7. The Kalman Filter as State Observer

The Kalman Filter can be used to estimate states that are not directly accessible through a sensor measurement; that is, states that do not explicitly appear in the measurement equation for $z(k)$. In last lecture's example, for instance, position and velocity estimates are obtained from a position measurement only. The KF can infer information about states that are not directly measured by exploiting possible couplings of the states through the system dynamics. In this sense, the KF is sometimes referred to as a *state observer*.

In this lecture, we discuss *detectability* and *stabilizability* as conditions that guarantee that all states of a linear, time-invariant system can be estimated reliably by a KF (in the sense that the error variances of all states converge). In addition, the discussion of asymptotic properties of the KF will allow us to derive the *Steady-State Kalman Filter*, which is a time-invariant implementation of the KF.

7.1 Model

For this lecture, we restrict the model of Sec. 6.1 to time-invariant systems and stationary distributions. That is, $A(k) = A$, $H(k) = H$, $Q(k) = Q$, and $R(k) = R$ are constant.

$$
\begin{aligned}
x(k) &= Ax(k\text{-}1) + u(k\text{-}1) + v(k\text{-}1) \qquad & x(0) &\sim \mathcal{N}(x_0, P_0), v(k\text{-}1) \sim \mathcal{N}(0, Q) \\
z(k) &= Hx(k) + w(k) & w(k) &\sim \mathcal{N}(0, R)
\end{aligned}
$$

where $x(k) \in \mathbb{R}^n$, $z(k) \in \mathbb{R}^m$.

7.2 Asymptotic Properties of the Kalman Filter

For constant A, H, Q, and R, the KF is still time-varying:

$$
\begin{aligned}
P_p(k) &= AP_m(k\text{-}1)A^T + Q \\
K(k) &= P_p(k)H^T(HP_p(k)H^T + R)^{-1} \\
P_m(k) &= (I - K(k)H)P_p(k).
\end{aligned}
$$

In the following we examine the estimation error $e(k) = x(k) - \hat{x}_m(k)$ as $k \to \infty$. We already know that the filter is unbiased, i.e. $\mathrm{E}[e(k)] = 0$ for all k if $\hat{x}_m(0) = x_0$.

We now consider the variance $P_p(k)$ and combine the equations above:

$$
P_p(k\text{+}1) = AP_p(k)A^T + Q - AP_p(k)H^T\left(HP_p(k)H^T + R\right)^{-1}HP_p(k)A^T. \tag{7.1}
$$

We will highlight different behaviors of $P_p(k)$ as k tends to infinity with the following examples. *(Matlab files to simulate the following examples are provided on the class website.)*

