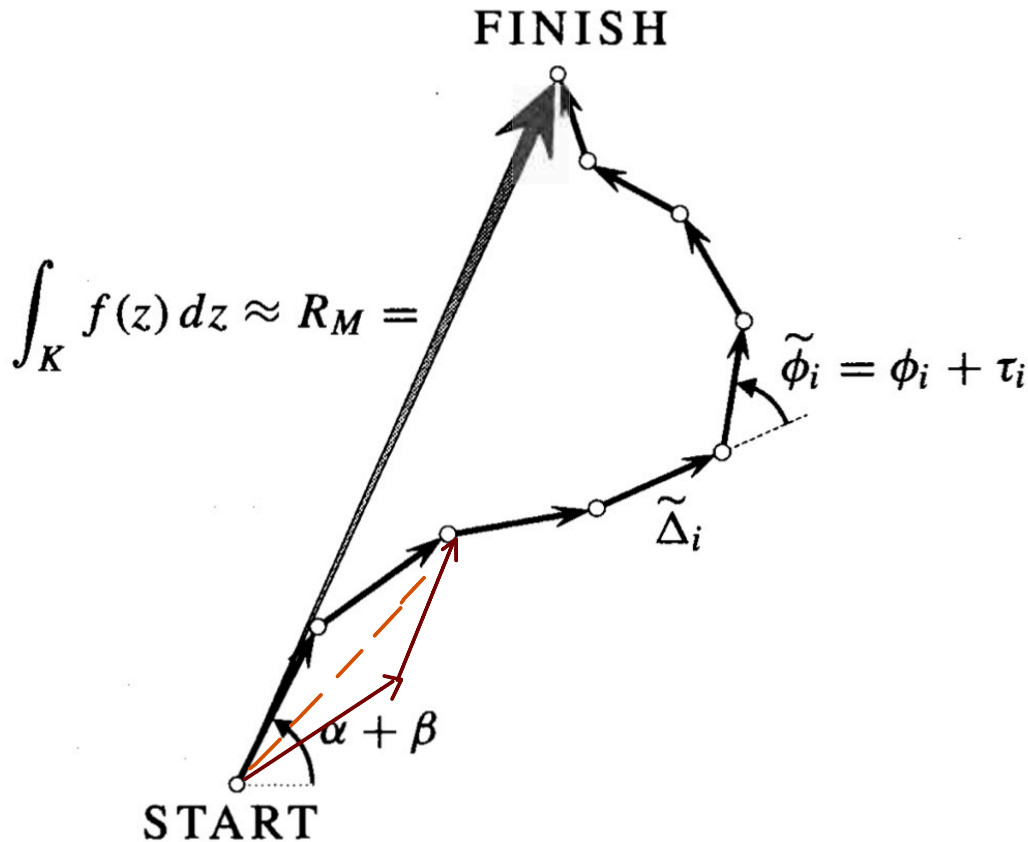




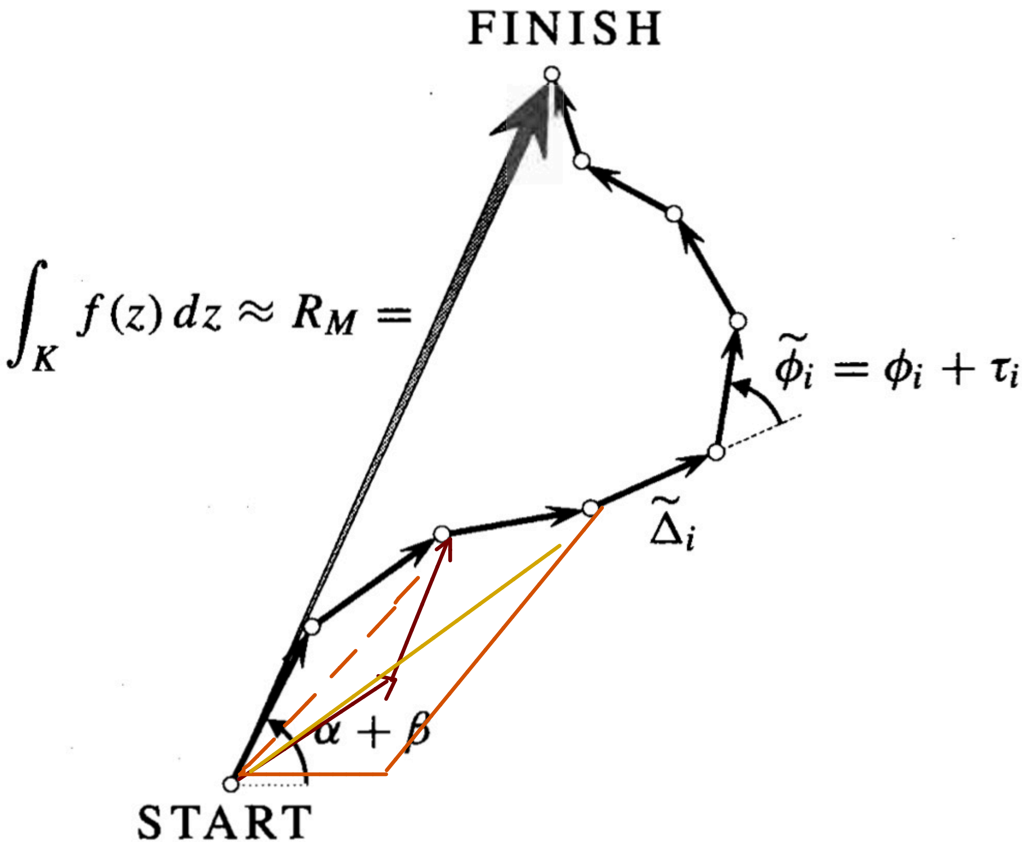
Eg: $|(1+i) - (2+3i)| = |-1-2i|$



$$\tan\theta = \frac{-2}{-1} = 2$$



FINISH
$\int_K f(z)\,dz \approx R_M =$
$\widetilde{\phi}_i = \phi_i + \tau_i$
$\widetilde{\Delta}_i$
$\alpha + \beta$
START



