Homework-3

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Problem 1:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & & & \\ 0 & 0 & a_{33} & & \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots\ 0 & a_{nn} \end{bmatrix}$

$A \in \mathbb{R}^{n \times n}$, an upper triangular matrix.

For an eigenvalue λ

$(A - \lambda I) v = 0$

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & & & \vdots \\ 0 & 0 & a_{33}-\lambda & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}-\lambda \end{bmatrix}$$

We know determinant of a triangular matrix is the product of its diagonal entries so

$\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) \ldots (a_{nn} - \lambda)$

Also $P_n(\lambda)$ (the characteristic equation),

$P_n(\lambda) = \det(A - \lambda I)$

$\rightarrow P_n(\lambda) = \prod_{i=1}^{n} (a_{ii} - \lambda_i)$

Eigenvalues are the root of the characteristic polynomial.

Thus eigenvalues for this matrix are $a_{11}, a_{22} \ldots a_{nn}$

Same is for a lower triangular matrix.

Hence proved.

Problem 2:

We know A is nondefective.

~~[illegible]~~

We know a nondefective matrix is diagonalizable

Thus A is diagonalizable.

Since it is diagonalizable,

$$A = X \Lambda X^{-1}$$

where X is a non singular matrix

Λ is a diagonal matrix.

X is invertible so

$$X A X^{-1} = \Lambda$$

We know multiplication of ~~a~~ matrix with an invertible matrix does not change its rank.

so $\text{rank}(AX) = \text{rank}(A)$

$$\text{rank}(X^{-1}AX) = \text{rank}(A)$$

$$\text{rank}(\Lambda) = \text{rank}(A)$$

Λ is a diagonal matrix and rank of a diagonal is equal to the non zero entries and the entries of a diagonal matrix are its eigenvalues (Problem 1)

~~so~~

rank(A) = non zero entries in Λ = non zero eigenvalues

Hence proved

Problem 3:

Claim: $A = uv^T$ is a rank-1 matrix.

Proof: $u \in \mathbb{R}^n$, $v \in \mathbb{R}^n$

The product uv^T will be a matrix $\in \mathbb{R}^{n \times n}$ containing multiples of u, with the multipliers as the elements of v. Since uv^T has all columns of u, its rank = 1

Thus $\text{rank}(A) = \text{rank}(uv^T) = 1$

Using the result of problem 2 since $\text{rank}(A) = 1$, the number of non-zero eigenvalues is also 1.

Since $v \in \text{col}(A)$ and $\text{col}(A)$ is 1 dimensional, u must be an eigenvector.

$$Au = (uv^T)u = u(v^Tu) = (v^Tu)\,u$$

v^Tu is a constant and also the non-zero eigenvalue of A.

Since $\dim(\text{col}(A)) = 1$; $\text{null}(A) = n-1$

where $A \in \mathbb{R}^{n \times n}$.

so 0 is an eigenvalue with multiplicity $n-1$.

$\Rightarrow$ Eigenvalue of $A \Rightarrow$ v^Tu, 0.

Column space of $A = uv^T$ is 1-dimensional and spanned by u. Thus the power method finds the eigenvector-eigenvalue pair in one step.

Problem 4:

Let A be the real symmetric positive definite matrix. $A \in R^{n \times n}$.

$\bar{A}$ = conjugate of A

$\bar{A} = A$ as $A \in R^{n\times n}$

$A = A^T$ (A is symmetric)

$A = A^T = \bar{A}$ — ①

Let (λ, x) be an eigen pair for A, $x \neq 0$

$$\lambda \bar{x}^T x = \bar{x}^T(\lambda x)$$
$$= \bar{x}^T(Ax)$$
$$= (A^T \bar{x})^T x$$
$$= (\bar{A}\bar{x})^T x \quad (\because ①)$$
$$= (\bar{\lambda}\bar{x})^T x \quad (\because \bar{A}\bar{x} = \bar{\lambda}\bar{x})$$
$$\lambda \bar{x}^T x = \bar{\lambda}\bar{x}^T x$$

Since $x \neq 0$, $\bar{x}^T x \neq 0$ so
$$\lambda = \bar{\lambda}$$

Hence eigenvalues are real.