Date compiled: May 2, 2017

Example 1

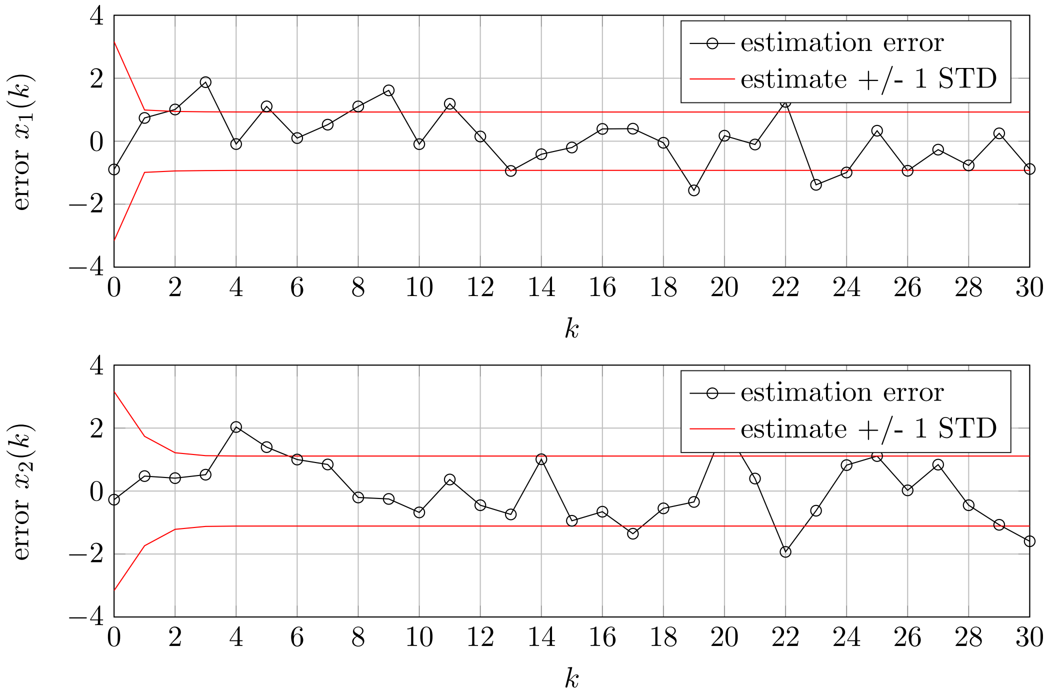
Consider the system

$$x(k) = Ax(k-1) + v(k-1) \qquad x(0) \sim \mathcal{N}(0, 10 \cdot I), v(k-1) \sim \mathcal{N}(0, I)$$
$$z(k) = Hx(k) + w(k) \qquad w(k) \sim \mathcal{N}(0, 1).$$

Case 1:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

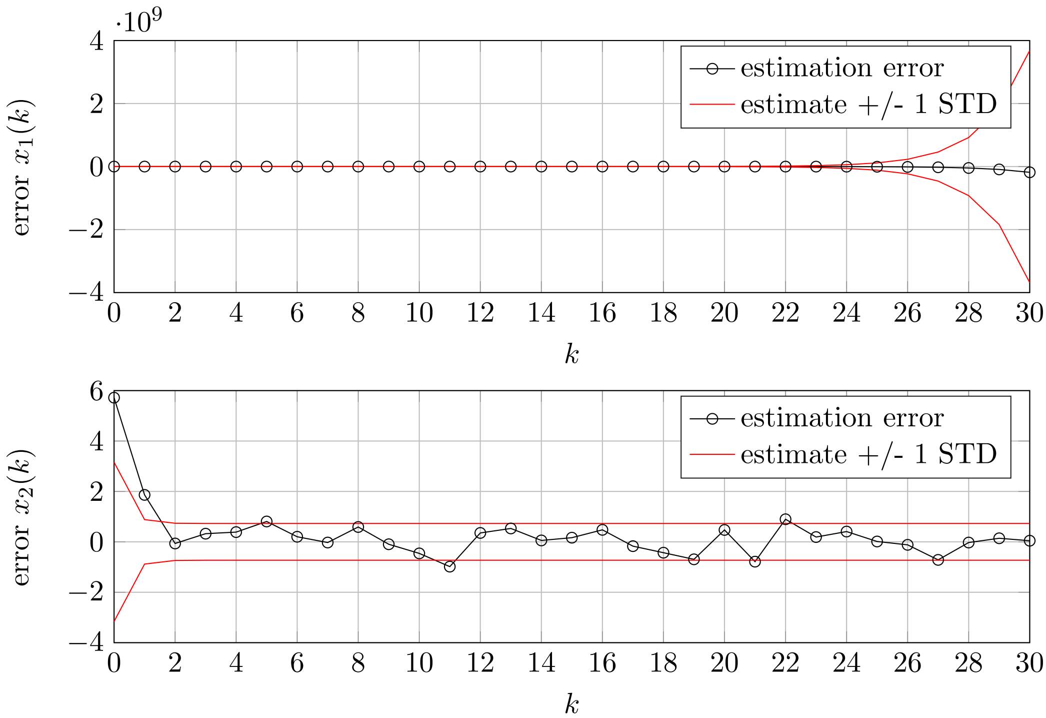
Here, the variance $P_p(k)$ converges, see figure below.



