

LECTURE- 6

Special Power Series
Exponential,
Logarithm &
Trigonometric fns



Lecture 6 :

§ EXPONENTIAL FUNCTION:

Recall, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $x \in \mathbb{R}$.

We define the complex exponential analogously:

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Radius of convergence: $\lim_{n \rightarrow \infty} \frac{|n!|}{|(n+1)!|} = 0 = \infty$

So, e^z is a genuine analytic fn
on \mathbb{C} .

Some of the properties of e^z :

$$\textcircled{1} \quad e^{x+y} = e^x e^y$$

$$\textcircled{2} \quad \frac{d(e^x)}{dx} = e^x$$

$$\textcircled{3} \quad e^0 = 1$$

We try to verify if these hold for e^z .

REMARKS:

(2)

1) Recall, we already set $e^{i\theta} = \cos \theta + i \sin \theta$

we now verify that there is no clash
of notation:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{k=0}^{\infty} \frac{(i\theta)^{2k}}{(2k)!}$$

$E_k = k$ th partial sum

$S_k = k$ th partial sum

$C_k = k$ th partial sum

$$+ \sum_{k=1}^{\infty} \frac{(i\theta)^{2k+1}}{(2k+1)!}$$

then

$$E_{2k} = C_k + iS_{k-1} = \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{2k!} + i \sum_{k=1}^{\infty} (-1)^k \frac{\theta^{2k+1}}{(2k+1)!}$$

$$E_{2k+1} = C_k + iS_k$$

$$\downarrow \quad \downarrow \quad \downarrow \quad e^{i\theta} = \cos \theta + i \sin \theta \quad \blacksquare$$

2) e^z is analytic on \mathbb{C}

$$\text{and } (e^z)' = \sum_{n=1}^{\infty} n \frac{(z)^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = e^z$$

(3) $e^0 = 1$

(3)

Properties of e^z :

$$\textcircled{1} \quad e^{z_1+z_2} = e^{z_1} \cdot e^{z_2} = e^{z_1} e^{z_2}$$

$$\text{Let } g(z) := e^z e^{z_1+z_2-z}$$

$$\begin{aligned} \text{Then } g'(z) &= (e^z)' e^{z_1+z_2-z} + e^z (e^{z_1+z_2-z})' \\ &= e^z e^{z_1+z_2-z} - e^z e^{z_1+z_2-z} = 0 \end{aligned}$$

$$\text{Hence } g(z) = \text{constant} = g(z_1+z_2)$$

$$= e^{z_1+z_2}$$

$$\text{ie } e^z e^{z_1+z_2-z} = e^{z_1+z_2}$$

$$\text{take } z = z_2. \text{ Then } e^{z_2} e^{z_1} = e^{z_1+z_2}$$

$$\begin{aligned} &\text{take } z = z_1. \text{ Then } e^{z_1} e^{z_2} = e^{z_1+z_2} \\ \text{In particular, } (e^{z_1})^{-1} &= e^{-z_1} \end{aligned}$$

$$\textcircled{2} \quad e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi = 1$$

$$\therefore e^{z+2n\pi i} = e^z \quad \text{ie } e^z \text{ has period } 2n\pi i$$

Conversely,

$$e^z = 1 \Rightarrow e^{x+iy} = 1 \Rightarrow e^x (\cos y + i \sin y) = 1$$

$$\Rightarrow e^x \cos y = 1, \quad e^x \sin y = 0$$

④

This is possible only if $\sin y = 0$

$$\Rightarrow y = n\pi$$

$$\Rightarrow \cos y = \pm 1; e^x > 0 \text{ & } e^x \cos y = 1$$

$$\Rightarrow \cos y = 1 \Rightarrow y = 2n\pi$$

$$\text{and } e^x = 1 \Rightarrow x = 0$$

$$\therefore e^z = 1 \Rightarrow z = 2n\pi i + n$$

$$③ e^{z_1} = e^{z_2} \Rightarrow z_1 - z_2 = 2n\pi i$$

In particular, the complex exponential is not 1-1.

Qn: Is it surjective?

