Hindawi
Discrete Dynamics in Nature and Society
Volume 2022, Article ID 2333965, 12 pages
https://doi.org/10.1155/2022/2333965

Research Article

Analysis of the Queueing-Inventory System with Impatient Customers and Mixed Sales

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Received 23 September 2021; Revised 28 February 2022; Accepted 18 March 2022; Published 6 May 2022

Academic Editor: Bo Yang

A queueing-inventory system with server's vacations and impatient customers is investigated. Continuous review (s, S) policy is considered for the queueing-inventory system. The server leaves for a vacation once the inventory is zero. The customer arrived waits for service with a probability p or is lost forever with a probability $1 - p$ when the server is off due to a vacation. Customers are impatient during period when the customer is waiting for service in the queue or when the server is off due to a vacation. The matrix geometric solution of the steady-state probability is obtained using the method of truncated approximation. Furthermore, some performance measures are obtained using the steady-state probability. Finally, the effect of system parameters on the performance measures, the optimal policy and the optimal cost are presented by numerical examples.

1. Introduction

Over the last decades, many researchers have captured much attention on queueing systems with an attached inventory. Such a queueing system is called queueing-inventory system (QIS). A QIS is different from both the traditional inventory system and the traditional queuing system (see Zhao and Lian [1]). The QIS has a wide range of applications in fields such as integrated supply chain management [2], vehicle maintenance [3] and medical service [4].

The QIS was first introduced by Sigman and Simchi-Levi [5] and Melikov and Molchanov [6]. Sigman and Simchi-Levi [5] studied an M/G/1 QIS, where customers arrived when the inventory is zero were backlogged. A light traffic heuristic was developed to obtain the performance measures. Melikov and Molchanov [6] introduced QIS in investigating transportation/storage system (TSS), in which user requests were patient that they waited in the queue until the inventory was replenished. Besides, both exact and approximate solution methods were developed to derive the performance measures. Subsequently, more and more scholars are interested in QISs. Schwarz et al. [2] and Schwarz and Daduna [7] studied the M/M/1 QISs for cases of lost sales and backordering, respectively. Yue et al. [8] analyzed a QIS with batch demands for two cases, namely partial-lost sales and full-lost sales.

To utilize the server's idle time, the QIS with servers' vacation were studied by some scholar. Lakshmanan et al. [9] considered the multiple working vacations in QIS model, where any customer enters a single finite retrial orbit when the demanding item is stocked out or the server is busy. They obtained the joint steady-state distributions of the system, and the performance measures and long run expected total cost. Manikandan and Nair [10] studied an M/M/1 QIS model with working vacations, vacation interruptions and lost sales. They obtained various performance measures and the stationary waiting time distribution. Zhang et al. [11] investigated an M/M/1 QIS with server's multiple vacations and a random order size policy, where lost sales was considered. The explicit product form solution for the steady-state distribution was obtained. The effect of parameters on performance measures was studied by using numerical examples and simulation method. For more results on this topic, we refer to the two survey papers given by Krishnamoorthy et al. [12, 13].

Vacation queueing models with customers' impatience have been greatly studied. Altman and Yechiali [14] studied the M/M/1, M/G/1 and M/M/c queues with server's vacation and customers' impatience. They assumed customers become impatient when the server goes on vacation. Furthermore, they compared the proportion of customer abandonments under the single-vacation and multiple-vacation. And then, Altman and Yechiali [15] extended the impatience phenomenon studied by Altman and Yechiali [14] in an infinite-server queue system. Furthermore, more results on queue system with servers' vacation and customer's impatience can be refer to Bouchentouf et al. [16, 17], Bouchentouf and Guendouzi [18], Kadi et al. [19], Padmavathy et al. [20], Yue et al. [21–23].

The early analytical results of server's vacations on QIS with impatient customers appeared in Viswanath et al. [24] who studied a very general model, where the customer demands followed the Markovian arrival process and the lead time followed a correlated process similar to it. Besides, the service times and the vacation time followed the phase type distributions. They assumed that whenever there were no on-hand inventory items or no customers in the system, the server took a vacation. Melikov et al. [25] studied a perishable QIS with early and delayed server vacations and they considered a finite queue of impatient claims. An asymptotic method was developed to obtain the performance measures. Koroliuk et al. [26] discussed a perishable QIS with impatient customers and a server's vacation, where the restocking follows the two-level policy. The server took a vacation if either there was no on-hand inventory item or no customer or both none. They developed an asymptotic method to obtain the performance measures.

In this paper, we consider an M/M/1 QIS with server's vacation and impatient customers. The server's vacation and the sales policy of this model differ from the models that were studied in [24–26]. They all assumed that the server went on vacation if either there were no on-hand inventory items or no customers in the system or both none, and also assumed that the customers became impatient during the time of waiting for service. In our model, when the on-hand inventory is zero, the server leaves for a vacation. If the server finds that the inventory is not empty at the end of a vacation, the server returns back to the system and serves a customer waiting for service. Otherwise, the server takes another vacation immediately. Besides, we consider a mixed sales policy, i.e., the customer arrived waits for service with a probability p or is lost forever with a probability $1-p$ when the server is off due to a vacation. Such mixed sales policy can be seen as a mixed of the lost sales and the backordering. If $p=0$, this mixed sales becomes the lost sales, and if $p=1$, it becomes the backordering case. To the best of our knowledge, all the research work on QIS with server's vacation and impatient customer considers the lost sales and backordering separately. In presented model, we consider a mixed sales policy which integrates the lost sales and the backordering in a QIS with server's vacation and impatient customers. How does the mixed sales policy affect the performance measures? In contrast with the lost sales policy ($p=0$) and the backordering policy ($p=1$), is there a better mixed sales policy ($0<p<1$)? This motivates us to the study of the present model.

A representative example of an application of our model is online sales of electronic products. The upgrading of electronic products such as mobile phone, smart TV, CPU, Graphics card is very fast because of their own particularity. Online retailers often sell their new products by online reservation. Some customers like to make reservation to buy a new generation of products, while some customers do not like to make reservation buy new products. This corresponds to the mixed sales policy of our model. The period of reservation can be regards as servers' vacation. In this period of time, the servers do not provide service for customers until the deadline of reservation. Besides, some customers who make reservation may cancel their reservation. This customer is impatient customer of our model.

The main contributions of this paper are summarized as follows: (a) We derive the matrix geometric solution of the steady-state probability using the method of truncated approximation. (b) We investigate the effect of the probability and impatient rate on some performance measures. (c) The optimal policy and the optimal cost are computed by using the genetic algorithm. (d) We derive the queuing performance measures including the mean number of customers in system, the mean number of customers in queue, the mean sojourn time and the mean waiting time.

