Lecture - 9 (L-13, 14)

Applications of Cauchy's Integral Formula.

1. <u>Cauchy's estimate</u>: Let f be analytic on a <u>simply</u> connected domain D. and $\overline{B_R(z_0)} \subset D$ for some $R > 0$. If $|f(z)| \leq M$ $\forall z \in S_R(z_0)$ then for all $n \geq 0$

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!\, M}{R^n}.$$

<u>Pf</u>: CIF & ML inequality

$$\Rightarrow \left| f^{(n)}(z_0) \right| = \left| \frac{n!}{2\pi i} \int_{S_R(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right|$$

$$\leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} \cdot 2\pi R = \frac{n!\, M}{R^n}.$$

2. <u>Liouville's theorem</u>: If f is analytic and bounded on $\mathbb{C}$ then f is constant

<u>Pf</u>:

$$\left| f^{(1)}(z_0) \right| \leq \frac{M}{R} \quad \text{for } z_0 \in \mathbb{C}$$

Since R can be made arbitrarily large, we get $f'(z_0) = 0$.

Cor: $\cos z$ and $\sin z$ are not bounded in $\mathbb{C}$.

3) Fundamental theorem of algebra:

Every non-constant polynomial $p(z)$ of degree $n \geq 1$ has a root (in $\mathbb{C}$)

Pf: Suppose not, then $\frac{1}{p(z)}$ is analytic on $\mathbb{C}$.

Also, $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ $(a_n \neq 0)$

$$\Rightarrow \lim_{|z| \to \infty} \frac{p(z)}{z^n} = a_n .$$

$\left(\text{Indeed, as given } \varepsilon > 0,\ \left| \frac{a_i z^i}{z^n} \right| = \left| \frac{a_i}{z^{n-i}} \right| < \varepsilon \quad \forall\ |z| > \frac{\sqrt[j]{|a_i|}}{\sqrt[j]{\varepsilon}} \quad (j = n-i) \right)$

$$\therefore \lim_{|z| \to \infty} \left| z^n \cdot \frac{p(z)}{z^n} \right| = \infty \quad \left(\because \lim_{|z| \to \infty} |z|^n = \infty \right)$$

$$\left| z^n \cdot \frac{p(z)}{z^n} \right| = |p(z)|$$

$$\Rightarrow \lim_{|z| \to \infty} \frac{1}{|p(z)|} = 0 \Rightarrow \left| \frac{1}{p(z)} \right| < M \quad \forall\ |z| > R$$

and $\left\{ \left| \frac{1}{p(z)} \right| \Big/ z \in B_{R+1}(0) \right\}$ is closed & bdd

$\Rightarrow 1/p(z)$ is bdd on $\mathbb{C}$ hence constant $\times$

4 MORERA'S theorem: (Converse of Cauchy's theorem)

If f is continuous in a simply connected domain and if $\int_C f(z)\,dz = 0$ for every simple closed contour C, then f is analytic.

Pf: $\int_C f(z)\,dz = 0 \Rightarrow \int_{z_0}^{z} f(\zeta)\,d\zeta$ is independent of path

$$\Rightarrow F(z) := \int_{z_0}^{z} f(\zeta)\,d\zeta \text{ is analytic}$$

Hence f is analytic.

(5) TAYLOR'S THEOREM: $\left\{\begin{array}{c}\text{Power series}\\ \text{at } z_0\end{array}\right\} = \left\{\begin{array}{c}\text{analytic}\\ \text{fn at } z_0\end{array}\right\}$

Let f be analytic on $D = \{z : |z - z_0| < R_0\}$. $= B_{R_0}(0)$

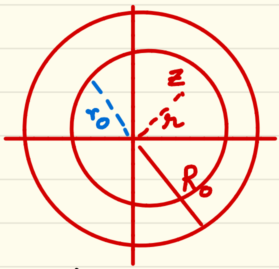
Then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \; \forall z \in D,$

where $a_n = \dfrac{f^{(n)}(z_0)}{n!}, \quad n = 0, 1, 2, \cdots$

Pf: (w.l.o.g $z_0 = 0$; refer "Useful Remark; Slides Lecture 5")

$$\frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$$

$$\therefore \frac{1}{1-q} = 1 + q + \cdots + q^{n-1} + \frac{q^n}{1-q} \quad \text{—(A)}$$