Case 2:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

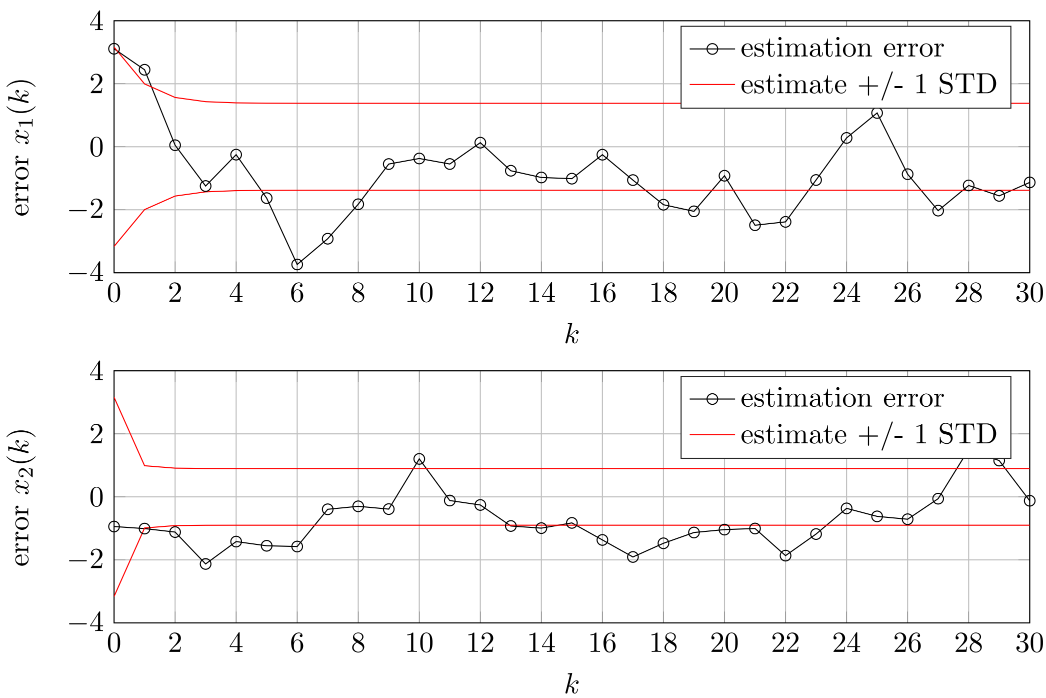
Compared to Case 1, the measurement $z(k)$ consists of $x_2(k)$ corrupted by noise. The variance corresponding to $x_1(k)$ grows unbounded: $\mathrm{Var}[e_1(k)] = P_p^{11}(k) \to \infty$ as $k \to \infty$.



Case 3:

$$A = \begin{bmatrix} 0.5 & 1 \\ 0 & 2 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Here, the variance $P_p(k)$ converges. Compared to Case 2, the diagonal elements of the matrix A are interchanged, resulting in stable dynamics of the state $x_1(k)$.



Example 1 indicates that the convergence of $P_p(k)$ is not only influenced by the sensor placement (the matrix H), but is also dependent on the system dynamics.

