MSO202A-Complex Analysis

Quiz

23 August, 2018

Name: ______________________________ **Set A**

Roll Number: _______________ **Max. time: 45 minutes**

Instruction:

- **Please enter your NAME and ROLL NUMBER in the space provided. Booklets without name and roll number will not be graded.**
- **This answer booklet has four printed pages with three questions. Space for rough work is provided on the fifth and sixth pages. Check to see if any of the print is faulty or missing. In such a case, ask for a replacement immediately.**
- **Please answer each question in the space provided only. Answers written outside the space provided for it WILL NOT be considered for grading. So remember to use space judiciously.**
- **Rough work must be done in the space provided for it. Extra rough sheets will be provided on request. Enter your name and roll number on rough sheets as well. However, they WILL NOT be accepted back along with the answer booklet.**

Answer all the questions.

1. (a) Using root test, determine the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ where

$$a_{2k} = \left(\frac{2+i}{3\sqrt{5}}\right)^{2k}, \quad a_{2k+1} = \left(\frac{2-i}{\sqrt{15}}\right)^{2k+1}, \text{ for } k = 0, 1, \ldots$$

(4)

Proof:

Step 1: $\sqrt[2k]{\left|\frac{2+i}{3\sqrt{5}}\right|^{2k}} = \left|\frac{2+i}{3\sqrt{5}}\right| = \frac{\sqrt{5}}{3\sqrt{5}} = \frac{1}{3}$ (1 mark)

Step 2: $\sqrt[2k+1]{\left|\frac{2-i}{\sqrt{15}}\right|^{2k+1}} = \left|\frac{2-i}{\sqrt{15}}\right| = \frac{\sqrt{5}}{\sqrt{3}\sqrt{5}} = \frac{1}{\sqrt{3}}$ (1 mark)

Step 3: Every convergent subsequence of $\{\sqrt[n]{|a_n|}\}$ is eventually $\{1/3\}$ or $\{1/\sqrt{3}\}$. Since $\frac{1}{3} < \frac{1}{\sqrt{3}}$, we have $\limsup \sqrt[n]{|a_n|} = \frac{1}{\sqrt{3}}$. (1 mark)

Step 4: By root test the given series converges if $\limsup \sqrt[n]{|a_n|}|z| < 1$, *i.e.*, $|z| < \sqrt{3}$ and diverges if $\limsup \sqrt[n]{|a_n|}|z| > 1$, *i.e.*, , $|z| < \sqrt{3}$. Hence the radius of convergence of the given power series is $\sqrt{3}$. (1 mark)

(b) **Notation:** $\mathbb{Z}$ *is the set of integers,* $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{R}$ *is the set of real numbers.* Let $\alpha \in \mathbb{Z}, \beta \in \mathbb{R} \setminus \mathbb{Z}$. For each such α, β, define

$$P_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n)}{\beta(\beta+1)\cdots(\beta+n)} z^n.$$

i) Determine the radius of convergence of $P_{\alpha,\beta}(z)$ when $\alpha \in \mathbb{N}$. (3)

ii) Determine the radius of convergence of $P_{\alpha,\beta}(z)$ when $\alpha \notin \mathbb{N}$. (3)

Proof: (i) $\alpha \in \mathbb{N}$

Step 1: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\alpha(\alpha+1)\cdots(\alpha+n+1)}{\beta(\beta+1)\cdots(\beta+n+1)} \frac{\beta(\beta+1)\cdots(\beta+n)}{\alpha(\alpha+1)\cdots(\alpha+n)}\right| = \frac{|\alpha+n+1|}{|\beta+n+1|}$. (1 mark)

Step 2: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|(\alpha/(n+1))+1|}{|(\beta/(n+1))+1|} = 1$ (1 mark)

Step 3: Radius of convergence by Ratio test is 1. (1 mark)

(ii) $\alpha \notin \mathbb{N}$

Step 1: $\alpha = -m$ for some $m \in \mathbb{N}$, hence $\alpha + m = 0 \Rightarrow \frac{\alpha(\alpha+1)\cdots(\alpha+m)}{\beta(\beta+1)\cdots(\beta+m)} = 0$. (1 mark)

Step 2: $\frac{\alpha(\alpha+1)\cdots(\alpha+n)}{\beta(\beta+1)\cdots(\beta+n)} = 0$ for all $n \geq m$, hence $P_{\alpha,\beta}(z)$ is a polynomial in z. (1 mark)

Step 3: Hence the radius of convergence is ∞. (1 mark)

Comments:

- Solutions to $1(a)$ which directly go to Step 3 or Step 4 skipping earlier steps will not be awarded marks for the preceding steps.

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2. (a) Use De Moivre's formula to obtain all the **distinct** roots of $w^5 - 1$. Express them in polar form. (4)

(b) Use (a) to obtain all the roots of $(1-z)^5 = (1+z)^5$. Express them in the form $x + iy$. (6)

Proof:(a) Let $w = r(\cos\theta + i\sin\theta)$.

