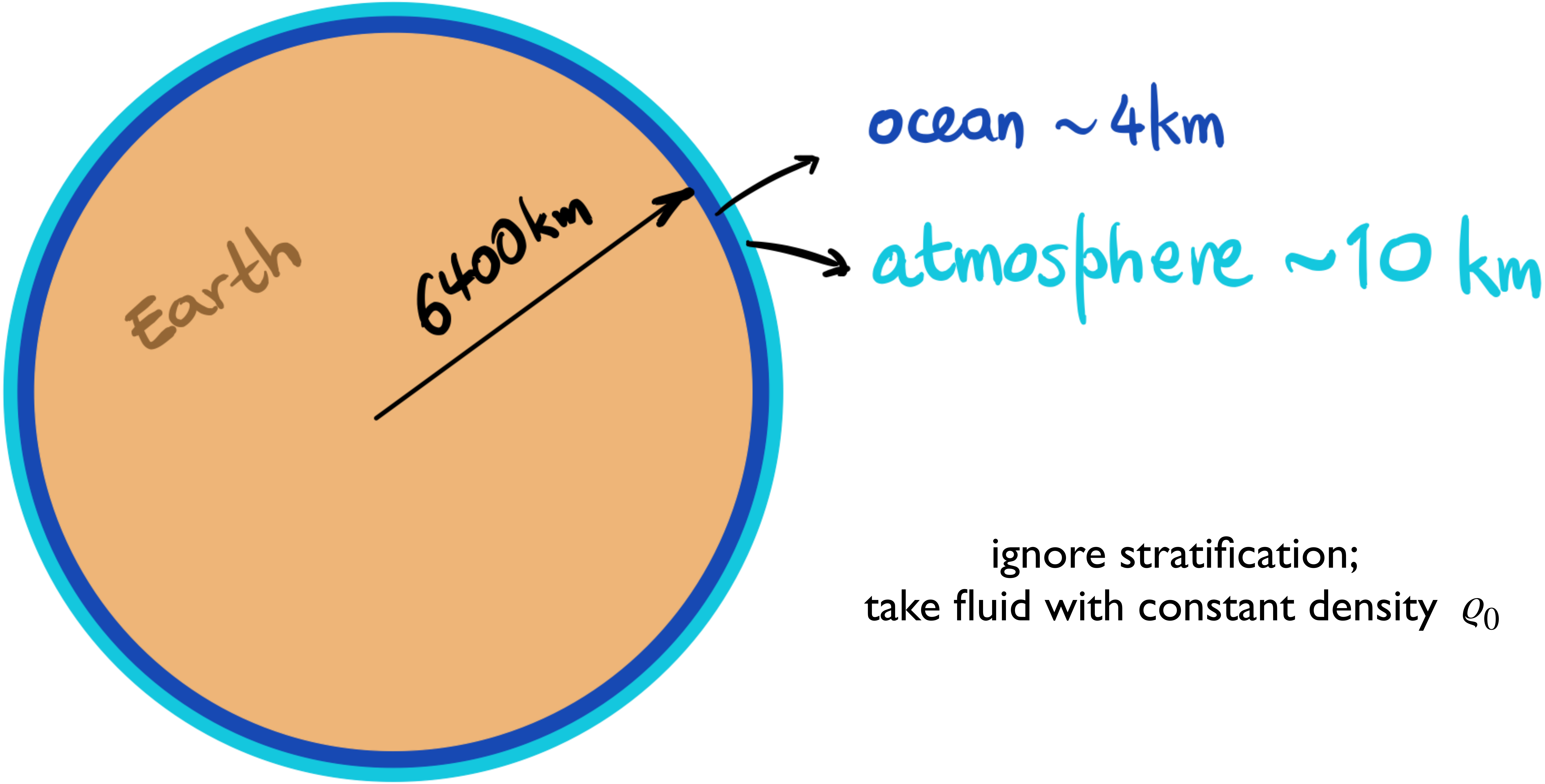
A taste of Quasi-Geostrophy

CLEX Winter School 2020
Atmosphere & Ocean Dynamics

(Teaser version via zoom)

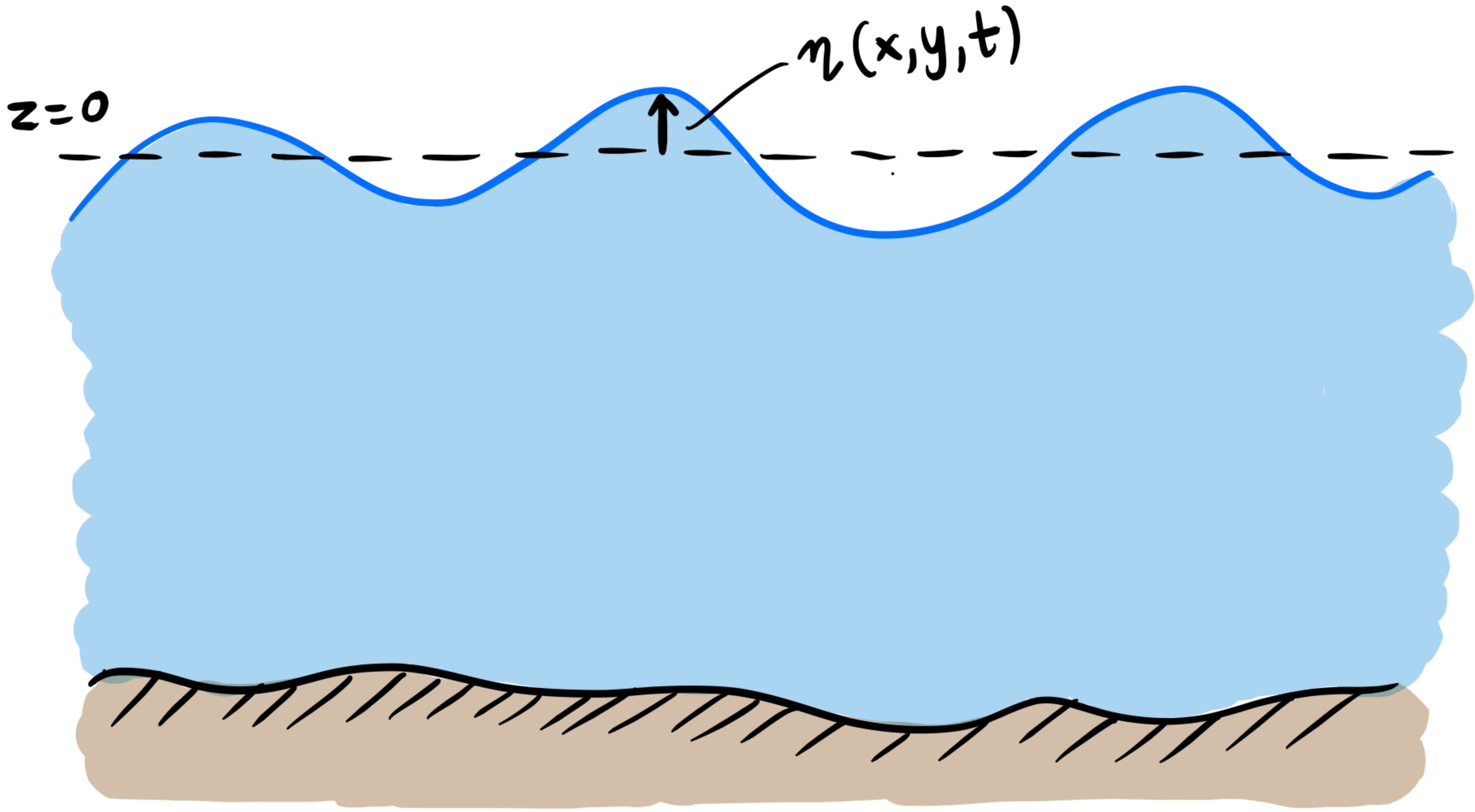
Navid Constantinou
ANU

Shallow-water dynamics



ignore stratification;
take fluid with constant density ϱ_0

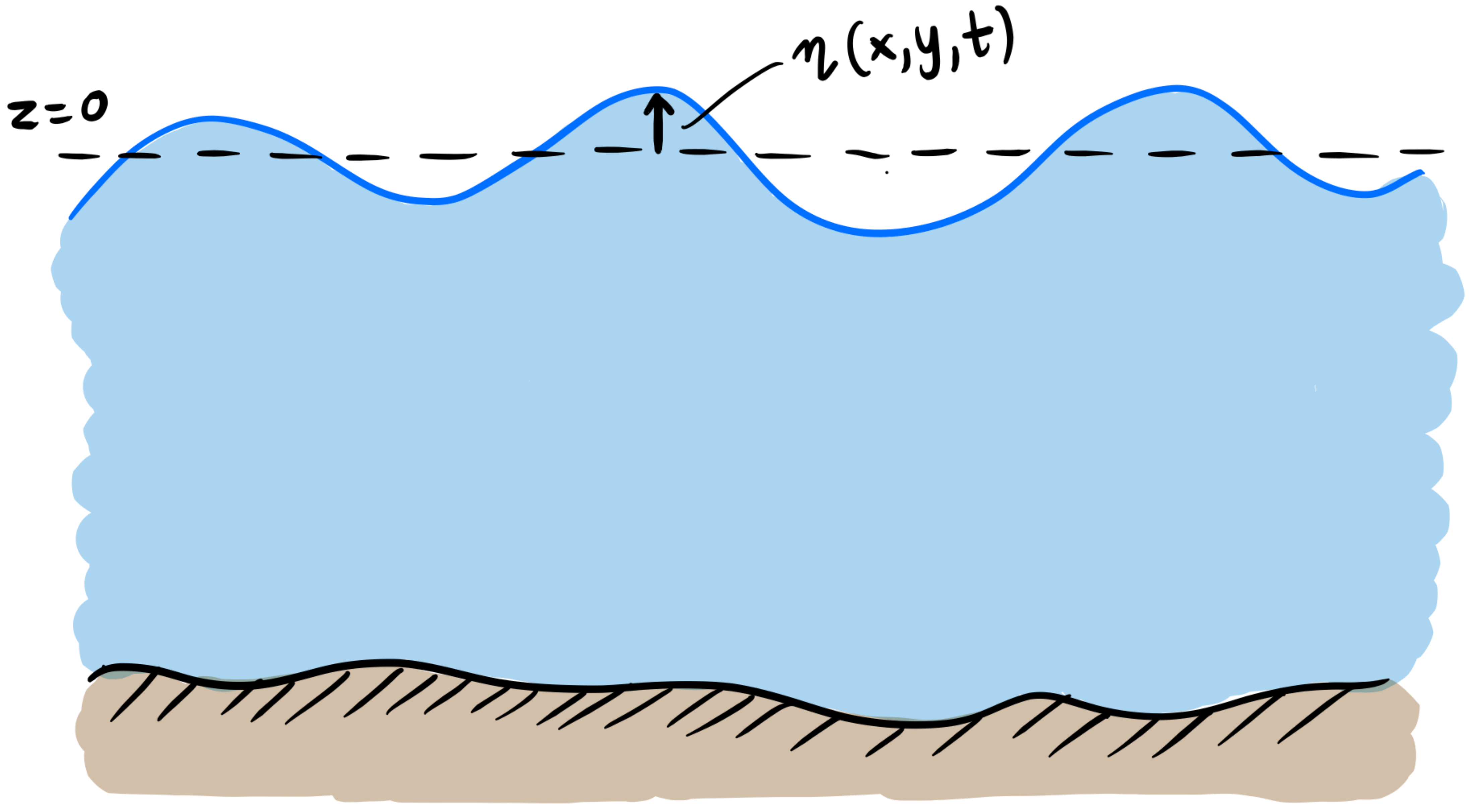
Shallow-water dynamics



vertical momentum equation

$$\underbrace{\frac{\partial w}{\partial t}}_{\text{very small}} + \underbrace{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}}_{\text{very small}} = \underbrace{-\frac{1}{\varrho_0}\frac{\partial p}{\partial z}}_{\text{LARGE}} \underbrace{-\, g}_{\text{LARGE}}$$

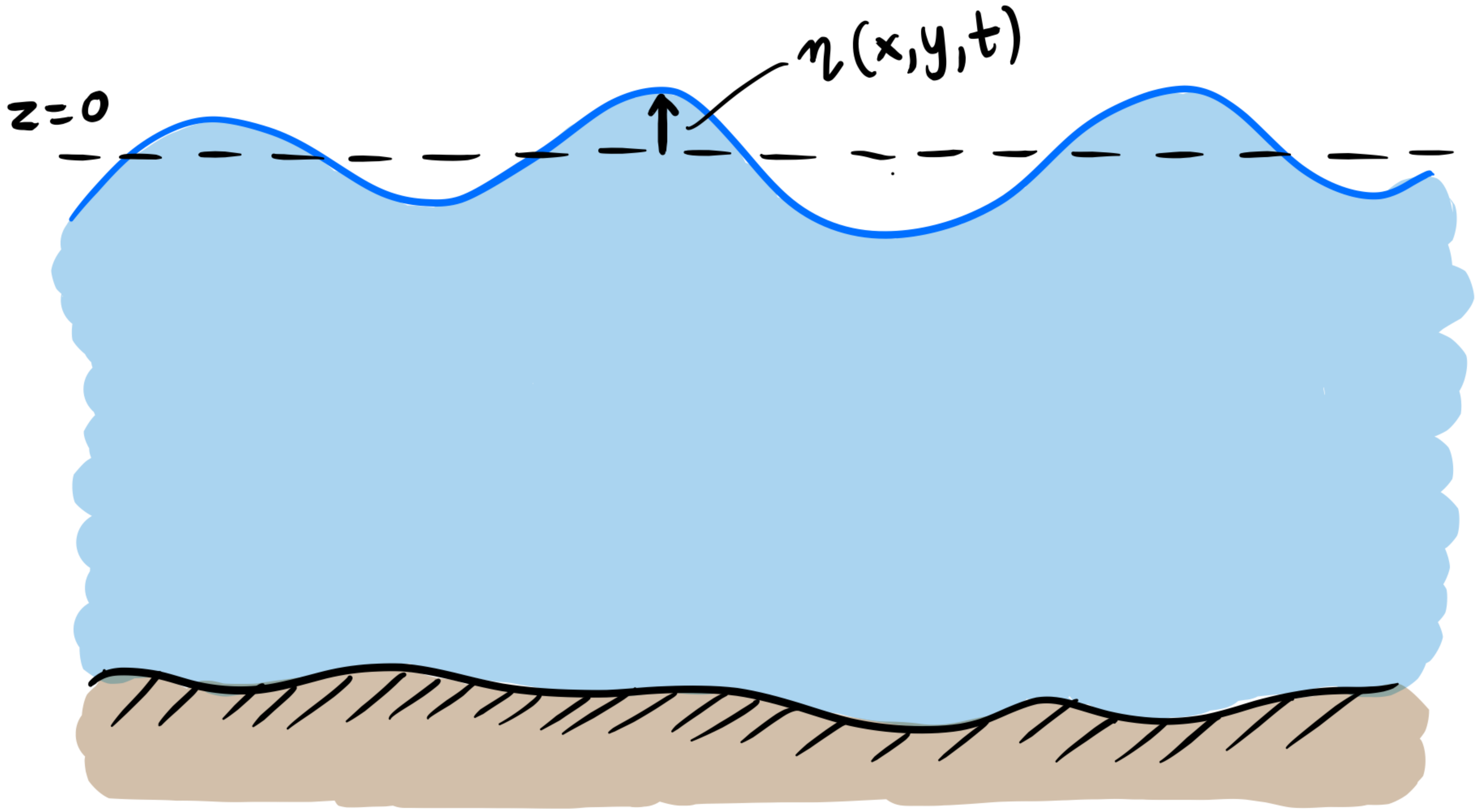
Shallow-water dynamics



vertical momentum equation

$$\underbrace{\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}}_{\text{very small}} = \underbrace{-\frac{1}{\varrho_0}\frac{\partial p}{\partial z}}_{\text{LARGE}} \underbrace{- g}_{\text{LARGE}}$$

