MSO202A COMPLEX ANALYSIS
Assignment 2

Exercise Problems:

1. Let $z = x + iy$ and $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$. Write $f(z)$ as a function of z and $\overline{z}$.

   Proof: Using $x = \frac{z+\overline{z}}{2}, y = \frac{z-\overline{z}}{2i}$ we get $f(z) = \overline{z} + 2iz$.

2. Verify Cauchy-Riemann equation for z^2, z^3.

   Proof: For z^2, $u = x^2 - y^2, v = 2xy \Rightarrow u_x = 2x, u_y = -2y, v_x = 2y, v_y = 2x$. Similarly for z^3.

3. Using the relations $x = \dfrac{z+\overline{z}}{2}, y = \dfrac{z-\overline{z}}{2i}$ and the chain rule show that $\dfrac{\partial}{\partial z} = \dfrac{1}{2}(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y})$; $\dfrac{\partial}{\partial \overline{z}} = \dfrac{1}{2}(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y})$.

   Proof: Straight forward.

4. Let $z, w \in \mathbb{C}, |z|, |w| < 1$ and $\overline{z}w \neq 1$. Prove that $\frac{|w-z|}{|1-\overline{w}z|} < 1$. Further, show that the equality holds if either $|z| = 1$ or $|w| = 1$.

   Proof: Suffices to show that $|w - z|^2 < |1 - \overline{w}z|^2$, i.e. $w\overline{w} + z\overline{z} - w\overline{z} - z\overline{w} < 1 - w\overline{z} - \overline{w}z + w\overline{w}z\overline{z}$. Since $(1 - z\overline{z})(1 - w\overline{w}) > 0$, the above is true.

   In case of equality, we see that either $(1 - z\overline{z})$ or $(1 - w\overline{w})$ is zero. Hence, in this case either $|z| = 1$ or $|w| = 1$.

5. Determine *all* $z \in \mathbb{C}$ for which each of the following power series is convergent.
   a)$\sum \frac{z^n}{n^2}$ b)$\sum \frac{z^n}{n!}$ c)$\sum \frac{z^n}{2^n}$ d)$\sum \frac{1}{2^n}\frac{1}{z^n}$.

   Proof:

   (a) Here $\frac{a_{n+1}}{a_n} \to 1 \Rightarrow R = 1$. The series converges for $|z| < 1$ and diverges for $|z| > 1$. For $|z| = 1$, by Comparison test it follows that the series converges since $\frac{|z|^n}{n^2} = \frac{1}{n^2}$.

   (b) As $\frac{a_{n+1}}{a_n} \to 0 \Rightarrow R = \infty$ and so the series converges for all z.

   (c) As $\frac{a_{n+1}}{a_n} \to \frac{1}{2} \Rightarrow R = 2$. The series converges for $|z| < 2$ and diverges for $|z| > 2$. Also it diverges for $|z| = 2$ as the n-th term sequence does not converge to zero.

   (d) Let $w = \frac{1}{z}$, where $z \neq 0$ and apply previous solution to conclude that the series converges for $|z| > 1/2$, and diverges for all other values.

6. Find all $z \in \mathbb{C}$ such that $|e^z| \leq 1$.

   Proof: For $z = x + iy$, $|e^z| = e^x \leq 1 \Leftrightarrow x \leq 0$.

7. Show that the CR-equations in polar form are given by: $u_r = \frac{1}{r} v_\theta$ and $u_\theta = -r v_r$.

   Proof: Expressing x, y in polar co-ordinates we have

$$x = r\cos\theta, \quad y = r\sin\theta.$$

   So,

$$\begin{aligned}
\frac{\partial}{\partial r} u &= u_x \frac{\partial}{\partial r} x + u_y \frac{\partial}{\partial r} y = u_x \cos\theta + u_y \sin\theta; \\
\frac{\partial}{\partial r} v &= v_x \frac{\partial}{\partial r} x + v_y \frac{\partial}{\partial r} y = v_x \cos\theta + v_y \sin\theta; \\
\frac{\partial}{\partial \theta} u &= u_x \frac{\partial}{\partial \theta} x + u_y \frac{\partial}{\partial \theta} y = r(-u_x \sin\theta + u_y \cos\theta), \\
\frac{\partial}{\partial \theta} v &= v_x \frac{\partial}{\partial \theta} x + v_y \frac{\partial}{\partial \theta} y = r(-v_x \sin\theta + v_y \cos\theta).
\end{aligned}$$

   Now it is easy to see that the CR-equations hold if and only if $u_r = \frac{1}{r} v_\theta$ and $u_\theta = -r v_r$.

