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Q1 a) Let $g(x) = x - \frac{f(x)}{d}$

d is constant in place of $f'(x)$

Now, $g'(x) = 1 - \frac{f'(x)}{d}$

The convergent will be local if,

$|g'(x)| < 1$

$\left|1 - \frac{f'(x)}{d}\right| < 1$

$-1 < 1 - \frac{f'(x)}{d} < 1$

-1 from both sides

$-2 < -\frac{f'(x)}{d} < 0$

So, $\frac{f'(x)}{d} < 2$, $\frac{f'(x)}{d} > 0$

$$\boxed{0 < \frac{f'(x)}{d} < 2}$$

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(b) consider $f(x)=0$ and α as one of the solution.

$f(\alpha)=0$

If we do a small perturbation of α by ε

then, $x=\alpha+\varepsilon$

for all $i=0$ to $i=n$

$x_i=\alpha+\varepsilon_i$

Now we know,

$$x_{i+1}=x_i-\frac{f'(x_i)}{d} \quad (\text{for all from } i:1 \text{ --- } n)$$

$$\cancel{\alpha}+\varepsilon_{i+1}=\cancel{\alpha}+\varepsilon_i-\frac{f(\alpha+\varepsilon_i)}{d}$$

$$\varepsilon_{i+1}=\varepsilon_i-\frac{f(\alpha+\varepsilon_i)}{d}$$

Using Taylor series expansion,

$$\varepsilon_{i+1}=\varepsilon_i-\left(\cancelto{0}{f(\alpha)}+\frac{f'(\alpha)}{d}\times\varepsilon_i\right)$$

$\boxed{i=1 \text{ to } i=n}$

$$\varepsilon_{i+1}=\varepsilon_i-\varepsilon_i\left(\frac{f'(\alpha)}{d}\right)$$

$$\varepsilon_{i+1}=\varepsilon_i\left(1-\frac{f'(\alpha)}{d}\right) \quad \text{--- (1)}$$

We needed an equation of form $\varepsilon_{i+1}=c\,\varepsilon_i^{p}$ --- (2), where $c=1-\frac{f'(x)}{d}$, p = rate

can we got it.

from (1) and (2) we can deduce that

$\boxed{p=1}$ = rate

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c) We need an equation of the form

$$\epsilon_{i+1} = C\epsilon_i^{p}$$

where $p = 2$

$$\epsilon_{i+1} = C\epsilon_i^{2}$$

and $C = \frac{1 - g'(x)}{d}$

from (i) we have.

$$\epsilon_{i+1} = \epsilon_i \left(\frac{1 - g'(\alpha)}{a}\right)$$

for $p = 2$, we have to do

~~Sol. arrang ans.~~

$$c\epsilon_i^2 = \epsilon_i \left(\frac{1 - g'(\alpha)}{a}\right)$$

α is solution of $g(\alpha)$.

Not possible for ~~any~~ any value.

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Q4 (b)

(1) $x^2-1=0$, Roots: 1.0000---53
Convergence Rate ~ 2

(2) $(x-1)^7=0$, Roots: 1.00090407----
Convergence Rate ~ 1.5

(3) $x-\cos(x)=0$, Roots = 0.7390-----
Convergence Rate ~ 2

for (2) the actual root is 1, and the computed root is 1.000---53
x-prime = 1, c = convergence rate
So, the error is = 0.0000---53 (E_k)

So, it will not possible to omit the higher terms in the taylor expansion of f(x-prime + E_k) & f(x-prime + E_k), and the value got from it will be multiplied with E_k^c and that will be a much greater value, and hence the convergence Rate gets drop to ~ 1.5

for all the remaining parts, computing higher values in the taylor series is the reason for computed convergence rate.

