MTH 102: Probability and Statistics

Homework

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Question 1. There are two Bernoulli random variables X and Y. Their PMF is such that they take the value 1 with probability 0.5 and the value 0 with probability 0.5. While we don't know much about them, it is known that X^2 and Y^3 are independent. Calculate $E[(X+Y)^2]$. How would you modify your answer were you told that

(a) $P[X=1, Y=1] = 0$.

(b) Instead, of the above, you were told that $P[X=1, Y=1] = 0.25$.

Question 2. We have two random variables X and Y. Their joint PDF is given by

$$f_{X,Y}(x,y) = \begin{cases} cxy & 0 \le y \le x \le 1, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where c is a normalizing constant. Answer the following questions.

(a) Calculate c.

(b) Calculate the marginal PMFs of X and Y.

(c) Are X and Y independent? Support your answer using the definition of independence.

(d) Calculate $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

(e) Define an alternate joint PMF for the marginal PMFs you calculated earlier.

Question 3. You arrive at a sweet bot to find one jalebi and three donuts in its display. The bot chooses a sweet uniformly and randomly from all available sweets and gives the chosen sweet to you. Soon after you receive your sweet another customer arrives at the bot. The bot chooses a sweet, as described above, from those remaining and gives it to the customer. What is the probability that you both get the same type of sweet?

Let random variable X and Y respectively be the type of sweet that you and the other customer get. Are X and Y independent? Support your answer using the definition of independence of random variables.

Let C be the difference in the numbers of donuts and jalebi left after both you and the other customer have received your sweets. Calculate $E[C]$ and $Var[C]$.

Now assume that the bot replaces the sweet you received before the other customer arrives. Thus the other customer sees the same sweets as you in the bot's display. Calculate $E[C]$ and $Var[C]$.

Question 4. The length of a radio advertisement is uniformly distributed over the continuum $(0, 1)$ minutes. What is the probability that an advertisement is exactly 0.5 minutes long? Calculate the probability that an advertisement that has already run for 0.5 minutes will not exceed 0.8 minutes in length.

Question 5. You add two independent Gaussian noise sources. One of them is zero mean and has unit standard deviation. The other has unit mean and a standard deviation of 4. What is the probability that the result of the addition is greater than 1? Explain your answer.

Calculate the probability that the addition results in a positive value. Leave your answer in terms of the CCDF of the standard Gaussian.

Question 6. You are given two independent sources A and B that generate bits (0 or 1). Source A generates a 0 with probability 0.6 and source B generates a 0 with probability 0.3. The generated bits are sent through a logical AND operator whose output may be observed. Consider an experiment in which sources A and B generate 10 bits each and we observe the corresponding outputs of the AND operator. Answer the following questions.

(a) Calculate the expected value of the sum of bits generated by source A.
(b) Calculate the expected value of the sum of bits obtained at the output of the AND operator.
(c) Calculate the probability that the sum of bits generated by A is greater than the sum of bits generated by B.
(d) Suppose the first five outputs of the AND gate are 0. Calculate the probability that the sum of all 10 outputs of the AND operator is greater than or equal to 5.

Question 7. Two employees A and B man the counter of IIIT's favorite cafe. Often they take breaks. Employee A's break has the following probabilistic description. A's break begins with a coffee break of length that is well described by a continuous uniform random variable that takes values over the continuum $(0, 1)$ minutes. The coffee break is followed by an exponentially distributed time independent of the coffee break during which A plays badminton. This play time is well described by an exponential distribution with mean 1 minutes.

Employee B goes on a break during which B either watches a YouTube video or reads his twitter feed. B chooses one of the two activities without any bias. Watching a video takes a time that is described by a continuous uniform random variable over $(0, 1)$ minutes. Reading the twitter feed takes time that is best described by an exponential random variable with mean 1 minutes. Answer the following questions. [Hint: An exponentially distributed X has the PDF $f_X(x) = \mu e^{-\mu x}, x \geq 0$ and its expected value is $1/\mu$.]

(a) Calculate the probability that A takes a break that is longer than 1 minutes.
(b) Calculate the probability that B takes a break longer than 1 minutes.
(c) Calculate the expected value of the time taken by A.
(d) Calculate the expected value of the time taken by B.
(e) Suppose both A and B go on their break at the same time. Calculate the probability that A returns from his break after B. Assume that the lengths of their breaks are independent of each other.