Contents

A function f is said to be *analytic* at z if there exists an open neighbourhood U around z such that f is differentiable at every point of U.

Eg: $|z|^2$ is differentiable at 0 but not analytic at 0.

z^2 is analytic at 0 since it is differentiable everywhere.

More examples???

Power Series

A series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is called a power series at z_0. Here $a_n \in \mathbb{C}$ and z is an indeterminate.

Eg: $\sum_{n=0}^{\infty} z^n$.

$$k^{th} \text{ partial sum} = s_k = \sum_{n=0}^{k} z^n = \frac{1-z^{k+1}}{1-z}$$

$$\lim_{k\to\infty} s_k = \frac{1}{1-z} - \lim_{k\to\infty} \frac{z^{k+1}}{1-z}$$

$\{s_k\}$ converges to $\frac{1}{1-z}$ for $|z|<1$ and diverges otherwise. (At $|z|=1$, we know that the n-th term of the series does not converge to 0 hence it diverges.)

Alternately, lets consider the root test for determining the convergence of $\sum_{n=0}^{\infty} z^n$.

$\lim_{n\to\infty} \sqrt[n]{|z|^n}$
$=$

$$\lim_{n\to\infty} |z| = |z|$$

By Root test, we conclude that $\sum_{n=0}^{\infty} z^n$ converges when $|z| < 1$ and diverges if $|z| > 1$. At $|z| = 1$, we know that the n-th term of the series does not converge to 0 hence it diverges.

The number 1 in the above example is called the *radius of convergence* of $\sum_{n=0}^{\infty} z^n$

Radius of convergence of a series

Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series around z_0. The radius of convergence is defined as

$$\sup\{|z-z_0| : \sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ converges.}\}$$

Let $\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ converges for some $z_1 \in \mathbb{C}$. Then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges $\forall z \ni |z-z_0| < |z_1-z_0|$.

Radius of convergence (Equivalent definition)

Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series around z_0. The radius of convergence is defined as $R \geq 0$ such that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for all $|z-z_0| < R$ and diverges for all $|z-z_0| > R$.

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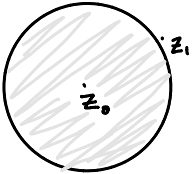
Let $\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$ converges for some $z_1 \in \mathbb{C}$. Then $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges $\forall z \ni |z - z_0| < |z_1 - z_0|$.



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Proof: Let $z \ni |z - z_0| < |z_1 - z_0| \Rightarrow \frac{|z - z_0|}{|z_1 - z_0|} < 1$

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<u>Proof</u>: Let $z \ni |z - z_0| < |z_1 - z_0| \Rightarrow \dfrac{|z - z_0|}{|z_1 - z_0|} < 1$

Let $\alpha \ni \dfrac{|z - z_0|}{|z_1 - z_0|} < \alpha < 1$

$$|a_n(z - z_0)^n| = |a_n||z - z_0|^n = |a_n| \frac{|z - z_0|^n}{|z_1 - z_0|^n} \cdot |z_1 - z_0|^n$$

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$\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ converges $\Rightarrow \{a_n(z_1-z_0)^n\}$ is a null sequence

$\Rightarrow$ given $\varepsilon>0$ $\exists N>0 \ni |a_n(z_1-z_0)^n| < \varepsilon \; \forall n>N$

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$\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ converges $\Rightarrow \{a_n(z_1-z_0)^n\}$ is a null sequence

$\Rightarrow$ given $1/2 > 0$ $\exists N > 0 \ni |a_n(z_1-z_0)^n| < 1/2 \ \forall n > N$.

$\therefore |a_n(z_1-z_0)^n| \leq M \ \forall n.$

Let $\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$ converges for some $z_1 \in \mathbb{C}$. Then $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges $\forall z \ni |z - z_0| < |z_1 - z_0|$.