Problem for Tutorial:

1. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. For a fixed w in $\mathbb{D}$, with $|w| < 1$, define the mapping $F : z \mapsto \frac{w-z}{1-\overline{w}z}$. Show that

   (a) F is a map from $\mathbb{D}$ to itself;

   (b) $F(0) = w$ and $F(w) = 0$;

   (c) $|F(z)| = 1$ if $|z| = 1$;

   (d) $F : \mathbb{D} \to \mathbb{D}$ is bijective.

   Proof:

   (a) Since $|w| < 1$, $|w^{-1}| > 1$ while $|z| \leq 1$ for all $z \in \mathbb{D}$, so $z\bar{w} \neq 1 \forall z \in \mathbb{D}$. Thus (a) follows by applying Ex. 4 above.

   (b) direct verification.

   (c) Since $|w| < 1$ it follows once again from Ex. 4 that $|F(z)| = 1$ only of $|z| = 1$.

   (d) Check that $F \circ F(z) = z$.

2. Let R be the radius of convergence of $\sum_n a_n z^n$. For a fixed $k \in \mathbb{N}$, find the radius of convergence of (a) $\sum {a_n}^k z^n$, (b) $\sum a_n z^{kn}$.

Proof: (a) $\frac{1}{\limsup \sqrt[n]{|a_n|^k}} = \left(\frac{1}{\limsup \sqrt[n]{|a_n|}}\right)^k = R^k$ (b) $\sum a_n (z^{\frac{1}{k}})^{kn}$ is convergent (resp. divergent) for $|z| < R$ (resp. $|z| > R$); take $w = z^{1/k}$ then $\sum a_n w^{kn}$ converges (resp. diverges) whenever $|w| < R^{\frac{1}{k}}$ (resp. $|w| > R^{\frac{1}{k}}$)*.

3. (a) Show that f satisfies the CR-equations if and only if $\frac{\partial}{\partial \bar{z}} f = 0$. (Recall from Ex. 3 above that $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$.) Moreover, if f is analytic then $f'(z) = \frac{\partial}{\partial z} f$.

Proof: (a) Let $f = u + iv$. We have $\frac{\partial}{\partial \bar{z}} f = \frac{1}{2}[(u_x + iv_x) + i(u_y + iv_y)]$. Thus, CR-equations hold iff $\frac{\partial}{\partial \bar{z}} f = 0$. Also, $f'(z) = u_x + iv_x$ while $\frac{\partial}{\partial z} f = \frac{1}{2}[(u_x + iv_x) - i(u_y + iv_y)]$. Applying CR-equations we get $f'(z) = \frac{\partial}{\partial z} f$.

4. Consider the following functions

   (a)

$$f(x+iy) = \begin{cases} \dfrac{xy(x+iy)}{x^2+y^2} & \text{if } x+iy \neq 0 \\ 0 & \text{if } x+iy = 0 \end{cases}$$

   (b) $f(x+iy) = \sqrt{|xy|}$

   Show that f satisfies the CR-equations but it is not differentiable at the origin.

   Proof:

   (a) $u(x,y) = \frac{x^2 y}{x^2+y^2}$ and $v(x,y) = \frac{xy^2}{x^2+y^2}$. So

$$u_x(0,0) = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = 0, \qquad u_y(0,0) = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = 0;$$

$$v_x(0,0) = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = 0, \qquad v_y(0,0) = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = 0.$$

   Thus the CR-equations are satisfied. However, along the x-axis, f takes the value 0. So, $\lim_{h \to 0} \frac{f(h+i0) - f(0)}{h}$ is 0, while

$$\lim_{h(1+i) \to 0} \frac{f(h+hi) - f(0)}{h+hi} = \lim_{h(1+i) \to 0} \frac{(h^3 + ih^3)}{(h^2+h^2)(h+hi)} = \frac{1}{2}.$$

   (b) $u_x(0,0) = 0 = u_y(0,0)$; $v_x(0,0) = v_y(0,0)$, hence CR equations are satisfied. $\lim_{h+i.0 \to 0} \frac{f(h)-f(0)}{h}$ is 0, while $\lim_{h(1+i) \to 0} \frac{f(h+hi)-f(0)}{h+hi} = \frac{1}{1+i}$, hence f is not differentiable.

*Note that $|x|^k \leq |y|^k \iff |x| \leq |y|$