Department of Mathematics and Statistics, IIT Kanpur

MSO202A Final Examination 2017-I

Maximum Marks: 70 Time: 2 Hours

Note: 1. Write your Name and Roll Number on the answer sheet, otherwise 3 marks will be deducted.
2. Number the pages of your answer booklet.
3. Answer all parts of a question in one place.
4. Make a table on the front page of the answer booklet indicating the page number on which each question has been answered.

1a. Find the principal value of $(-i)^{i/3}$. [4]

1b. Suppose that the functions $z \to f(z)$ and $z \to \overline{f(z)}$ are both analytic on an open connected subset D in $\mathbb{C}$. Show that f is identically constant on D. [5]

1c. Find the the Laurent series expansion for the function $f(z) = \dfrac{z}{(z-1)(z-3)}$ valid in $\{z \in \mathbb{C} : 0 < |z-1| < 2\}$, mentioning the general m-term. [5]

2a. Is it possible to find a polynomial $P(z)$ with complex coefficients such that $P(n) = (-1)^n$? Give reasons for your answer. Does there exist an entire function with the same property ? Justify your answer. [4]

2b. Let C be the positively oriented circle $|z| = 3$. For $w \in \mathbb{C}$, such that $|w| \neq 3$, let $g(w) = \int_C \frac{2z^3 - z - 2}{(z-w)^3}\, dz$. Find the value of $g(2)$ and the value of $g(w)$, for w such that $|w| > 4$. [5]

2c. Give an example of a function with removable singularity at $z = \pi$, a pole of order 2 with residue equal to 1 at $z = 0$, and an essential singularity with residue equal to 0 at $z = -1$. Explain it briefly. [5]

3a. Let $f(z) = \sin z$, $g(z) = \sin 2z$ and $a_n = n\pi$. Then $f(a_n) = g(a_n)$ for $n = 1, 2, 3, \cdots$ but $f \neq g$ on $\mathbb{C}$. Does this contradict the identity theorem. Justify your answer. [4]

3b. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Show that the image of f intersects the set $A = \{z : |z| \leq 1\}$. [5]

3c. Prove or disprove. $\displaystyle\int_0^{2\pi} e^{e^{i\theta}}\, d\theta = 2\pi.$ [5]

$e^{e^{i\theta}} d\theta$ $e^{i\theta} = z$ $\oint \frac{e^z}{iz}\, dz$

$dz = i e^{i\theta} d\theta$

$dz = iz\, d\theta$

$z^2 + 2az + 1$

4a. For $a > 1$, evaluate the integral $\displaystyle\int_C \frac{dz}{z^2+2az+1}$: around the counter-clockwise oriented circle $C : |z| = 1$. [4]

4b. Evaluate the integral $\displaystyle\int_C \frac{e^{-1/z^2}}{z^3}\, dz$: around the counter-clockwise oriented circle C : $|z - \frac{1}{2}| = 1$. [3]

4c. Let D be a simply connected domain and let $f : D \to \mathbb{C}$ be analytic except for a pole of order m at $z = a$. Let C_r be the circle $|z - a| = r$, oriented counterclockwise. Show that the function $\frac{f'(z)}{f(z)}$ has a simple pole at $z = a$ and hence conclude that $\lim_{r\to 0} \int_{C_r} \frac{f'(z)}{f(z)}\, dz = -2\pi i m$. [7]

5a. Let $p(z)$ be a non-constant polynomial with complex coefficients satisfying $|p(z)| = 1$ whenever $|z| = 1$. Show that $p(z)$ has a root in the unit disc $\mathbb{D} = \{z : |z| < 1\}$. [6]

5b. For $a > 0$ and $\xi > 0$, show that $\displaystyle\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi}\, dx = e^{-2\pi a\xi}$. [8]

$\frac{g}{(z-a)^m}$ $\qquad z = Re^{i\theta}$ $\qquad f'(a) = \int \frac{f(z)}{(z-a)^2}$

$f(z) = \frac{g(z)}{(z-a)^m}$

$f'(z) = \frac{(z-a)^m g'(z) - m(z-a)^{m-1} g(z)}{(z-a)^{2m}}$

$= \frac{g'(z)}{(z-a)^m} - \frac{m\, g(z)}{(z-a)^{m+1}}$

f

$\frac{f'}{f} = \frac{g'(z)}{g(z)} - \frac{m}{z-a}$