The rest of this paper is organized as follows. Section 2 describes the system model. In Section 3, we obtain the matrix geometric solution of the steady-state probability by using truncated approximation method. Furthermore, we derive some performance measures. Section 4 investigates the impact of the probability p and impatient rate ξ on some important performance measures, the optimal policy and the optimal cost. Section 5 concludes the paper.

2. System Model

Consider a QIS with impatient customers and multiple vacations. The arrival process of customers is Poisson process with rate $\lambda\,(\lambda>0)$. Customers are served one by one by a server under a First-Come, First-Served (FCFS) discipline. The service time follows exponential distribution with parameter $\mu\,(\mu>0)$.

Multiple vacations policy is considered. When the on-hand inventory is zero, the server leaves for a vacation. If the server finds that the inventory is not empty at the end of a vacation, the server returns back to the system and serves a customer waiting for service. Otherwise, the server takes another vacation immediately. The vacation time follows an exponential distribution with parameter $\theta\,(\theta>0)$.

The new arriving customer waits for service with probability $p\,(p>0)$ or is lost forever with probability $1-p$ when the server is off due to a vacation. Customers are impatient during period when the customer is waiting for service in the queue or when the server is off due to a vacation. The impatient times follows an exponential distribution with parameter $\xi\,(\xi>0)$.

An (s,S) ordering policy is adopted. When on-hand inventory level drops to s, an order is placed such that the on-hand inventory level is restocked to level $S(s<S)$. The lead time follows an exponential distribution with parameter $\eta(\eta>0)$.

3. Steady-State Analysis

In this section, we develop a three-dimensional Markov process and derive the steady-state probability using method of matrix analysis. Then, we calculate the steady-state probability by using truncated approximation method and further derive some performance measures.

3.1. Steady-State Distribution. Let $\Phi(t)=\{(X(t),Y(t),Z(t)),t\geq 0\}$ be the state process of the system, where $X(t)$ denotes the number of customers at time t, $Y(t)$ denotes the inventory level at time t, $Z(t)$ denotes the inventory level at time t, which is defined as follows: $Z(t)=0$ if the server is off due to a vacation, and $Z(t)=1$ if the server is on. Then, the process $\Phi(t)$ is a Markov process with state space:

$$\Omega=\bigcup_{n=0}^{\infty}\{(n,0,0),(n,1,1),(n,2,1),\ldots,(n,S,1),(n,S,0)\},\quad n=0,1,\ldots, \tag{1}$$

is the collection of states with $X(t)=n$, $n\geq 0$, called the level n. The state-transition diagram of the QIS is presented in Figure 1.

The state process $\Phi(t)$ is a level-dependent quasi-birth-and-death (LDQBD) process, and the infinitesimal generator of the process $\Phi(t)$ is as follows:

$$\mathbf{Q}=\begin{pmatrix}\mathbf{A}_0 & \mathbf{C} & & & & & \\ \mathbf{B}_1 & \mathbf{A}_1 & \mathbf{C} & & & & \\ & \mathbf{B}_2 & \mathbf{A}_2 & \mathbf{C} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \mathbf{B}_n & \mathbf{A}_n & \mathbf{C} & \\ & & & & \ddots & \ddots & \ddots\end{pmatrix}, \tag{2}$$

where $\mathbf{A}_0$, $\mathbf{A}_n$, $\mathbf{B}_n$ and $\mathbf{C}$ are all square matrices of the order $S+2$, and they are given by

$$\mathbf{A}_0=\begin{pmatrix}-(p\lambda+\eta) & 0 & \cdots & 0 & \cdots & 0 & \eta\\ 0 & -(\lambda+\eta) & \cdots & 0 & \cdots & \eta & 0\\ \vdots & \vdots & & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & -(\lambda+\eta) & \cdots & \eta & 0\\ \vdots & \vdots & & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & \cdots & -\lambda & 0\\ 0 & 0 & \cdots & 0 & \cdots & \theta & -(p\lambda+\theta)\end{pmatrix},$$
$$\mathbf{A}_n=\begin{pmatrix}h_{n00} & 0 & \cdots & 0 & \cdots & 0 & \eta\\ 0 & h_{n11} & \cdots & 0 & \cdots & \eta & 0\\ \vdots & \vdots & & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & h_{ns1} & \cdots & \eta & 0\\ \vdots & \vdots & & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & \cdots & h_{nS1} & 0\\ 0 & 0 & \cdots & 0 & \cdots & \theta & h_{nS0}\end{pmatrix}, \tag{3}$$

where

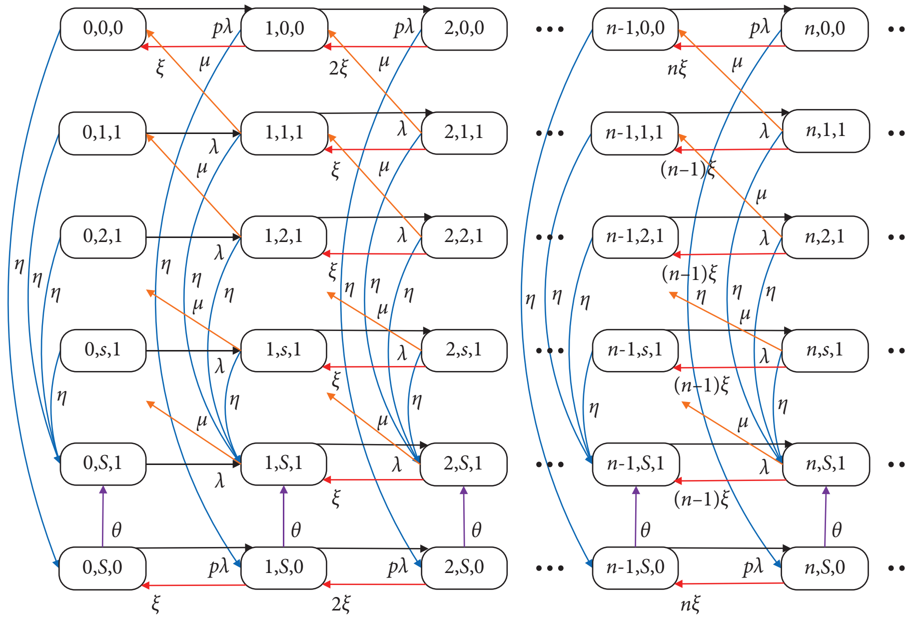

Figure 1: State-transition diagram of the QIS model.

$$h_{ni0} = \begin{cases} -(p\lambda + \eta + n\xi), & i = 0,\ n = 1, 2, \ldots, \\ -(p\lambda + \theta + n\xi), & i = S,\ n = 1, 2, \ldots, \end{cases} \tag{4}$$

and

$$h_{ni1} = \begin{cases} -[\lambda + \eta + \mu + (n-1)\xi], & i = 1, 2, \ldots, s,\ n = 1, 2, \ldots, \\ -[\lambda + \mu + (n-1)\xi], & i = s+1, s+2, \ldots, S,\ n = 1, 2, \ldots, \end{cases}$$

$$\mathbf{B}_n = \begin{pmatrix} n\xi & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \mu & (n-1)\xi & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mu & (n-1)\xi & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (n-1)\xi & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots & \mu & (n-1)\xi & 0 \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & n\xi \end{pmatrix}, \tag{5}$$

$$\mathbf{C} = (p\lambda, \lambda, \ldots, \lambda, p\lambda)\mathbf{I},$$

where $\mathbf{I}$ is an identity matrix of the order $S+2$.