Shallow-water dynamics



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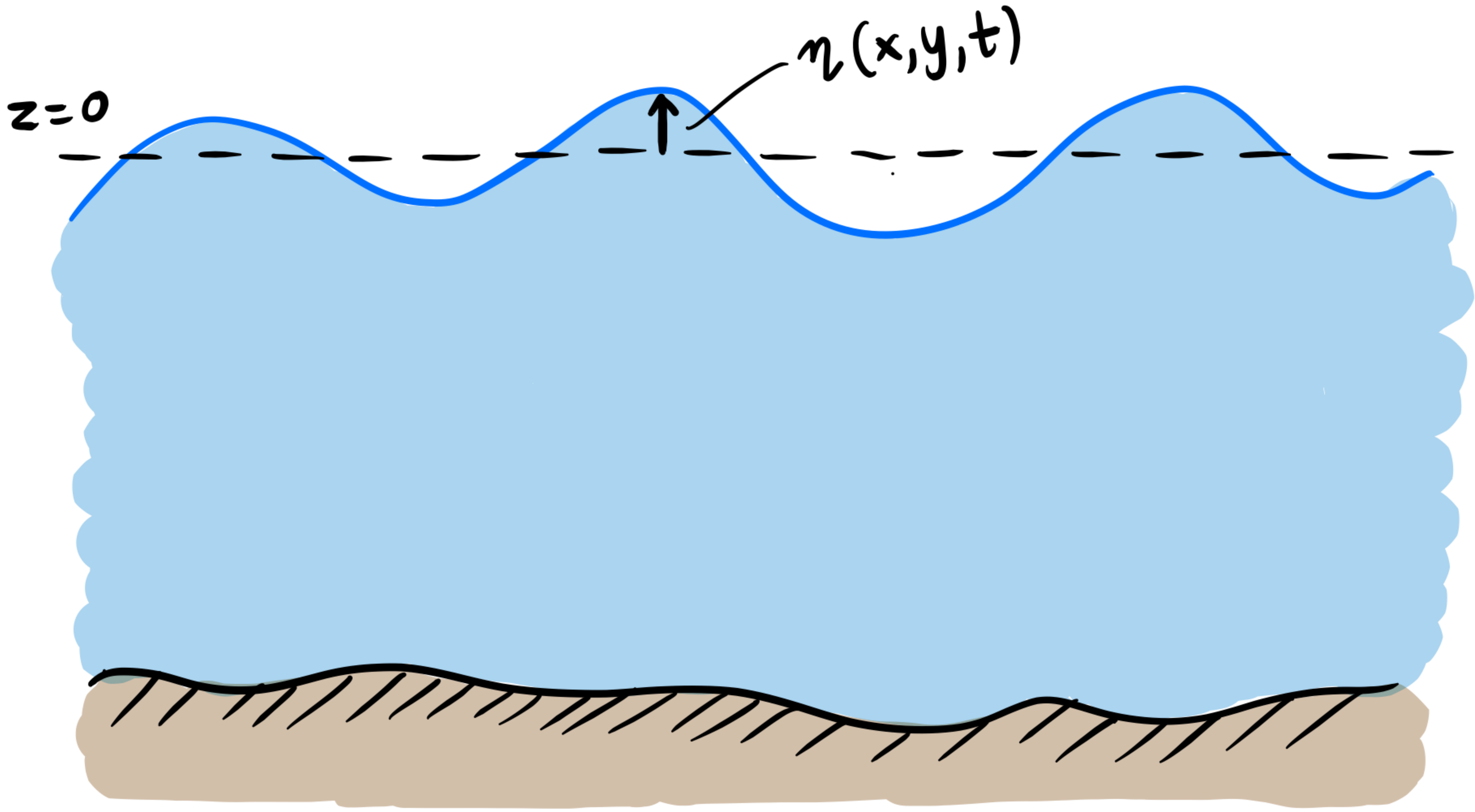
hydrostatic balance

$$\frac{\partial p}{\partial z} = -\varrho_0 g$$

Shallow-water dynamics



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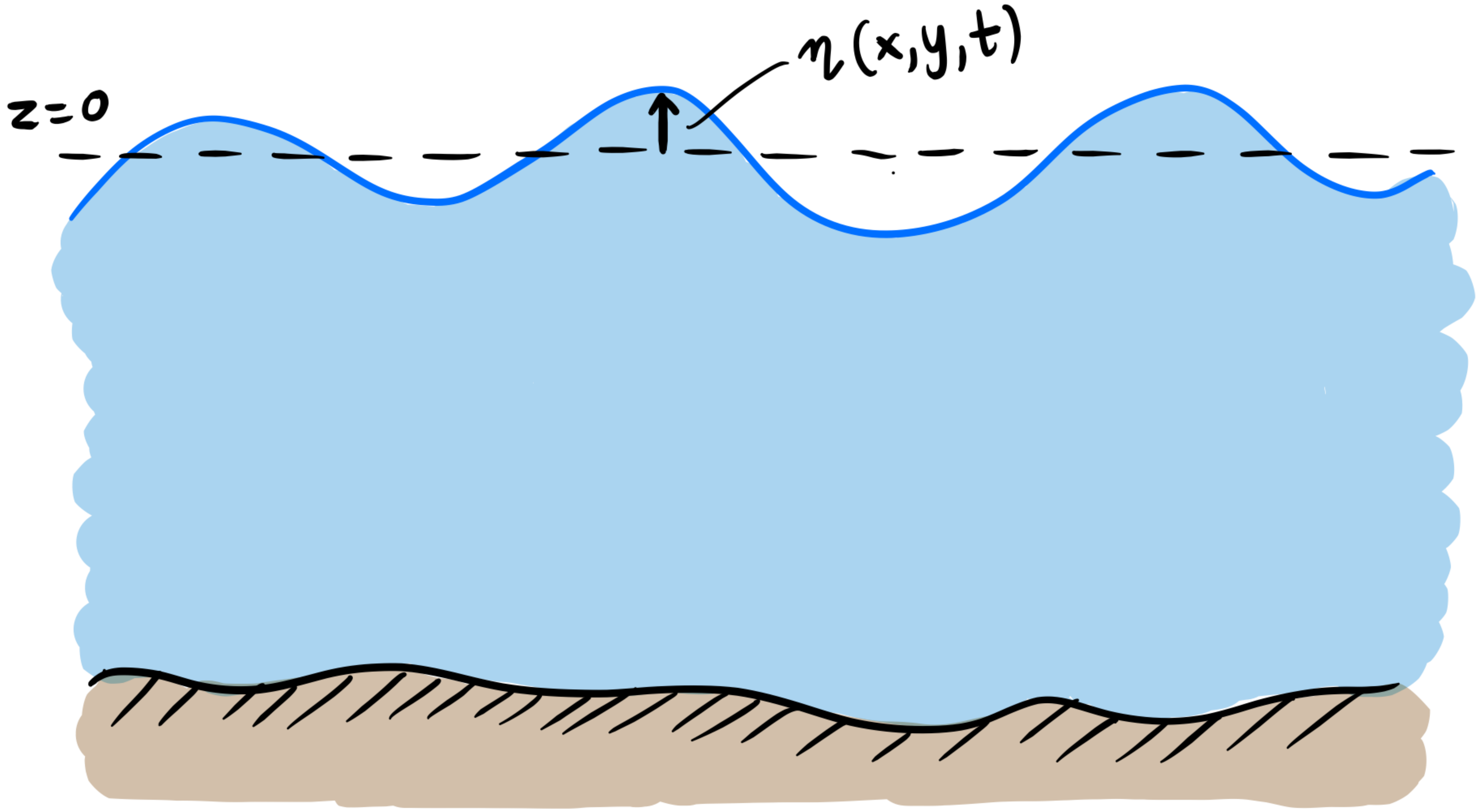
hydrostatic balance

$$\frac{\partial p}{\partial z} = -\varrho_0 g$$

$$p(x, y, z = \eta, t) = 1 \text{ atm}$$

$$\Longrightarrow p(x, y, z, t) = \varrho_0 g(\eta(x, y, t) - z) + 1 \text{ atm}$$

Shallow-water dynamics



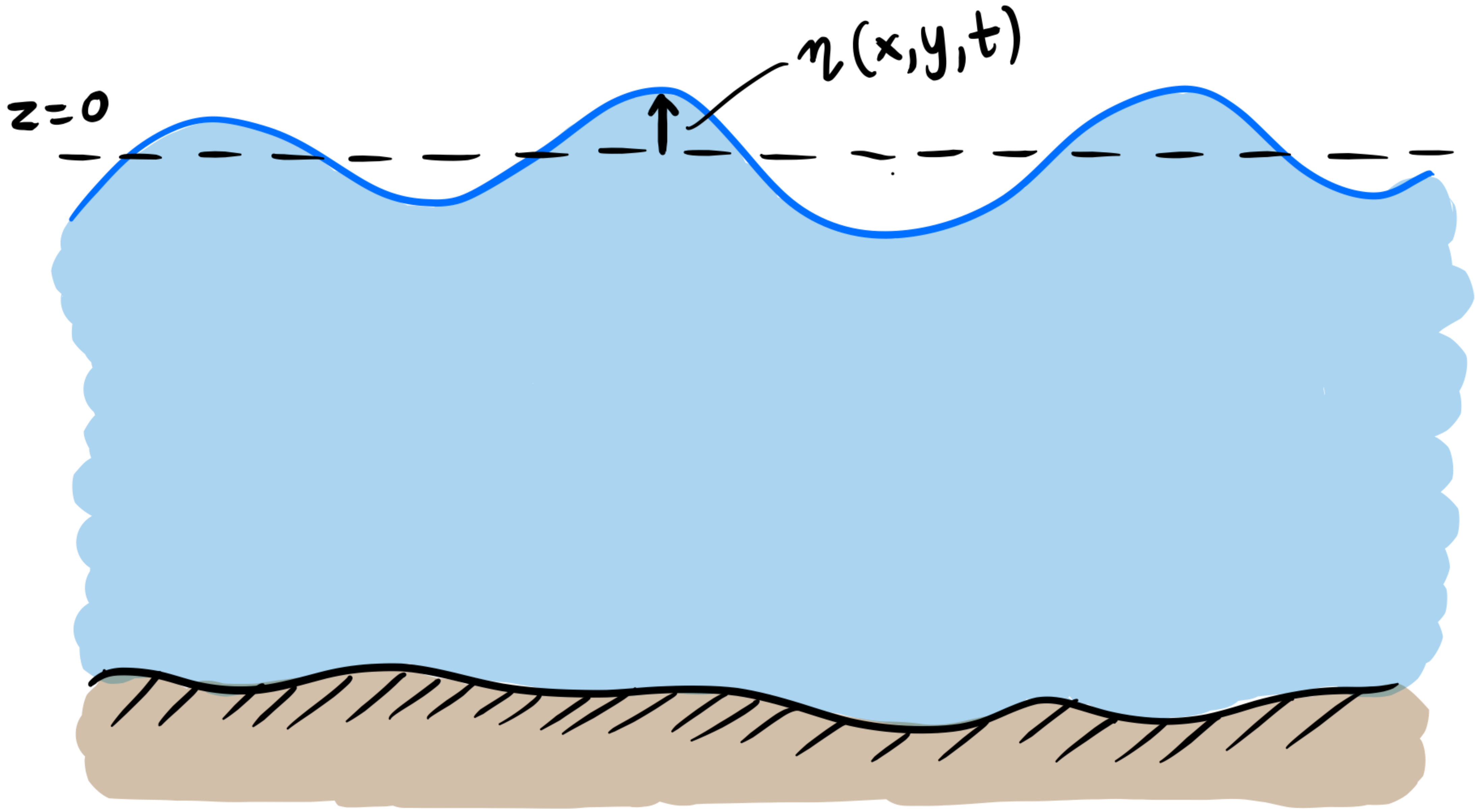
horizontal momentum equations

$$\underbrace{\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}}_{\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}} + f\hat{z}\times\boldsymbol{u} = -\frac{1}{\varrho_0}\nabla p$$

$$\boldsymbol{u} = (u(x,y,t), v(x,y,t))$$

$$\nabla = \frac{\partial}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial}{\partial y}\hat{\boldsymbol{y}}$$

Shallow-water dynamics



horizontal momentum equations

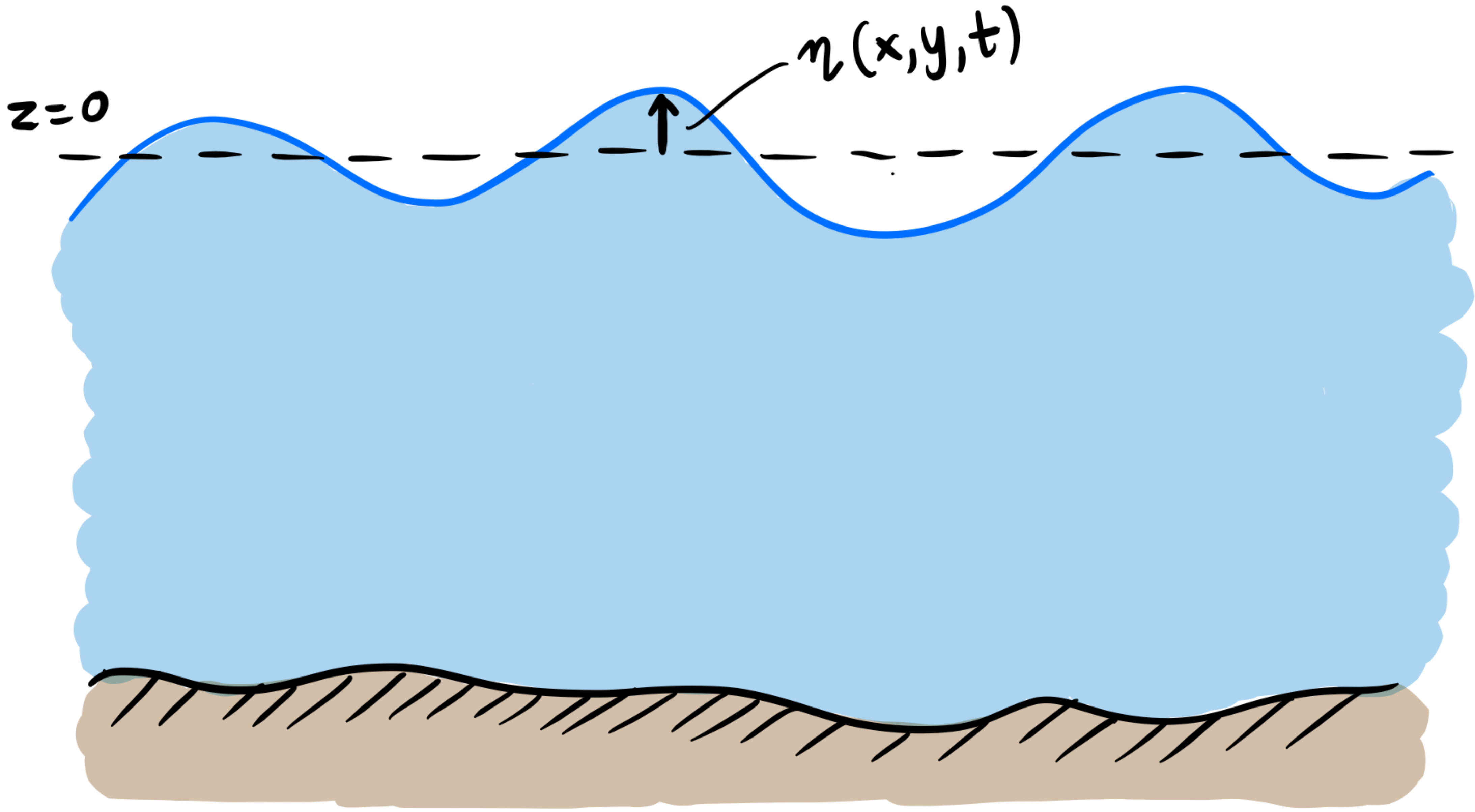
$$\underbrace{\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}}_{\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}} + f\hat{\boldsymbol{z}}\times\boldsymbol{u} = -\underbrace{\frac{1}{\varrho_0}\nabla p}_{\text{use hydrostatic balance}}$$

$$\boldsymbol{u} = (u(x,y,t), v(x,y,t))$$

$$\nabla = \frac{\partial}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial}{\partial y}\hat{\boldsymbol{y}}$$

$$p(x,y,z,t) = \varrho_0 g(\eta(x,y,t) - z) + p_0$$

Shallow-water dynamics



horizontal momentum equations

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$$\nabla = \frac{\partial}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial}{\partial y}\hat{\boldsymbol{y}}$$

