Lecture 1

<u>Complex number</u> :- An ordered pair of real numbers (x, y) is a complex number.
Since $(x,y) = x(1,0) + y(0,1)$, denoting $(0,1)$ as i and $(x,0)$ as x, we represent (x,y) as $x+iy$

$$\underset{\text{real part}}{x} + i \underset{\text{imaginary part}}{y}$$

" $\mathbb{C}$ "
complex numbers

$\mathbb{C} = \mathbb{R} + i\mathbb{R}$

<u>How are complex numbers different from $\mathbb{R}^2$?</u>
From the above definition, it is not clear what the purpose of representing $\mathbb{R}^2$ differently is?
The answer is in the following statement

" $\mathbb{C}$ is a field w.r.t $+$ and $\cdot$ defined as follows :
$$(x_1 + iy_1) + (x_2 + iy_2) := x_1 + x_2 + i(y_1 + y_2)$$

$$(x_1 + iy_1) \cdot (x_2 + iy_2) := (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

In particular, $i^2 = -1$.

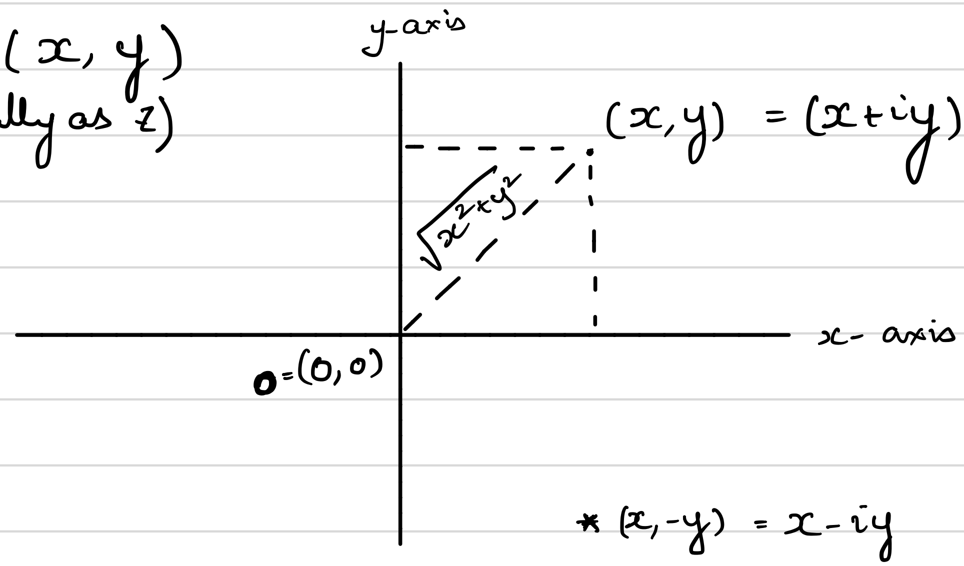
$$\frac{1}{(x_1 + iy_1)} = \frac{x_1 - iy_1}{x_1^2 + y_1^2} \qquad \text{(Check!)}$$

Geometry of $\mathbb{C}$

Remember that $\mathbb{C}$, physically is just $\mathbb{R}^2$.
It is different from $\mathbb{R}^2$, algebraicblly!

So, the geometry of $\mathbb{C}$ is that of $\mathbb{R}^2$

$x+iy = (x, y)$
(denoted usually as z)



length of $x+iy := +\sqrt{x^2+y^2} = |x+iy|$
Also called modulus of $x+iy$

conjugate of $x+iy$ = reflection of $x+iy$ about x-axis
$= x-iy$.

we denote the conjugate of $x+iy$ by $\overline{x+iy}$

Properties of $|x+iy|$, $\overline{x+iy}$

(1) $z\bar{z} = |z|^2 = \bar{z}z$

(2) $\operatorname{Re} z = \dfrac{z+\bar{z}}{2}$; $\operatorname{Im} z = \dfrac{z-\bar{z}}{2i}$

