LECTURE - 5

Analytic functions
and power series

Lecture 5 : Analytic functions and power series

A function $f: U \to \mathbb{C}$ is said to be <u>analytic</u> at $z_0 \in U$ if it is differentiable in a neighbourhood of z_0.

<u>Eg</u>:- ① $|z|^2$ is differentiable at '0' but not at any other point. So, it is not analytic at '0'.

② z^n $(n>0)$ is differentiable everywhere in $\mathbb{C}$ hence it is analytic at any pt of $\mathbb{C}$.

§§ Are there examples of analytic functions other than z^n??

The answer is YES!! And the recipe is via POWER SERIES

A series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, where $a_n \in \mathbb{C}$ $\forall n$, $z_0 \in \mathbb{C}$ and z is an indeterminate is called a <u>power series around z_0</u>

Eg: $\sum_{n=0}^{\infty} z^n$ is a power series around '0'.

Recall that $\sum_{n=0}^{\infty} z^n$ is convergent if $|z|<1$

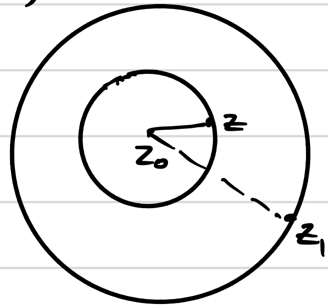
and it is divergent if $|z| \geq 1$

We say that 1 is the radius of convergence of $\sum_{n=0}^{\infty} z^n$.

Defn: Radius of convergence of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n := \sup\{|z_1 - z_0| \,/\, \sum_{n=0}^{\infty} a_n(z_1-z_0)^n \text{ converges}\}$

Why call it "radius of convergence" ??

Proposition: Let the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ convergent at z_1. Then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is convergent $\forall\, z \ni |z-z_0| < |z_1 - z_0|$.



Pf: Let $\dfrac{|z-z_0|}{|z_1-z_0|} = \rho < 1$

$$\Rightarrow \frac{|z-z_0|^n}{|z_1-z_0|^n} = \rho^n$$

Further, $\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ is cgt $\Rightarrow a_n(z_1-z_0)^n \to 0$ as $n\to\infty$

$\therefore\ |a_n(z_1-z_0)^n| < K$ for some $K>0$

$$|a_n(z-z_0)^n| = |a_n| \frac{|z-z_0|^n}{|z_1-z_0|^n} |z_1-z_0|^n$$

$$= \rho^n |a_n||z_1-z_0|^n < k\rho^n \quad \forall n.$$

Since $\sum_{n=0}^{\infty} k\rho^n$ is cgt, by comparison test

$\sum a_n(z-z_0)^n$ is cgt (absolutely), as required. $\blacksquare$

Thus, we get an equivalent definition of Radius of convergence (which also justifies its name).

<u>Radius of convergence</u> of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is $R>0$ (including ∞) such that

$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is cgt (absolutely) $\forall z \ni |z-z_0|<R$

<u>and</u> $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is dgt $\forall z \ni |z-z_0|>R$.

Hadamard's formula for Radius of convergence of a power series.

Recall, $\sum_{n=0}^{\infty} a_n$ be a series of complex number then if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = l$

then $\sum_{n=0}^{\infty} a_n$ is cgt if $l < 1$

.... dgt if $l > 1$.

Ratio Test

Consider the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

Let $\lim_{n\to\infty} \frac{|a_{n+1}(z-z_0)^{n+1}|}{|a_n(z-z_0)^n|} = l$ (ie limit exists)

Then $l = |z-z_0| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

Applying Ratio test, we get $\sum a_n(z-z_0)^n$ that the series converges (absolutely) if $l < 1$

$$\Leftrightarrow |z-z_0| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$$

$$\Leftrightarrow |z-z_0| < \frac{1}{\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}}$$

(with the convention $1/0 = \infty$ & $1/\infty = 0$)

the series diverges if $l > 1$

ie $|z - z_0| \lim \frac{|a_{n+1}|}{|a_n|} > 1$

ie $|z - z_0| > \frac{1}{\lim \frac{|a_{n+1}|}{|a_n|}}$ (again with the same convention as above: $1/0 = \infty$, $1/\infty = 0$)

Thus, by defn of Radius of convergence, the radius of convergence of $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is

$$\boxed{1 \Big/ \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}}$$

whenever the limit exists (Convention: $1/\infty = 0$, $1/0 = \infty$).