Proof: Let $z \ni |z - z_0| < |z_1 - z_0| \Rightarrow \dfrac{|z-z_0|}{|z_1-z_0|} < 1$

Let $\alpha \ni \dfrac{|z-z_0|}{|z_1-z_0|} < \alpha < 1$

$$|a_n(z-z_0)^n| = |a_n||z-z_0|^n = |a_n|\frac{|z-z_0|^n}{|z_1-z_0|^n}|z_1-z_0|^n$$

$$\leq M\cdot\alpha^n$$

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Let $\alpha \ni \dfrac{|z - z_0|}{|z_1 - z_0|} < \alpha < 1$

$$|a_n(z - z_0)^n| = |a_n| |z - z_0|^n = |a_n| \frac{|z - z_0|^n}{|z_1 - z_0|^n} |z_1 - z_0|^n$$

$$\leq M \cdot \alpha^n$$

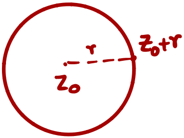
By Comparison test, $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges absolutely

Radius of convergence of a series

Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series around z_0. The radius of convergence is defined as

$$\sup\{|z-z_0| : \sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ converges.}\}$$

$\sum a_n r^n$ converges $\Rightarrow$ $R \geq r$



R is unique

Radius of convergence (Equivalent definition)

Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series around z_0. The radius of convergence is defined as $R \geq 0$ such that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for all $|z-z_0| < R$ and diverges for all $|z-z_0| > R$.

Useful remark

Radius of convergence of $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is R if and only if radius of convergence of $\sum_{n=0}^{\infty} a_n w^n$ is R.

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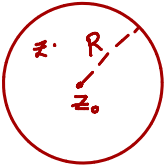

$\sum_{n=0}^{\infty} a_n(z-z_0^n)$

$\sum_{n=0}^{\infty} a_n \omega^n$

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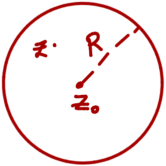

$$\sum_{n=0}^{\infty} a_n(z-z_0^n)$$
$$\|$$
$$f(z)$$

$$\sum_{n=0}^{\infty} a_n \omega^n = g(\omega)$$

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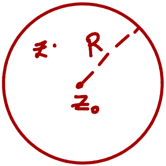

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$
$$\parallel$$
$$f(z)$$

$$\boxed{f(z) = g(w+z_0)}$$

$$\sum_{n=0}^{\infty} a_n w^n = g(w)$$

Useful remark

$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for $|z-z_0|<R$ if and only if $\sum_{n=0}^{\infty} a_n w^n$ converges for $|w|<R$.

Useful remark

$\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges for $|z - z_0| < R$ if and only if $\sum_{n=0}^{\infty} a_n w^n$ converges for $|w| < R$.

Set $\omega := z - z_0$

$\sum a_n(z-z_0)^n$ cgs for $|z-z_0|<R$

$\Rightarrow \sum a_n \omega^n$ cgs for $|\omega|<R$

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Set $\omega := z - z_0$

$\sum a_n(z-z_0)^n$ cgs for $|z-z_0|<R$

$\Leftrightarrow \sum a_n \omega^n$ cgs for $|\omega|<R$

Set $z := \omega + z_0$

Useful remark

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Eg: $\sum_{n=0}^{\infty} z^n$ cgs $\forall$ $|z|<1$

$\Updownarrow$

$\sum_{n=0}^{\infty} (z-1)^n$ cgs $\forall$ $|z-1|<1$

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$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for $|z-z_0|<R$ if and only if $\sum_{n=0}^{\infty} a_n w^n$ converges for $|w|<R$.

Eg: $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ cgs $\forall\ |z|<1$

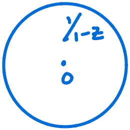
$\Updownarrow$

$\sum_{n=0}^{\infty} (z-1)^n = \frac{1}{1-(z-1)} = \frac{-1}{z}$ cgs $\forall\ |z-1|<1$

Useful remark

$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for $|z-z_0| < R$ if and only if $\sum_{n=0}^{\infty} a_n w^n$ converges for $|w| < R$.

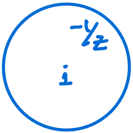
Eg: $\sum_{n=0}^{\infty} z^n$ cgs $\forall\ |z| < 1$ $= \frac{1}{1-z}$



$\Updownarrow$

$\sum_{n=0}^{\infty} (z-1)^n$ cgs $\forall\ |z-1| < 1$

$= \frac{1}{1-(z-1)} = \frac{-1}{z}$



Hadamard's formula

Ratio Test

$l = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

For a series $\sum_{n=0}^{k} a_n z^n$. The sequence of ratios $|a_{n+1}||z|/|a_n|$ has a limit $l|z|$. Then

i $\sum_{n=0}^{k} a_n z^n$ converges absolutely if $l|z| < 1$;

ii $\sum_{n=0}^{k} a_n z^n$ diverges if $l|z| > 1$;

iii if $l|z| = 1$ then the series may or may not converge.