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3(a) $F_n(t) = \cos(n\cos^{-1}(t))$

$F_0(t) = \cos(0) = 1$ — ①

$F_1(t) = \cos(\cos^{-1}(t))$
$= t$ — ②

$F_{n+1}(t) = \cos((n+1)\cos^{-1}t)$
$= \underbrace{\cos(n\cos^{-1}(t))}\underbrace{\cos(\cos^{-1}t)} - \sin(n\cos^{-1}(t))\sin(\cos^{-1}t)$

$= F_n(t) \times t - \sin(n\cos^{-1}(t))\sin(\cos^{-1}(t))$ — ⓐ

$F_{n-1}(t) = \cos((n-1)\cos^{-1}(t))$
$= \cos(n\cos^{-1}(t))\underbrace{\cos(\cos^{-1}(t))} + \sin(n\cos^{-1}(t)) \times \sin(\cos^{-1}(t))$

$= F_n(t) \times t + \sin(n\cos^{-1}(t)) \times \sin(\cos^{-1}(t))$ — ⓑ

a+b

$F_{n+1}(t) + F_{n-1}(t) = 2t F_n(t)$

$F_{n+1}(t) + F_{n-1}(t) = 2t F_n(t)$
$F_{n+1}(t) = 2t F_n(t) - F_{n-1}(t)$ — ③

from ①, ② & ③, we can say $F_n(t)$ satisfies the chebyshev three term recurrence relation.

3b we can prove $F_{n+1}(t)$ is a polynomial by induction.

Base case: $F_0(t)$ and $F_1(t)$ are polynomials in t.
(shown in 3a).)

Induction hypothesis: ~~~~ let $F_i(t)$ be polynomials for all $0 \le i \le n$.

Induction step:
From our I.H we can say ~~~~ $F_n(t)$ and $F_{n-1}(t)$ are polynomials. Also, $2tF_n(t)$ is also a polynomial in t, because we are just multiplying t to a polynomial only, ~~~~ now we can conclude that $2tF_n(t) - F_n(t)$ is also a polynomial as both of them individually are polynomials. So, as we proved $F_{n+1}(t) = 2tF_n(t) - F_n(t)$, ~~~~
we can say, $F_{n+1}(t)$ is a polynomial for all $n \ge 0$ by induction.

Q4 a) The interpolating polynomial can be given by

$$P_n(t) = a_0 + a_1(t-t_0) + a_2(t-t_0)(t-t_1) + \cdots + a_n(t-t_0)(t-t_1)\cdots(t-t_n)$$

t_i is interpolating node

First of all let's prove that a_k is a function of all the interpolating nodes, ie $a_k = f[t_0, t_1, \ldots, t_k]$

Proof by Induction

Base case $k=0$

For $t = t_0$

$P_0(t_0) = a_0$ (1)

Since P interpolates f in t_0,

$P(t_0) = f(t_0)$ (2)

from (1) & (2)

$a_0 = f[t_0]$

~~Induction Step~~:

Induction hypothesis:

Let $a_i = f[t_0, t_1, \ldots, t_i]$ $\forall i \le (k-1)$

Induction step:

To prove $a_k = f[t_0, t_1, \ldots t_k]$

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Let $P_k(t) = a_0 + a_1(t-t_0) + \cdots + a_k(t-t_0)(t-t_1)\cdots(t-t_{k-1})$

$P_k(t)$ is interpolating f on $t_0, \cdots t_k$

$f(t_k) = P_k(t_k)$

$f(t_k) = a_0 + a_1(t_k - t_0) + \cdots + a_k(t_k - t_0) \times \cdots (t_k - t_{k-1})$

$f(t_k) - a_0 = a_1(t_k - t_0) + \cdots + a_k (t_k-t_0)(t_k-t_1) \times (t_k - t_2) \cdots (t_k - t_{k-1})$

$$\frac{f(t_k) - a_0}{t_k - t_0} = a_1 + a_2(t_k - t_1) + \cdots + a_k(t_k - t_1)\cdots(t_k - t_{k-1}) \quad \text{—} (A)$$