On the other hand, if we apply Root test to $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ then the series converges (absolutely) if $\limsup \sqrt[n]{|a_n (z-z_0)^n|} < 1$ and it diverges if $\limsup \sqrt[n]{|a_n (z-z_0)^n|} > 1$.

Since $\limsup \sqrt[n]{|a_n||z-z_0|^n}$

$$= \left(\limsup \sqrt[n]{|a_n|}\right)|z-z_0|$$

We get that $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges

if $\quad |z-z_0| < \dfrac{1}{\limsup \sqrt[n]{|a_n|}}$

diverges if $|z-z_0| > \dfrac{1}{\limsup \sqrt[n]{|a_n|}}$

Convention: $1/0 = \infty$ $\quad 1/\infty = 0$

So, radius of convergence of $\sum_{n=0}^{\infty} a_n (z-z_0)^n$

is $\dfrac{1}{\limsup \sqrt[n]{|a_n|}}$, with the convention that $1/0 = \infty$ & $1/\infty = 0$.

Eg: $\sum_{n=0}^{\infty} z^n$, $\quad \lim_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = 1$

$$\limsup_n \sqrt[n]{|a_n|} = \limsup_n 1 = 1$$

$\therefore \sum_{n=0}^{\infty} z^n$ has radius of convergence 1.

§ Power series are analytic functions

THEOREM: Consider the function given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \forall z \ni |z - z_0| < R$$

where R = radius of convergence of the power series

Then f is analytic in $B_R(z_0)$ and

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

__Lemma:__ Let $\sum_{n=0}^{\infty} a_n z^n$ be cgt for $|z| < R$. Then

$\sum_{n=1}^{\infty} n a_n z^{n-1}$ is also cgt for $|z| < R$.

__Pf:__ Let $|z| < r < R \Rightarrow \frac{|z|}{r} = \rho < 1$

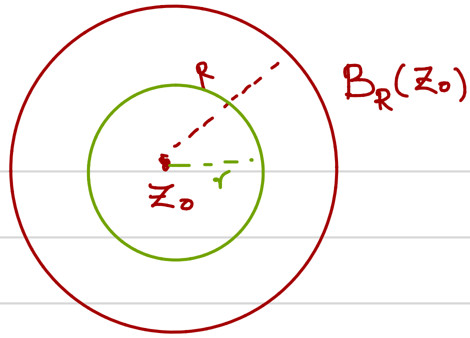
$$|n a_n z^{n-1}| = |n a_n z^{n-1}|$$

$$= n |a_n| |z|^{n-1}$$

$$< n |a_n| r^{n-1}$$

The real series $\sum_{n=0}^{\infty} |a_n| r^n$ has derivative $\sum_{n=0}^{\infty} n|a_n| r^{n-1}$ (Ref: Rudin §8.1)

By the real case, $\sum_{n=0}^{\infty} n|a_n| r^{n-1}$ is cgt



NOTE : $\sum_{n=0}^{\infty} a_n z^n$ cgs absolutely for all $z \ni |z - z_0| < R$

In particular, for

$z = z_0 + x \qquad -r < x < r \qquad (r < R)$

$\Rightarrow \sum_{n=0}^{\infty} |a_n| |z - z_0|^n$ converges

ie $\sum_{n=0}^{\infty} |a_n| x^n$ converges $\forall \; |x| < r$.

By Comparison test,

$$\sum_{n=0}^{\infty} n a_n z^{n-1} \text{ is convergent } \forall \; |z| < R. \; \boxtimes$$

PROOF OF THEOREM :

First of all, note that we may assume that $z_0 = 0$

$\left(\because f(z + z_0) = \sum_{n=0}^{\infty} a_n z^n \right.$ converges (absolutely) for $|z| < R$

iff $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges (absolutely) for $|z - z_0| < R$.