The process $\Phi(t)$ is a level dependent quasi-birth-and-birth process, the steady-state probability distribution is defined as follows:

$$\begin{aligned} x_n(i,0) &= \lim_{t\longrightarrow\infty} p\{X(t) = n, Y(t) = i, Z(t) = 0\}, \quad (n,i,0) \in \Omega, \\ x_n(i,1) &= \lim_{t\longrightarrow\infty} p\{X(t) = n, Y(t) = i, Z(t) = 1\}, \quad (n,i,1) \in \Omega. \end{aligned} \tag{6}$$

Let $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n, \ldots)$ be the steady-state probability vector of the process $\Phi(t)$, where

$$x_n = \left(x_n(0,0), x_n(1,1), \ldots, x_n(s,1), \ldots, x_n(S,1), x_n(S,0)\right), \quad n = 0, 1, \ldots. \tag{7}$$

Then, $\mathbf{x}$ satisfies the following set of equations:

$$\begin{cases} \mathbf{xQ} = 0, \\ \mathbf{xe} = 1. \end{cases} \tag{8}$$

The balance equations of the system is given by

$$\begin{aligned}
(p\lambda + \eta)x_0(0,0) &= \xi x_1(0,0) + \mu x_1(1,1), \quad n = 0, \\
(\lambda + \eta)x_0(i,1) &= \mu x_1(i+1,1), \quad i = 1, 2, \ldots, s, \ n = 0, \\
\lambda x_0(i,1) &= \mu x_1(i+1,1), \quad i = s+1, \ s+2, \ldots, S-1, \ n = 0, \\
(p\lambda + \theta)x_0(S,0) &= \eta x_0(0,0) + \xi x_1(S,0), \quad n = 0, \\
\lambda x_0(S,1) &= \theta x_0(S,0) + \eta \sum_{i=1}^{s} x_0(i,1), \quad n = 0, \\
(p\lambda + \theta)x_0(S,0) &= \eta x_0(0,0) + \xi x_1(S,0), \quad n = 0, \\
[\lambda + \eta + (n-1)\xi]x_n(i,1) &= n\xi x_{n+1}(i,1) + \mu x_{n+1}(i+1,1) + \lambda x_{n-1}(i,1), \quad i = 1, 2, \ldots s, \ n = 1, 2, \ldots, \\
[\lambda + (n-1)\xi]x_n(i,1) &= n\xi x_{n+1}(i,1) + \mu x_{n+1}(i+1,1) + \lambda x_{n-1}(i,1), \quad i = s+1, \ldots, S-1, \ n = 1, 2, \ldots, \\
[\lambda + (n-1)\xi]x_n(S,1) &= n\xi x_{n+1}(S,1) + \lambda x_{n-1}(S,1) + \theta x_n(S,0) + \eta \sum_{i=1}^{s} x_n(i,1), \quad n = 1, 2, \ldots, \\
(p\lambda + \theta + n\xi)x_n(S,0) &= (n+1)\xi x_{n+1}(S,0) + \lambda x_{n-1}(S,0) + \eta x_n(0,0), \quad n = 1, 2, \ldots.
\end{aligned} \tag{9}$$

The common method to solve the above balance equations is the probability generating function (PGF) method. However, these balance equations are so complex that they are difficult to obtain the explicit solution of the system by using PGF method to solve these equations. Alternatively, we solve these equations by using the matrix geometric solution (MGS) method. Comparing the PGF method, the advantage of MGS method is that we can obtain the steady-state probabilities of the system in matrix form, and then we can computer these steady-state probabilities in algorithmic approach.

Theorem 1. *The components of the steady-state probability vector* $\mathbf{x}$ *is given by*

$$\mathbf{x}_n = \mathbf{x}_0 \prod_{k=0}^{n-1} \mathbf{R}_k, \quad n = 0, 1, \ldots, \tag{10}$$

where $\mathbf{x}_0$ *satisfies the set of equations:*

$$\mathbf{x}_0\left(\mathbf{A}_0 + \mathbf{R}_0\mathbf{B}_1\right) = 0, \tag{11}$$

$$\mathbf{x}_0 \sum_{n=0}^{\infty} \left(\prod_{k=0}^{n-1} \mathbf{R}_k\right)\mathbf{e} = 1, \tag{12}$$

The matrix $\mathbf{R}_n$, $n \geq 0$ *is the minimal nonnegative solution of the following equations:*

$$\mathbf{C} + \mathbf{R}_{n-1}\mathbf{A}_n + \mathbf{R}_{n-1}\mathbf{R}_n\mathbf{B}_{n+1} = 0, \quad n = 1, 2, \ldots. \tag{13}$$

Proof of Theorem 1. From equation (8), we have

$$\mathbf{x}_0\mathbf{A}_0 + \mathbf{x}_1\mathbf{B}_1 = 0, \tag{14}$$

$$\mathbf{x}_{n-1}\mathbf{C} + \mathbf{x}_n\mathbf{A}_n + \mathbf{x}_{n+1}\mathbf{B}_{n+1} = 0, \quad n = 1, 2, \ldots. \tag{15}$$

Let

$$\mathbf{x}_n = \mathbf{x}_0 \prod_{k=0}^{n-1} \mathbf{R}_k, \quad n = 0, 1, \ldots. \tag{16}$$

where the matrix $\mathbf{R}_n$, $n \geq 0$ satisfies equation (13). Substituting equation (16) into the left side of equation (14), we have

$$\mathbf{x}_0\mathbf{A}_0 + \mathbf{x}_1\mathbf{B}_1 = \mathbf{x}_0\left(\mathbf{A}_0 + \mathbf{R}_0\mathbf{B}_1\right). \tag{17}$$

Equation (14) is verified by using equation (11).

Substituting equation (16) into the left side of equation (15), we have

$$\begin{aligned}&\mathbf{x}_{n-1}\mathbf{C}+\mathbf{x}_n\mathbf{A}_n+\mathbf{x}_{n+1}\mathbf{B}_{n+1}\\&=\mathbf{x}_0\prod_{k=0}^{n-2}\mathbf{R}_k\mathbf{C}+\mathbf{x}_0\prod_{k=0}^{n-1}\mathbf{R}_k\mathbf{A}_n+\mathbf{x}_0\prod_{k=0}^{n}\mathbf{R}_k\mathbf{B}_{n+1}\\&=\mathbf{x}_0\prod_{k=0}^{n-2}\mathbf{R}_k\left(\mathbf{C}+\mathbf{R}_{n-1}\mathbf{A}_n+\mathbf{R}_{n-1}\mathbf{R}_n\mathbf{B}_{n+1}\right).\end{aligned}\tag{18}$$

Thus, equation (15) is verified by using equation (13).