$-|z| \leq \operatorname{Re} z, \operatorname{Im} z \leq |z|$

3. $|z_1 z_2| = |z_1|\,|z_2| = |z_2 z_1|$

4. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \;;\; \overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

5. Triangle inequality:

$|z_1 + z_2| \le |z_1| + |z_2|$: Pf:
$$
\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\,\overline{(z_1 + z_2)} \\
&= (z_1 + z_2)\,(\bar{z}_1 + \bar{z}_2) \\
&= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 \\
&= |z_1|^2 + 2\,\mathrm{Re}\, z_1\bar{z}_2 + |z_2|^2
\end{aligned}
$$

On the otherhand,
$$(|z_1| + |z_2|)^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

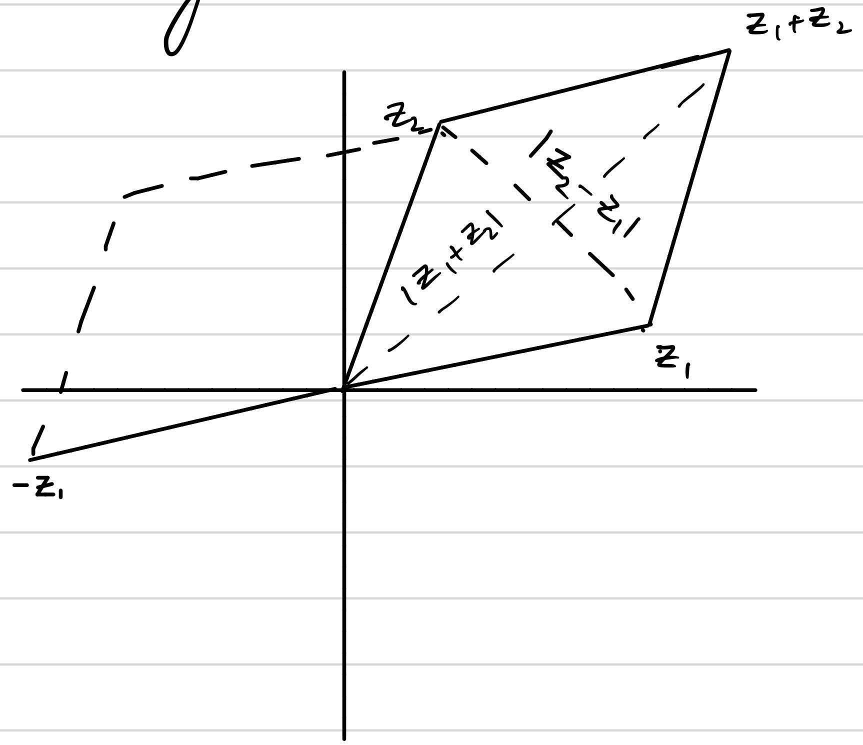
$$
\begin{aligned}
\therefore |z_1 + z_2|^2 - (|z_1| + |z_2|)^2 &= 2\,\mathrm{Re}\, z_1\bar{z}_2 - 2|z_1||z_2| \\
&= 2\,\mathrm{Re}(z_1\bar{z}_2) - 2\,(z_1\bar{z}_2\,\bar{z}_1 z_2) \\
&= 2\left(\mathrm{Re}(z_1\bar{z}_2) - |z_1\bar{z}_2|\right) \le 0
\end{aligned}
$$

Hence, $|z_1 + z_2| \le |z_1| + |z_2|$

6. $|z_1| = |z_1 - z_2 + z_2| \le |z_1 - z_2| + |z_2|$

$$
\left.
\begin{aligned}
\therefore\ & |z_1| - |z_2| \le |z_1 - z_2| \\
\text{|||}^{\text{ly}}\ & |z_2| - |z_1| \le |z_1 - z_2|
\end{aligned}
\right\}
\therefore\ \big||z_1| - |z_2|\big| \le |z_1 - z_2|.
$$

(7) Parallelogram law:



$$\boxed{|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)}$$

Pf: $|z_1+z_2|^2 = (z_1+z_2)(\bar{z}_1+\bar{z}_2) = |z_1|^2 + 2\operatorname{Re} z_1\bar{z}_2 + |z_2|^2$

$|z_1-z_2|^2 = (z_1-z_2)(\bar{z}_1-\bar{z}_2) = |z_1|^2 - 2\operatorname{Re} z_1\bar{z}_2 + |z_2|^2$

$\Rightarrow |z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$

Polar coordinates of a complex number

Given a complex number $z = x+iy \ (\neq 0)$.