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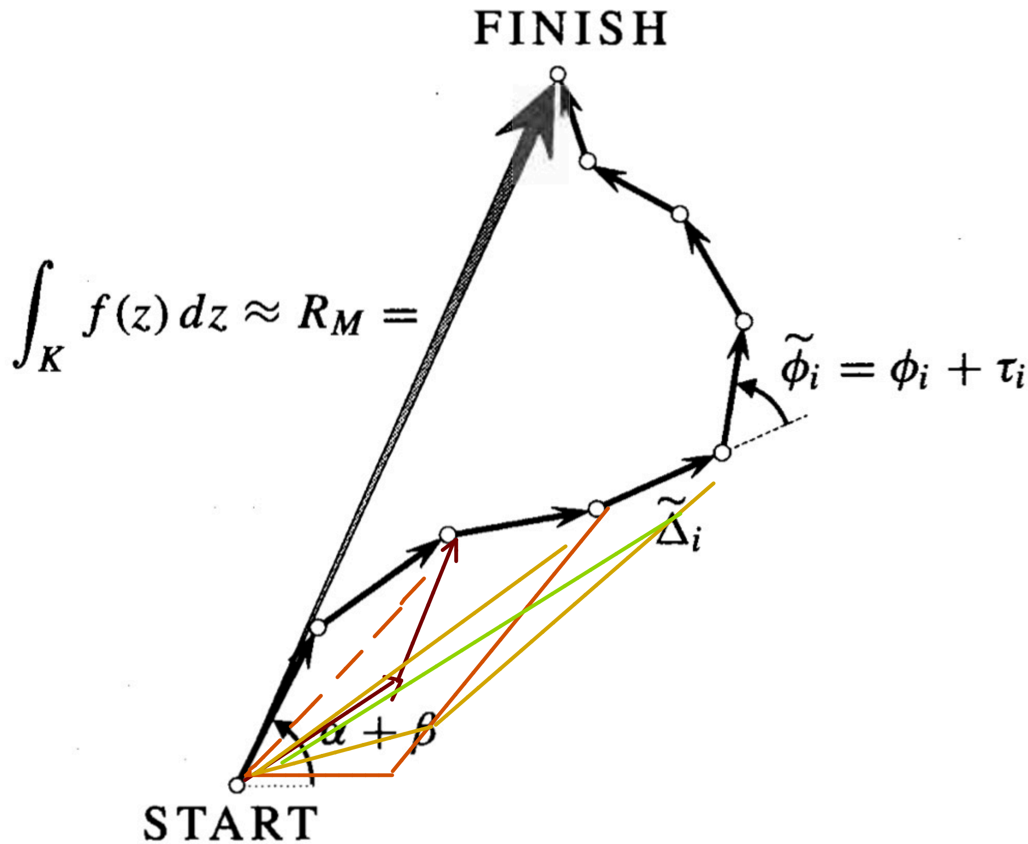
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