Applying the normalizing condition $\mathbf{xe}=1$, we obtained equation (12). Thus, we complete the Proof of Theorem 1.

It is difficult to obtain the exact expressions of the matrix $\mathbf{R}_n$, $n\geq 0$, from (13). However, based on Theorem 1, the steady-state probability $\mathbf{x}_n$ can be computed approximately by using the method of truncated approximation given by Bright and Taylor [27]. The approximate steady-state probability obtained is denoted by $\mathbf{x}_n^*$. For more details on the procedure of the approximate algorithm for computing the steady-state probability, we refer the reader to Bright and Taylor [27]. The specific steps of method of truncated approximation are as follows:

If the condition of the process $\Phi(t)$ is normal, the $\mathbf{R}_n$ is given by

$$\mathbf{R}_n=\sum_{l=0}^{\infty}\mathbf{U}_n^l\prod_{i=0}^{l-1}\mathbf{D}_{n+2^{l-i}}^{l-1-i},\quad n\geq 0.\tag{19}$$

where

$$\begin{aligned}\mathbf{U}_n^0&=\mathbf{C}\left(-\mathbf{A}_{n+1}\right)^{-1},\quad n\geq 1,\\\mathbf{D}_n^0&=\mathbf{B}_n\left(-\mathbf{A}_{n-1}\right)^{-1},\quad n\geq 1,\\\mathbf{U}_n^{l+1}&=\mathbf{U}_n^l\mathbf{U}_{n+2^l}^l\left[\mathbf{I}-\mathbf{U}_{n+2^{l+1}}^l\mathbf{D}_{n+3.2^l}^l-\mathbf{D}_{n+2^{l+1}}^l\mathbf{U}_{n+2^l}^l\right]^{-1},\\\mathbf{D}_n^{l+1}&=\mathbf{D}_n^l\mathbf{D}_{n-2^l}^l\left[\mathbf{I}-\mathbf{U}_{n-2^{l+1}}^l\mathbf{D}_{n-2^l}^l-\mathbf{D}_{n-2^{l+1}}^l\mathbf{U}_{n-3.2^l}^l\right]^{-1},\quad n=1,2,\ldots.\end{aligned}\tag{20}$$

Step 1: Solving N. First, let $N_{\text{old}}=0$, and then given N_{new} such that $N_{\text{new}}>N_{\text{old}}$.

Step 2: Determining the value of L. $\mathbf{R}_{N_{\text{new}}}(L)$ is the sum of the first $L+1$ terms in the sum of equation (19). We take $\mathbf{R}_{N_{\text{new}}-1}=\mathbf{R}_{N_{\text{new}}}(L)$ where L is chosen such that $(\mathbf{R}_{N_{\text{new}}}(L)-\mathbf{R}_{N_{\text{new}}}(L-1))_{\max}<\varepsilon$, where $(\mathbf{R}_{N_{\text{new}}}(L)-\mathbf{R}_{N_{\text{new}}}(L-1))_{\max}$ denotes the largest element in matrix $(\mathbf{R}_{N_{\text{new}}}(L)-\mathbf{R}_{N_{\text{new}}}(L-1))$.

Step 3: we can obtain $\mathbf{R}_{N_{\text{new}}-1}$ by equation (19). Solve $\mathbf{x}_0(\mathbf{A}_0+\mathbf{R}_0\mathbf{B}_1)=0$ subject to $\mathbf{x}_0\sum_{n=0}^{N_{\text{new}}}(\prod_{k=0}^{n-1}\mathbf{R}_k)\mathbf{e}=1$. Then, we obtain $\mathbf{x}_0$. Compute $\mathbf{x}_{N_{\text{new}}}=\mathbf{x}_0\prod_{n=0}^{N_{\text{new}}-1}\mathbf{R}_n$. Normalise $\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_{N_{\text{new}}}$, s.t. $\mathbf{x}_0\sum_{n=0}^{N_{\text{new}}}(\prod_{k=0}^{n-1}\mathbf{R}_k)\mathbf{e}=1$ until $\mathbf{x}_{N_{\text{new}}}e<\varepsilon$. Set $N=N_{\text{new}}$.

Step 4: solving $\mathbf{R}_n$, $n=0,1,\ldots,N$. From equation (13), we have

$$\mathbf{R}_{n-1}=-\mathbf{C}\left(\mathbf{A}_n+\mathbf{R}_n\mathbf{B}_{n+1}\right)^{-1},\quad n=0,1,\ldots.\tag{21}$$

Thus, we can obtain $\mathbf{R}_{N-2},\mathbf{R}_{N-3},\ldots,\mathbf{R}_0$ by equation (21).

3.2. Performance Measures. In this subsection, we derive the following performance measures of the system by using the approximate steady-state probabilities $\mathbf{x}_n^*$.

(1) The mean inventory level is given by

$$\begin{aligned}E_i&=\sum_{i=1}^{S}i\sum_{n=0}^{\infty}x_n^*(i,1)+S\sum_{n=0}^{\infty}x_n^*(S,0)\\&=\sum_{i=1}^{S}i\mathbf{e}_i\sum_{n=0}^{\infty}\mathbf{x}_n^*+S\sum_{n=0}^{\infty}\mathbf{x}_n^*\mathbf{e}_{S+1},\end{aligned}\tag{22}$$

where
$e_i=(0,\ldots,0,1,0,\ldots,0)^T$, $i=1,2,\ldots,S+1$.

(2) The mean number of replenishment per unit of time is given by

$$\begin{aligned}E_o&=\eta\sum_{i=1}^{s}\sum_{n=0}^{\infty}x_n^*(i,1)+\eta\sum_{n=0}^{\infty}x_n^*(0,0)\\&=\eta\sum_{i=1}^{s}\mathbf{e}_i\sum_{n=0}^{\infty}\mathbf{x}_n^*+\eta\sum_{n=0}^{\infty}\mathbf{x}_n^*\mathbf{e}_0,\end{aligned}\tag{23}$$

where $e_0=(1,0,\ldots,0,\ldots,0)^T$.