Let $|w| = r_0 \quad (r = |z| < r_0 < R_0)$

ie $w \in S_{r_0}(0)$

Then $|z| < r_0 = |w| \quad \therefore \dfrac{|z|}{|w|} < 1$

$\therefore$ For $q = z/w$

the above identity (A) becomes

$$\frac{1}{1-z/\omega} = 1 + \frac{z}{\omega} + \cdots + \left(\frac{z}{\omega}\right)^{n-1} + \frac{z^n}{\omega^n(1-z/\omega)}$$

$$\therefore \quad \frac{1}{\omega - z} = \frac{1}{\omega} + \frac{z}{\omega^2} + \cdots + \frac{z^{n-1}}{\omega^n} + \frac{z^n \cdot \omega}{\omega^{n+1}(\omega - z)}$$

By CIF, $\quad f(z) = \frac{1}{2\pi i} \int_{S_{r_0}} \frac{f(\omega)}{\omega - z}\, d\omega$

$$\frac{2\pi i}{n!} f^{(n)}(z_0) = \int_C \frac{f(\omega)}{(\omega - z_0)^{n+1}}\, d\omega$$

$$\underbrace{\frac{f(\omega)}{\omega - z} = \sum_{n=1}^{\infty} \frac{f(\omega) \cdot z^{n-1}}{\omega^n}}$$

we are trying to justify that $\int_{S_{r_0}}$ can be taken inside $\sum_{n=1}^{\infty}$!!

$$= \frac{1}{2\pi i} \int_{S_{r_0}} \left[\frac{f(\omega)}{\omega} + \frac{z f(\omega)}{\omega^2} + \cdots + \frac{z^{n-1} f(\omega)}{\omega^n} + \frac{z^n f(\omega)}{\omega^n (\omega - z)} \right] d\omega$$

$$= \frac{1}{2\pi i} \left[f(0) + z f'(0) + z^2 \frac{f''(0)}{2!} + \cdots + z^{n-1} \frac{f^{(n)}(0)}{n!} + \underbrace{z^n \int_{S_{r_0}} \frac{f(\omega)}{\omega^n (\omega - z)}\, d\omega}_{P_n(z)} \right]$$

For justifying this, we have to show that partial sum of $\sum_{n=1}^{\infty} \int \ldots$ cgs to $f(z)$.

$$\left| f(z) - \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} z^i \right| \longrightarrow \left| P_n(z) \right| = \left| z^n \int_{S_{r_0}} \frac{f(\omega)}{\omega^n (\omega - z)}\, d\omega \right|$$

$\frac{f(w)}{w-z}$ is analytic on $B_{R_0}(0)\setminus\{z\}$

$\Rightarrow$ it is continuous on S_{r_0}

$\therefore \left|\frac{f(w)}{w-z}\right| \leq K \quad (\because S_{r_0}$ - closed & bdd$)$
$\forall w \in S_{r_0}$

$$\Rightarrow \left|\frac{f(w)}{w^n(w-z)}\right| \leq \frac{K}{r_0^n}$$

$\therefore$ By ML-inequality,

$$|P_n(z)| = \left|z^n \int_{S_{r_0}} \frac{f(w)}{(w-z)w^n} \, dw\right|$$

$$\leq \frac{K}{r_0^n} \cdot 2\pi r_0 \cdot |z|^n.$$

$\frac{|z|}{r_0} < 1 \quad \therefore \quad \left|\frac{z}{r_0}\right|^n \longrightarrow 0$ as $n \to \infty$

$$\Rightarrow f(z) = f(0) + f'(0)\cdot z + \frac{f''(0)}{2!}\cdot z^2 + \cdots$$

$\blacksquare$

REMARK: The a_n's are uniquely determined!
The Taylor series of $\sum_{n=0}^{\infty} a_n z^n$ in its radius of cgs is itself.