Let $x = r\cos\theta$

$y = r\sin\theta$. Then $z = r(\cos\theta + i\sin\theta)$.

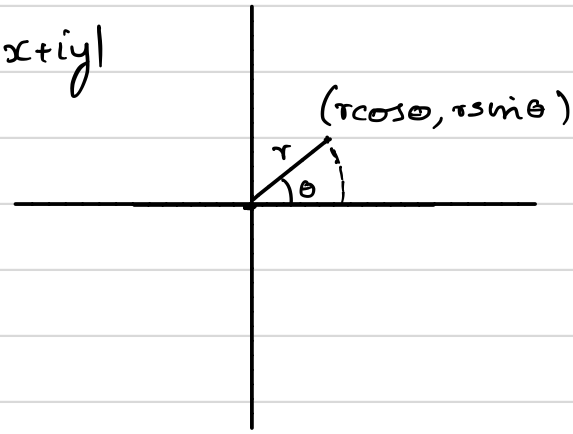
Set $\cos\theta + i\sin\theta$ as $e^{i\theta}$.

By polar coordinate for $x+iy$, we mean the expression $re^{i\theta}$.

$r = |r\cos\theta + i\sin\theta| = |x+iy|$

and

$\frac{y}{x} = \tan\theta$



θ is called the argument of $z = x+iy = re^{i\theta}$.

It is denoted as $\arg(z)$

$\arg : \mathbb{C} \to \mathbb{R}$ is a multivalued fn

$\because e^{i\theta} = e^{i(\theta+2n\pi)} \quad \forall n \in \mathbb{Z}$.

We associate to arg a single-valued function by restricting θ between $-\pi$ and π (or any interval of length 2π).

This single-valued function is denoted as $\mathrm{Arg} : \mathbb{C} \to (-\pi, \pi]$

$z \mapsto \arg z \ni -\pi < \arg z \leq \pi$

Arg z is called the principal argument of z.

Properties of the polar form:

① $(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$

② $\dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

From the above, we have the following properties of arg and Arg:

① $\arg(z_1 z_2) = \arg z_1 + \arg z_2 \pmod{2\pi}$

② $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \pmod{2\pi}$

③ $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$ if the sum is $\leq \pi$

<u>Remark</u>: $r_1 e^{i\theta_1} = r_2 e^{i\theta_2}$

$\Leftrightarrow$ $r_1 = r_2$ and $\theta_1 = \theta_2 + 2n\pi$

(Indeed, $|r_1 e^{i\theta_1}| = |r_2 e^{i\theta_2}|$ and $e^{i(\theta_1-\theta_2)} = 1$.
ie $r_1 = r_2$
& $(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)) = 1$

$\Leftrightarrow$ $\cos(\theta_1-\theta_2) = 1$ & $\sin(\theta_1-\theta_2) = 0$

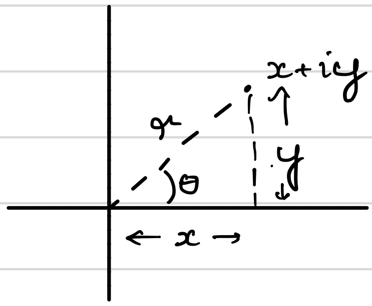
$\Leftrightarrow$ $\theta_1 - \theta_2 = 2n\pi$

Given $x+iy$, can we express r and θ in terms of x, y?.

