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# Discrete Structures

## Midsem - 2

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1.(i) $f(\phi) = \phi$

$\phi \Rightarrow$ no elements in the subset.

$f(\phi)$ will also have no elements

Same holds for $f^{-1}(\phi)$

$\therefore f^{-1}(\phi) = \phi$

(ii) Given: $S, T \subseteq B$

To prove: $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

Case 1: Let $x \in S \cap T$

$\therefore x \in S$ and $x \in T$

$f^{-1}(x) \in f^{-1}(S)$ and $f^{-1}(x) \in f^{-1}(T)$

But our assumption was $f^{-1}(x) \in f^{-1}(S \cap T)$

Since $x$ is arbitrary,

$f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$

Case 2: Let $x \in S$ and $x \in T$

$\therefore x \in S \cap T$

$f^{-1}(x) \in f^{-1}(S \cap T)$

but $f^{-1}(x) \subseteq f^{-1}(S)$ and $f^{-1}(x) \subseteq f^{-1}(T)$

$\therefore$ Since $x$ is any arbitrary element in S and T.

From both the cases we conclude,

$f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

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(iii) Let $y \in f(U \wedge V)$

$\therefore \exists\, x \in U \cap V$ such that $f(x) = y$

so $y = f(x) \in f(U)$ and $f(x) \in f(V)$

$\therefore y \in f(U) \cap f(V)$

$\therefore f(U \cap V) \in f(U) \cap f(V)$ — ①

Let $y \in f(U) \cap f(V)$

$\therefore y \in f(U)$ and $y \in f(V)$

WLOG, let $y \in f(U)$

$\therefore x \in U$ such that $y = f(x)$

Now $U \subseteq U \cap V$ so $x \in U \cap V$

so $y = f(x) \in f(U \wedge V)$

$\therefore f(U) \cap f(V) \subseteq f(U \cap V)$ — ②

From ① & ②

$f(U \wedge V) = f(U) \cap f(V)$

Q2. (i) $f(f^{-1}(T)) = T$

We DISAPPROVE

let $f : \mathbb{N} \to \mathbb{Z}$

$A = \mathbb{N}, \quad B = \mathbb{Z}$

let $f(x) = 2x$

$f^{-1}(x) = \frac{x}{2}$

let $t = 3$ where $t \in T \subseteq \mathbb{Z}$

$f^{-1}(t) = t/2$

$= 3/2 \notin T$

$\notin \mathbb{Z}$

$f^{-1}(t)$ doesn't exist

$\therefore f(f^{-1}(t)) \neq t$ as $f^{-1}(t)$ doesn't exist

because function is not onto / surjective.

→ If we assume that $f : A \to B$ is surjective then there must be an inverse image $s \in S$ of $f^{-1}(t)$, $t \in T$ for which $f^{-1}(t) = s$.

Now $f(s) = t$ and since

$f(f^{-1}(t)) = f(s) = t$

$\therefore f(f^{-1}(T)) = T$

(ii) $f^{-1}(f(S)) = S$

We DISAPPROVE

Let $f : \mathbb{Z} \to \mathbb{N}$

$A = \mathbb{Z}$, $B = \mathbb{N}$

$f(x) = |x|$

let $s_1 = 2$, $s_1 \in S \subseteq \mathbb{Z}$

let $s_2 = -2$, $s_2 \in S \subseteq \mathbb{Z}$

$f(s_1) = |2| = 2$

$f(s_2) = |-2| = 2$

Now $f^{-1}(f(s_2)) = f^{-1}(2)$

$f^{-1}(2)$ was 2 inverse images

Hence $f^{-1}(f(S)) \neq S$ as the function is not one to one.

$\Rightarrow$ Now if we assume $f : A \to B$ is an injective function then for every $s \in S$ $\exists$ $t \in T$ such that

$f(s) = t$

$f^{-1}(f(s)) = f^{-1}(t) = s$

as the function is injective

Hence

$f^{-1}(f(S)) = S$

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3. We have been given that $a, b$ are positive numbers
We need to prove $(a+b)^n \geq a^n + b^n$ for $\forall\, n \in \mathbb{N}$
We use the Principle of Induction.

<u>Base case</u> : $n = 1$

$(a+b)^1 = a^1 + b^1$

$a + b = a + b$

LHS = RHS

Hence proved.

Induction hypothesis
Let $(a+b)^k \geq a^k + b^k$ be true for $k \in \mathbb{N}$

Induction Step :
We need to show $(a+b)^{k+1} \geq a^{k+1} + b^{k+1}$

$(a+b)^{k+1} = (a+b)^k (a+b)$
$= (a^k + b^k)(a+b)$

(Assuming equality from induction hypothesis step)

$(a+b)^{k+1} = a^{k+1} + b^{k+1} + ab^k + ba^k$

$a^{k+1} + b^{k+1} + ab^k + ba^k \geq a^{k+1} + b^{k+1}$

$\Rightarrow (a+b)^{k+1} \geq a^{k+1} + b^{k+1}$

Hence $P(n)$ is true for $n = k+1$

$\Rightarrow$ By Principle of Induction, the above inequality is true for all positive integers $\in \mathbb{N}$.

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4. X is a non-empty set (Given)

R is a relation, ie equivalence and ordering

An equivalence relation is symmetric, transitive and reflexive.

An ordering relation is transitive, anti-symmetric and reflexive

A relation R is transitive for a, b, c $\in X$ if a is related to b, b is related to c and c is related to a.

Relation R on X is symmetric if and only if an element a is related to b and b is also related to a $\forall$ a, b $\in X$

Relation R is antisymmetric if $\forall$ $x, y \in X$, $xRy$ and $yRx$ implies $x = y$

$$((x,y), (y,x) \in R) \Rightarrow (x = y)$$

Relation R is reflexive if $xRx$ for every $x \in X$.