Example 2

Consider the system

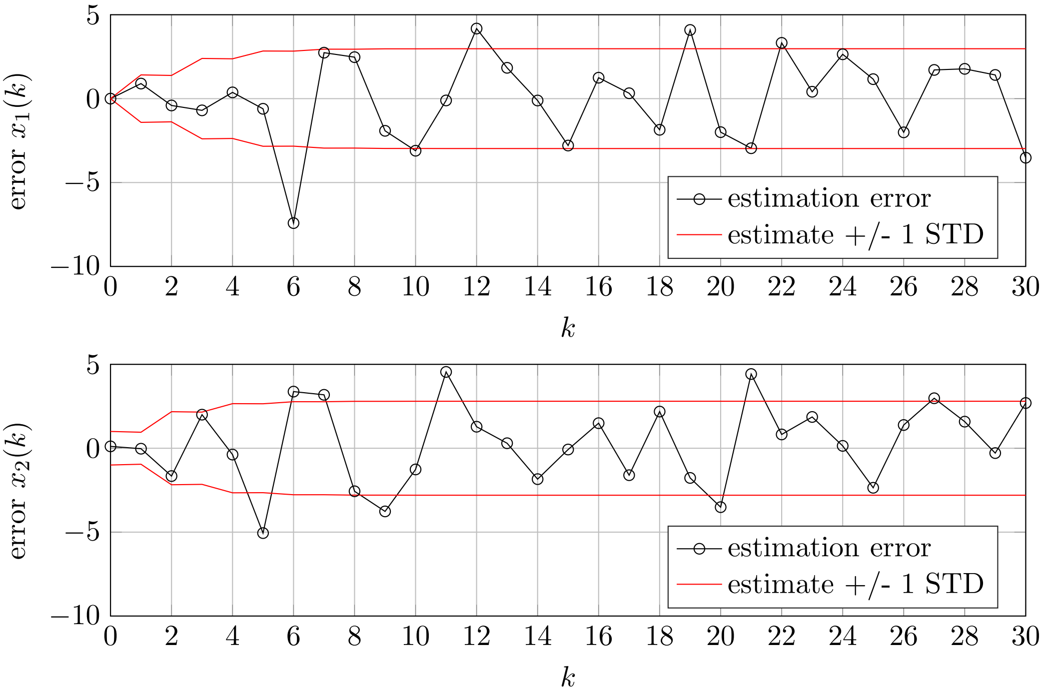
$$x(k) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x(k\text{–}1) + v(k\text{–}1) \qquad\qquad x(0) \sim \mathcal{N}(0, P_0), v(k\text{–}1) \sim \mathcal{N}(0, Q)$$

$$z(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(k) + w(k) \qquad\qquad w(k) \sim \mathcal{N}(0, 10).$$

Case 1:

$$P_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q = I$$

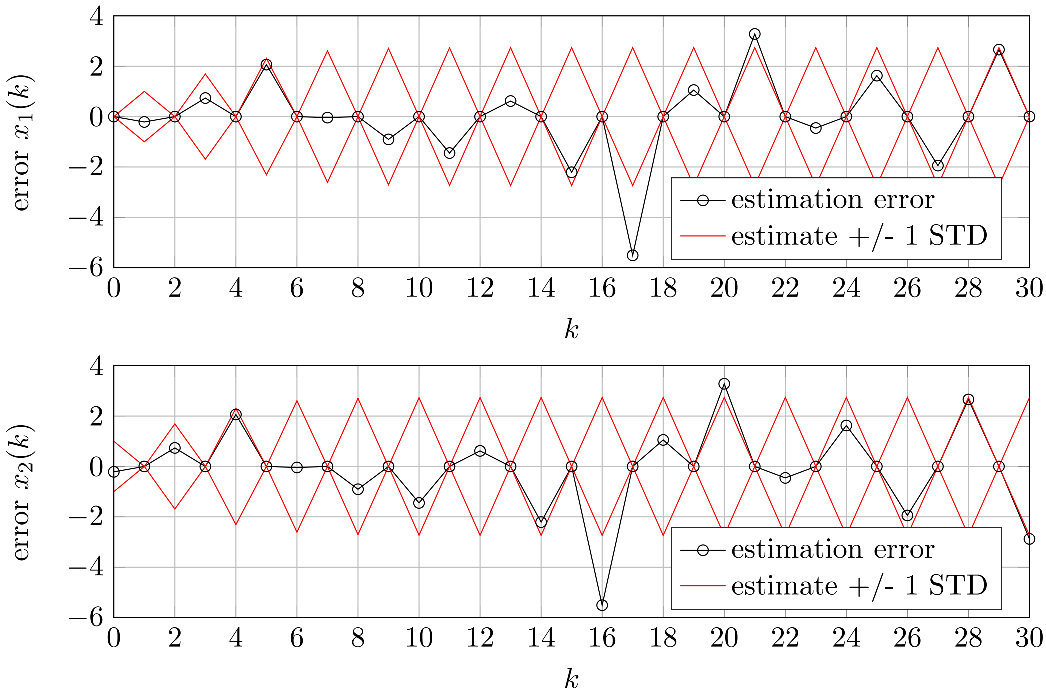
Here, $P_p(k)$ converges, see figure below.



