LECTURE - 7

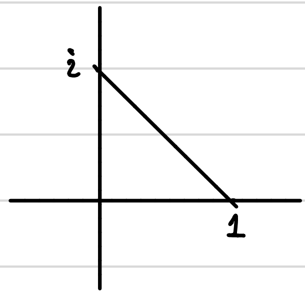
Contour integration

Lecture - 7 : Contours, contour integration:

A **curve** α is a continuous function from a bounded, closed interval $[a,b]$ to $\mathbb{C}$. The initial and final point of the curve are $\alpha(a)$ and $\alpha(b)$, respectively.

Eg: $\alpha : [0,1] \to \mathbb{C}$

$\qquad t \mapsto ti + (1-t)$



at $t=0$, $\alpha(0) = 1$

$t = 1$, $\alpha(1) = i$

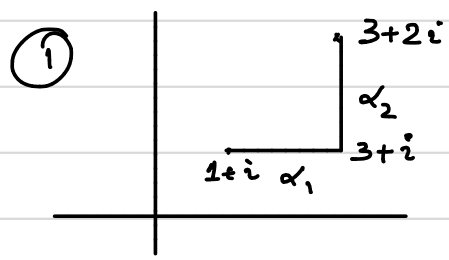
Abuse of notation: The image $\{\alpha(t) / t \in [a,b]\}$ is often also called a curve.

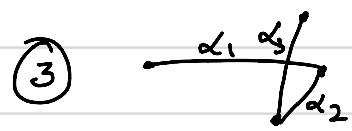
A curve α is said to be **smooth** if α' exists and is continuous, ie if $\alpha(t) = x(t) + iy(t)$ then $\alpha'(t) := x'(t) + iy'(t)$ exists and x', y' are continuous.

Eg: $\alpha'(t)$ in the above example is $-1+i$

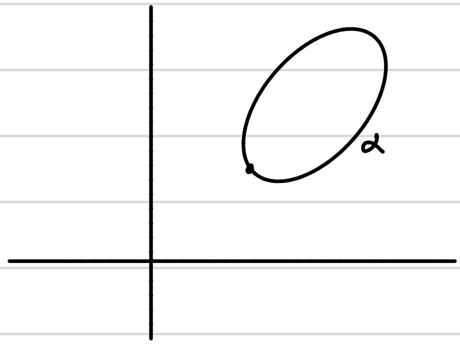
A **contour** in $\mathbb{C}$ is obtained by joining finitely many smooth curves.

Eg:





A closed curve (contour) is a curve (contour) whose initial and final points are the same.



A simple closed curve (contour) is a curve (contour) which does not cross itself with the exception of its initial and final points, ie

$\alpha : [a,b] \to \mathbb{C}$ is simple closed if α is a curve such that

$$\alpha(t) \neq \alpha(t') \text{ for } t \neq t' \in (a,b)$$

and $\alpha(a) = \alpha(b)$

not simple. simple

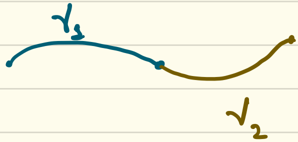
Opposite curve: Let $\gamma : [a,b] \to \mathbb{C}$ be a curve. Then its opposite is the curve

$$-\gamma : [a,b] \longrightarrow \mathbb{C}$$

given by $t \longmapsto \gamma(a+b-t)$

Join of two curve γ_1 & γ_2:

$$\gamma_1 : [a_1, b_1] \longrightarrow \mathbb{C}$$



$$\gamma_2 : [a_2, b_2] \longrightarrow \mathbb{C}$$

$$\ni \gamma_1(b_1) = \gamma_2(a_2)$$

$$\gamma_1 + \gamma_2 = \gamma : [a_1, b_1 + (b_2 - a_2)] \longrightarrow \mathbb{C}$$

$$t \longmapsto \gamma_1(t) \quad \text{if } t \in [a_1, b_1]$$

$$t \longmapsto \gamma_2(t - b_1 + a_2)$$

$$\text{if } t \in [b_1, b_1 + (b_2 - a_2)]$$

Integration of a complex-valued function of a real variable:

Let $f : [a, b] \longrightarrow \mathbb{C}$ be a continuous function.

Then $$\int_a^b f(t)\,dt := \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt$$

where $f(t) = x(t) + iy(t)$.

Integration of a function along a contour γ:

Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth curve.

