Q1 (i) $\forall x (x < 0) \to \exists y \forall z (y > z)$

Our proposition is $\forall x (x < 0)$ and our conclusion is $\exists y \forall z (y > z)$

$\forall x (x < 0)$ is ~~true~~ false, $x \in R$ as not all $x \in R$ is negative

$\exists y \forall z (y > z)$ is true, $y, z \in R$

Hence the implication is ~~false~~ true.

(ii) $\exists x (x < 0) \to \forall y \exists z (y > z)$

Proposition: $\exists x (x < 0)$ is true, $x \in \mathbb{R}$

Conclusion: $\forall y \exists z (y > z)$ is true, $y, z \in \mathbb{R}$

Hence the implication is true.

(iii) $\forall x \exists y (x \leq 0 \wedge xy < 0)$

First part of the 'and' statement is true ie $x \leq 0$.

However for $x = 0$ the second part becomes false $xy \not< 0$

and thus the statement, $\forall x \exists y (x \leq 0 \wedge xy < 0)$ becomes false for $x = 0$

Truth value = FALSE.

(iv) $\forall x \forall y (x \leq y \to \exists z (x \leq z \wedge z \leq y))$

$\forall x \forall y$; $x, y \in \mathbb{R}$

if $x \leq y$ then there can exist a $z \in \mathbb{R}$ such that

$x \leq z \wedge z \leq y$. (x, y) simply becomes the range for $z \in \mathbb{R}$.

Truth value = TRUE

Q2. Let $(x,y) \in A$ such that $(x,y) \in (R \circ S)^{-1}$.

Then we have

$$(y, x) \in R \circ S$$

So, there exists a $z \in A$ such that $(y,z) \in R$ and $(z,x) \in S$.

Now,

$$((y,z) \in R) \rightarrow ((z,y) \in R^{-1})$$

and

$$((z,x) \in S) \rightarrow ((x,z) \in S^{-1})$$

Then,

$(x,z) \in S^{-1}$ and $(z,y) \in R^{-1}$ implies that $(x,y) \in S^{-1} \circ R^{-1}$.

Thus $(R \circ S)^{-1} \subseteq S^{-1} \circ R^{-1}$. —— ①

Conversely,

let $(x,y) \in A$ such that $(x,y) \in S^{-1} \circ R^{-1}$

Then there exists $z \in A$ such that $(x,z) \in S^{-1}$ and $(z,y) \in R^{-1}$.

Now, $(x,z) \in S^{-1}$ implies $(z,x) \in S$ and $(z,y) \in R^{-1}$ implies that $(y,z) \in R$.

Then, $(y,z) \in R$ and $(z,x) \in S$ implies $(y,x) \in R \circ S$.

So, $(x,y) \in (R \circ S)^{-1}$

Thus $S^{-1} \circ R^{-1} \subseteq (R \circ S)^{-1}$ —— ②

From ① & ②:

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

Hence proved

Q3. If $X = \phi$, then the empty function is the only function from X to X.

The empty function is injective, since if it is not then

$\exists (x_1 \neq x_2) \in X$ such that $f(x_1) = f(x_2)$

But that is not possible for $X = \phi$.

The empty function is surjective, since if it is not then $\exists y \in X$ such that $y \neq f(x)$ $\forall$ $x \in X$. But X is empty so there cannot exist such a y.

Thus, the only function from X to X where $X = \phi$ is injective and surjective. Hence a function from $X = \phi$ is injective iff it is surjective.

Now, suppose $X \neq \phi$.

If $X = \{x\}$ then the only injective function from X to X is the identity function that maps x to itself.

This is also the only surjective function.

Hence a function from X to itself, where $|X| = 1$ is injective iff it is surjective.

$\Rightarrow$ $|X| \geq 2$, $X = \{x_1 \dots x_n\}$

Let $f: X \to X$ be injective, To Prove: f is surjective

Proof by contradiction: f is not surjective.

Then there exists $x_i \in X$, $1 \leq i \leq n$ such that $f(x_j) \neq x_i$ for all $1 \leq j \leq n$.

f maps the n distinct elements of X into the $n-1$ distinct elements in $X \setminus \{x_i\}$. So atleast 2 elements in X must be mapped to one element of $X \setminus \{x_i\}$. So f is not injective.

This is a contradiction.

Hence f is surjective.

Converse :

$f: X \to X$ is surjective. , To prove : f is injective

Proof by Contradiction : f is not injective.

Then there exists $x_i, x_j \in X$ where $1 \leq i < j \leq n$ such that

$f(x_i) = f(x_j)$.

Let $f(x_i) = f(x_j) = x_k \in X$, $1 \leq k \leq n$.