Case 2:

$$P_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q = 0 \quad \text{(no process noise)}$$

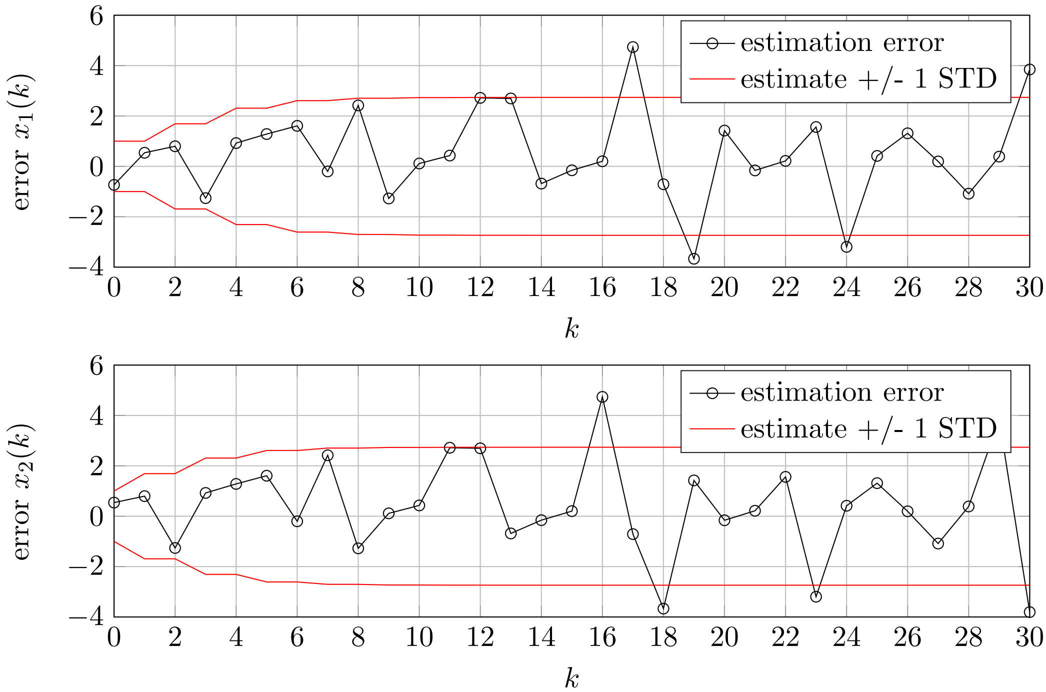
The variance $P_p(k)$ does not converge, but remains bounded.



Case 3:

$$P_0 = I, Q = 0 \quad \text{(no process noise)}$$

The variance $P_p(k)$ converges.



Example 2 indicates that the convergence of $P_p(k)$ is also a function of the process noise and the variance of $x(0)$.
Summarizing, the asymptotic properties of the Kalman Filter are influenced by the sensor placement (the matrix H), the system dynamics (the matrix A), the process noise variance (the matrix Q), and may depend on the variance of the initial condition $x(0)$.

In order to discuss the asymptotic properties of the Kalman Filter rigorously, the concepts of detectability and stabilizability are required. These are introduced in the following.