For r, the expression is immediate, since $re^{i\theta} = x+iy \Rightarrow |re^{i\theta}| = |x+iy|$

$$\Rightarrow \quad r = +\sqrt{x^2+y^2}$$

How about $\theta = \operatorname{Arg} z$?

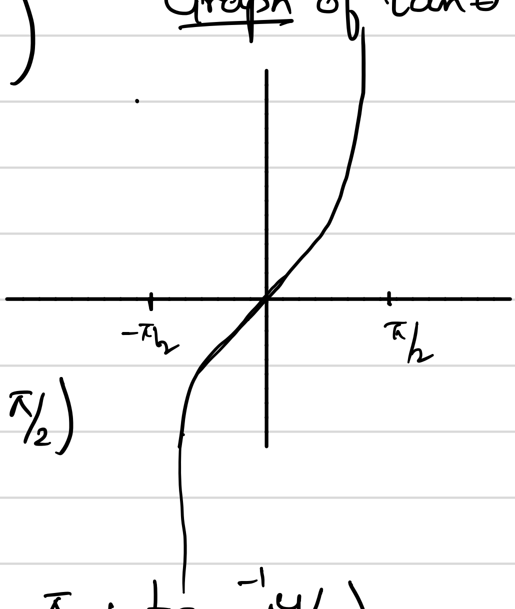


$\tan\theta = y/x$

Recall, that tan is invertible in $(-\pi/2, \pi/2)$

So $\tan^{-1}(y/x) \in (-\pi/2, \pi/2)$

Graph of $\tan\theta$



We know that $y/x = \frac{-y}{-x}$.

So, for $x>0, y>0$ and $\underbrace{x<0, y<0}_{3^{rd}\text{ quadrant}}$ $\Big\}$ $\tan^{-1} y/x \in [0, \pi/2)$

to get $\operatorname{Arg} z$, we take $-\pi + \tan^{-1}(y/x)$ for z in 3^{rd} quadrant.

Similarly, for $x>0, y<0$ and $\underbrace{x<0, y>0}_{2^{nd}\text{ quadrant}}$ $\Big\}$ $\tan^{-1}(y/x) \in (-\pi/2, 0)$

So to get $\operatorname{Arg} z$ for z in 2^{nd} quadrant, we take $\pi + \tan^{-1}(y/x)$

When $x=0, y>0$, $\operatorname{Arg} z = \pi/2$.
$x=0, y<0$ $\operatorname{Arg} z = -\pi/2$.

§ Continuity of Arg z in $\mathbb{C} \setminus \{x+iy \,/\, x \le 0, y=0\}$

Recall that $\tan^{-1}(y/x) : \mathbb{R}^2 \setminus (x=0) \longrightarrow \mathbb{R} \to (-\pi/2, \pi/2)$

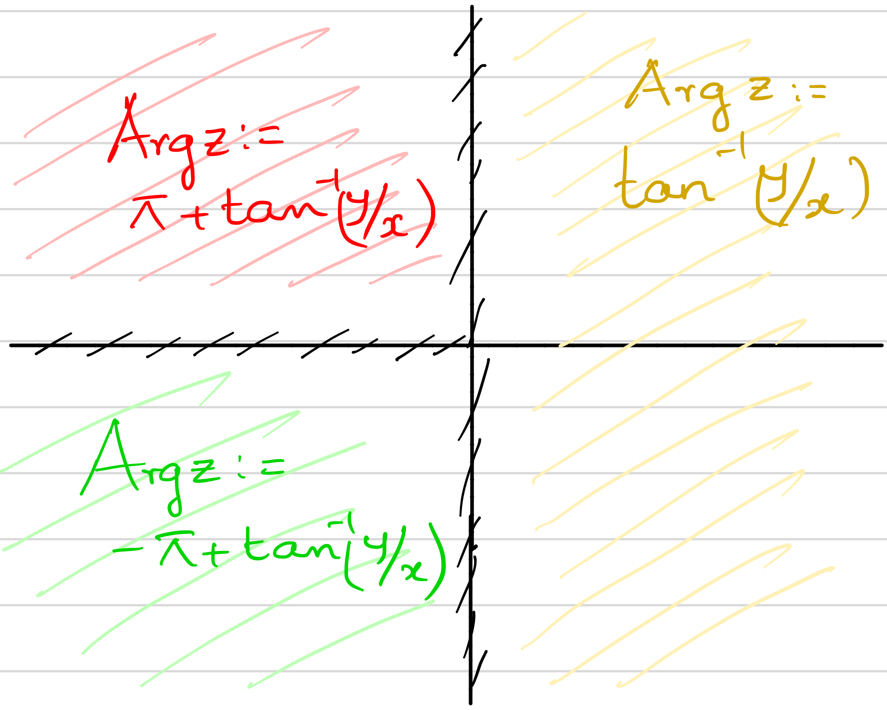
is a composition of $(x,y) \longmapsto y/x$

