Homework 4
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2020223

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Problem 1:

x	-2	0	1	3
y	15	-1	0	-2

(i) Monomial Basis:

$$P_3(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$

$$\begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 15 \\ -1 \\ 0 \\ -2 \end{bmatrix}$$

Solving this system, we get:

$a_1 = -1$

$a_2 = -8/15$

$a_3 = 34/15$

$a_4 = -11/15$

so $P_3(x) = -1 - \frac{8}{15}x + \frac{34}{15}x^2 - \frac{11}{15}x^3$

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(ii) Lagrange Basis

$$P_3(x) = y_1\frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} + y_2\frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}$$

$$+ y_3\frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} + y_4\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$

Using the given points:

$$P_3(x) = \frac{\cancel{15}(x-0)(x-1)(x-3)}{(-2)(\cancel{-3})(\cancel{-5})} + \frac{\cancel{1}(x+2)(x-1)(x-3)}{(2)(\cancel{-1})(-3)}$$

$$+ 0 + \frac{-\cancel{7}(x+2)(x-0)(x-1)}{(5)(3)(\cancel{7})}$$

$$P_3(x) = \frac{x(x-1)(x-3)}{-2} + \frac{(x+2)(x-1)(x-3)}{-6} - \frac{(x+2)x(x-1)}{15}$$

$$= \frac{(x-1)}{30}(-22x^2 + 50x + 26)$$

$$= \frac{(x-1)}{15}(-11x^2 + 25x + 13)$$

$$P_3(x) = \frac{-11x^3}{15} + \frac{34}{15}x^2 - \frac{8}{15}x - 1$$

(iii) Newton Basis

$$P_3(x) = \sum_{i=1}^{n} a_i \prod_{j=1}^{i-1} (x - x_j)$$

$$P_3(x) = a_1 + a_2(x-x_1) + a_3(x-x_1)(x-x_2) + a_4(x-x_1)(x-x_2)(x-x_3)$$

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & x_2-x_1 & 0 & 0 \\ 1 & x_3-x_1 & (x_3-x_1)(x_3-x_2) & 0 \\ 1 & x_4-x_1 & (x_4-x_1)(x_4-x_2) & (x_4-x_1)(x_4-x_2)(x_4-x_3) \end{bmatrix}$$

$Na = y$ where

$a = [a_1 \ a_2 \ a_3 \ a_4]^T$

$y = [15 \ -1 \ 0 \ -2]^T$

Using the given points

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & 5 & 15 & 30 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 15 \\ -1 \\ 0 \\ -2 \end{bmatrix}$$

$a_1 = 15$, $a_2 = -8$, $a_3 = 3$, $a_4 = -11/15$

$$P_3(x) = 15 + (-8)(x+2) + 3(x+2)(x) - \frac{11}{15}(x+2)x(x-1)$$

$$P_3(x) = \frac{-11x^3}{15} + \frac{34}{15}x^2 - \frac{8x}{15} - 1$$

(iv) In all the above 3 parts we got the same equations

ie $P(x) = \frac{-11x^3}{15} + \frac{34x^2}{15} - \frac{8x}{15} - 1$

$\Rightarrow$ All the representations give the same polynomial.

Problem 2:

(a) $\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$, $f(x) = \frac{1}{1+x^2}$

(i) Midpoint Rule

$I(f) \approx \int_a^b f(t)\,dt = (b-a)\, f\left(\frac{a+b}{2}\right)$

$a = 0$, $b = 1$

$f(x) = \frac{1}{1+x^2}$

$\frac{a+b}{2} = \frac{1}{2}$

$f\left(\frac{a+b}{2}\right) = f\left(\frac{1}{2}\right) = \frac{1}{1+1/4} = \frac{1}{5/4}$

$f\left(\frac{1}{2}\right) = \frac{4}{5}$

$\frac{4}{5} = 0.8$ and $\frac{\pi}{4} = 0.78$

$b - a = 1$

By midpoint rule

$I(f) \approx 1 \cdot f\left(\frac{1}{2}\right) = \underline{\underline{0.8}}$

(ii) Trapezoid Rule

$I(f) \approx \left(\frac{b-a}{2}\right)(f(a) + f(b))$

$a = 0$, $b = 1$

$f(a) = f(0) = \frac{1}{1+0^2} = 1$

$f(b) = f(1) = \frac{1}{1+1} = \frac{1}{2}$

$I(f) = \frac{1}{2} \times \left(1 + \frac{1}{2}\right) = \frac{1}{2} \times \frac{3}{2}) = \frac{3}{4}$