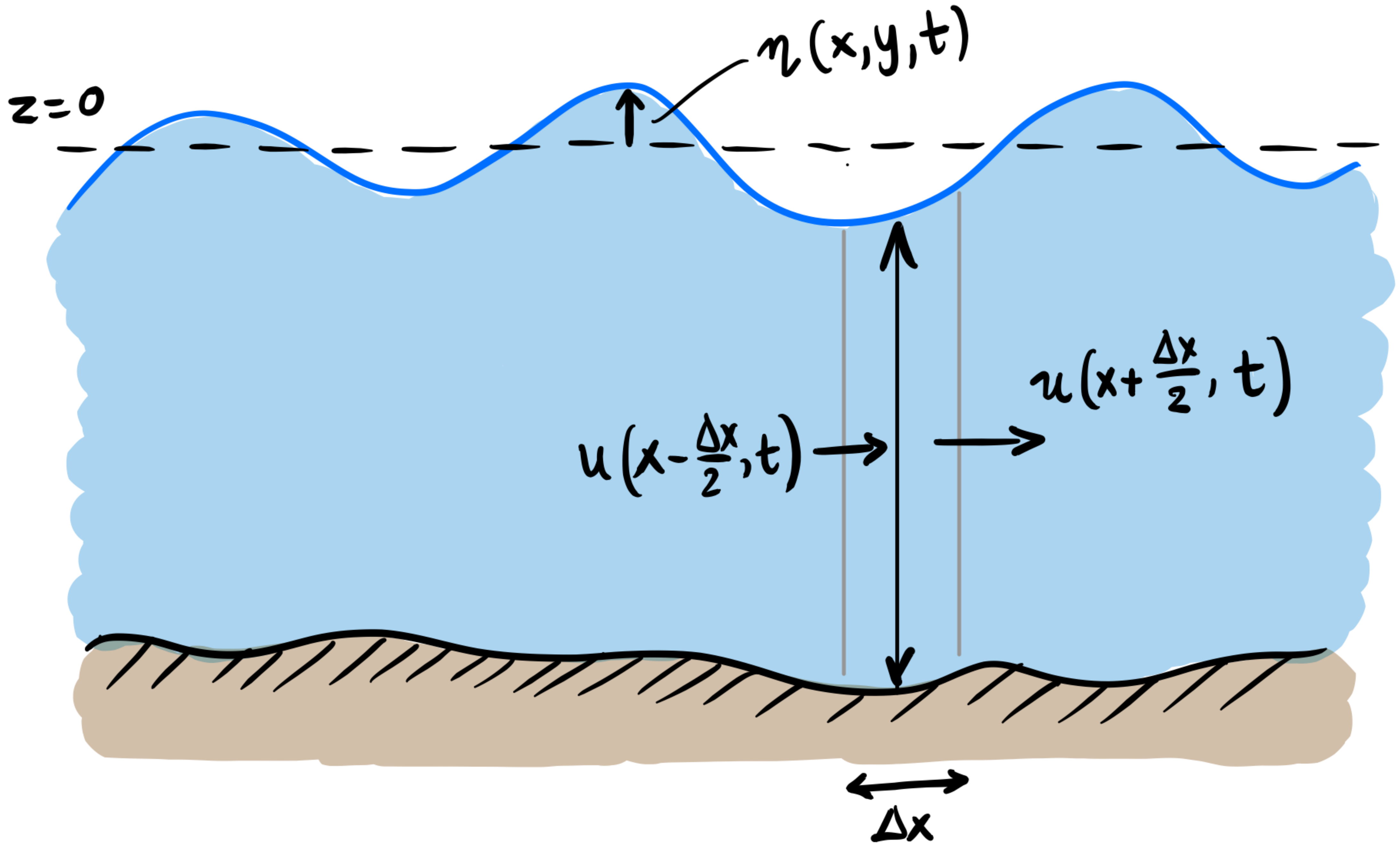
$$p(x,y,z,t) = \varrho_0 g(\eta(x,y,t) - z) + p_0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + f\hat{z}\times\boldsymbol{u} = -\,g\,\nabla\eta$$

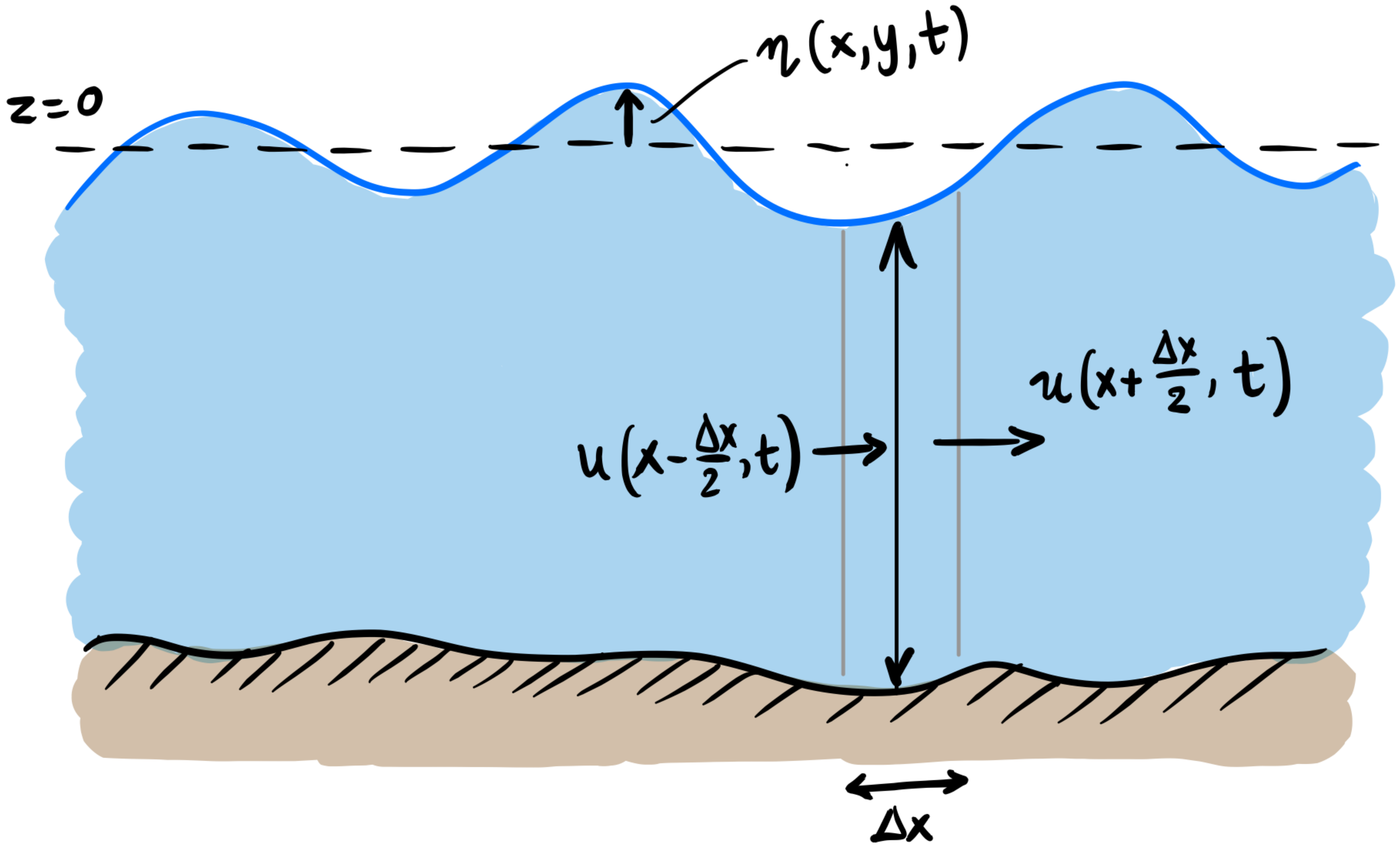
Shallow-water dynamics

mass conservation

constant density $\Longrightarrow$ volume conservation



Shallow-water dynamics

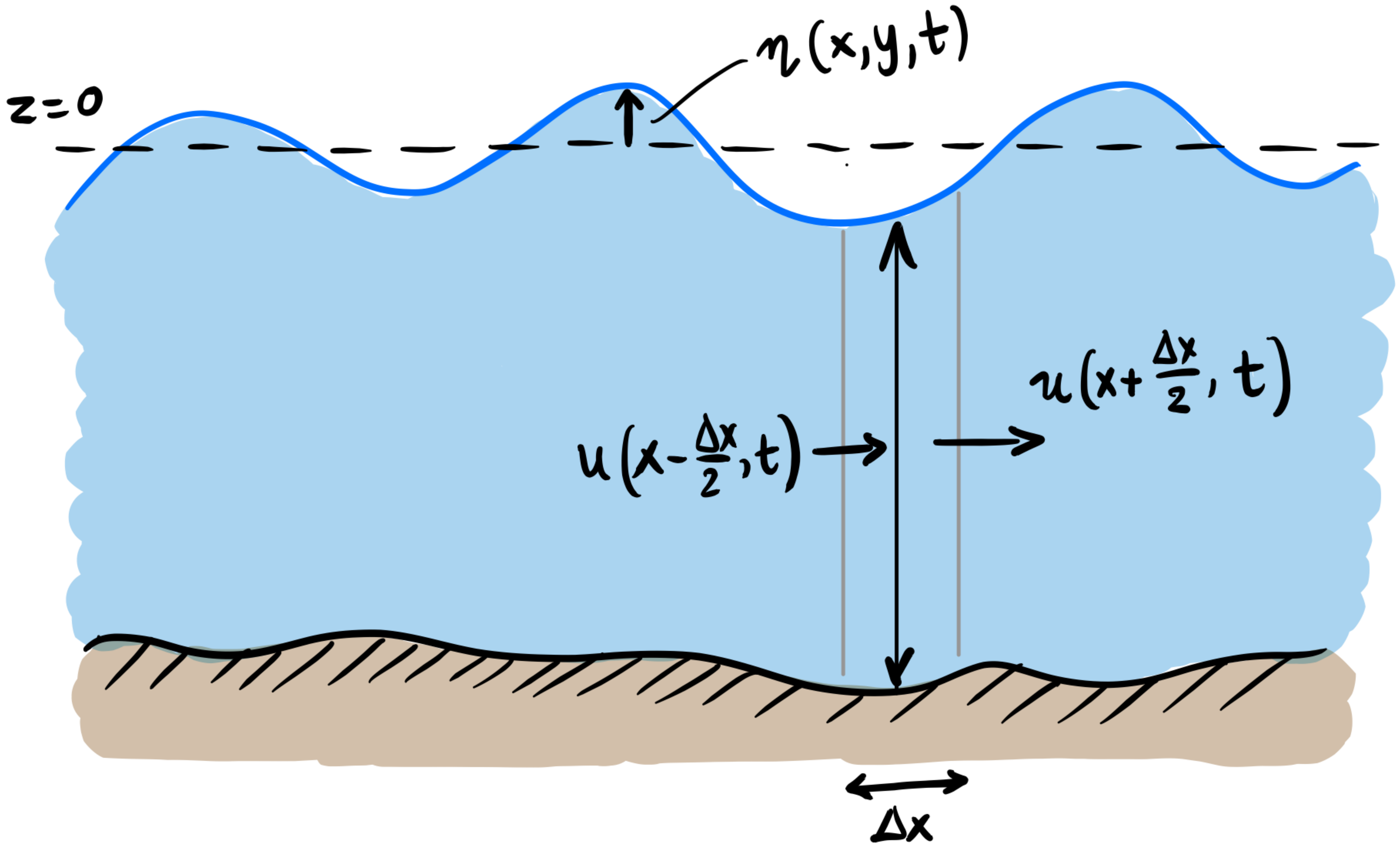


mass conservation

constant density $\Longrightarrow$ volume conservation

volume of  $= \; h(x,t)\ \Delta x$

Shallow-water dynamics



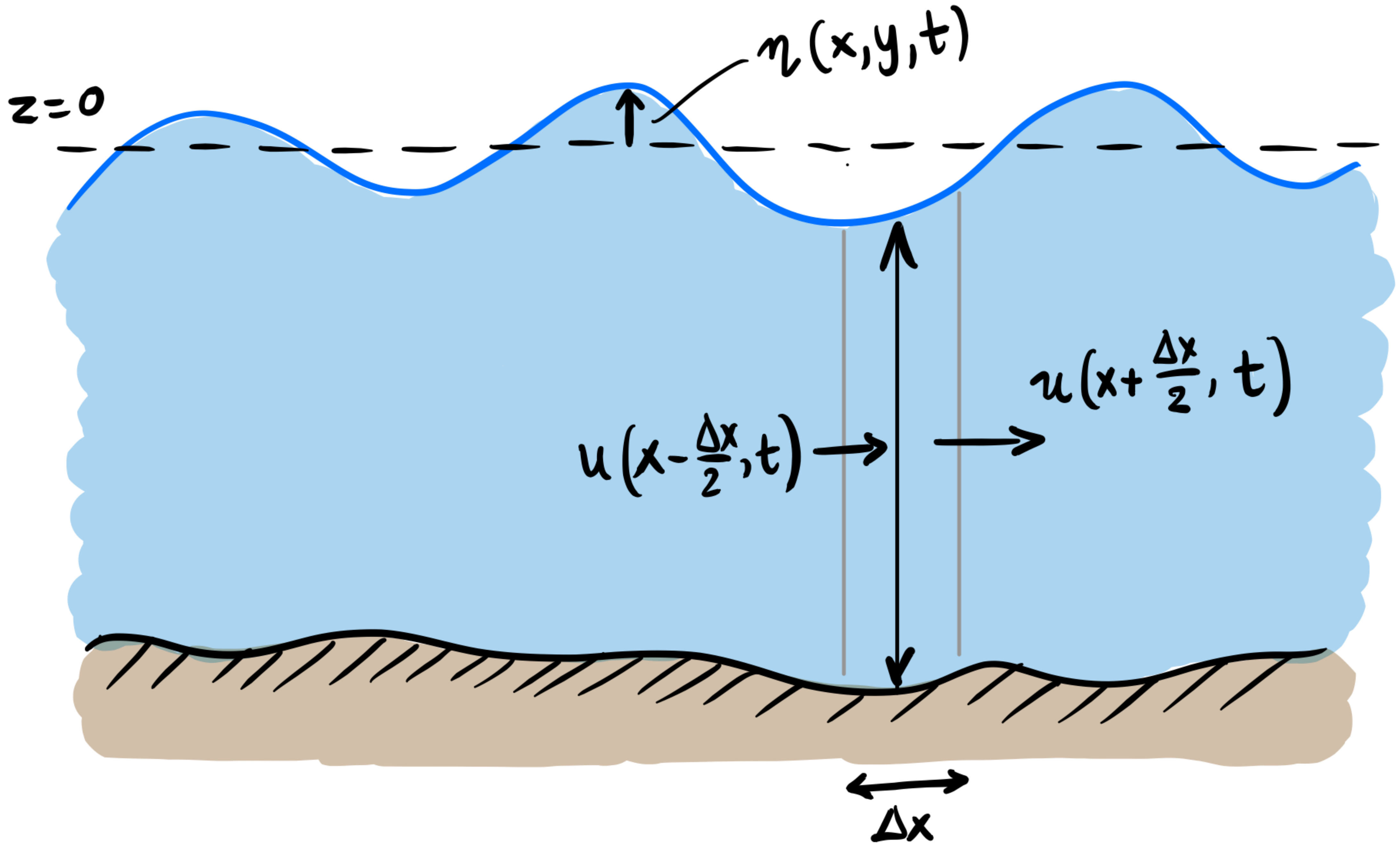
mass conservation

constant density $\Longrightarrow$ volume conservation

volume of $=$ $h(x,t)\ \Delta x$

change of volume $=$ fluid flux into $-$ fluid flux out of

Shallow-water dynamics



mass conservation

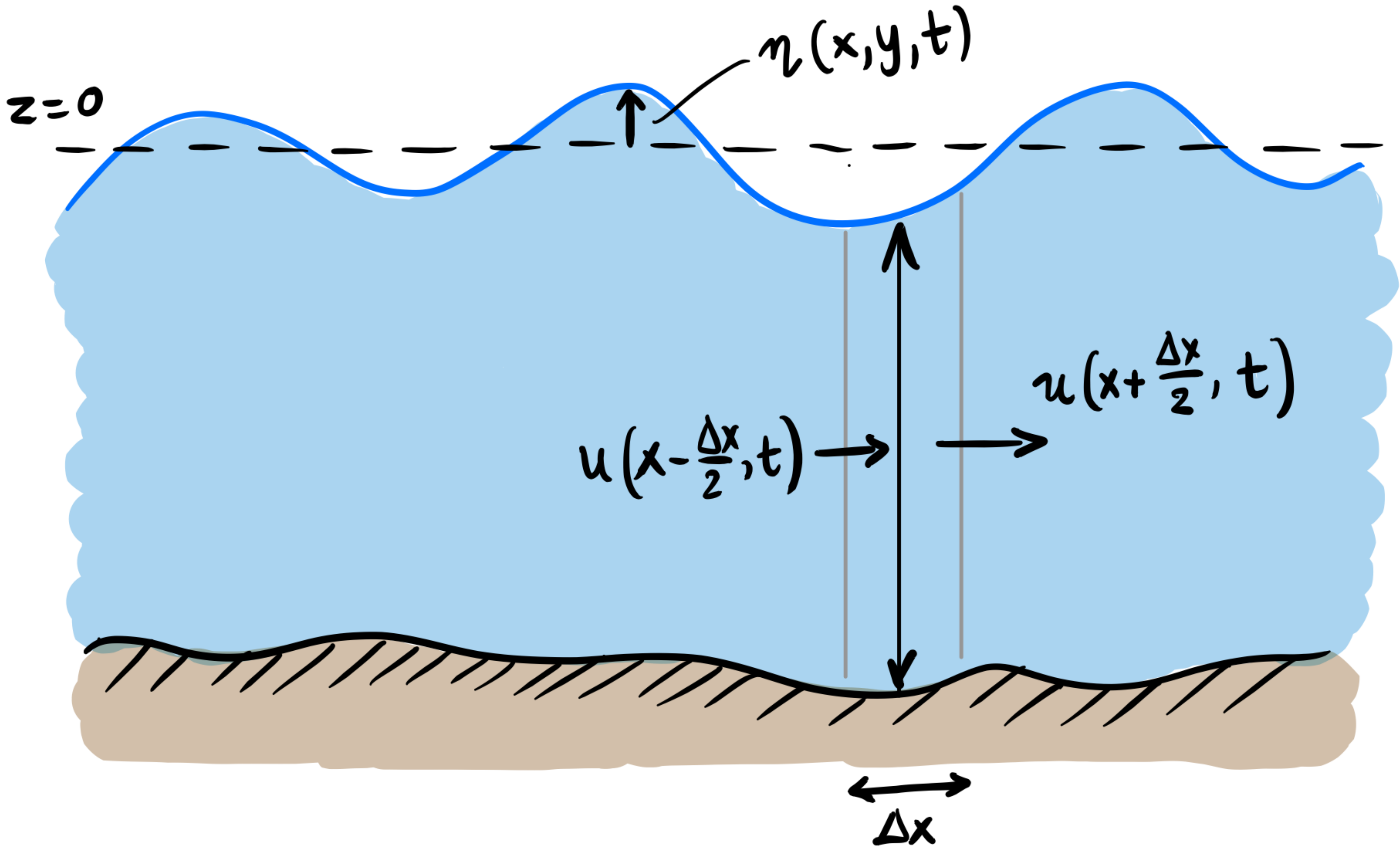
constant density $\Longrightarrow$ volume conservation

volume of $=$ $h(x,t)\,\Delta x$

change of volume $=$ fluid flux into $-$ fluid flux out of

$$\Delta x \, \frac{\partial h(x,t)}{\partial t} = u\Big(x-\frac{\Delta x}{2},t\Big)\,h\Big(x-\frac{\Delta x}{2},t\Big) - u\Big(x+\frac{\Delta x}{2},t\Big)\,h\Big(x+\frac{\Delta x}{2},t\Big)$$