and $u \longmapsto \tan^{-1} u$.

When $x \ne 0$, $(x,y) \mapsto y/x$ is continuous

and $u \longmapsto \tan^{-1} u$ is differentiable.

hence, the map $\tan^{-1}(y/x)$ is continuous on $\mathbb{R}^2 \setminus \{y\text{-axis}\}$.



Define

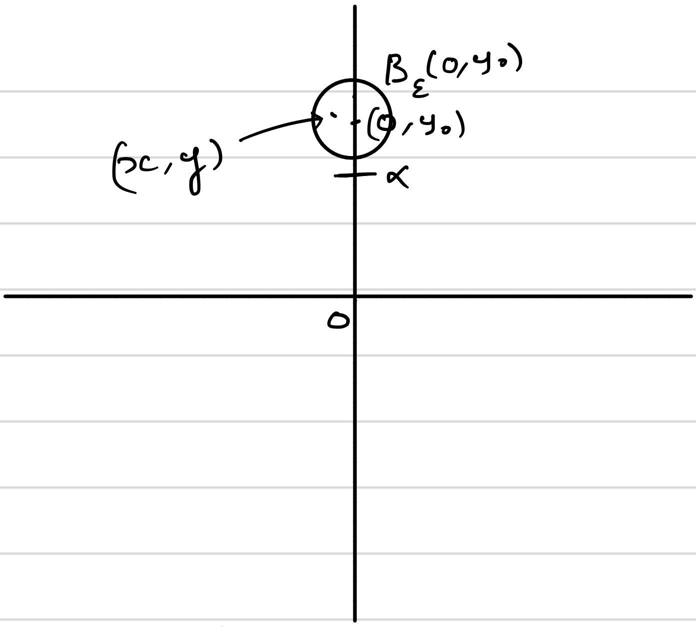
$A : \mathbb{R}^2 \setminus (0,0) \to \mathbb{R}$

$(x,y) \mapsto \operatorname{Arg}(x+iy)$

By continuity of $\tan^{-1}(y/x)$, $A(x,y)$ is clearly continuous in each of right-half plane, 2^{nd} quadrant, 3^{rd} quadrant.

We look at the axes separately

6.3



Let's consider $(0,y_0) \in$ x-axis.

Since $y_0 \neq 0$, $|y_0| > \alpha > 0$. Choose $\varepsilon << 0$

$\ni (x,y) \in B_\varepsilon(0,y_0) \Rightarrow |y| > \alpha$.

Then $\left|\frac{y}{x}\right| > \frac{\alpha}{|x|} \quad \forall (x,y) \in B_\varepsilon(0,y_0)$

and $\frac{\alpha}{|x|} \to \infty$ as $x \to 0$

$\therefore \tan^{-1}(y/x) \to \frac{\pi}{2}$ if $x \to 0^+$

$\left.\begin{array}{l} \tan^{-1}(y/x) \to -\pi/2 \\ \Rightarrow \pi + \tan^{-1}(y/x) \to \pi/2 \end{array}\right\}$ if $x \to 0^-$

