


LECTURE - 13 (L-17)

Calculus of residues
Evaluation of integrals.



① EVALUATING INTEGRALS

$$\int_0^{2\pi} F(\cos\theta, \sin\theta) d\theta$$

$$\frac{z+z^{-1}}{2} \quad \frac{z-z^{-1}}{2i}$$

$$z = e^{i\theta}$$

eg: $\int_0^{2\pi} \frac{1}{1+8\cos^2\theta} d\theta$

$$\gamma(\theta) = e^{i\theta}, \quad \gamma'(\theta) = ie^{i\theta}, \quad \text{if } z = e^{i\theta}, \quad \gamma'(\theta) = iz$$

$$\cos\theta = \frac{z+z^{-1}}{2}$$

$$\int_{\gamma(0,1)} \frac{1}{1+8\left[\frac{z+z^{-1}}{2}\right]^2} \cdot \frac{dz}{iz} = \int_{\gamma} \frac{1}{1+2(z^2+2+z^{-2})} \frac{dz}{iz}$$

$$\int_{\gamma(0)} f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) dz = \int_{\gamma} \frac{z}{z^2+2z^4+4z^2+2} \frac{dz}{i}$$

$$\int_0^{2\pi} \underbrace{f(\cos\theta, \sin\theta)}_{F(\cos\theta, \sin\theta)} ie^{i\theta} d\theta = \int_{\gamma} \frac{z}{(2z^4+5z^2+2)i} dz$$

$$\therefore f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)$$

$$= F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)$$

iz

$$= \int_{\gamma} \frac{z}{i(2z^2+1)(z^2+2)} dz$$

$$z = \pm \frac{i}{\sqrt{2}}$$

$$z = \pm i\sqrt{2}$$

$$2z^4+4z^2+z^2+2$$

$$2z^2(z^2+2)+1(z^2+2)$$

$$\underline{E_g}: \int_0^{2\pi} \frac{1}{1+8\cos^2\theta} d\theta$$

$$= \int_{C_1(0)} \frac{1}{1+8\left(\frac{z+z^{-1}}{2}\right)^2} \cdot \frac{dz}{iz}$$

$$= \int_{C_1(0)} \frac{z^2}{2z^4 + 2 + 4z^2 + z^2} \cdot \frac{dz}{iz}$$

$$= \int_{C_1(0)} \frac{-iz}{2z^4 + 5z^2 + 2} dz$$

$$(2z^2+1)(z^2+2)$$

$$\boxed{\text{poles: } z = \frac{\pm i}{\sqrt{2}}, \pm i\sqrt{2}}$$

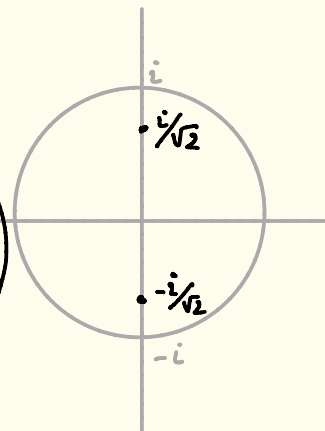
$$= \frac{2\pi i}{2} \left(\frac{-i(i/\sqrt{2})}{\left(2\frac{i}{\sqrt{2}}\right)(2-\frac{1}{2})} + \frac{-i(-i/\sqrt{2})}{\frac{2(-i)}{\sqrt{2}}(2-\frac{1}{2})} \right)$$

$$= 2\pi/3$$

Tip:

$$\frac{f}{k} \cdot \frac{(-iz/z^2+2)}{2z^2+1}$$

$$= \frac{f(a)}{k'(a)}$$



②

EVALUATING REAL INTEGRAL $\int_0^{\infty} f(x) dx$.