Step 1: By De Moivre's theorem, $w^5 = r^5(\cos 5\theta + i \sin 5\theta)$. (1 mark)

Step 2: So $w^5 = 1 \Rightarrow r^5(\cos 5\theta + i\sin 5\theta) = 1 \Rightarrow r^5 \cos 5\theta = 1,\ r^5 \sin 5\theta = 0$. (1 mark)

Step 3: Hence $r^5 = 1,\ 5\theta = 2n\pi \Rightarrow \theta = 2n\pi/5$ where $n \in \mathbb{Z}$ and $r = 1$. (1 mark)

Step 4: Distinct roots in polar form: $e^{i2n\pi/5}$, $n = 0, 1, 2, 3, 4$. (1 mark)

(b) Let $\left(\frac{1-z}{1+z}\right)^5 = 1$.

Step 1: $\frac{1-z}{1+z} = e^{i2n\pi/5}$ (1 mark)
where $n = 0, 1, 2, 3, 4$. (1 mark)

Step 2:

$$(1-z) = (1+z)e^{i2n\pi/5} \Rightarrow z = \frac{1 - e^{i2n\pi/5}}{1 + e^{i2n\pi/5}}$$

(2 mark)

Step 3:

$$z = \frac{e^{in\pi/5}(e^{-in\pi/5} - e^{in\pi/5})}{e^{in\pi/5}(e^{-in\pi/5} + e^{in\pi/5})} = \frac{-2i\sin(n\pi/5)}{2\cos(n\pi/5)} = -i\tan(n\pi/5)$$

(1 mark)

(OR) Other methods of simplifying may also be used to arrive at the alternate expressions in Step 3.

Step 4: Roots are $-i\tan(n\pi/5)$ for $n = 0, 1, 2, 3, 4$. (1 mark)

(OR) Alternately: Let $\left(\frac{1+z}{1-z}\right)^5 = 1$.

Step 1: $\frac{1+z}{1-z} = e^{i2n\pi/5}$ (1 mark)
where $n = 0, 1, 2, 3, 4$. (1 mark)

Step 2:

$$(1+z) = (1-z)e^{i2n\pi/5} \Rightarrow z = \frac{e^{i2n\pi/5} - 1}{1 + e^{i2n\pi/5}}$$

(2 mark)

Step 3:

$$z = \frac{e^{in\pi/5}(e^{in\pi/5} - e^{-in\pi/5})}{e^{in\pi/5}(e^{-in\pi/5} + e^{in\pi/5})} = \frac{2i\sin(n\pi/5)}{2\cos(n\pi/5)} = i\tan(n\pi/5)$$

(1 mark)

Step 4: Roots are $i\tan(n\pi/5)$ for $n = 0, 1, 2, 3, 4$. (1 mark)

Comments:

- In 2(a), if De Moivre's formula is stated wrongly or not stated, marks assigned to Step 1 will be deducted. Similarly, if one directly states that De Moivre's formula implies Step 3, missing Step 1 and Step 2, marks are awarded only from Step 3 onwards.
- In 2(b), the question insists on using (a), so solutions obtained by alternate methods not using (a) are given only a maximum of 2 marks based on the correctness of the solution.
- In 2(b) the final answer is expected to be in $x + iy$ form. So marks assigned to Step3 and Step 4 are not awarded in such a case.

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3. Consider $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = zRe(z)$. Determine the set S of **all** $z \in \mathbb{C}$ such that f is differentiable at z. Find the derivative $f'(z)$ for every $z \in S$. *Statements used should be written clearly and precisely. They carry marks as well.* (10)

Proof:

Step 1: $f(z) = z\ Re(z) \Rightarrow u(x, y) = x^2$, (1 mark)
$v(x, y) = xy$. (1 mark)

Step 2: Partial derivatives $u_x = 2x$, $u_y = 0$ (1 mark)
$v_x = y$, $v_y = x$. (1 mark)

Step 3: If f is differentiable at z then CR-equations hold, so $u_x = v_y$, $v_x = -u_y$. (1 mark)

Step 4: Thus $2x = x$, $y = 0 \Rightarrow x = 0 = y$. (1 mark)

Step 5: Hence f is not differentiable at $x + iy \neq 0$. (1 mark)

Step 6: At $z = 0$, $\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{x+iy\to 0} x = 0$. (2 mark)
(OR) If the partial derivatives of u and v exist , are continuous and satisfies CR-equations at a then f is differentiable at a. Here f satisfies CR equation at 0 and the partial derivatives exist and are continuous at 0, so f is differentiable at 0.

Step 7: Hence f is differentiable at 0 and the derivative $f'(0) = 0$. (1 mark)
(OR) The derivative at 0 is given by $u_x(0, 0) + iv_x(0, 0) = 0$

Step 8: $S = \{0\}$.

Comments:

- Alternate solutions in which $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ is shown to not exist at $z_0 = 0$ are awarded marks assigned to Steps 1-5.
- In Step 5, the statement "Hence f is differentiable only at $z = 0$" is not regarded as the same as saying "Hence f is not differentiable at $z \neq 0$." They are two different statements. No marks are awarded to the wrong statement (though you may claim, you meant the latter!).
- *Statements used should be stated clearly and precisely"*. So, missing statements in Step 3 or Step 6 would amount to a penalty of 1 mark each.