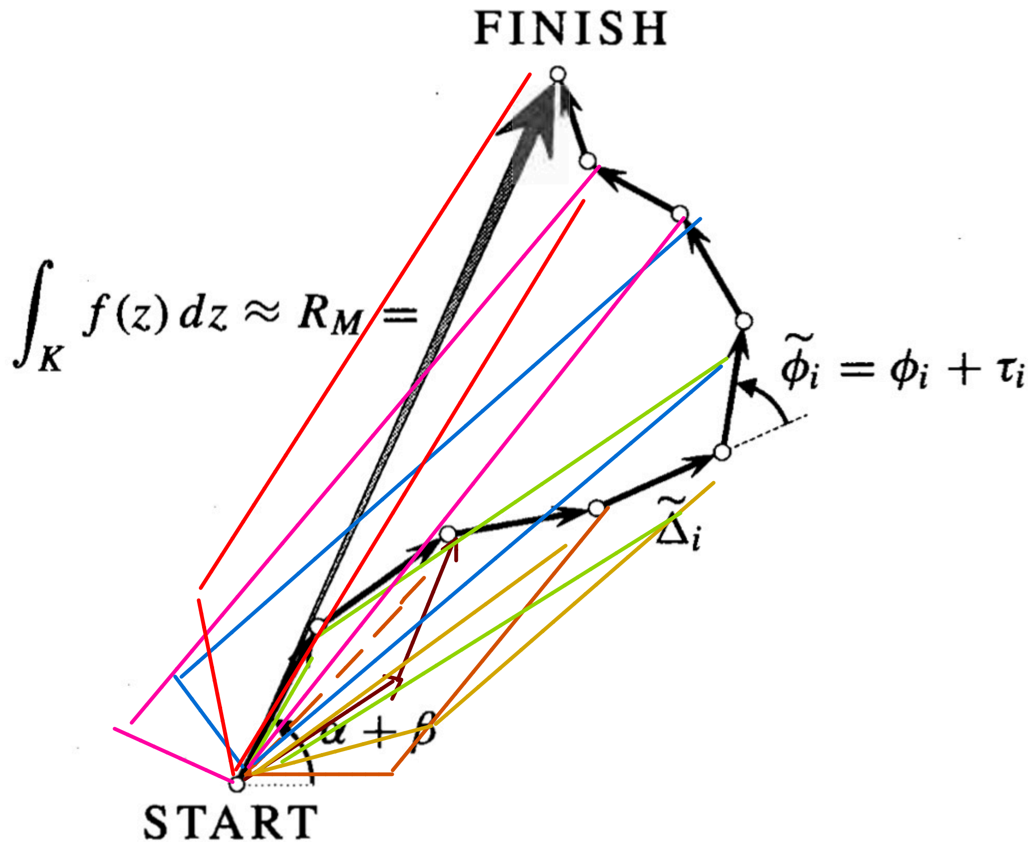
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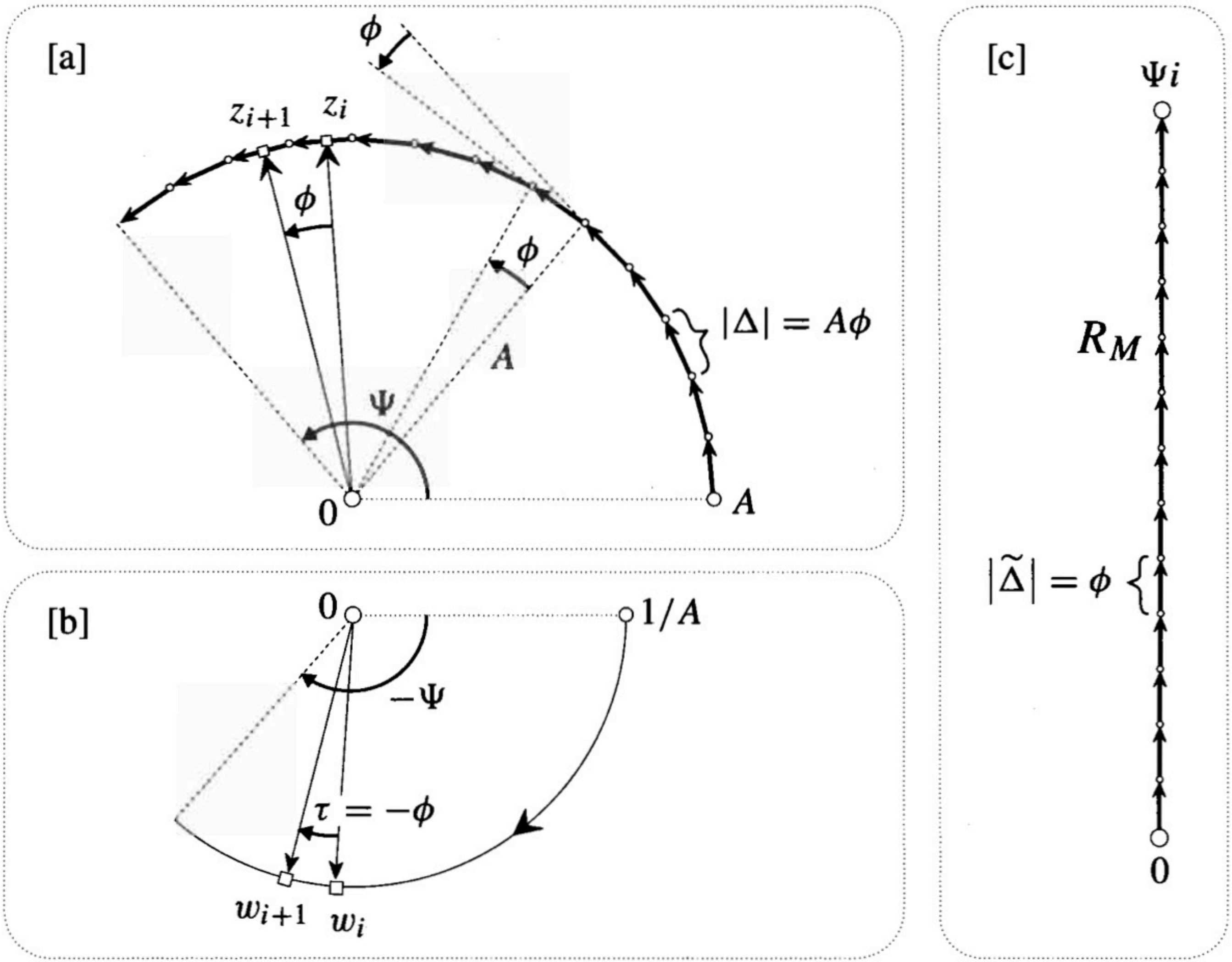