Shallow-water dynamics



mass conservation

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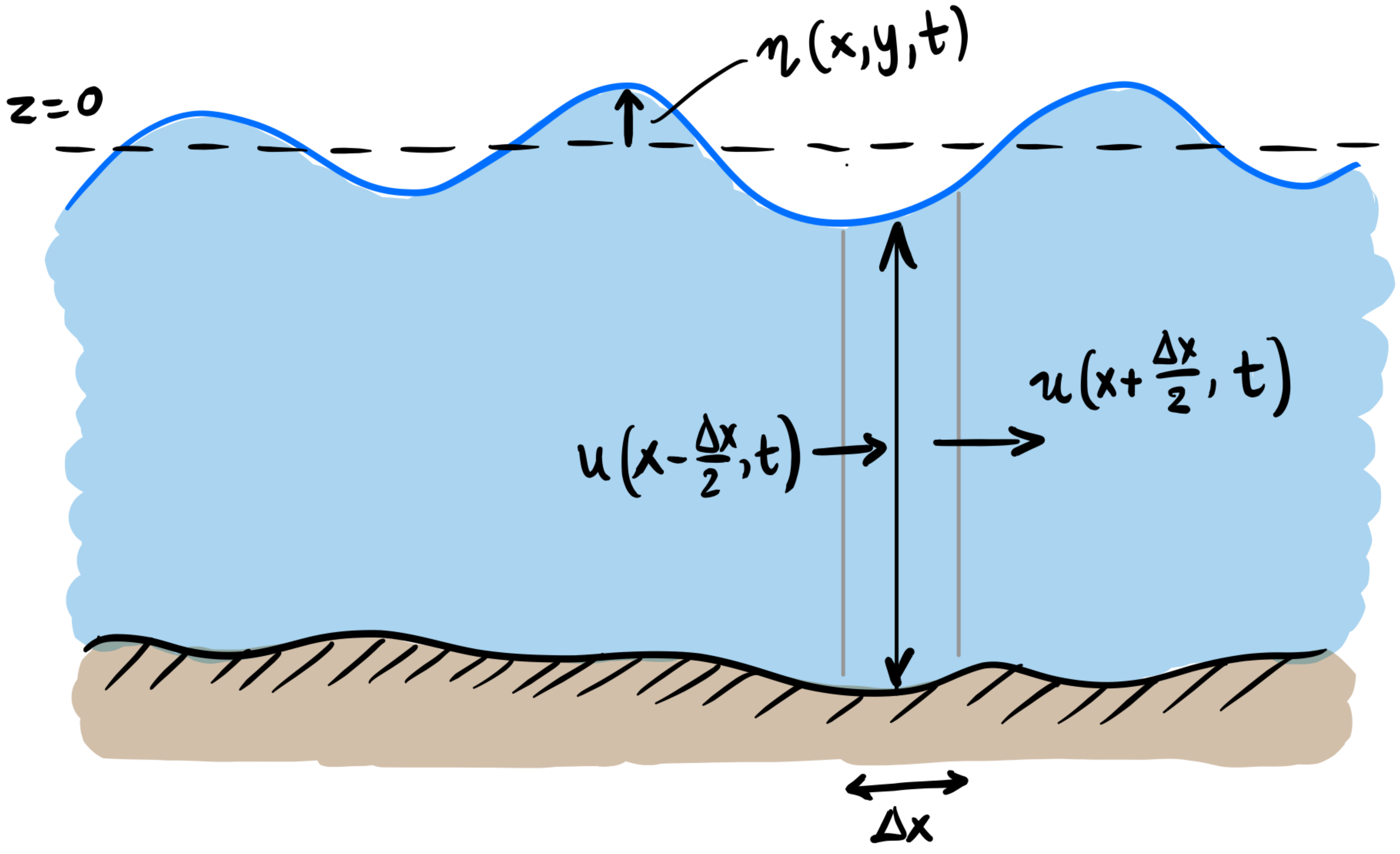
$$= \Delta x\,\frac{u\big(x-\frac{\Delta x}{2},t\big)\,h\big(x-\frac{\Delta x}{2},t\big) - u\big(x+\frac{\Delta x}{2},t\big)\,h\big(x+\frac{\Delta x}{2},t\big)}{\Delta x}$$

Shallow-water dynamics



mass conservation

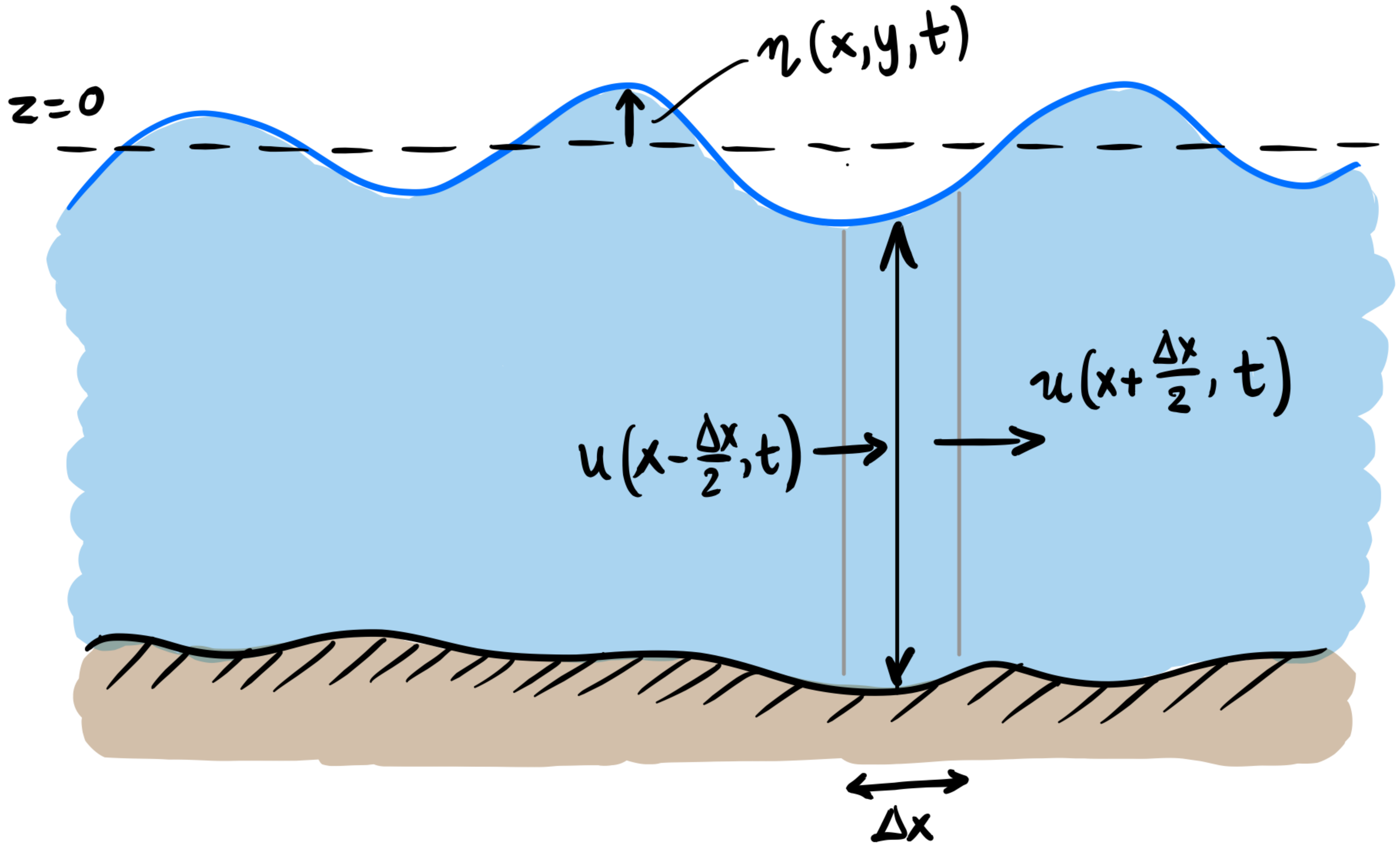
constant density $\implies$ volume conservation

volume of ▯ $=$ $h(x,t)\ \Delta x$

change of volume ▯ $=$ fluid flux into ▯ $-$ fluid flux out of ▯

$$
\begin{aligned}
\Delta x\,\frac{\partial h(x,t)}{\partial t} &= u\big(x-\frac{\Delta x}{2},t\big)\,h\big(x-\frac{\Delta x}{2},t\big) - u\big(x+\frac{\Delta x}{2},t\big)\,h\big(x+\frac{\Delta x}{2},t\big) \\
&= \Delta x\,\frac{u\big(x-\frac{\Delta x}{2},t\big)\,h\big(x-\frac{\Delta x}{2},t\big) - u\big(x+\frac{\Delta x}{2},t\big)\,h\big(x+\frac{\Delta x}{2},t\big)}{\Delta x} \\
&\approx -\,\Delta x\,\frac{\partial}{\partial x}\big[h(x,t)u(x,t)\big]
\end{aligned}
$$

Shallow-water dynamics



mass conservation

constant density $\implies$ volume conservation

volume of $=$ $h(x,t)\,\Delta x$

change of volume $=$ fluid flux into $-$ fluid flux out of

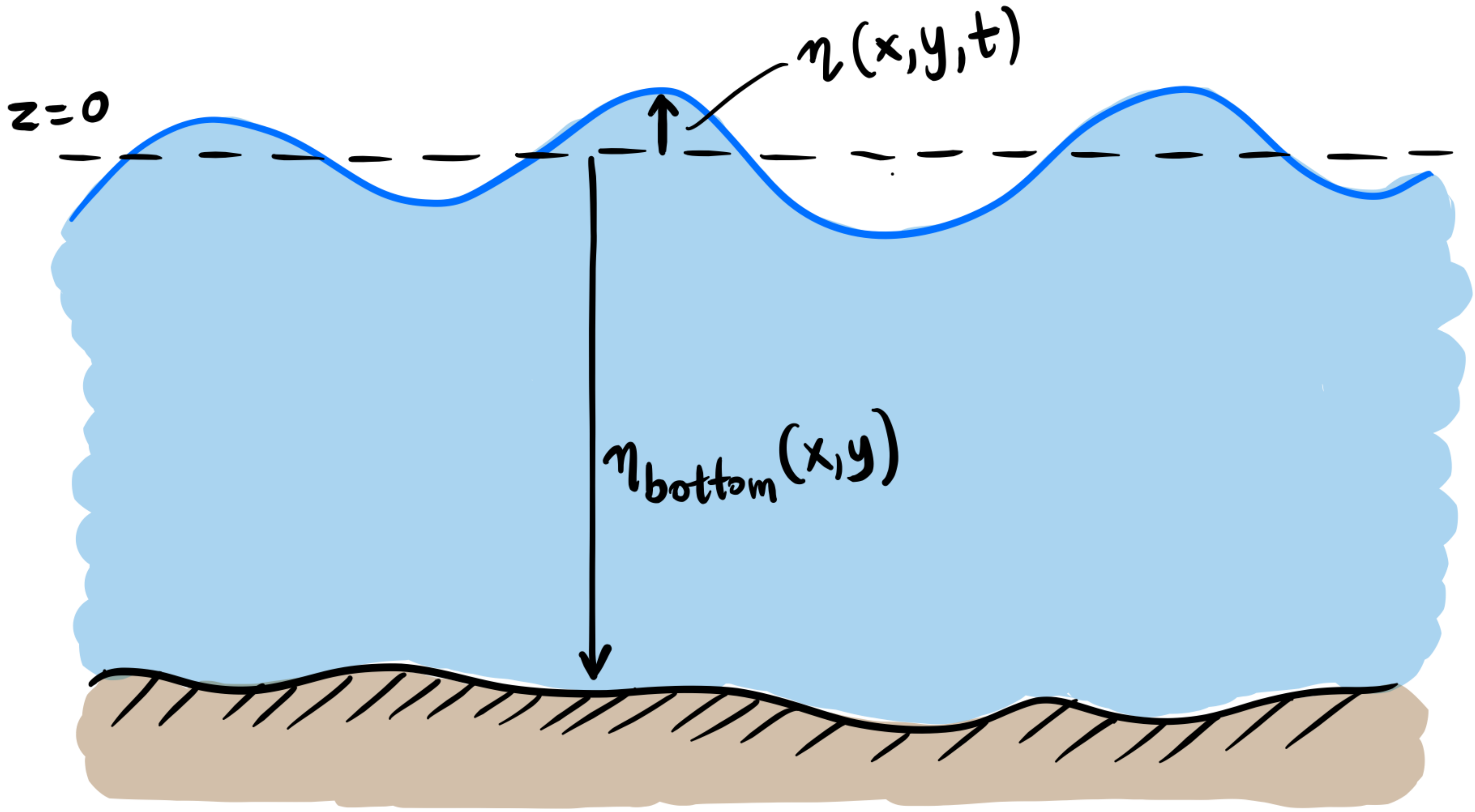
$$\Delta x \frac{\partial h(x,t)}{\partial t} = u\left(x-\frac{\Delta x}{2},t\right) h\left(x-\frac{\Delta x}{2},t\right) - u\left(x+\frac{\Delta x}{2},t\right) h\left(x+\frac{\Delta x}{2},t\right)$$

$$= \Delta x \frac{u\left(x-\frac{\Delta x}{2},t\right) h\left(x-\frac{\Delta x}{2},t\right) - u\left(x+\frac{\Delta x}{2},t\right) h\left(x+\frac{\Delta x}{2},t\right)}{\Delta x}$$

$$\approx -\Delta x \frac{\partial}{\partial x}\left[h(x,t)u(x,t)\right]$$

$$\boxed{\frac{\partial h}{\partial t} + \nabla \cdot \left(\boldsymbol{u}h\right) = 0}$$

Shallow-water dynamics



$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + f\hat{\boldsymbol{z}} \times \boldsymbol{u} = -\, g \nabla \eta$$

$$\frac{\partial h}{\partial t} + \nabla \cdot \left(\boldsymbol{u} h\right) = 0$$

$$h(x, y, t) = \eta(x, y, t) - \eta_{\text{bottom}}(x, y)$$

$$\boldsymbol{u} = (u(x, y, t), v(x, y, t))$$

$$\nabla = \frac{\partial}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial}{\partial y}\hat{\boldsymbol{y}}$$

admit wave solutions
(waves that “live” on the fluid’s surface)

