INDRAPRASTHA INSTITUTE OF INFORMATION TECHNOLOGY, DELHI

MONSONON 2020

End Semester Examination

Course name: Real Analysis I Course Code: MTH 240 **Total marks: 36**

Date: December 15, 2020 **Time: 9:00-12:00**

1 Single Variable Calculus

Answer any 2 questions from below (marks $\mathbf{2} \times \mathbf{7} = \mathbf{14}$).

1. a) Show that the equation $x^2 - 10 = x \sin x$ has a real solution. **(3 marks)**
   b) Let $f(x) = x^{1/3}$ for $x \in \mathbb{R}$ and use the definition of derivative to prove $f'(x) = \frac{1}{3}x^{-2/3}$ for $x \neq 0$. **(3 marks)**
   c) Find the limit of $\lim_{x\to 0}(\frac{1}{\sin x} - \frac{1}{x})$. **(1 mark)**

2. a) Let f be a differentiable function defined on an interval with bounded derivative, then f is uniformly continuous. Is the converse true ? **(3 + 1 marks)**
   b) Is the given function uniformly continuous $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x+1} \cos x^2$? **(3 marks)**

3. a) Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = f(x)g(x)$ where f and g are non-negative functions. Show that h has a local maximum at a if f and g have a local maximum at a. Is the converse of the above true? **(3 + 1 marks)**
   c) Find the points of local maximum and local minimum of $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x^4 e^{-x^2}$. **(3 marks)**

Answer any 1 of the following (marks $\mathbf{1} \times \mathbf{8} = \mathbf{8}$).

1. Let f be a continuous function.
   a) Find $f(4)$ if $\int_0^{x^2} f(t)dt = x \sin x\pi$ for all x. **(4 marks)**
   b) For what values of p the integral $\int_0^\infty \frac{t^{p-1}}{t+1}\, dt$ converges. **(4 marks)**

2. a) Suppose that f is a continuous function satisfying

$$f(x) = x \int_0^x f(t)dt + x^3$$

   for all x, and c is a real number such that $f(c) = 1$. Express $f'(c)$ in terms of c only. **(4 marks)**
   b) Evaluate the limit of $\lim_{n\to\infty} \left(n \left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+3)^2} + \text{.........} + \frac{1}{(4n-1)^2} \right) \right)$ **(4 marks).**

2 Multivariable Calculus

Answer any 2 questions from below (marks $\mathbf{2} \times \mathbf{7} = \mathbf{14}$).

1. Determine all values of the positive constant k for which the limit $\lim_{(x,y)\to(0,0)} \frac{|x|}{(x^2+y^2)^k}$ exists.

2. Assume that z and w are differentiable functions of x and y satisfying the equations $xw^3 + yz^2 + z^3 = -1$ and $zw^3 - xz^3 + y^2 w = 1$. Find $\frac{\partial z}{\partial x}$ at (x, y, z, w) at $(1, -1, -1, 1)$.

3. Check whether the following function is differentiable at $(0, 0)$ $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{2x^2 y}{\sqrt{x^2+y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$