(3) The mean number of order size per unit of time is given by

$$\begin{aligned}E_q&=S\sum_{n=0}^{\infty}x_n^*(0,0)+\sum_{i=1}^{s}(S-i)\sum_{n=0}^{\infty}x_n^*(i,1)\\&=S\sum_{n=0}^{\infty}\mathbf{x}_n^*\mathbf{e}_0+\sum_{i=1}^{s}(S-i)\mathbf{e}_i\sum_{n=0}^{\infty}\mathbf{x}_n^*.\end{aligned}\tag{24}$$

(4) The probability that the server is off due to a vacation is given by

$$\begin{aligned}\overline{P}_v&=\sum_{n=0}^{\infty}\mathbf{x}_n^*(0,0)+\sum_{n=0}^{\infty}\mathbf{x}_n^*(S,0),\\&=\sum_{n=0}^{\infty}\mathbf{x}_n^*\mathbf{e}_0+\sum_{n=0}^{\infty}\mathbf{x}_n^*\mathbf{e}_{S+1}.\end{aligned}\tag{25}$$

(5) The mean number of lost customers due to customer impatient is given by

$$E_l^1 = \xi \overline{L}_w. \tag{26}$$

(6) The mean number of lost customers due to server's vacation is given by

$$E_l^2 = (1-p)\lambda \overline{p}_v. \tag{27}$$

(7) The mean total number of lost customers per unit of time is given by

$$E_l = E_l^1 + E_l^2. \tag{28}$$

(8) The mean arrival rate of customers who are admitted to the system per unit of time is given by

$$\lambda_A = \lambda - E_l. \tag{29}$$

(9) The mean number of customers in the system is given by

$$\begin{aligned} L &= \sum_{n=0}^{\infty} n x_n^* (0,0) + \sum_{n=0}^{\infty} n \sum_{i=1}^{S} x_n^* (i,1) + \sum_{n=0}^{\infty} n x_n^* (S,0), \\ &= \sum_{n=0}^{\infty} n \mathbf{x}_n^* \boldsymbol{\delta}_1 + \sum_{n=0}^{\infty} n \mathbf{x}_n^* \boldsymbol{\delta}_2, \end{aligned} \tag{30}$$

where $\delta_1 = (1,0,\ldots,0,1)^T$ and $\delta_2 = (0,1,\ldots,1,0)^T$.

(10) The mean number of customers in the queue is given by

$$\begin{aligned} L_w &= \sum_{n=1}^{\infty} n x_n^* (0,0) + \sum_{n=1}^{\infty} (n-1) \sum_{i=1}^{S} x_n^* (i,1) + \sum_{n=1}^{\infty} n x_n^* (S,0), \\ &= \sum_{n=1}^{\infty} n \mathbf{x}_n^* \boldsymbol{\delta}_1 + \sum_{n=1}^{\infty} (n-1) \mathbf{x}_n^* \boldsymbol{\delta}_2. \end{aligned} \tag{31}$$

(11) The mean sojourn time is given by

$$W = \frac{\overline{L}}{\lambda_A}. \tag{32}$$

(12) The mean waiting time is given by

$$W_q = \frac{\overline{L}_w}{\lambda_A}. \tag{33}$$

4. Numerical Analysis

In this section, we conduct a numerical investigation to explore how the probability p and the impatient rate ξ affect some performance measures. Besides, we develop a cost model and compute the optimal policy and the optimal cost by using a genetic algorithm. The effects of system parameters on the optimal value and the optimal cost are also investigated.

4.1. Effects of the Mixed Probability p and the Impatient Rate ξ. We conduct a sensitivity analysis to show the effect of the two system parameters p and ξ on some performance measures. We plot the curves for E_i, E_o, E_q, E_l^1, E_l^2 and W_q by varying the probability p and the impatient rate ξ, respectively. The results are presented by Figures 2–6. The system parameters are fixed as $\lambda = 3$, $\eta = 5$, $\theta = 0.2$, $\mu = 30$, $s = 3$ and $S = 10$.

The main observations can be summarized as follows:

(i) Figure 2 shows that the mean inventory level E_i first decreases slightly with an increase in the probability p, and attains a minimum, and then increases significantly. This means there exists a mixed sales policy that minimizes the mean inventory level. Also, we observe that when p is small, e.g. $p < 0.2$, the effect of ξ on E_i is much less than the case of large p. Besides, Figure 2 shows that the effect of ξ on E_i for the case of small value $\xi = 0.1$ is much higher than the cases of large values $\xi = 0.3, 0.5$.This shows the sensitivity of ξ on E_i for the case of small value of p is much less than the case of large value of p, and that the sensitivity of E_i to p is much higher for the case of small value of ξ than for the case of large value of ξ.

(ii) From Figure 3, we find that the mean number of order size E_q increases with p and decreases with ξ. This can be explained as follows: The probability of customers willing to wait when the inventory level is zero increases with the probability p , and the mean impatient time of customers increases with increasing of ξ. This leads to the demands for inventory items which results in the increasing the mean number of order size E_q. So, $p = 0$,i.e., the lost sales policy is best policy among the mixed sales policy from the point of minimizing the mean number of order size E_q.

(iii) It is observed from Figures 4 and 5 tha E_l^1 decreases with either the probability p or the impatient rate ξ. However, E_l^2 increases with either the probability p or the impatient rate ξ. This agrees with our expectation. Since the probability of customers' willing to wait increases with p, and the mean impatient time increases with ξ. Thus, the probability that the server go for vacations decreases with p and ξ. This explains why E_l^1 increase with p and ξ, and why E_l^2 decreases with p and ξ. This observation shows that the lost sales policy $(p = 0)$ minimizes E_l^1, and backordering policy $(p = 1)$ minimizes E_l^2.

(iv) From Figure 6, we observe that the mean waiting time of customer increases with p and decreases with ξ. The reason for this observation is the same as we explained for our observation from Figure 3.

In summary, we have the following conclusion from the observations above:

(i) There exists a mixed sales policy that minimizes the mean inventory level.

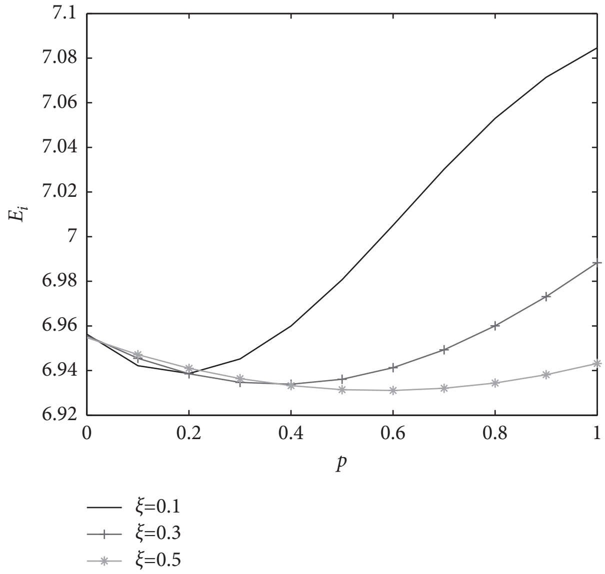

Figure 2: Effect of p on mean inventory level E_i for various impatient rate ξ.