[a]
ϕ
z_{i+1}
z_i
ϕ
ϕ
$|\Delta| = A\phi$
A
Ψ
0
A
[b]
0
$1/A$
$-\Psi$
$\tau = -\phi$
w_{i+1}
w_i
[c]
Ψi
R_M
$|\widetilde{\Delta}| = \phi$
0

ANTI-DERIVATIVE

Fundamental theorem of analytic functions

Let f be continuous in a domain D. Suppose there exists F on D such that $F' = f$. Then $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$.

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open + connected

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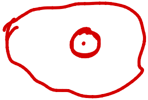
open + connected

$z \in D$



not disconnected.

Fundamental theorem of analytic functions

Let f be continuous in a domain D. Suppose there exists F on D such that $F' = f$. Then $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$.

independent of contour.

UPSHOT: if f has an antiderivative

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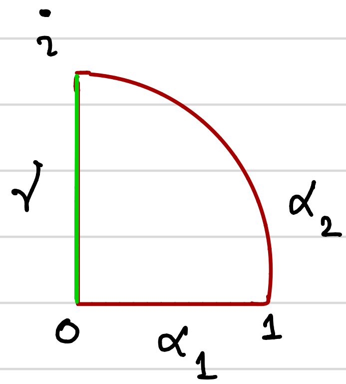
independent of contour.

if f has an antiderivative

NOT TRUE IN GENERAL

Eg:

$f(z) = |z|$ is not diff'ble



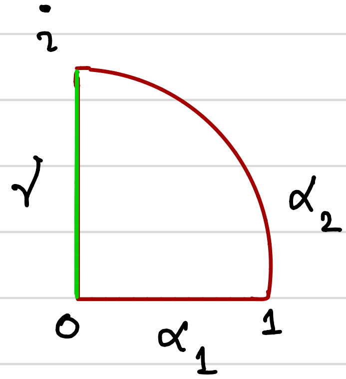
$$\int_{\gamma} f(z)\,dz$$

$$= \int_0^1 f(it)\, i\, dt$$

$$= \int_0^1 |t|\, i\, dt = \frac{i}{2}$$

$f(z) = |z|$ is not diff'ble

$$\int_\gamma f(z)\,dz$$



$$= \int_0^1 f(it)\, i\,dt$$

$$= \int_0^1 |t|\, i\,dt \quad = \frac{i}{2}$$

$$\int_{\alpha_1+\alpha_2} f(z)\,dz = \int_{\alpha_1} f(z)\,dz + \int_{\alpha_2} f(z)\,dz$$

$$= \int_0^1 f(t)\cdot 1\,dt + \int_0^{\pi/2} f(e^{it})\, i e^{it}\,dt$$

$$= \int_0^1 t\,dt + \int_0^{\pi/2} i e^{it}\,dt$$

$$= \frac{t^2}{2}\Big|_0^1 + i\left(\int_0^{\pi/2} \cos t\,dt + i\int_0^{\pi/2} \sin t\,dt\right)$$

$$= i - \tfrac{1}{2}.$$

Fundamental theorem of analytic functions

Let f be continuous in a domain D. Suppose there exists F on D such that $F' = f$. Then $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$.

Are there more fns for which ↑ is true?

Cauchy's theorem

Let f be "analytic" in a domain D. Let C be a simple closed curve then $\int_C f(z)dz = 0$.

Fundamental theorem of analytic functions

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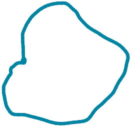

closed curve: $\alpha : [a,b] \to \mathbb{C}$

$\alpha(a) = \alpha(b)$

Fundamental theorem of analytic functions

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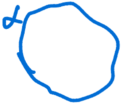

simple closed curve: $\alpha : [a,b] \to \mathbb{C}$

$\alpha(a) = \alpha(b)$

$\alpha(t) \neq \alpha(t')$ for $a < t \neq t' < b$

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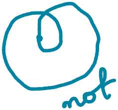

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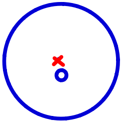

$$\int_C \frac{1}{z} dz = 2\pi i$$

Fundamental theorem of analytic functions

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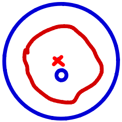
Cauchy's theorem ? ?

Let f be analytic in a domain D. Let C be a simple closed curve then $\int_C f(z)dz = 0$.

?



?

$\int_C \frac{1}{z} dz \neq 0$?

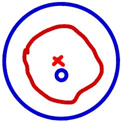
Fundamental theorem of analytic functions

Let f be continuous in a domain D. Suppose there exists F on D such that $F' = f$. Then $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$.

Cauchy's theorem

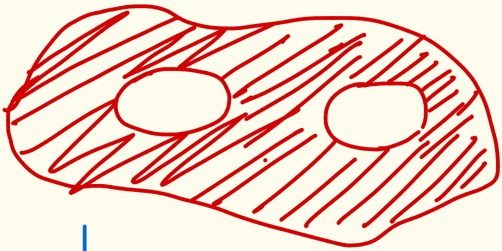
Let f be analytic in a "simply connected" domain D. Let C be a simple closed curve then $\int_C f(z)dz = 0$.