7.3 Observability and detectability

Observability

We first introduce the concept of observability, which is closely related to detectability. We consider a deterministic system, i.e. without noise ($v(k-1) = 0$, $w(k) = 0$), where the goal is to reconstruct $x(0)$ from measurements $z(0)$, $z(1)$, $z(2)$, etc.[1]

We have

$$\begin{aligned} z(0) &= Hx(0) \\ z(1) &= Hx(1) = HAx(0) + Hu(0) \\ z(2) &= Hx(2) = HA^2x(0) + Hu(1) + HAu(0) \\ &\vdots \\ z(n-1) &= Hx(n-1) = HA^{n-1}x(0) + Hu(n-2) + \cdots + HA^{n-2}u(0), \end{aligned}$$

which we rewrite as:

$$\underbrace{\begin{bmatrix} H \\ HA \\ \vdots \\ HA^{n-1} \end{bmatrix}}_{=:\mathcal{O}} x(0) = \begin{bmatrix} z(0) \\ z(1) \\ \vdots \\ z(n-1) \end{bmatrix} - \begin{bmatrix} 0 & 0 & \cdots & 0 \\ H & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ HA^{n-2} & HA^{n-3} & \cdots & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(n-1) \end{bmatrix}.$$

Notice that the matrix $\mathcal{O}$ is constructed using powers of A only up to A^{n-1}. The reason follows from the Cayley-Hamilton theorem, which states that A^n (and higher orders) will be a linear combination of A^0 through A^{n-1}, meaning that adding further powers of A cannot increase the rank of $\mathcal{O}$.
The right-hand side (RHS) is known. Hence, we can uniquely solve for $x(0)$ if and only if $\text{rank}(\mathcal{O}) = n$ (full column rank). In that case we can use the following least squares approach:

$$x(0) = (\mathcal{O}^T\mathcal{O})^{-1}\mathcal{O}^T \cdot \text{RHS}.$$

If $\text{rank}(\mathcal{O}) = n$, we say that the pair (A, H) is *observable*.

Observability conditions[2]

The pair (A, H) is observable.

- $\Leftrightarrow$ For a deterministic LTI system ($x(k) = Ax(k-1) + u(k-1)$, $z(k) = Hx(k)$), knowledge of $z(0{:}n-1)$ and $u(0{:}n-1)$ suffices to determine $x(0)$.
- $\Leftrightarrow$ $\text{rank}(\mathcal{O}) = n$.
- $\Leftrightarrow$ $\begin{bmatrix} A - \lambda I \\ H \end{bmatrix}$ is full rank for all $\lambda \in \mathbb{C}$ (PBH-Test).
- $\Leftrightarrow$ The eigenvalues of $A - KH$ can be placed arbitrarily by a suitable choice of the matrix $K \in \mathbb{R}^{n\times m}$.

[1] Note that once we know $x(0)$, we can reconstruct $x(k)$ for all k (for the deterministic case).
[2] *Anderson, Moore, Optimal Filtering, Dover Publications, 2005.*

- For the PBH-Test (PBH = Popov-Belevitch-Hautus), one only needs to check those λ that are eigenvalues of A. For all other values of λ, $A - \lambda I$ has full rank.

Detectability

A system is detectable if all its unstable modes are observable. This leads to the following conditions for detectability.

Detectability conditions

The pair (A, H) is detectable.

$\Leftrightarrow$ $\begin{bmatrix} A - \lambda I \\ H \end{bmatrix}$ is full rank for all $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$ (PBH-Test).

$\Leftrightarrow$ The eigenvalues of $A - KH$ (or equivalently $(I - KH)A$) can be placed within the unit circle by a suitable choice of the matrix $K \in \mathbb{R}^{n \times m}$.

Furthermore, if (A, H) is not observable, then there exists a state transformation T such that

$$TAT^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad HT^{-1} = \begin{bmatrix} H_1 & 0 \end{bmatrix}, \quad \text{and } (A_{11}, H_1) \text{ observable.}$$

Then: (A, H) is detectable $\Leftrightarrow$ A_{22} is stable (all eigenvalues have magnitude less than 1).

- The main idea behind detectability:

  Assume that there exists a λ with $|\lambda| \geq 1$ such that $\left[\begin{smallmatrix} A-\lambda I \\ H \end{smallmatrix}\right]$ is not full column rank. Then there exists a vector v, $v \neq 0$, such that

  $$(A - \lambda I)v = 0, \quad Hv = 0 \quad \Leftrightarrow \quad Av = \lambda v, \quad Hv = 0.$$

  That is, the vector v is a natural mode of the system that does not decay and that is not seen in the output of the system.