$I(f) = 0.75$

According to trapezoid Rule

$I(f) \approx 0.75$

(iii) Simpson's Rule:

$I(f) \approx Q(f) = \left(\frac{b-a}{6}\right)\left(f(a) + 4\left(f\left(\frac{a+b}{2}\right)\right) + f(b)\right)$

$a = 0$, $b = 1$

$f(a) = f(0) = 1$

$f\left(\frac{a+b}{2}\right) = f\left(\frac{1}{2}\right) = \frac{4}{5}$

$f(b) = f(1) = \frac{1}{2}$

$Q(f) = \frac{1}{6}\left(1 + 4 \times \frac{4}{5} + \frac{1}{2}\right)$

$= \frac{1}{6} \times \frac{47}{10} = \frac{47}{60} = 0.78$

According Simpson's Rule

$I(f) \approx 0.78$

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(iv) 2 point Gaussian Rule

We know (from lecture) that

$I(f) \approx Q(f) = w_1 \cdot f(x_1) + w_2 f(x_2)$

$w_1 = 1, \; w_2 = 1$

$x_1 = -1/\sqrt{3}, \; x_2 = 1/\sqrt{3}$

$f(x_1) = \frac{3}{4}, \; f(x_2) = \frac{3}{4}$

$Q(f) = 1 \cdot \frac{3}{4} + 1 \cdot \frac{3}{4}$

$I(f) \approx Q(f) = \frac{6}{4} = 1.5$

According to 2 point Gaussian Rule

$I(f) \approx 1.5$

(b) $\int_0^1 \sqrt{x} \log x \, dx = \frac{-4}{9}$

$f(x) = \sqrt{x} \log x$

(i) Midpoint Rule:

$I(f) \approx \int_a^b f(t)\,dt = (b-a) f\left(\frac{a+b}{2}\right)$

$a = 0, b = 1$

$f\left(\frac{a+b}{2}\right) = f\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}} \log \frac{1}{2} = \frac{-1}{\sqrt{2}} \times 0.301$

$f\left(\frac{1}{2}\right) = -0.212$

$I(f) \approx -1 \times 0.212$

$\frac{-4}{9} = 0.44$

According to midpoint rule:

$I(f) \approx -0.212$

(ii) Trapezoid Rule:

$$I(f) \approx \left(\frac{b-a}{2}\right)\left(f(a)+f(b)\right)$$

$$a=0,\ b=1$$

$f(a) = f(0) =$ ~~ec~~ not defined.

$f(b) = f(1) = 1 \cdot \log 1 = 0$

According Trapezoid rule:

~~I(f)~~ $I(f)$ is not defined.

(iii) Simpson's Rule:

$$I(f) \approx \left(\frac{b-a}{6}\right)\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

$$a=0,\ b=1$$

$$f(a) = 0,\ f(b) = 0,\ f\left(\frac{a+b}{2}\right) = -0.212$$

$$I(f) \approx \frac{1}{\cancel{6}\,3}\left(\cancel{4}^{2} \times 0.212\right) = (-0.141)$$

According to Simpson's Rule:

$I(f) \approx -0.141$

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(iv) 2 point Gaussian Rule:

We know (from lectures) that:

$$I(f) \approx Q(f) = w_1 f(x_1) + w_2 f(x_2)$$

$$w_1 = 1, \ w_2 = 1$$

$$x_1 = \frac{-1}{\sqrt{3}}, \ x_2 = \frac{1}{\sqrt{3}}$$

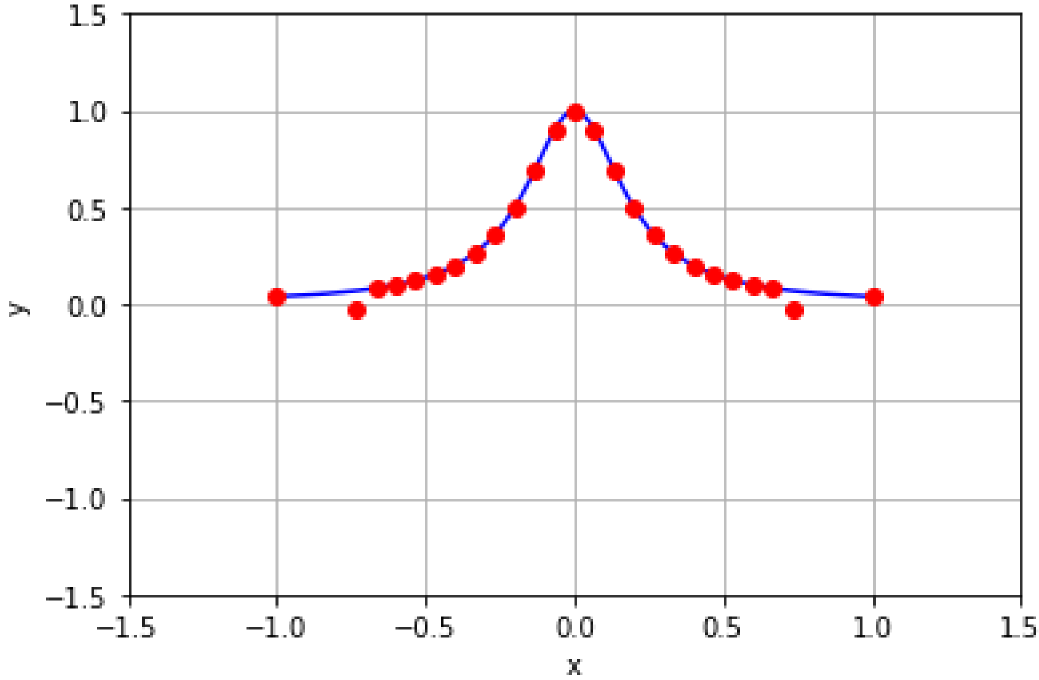
$$f\left(\frac{-1}{\sqrt{3}}\right) = \sqrt{\frac{-1}{\sqrt{3}}} \log \frac{-1}{3} = \text{not defined}$$

According to 2 point Gaussian rule

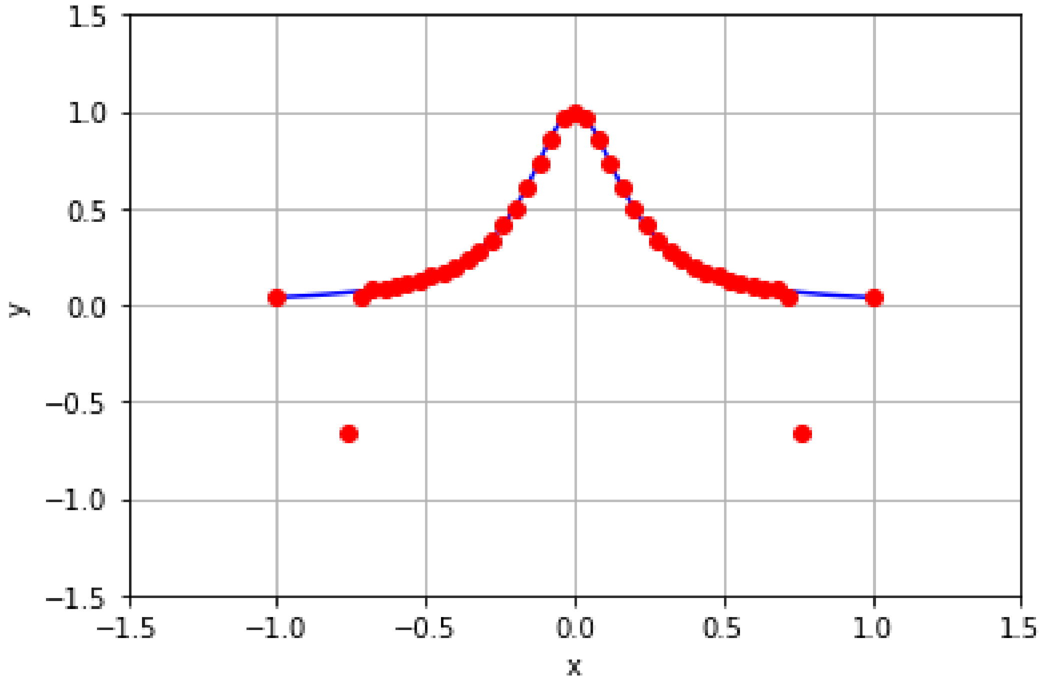
$I(f)$ is not defined for the given function.

Problem 3:

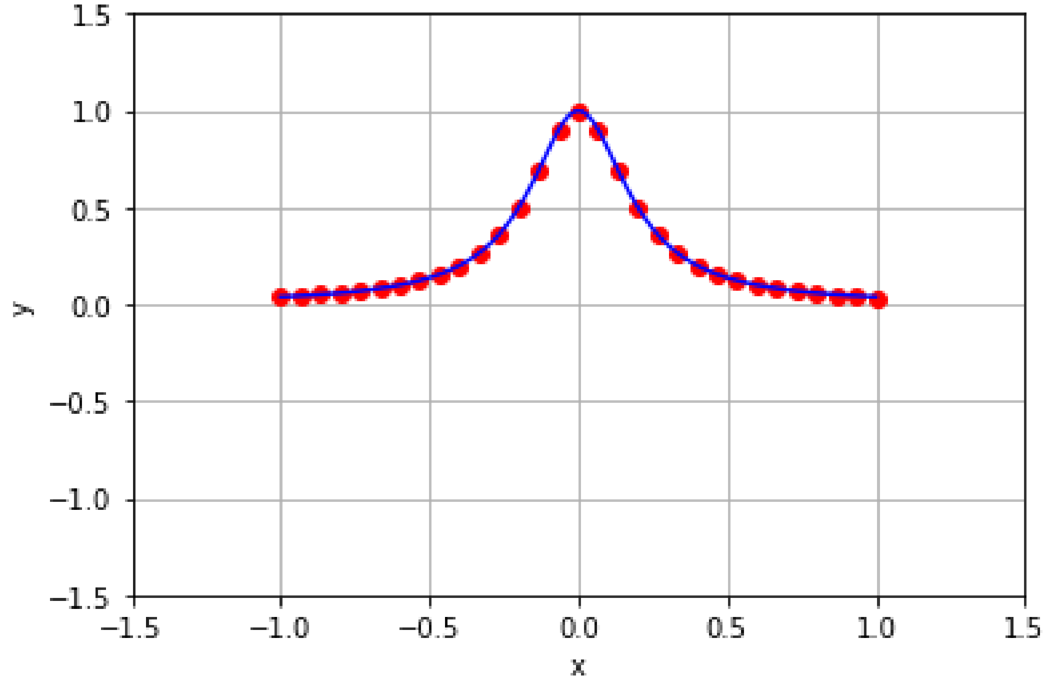
a) N=11:



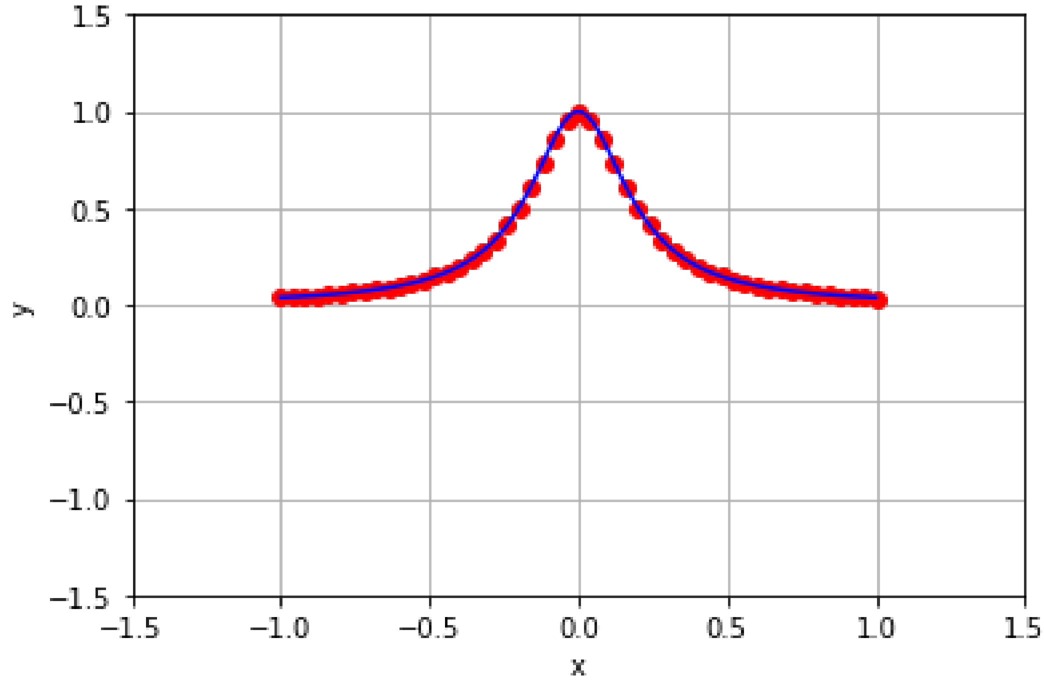
N=21:



b) N=11:


1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
y
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x

N=21:


1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
y
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x

Problem 4:
For n= 2
the value of the integral is 0
absolute error is 18.79829683678703
relative err is 1.0

For n= 4
the value of the integral is 24.943415371414666
absolute error is 43.7417122082017
relative err is 2.3268976220548896

For n= 6
the value of the integral is 19.2884807352198
absolute error is 38.08677757200683
relative err is 2.026075973940017

For n= 8
the value of the integral is 6.7521505349988615
absolute error is 25.550447371785893
relative err is 1.359189483686913