~~from Indu~~

from our I.H, since $a_0 = f(t_0)$

So, $$\frac{f(t_k) - f(t_0)}{t_k - t_0} = f[t_0, t_k] \quad \text{—} (B)$$

from A & B

$$\frac{f[t_0, t_k] - a_1}{(t_k - t_1)} = a_2 + a_3(t_k - t_2) + \cdots a_k(t_k - t_2)\cdots(t_k - t_{k-1})$$

Again from I.H, since $a_1 = f[t_0, t_1]$

$$\frac{f[t_0, t_k] - a_1}{t_k - t_1} = \frac{f[t_0, t_k] - f[t_0, t_1]}{t_k - t_1}$$

$$= f[t_0, t_1, t_k]$$

Doing the same computation $(k-1)$ times,

$$f[t_0, t_1, \ldots, t_{k-2}, t_k] = a_{k-1} + a_k(t_k - t_{k-1})$$

$$\frac{f[t_0, t_1, \ldots, t_k] - a_{k-1}}{t_k - t_{k-1}} = a_k \quad \text{--- } (*)$$

Using Induction hypothesis, putting value of a_{k-1} we can compute the LHS of $(*)$ as $f[t_0, \ldots, t_k]$

$$\therefore f[t_0, \ldots t_k] = a_k$$

proved

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4(b)(i) for degree 2

~~$P_2(x) = a_1 + a_2x + a_3x^2$~~

$P_2(x) = a_1 + a_2 x + a_3 x^2$

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Given

$(x_1, y_1) = (-1, 1)$

$(x_2, y_2) = (0, 0)$

$(x_3, y_3) = (1, 1)$

Now,

$$\begin{bmatrix} 1 & -1 & (-1)^2 \\ 1 & 0 & (0)^2 \\ 1 & 1 & (1)^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

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$a_1 - a_2 + a_3 = 1$

$a_1 = 0$

$2a_2 = 0$

$a_1 = a_2 = 0$

$a_3 = 1$

So, $P_2(x) = 0 + 0 + x^2$

$P_2(x) = x^2$

(ii) Lagrange way

$$l_j(x) = \frac{\prod_{\substack{i=1 \\ i \neq j}}^{n} (x - x_i)}{\prod_{\substack{i=1 \\ i \neq j}}^{n} (x_j - x_i)} \quad (j = 0, 1, \dots, n)$$

$$P_{n-1}(x) = \sum_{j=1}^{n} y_j l_j(x)$$

degree = 2

$$P_2(x) = y_1 \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + y_2 \times \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

$$+ y_3 \times \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

Now again using the points given.

$$P_2(x) = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} + 0 + \frac{(x+1)(x-0)}{(1+1)(1)}$$

$$P_2(x) = \frac{x(x-1) + x(x+1)}{2}$$

$$P_2(x) = \frac{x^2 - \cancel{x} + x^2 + \cancel{x}}{2}$$

$$\boxed{P_2(x) = x^2}$$

(iii) Using Newton basis

$$\pi_j(x) = \prod_{k=1}^{j-1}(x-x_k) \quad , j=1 \dots n$$

$$\cancel{P_{n-1}(t) = \sum \pi_j(x) * a_k}$$

$$P_2(t) = a_1 + a_2(x-x_1) + a_3(x-x_2)(x-x_1)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & x_2-x_1 & 0 \\ 1 & x_3-x_1 & (x_3-x_1)(x_3-x_2) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

For the given datapoints,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

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$a_1 = 1$

$a_2 = -1$

$a_1 + 2a_2 + 2a_3 = 1$

$1 + (-2) + 2a_3 = 1$

$= a_3 = 1$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$P_2(x) = 1 + (-1)(x+1) + (x-0)(x+1)$

$P_2(x) = \cancel{1} - \cancel{x} - \cancel{1} + x^2 + \cancel{x}$

$P_2(x) = x^2$

(iv) In all the above 3 parts we got the same ~~eq~~ equation i.e $P_2(x) = x^2$. ~~So~~ hence, all the three representations give out the same polynomial.