So, by definition of the radius of convergence $1/l$ is the radius of convergence of the series.

Hadamard's formula

Radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is

$$1/\lim_{n\to\infty} |a_{n+1}|/|a_n|$$

with the convention that $1/\infty = 0$ and $1/0 = \infty$.

Hadamard's formula

$l = \limsup \sqrt[n]{|a_n|}$

Root Test: For a series $\sum_{n=0}^{k} a_n z^n$ let the lim sup $\sqrt[n]{|a_n|}|z|$ be $l|z|$. Then

i $\sum_{n=0}^{k} a_n z^n$ converges absolutely if $l|z| < 1$;

ii $\sum_{n=0}^{k} a_n z^n$ diverges if $l|z| > 1$;

iii if $l|z| = 1$ then the series may or may not converge.

So, by definition of the radius of convergence $1/l$ is the radius of convergence of the series.

Hadamard's formula

Radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is

$$1/\ \lim_{n\to\infty}\sup \sqrt[n]{|a_n|}$$

with the convention that $1/\infty = 0$ and $1/0 = \infty$.

If $\sum_{n=0}^{\infty} a_n z^n$ converges in a disc of radius R then so does the derived series $\sum_{n=0}^{\infty} n a_n z^{n-1}$.

Theorem: Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ in the disc of convergence $B_R(0)$. Then $f(z)$ is analytic in $B_R(0)$. Its derivative is given by $\sum_{n=0}^{\infty} n a_n z^{n-1}$.

Power series are analytic in their disc of convergence.

$$\left\{ \begin{array}{c} \text{Power series} \\ \text{at } z_0 \end{array} \right\} \subseteq \left\{ \begin{array}{c} \text{analytic fn} \\ \text{at } z_0 \end{array} \right\}$$

z_0

Pf of theorem:

Consider

$$\left| \frac{f(z) - f(w)}{z - w} - g(w) \right|.$$

For $N \in \mathbb{N}$, $\displaystyle\sum_{n=0}^{\infty} a_n z^n = \underbrace{\sum_{n=0}^{N} a_n z^n}_{S_N(z)} + \underbrace{\sum_{n=N+1}^{\infty} a_n z^n}_{E_N(z)}$

$S_N(z)$ is a polynomial whose derivative is

$$S_N'(z) = \sum_{n=0}^{N} n a_n z^{n-1} \left(= N\text{-th partial sum of } \sum_{n=0}^{\infty} n a_n z^{n-1} \right)$$

$$\frac{f(z) - f(w)}{z - w} - g(w) = \frac{\sum_{n=0}^{\infty} a_n z^n - \sum a_n w^n}{z - w} - g(w)$$

$$= \frac{S_N(z) + E_N(z) - (S_N(w) - E_N(w))}{z - w} - g(w)$$

$$= \frac{S_N(z) - S_N(w)}{z - w} + \frac{E_N(z) - E_N(w)}{z - w} - S_N'(w)$$

$$+ S_N'(w) - g(w)$$

$$\therefore \left| \frac{f(z) - f(w)}{z - w} - g(w) \right| \le$$

$$\underbrace{\left| \frac{S_N(z) - S_N(w)}{z - w} - S_N'(w) \right|} + \left| S_N'(w) - g(w) \right|$$

$$+ \left| \frac{E_N(z) - E_N(w)}{z - w} \right|$$

$$\therefore \left| \frac{f(z)-f(\omega)}{z-\omega} - g(\omega) \right| \le$$

$$\underbrace{\left| \frac{S_N(z)-S_N(\omega)}{z-\omega} - S_N'(\omega) \right|} + \left| S_N'(\omega) - g(\omega) \right|$$