Let $f : \mathbb{C} \to \mathbb{C}$ be continuous.

Then $\displaystyle\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$.

Let $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_r$ be a contour.

Then $\displaystyle\int_\gamma f(z)\,dz := \int_{\gamma_1} f(z)\,dz + \cdots + \int_{\gamma_r} f(z)\,dz$.

Evaluating $\displaystyle\int_\gamma f(z)\,dz$

first principles

anti-derivatives
(Fundamental thm)

Cauchy's thm

Example: (The fundamental integral)

Let $z_0 \in \mathbb{C}$, $r > 0$ and $n \in \mathbb{Z}$.

$C_{z_0, r}$ = circle of radius r centered at z_0

given by $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$
$t \longmapsto z_0 + re^{it}$

$$\int_{C_{z_0,r}} (z - z_0)^n \, dz = \int_0^{2\pi} (z_0 + re^{it} - z_0)^n \cdot ire^{it} \, dt$$

$$= \int_0^{2\pi} (re^{it})^n \, ire^{it} \, dt = ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt$$

$$= ir^{n+1} \left(\int_0^{2\pi} \cos(n+1)t \, dt + i \int_0^{2\pi} \sin(n+1)t \, dt \right)$$

if $n + 1 \neq 0$

$$= ir^{n+1} \left(\frac{-\sin(n+1)t}{n+1} + \frac{i(\cos(n+1)t)}{n+1} \right) \Bigg|_0^{2\pi}$$

$$= \frac{ir^{n+1}}{n+1} \times 0 = 0$$

if $n+1=0$ i.e. $n=-1$

then $\int_0^{2\pi} (re^{it})^n \, ire^{it} dt$

$= i\int_0^{2\pi} dt = 2\pi i$

Thus, $$\boxed{\int_{C_{z_0,r}} (z-z_0)^n dz = 0 \quad \text{if } n \neq -1 \\ = 2\pi i \quad \text{if } n = -1}$$

§ Integral via anti-derivatives:

domain = open + connected.

Theorem: Let f be a continuous function defined on a domain D. Suppose there exists F on D $\ni$ $\boxed{F' = f}$.

Let $z_1, z_2 \in D$. Then for any contour C starting at z_1 and ending at z_2, the integral $\int_C f(z)\,dz = F(z_2) - F(z_1)$

In particular, the integral is independent of the contour.

Proof: Let the contour C be given by $\gamma : [a,b] \longrightarrow \mathbb{C}$. $(\gamma(a) = z_1;\ \gamma(b) = z_2)$.

Then $\frac{d}{dt}(F(\gamma(t))) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t)$

Hence $$\int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b \frac{d}{dt}(F(\gamma(t))\,dt$$

$$= F(\gamma(b)) - F(\gamma(a))$$

$$= F(z_2) - F(z_1)$$

$\blacksquare$

NOTE: If $z_1 = z_2$ ie if C is a closed contour then $\int_C f(z)\,dz = 0$

Examples: (i) $\displaystyle\int_{z_1}^{z_2} z\,dz = \frac{z^2}{2}\Big|_{z_1}^{z_2} = \frac{z_2^2 - z_1^2}{2}$

(ii) $\displaystyle\int_{z_1}^{z_2} \frac{1}{z}\,dz = \operatorname{Log} z_2 - \operatorname{Log} z_1$

Properties of contour integrals:

① $\displaystyle\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz$

② $\displaystyle\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$

③ (ML-inequality): Let $f: D \to \mathbb{C}$ be a continuous function on a domain D.

Let C be a contour in D given by $\gamma: [a,b] \to \mathbb{C}$

If $|f(z)| \leq M \;\forall\, z \in C$, then $\left|\displaystyle\int_C f(z)\,dz\right| \leq M\ell$

where $\ell = \underbrace{\displaystyle\int_a^b |\gamma'(t)|\,dt}_{\text{length of } \gamma}$.

1 & 2 are consequences of the corresponding statement for real-valued functions of a real variable.

Pf of 3:

$$\left| \int_C f(z)\,dz \right| = \left| \int_a^b f(\gamma(t))\,\gamma'(t)\,dt \right|$$

$$\leq \int_a^b \left| f(\gamma(t))\,\gamma'(t) \right| dt$$

$$= \int_a^b \left| f(\gamma(t)) \right| \left| \gamma'(t) \right| dt \leq M\,\ell .$$