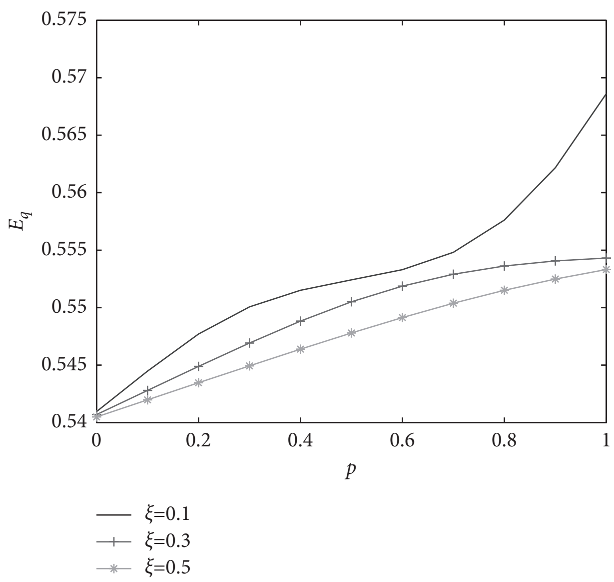

Figure 3: Effect of p on mean number of order size E_q for various impatient rate ξ.

(ii) The sensitivity of E_i on ξ for the case of small value of p is much less than the case of large value of p, and the sensitivity of E_i on p is much higher for the case of small value of ξ than for the case of large value of ξ.

(iii) The lost sales policy is best policy among the mixed sales policy from the point of minimizing the mean number of order size E_q.

(iv) The lost sales policy minimizes E_l^1, and back-ordering policy minimizes E_l^2.

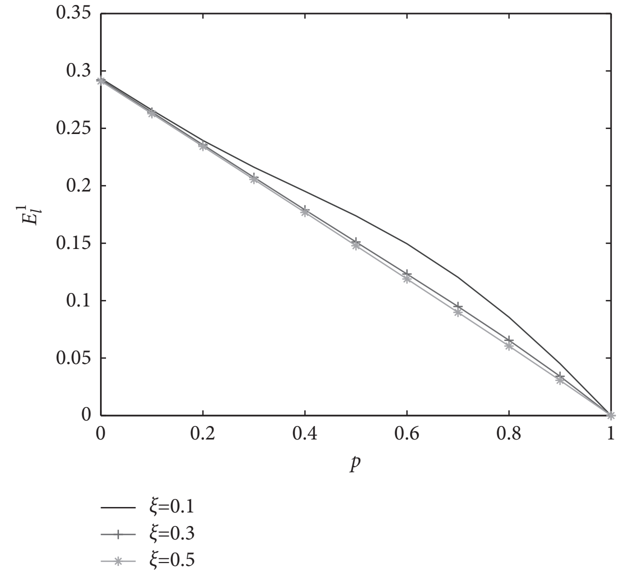

Figure 4: Effect of p on mean number of lost customers due to customer impatience E_l^1 for various impatient rate ξ.

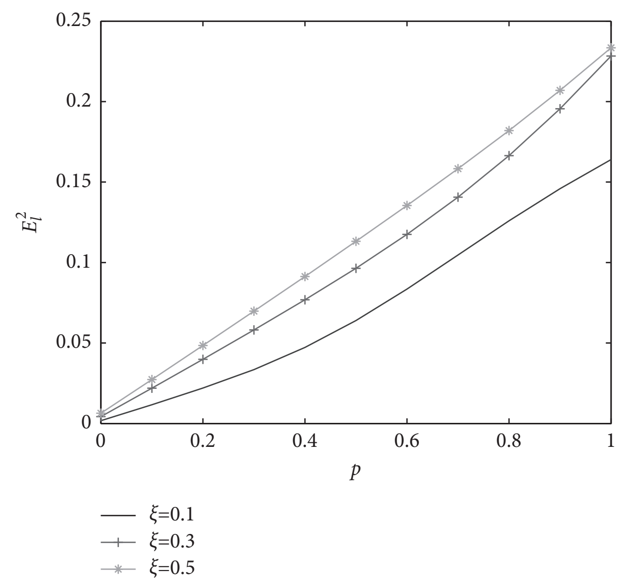

Figure 5: Effect of p on mean number of lost customers due to server's vacation E_l^2 for various impatient rate ξ.

4.2. Optimization. In this section, we develop the following expected cost function by performance measures given in Subsection 3.2. The expected cost function per unit time can be given by

$$F(s,S) = C_1E_i + C_2E_o + C_3E_qE_o + C_4E_l + C_5W_q, \quad (34)$$

where C_1 is a holding cost of inventory per unit time, C_2 is a fixed cost for placing an order, C_3 is a cost of per unit item for each order per unit time, C_4 is a cost incurred due to lost sales and impatience of customers per unit time, C_5 is a waiting cost per unit of time for each waiting customer in queue.

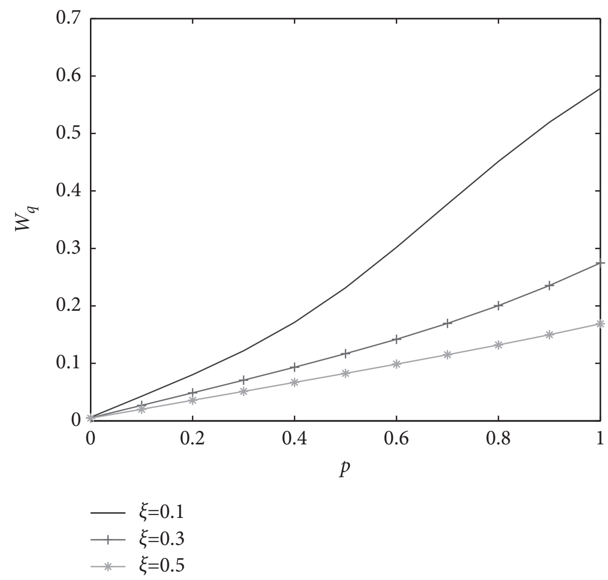

Figure 6: Effect of p on mean waiting time W_q for various impatient rate ξ.

Table 1: The effect of the parameter λ on the optimal policy and the optimal cost.

λ	5	6	7	8	9	10	11	12
(s^*, S^*)	(3 34)	(3 37)	(4 41)	(4 45)	(5 51)	(5 51)	(6 55)	(6 57)
$F(s^*, S^*)$	182.33	201.82	220.32	237.77	255.00	270.75	286.55	301.75

Table 2: The effect of the parameter η on the optimal policy and the optimal cost.

η	1	2	3	4	5	6	7	8
(s^*, S^*)	(5 77)	(9 72)	(8 70)	(7 66)	(7 65)	(6 64)	(6 63)	(6 63)
$F(s^*, S^*)$	404.15	377.10	357.93	345.60	337.01	330.84	325.97	322.35

Table 3: The effect of the parameter θ on the optimal policy and the optimal cost.