not →

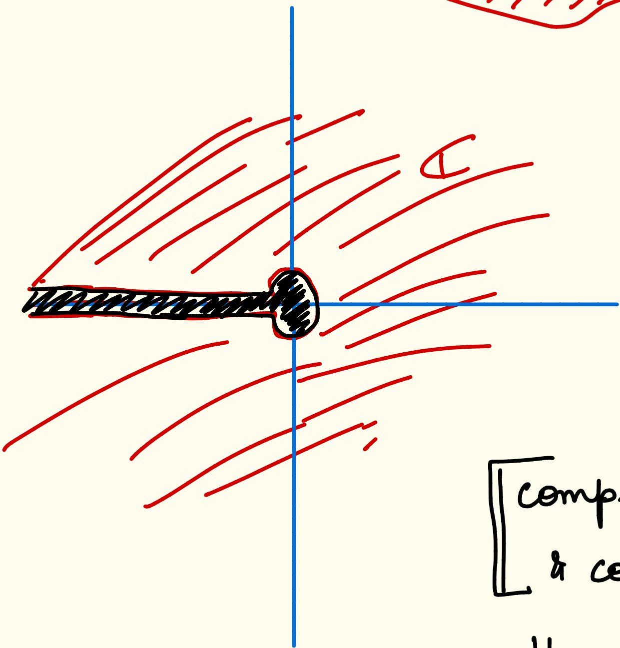


$$\int_C \frac{1}{z} dz \neq 0$$

SIMPLY - CONNECTED



not simply connected



simply connected.

complement is unbdd & connected (or empty)

||

every closed curve in S ONLY contains points in S

Fundamental theorem of analytic functions

Let f be continuous in a domain D. Suppose there exists F on D such that $F' = f$. Then $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$.

Cauchy's theorem

simply connected

Let f be analytic in a domain D. Let C be a simple closed curve then $\int_C f(z)dz = 0$.

Fundamental theorem of analytic functions

Let f be continuous in a domain D. Suppose there exists F on D such that $F' = f$. Then $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$.

$$\Downarrow$$

Cauchy's theorem

Let f be analytic in a simply connected domain D. Let C be a simple closed curve then $\int_C f(z)dz = 0$.

Fundamental theorem of analytic functions

Let f be continuous in a domain D. Suppose there exists F on D such that $F' = f$. Then $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$.

$\Downarrow$

Cauchy's theorem

simply connected

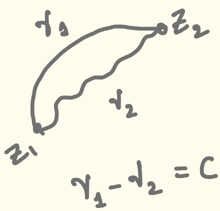
Let f be analytic in a domain D. Let C be a simple closed curve then $\int_C f(z)dz = 0$.

(*) Simply-connected domain + simple closed contours.

f has antiderivative, F

$$\overset{\text{Fund. Thm.}}{\Rightarrow} \int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1)$$

$$f \text{ is analytic} \underset{\text{(Cauchy's thm)}}{\Rightarrow} \int_C f(z)\,dz = 0$$



$$\Rightarrow \int_{z_1}^{z_2} f(z)\,dz \text{ is independent of path}$$

$$\therefore F(z) := \int_{z_1}^{z} f(\zeta)\,d\zeta \quad \text{(is well-defined)}$$

We prove that $F'(z) = f$.

UPSHOT: analytic $\Rightarrow$ antiderivative exists.

We will soon see that the converse is also true. (a consequence of CIF)

Fundamental theorem of analytic functions

Let f be continuous in a domain D. Suppose there exists F on D such that $F' = f$. Then $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$.

(Consequence of CIF)