- Detectability is weaker than observability: (A, H) observable $\Rightarrow$ (A, H) detectable.

7.4 Controllability and stabilizability

Controllability

The system

$$x(k) = Ax(k-1) + Bu(k-1)$$

is called controllable (from the origin) if there exists a sequence of inputs $u(k)$, $k = 0, 1, \ldots, n-1$ that drive the system from $x(0) = 0$ to any location $x(n)$ in the state space.[3]
Controllability is the dual of observability, that is, (A, B) is controllable if and only if (A^T, B^T) is observable. As a result, the following controllability conditions follow, where the controllability matrix $\mathcal{C}$ is defined as

$$\mathcal{C} := \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix}.$$

[3] In some textbooks, such as *Anderson, Moore, Optimal Filtering, Dover Publications, 2005*, this is called *reachability*. In *Anderson, Moore, Optimal Filtering, Dover Publications, 2005* controllability (to the origin) is then defined to mean that one can steer an LTI system from any initial state to 0 in finite time; whereas reachable means that one can reach an arbitrary state from 0 in finite time. In this sense, reachable and controllable are not equivalent for discrete-time systems. Consider, for example, $A = 0$, which is controllable to the origin, but not reachable.

Controllability conditions

The pair (A, B) is controllable (from the origin).

$\Leftrightarrow$ The deterministic LTI system $(x(k) = Ax(k-1) + Bu(k-1))$ can be driven from $x(0) = 0$ to any $x(n) \in \mathbb{R}^n$.

$\Leftrightarrow$ $\text{rank}(\mathcal{C}) = n$.

$\Leftrightarrow$ $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ is full rank for all $\lambda \in \mathbb{C}$ (PBH-Test).

$\Leftrightarrow$ The eigenvalues of $A - BK$ can be placed arbitrarily by a suitable choice of the matrix K.

Stabilizability

A system is stabilizable if all its unstable modes are controllable. This leads to the following conditions for stabilizability.

Stabilizability conditions

The pair (A, B) is stabilizable.

$\Leftrightarrow$ $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ is full rank for all $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$ (PBH-Test).

$\Leftrightarrow$ The eigenvalues of $A - BK$ can be placed within the unit circle by a suitable choice of the matrix K.

Furthermore, if (A, B) is not controllable, then there exists a state transformation T such that

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad TB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \text{and } (A_{11}, B_1) \text{ controllable.}$$

Then: (A, B) is stabilizable $\Leftrightarrow$ A_{22} is stable (all eigenvalues have magnitude less than 1).

- Stabilizability is weaker than controllability: (A, B) controllable $\Rightarrow$ (A, B) stabilizable.
- Stabilizability is the dual of detectability, that is, (A, B) is stabilizable if and only if (A^T, B^T) is detectable.

7.5 The Steady-State Kalman Filter

Motivation: if the KF variance converges, then so does the KF gain: $\lim_{k\to\infty} K(k) = K_\infty$. Using the constant gain K_∞ instead of the time-varying gain $K(k)$ simplifies the implementation of the filter (there is no need to compute or store $K(k)$). This filter is called the *Steady-State KF*.

Computation of K_∞

Assume $P_p(k)$ converges to P_∞. Then, (7.1) reads

$$P_\infty = AP_\infty A^T + Q - AP_\infty H^T \big(HP_\infty H^T + R\big)^{-1} HP_\infty A^T.$$

- This is an algebraic equation in P_∞, called the *Discrete Algebraic Riccati Equation* (DARE).
- Efficient methods exist for solving it (under certain assumptions on the problem parameters, see below); Matlab implementation: `dare(A',H',Q,R)`.
- The steady-state KF gain then is: $K_\infty = P_\infty H^T (HP_\infty H^T + R)^{-1}$.