Since f is a function the pre-images of distinct images are distinct, $f^{-1}(\{x_s\}) \cap f^{-1}(\{x_t\}) = \phi$, $1 \leq s \neq t \leq n$.

Since f is surjective, each element of X has atleast one pre-image in X.

Now, since x_k has atleast 2 pre-image in X, this implies that there are atmost $n-2$ distinct pre-image available for the $n-1$ elements in $X \setminus \{x_k\}$. This implies that not all elements in $X \setminus \{x_k\}$ can have distinct pre-image.

This is a contradiction that f is a function.

Hence f must be injective.

Thus, for any finite X, a function $f: X \to X$ is injective iff it is surjective.

$\Rightarrow$ Suppose X is an infinite set. Then there is a countably infinite subset $\{d_1, d_2 \ldots\}$ of X

Let $f_1, f_2 : X \to X$

$f_1(x) = \begin{cases} d_{i+1} , & x = d_i \text{ for some } i \\ x , & \text{otherwise} \end{cases}$; $f_2(x) = \begin{cases} d_{i-1} , & x = d_i \text{ for some } i > 1 \\ x , & \text{otherwise} \end{cases}$

f_1 is an injection , f_2 is a surjection

but neither is a bijection. ie, injection and surjection.

Q4 We know

$$\binom{p}{k} = \frac{\prod_{i=0}^{k-1}(p-i)}{k!}$$

Since $k \geq 1$, $\prod_{i=0}^{k-1}(p-i) \neq 1$. Thus $p = (p-0)$ is a factor in the product in the numerator.

Now, $\binom{p}{k}$ is the number of k-element subsets of a p-element set, hence it is a natural number.

Thus $k!$ divides $\prod_{i=0}^{k-1}(p-i)$.

Since p is prime and $k < p$, $k, k-1, \ldots, 3, 2$ do not divide p. So $k!$ must divide $\prod_{i=0}^{k-1}(p-i)$

Let $\frac{\prod_{i=0}^{k-1}(p-i)}{k!} = l \in \mathbb{Z}^+$. Thus $\binom{p}{k} = p \cdot l$.

Thus $\binom{p}{k}$ is divisible by p.

Q5

$x^2(y^2)^4 z = x^2 y^8 z$

In $(2x + y^2 - 5z)^7$ we have

$$\frac{7!}{2!\,4!\,1!} = 105$$

So $x^2(y^2)^4 z$ term is

$105(2x)^2(y^2)^4(-5z) = (105)(4)(-5)\,x^2y^8z$

$= -2100\,x^2y^8z$

Hence coefficient is $\underline{\underline{(-2100)}}$.

Q6. Yes, it is true that if 2 graphs are isomorphic then there exists a bijection $f: E(G) \to E(H)$ where G and H are the 2 graphs.

For two isomorphic graphs we know that the number of edges and vertices are the same.

$$|E(G)| = |E(H)| \quad , \quad |V(G)| = |V(H)|$$

→ For every edge in G, there is a corresponding edge in H.
Hence $f: E(G) \longrightarrow E(H)$ is injective

→ For every edge in H, we can find the corresponding edge in G, hence every element in E(H) has a pre-image in E(G).
Thus $f: E(G) \to E(H)$ is surjective.

As $f: E(G) \to E(H)$ is injective and surjective, it is bijective.

Moreover,
2 graphs are isomorphic if a bijection $f: V(G) \to V(H)$ exists such that $\{x, y\} \in E(G)$ iff $\{f(x), f(y)\} \in E(H)$ holds for all $x, y \in V(G)$, $x \neq y$.

Q7 We will first note the sum of the degree:

$1 + 2 + 2 + 3 = \underbrace{8}_{\text{even}}$,

We reduce the sequence by a repeated use of the Score's Theorem.

Sequence $(1, 2, 2, 3) = D$

Here $n = 4$, $d_n = 3$, $n - d_n = 1$

$D' = (0, 1, 1)$

Again, sequence $(0, 1, 1)$

$n = 3$, $d_n = 1$, $n - d_n = 2$

$D'' = (0, 0)$

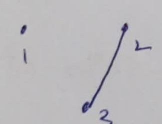
D'' is a graph score with, G_1 = 1 2 is a graph

The given sequence must be a graph score by the Score Theorem.

Now, to find a graph with score D we reduce our steps from G_1 using the scores obtained.

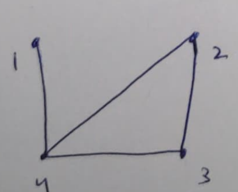
G_1 = 

(Score $= D''$)

G_2 = 

(Score $= D'$)

$D' = (0, 1, 1)$ was obtained

G_3 = 

$(1, 2, 2, 3)$

→ Graph required.

Yes, by using the score theorem, we proved that the sequence can be drawn into a graph using score $(1, 2, 2, 3)$.