MSO202A COMPLEX ANALYSIS
Assignment 3

Exercise Problems:

1. (a) The hyperbolic functions $\cosh z$ and $\sinh z$ are defined as $\cos iz$ and $-i \sin iz$, respectively. Show that $\cosh^2 z - \sinh^2 z = 1$.

   (b) Show that $|\cos z|^2 = \cos^2 x + \sinh^2 y$. Conclude that $\cos z$ is not bounded in $\mathbb{C}$.

   (c) Show that $\cos z = 0 \iff z = (2n+1)\pi/2$ for $n \in \mathbb{Z}$.

2. Find the roots of the equation $\sin z = 2$.

3. Express the following complex numbers in the standard form $x + iy$ and find their principal value. (a) i^{-i} (b) $(-1 + i\sqrt{3})^i$. (*Note: For* $c \in \mathbb{C}$, $z^c = e^{c \log z}$, *and for principal value of* z^c *we take* $z^c = e^{c \operatorname{Log} z}$, *where* $\operatorname{Log}(z) = \ln |z| + i\operatorname{Arg}(z)$, *with* $\operatorname{Arg}(z) \in (-\pi, \pi]$ *and* $\log(z) = \operatorname{Log}(z) + i2\pi k$.)

4. Using the method of parametric representation, evaluate $\oint_C f(z)\, dz$ for (a) $f(z) = \overline{z}$, (b) $f(z) = z + \frac{1}{z}$, (c) $f(z) = \operatorname{Re} z$ (d) $f(z) = \sin z / z$ and C is the unit circle centered at origin oriented counterclockwise.

5. Evaluate the integral $\int_\Gamma z e^{z^2} dz$ where Γ is the curve from 0 to $1+i$ along the parabola $y = x^2$.

6. (a) Assign an appropriate meaning to the integral $\displaystyle\int_{-i}^{i} \frac{1}{z}\, dz$ and find its value.

   (b) $\int_C \sin^2 z\, dz$, C is the curve from $-\pi i$ to πi along $|z| = \pi$ taken counter -clockwise.

Problem for Tutorial:

1. A function $u : U \to \mathbb{R}$ is said to be *harmonic* on an open subset $U \subset \mathbb{R}^2$ if its 1st and 2nd order partial derivatives w.r.t x and y exist, are continuous and satisfy the equation $u_{xx}+u_{yy} = 0$ on U. A harmonic function $v : U \to \mathbb{R}$ is said to be a *harmonic conjugate* of u if the function $f(z) := u(x, y) + iv(x, y)$ is analytic (equivalently, if the CR equations hold for u and v).

   (a) Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be a twice* continuously differentiable function on a domain D. Then show that
   (i) u, v are harmonic functions and v is a harmonic conjugate of u;
   (ii) v is unique upto a constant, *i*.e., if v' is another harmonic conjugate of u then $v' = v + c$ for some $c \in \mathbb{R}$;
   (iii) further, if u is a harmomic conjugate of v as well, then u and v are constants.

   (b) Find a harmonic conjugate of $u(x, y) = 3xy^2 - x^3$ on $\mathbb{C}$.

2. Show that $u(x, y) := \log(|\sqrt{x^2 + y^2}|)$ is harmonic on $\mathbb{R}^2 \setminus \{0\}$ (*i*.e., $\mathbb{C} \setminus \{0\}$, also denoted as $\mathbb{C}^*$) but it does not have any harmonic conjugates there.

3. Express i^i in the standard form $x + iy$ and find its principal value.

4. Evaluate the following integrals by parametrizing the contour

   (a) $\int_{\mathcal{C}} Re\, z\, dz$ where $\mathcal{C}$ is the line segment joining 1 to i.

   (b) $\int_{\mathcal{C}} (z - 1)dz$ where $\mathcal{C}$ is the semicircle (in the lower half plane) joining 0 to 2.

5. Let $\bar{\mathbb{D}} = \{z \in \mathbb{C} \; : \; |z| \leq 1\}$ and f be analytic on $\mathbb{D}$. Let $a,\ b \in \mathbb{D}$ and $\gamma(t) = a + t(b - a)$, $t \in [0, 1]$ be the straight line joining a and b.

   (a) Prove that $\frac{f(b)-f(a)}{b-a} = \int_0^1 f'(\gamma(t))dt$.

   (b) Using the above, if required, show that if $Re\ f'(z) > 0$ for all $z \in \mathbb{D}$ then f is injective.

*A function that is analytic in a domain is infinitely differentiable.