LECTURE-3

Functions, limits
Continuity

Lecture 3: Functions, limits and continuity.

Our beacon light once again is $\mathbb{R}$.

Function : $f : \mathbb{C} \to \mathbb{C}$
$$x+iy \mapsto u+iv.$$

Recall, $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$; $\mathrm{Im} : \mathbb{C} \to \mathbb{R}$
$$x+iy \mapsto x \qquad\qquad x+iy \mapsto y$$

Composing f with these fns we get,

$$\mathrm{Re}(f) := \mathrm{Re} \circ f \quad ; \quad \mathrm{Im}(f) := \mathrm{Im} \circ f$$

We may think of $f : \mathbb{C} \to \mathbb{C}$ as a fn from $\mathbb{R}^2 \to \mathbb{R}^2$, in fact as two functions from $\mathbb{R}^2 \to \mathbb{R}$,

namely,
$$u : \mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \mapsto \mathrm{Re}(f)(x+iy)$$

$$v : \mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \mapsto \mathrm{Im}(f)(x+iy)$$

Thus, $f(x+iy) = u(x,y) + i\,v(x,y)$

(WARNING: the domain of f is $\mathbb{C}$, while of u, v is $\mathbb{R}^2$. The above equality is not equality of functions!! It is under the identification of $\mathbb{C}$ with $\mathbb{R}^2$.)

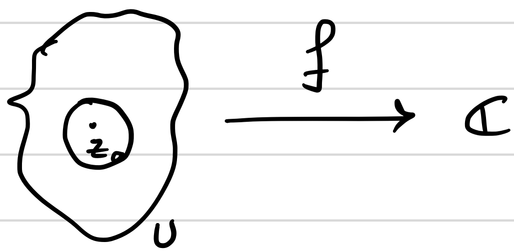
Eg: $f(z) = |z|^2$

$f(x+iy) = x^2+y^2$

$u(x,y) = x^2+y^2$; $v(x,y) = 0$

Limit of a function at a point z_0:

Let f be a function defined in a neighbourhood around z_0 (except possibly at z_0)



We say that f has a limit at z_0 if "there exists $l \in \mathbb{C} \ni$ given any $\varepsilon > 0$ $\exists\ \delta > 0$ such that $|f(z) - l| < \varepsilon$ $\forall\ 0 < |z - z_0| < \delta$".

We denote this information by writing $\lim\limits_{z \to z_0} f(z) = l$.

REMARK: $\lim\limits_{z \to z_0} f(z) = l \iff \lim\limits_{z \to z_0} u(x,y) = \operatorname{Re} l$ & $\lim\limits_{z \to z_0} v(x,y) = \operatorname{Im} l$ ☒

Eg: $f(z) = |z|$

$$\lim_{z \to 0} |z| = 0$$

($\because$ given $\varepsilon > 0$, choose $\delta = \varepsilon$ then
$\big||z| - 0\big| = |z| < \varepsilon \quad \forall\ 0 \le |z| < \varepsilon$)

<u>Arithmetic of limits</u>

Let f, g be two functions defined in a neighbourhood of z_0. Let $\lim_{z \to z_0} f(z) = l$ and $\lim_{z \to z_0} g(z) = l'$.

Then (i) $\lim_{z \to z_0} (f+g)(z) = l + l'$

(ii) $\lim_{z \to z_0} (f \cdot g)(z) = l l'$

(iii) if $l' \neq 0$ then $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{l}{l'}$

(iv) $\lim_{z \to z_0} c f(z) = c l$

Pf (as in the real case) $\blacksquare$

Continuity: Let f be a function defined in a neighbourhood of z_0 (including at z_0). We say that f is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

REMARK: ① f is continuous at z_0 iff u, v are continuous at (x_0, y_0).

② f is continuous at z_0 iff whenever $z_n \to z$, the sequence $f(z_n) \to f(z)$.
(Follows, since this is true for u, v.)

③ Let f be continuous at $g(z_0)$ and g be continuous at z_0 then
$f \circ g$ is continuous at z_0.
(Proof same as in the real case).

④ By the arithmetic of limits, it follows that the if f, g are continuous at z_0 then so is $f \pm g$, fg, cf, f/g (if $g(z_0) \neq 0$).

Some examples:

① $f(z) = \bar{z}$ $\quad \lim_{z \to z_0} \bar{z} = \lim_{z \to z_0} x - iy = \lim_{(x,y)\to(x_0,y_0)} x - i \lim y$

$$= x_0 - iy_0 .$$

② $f(z) = |z|$ $\quad \underbrace{\lim_{z \to z_0} |z| = |z_0|}$

$$\big||z| - |z_0|\big| \leq |z - z_0| \quad (\text{see, triangle inequality})$$

So, given $\varepsilon > 0$, choose $\delta = \varepsilon$ then

$$\big||z| - |z_0|\big| < \varepsilon \quad \forall\ |z - z_0| < \varepsilon .$$

③ $f(z) = \dfrac{|z|}{z}$, $\lim_{z \to 0} \dfrac{|z|}{z}$ does not exist, since

along the x-axis, $z = x$, $|z| = |x|$

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1 \neq \lim_{x \to 0^-} \frac{|x|}{x} = -1 .$$

However, since $|z|$ & $\dfrac{1}{z}$ are continuous

at $z \neq 0$ we get by the arithmetic of

continuous fun that $\dfrac{|z|}{z}$ is continuous at $z \neq 0$. ∎