Layer approximation

Navid C. Constantinou

May 15, 2020

Abstract

We derive an optimal way to split a continuous stratified fluid into layers of constant density.

Discretization of a continuously stratified fluid into N layers

Consider the continuous density profile

$$\varrho(z) = \varrho_0 + (\varrho_\mathrm{b} - \varrho_0)(1 - \mathrm{e}^{\alpha z}) \quad \text{for } -H \le z \le 0\,. \tag{1}$$

Say that we want to best approximate the above density profile with N fluid layers. What is the best choice of fluid layer depths, $H_1, \dots, H_N$, given of course the constraint that $\sum_{j=1}^{N} H_j = H$?

A layer discretization $H_1, \dots, H_N$ implies that the fluid layers have mean densities:

$$\varrho_j = \frac{1}{H_j} \int_{-h_j}^{-h_{j-1}} \varrho(z)\,\mathrm{d}z\,, \tag{2}$$

where $h_j = \sum_{n=1}^{j} H_j$ and $h_0 \stackrel{\text{def}}{=} 0$. We want to choose H_j so that the squared difference from the continuous density profile is minimized. That is we want to minimize the cost function:

$$\begin{aligned}
\mathcal{C}(H_1, \dots, H_N) &= \sum_{j=1}^{N} \int_{-h_j}^{-h_{j-1}} [\varrho(z) - \varrho_j]^2\,\mathrm{d}z & (3)\\
&= \sum_{j=1}^{N} \int_{-h_j}^{-h_{j-1}} \varrho(z)^2\,\mathrm{d}z + \sum_{j=1}^{N} H_j \varrho_j^2 - 2\sum_{j=1}^{N} \varrho_j \int_{-h_j}^{-h_{j-1}} \varrho(z)\,\mathrm{d}z & (4)\\
&= \underbrace{\int_{-H}^{0} \varrho(z)^2\,\mathrm{d}z}_{\text{independent of } H_j} - \sum_{j=1}^{N} H_j \varrho_j^2\,. & (5)
\end{aligned}$$

To do that, we demand that

$$\frac{\partial \mathcal{C}}{\partial H_j} = 0\,. \tag{6}$$

We can use the constraint $H = \sum_{j=1}^{N} H_j$ to eliminate one of the depths, e.g., $H_N = H - \sum_{j=1}^{N-1}$. Thus, for an N-layer discretization we are get $N-1$ equations of the form of (6) that allow us to determine the depths $H_1, \dots, H_{N-1}$.

An example with 2 layers

Pick: $1/\alpha = 703.735\,\mathrm{m}$, $H = 4\,\mathrm{km}$, and also $\varrho_0 = 1026.94\,\mathrm{kg\,m^{-3}}$, $\varrho_\mathrm{b} = 1028.01\,\mathrm{kg\,m^{-3}}$.

For two layers we only need to choose H_1 so that $\partial\mathcal{C}/\partial H_1 = 0$, i.e.:

$$\frac{\partial}{\partial H_1}\left[H_1\varrho_1^2 + (H - H_1)]\varrho_2^2\right] = 0\,. \tag{7}$$

Using the numbers above we get that (7) is solved for $H_1 = 750\,\mathrm{m}$, which further implies that $\varrho_1 = 1027.35\,\mathrm{kg\,m^{-3}}$ and $\varrho_1 = 1028\,\mathrm{kg\,m^{-3}}$.