So, without loss of generality, we assume $\left. z_0 = 0 \right)$

$f(z) = \sum_{n=0}^{\infty} a_n z^n$. where $|z| < R$ is convergent (absolutely)

Define $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$

$$\left| \frac{f(z)-f(w)}{z-w} - g(w) \right|$$

$$= \left| \frac{\sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} a_n w^n}{z-w} - \sum_{n=0}^{\infty} n a_n w^{n-1} \right|$$

$$= \left| \sum_{n=0}^{\infty} a_n \left(z^{n-1} + z^{n-2} w + \cdots + w^{n-1} - n w^{n-1} \right) \right| \quad -(1)$$

Since $\sum_{n=0}^{\infty} n a_n w^{n-1}$ is convergent (absolutely) for $|w| < R$ (by Lemma)

In particular for $r \ni |w| < r < R$

$\exists N > 0 \ni \left| \sum_{n=N}^{\infty} n |a_n| r^{n-1} \right| < \varepsilon/4$.

So, we split (1) into 2 parts

$$\left| \sum_{n=0}^{N} a_n \left(z^{n-1} + \cdots + w^{n-1} - n w^{n-1} \right) \right|$$

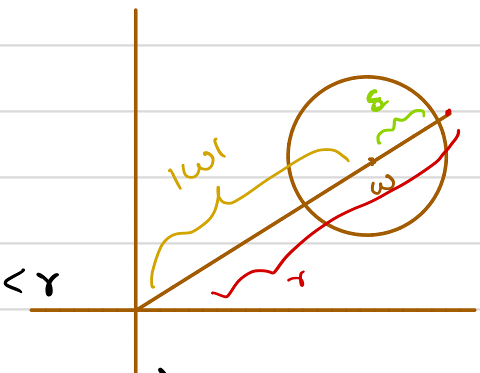
$$+ \left| \sum_{n=N}^{\infty} a_n \left(z^{n-1} + \cdots + w^{n-1} - n w^{n-1} \right) \right|$$

$$\le \left| \sum_{n=0}^{N} a_n \underbrace{\left(z^{n-1}+\cdots+w^{n-1}-nw^{n-1}\right)}_{A} \right|$$

$$+ \underbrace{\sum_{n=N+1}^{\infty} |a_n| \left(|z|^{n-1}+\cdots+|w|^{n-1}+n|w^{n-1}|\right)}_{B}$$

$$\left(\overset{\in\mathbb{R}}{s_k} \le \overset{\in\mathbb{R}}{t_k} \Rightarrow \lim_{k\to\infty} s_k \le \lim_{k\to\infty} t_k \right)$$

Choose $\delta \ni |z-w| < \delta$ & $|w| + \delta < r$



Then $|z| \le |z-w| + |w| < r$

Further, $\displaystyle\lim_{z\to w} \sum_{n=0}^{N} a_n \left(z^{n-1}+\cdots+w^{n-1}-nw^{n-1}\right) = 0$

$\therefore$ Given $\varepsilon > 0$ $\exists\, \delta' > 0 \ni \left| \displaystyle\sum_{n=0}^{N} a_n \left(z^{n-1}+\cdots+w^{n-1}-nw^{n-1}\right) \right| < \varepsilon/2$ $\forall\, w \ni |z-w| < \delta$

Choose $\delta_0 = \min\{\delta', \underbrace{r-|w|}_{=\delta}\}$.

Then $A + B \le \varepsilon/2 + \displaystyle\sum_{n=N+1}^{\infty} |a_n| \left(r^{n-1}+\cdots+r^{n-1}+nr^{n-1}\right)$

$$= \varepsilon/2 + 2\sum_{n=N+1}^{\infty} |a_n|\, n\, r^{n-1}$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Thus, $f'(z) = g(z)$ $\forall\, z \in B_R(0)$. $\blacksquare$

Rmk: If that was not understandable then try this!

Lemma: Let R be the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$.