Strategies to compute Taylor series:

1. Compute derivatives $f^{(n)}(a)$. (Rarely recommended).

2. Use known power series to get Taylor series of more complicated fns. (The uniqueness part of the Taylor's expansion says that the power series so obtained will infact be its Taylor series) Eg: $z^5 \sin z$

$$\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \quad \forall\, z \in \mathbb{C}$$

$$\therefore\ z^5 \sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+6}}{(2k+1)!}$$

3. Multiplication of power series:

Lemma: Let $\sum_{n=0}^{\infty} a_n z^n$ be cgt $\forall\, |z| < R_1$ and $\sum_{n=0}^{\infty} b_n z^n$ be cgt $\forall\, |z| < R_2$

Then $\sum_{k=0}^{\infty} c_k z^k$, where $c_k := \sum_{i=0}^{k} a_i b_{k-i}$

is cgt $\forall\, |z| < \min\{R_1, R_2\}$.

Pf: $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$

then $(f\cdot g)^{(n)}(0) = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} f^{(r)}(0)\, g^{(n-r)}(0)$

$$= n! \sum_{r=0}^{n} \frac{r!\, a_r \cdot (n-r)! \cdot b_{n-r}}{r!(n-r)!}$$

Since fg is holo in $B_R(0)$, $R = \min\{R_1, R_2\}$

its Taylor series is given by

$$\sum_{n=0}^{\infty} \frac{(fg)^{(n)}(0)}{n!} z^n \qquad \blacksquare$$

Eg: $\frac{e^z}{1+z}$ has Taylor series in $B_1(0)$

given by $\left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}(-z)^n\right)$

$$= \sum_{k=0}^{\infty} c_k z^k$$

where $c_k = \sum_{i=0}^{k} \frac{(-1)^{k-i}}{i!}$

$$\therefore \frac{e^z}{1+z} = 1 + \frac{z^2}{2} - \frac{z^3}{3} + \cdots$$

(4) Eg: $f(z) = \sin^3 z$

$$= \frac{3\sin z - \sin 4z}{4}$$

gives Taylor series of $\sin^3 z$ without having to multiply.

(5) Change of variable:-

Eg: $f(z) = \operatorname{Log} z$ is analytic in $\mathbb{C} \setminus \{$neg real axis$\}$

Let's try to obtain the Taylor series of $\operatorname{Log} z$ around 1.

We know, $f'(z) = \frac{1}{z}$

$$f'(z) = \frac{1}{1+(z-1)} = \sum_{n=0}^{\infty} (z-1)^n (-1)^n$$

We know from results on "Power series"

if $f(z) = \sum_{n=0}^{\infty} a_n (z-1)^n$ then $f'(z) = \sum_{n=1}^{\infty} n a_n (z-1)^{n-1}$

By uniqueness of Taylor series,

$$n a_n = (-1)^n \Rightarrow a_n = \frac{(-1)^n}{n} \quad \forall\, n \geq 1$$

$$\therefore \operatorname{Log} z = a_0 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (z-1)^n \; ; \; \operatorname{Log} 1 = 0 \Rightarrow a_0 = 0.$$

→ Zeros of analytic functions

→ Identity theorem

→ Maximum Modulus principle.

§ Zeros of analytic functions:

Let f be an analytic function in a domain $D \supset B_R(z_0)$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

where $a_n = \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$.