Steady-state estimator

The steady-state KF equations with $\hat{x}(k) := \hat{x}_m(k)$:

$$\begin{aligned}\hat{x}(k) &= \big(I - K_\infty H\big) A\,\hat{x}(k\text{-}1) + \big(I - K_\infty H\big)\,u(k\text{-}1) + K_\infty\, z(k)\\ &= \hat{A}\,\hat{x}(k\text{-}1) + \hat{B}\,u(k\text{-}1) + K_\infty\, z(k),\end{aligned}$$

a linear time-invariant system.

Estimation error:

$$e(k) = x(k) - \hat{x}(k) = \underbrace{(I - K_\infty H)A}_{\substack{\text{stability}\\ \text{important!}}}\, e(k\text{-}1) + (I - K_\infty H)\, v(k\text{-}1) - K_\infty\, w(k).$$

- Want $(I - K_\infty H)A$ to be stable (i.e. all eigenvalues inside the unit circle) for the error not to diverge.
- Mean: $\mathrm{E}[e(k)] = (I - K_\infty H)A\,\mathrm{E}[e(k\text{-}1)]$.
  Consider the case if $\mathrm{E}[e(0)] = x_0 - \hat{x}(0) \neq 0$ (we may not know $x_0 = \mathrm{E}[x(0)]$). We have: $\mathrm{E}[e(k)] \to 0$ as $k \to \infty$ for any initial $\mathrm{E}[e(0)]$ if and only if $(I - K_\infty H)A$ is stable.

What can go wrong?

- $P_p(k)$ does not converge as $k \to \infty$. (Recall the examples in Sec. 7.2.)
- $P_p(k)$ converges, but to different solutions for different initial $P_p(1)$. This is not desirable – which one should we use to compute K_∞? (Recall the examples in Sec. 7.2.)
- $(I - K_\infty H)A$ is unstable.

All these are addressed by the following theorem.

Theorem[4]

Assume $R > 0$ and $Q \geq 0$, and let G be any matrix such that $Q = GG^T$ (can always be done for symmetric positive semidefinite matrices).
Remark: we can now write the system dynamics as

$$x(k) = Ax(k\text{-}1) + u(k\text{-}1) + G\bar{v}(k\text{-}1)$$

where $\bar{v}(k) \sim \mathcal{N}(0, I)$.
Then the following two statements are equivalent:

1. (A, H) is detectable and (A, G) is stabilizable.
2. The DARE has a unique positive semidefinite solution $P_\infty \geq 0$, the resulting $(I - K_\infty H)A$ is stable, and

$$\lim_{k\to\infty} P_p(k) = P_\infty \qquad \text{for any initial } P_p(1) \geq 0 \text{ (and, hence, any } P_m(0) = P_0 \geq 0).$$

Interpretation of the two conditions:

[4] Adapted from *Simon, Optimal State Estimation, Wiley, 2006* and *Anderson, Moore, Optimal Filtering, Dover Publications, 2005.* See these and references therein for proofs.

- (A, H) is detectable: can observe all unstable modes.
- (A, G) is stabilizable: noise excites unstable modes. Note that if $Q > 0$ then (A, G) is always stabilizable.
- Additional examples where one of these is not satisfied are discussed in the recitation.

7.6 Remarks

- The KF is the optimal state estimator (for a linear system and Gaussian distributions) irrespective of whether the system is detectable, stabilizable (with respect to the process noise), or neither. The KF does the best it can, even if the measurements do not provide sufficient information for reliably estimating all states. This is why detectability/stabilizability was not discussed when we derived the Kalman filter.
- Detectability and stabilizability (with respect to the process noise) are properties of the system, and not of the estimation algorithm. Hence, they cannot be altered by using a different state estimation algorithm, but only by modifying the system (for example, by placing an additional sensor).
- Observability, detectability, controllability, and stabilizability can also be defined for time-varying or nonlinear systems. However, conditions for checking them are usually not as straightforward as for the linear time-invariant case.