$f = 0$

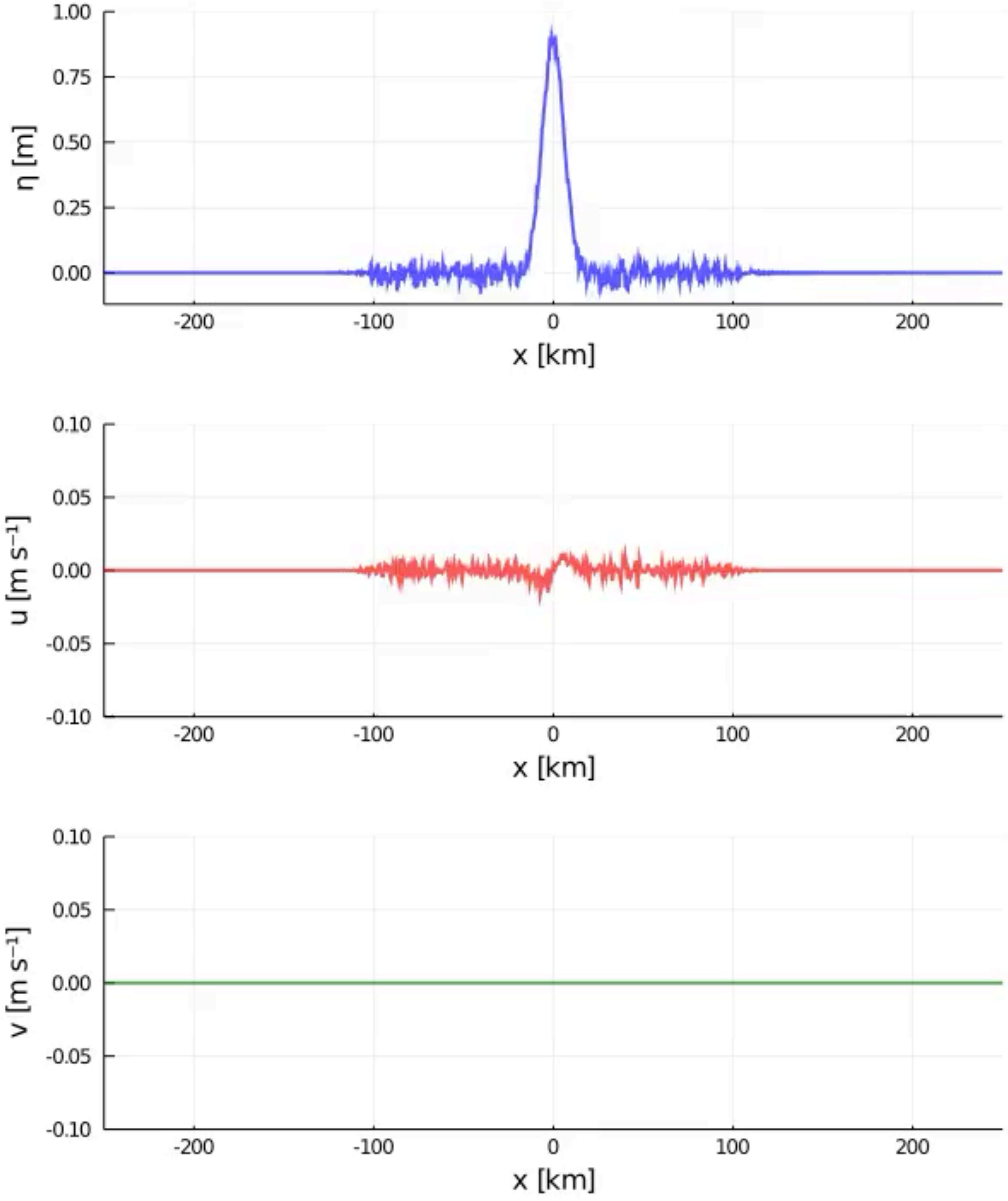
non-rotating

$\sqrt{gH} = 160 \text{ km h}^{-1}$

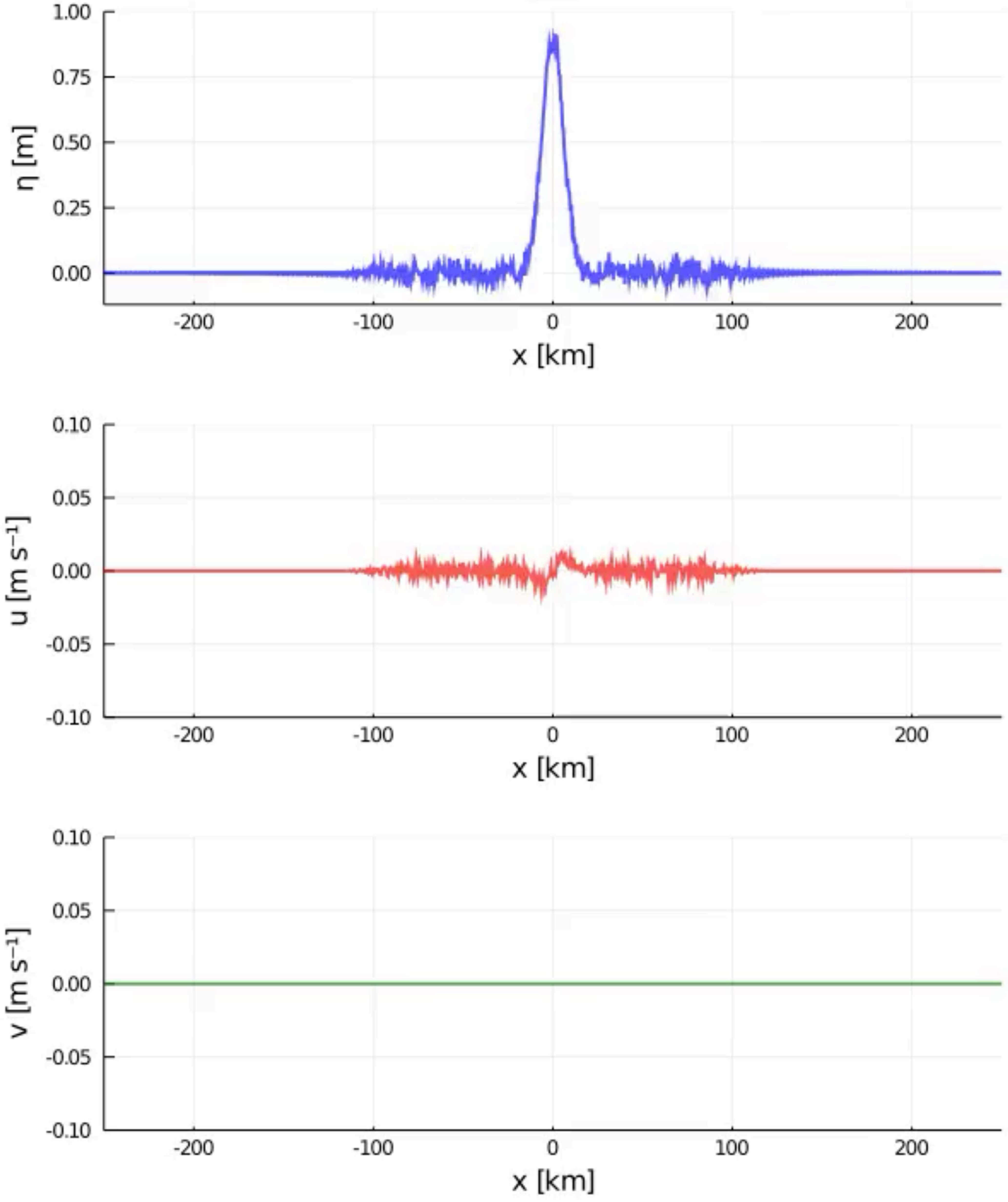
rotating

$f = 10^{-2} \text{ s}^{-1}$

t = 0.0 min

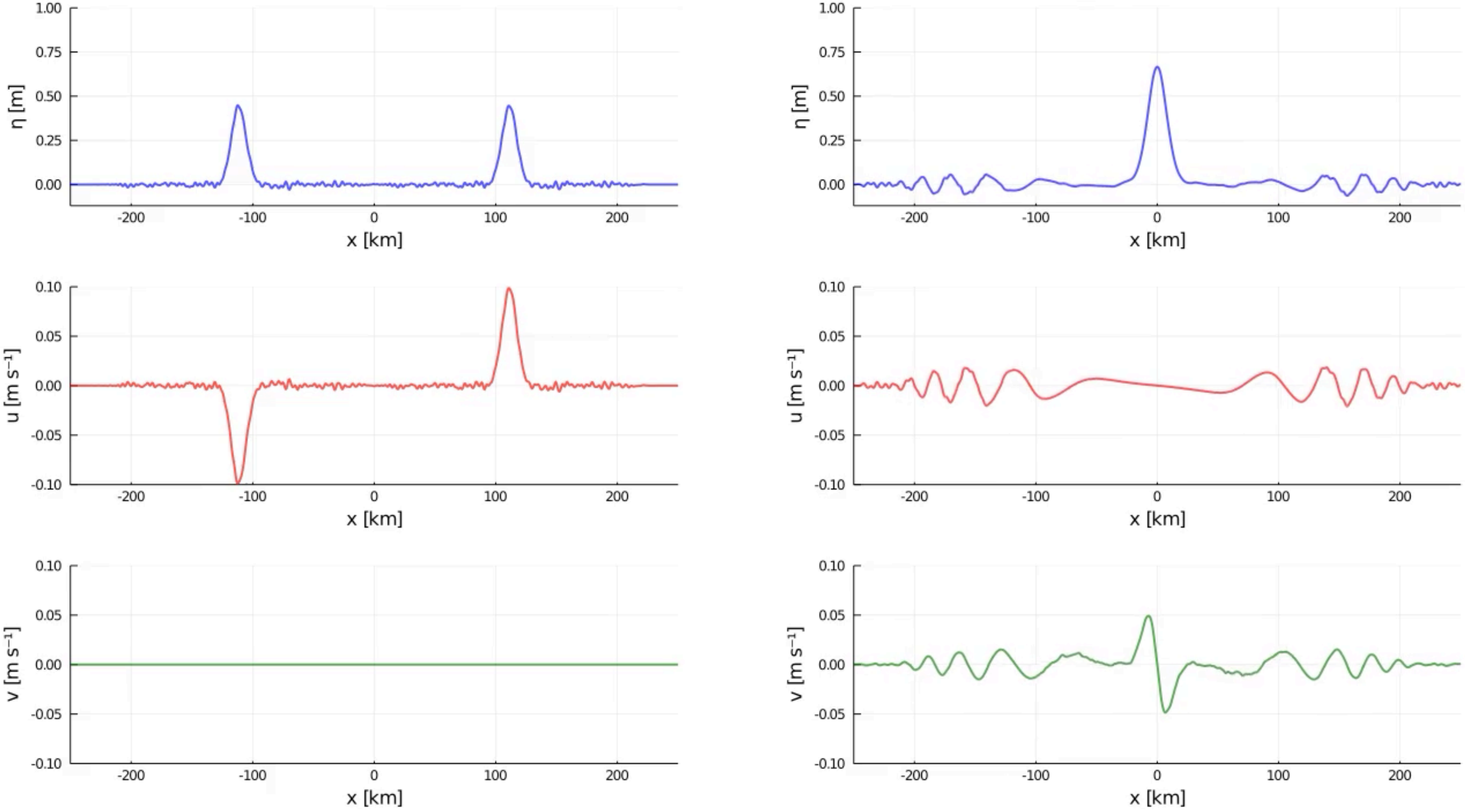


t = 0.0 min



$f = 0$ **non-rotating** $\sqrt{gH} = 160 \text{ km h}^{-1}$ **rotating** $f = 10^{-2} \text{ s}^{-1}$

t = 41.7 min t = 89.0 min



Rotating shallow-water dynamics

horizontal momentum eqs

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + f\hat{z} \times \boldsymbol{u} = -\, g \nabla \eta$$

after the dust settles…

$$f\hat{z} \times \boldsymbol{u} \approx -\, g \nabla \eta$$

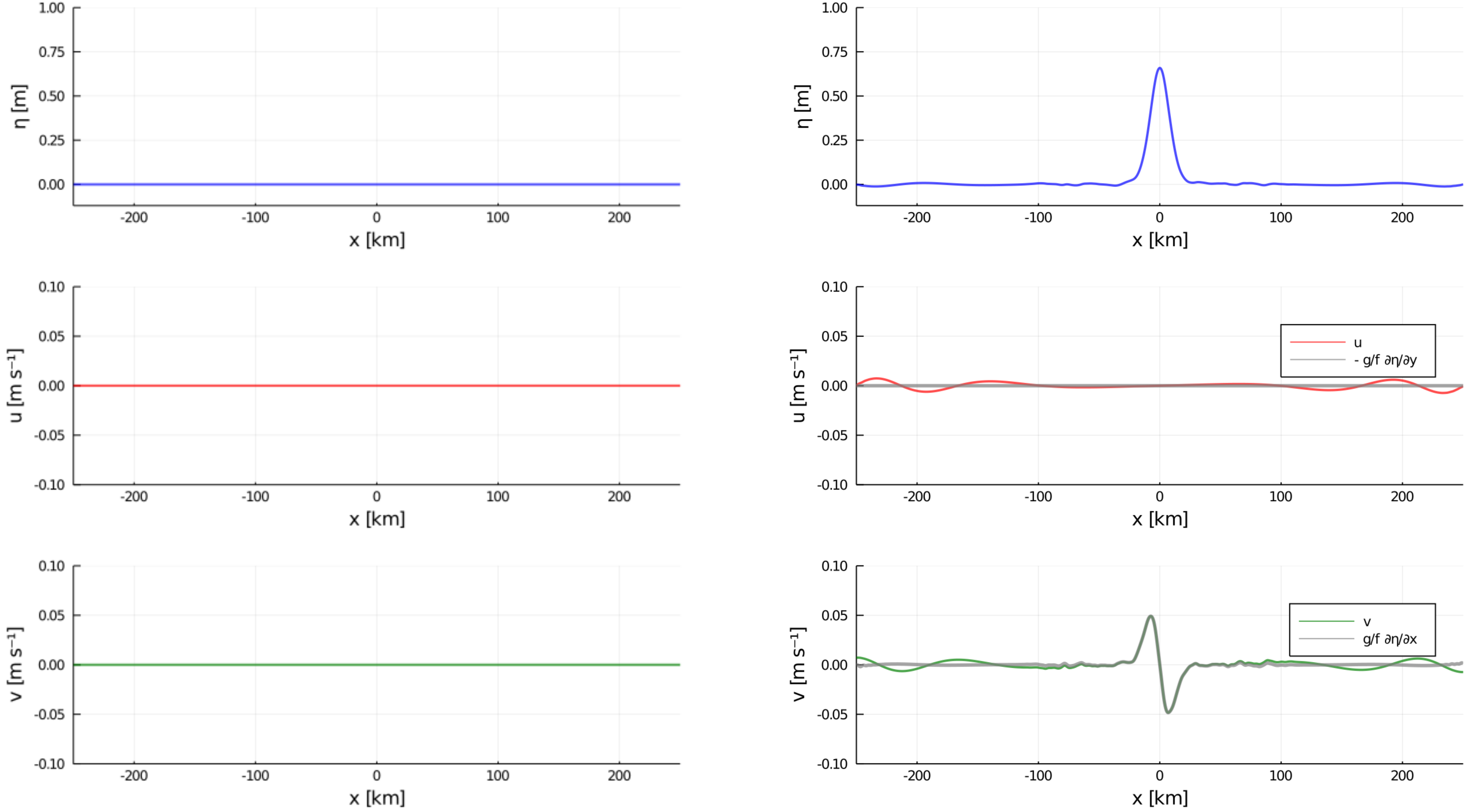
Coriolis $\approx$ pressure gradient

geostrophic balance

$$u_{\text{geostrophic}} = -\frac{g}{f}\frac{\partial \eta}{\partial y} \qquad v_{\text{geostrophic}} = +\frac{g}{f}\frac{\partial \eta}{\partial x}$$

$f = 0$ **non-rotating** t = 200.0 min

rotating t = 250.0 min $f = 10^{-2}\ \mathrm{s}^{-1}$



Rotating shallow-water dynamics

Slow motions approximately in geostrophic balance

~ days (atmos)/weeks (ocean)

"weather"

$$u_{\text{geostrophic}} = -\frac{\partial}{\partial y}\left(\frac{p}{\varrho f}\right)$$

$$v_{\text{geostrophic}} = +\frac{\partial}{\partial x}\left(\frac{p}{\varrho f}\right)$$

Fast-travelling motions

~ hours

"noise"

Flows in Geostrophic Balance

$$u_{\text{geostrophic}} = -\frac{\partial}{\partial y}\left(\frac{p}{\varrho f}\right) \qquad v_{\text{geostrophic}} = +\frac{\partial}{\partial x}\left(\frac{p}{\varrho f}\right)$$

Evolve much slower than gravity waves

Flows in Geostrophic Balance

$$u_{\text{geostrophic}} = -\frac{\partial}{\partial y}\left(\frac{p}{\varrho f}\right) \qquad v_{\text{geostrophic}} = +\frac{\partial}{\partial x}\left(\frac{p}{\varrho f}\right)$$

Evolve much slower than gravity waves

Incompressible: $$\frac{\partial}{\partial x}u_{\text{geostrophic}} + \frac{\partial}{\partial y}v_{\text{geostrophic}} = 0$$

