MTH 102: Probability and Statistics

Quiz 4

21/04/2020

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Question 1. A radio receives three independent signals W, X, and Y. The signal W is Gaussian distributed with mean 0 and variance 1. The signal X is Gaussian distributed with mean 1 and variance 4. The signal Y is a Bernoulli random variable with parameter $p = 0.5$. The total power is approximated by $Z = W^2 + X^2 + Y^2$. We are interested in $E[Z]$, $E[Z^2]$, and $\text{Var}[Z]$. Do the following.

1) Using information given about W, X, Y and Z calculate as many of the three above stated moments of Z as you can. State the moments you can calculate and your answers for the same in the provided text box. Provide detailed steps that led to the answer later in the upload.
2) Use the moments you were able to calculate to find as good an upper bound as you can for the probability $P[Z > 2]$. In the text box state the inequality you used and the value of the upper bound. Provide detailed steps that led to the answer later in the upload.

Question 2. You would like to come up with a relative frequency estimate R for whether a randomly selected person is infected by the Novel Coronavirus. You would like to design an experiment such that with probability at least as large as $1 - 10^{-6}$ the relative frequency estimate is within $\pm 10^{-6}$ of the true unknown probability r that a person is infected by the virus. Do the following.

1) You must decide on the number of tests n of randomly chosen people you plan to conduct. Provide the range of values that n may take to satisfy the above mentioned conditions. In the text box state the range. Provide detailed steps that led to the answer later in the upload.
2) For any given value of n chosen from the desired range calculate the resulting mean squared error. In the text box state the choice of n and the mean squared error. Provide detailed steps that led to the answer later in the upload.

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$$E[Z] = E[W^2] + E[X^2] + E[Y^2]$$

$$E[W^2] = \text{Var}[W] + (E[W])^2 = 1 + 0 = 1.$$

$$E[X^2] = \text{Var}[X] + (E[X])^2 = 4 + (1)^2 = 5.$$

$$E[Y^2] = \text{Var}[Y] + (E[Y])^2 = (0.5)(1-0.5) + (0.5)^2$$
$$= 0.5.$$

$$E[Z] = 1 + 5 + 0.5 = 6.5$$

Since we are given that W, X, Y are independent RV(s), we know that W^2, X^2 and Y^2 are independent.

$$\text{Var}[Z] = \text{Var}[W^2 + X^2 + Y^2] = \text{Var}[W^2] + \text{Var}[X^2] + \text{Var}[Y^2]$$

$$E[Z^2] = \text{Var}[Z^2] + (E[Z^2])^2$$
$$= \text{Var}[W^2] + \text{Var}[X^2] + \text{Var}[Y^2]$$
$$+ 2E[W^2]E[X^2] + 2E[W^2]E[Y^2] + 2E[X^2]E[Y^2]$$

We can't calculate these using just the given information, as we don't have $E[W^4]$, $E[X^4]$, $E[Y^4]$.
Note that: $\text{Var}[W^2] = E[W^4] - (E[W^2])^2$ and so on. While the Bernoulli PMF is straightforward and allows easy calculation of $E[Y^4]$, I didn't assume knowledge of the Gaussian PDF/MGF.

Some students used the Gaussian MGF, which is of course fine, but more than what was expected in the question. The next page has the soln and a different rubric.

For students who calculated $E[Z]$ and showed that $E[Z^2]$ and $\text{Var}[Z]$ can't be calculated (as shown above):

Part (1): Total 20 marks — 10 marks for calculation of $E[Z]$; 10 for showing the expressions of $E[Z^2]$ and $\text{Var}[Z]$ to argue that they can't be calculated.

Part (2): 30 marks

(2). Since we know only $E[Z]$, we must use the Markov Inequality.

$$P[Z > 2] \leq \frac{E[Z]}{2} = 3.25.$$

Clearly, a useless bound, as we know that any probability ≤ 1.

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$$E[Z] = E[W^2] + E[X^2] + E[Y^2]$$

$$E[W^2] = \mathrm{Var}[W] + (E[W])^2 = 1 + 0 = 1.$$

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$$E[Y^2] = \mathrm{Var}[Y] + (E[Y])^2 = (0.5)(1-0.5) + (0.5)^2 = 0.5.$$

$$E[Z] = 1 + 5 + 0.5 = 6.5$$

Now suppose, you decided to partake in the adventure that is calculating $E[W^4]$, $E[X^4]$, $E[Y^4]$.

$$E[Y^4] = 0.5\,(1)^4 + 0.5\,(0)^4 = 0.5.$$

The MGF of $K \sim N(\mu, \sigma)$ is $e^{s\mu + s^2\sigma^2/2}$

$$E[K^4] = 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4$$

[You may have used this result without showing it or you may have derived it. Either is fine]

$$\therefore E[W^4] = 3.$$

$$E[X^4] = 3(2)^4 + 6(1)(4) + (1)^4 = 48 + 24 + 1 = 73.$$

Part 1: $E[Z] \to 10$

$E[Z^2]$, $\mathrm{Var}[Z] \to 20$.

Part 2: 20.

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1) You must decide on the number of tests n of randomly chosen people you plan to conduct. Provide the range of values that n may take to satisfy the above mentioned conditions. In the text box state the range. Provide detailed steps that led to the answer later in the upload.
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(1) 25/50 (2) 25/50.

We desire $P[|R-r| \le 10^{-6}] \ge 1-10^{-6}$

Note that whether a person is infected can be modelled as a Bernoulli random variable with parameter r, where r is the probability that a person is infected.

R, the relative freq estimate, is simply the sample mean

$$R = M_n(X) = \frac{X_1 + X_2 + \dots + X_n}{n}$$

where $X_i \sim \text{Bernoulli}(r)$, X_i, for all i, are mutually independent. Note $E[R] = r$.

$$P[|R-r| \le 10^{-6}] = P[|R-E[R]| \le 10^{-6}]$$

$$\ge 1 - \frac{Var[R]}{(10^{-6})^2}$$

$$= 1 - \frac{Var[X]}{n(10^{-6})^2}$$

We require $1 - \frac{Var[X]}{n(10^{-6})^2} = 1 - 10^{-6}$

$$\therefore \frac{Var[X]}{n(10^{-6})^2} = 10^{-6}$$

The value of n obtained is the smallest desired value.

$$\therefore \text{We require } n \ge \frac{Var[X]}{(10^{-6})^3}$$

The largest value of $Var[X] = (0.5)(1-0.5)$
$= 0.25$.

Therefore, since we would like our selection of n to work for any Bernoulli RV, we require

$$n \ge \frac{(0.25)}{(10^{-6})^3} = \frac{10^{+18}}{4}$$

(2) The MSE is

$$E[(R-r)^2] = E[(R-E[R])^2],$$

since $E[R] = r$ as R is the sample mean.

$$\therefore MSE = Var[R] = \frac{Var[X]}{n} = \frac{0.25}{(10^{18})/4} = \frac{1}{10^{18}}$$

$$= 10^{-18}$$

[This for $n = (10^{18})/4$]