Let $w \in \mathbb{C}$, ie $w = re^{i\theta}$. Does there exist $x+iy \Rightarrow e^x e^{iy} = re^{i\theta}$

$$\Rightarrow e^x = r, e^{iy} = e^{i\theta}$$

$$\Rightarrow y = \theta + 2n\pi.$$

If $w \neq 0$, then $x = \ln(r)$, $y = \theta + 2n\pi$

Satisfies $e^{x+iy} = re^{i\theta}$.

Thus, e^z is surjective onto \mathbb{C}^* .

In fact, if we restrict e^z to

$$\mathbb{H} = \{x+iy \mid -\pi < y \leq \pi\} \text{ then}$$

$e^z : \mathbb{H} \rightarrow \mathbb{C}^*$ is bijective.

$$\left(\because e^{z_1} = e^{z_2} \Rightarrow z_1 - z_2 = 2n\pi i \right.$$

but $z_1, z_2 \in \mathbb{H} \Rightarrow \operatorname{Im} z_1 - \operatorname{Im} z_2 < 2\pi$

$$\left. \therefore z_1 - z_2 = 0 \right)$$

$\Rightarrow z_1 = z_2$
 $e^{\ln|w| + i\operatorname{Arg} w} = w$

Define the inverse map as Logarithm

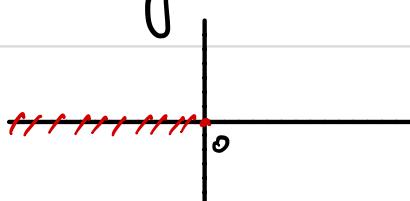
$$\begin{aligned} \operatorname{Log} : \mathbb{C}^* &\longrightarrow \mathbb{C} \\ w &\longmapsto \ln|w| + i\operatorname{Arg} w. \end{aligned}$$

$$\left(\begin{aligned} \log : \mathbb{C}^* &\longrightarrow \mathbb{C} \\ w &\longmapsto \ln|w| + i\arg w \end{aligned} \right)$$

is multivalued !!

REMARK: 1) Log is not continuous on the

negative real axis ie $\{x+iy \mid x \leq 0, y=0\}$



(6)

To see this, consider $\alpha \in \mathbb{R}^+$

$$\text{Let } a_n = |\alpha| e^{i(\pi - \gamma_n)} \text{ and } b_n = |\alpha| e^{i(-\pi + \gamma_n)}$$



$$\log(a_n) = \ln|\alpha| + i(\pi - \gamma_n)$$

$$\log(b_n) = \ln|\alpha| + i(-\pi + \gamma_n)$$

$$\lim_{n \rightarrow \infty} \log(a_n) = \ln|\alpha| + i\pi$$

//

$$\lim_{n \rightarrow \infty} \log(b_n) = \ln|\alpha| - i\pi$$

2) However, on the rest of the complex plane

i.e. $\mathbb{C} \setminus (\mathbb{R} \cup \{0\})$ $\log z$ is not just continuous but also analytic.

Continuity of $\log z$ on $\mathbb{C}^* - \mathbb{R}^-$

$\operatorname{Arg} z : \mathbb{C}^* - \mathbb{R}^- \rightarrow (-\pi, \pi)$
is continuous.

hence $\underbrace{\ln|z| + i\operatorname{Arg} z}_{= \operatorname{Log} z}$ is continuous.
on $\mathbb{C}^* - \mathbb{R}^-$

Analyticity of $\operatorname{Log} z$ on $\mathbb{C}^* - \mathbb{R}^-$

Let $z = r e^{i\theta}$ $r \in \mathbb{R}^{>0}$ & $\theta \in (-\pi, \pi)$

then $\operatorname{Log} z = \ln r + i\theta$

$$\therefore u(r, \theta) = \ln r$$

$$v(r, \theta) = \theta$$

$$u_r = \frac{1}{r}, \quad u_\theta = 0$$

$$v_r = 0, \quad v_\theta = 1$$

CR eqns in polar form: $u_r = \frac{1}{r} v_\theta$

& $v_r = -\frac{1}{r} u_\theta$
are satisfied.