Flows in Geostrophic Balance

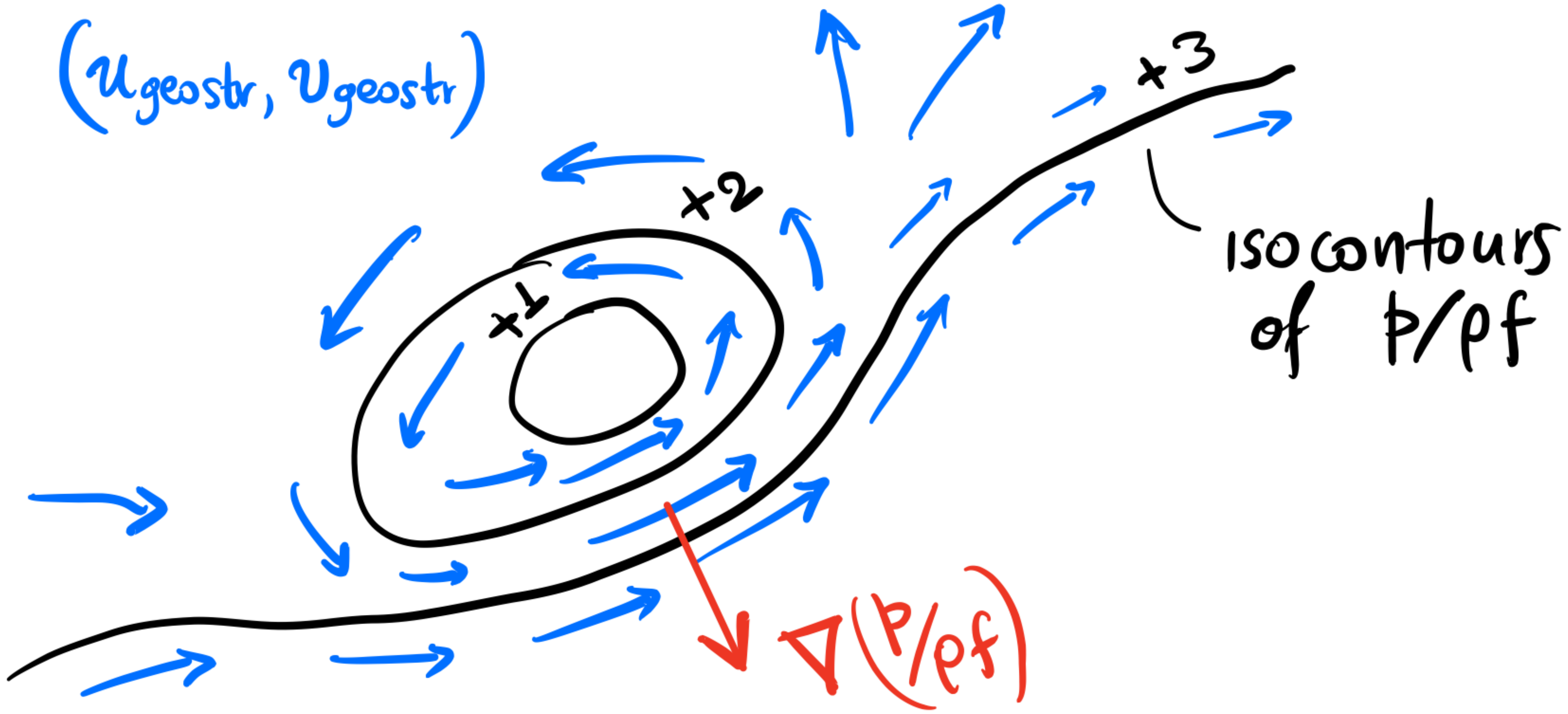
$$u_{\text{geostrophic}} = -\frac{\partial}{\partial y}\left(\frac{p}{\varrho f}\right) \qquad v_{\text{geostrophic}} = +\frac{\partial}{\partial x}\left(\frac{p}{\varrho f}\right)$$

Evolve much slower than gravity waves

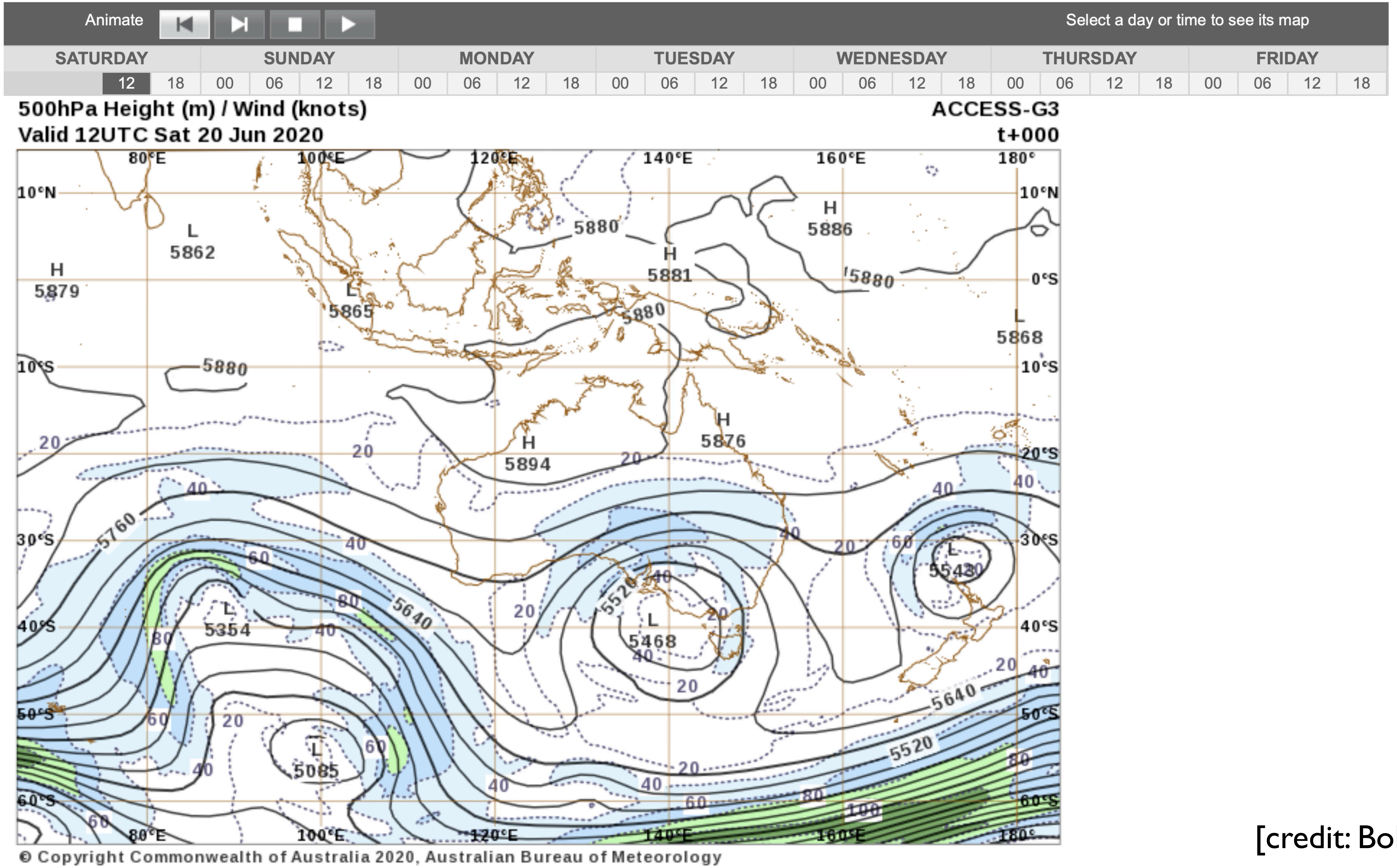
Incompressible: $$\frac{\partial}{\partial x}u_{\text{geostrophic}} + \frac{\partial}{\partial y}v_{\text{geostrophic}} = 0$$

Flow follows contours of constant $p/\varrho f$

$$\nabla\left(\frac{p}{\varrho f}\right) \cdot \boldsymbol{u}_{\text{geostrophic}} = 0$$



Weather maps are all about **Q**uasi-**G**eostrophy



[credit: BoM]

What if
we don’t care about “noise” (=gravity waves)
and we just want to know
how the “weather” (almost geostrophically balanced flow)
evolves?

Rotating
shallow-water
dynamics

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \cdots$$

filter out
fast gravity waves

Quasi-Geostrophic
dynamics

$$\frac{\partial}{\partial t} p = \cdots$$

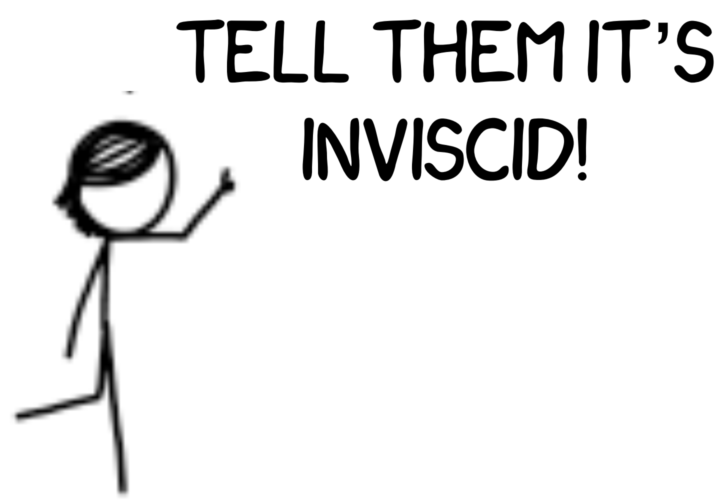
One variable suffices
to obtain the flow

Let's change pace.

How does QG dynamics relates
to 2D turbulence?

(Incompressible 2D flow = Quasi-Geostrophy **without** rotation)

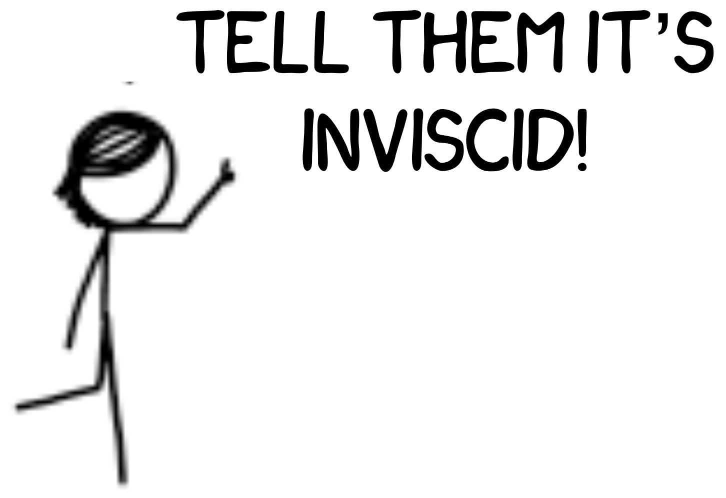
Incompressible 2D flow



$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\varrho_0} \nabla p \qquad \boldsymbol{u} = (u(x,y,t), v(x,y,t)) \qquad p(x,y,t)$$

$$\nabla \cdot \boldsymbol{u} = 0$$

Incompressible 2D flow



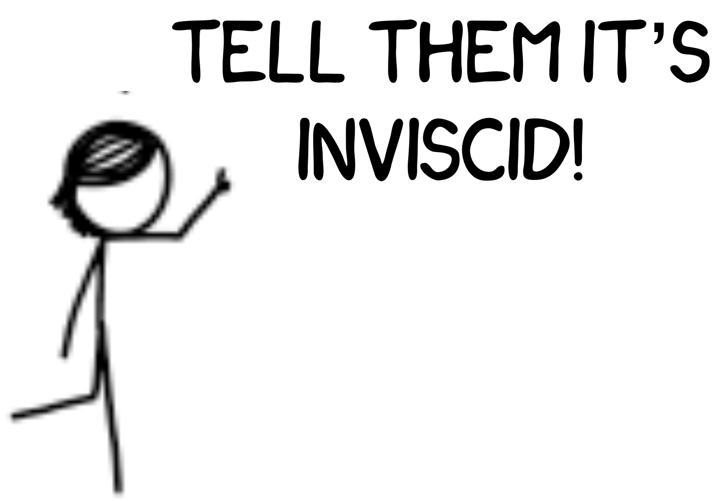
$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\varrho_0} \nabla p \qquad \boldsymbol{u} = (u(x, y, t), v(x, y, t)) \qquad p(x, y, t)$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$(u, v) = (-\partial_y \psi, \partial_x \psi)$$

vorticity $\quad (\nabla \times \boldsymbol{u}) \cdot \hat{z} = \nabla^2 \psi$

Incompressible 2D flow



$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\varrho_0} \nabla p \qquad \boldsymbol{u} = (u(x,y,t), v(x,y,t)) \qquad p(x,y,t)$$

$$\nabla \cdot \boldsymbol{u} = 0$$

take the curl $\nabla \times$

$$(u, v) = (-\partial_y \psi, \partial_x \psi)$$

vorticity $\quad (\nabla \times \boldsymbol{u}) \cdot \hat{z} = \nabla^2 \psi$

incompressible 2D flow

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla \right) \nabla^2 \psi = 0$$

$$(u, v) = (-\partial_y \psi, \partial_x \psi)$$

Incompressible 2D flow = Quasi-Geostrophy **on f-plane**

Incompressible 2D flow

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\nabla^2\psi = 0$$

$$(u, v) = (-\partial_y\psi, \partial_x\psi)$$

QG **without** rotation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\nabla^2\left(\frac{p}{\varrho_0 f}\right) = 0$$

$$(u, v) = \left(-\frac{\partial}{\partial y}\left(\frac{p}{\varrho_0 f}\right), \frac{\partial}{\partial x}\left(\frac{p}{\varrho_0 f}\right) \right)$$

Note similarity with
passive tracer equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\theta = 0$$

***f*-plane**

$$f = f_0 = \text{const}\,.$$

(Flat Earth)

β-plane

$$f = f_0 + \beta y$$

(Spherical Earth)

[Btw you may find this entertaining: Can the earth be flat ? A physical oceanographer's perspective, arXiv:2001.01521]

Incompressible 2D flow = Quasi-Geostrophy **on f-plane**

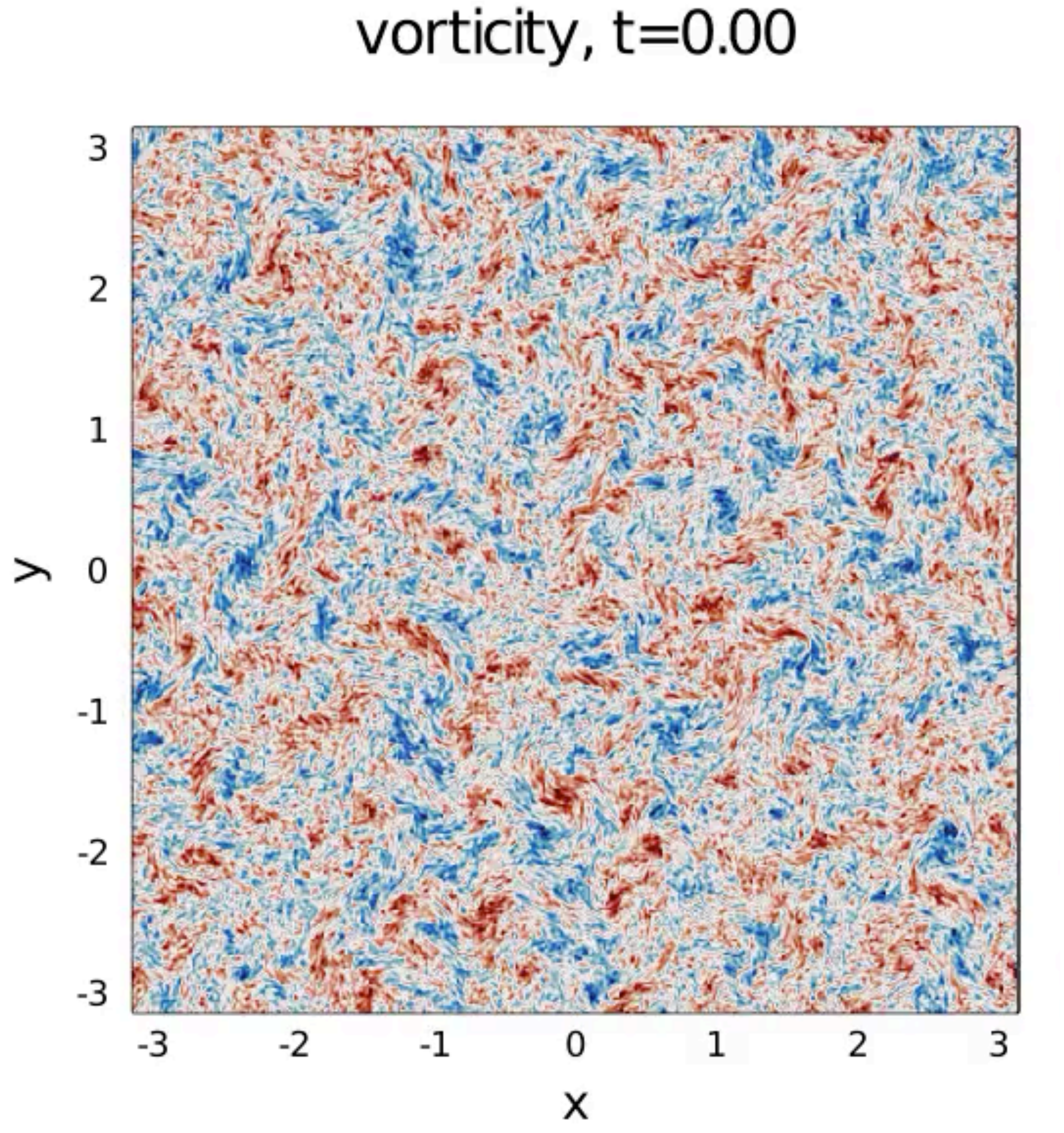
$$(\nabla \times \boldsymbol{u}) \cdot \hat{z} = \nabla^2 \psi$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla \right) \nabla^2 \psi = 0$$

$$(u, v) = (-\partial_y \psi, \partial_x \psi)$$

Note similarity with
passive tracer equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla \right) \theta = 0$$