$$+ \left| \frac{E_N(z)-E_N(\omega)}{z-\omega} \right|$$

$$\lim_{z \to \omega} \frac{S_N(z)-S_N(\omega)}{z-\omega} = S_N'(\omega)$$

Choose $\delta_1 > 0 \ni \Big|\big(\qquad\quad \big)\Big| < \varepsilon/3$

$$\therefore \left| \frac{f(z)-f(w)}{z-w} - g(w) \right| \leq$$

$$\left| \frac{S_N(z)-S_N(w)}{z-w} - S_N'(w) \right| + \left| S_N'(w) - g(w) \right|$$

$$+ \left| \frac{E_N(z)-E_N(w)}{z-w} \right|$$

$$\lim_{z \to w} \frac{S_N(z)-S_N(w)}{z-w} = S_N'(w)$$

Choose $\delta_1 > 0 \ni \left| \left(\qquad \right) \right| < \varepsilon/3$

$S_N'(w) = N^{th}$ partial sum of $g(w)$

$\therefore$ as $N \to \infty$

$S_N'(w) \longrightarrow g(w)$

Choose $N_1 > 0$

$\left| \left(\qquad \right) \right| < \varepsilon/3$

$$\therefore \left| \frac{f(z)-f(\omega)}{z-\omega} - g(\omega) \right| \leq$$

$$\underbrace{\left| \frac{S_N(z)-S_N(\omega)}{z-\omega} - S_N'(\omega) \right|} + \left| S_N'(\omega) - g(\omega) \right|$$

$$+ \left| \frac{E_N(z)-E_N(\omega)}{z-\omega} \right|$$

$$\lim_{z\to\omega} \frac{S_N(z)-S_N(\omega)}{z-\omega} = S_N'(\omega)$$

Choose $\delta_1 > 0 \ni \left| \left(\quad \right) \right| < \varepsilon/3$

$S_N'(\omega) = N^{th}$ partial sum of $g(\omega)$

$\therefore$ as $N \to \infty$

$S_N'(\omega) \longrightarrow g(\omega)$

Choose $N_1 > 0$

$\left| \left(\quad \right) \right| < \varepsilon/3$

$$\left| \sum_{n=N+1}^{\infty} a_n \frac{(z^n - \omega^n)}{(z-\omega)} \right|$$

$$= \left| \sum_{n=N+1}^{\infty} a_n \left(z^{n-1} + z^{n-2}\omega + \cdots + \omega^{n-1} \right) \right|$$

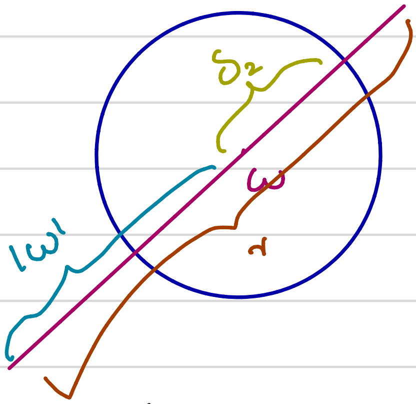
$$\leq \sum_{n=N+1}^{\infty} |a_n| \left(|z|^{n-1} + |z|^{n-2}|\omega| + \cdots + |\omega|^{n-1} \right)$$

($\because$ partial sums satisfy this inequality).

Choose $\delta_2 \ni |\omega| + \delta_2 < r$

then for $z \in B_\omega(\delta)$

$$|z| \le |z-\omega| + |\omega| < r$$



$$\therefore \sum_{n=N+1}^{\infty} |a_n| \left(|z|^{n-1} + \cdots + |\omega|^{n-1} \right) \le \sum_{n=N+1}^{\infty} |a_n| \, n \cdot r^{n-1}$$

Since $r < R$, & $\sum_{n=0}^{\infty} n a_n z^{n-1}$ converges absolutely $\forall\, |z| < R$

we get a $N_2 >> 0 \ni \sum_{n=N_1+1}^{\infty} n |a_n| r^{n-1} < \varepsilon/3$

$$\left| \frac{E_{N_2}(z) - E_{N_2}(\omega)}{z - \omega} \right| < \varepsilon/3$$

Choose $N \ge \{N_1, N_2\}$

$\delta < \{\delta_1, \delta_2\}$

then,

$|\text{Term } 1| + |\text{Term } 2| + |\text{Term } 3| < \varepsilon$.