Mid Sem Assignment

Full marks-24

October 17, 2020

1. Discuss the convergence and divergence of

$$x_n = \frac{[\alpha] + [2\alpha] + \cdots\cdots + [n\alpha]}{n^2}$$

where $[x]$ denotes the greatest integer less than or equal to the real number x, and α is an arbitrary real number. (6 marks)

2. Let $\{x_n\}$ be a real sequence which converges to l.
a) Show that the sequence

$$y_n = \frac{x_1 + x_2 + \cdots\cdots + x_n}{n}$$

also converges to l. (3 marks)
b) What about the converse? (1 mark)
c) As an application of this, show that if $\{x_n\}$ is such that $\lim_{n\to\infty}(x_{n+1} - x_n) = l$, then

$$\lim_{n\to\infty} \frac{x_n}{n} = l \qquad \text{(3 marks)}$$

3. a) Show that there does not exist a function $f : [0,1] \to \mathbb{R}$ which is differentiable on $(0,1)$ such that

$$f'(x) = \begin{cases} 0 & \text{if } 0 < x < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq x < 1. \end{cases} \qquad \text{(3 marks)}$$

b) Show that the following inequality $\log(1+x) \leq x \ \forall x \geq 0$.
(3 marks)

4. a) Test the uniform continuity of $f(x) = x \sin x$ for $x \in [0, \infty)$. (2.5 marks).

   b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic, then it attains its supremum and infimum. (3.5 marks)