LECTURE - 4

Differentiabity

Cauchy-Riemann equations

Lecture 4 : Differentiability and CR-eqns

Let f be a fn defined in a neighbourhood of z_0. We say f is differentiable at z_0 if the function

$$\frac{f(z)-f(z_0)}{z-z_0}$$ defined in the deleted neighbourhood of z_0

has a limit.

In other words, there exists $l \in \mathbb{C}$ such that given any $\varepsilon > 0$ $\exists$ $\delta > 0$

$$\ni \left| \frac{f(z)-f(z_0)}{z-z_0} - l \right| < \varepsilon \quad \forall\, 0 < |z-z_0| < \delta.$$

The limit is denoted as $f'(z_0)$.

Eg: $f(z) = z^n$

$$\lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0} = ??$$

$$\frac{z^n - z_0^n}{z - z_0} = z^{n-1} + z^{n-2} z_0 + \cdots + z_0^{n-1}$$

$$\therefore \lim_{z \to z_0} z^{n-1} + z^{n-2} z_0 + \cdots + z_0^{n-1} = n z_0^{n-1}$$

<u>Useful Remark</u>: If f is differentiable at z_0 then the function η defined as follows

$$\eta(z) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \quad \text{for } z \neq z_0$$

$$\eta(z_0) = 0$$

is continuous at z_0.

$$\left(\because |\eta(z) - \eta(z_0)| = \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| \right)$$

Proposition: If f is differentiable at z_0 then f is continuous at z_0, as well.

<u>Pf</u>:

We use the above remark, to get

$$f(z) = f(z_0) + \eta(z)(z - z_0) + f'(z_0)(z - z_0)$$

each of this is continuous hence $f(z)$ is continuous at z_0.

Arithmetic of differentiability:

Let f and g be differentiable at z_0. Then $f \pm g$, fg, f/g (if $g(z_0) \neq 0$) and cf are differentiable.

Further,
$$(f \pm g)'(z_0) = f'(z_0) + g'(z_0)$$
$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$$
$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$$
$$(cf)'(z_0) = cf'(z_0).$$

Chain rule:

If f is differentiable at $g(z_0)$ and g is differentiable at z_0 then $(f \circ g)'(z_0) = f'(g(z_0))\,g'(z_0)$.

(Proofs of all the above is exactly as in the real case).

Eg: ① $f(z) = |z|$: $\lim\limits_{z \to 0} \frac{|z|}{z}$ does not exist.

So f is not differentiable at $z = 0$

② $f(z) = |z|^2$: $\lim\limits_{z \to 0} \frac{|z|^2}{z} = \lim\limits_{z \to 0} \bar{z} = 0$

So f is differentiable at 0.

$$\lim_{h\to 0} \frac{|z+h|^2 - |z|^2}{h} = \lim_{h\to 0} \frac{(z+h)(\bar{z}+\bar{h}) - z\bar{z}}{h}$$

$$= \lim_{h\to 0} \frac{z\bar{h} + h\bar{z}}{h} = \bar{z} + \underbrace{\lim_{h\to 0} z\frac{\bar{h}}{h}}_{\text{does not exist.}}$$

$\therefore$ does not exist. ▨

Remark: 1) By example 1 it follows that even if u, v are differentiable, f need not be differentiable

2) By Example 2, it is seen that even if f is differentiable at z_0 it need not be differentiable anywhere around z_0.

Remark 1, in particular, means that differentiability of u and v is not sufficient to conclude differentiability of $u+iv$.

So, we try to investigate a sufficiency criteria for f to be differentiable.

For this we work backwards:

If f is differentiable then $f'(z)$ exists

So, $$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

Take $h = h_1 + i \cdot 0$. Then

$$f'(z) = \lim_{h_1 \to 0} \frac{u(x+h_1, y) + iv(x+h_1, y) - (u(x,y) + iv(x,y))}{h_1}$$

$$= \lim_{h_1 \to 0} \frac{u(x+h_1, y) - u(x,y)}{h_1} + i \lim_{h_1 \to 0} \frac{v(x+h_1, y) - v(x,y)}{h_1}$$

$$= u_x(x,y) + i v_x(x,y)$$

Take $h = 0 + ih_2$. Then,

$$f'(z) = \lim_{ih_2 \to 0} \frac{u(x, y+h_2) + iv(x, y+h_2) - (u(x,y) + iv(x,y))}{ih_2}$$

$$= \lim_{h_2 \to 0} \frac{u(x, y+h_2) - u(x,y) + i(v(x, y+h_2) - v(x,y))}{ih_2}$$