Suppose $f(z_0) = 0 \Rightarrow a_0 = 0$

($f \not\equiv 0$ on $B_R(z_0)$) Let $a_m \neq 0 \ni a_i = 0 \ \forall \ i < m$

Then $f(z) = \sum_{n=m}^{\infty} a_n (z-z_0)^n$

$$= (z-z_0)^m \underbrace{\sum_{n=0}^{\infty} a_{n+m} (z-z_0)^n}_{g(z)}.$$

$$f(z) = (z-z_0)^m g(z), \quad g(z_0) = a_m \neq 0$$

Now, $g(z)$ is analytic, hence continuous

$$\Rightarrow \quad g(z_0) \neq 0 \quad \Rightarrow g(z) \neq 0 \ \forall \ z \in B_\varepsilon(z_0)$$

$$\therefore \ f(z) \neq 0 \quad \forall \ z \neq z_0, \ z \in B_\varepsilon(z_0)$$

ie zeros of f are isolated.

§ Identity theorem

Suppose f is analytic on D (domain). If

$$\{z_n\} \subset D \ \ni \ z_n \longrightarrow z_0 \in D \ \ni \ f(z_n) = 0 \ \forall n$$

Then $f \equiv 0$ on D.

Pf: $\quad z_n \to z_0 \Rightarrow f(z_n) \to f(z_0)$

$$\therefore f(z_0) = 0.$$

Suppose $f \not\equiv 0$ on $B_R(z_0)$

By previous result, $\exists \ \varepsilon > 0 \ \ni \ f(z) \neq 0$

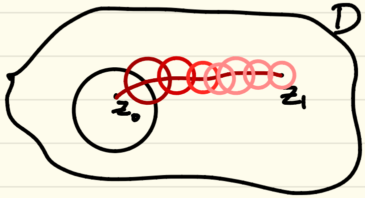
$$\forall \ z \neq z_0, \ z \in B_\varepsilon(z_0)$$

but $z_n \to z_0 \Rightarrow \exists \ N > 0 \ \ni \ z_n \in B_\varepsilon(z_0)$

$$\forall n > N$$

$\Rightarrow\Leftarrow$

Thus, $f \equiv 0$ on $B_R(z_0)$



Idea of proof.

Take $|z|=\varepsilon$, $\exists \{\zeta_n\} \to z \ni f(\zeta_n)=0 \ \forall n$

by similar argument as above

$f\equiv 0$ on $B_{\varepsilon_2}(z)$. Proceeding thus

we get $f(z_1)=0$ ∎

<u>Cor</u>: (Uniqueness theorem) Let f, g be analytic on $B_R(z_0) \ni f(z_n)=g(z_n)$ for $\{z_n\} \to z$.

Then $f=g$ on $B_R(z_0)$. (Pf: apply above to $f-g$)

§ Maximum-Modulus principle.

Let f be a non-constant analytic fn. on a domain G. Then $|f|$ does not attain a local maximum "in" G.

<u>Pf:</u> Suppose $\exists z_0 \in G \ni |f(z)| \leq |f(z_0)| \ \forall z \in B_R(z_0)$

Now $$f(z_0) = \frac{1}{2\pi i} \int_{S_r(z_0)} \frac{f(\omega)\,d\omega}{\omega - z_0}$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0+re^{it})}{re^{it}}\, rie^{it}\, dt$$

$$\boxed{f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0+re^{it})\, dt}$$ "mean value property".

$B_r(z_0) \subset B_R(z_0)$

$$\Rightarrow \quad |f(z_0)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0+re^{it})|\, dt$$

$$\le |f(z_0)|$$

$$\therefore \quad \frac{1}{2\pi} \int_0^{2\pi} \Big|\, |f(z_0)| - |f(z_0+re^{it})| \,\Big|\, dt = 0$$

$$\Rightarrow \quad |f(z_0)| = |f(z_0+re^{it})|$$

$\Rightarrow$ $|f|$ is constant on $B_R(z_0)$

$\Rightarrow$ f is constant on $B_R(z_0)$

$\Rightarrow$ f = constant on G (by Uniqueness thm).

<u>Cor</u>: If f is analytic inside and on a simple closed curve C, then $|f|$ attains its maximum on the boundary.