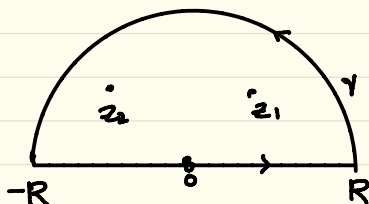
$$\int_0^{\infty} \frac{1}{x^4+1} dx$$

~~odd~~
even

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{x^4+1} dx$$

$$= \frac{1}{2} \lim_{R \rightarrow \infty} \int_{-R}^R \frac{1}{x^4+1} dx.$$

Find a suitable contour



Find the complex-valued f.z.

$$\int_0^{\pi} \frac{1}{z^4+1} dz + \int_{-R}^R \frac{1}{1+x^4} dx$$

$$z^4+1=0$$

$$z^4 = -1$$

$$z_1 = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$$

$$z_2 = \frac{-1+i}{\sqrt{2}}$$

$$= 2\pi i [\text{Res}(f; z_1) + \text{Res}(f; z_2)]$$

$$= \frac{\pi}{\sqrt{2}}$$

estimate

$$\left| \int_0^{\pi} \frac{1}{Re^{i4t}+1} \cdot Rie^{it} \cdot dt \right| \leq \frac{2\pi \cdot R}{R(R^3-1)} = \frac{2\pi}{R^3-1} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

$$|R^4 e^{i4t} + 1| \geq |R^4 e^{i4t}| - |1| \geq R^4 - R$$

$$\text{Res}\left(\frac{1}{1+z^4}, \frac{1+i}{\sqrt{2}}\right)$$

$$\frac{h''(z)}{k'(z)} \rightarrow h(a) \neq 0$$

$$k(a) = 0, k'(a) \neq 0$$

$$\text{Res}\left(\frac{h}{k}; a\right) = \frac{h(a)}{k'(a)} = \frac{1}{4z^3} \Big|_{z=a}$$

OBSERVE!:- $z^4 = -1$

$$\Rightarrow z = -\frac{1}{z^3}$$

$$\leq \frac{R^R}{R^4-1} \rightarrow 0 \text{ as } R \rightarrow \infty$$

$$= -\frac{1}{4}z \Big|_{z=a}$$

$$\left| \int_0^\pi \frac{1}{1+R^4 e^{i4\theta}} R i e^{i\theta} d\theta \right| + \int_{-R}^R \frac{1}{1+x^4} dx = 2\pi i \left(-\frac{1}{4} \left(\frac{-1+i}{\sqrt{2}} + \frac{1+i}{\sqrt{2}} \right) \right)$$

$$= \frac{2\pi i}{4} \left(-\frac{2i}{\sqrt{2}} \right) = \frac{\pi\sqrt{2}}{2}$$

$$\therefore \lim_{R \rightarrow \infty} \frac{1}{2} \int_{-R}^R \frac{1}{1+x^4} dx = \frac{\pi}{2\sqrt{2}}$$

§ Integrals of rational functions

$$\int_0^{\infty} \frac{1}{(x^2+1)^2(x^2+4)} dx$$

(this works since $f(x)$ is even)

$$= 4 \lim_{R \rightarrow \infty} \int_{-R}^R \dots$$

For even functions, this approach is effective!

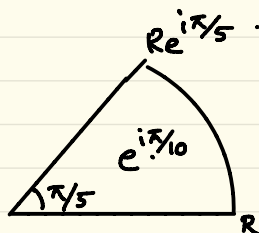
if we took semi circle then we would have at least 5 poles to take care of!

Consider $\lim_{R \rightarrow \infty} \int_{\gamma_R(0)} \dots$

$$\int_0^{\infty} \frac{1}{1+x^{10}} dx$$

roots: $e^{i(2k+1)\pi/10}$

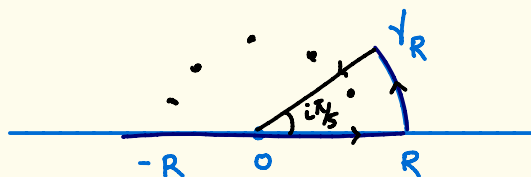
If we took the semicircular contour, then we would have to consider more poles. Choosing sector of angle $\pi/5$, we ensure that there is only one pole inside.