Thus, we get

$$f'(z) = u_x(x,y) + iv_x(x,y) = u_x + iv_x.$$

$$= \frac{u_y(x,y) + iv_y(x,y)}{i} = v_y - iu_y$$

<u>Cauchy-Riemann equations</u> : $\boxed{\begin{array}{l} u_x = v_y \\ v_x = -u_y \end{array}}$

We have just seen that: (Necessary condition for differentiability)

If f is differentiable at z_0 then

$$u_x(x_0,y_0) = v_y(x_0,y_0) \ \& \ v_x(x_0,y_0) = -u_y(x_0,y_0).$$

Does this give a sufficiency condition as well?? No! Example (see exercise sheet 2).

Let $z_0 = x_0 + iy_0$

THEOREM: If the partial derivatives u_x, u_y, v_x, v_y exist at (x_0,y_0) and are continuous. Then f is differentiable at z_0 if the CR equations are satisfied at (x_0,y_0).

(Proof skipped).

Applications of CR-equations

1. Let f be differentiable on an open set U. Let $f'(z)=0$ $\forall z\in U$. Then f is constant on U.

Pf: $f'(z_0) = u_x(x_0,y_0) + i v_x(x_0,y_0) = 0 \quad \forall z_0\in U$

$$\left.\begin{aligned} &\Rightarrow u_x(x,y) = 0 = v_x(x,y) \\ &\Rightarrow u = u(y),\ v = v(y) \end{aligned}\right\} \text{ on } U$$

$$\left.\begin{aligned} \text{But, } f'(z_0) &= v_y(x_0,y_0) - i u_y(x_0,y_0) \\ \text{So, } \quad u_y &= 0 = v_y \end{aligned}\right\} \text{ on } U$$

Thus, $u = \text{constant} = v$ ∎

2. Let f be a differentiable function on U

If $|f|$, $\operatorname{Re} f$ or $\operatorname{Im} f$ is constant on U then so is f.

Pf: $|f|^2 = \text{constant}$ ($\neq 0$ w.l.o.g)

$\Rightarrow u^2+v^2 = \text{constant}$

$\therefore \dfrac{\partial}{\partial x}(u^2+v^2) = 0 \Rightarrow 2uu_x + 2vv_x = 0$

By CR eqns $\quad 2uv_y - 2vu_y = 0$

$$\frac{\partial}{\partial y}(u^2+v^2)=0 \Rightarrow 2uu_y+2vv_y=0$$

from the above 2 equations, we get

$$(u^2+v^2)u_y=0$$

$$\Rightarrow u_y=0 \quad |||^{ly}\ v_y=0$$

$$\Rightarrow f'(z)=0 \quad \forall\ z\in U$$

hence f is constant on U.

Others, similarly argued.

(3) Let f be differentiable on U. If f is real valued (or purely imaginary) then f is constant.

Polar form of CR equations:

$x(r,\theta)=r\cos\theta, \quad y(r,\theta)=r\sin\theta$

$$\therefore \frac{\partial x}{\partial r}=\cos\theta, \quad \frac{\partial x}{\partial\theta}=-r\sin\theta$$

$$\frac{\partial y}{\partial r}=\sin\theta, \quad \frac{\partial y}{\partial\theta}=r\cos\theta$$

Consider, $u = u(x, y)$

$$u_r = \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$= u_x \cos\theta + u_y \sin\theta \qquad -(1)$$

lll^{ly} $\quad V_r = V_x \cos\theta + V_y \sin\theta \qquad -(2)$

$$u_\theta = -r u_x \sin\theta + r u_y \cos\theta \qquad -(3)$$

$$V_\theta = -r V_x \sin\theta + r V_y \cos\theta \qquad -(4)$$

Applying CR equation to (4), we get

$$V_\theta = r u_x \cos\theta + r u_y \sin\theta$$

$$= r u_r$$

lll^{ly}, $\quad u_\theta = -r V_r$

$$\left.\begin{array}{l} V_\theta = r u_r \\ u_\theta = -r V_r \end{array}\right\} \text{CR- equations in POLAR FORM.}$$

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