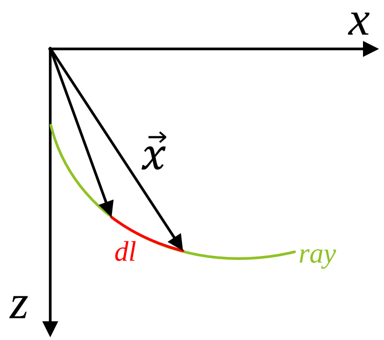
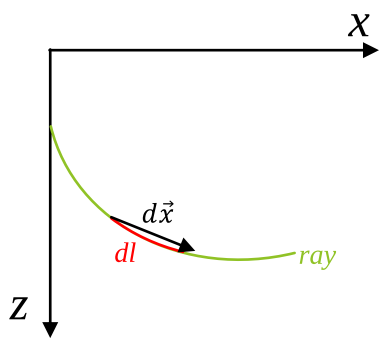
Ray tracing equations





$\vec{x}$ is position vector along the ray path

dl is a small segment of ray

Elapsed time in segment with slowness $s(\vec{x})$ is dT

$$\boxed{\frac{dT}{dl} = s(\vec{x})} \quad \text{Eq. 1}$$

$\vec{p}$ is slowness vector that shows ray direction: $\vec{p} = s(\vec{x})\hat{k}$

$$\boxed{\frac{d\vec{x}}{dl} = \frac{\vec{p}}{s(\vec{x})}} \quad \text{Eq. 2}$$

Ray tracing equations

$$\frac{d\vec{p}}{dl} = ?$$

Eikonal Eq. $\boxed{\nabla T = \vec{p}}$ Eq. 3

$$\Rightarrow \frac{d\vec{p}}{dl} = \frac{d}{dl}(\nabla T) = \frac{d}{dl}\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right)$$

$$\frac{d}{dl}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial^2 T}{\partial x^2}\frac{dx}{dl} + \frac{\partial^2 T}{\partial x \partial y}\frac{dy}{dl} + \frac{\partial^2 T}{\partial x \partial z}\frac{dz}{dl}$$ chain rule

Eq. 2 & Eq. 3 $\Rightarrow$

$$= \frac{1}{s}\left(\frac{\partial^2 T}{\partial x^2}\frac{\partial T}{\partial x} + \frac{\partial^2 T}{\partial x \partial y}\frac{\partial T}{\partial y} + \frac{\partial^2 T}{\partial x \partial z}\frac{\partial T}{\partial z}\right)$$

$$= \frac{1}{2s}\frac{1}{\partial x}\left(\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2\right)$$

$$= \frac{1}{2s}\frac{1}{\partial x}(s^2) = \frac{\partial s}{\partial x}$$

$$\Rightarrow \boxed{\frac{d\vec{p}}{dl} = \nabla s(\vec{x})}$$ Eq. 4

Ray tracing equations

Raytracing can be accomplished by solving the following ODE system:

$$\begin{cases} \dfrac{d\vec{x}}{dl} = \dfrac{\vec{p}}{s(\vec{x})} \\[2em] \dfrac{d\vec{p}}{dl} = \nabla s(\vec{x}) \\[2em] \dfrac{dT}{dl} = s(\vec{x}) \end{cases}$$

Reference:

Introduction to Seismology, P. M. Shearer, Cambridge University Press, 2009.