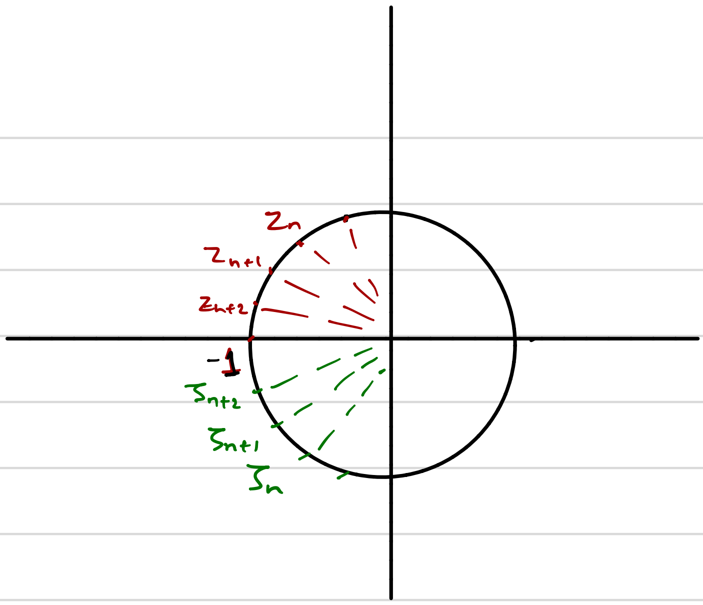
More precisely, given $\varepsilon > 0$ $\exists \delta > 0$ $\ni (x,y) \in B_\delta(0,y_0)$ $\Rightarrow |\tan^{-1} y/x - \pi/2| < \varepsilon$ if $(x,y) \in 1^{st}$ quadrant

$|\pi + \tan^{-1} y/x - \pi/2| < \varepsilon$ if $(x,y) \in 2^{nd}$ quadrant

Thus $\lim_{(x,y) \to (0,y_0)} A(x,y) = \pi/2 = A(0,y_0)$

Similarly, we check that A is continuous for $(0,y_0)$ $(y_0 < 0)$. Thus A is continuous on the y-axis.

Let's consider the negative x-axis now.



Consider $\{z_n\}$ and $\{\zeta_n\}$ as above

$\ni \quad A(z_n) = \pi - \pi/n \quad \& \quad z_n \to -1$

$A(\zeta_n) = -\pi + \pi/n . \quad \& \quad \zeta_n \to -1$

Then $\lim_{n\to\infty} A(z_n) = \pi = \operatorname{Arg}(-1)$

$\lim_{n\to\infty} A(\zeta_n) = -\pi \neq \operatorname{Arg}(-1)$

Thus, A is not continuous at -1.

Similarly, for any $\alpha < 0$ on the negative real axis.

$\blacksquare$

De Moivre's formula

$$(re^{i\theta})^n = r^n e^{in\theta} \quad \text{(by property 1 above)}$$

$$\therefore \left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$$

Application:

Find the distinct values of $\omega \ni \omega^n = z$ for a fixed $z \in \mathbb{C} \setminus \{0\}$.

Let $\omega = re^{i\theta}$ and $z = se^{i\phi}$.

Then $\omega^n = r^n e^{in\theta} = se^{i\phi}$

$\iff r^n = s$ and $n\theta = \phi + 2k\pi$

ie $r = \sqrt[n]{s}$ and $\theta = \dfrac{\phi + 2k\pi}{n}$

So, the "distinct" $\omega \ni \omega^n = z$ are given by

$$\sqrt[n]{s}\, e^{i\left(\frac{\phi}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0, 1, \ldots, n-1$$

(Indeed, for $m \in \mathbb{Z}$, $m = nq + k$ so $\dfrac{2m\pi}{n} = 2q\pi + \dfrac{2k\pi}{n}$

& $e^{i2q\pi} = 1 \quad \therefore e^{i\left(\frac{2m\pi}{n}\right)} = e^{i\frac{2k\pi}{n}}$)

Principal root of $\omega^n = z$ is $\sqrt[n]{s}\, e^{i\phi/n}$. where $-\pi < \phi \le \pi$

$\boxtimes$