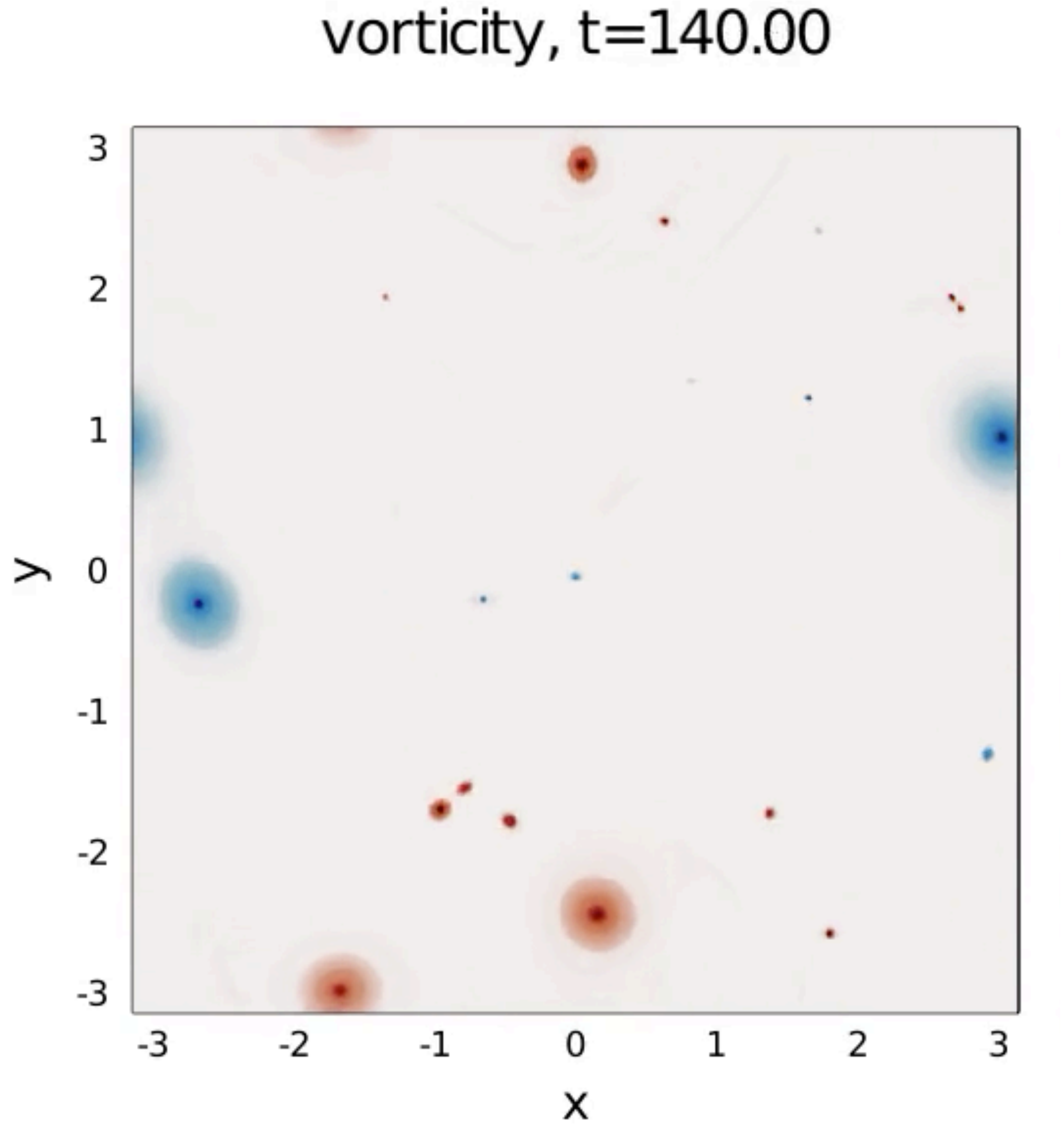
[simulation using GeophysicFlows.jl]

Incompressible 2D flow = Quasi-Geostrophy **on f-plane**

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right) \nabla^2 \psi = 0$$

$$(u, v) = (-\partial_y \psi, \partial_x \psi)$$

$$(\nabla \times \boldsymbol{u}) \cdot \hat{z} = \nabla^2 \psi$$



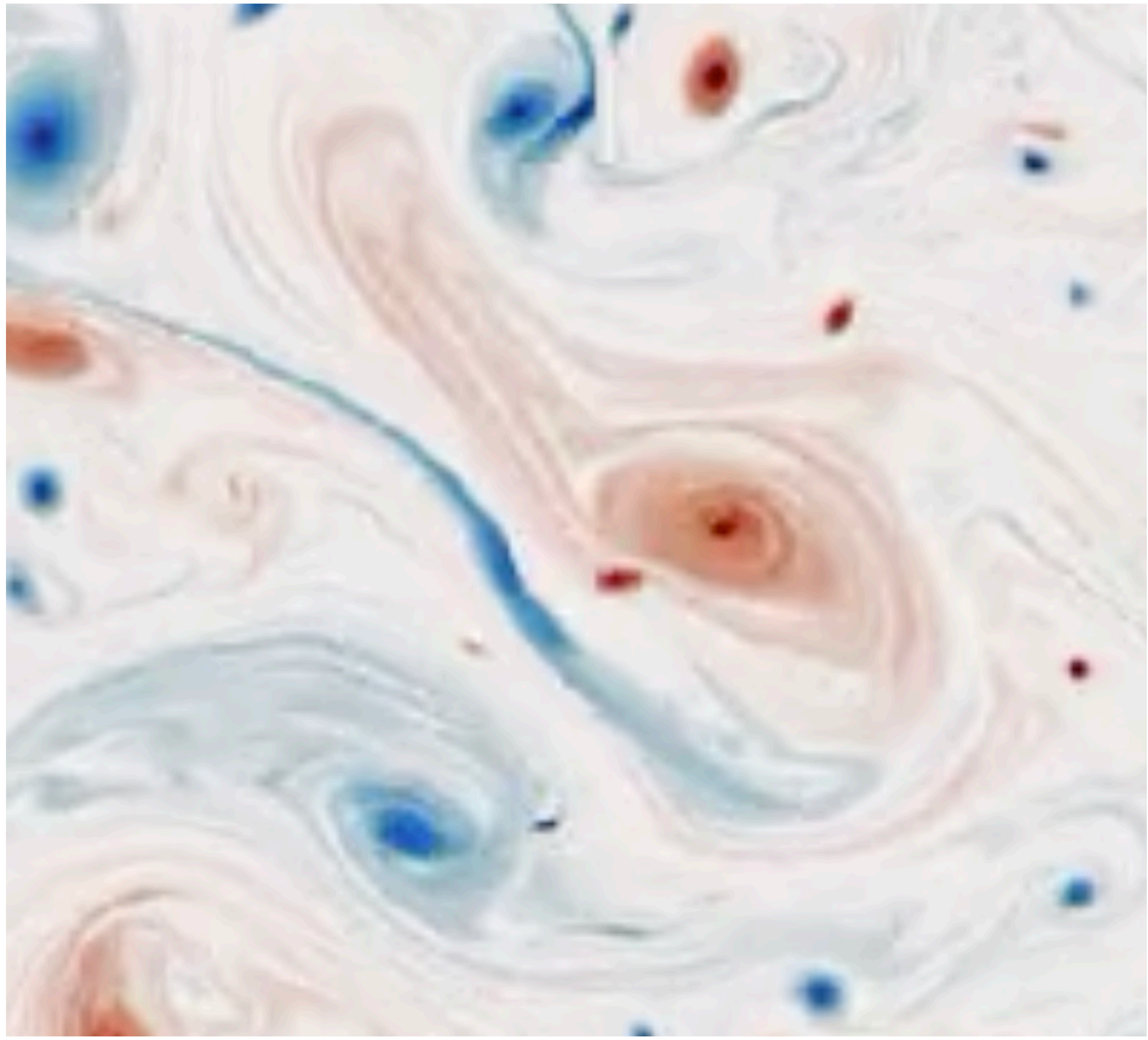
Note similarity with passive tracer equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right) \theta = 0$$

[simulation using GeophysicFlows.jl]

Rotating 3D fluids *resemble* 2D turbulence

2D turbulence **without rotation**



[simulation using GeophysicFlows.jl]

3D fluid in **rotating** tank

[MIT Weather in Tank]

Quasi-Geostrophy **with Earth's curvarture (β-plane)**

$$\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla\right)\underbrace{\left(\nabla^2\psi+f\right)}_{\text{PV}}=0$$

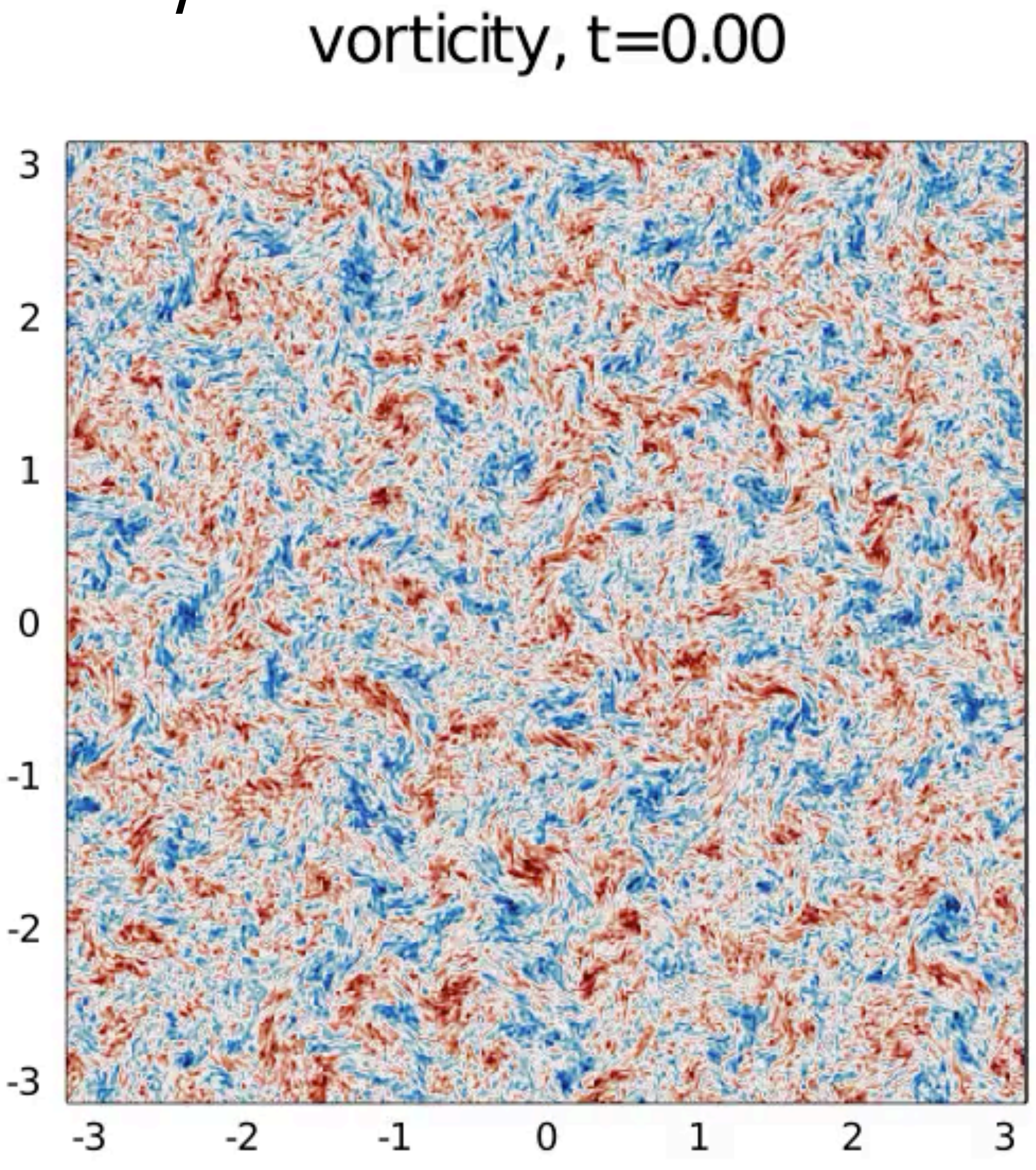
$$(u,v)=(-\partial_y\psi,\partial_x\psi)\ ,\qquad \psi=\frac{p}{\varrho_0 f}$$

What's materially conserved is the Potential Vorticity (PV)