Then the series $\sum_{n=0}^{\infty} n a_n z^{n-1}$ also converges $\forall\, |z| < R$.

Proof: Given R is the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$

Then $\sum_{n=0}^{\infty} |a_n z^n|$ also converges $\forall\, |z| < R$

In particular, the real power series

$\sum_{n=0}^{\infty} |a_n| x^n$ converges absolutely $\forall\, |x| < R$.

Recall that this implies that $\sum_{n=0}^{\infty} |a_n| n x^{n-1}$ also converges $\forall\, |x| < R$ (Ref: Rudin §8.1)

Let $|z| < r < R$ then, $\sum_{n=0}^{\infty} n |a_n| r^{n-1}$ converges.

Now, $|n a_n z^{n-1}| = n |a_n| |z|^{n-1} < n |a_n| r^{n-1}$

By Comparison test, therefore, $\sum_{n=0}^{\infty} n a_n z^{n-1}$ converges $\forall\, |z| < R$.

THEOREM: Let $f(z) := \sum_{n=0}^{\infty} a_n (z-z_0)^n \ \forall\ z \ni |z-z_0| < R$ where R = radius of convergence of the series. Then f is analytic in $B_R(z_0)$ and its derivative is given by $\sum_{n=0}^{\infty} n a_n (z-z_0)^{n-1}$.

<u>Pf</u>: First of all, taking $\omega = z - z_0$, we get

$$\sum_{n=0}^{\infty} a_n \omega^n \text{ converges } \forall\ |\omega| < R.$$

Thus, it suffices to show that

$$\left[\begin{array}{l} f(\omega) := \sum_{n=0}^{\infty} a_n \omega^n \text{ is analytic in } B_R(0) \\ \text{and its derivative is } \sum_{n=0}^{\infty} n a_n \omega^{n-1}. \end{array}\right.$$

$$\left| \frac{f(z)-f(\omega)}{z-\omega} - \sum_{n=0}^{\infty} n a_n \omega^{n-1} \right|$$

$$= \left| \sum_{n=0}^{\infty} a_n \frac{(z^n - \omega^n)}{z-\omega} - \sum_{n=0}^{\infty} n a_n \omega^{n-1} \right|$$

$$= \left| \sum_{n=0}^{\infty} a_n \left(z^{n-1} + z^{n-2}\omega + \cdots + \omega^{n-1}\right) - \sum_{n=0}^{\infty} n a_n \omega^{n-1} \right|$$

By earlier lemma, $\sum_{n=0}^{\infty} n a_n \omega^{n-1}$ converges $\forall\ |\omega| < R$.

Thus, given $\varepsilon > 0$ $\exists\ N > 0 \ni$

$$\left| \sum_{n=N+1}^{\infty} n a_n \omega^{n-1} \right| < \varepsilon \qquad \text{(Refer: Lecture)}$$

$$\left| \sum_{n=0}^{\infty} a_n \left(z^{n-1} + z^{n-2}\omega + \cdots + \omega^{n-1} \right) - n a_n \omega^{n-1} \right|$$

$$= \left| \sum_{n=0}^{N} \left(\cdots\cdots \right) + \sum_{n=N+1}^{\infty} \left(\cdots\cdots \right) \right|$$

$$\le \left| \cdots\cdots \right| + \left| \cdots\cdots \right|$$

$$\le \left| \cdots\cdots \right| + \left| \sum_{n=N+1}^{\infty} a_n \left(z^{n-1} + z^{n-2}\omega + \cdots + \omega^{n-1} \right) \right|$$

$$+ \left| \sum_{n=N+1}^{\infty} n a_n \omega^{n-1} \right|$$

$$\le \left| \cdots\cdots \right| + \sum_{n=N+1}^{\infty} n |a_n| r^{n-1} + \varepsilon$$

cgs $\downarrow$ as $z \to \omega$ $\wedge\!|$ ε

0

$$\le \varepsilon + \varepsilon + \varepsilon$$

$\forall\ z \ni |z - \omega| < \min\{\delta, \delta'\}$