MSO202A COMPLEX ANALYSIS
Assignment 2

Exercise Problems:

1. Let $z = x + iy$ and $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$. Write $f(z)$ as a function of z and $\overline{z}$.

2. Verify Cauchy-Riemann equation for $z^2,\ z^3$.

3. Using the relations $x = \dfrac{z+\overline{z}}{2}, y = \dfrac{z-\overline{z}}{2i}$ and the chain rule show that $\dfrac{\partial}{\partial z} = \dfrac{1}{2}(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y})$; $\dfrac{\partial}{\partial \overline{z}} = \dfrac{1}{2}(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y})$.

4. Let $z, w \in \mathbb{C}, |z|, |w| < 1$ and $\overline{z}w \neq 1$. Prove that $\frac{|w-z|}{|1-\overline{w}z|} < 1$. Further, show that the equality holds if either $|z| = 1$ or $|w| = 1$.

5. Determine *all* $z \in \mathbb{C}$ for which each of the following power series is convergent.
   a)$\sum \frac{z^n}{n^2}$ b)$\sum \frac{z^n}{n!}$ c)$\sum \frac{z^n}{2^n}$ d)$\sum \frac{1}{2^n}\frac{1}{z^n}$.

6. Find all $z \in \mathbb{C}$ such that $|e^z| \leq 1$.

7. Show that the CR-equations in polar form are given by: $u_r = \frac{1}{r}v_\theta$ and $u_\theta = -rv_r$.

Problem for Tutorial:

1. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. For a fixed w in $\mathbb{D}$, with $|w| < 1$, define the mapping $F : z \mapsto \frac{w-z}{1-\overline{w}z}$. Show that
   (a) F is a map from $\mathbb{D}$ to itself;
   (b) $F(0) = w$ and $F(w) = 0$;
   (c) $|F(z)| = 1$ if $|z| = 1$;
   (d) $F : \mathbb{D} \to \mathbb{D}$ is bijective.

2. Let R be the radius of convergence of $\sum_n a_n z^n$. For a fixed $k \in \mathbb{N}$, find the radius of convergence of (a) $\sum {a_n}^k z^n$, (b) $\sum a_n z^{kn}$.

3. (a) Show that f satisfies the CR-equations if and only if $\frac{\partial}{\partial \overline{z}}f = 0$. (Recall from Ex. 3 above that $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$.) Moreover, if f is analytic then $f'(z) = \frac{\partial}{\partial z}f$.

4. Consider the following functions
   (a)
$$f(x+iy) = \begin{cases} \dfrac{xy(x+iy)}{x^2+y^2} & \text{if } x+iy \neq 0 \\ 0 & \text{if } x+iy = 0 \end{cases}$$

(b) $f(x+iy) = \sqrt{|xy|}$

Show that f satisfies the CR-equations but it is not differentiable at the origin.