θ	0.1	0.3	0.5	0.7	0.9	1.1	1.3	1.5
(s^*, S^*)	(9 70)	(7 66)	(6 65)	(5 63)	(4 63)	(4 62)	(3 62)	(3 62)
$F(s^*, S^*)$	352.86	345.60	339.07	333.93	329.84	326.57	323.69	321.48

Table 4: The effect of the parameter μ on the optimal policy and the optimal cost.

μ	20	25	30	35	40	45	50	55
(s^*, S^*)	(8 66)	(8 66)	(7 66)	(7 66)	(7 66)	(7 66)	(7 66)	(7 66)
$F(s^*, S^*)$	348.46	346.70	345.60	344.81	344.25	343.84	343.53	343.29

Table 5: The effect of the parameter ξ on the optimal policy and the optimal cost.

ξ	1	5	9	13	17	21	25	29
(s^*, S^*)	(9 64)	(9 62)	(8 60)	(8 59)	(8 58)	(8 58)	(7 56)	(7 56)
$F(s^*, S^*)$	357.83	362.34	364.37	365.75	366.82	367.69	368.27	368.74

Table 6: The effect of the parameter p on the optimal policy and the optimal cost.

p	0	0.2	0.4	0.6	0.8	1.0
(s^*, S^*)	(10 66)	(9 65)	(8 66)	(7 66)	(6 67)	(5 67)
$F(s^*, S^*)$	357.92	354.85	348.33	342.81	337.48	332.05

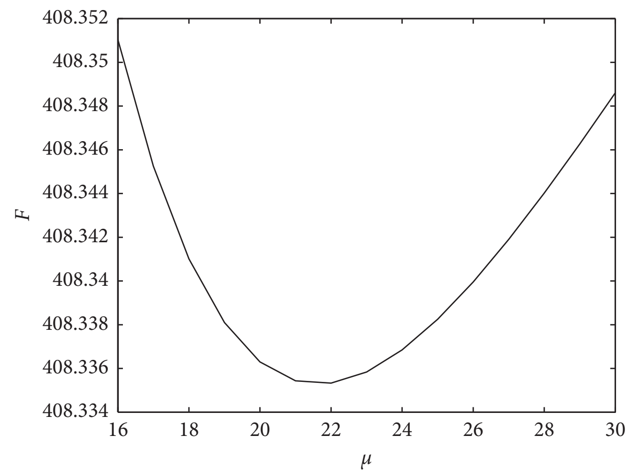

Figure 7: The effect of service rate μ on the cost function F.

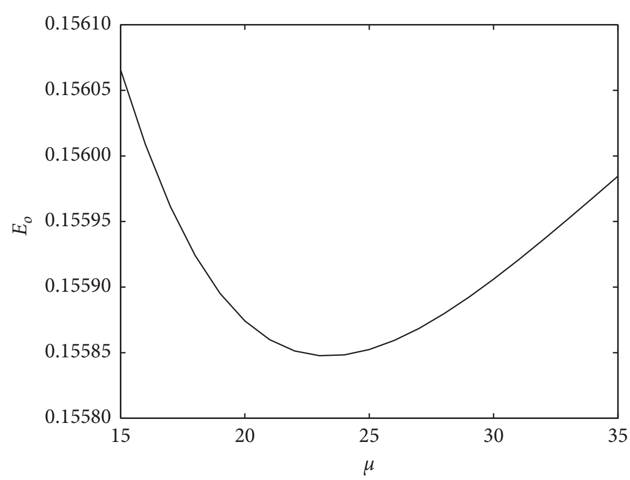

Figure 8: The effect of service rate μ on the mean number of replenishment E_o.

Unfortunately, we failed to obtain the explicit expressions for the expected cost function, and it is difficult to show the convexity of this cost function. Hence, we compute the optimal policy and the optimal cost by using the genetic algorithm (see [28]). The results are given in Tables 1–6. We assume the system parameters as $\lambda = 15$, $\mu = 30$, $\eta = 4$, $\theta = 0.3$, $\xi = 0.1$, $p = 0.5$, unless their values are mentioned in the respective tables. The cost parameters are set as $C_1 = 5$, $C_2 = 500$, $C_3 = 30$, $C_4 = 20$, $C_5 = 10$.

The main observations can be summarized as follows:

(i) Both the optimal policy (s^*, S^*) and the optimal cost $F(s^*, S^*)$ increase with an increase in the arrival rate λ.

(ii) The optimal value S^* and the optimal cost $F(s^*, S^*)$ decrease with an increase in the replenishment rate η. The optimal value s^* firstly increases and then decreases with the replenishment rate η.

(iii) The optimal policy (s^*, S^*) and the optimal cost $F(s^*, S^*)$ decrease with an increase in the vacation rate θ.

(iv) The optimal value s^* decreases slightly with an increase in the service rate μ, but the optimal value S^* is independent of the service rate μ. We also observe that the optimal cost $F(s^*, S^*)$ decreases with the service rate μ.

(v) The optimal policy (s^*, S^*) decreases with an increase in the impatient rate ξ, and the optimal cost $F(s^*, S^*)$ increases with the impatient rate ξ.

(vi) The optimal value s^* and the optimal cost $F(s^*, S^*)$ decrease with an increase in the probability p, and the optimal value S^* increases slightly with the probability p when p varies from 0.2 to 1.0.

Next, we explore the concavity and convexity of the cost function F and the mean number of replenishment E_o about the service rate. The curves are given by Figures 7 and 8. Figure 7 corresponds to the case of the varying parameter μ and fixed parameters $\lambda = 15$, $\eta = 0.5$, $\xi = 4$, $p = 0.5$, $\theta = 0.2$, $s = 5$, $S = 15$, $C_1 = 5$, $C_2 = 500$, $C_3 = 30$, $C_4 = 20$, $C_5 = 10$. Figure 8 corresponds to the case of the varying parameter μ and fixed parameters $\lambda = 20$, $\eta = 0.6$, $\xi = 4$, $p = 0.5$, $\theta = 0.2$, $s = 5$, $S = 15$.

From Figures 7 and 8, we observe that the cost function F and the mean number of replenishment E_o exhibit convexity under the fixed parameters.

5. Conclusions

In this paper, we considered a QIS model with server's multiple vacations and impatient customers, in which the new customer arrived waits for service with a probability when the server is off due to a vacation. We obtained the matrix geometric solution of the steady-state probability using the method of truncated approximation, and then derived some performance measures. We investigated the effect of the probability and the impatient rate on some performance measures using numerical analysis. The optimal policy and the optimal cost were computed using the genetic algorithm. The optimal service rate is derived. Our model can be further extended in the following directions: (i) studying non-Markovian QIS and (ii) investigating QIS with multiservers. Those directions become more challenging than the current one due to their analytical complexity.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare no conflicts of interest.