Quasi-Geostrophy **with Earth's curvature (β-plane)**

non-rotating

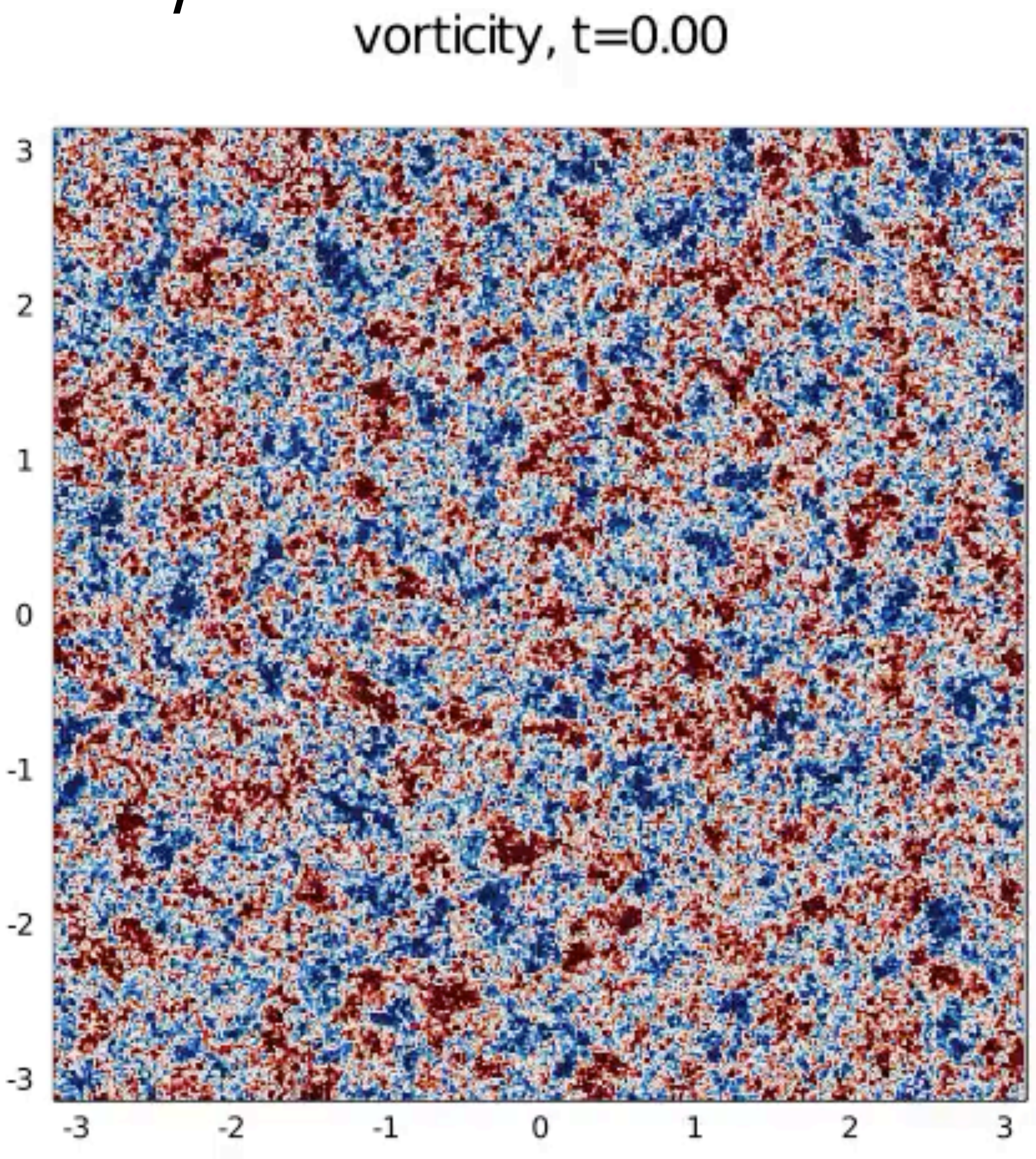
$\nabla^2\psi$



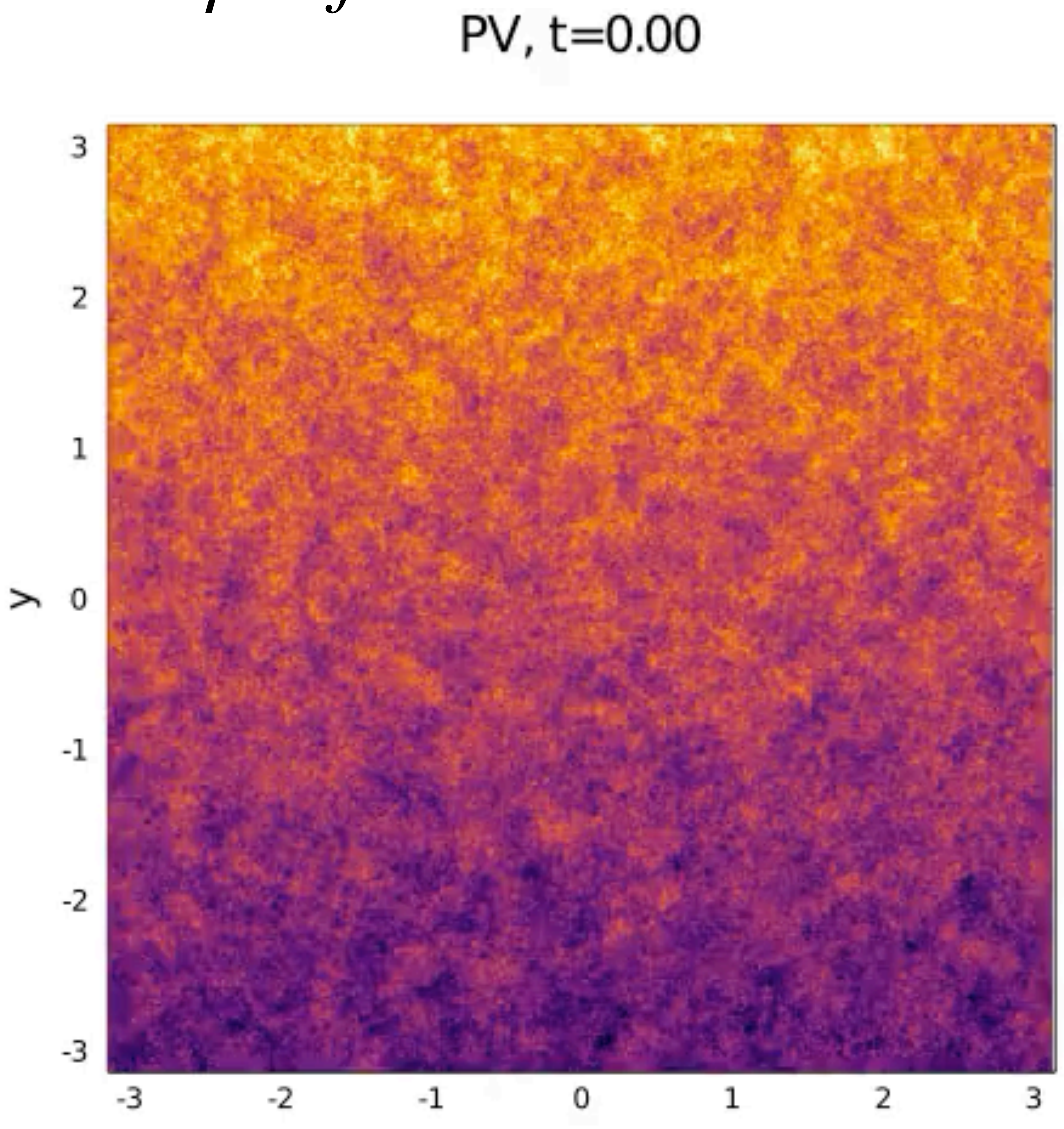
$f = 0$

rotating

$\nabla^2\psi$



$\nabla^2\psi + f$



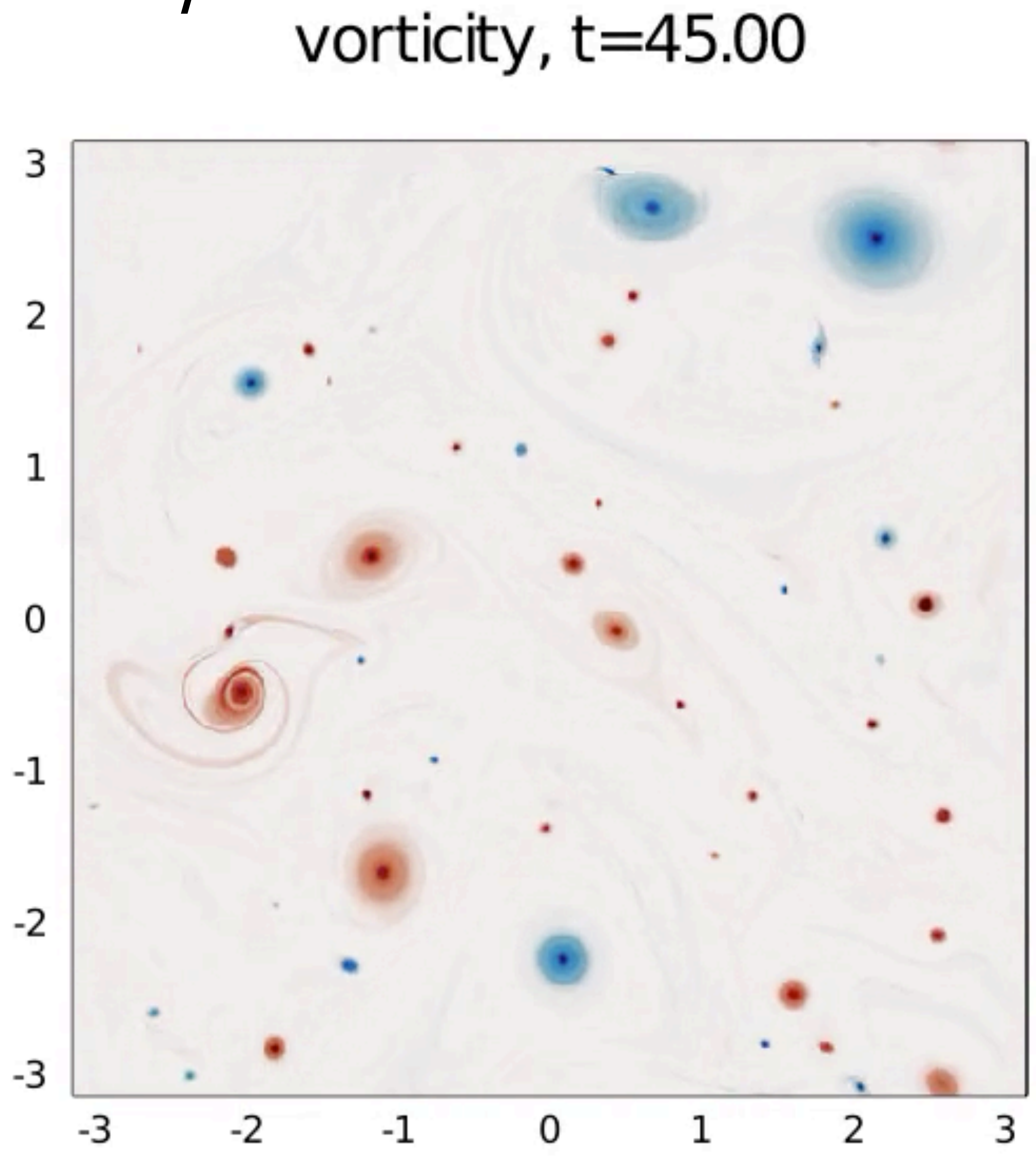
$f = f_0 + \beta y$

[simulations using GeophysicFlows.jl]

Quasi-Geostrophy **with Earth's curvarture (β-plane)**

non-rotating

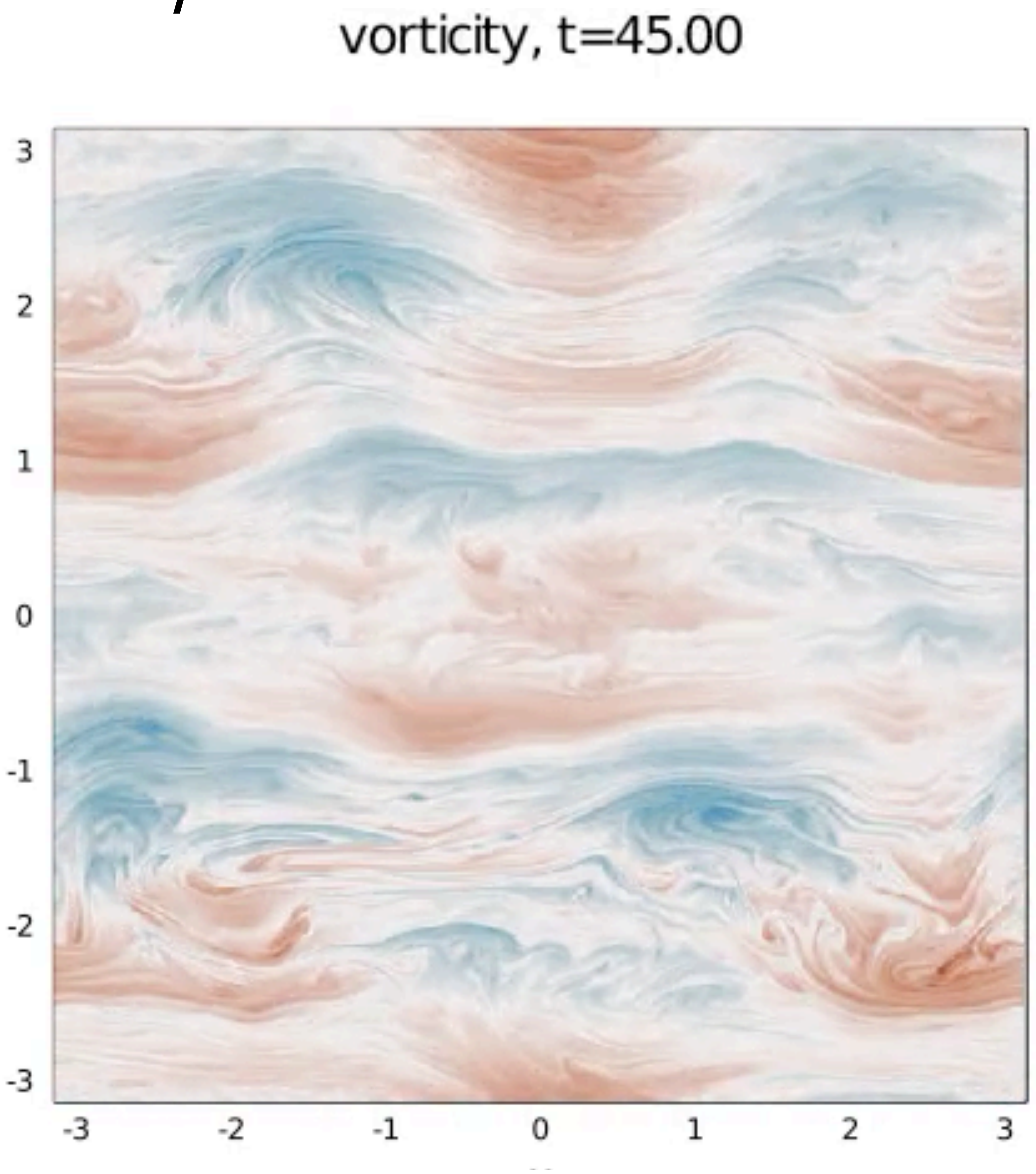
$\nabla^2\psi$



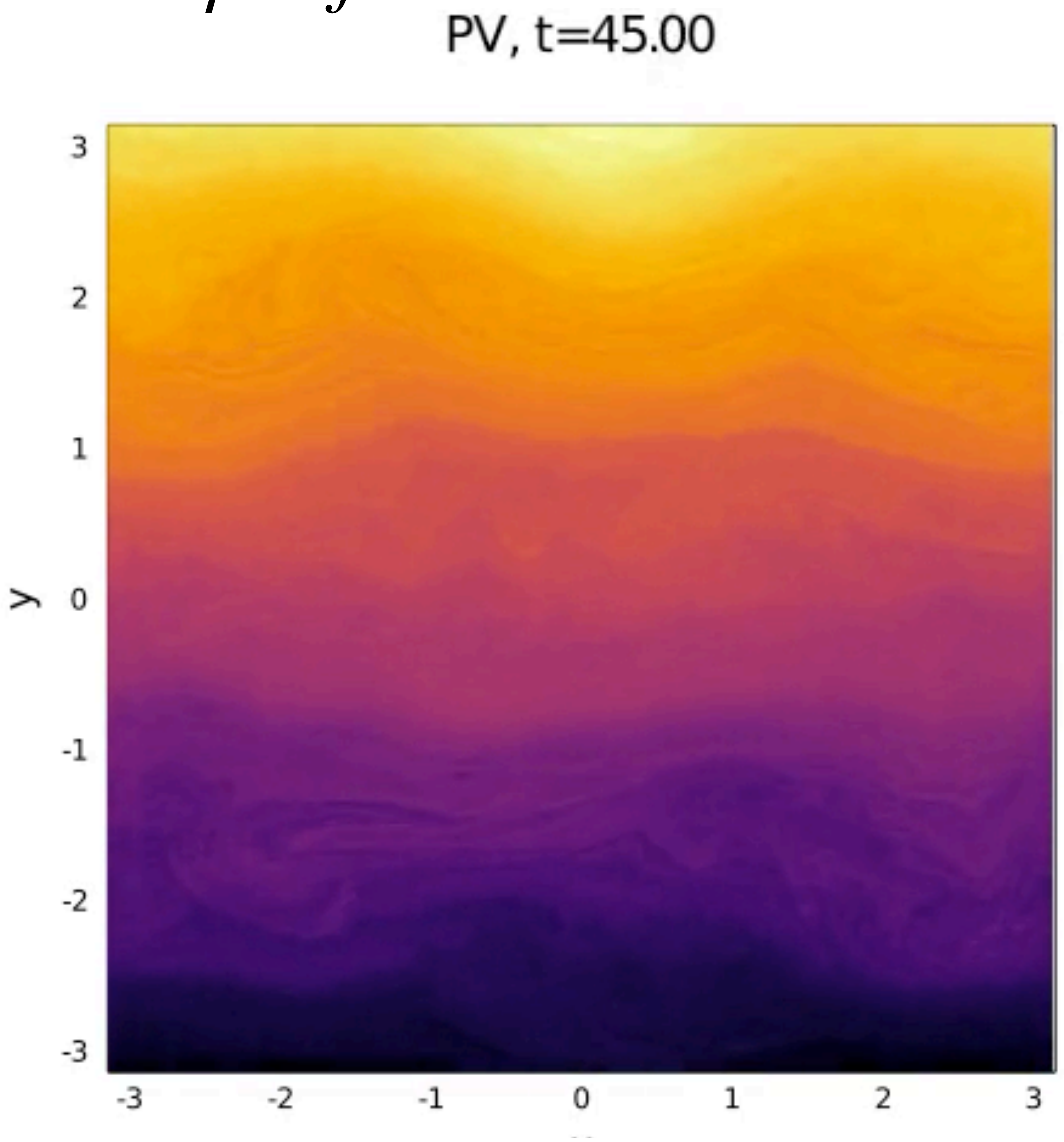
$f = 0$

rotating

$\nabla^2\psi$



$\nabla^2\psi + f$



$f = f_0 + \beta y$

[simulations using GeophysicFlows.jl]

Atmosphere & Ocean Dynamics
CLEx X School 2021 (?)

X = {Winter, Summer, Autum, Spring, Xmas, Easter, …}

THE FOLLOWING **PREVIEW** HAS BEEN APPROVED FOR

ALL AUDIENCES

BY THE MOTION PICTURE ASSOCIATION OF ~~AMERICA~~ ARC, INC.

Jupyter notebooks for reproducing animations can be found at:

github.com/navidcy/CLExWinterSchool2020