Cauchy's theorem

simply connected

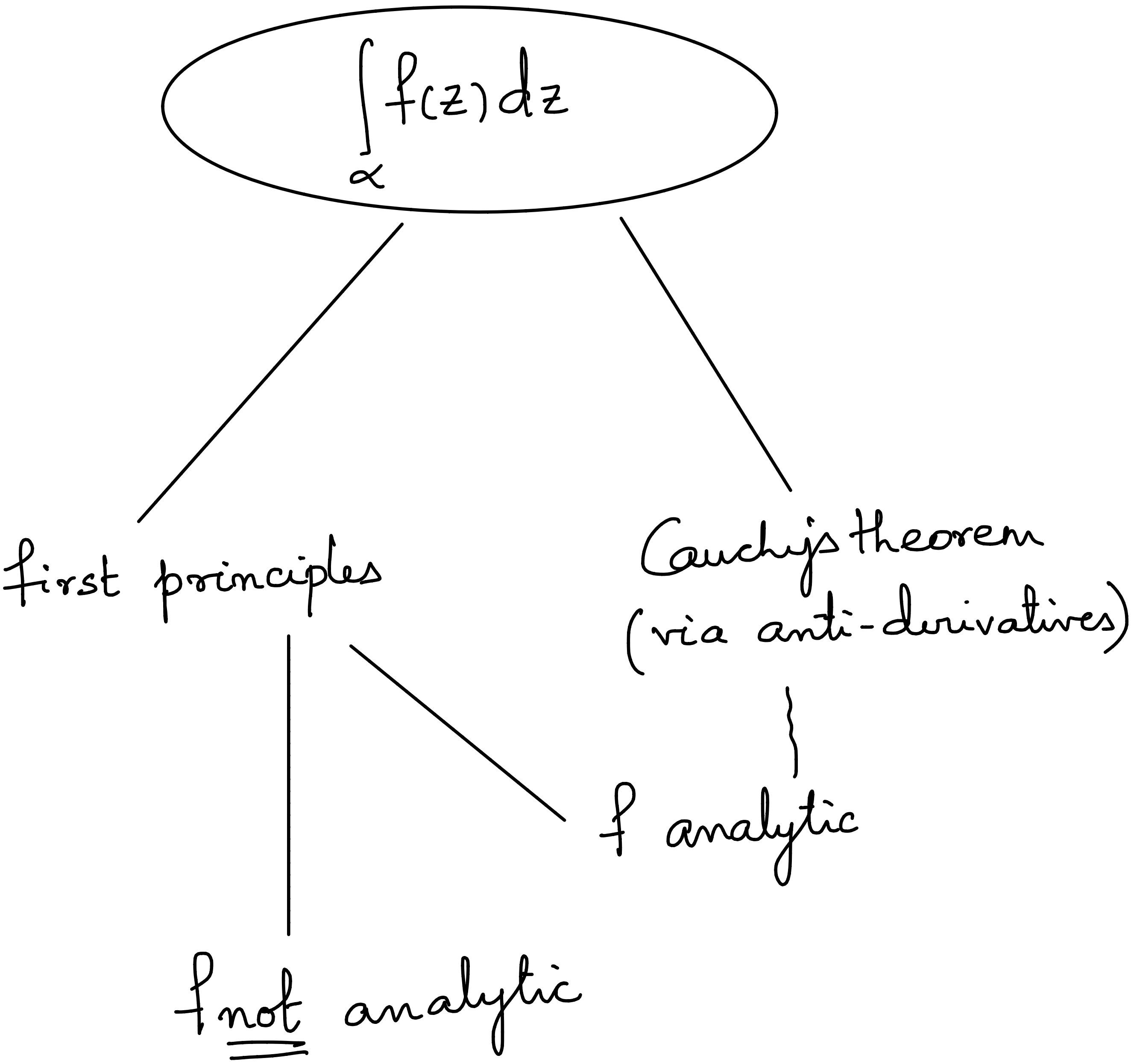
Let f be analytic in a domain D. Let C be a simple closed curve then $\int_C f(z)dz = 0$.