(8)

$$\begin{pmatrix} U_r \\ U_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} U_x \\ U_y \end{pmatrix} \Rightarrow \begin{array}{l} U_r, U_\theta, V_r, V_\theta \text{ cont} \\ \uparrow \downarrow \\ U_x, U_y, V_x, V_y \text{ cont.} \end{array}$$

$U_r, U_\theta, V_r, V_\theta$ are all continuous

$\Rightarrow \log z$ is analytic

Let's find the derivative of $\log z$ for $z \in \mathbb{C}^* \setminus \mathbb{R}^-$

$$(\log z)' = \lim_{w \rightarrow z} \frac{\log w - \log z}{w - z}$$

Since, $\log: \mathbb{C}^* \setminus \mathbb{R}^- \rightarrow \{x+iy \mid x \in \mathbb{R}, y \in (-\pi, \pi)\} = S$

is continuous and bijective with inverse $\exp: S \rightarrow \mathbb{C}^* \setminus \mathbb{R}^-$,

$$\text{so } w = e^\zeta, z = e^{\zeta_0} \left\{ \begin{array}{l} \Rightarrow w \rightarrow z \xrightarrow{\text{as } \zeta \rightarrow \zeta_0} \log w \rightarrow \log z \\ \text{for } \zeta, \zeta_0 \in S \end{array} \right.$$

$\left. \begin{array}{l} \log(e^\zeta) \rightarrow \log(e^{\zeta_0}) \\ \zeta \rightarrow \zeta_0 \end{array} \right.$

$$\text{Thus, } (\log z)' = \lim_{\zeta \rightarrow \zeta_0} \frac{\zeta - \zeta_0}{e^\zeta - e^{\zeta_0}}$$

$$= \lim_{\zeta \rightarrow \zeta_0} \frac{1}{\frac{e^\zeta - e^{\zeta_0}}{\zeta - \zeta_0}} = \frac{1}{e^{\zeta_0}} = \frac{1}{z}.$$



Trigonometric functions

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \dots$$

$$\sin z := z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots + (-1)^k \frac{z^{2k+1}}{(2k+1)!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^k \frac{x^{2k}}{(2k)!} + \dots$$

$$\cos z := 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots + (-1)^k \frac{z^{2k}}{(2k)!} + \dots$$

Each of the above series is absolutely convergent $\forall z \in \mathbb{C}$, since $\sum_{n=0}^{\infty} \frac{|iz|^n}{n!}$ is absolutely cgt.

Properties: ① $\cos(-z) = \cos z$; $\sin(-z) = -\sin z$

$$\textcircled{2} \quad e^{iz} = \cos z + i \sin z; \quad e^{-iz} = \cos(-z) + i \sin(-z) \\ = \cos z - i \sin z.$$

$$\text{Q} \quad (e^{iz})^n = \cos nz + i \sin nz.$$

$$\textcircled{3} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}; \quad \sin z = \frac{e^{iz} - e^{-iz}}{2} \quad (\text{By } \textcircled{2})$$

$$\textcircled{4} \quad \cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

Appendix

$$(*) \quad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$$

$$e^{|z|} = 1 + |z| + \frac{(|z|)^2}{2!} + \frac{(|z|)^3}{3!} + \frac{(|z|)^4}{4!} + \dots$$

$$\text{k-th partial sum of } \cos z = S_k = \sum_{n=0}^k (-1)^n \frac{z^{2n}}{(2n)!}$$

$$\dots \quad e^{|z|} = t_k.$$

$$|S_k - S_{k'}| = \left| \sum_{n=k}^{k'} (-1)^n \frac{z^{2n}}{(2n)!} \right| \leq \sum_{n=k}^{k'} \frac{|z|^{2n}}{(2n)!}$$

$$\leq \sum_{n=2k}^{2k'} \frac{|z|^n}{n!}$$

$$= |t_k - t_{k'}| < \varepsilon$$

$\forall k, k' > N$ (by abs
cgs of $e^{|z|}$)