Eg: $\int_0^{\infty} \frac{1}{1+x^{10}} dx$

$\frac{1}{1+z^{10}} = f(z)$

poles $e^{i\pi/10}, e^{i3\pi/10}, \dots, e^{i9\pi/10}$



$$\int_0^R f(x) dx + \int_0^{\pi/5} \frac{1}{1+R^{10}e^{i10\theta}} (iRe^{i\theta}) d\theta + \int_R^0 \frac{1}{1+te^{i10\theta}} (e^{i\pi/5}) dt$$

$$= \int_0^R \frac{1-e^{i\pi/5}}{1+t^{10}} dt + \left| \int_0^{\pi/5} \frac{iRe^{i\theta}}{1+R^{10}e^{i10\theta}} d\theta \right| \leq \frac{R}{R^{10}-1} \cdot \frac{\pi}{5} = -\frac{2\pi i e^{i\pi/10}}{10}$$

$$\lim_{R \rightarrow \infty} \int_0^R \frac{1}{1+t^{10}} dt = \frac{e^{i\pi/10}}{10(1-e^{i\pi/5})}$$

§ EVALUATION of $\int_0^{\infty} \phi(x) \begin{Bmatrix} \sin mx \\ \cos mx \end{Bmatrix} dx$

$$\int_{-\infty}^{\infty} \phi(x) \{ \dots \} dx$$

$$|\cos Re^{i\theta}|^2 = \cosh^2(R \sin \theta) - \sin^2(R \cos \theta)$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + x + 1} dx$$

$\frac{\cos z}{z^2 + z + 1}$ wouldn't have been easy to estimate

$$\int \frac{e^{iz}}{z^2 + z + 1} dz$$

$\Upsilon_R(0)$

(when $z \in \mathbb{R}$, real part of integral is the required integral as $R \rightarrow \infty$)

not even function

the integral evaluates the Principal value integral $\lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$.

the improper integral $\lim_{R, S \rightarrow \infty} \int_{-S}^R f(x) dx$

is equal to PV $\int_{-\infty}^{\infty} f(x) dx$, if it

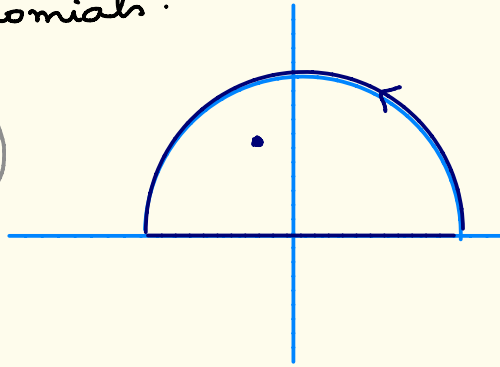
exists. It does, if $f(x) = O(x^{-p})$, $p > 1$

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cdot \begin{cases} \sin mx \\ \cos mx \end{cases} dx$$

p, q are polynomials.

Ex:

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + x + 1} dx$$



$$\gamma_R = Re^{i\theta}; \quad \frac{e^{iz}}{z^2 + z + 1}$$

$$\int_{\gamma_R} \frac{e^{iz}}{z^2 + z + 1} dz = 2\pi i \left(\frac{e^{iz}}{z^2 + z + 1}; e^{2\pi i/3} \right)$$

$$= 2\pi i \left. \frac{e^{iz}}{(z^2 + z + 1)} \right|_{z=e^{2\pi i/3}}$$

$$= 2\pi i \frac{e^{i(\cos 2\pi/3 + i \sin 2\pi/3)}}{2e^{2\pi i/3} + 1}$$

$$\int_{-R}^R \frac{\cos x + i \sin x}{x^2 + x + 1} dx + \int_0^{2\pi} \frac{e^{iRe^{i\theta}} \cdot Re^{i\theta} i d\theta}{Re^{2i\theta} + Re^{i\theta} + 1}$$

real part

$$= \frac{2\pi}{\sqrt{3}} e^{-1/2} \cos \frac{1}{2}$$

At real part
R → ∞

Indentation lemma (for simple poles).

Let f be analytic in $B_r(a) \setminus \{a\}$ with a simple pole of residue b at a .