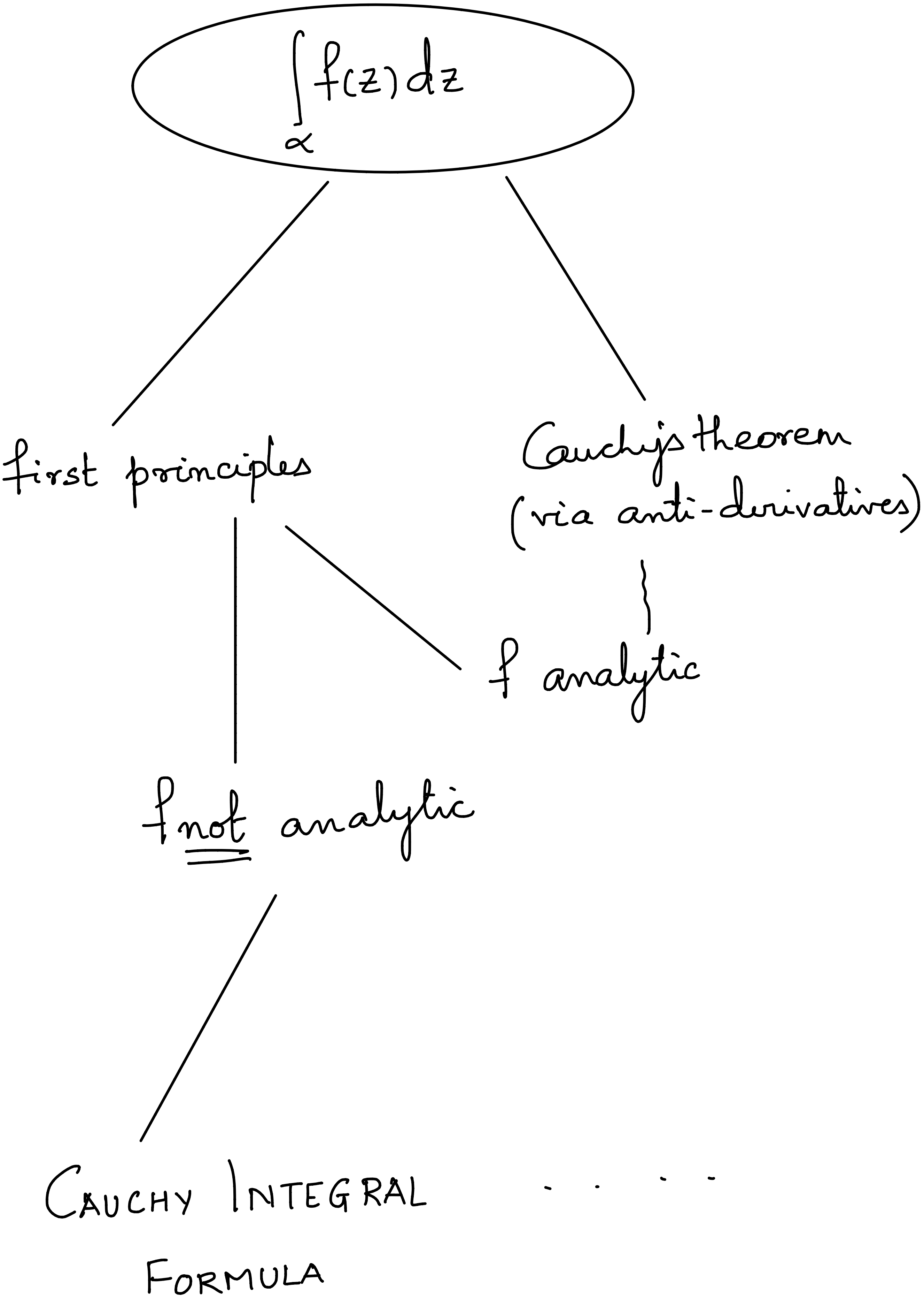
analytic $\Leftrightarrow$ anti-derivative

$$\int_{\alpha} f(z)\,dz$$

first principles

Cauchy's theorem
(via anti-derivatives)


$\int_{\alpha} f(z)\,dz$
first principles
Cauchy's theorem
(via anti-derivatives)
f analytic
f not analytic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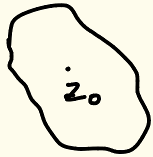
first principles
Cauchy's theorem
(via anti-derivatives)
f analytic
f not analytic
CAUCHY INTEGRAL
FORMULA
. . . .

CAUCHY INTEGRAL FORMULA

$f : D \to \mathbb{C}$ be analytic on a simply connected domain D.

Let C be a simple closed contour containing $z_0 \in D$



$$\int_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0)$$

Eg: $f \equiv 1$, Fundamental integral $= 2\pi i$.

Heuristic:

If $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$

Then $\displaystyle\int_{C_{1,z_0}} \frac{f(z)}{(z-z_0)^n} \underset{\text{hopefully}}{=} \sum_{n=0}^{\infty} \int_{C_{1,z_0}} a_n (z - z_0)^{n-1}\, dz$

$$= \sum_{n=0}^{\infty} a_n (2\pi i\, \delta_{n,0}) = a_0 \cdot 2\pi i$$

$$= f(z_0) \cdot 2\pi i.$$

This has many gaps so will definitely not pass off as a PROOF!