For n= 10
the value of the integral is -2.034625168477337
absolute error is 16.763671668309694
relative err is 0.8917654516181642

For n= 12
the value of the integral is -7.473353127804982
absolute error is 11.32494370898205
relative err is 0.602445200610934

For n= 14
the value of the integral is -10.827423084569027
absolute error is 7.970873752218004
relative err is 0.4240210600685658

For n= 16
the value of the integral is -12.951672112370545
absolute error is 5.846624724416486
relative err is 0.31101885320669187

For n= 18
the value of the integral is -14.344148922195334
absolute error is 4.454147914591697
relative err is 0.2369442270895107

For n= 20
the value of the integral is -15.289490940771019
absolute error is 3.5088058960160122
relative err is 0.18665552132092678

For n= 22
the value of the integral is -15.95291471812981
absolute error is 2.845382118657217
relative err is 0.15136382531682316

For n= 24
the value of the integral is -16.432891664684902
absolute error is 2.365405172102129
relative err is 0.12583082353892755

For n= 26
the value of the integral is -16.789861585609408
absolute error is 2.008435251177623
relative err is 0.10684134145851168

For n= 28
the value of the integral is -17.06202025031144
absolute error is 1.73627658647559
relative err is 0.09236350513828523

For n= 30
the value of the integral is -17.27418650351465
absolute error is 1.5241103332723824
relative err is 0.08107704365481658

For n= 32
the value of the integral is -17.44291140985203
absolute error is 1.3553854269350012
relative err is 0.07210150146595201

For n= 34
the value of the integral is -17.57950098980332
absolute error is 1.21879584698371
relative err is 0.0648354400170236

For n= 36
the value of the integral is -17.691851431748507
absolute error is 1.1064454050385244
relative err is 0.058858811234073265

For n= 38
the value of the integral is -17.785591126347146
absolute error is 1.012705710439885
relative err is 0.05387220551055916

For n= 40
the value of the integral is -17.864808021010987
absolute error is 0.9334888157760446
relative err is 0.0496581591343567

For n= 42

the value of the integral is -17.93252302218363
absolute error is 0.8657738146034006
relative err is 0.046055971033989535

For n= 44
the value of the integral is -17.991004401466714
absolute error is 0.807292435320317
relative err is 0.042944977533310294

For n= 46
the value of the integral is -18.04198057106806
absolute error is 0.7563162657189721
relative err is 0.04023323348309465

For n= 48
the value of the integral is -18.086786624128784
absolute error is 0.7115102126582471
relative err is 0.037849716856575455

For n= 50
the value of the integral is -18.126466918065027
absolute error is 0.671829918722004
relative err is 0.0357388716943376

For n= 52
the value of the integral is -18.16184798707918
absolute error is 0.6364488497078504
relative err is 0.033856729427868264

For n= 54
the value of the integral is -18.193591106878582
absolute error is 0.6047057299084493
relative err is 0.032168112630559166

For n= 56
the value of the integral is -18.2222306965338
absolute error is 0.5760661402532321
relative err is 0.030644592180601626

For n= 58
the value of the integral is -18.24820272434079
absolute error is 0.5500941124462422
relative err is 0.02926297617397679

For n= 60
the value of the integral is -18.27186596591786
absolute error is 0.5264308708691701
relative err is 0.028004179072168893

For n= 62
the value of the integral is -18.293518088123456

absolute error is 0.5047787486635755
relative err is 0.026852366097111346

For n= 64
the value of the integral is -18.313407943975122
absolute error is 0.4848888928119095
relative err is 0.025794299186882388

Problem 5;

for h = 0.1 error is 0.06940588094341621
for h = 0.01 error is 0.005227002682469728
for h = 0.001 error is 0.0005069717636996818
for h = 0.0001 error is 5.0541282400395904e-05
for h = 1e-05 error is 5.052570714092486e-06
for h = 1e-06 error is 5.052526364512921e-07
for h = 1e-07 error is 4.939506254020287e-08
for h = 1e-08 error is 6.096364579821767e-09
for h = 1e-09 error is 4.941478665143606e-08
for h = 1e-10 error is 6.16075158110796e-08
for h = 1e-11 error is 1.1718305404362361e-06
for h = 1e-12 error is 5.433932069082159e-05
for h = 1e-13 error is 0.0002763839256158529
for h = 1e-14 error is 0.004717276024116479
for h = 1e-15 error is 0.11573957848663208

The error decreases, for h = 1e-08 error is 6.096364579821767e-09 which is close to 0 and then the error increases as shown below. It attains minimum value at h = 1e-08