Let $\gamma_\varepsilon(\theta) = a + \varepsilon e^{i\theta}$, $0 \leq \theta_1 \leq \theta \leq \theta_2 \leq 2\pi$.

$$\text{Then } \boxed{\lim_{\varepsilon \rightarrow 0} \int_{\gamma_\varepsilon} f(z) dz = ib(\theta_2 - \theta_1).}$$

Pf:

$$b = \lim_{z \rightarrow a} (z-a)f(z)$$

$$\therefore \text{Given } \eta > 0 \exists \delta > 0 \ni |(z-a)f(z) - b| < \eta \quad \forall |z-a| < \delta$$

$$\begin{aligned} & \left| \int_{\gamma_\varepsilon} f(z) dz - ib(\theta_2 - \theta_1) \right| \\ &= \left| \int_{\theta_1}^{\theta_2} \underbrace{f(\gamma_\varepsilon(\theta))}_{z} (\underbrace{\varepsilon i e^{i\theta}}_{i(z-a)}) - ib \right) d\theta \right| \\ &\leq \eta \int_{\theta_1}^{\theta_2} d\theta = \eta(\theta_2 - \theta_1). \end{aligned}$$

$$\therefore \lim_{\varepsilon \rightarrow 0} \int_{\gamma_\varepsilon} f(z) dz = ib(\theta_2 - \theta_1)$$



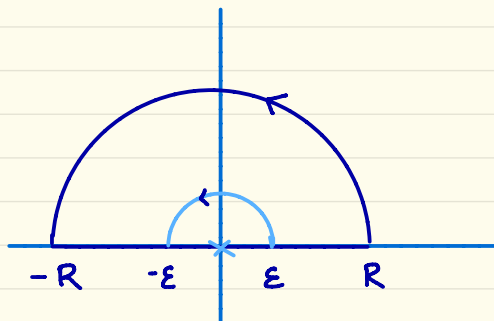
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$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx \rightarrow \frac{1 - \cos 2x}{2}$$

Γ_R : semicircle. $\frac{1}{2} \int_{\Gamma_R} \frac{1 - e^{i2z}}{z^2} dz$ real part

$\gamma_R(0)$ po \rightarrow pole.



$$\int_{-R}^{-\epsilon} + \int_{\Gamma_{\epsilon}} + \int_{\epsilon}^R + \int_{\Gamma_R} = 0$$

$$2 \int_{\epsilon}^R \frac{1 - e^{2ix}}{x^2} dx = 2 \int_{\epsilon}^R \frac{1 - \cos 2x - i \sin 2x}{x^2} dx$$

the real part

$\frac{1 - e^{i2x}}{x^2}$ has a
simple pole at '0'.

$$= \int_{\epsilon}^R \frac{4 \sin^2 x}{x^2} dx$$

Indentation lemma \Rightarrow

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon} f(z) dz = i(\pi - 0) \operatorname{res}(f; 0) = 2\pi.$$

Estimation:

$$\left| \int_{\Gamma_R} f(z) dz \right| \leq \int_0^\pi \left| \frac{1 + e^{-2R \sin \theta}}{R^2} \right| R d\theta \leq \frac{2}{R} \cdot \pi \rightarrow 0 \text{ as } R \rightarrow \infty$$

$$\therefore \lim_{\substack{R \rightarrow \infty \\ \epsilon \rightarrow 0}} \int_{\Gamma_\epsilon} \frac{\sin^2 x}{x^2} dx = \frac{2\pi}{4} = \frac{\pi}{2}.$$

$$\text{ii} \quad \int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$$

Up V/s Down.

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$$\int_{-\infty}^{\infty} \frac{e^{-2ix}}{1+x^4} dx$$

$$\int_{-\infty}^{\infty} \frac{e^{-2iz}}{1+z^4} dz$$

$$z = Re^{i\theta}$$

$$|e^{-2iz}| = |e^{2R\sin\theta}|$$

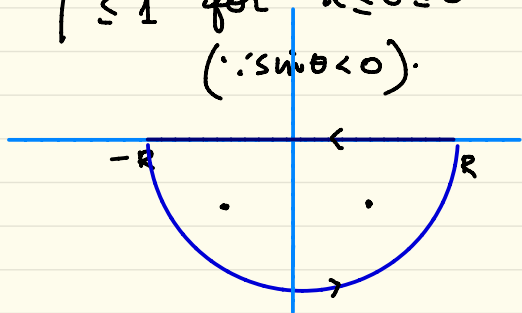
is not bounded.