Acknowledgments

This research was supported by the National Natural Science Foundation of China, grant no. 71971189, and the Key Project of Scientific Research in Higher Education of Hebei Province, China, grant no. ZD2018042.

References

[1] N. Zhao and Z. Lian, "A queueing-inventory system with two classes of customers," *International Journal of Production Economics*, vol. 129, no. 1, pp. 225–231, 2011.

[2] M. Schwarz, C. Sauer, H. Daduna, R. Kulik, and R. Szekli, "M/M/1 Queueing systems with inventory," *Queueing Systems*, vol. 54, no. 1, pp. 55–78, 2006.

[3] A. Krishnamoorthy, D. Shajin, and B. Lakshmy, "On a queueing-inventory with reservation, cancellation, common life time and retrial," *Annals of Operations Research*, vol. 247, no. 1, pp. 365–389, 2016.

[4] C. P. Arun, "Queueing and inventory theory in clinical practice: application to clinical toxicology," *Annals of the New York Academy of Sciences*, vol. 919, no. 1, pp. 284–287, 2010.

[5] K. Sigman and D. Simchi-Levi, "Light traffic heuristic for an M/G/1 queue with limited inventory," *Annals of Operations Research*, vol. 40, no. 1, pp. 371–380, 1992.

[6] A. Z. Melikov and A. A. Molchanov, "Stock optimization in transportation/storage systems," *Cybernetics and Systems Analysis*, vol. 28, no. 3, pp. 484–487, 1992.

[7] M. Schwarz and H. Daduna, "Queueing systems with inventory management with random lead times and with backordering," *Mathematical Methods of Operations Research*, vol. 64, no. 3, pp. 383–414, 2006.

[8] D. Yue, G. Zhao, and Y. Qin, "An M/M/1 queueing-inventory system with geometric batch demands and lost sales," *Journal of Systems Science and Complexity*, vol. 31, no. 4, pp. 1024–1041, 2018.

[9] K. Lakshmanan, S. Padmasekaran, and K. Jeganathan, "Mathematical analysis of queueing-inventory model with compliment and multiple working vacations," *International Journal of Engineering and Advanced Technology*, vol. 8, no. 6, pp. 4239-4240, 2019.

[10] R. Manikandan and S. S. Nair, "An M/M/1 queueing-inventory system with working vacations, vacation interruptions and lost sales," *Automation and Remote Control*, vol. 81, no. 4, pp. 746–759, 2020.

[11] Y. Zhang, D. Yue, and W. Yue, "A queueing-inventory system with random order size policy and server vacations," *Annals of Operations Research*, vol. 310, no. 2, pp. 595–620, 2020.

[12] A. Krishnamoorthy, B. Lakshmy, and R. Manikandan, "A survey on inventory models with positive service time," *Opsearch*, vol. 48, no. 2, pp. 153–169, 2011.

[13] A. Krishnamoorthy, D. Shajin, and C. N. Viswanath, "Inventory with positive service time: a survey," in *Advanced Trends in Queueing Theory*, V. Anisimov and N. Limnios, Eds., ISTE & Wiley, London, UK, 2019.

[14] E. Altman and U. Yechiali, "Analysis of customers' impatience in queues with server vacations," *Queueing Systems*, vol. 52, no. 4, pp. 261–279, 2006.

[15] E. Altman and U. Yechiali, "Infinite-server queues with system's additional tasks and impatient customers," *Probability in the Engineering and Informational Sciences*, vol. 22, no. 4, pp. 477–493, 2008.

[16] A. Bouchentouf, M. Boualem, M. Cherfaoui, and L. Medjahri, "Variant vacation queueing system with Bernoulli feedback, balking and server's states-dependent reneging," *Yugoslav Journal of Operations Research*, vol. 31, no. 4, pp. 557–575, 2021.

[17] A. A. Bouchentouf, M. Cherfaoui, and M. Boualem, "Analysis and performance evaluation of Markovian feedback multi-server queueing model with vacation and impatience," *American Journal of Mathematical and Management Sciences*, vol. 40, no. 3, pp. 261–282, 2021.

[18] A. A. Bouchentouf and A. Guendouzi, "Single server batch Arrival Bernoulli feedback queueing system with waiting server, K-variant vacations and impatient customers," *Operations Research Forum*, vol. 2, no. 1, 2021.

[19] M. Kadi, A. A. Bouchentouf, and L. Yahiaoui, "On a multi-server queueing system with customers' impatience until the end of service under single and multiple vacation policies," *Applications and Applied Mathematics: International Journal*, vol. 15, no. 2, pp. 740–763, 2020.

[20] R. Padmavathy, K. Kalidass, and K. Ramanath, "Vacation queues with impatient customers and a waiting server," *International Journal of Latest Trends in Software Engineering*, vol. 1, no. 1, pp. 10–19, 2011.

[21] D. Yue, Y. Zhang, and W. Yue, "Optimal performance analysis of an M/M/1 queue system with balking, reneging and server vacation," *International Journal of Pure and Applied Mathematics*, vol. 28, no. 1, pp. 101–115, 2006.

[22] D. Yue, "Analysis of an M/M/1 queueing system with impatient customers and a variant of multiple vacation policy," *Journal of Industrial and Management Optimization*, vol. 10, no. 1, pp. 89–112, 2014.

[23] D. Yue, W. Yue, and G. Zhao, "Analysis of an M/M/1 queue with vacations and impatience timers which depend on the server's states," *Journal of Industrial and Management Optimization*, vol. 12, no. 2, pp. 653–666, 2016.

[24] C. N. Viswanath, T. G. Deepak, A. Krishnamoorthy, and B. Krishnakumar, "On (s,S) inventory policy with service time, vacation to server and correlated lead time," *Quality Technology and Quantitative Management*, vol. 5, no. 2, pp. 129–144, 2008.

[25] A. Z. Melikov, A. M. Rustamov, and L. A. Ponomarenko, "Approximate analysis of a queueing-inventory system with early and delayed server vacations," *Automation and Remote Control*, vol. 78, no. 11, pp. 1991–2003, 2017.

[26] V. S. Koroliuk, A. Z. Melikov, L. A. Ponomarenko, and A. M. Rustamov, "Asymptotic analysis of the system with server vacation and perishable inventory," *Cybernetics and Systems Analysis*, vol. 53, no. 4, pp. 543–553, 2017.

[27] L. Bright and P. G. Taylor, "Calculating the equilibrium distribution in level dependent quasi-birth-and-death processes," *Communications in Statistics - Stochastic Models*, vol. 11, no. 3, pp. 497–525, 1995.

[28] J. R. Feng, Z. H. Liu, and Z. H. Liu, "A mixed integer genetic algorithms for solving the mixed integer programming problems and simulation implementing," *Journal of System Simulation*, vol. 16, no. 4, pp. 845–848, 2004.