So we use the lower semicircle.

$$|e^{-2iz}| = |e^{2R\sin\theta}| \leq 1 \text{ for } -\pi \leq \theta \leq 0$$

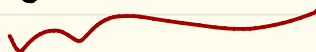
($\because \sin\theta < 0$).

$$\int_{\gamma_R(0)} \frac{e^{-i2z}}{1+z^4} dz$$



$$- \int_{-R}^R \frac{e^{-2ix}}{1+x^4} dx - \int_0^\pi \frac{e^{-iR(\cos\theta - i\sin\theta)}}{1+R^4 e^{-i4\theta}} (-d\theta)$$

$$\left| - \int_0^\pi \frac{e^{-iR(\cos\theta - i\sin\theta)}}{1 + R^4 e^{-i4\theta}} (-d\theta) \right|$$



bdd.

$$\leq \frac{e^{-R\sin\theta}}{R^4 - 1}$$

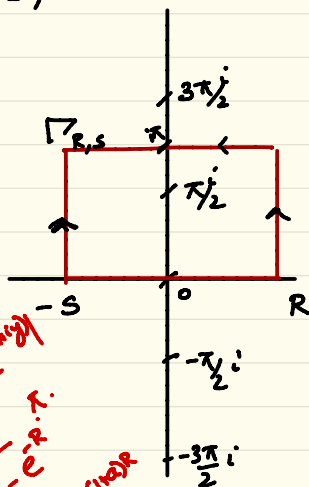
$$\therefore \lim_{R \rightarrow \infty} \int_{-R}^R \frac{e^{-2ix}}{1+x^4} dx = - \left[2\pi i \operatorname{Res} \left(\frac{e^{iz}}{1+z^4}; e^{-i\pi/4} \right) + 2\pi i \operatorname{Res} \left(\frac{e^{iz}}{1+z^4}; e^{-3\pi i/4} \right) \right]$$

A contour excluding infinitely many poles

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$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x} dx \quad (-1 < a < 1)$$

zeros of $\cosh z = \cos(iz)$
 $= (2n+1)\frac{\pi}{2}i$



$$\int_{R,S} \frac{e^{az}}{\cosh z} dz$$

$$= \int_{-S}^R \frac{e^{ax}}{\cosh x} dx + \int_0^{\pi} \frac{e^{a(R+iy)}}{\cosh(R+iy)} i dy$$

or $\frac{e^{aR} \cdot \pi}{e^{\pi i y} + e^{-\pi i y}}$
 $\leq \frac{e^{aR}}{e^R - e^{-R}}$
 $\leq \frac{e^{aR}}{e^{R/2} - e^{-R/2}}$

$$+ \int_{R, -\frac{\pi}{2}}^{-S} \frac{e^{a(\pi i + x)}}{\cosh(x + i\pi)} dx + \int_{\pi}^0 \frac{e^{a(-s+iy)}}{\cosh(s+iy)} dy$$

$\leq \frac{2e^{-aS}}{e^{-s+iy} + e^{s-iy}} \cdot \pi$
 $= \frac{2e^{-aS}}{e^{-s} + e^s} \cdot \pi$
 $= 2\pi i \operatorname{Res} \left(\frac{e^{az}}{\cosh z}; i\frac{\pi}{2} \right)$
 \uparrow simple pole
 $= 2\pi i \frac{e^{a(i\pi/2)}}{\sinh(i\pi/2)}$
 $= 2\pi e^{ia\pi/2} \sin(-\pi/2)$
 $= -2\pi e^{ia\pi/2}$

\therefore as $R \rightarrow \infty$ & $S \rightarrow \infty$

$$(1 + e^{\frac{a\pi i}{2}}) \int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x} dx = 2\pi e^{\frac{a\pi i}{2}}$$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x} dx = \frac{2\pi e^{\frac{a\pi i}{2}}}{e^{\frac{a\pi i}{2}} \left(e^{-\frac{a\pi i}{2}} + e^{\frac{a\pi i}{2}} \right)}$$

$$= \frac{\pi}{\cos\left(\frac{a\pi}{2}\right)}$$