A is positive definite so

$\forall x \in R^n \quad x^T A x > 0$ (definition)

$x^T A x = x^T(\lambda x) > 0$ ((λ, x) is an eigenpair)

$\Rightarrow \lambda x^T x > 0$

$\lambda \|x\|^2 > 0$

Length of x i.e., $\|x\|^2$ is positive

so $\underline{\lambda > 0}$ for $\lambda \|x\|^2 > 0$

Hence $\underline{\text{proved}}$.

Problem 5:

$$e^{A} = \sum_{k=0}^{\infty} \frac{1}{k!} A^{k}$$

$$e^{A} = I + A + \frac{1}{2!} A^{2} + \dots \quad \text{— (1)}$$

$$e^{A} = e^{A - \lambda I + \lambda I} = e^{\lambda I} e^{(A - \lambda I)} \quad \text{— (2)}$$

$$e^{A-\lambda I} = \sum_{k=0}^{\infty} \frac{1}{k!} (A - \lambda I)^{k}$$

Let (λ, v) be an eigen pair, so:

$$e^{A-\lambda I} v = \left(I + (A - \lambda I) + \frac{1}{2!} (A - \lambda I)^{2} + \dots\right) v$$

$$= v + (A - \lambda I) v + \frac{1}{2} (A - \lambda I) v (A - \lambda I) + \dots \Big) \quad \text{— (4)}$$

Since $Av = \lambda v$

$$(A - \lambda I) v = 0 \quad \text{— (3)}$$

(3) and (4):

$$e^{A-\lambda I} v = vI$$

$$\Rightarrow \underline{e^{A-\lambda I} = I} \quad \text{— (5)}$$

(5) & (2):

$$e^{A} = e^{\lambda I} e^{A-\lambda I} = e^{\lambda I} I = e^{\lambda I}$$

$$\text{So } e^{A} = e^{\lambda I} = e^{\lambda} = \sum_{k=0}^{\infty} \frac{1}{k!} (\lambda)^{k}$$

Problem 6:
Power Method:
Largest Eigenvalue (magnitude) is 11.0
eigenvector is

```
[[0.5 ]
 [1.  ]
 [0.75]]
```

Inverse Power Method:
Smallest Eigenvalue (magnitude) is 1.9999999999999996
eigenvector is

```
[[-0.2]
 [-0.4]
 [ 1. ]]
```

According to the library:
Eigenvalues are [11. -2. -3.]
eigenvectors are

```
[[ 3.71390676e-01  1.82574186e-01  2.17732649e-17]
 [ 7.42781353e-01  3.65148372e-01 -5.54700196e-01]
 [ 5.57086015e-01 -9.12870929e-01  8.32050294e-01]]
```

Problem 7:
Shifted inverse method:
Eigenvalue closest to 2: 2.133074475348525
eigenvector is

```
[[-0.60692002]
 [ 1.        ]
 [ 0.34691451]]
```

According to the library
Eigenvalues are [7.28799214 2.13307448 0.57893339]
Eigenvectors are

```
[[ 0.86643225  0.49742503 -0.0431682 ]
 [ 0.45305757 -0.8195891  -0.35073145]
 [ 0.20984279 -0.28432735  0.9354806 ]]
```

Problem 8:
convergence rate for 1 th iteration is -2.0105113838524735
convergence rate for 2 th iteration is 0.23830609100914138
convergence rate for 3 th iteration is 0.34811990082400446
convergence rate for 4 th iteration is 0.4336109012595084
convergence rate for 5 th iteration is 0.5883847596701156
convergence rate for 6 th iteration is 1.0
convergence rate for 7 th iteration is 1.0
convergence rate for 8 th iteration is 1.0
convergence rate for 9 th iteration is 1.0

rayleigh quotient iteration:
eigenvalue: 11.0
eigenvector:

```
[[0.5 ]
 [1.  ]
 [0.75]]
```

According to the library, 11. was the largest eigenvalue and we have obtained 11.0 eigenvalue out of the iteration.

Problem 9:
qr iteration matrix(Ak) for matrix in problem 6 is

```
[[11.          0.72199487 -6.18642278]
 [ 0.         -3.         -3.8996021 ]
 [ 0.          0.         -2.        ]]
```

qr iteration matrix(Ak) for matrix in problem 7 is

```
[[ 7.28799214e+00  5.53782393e-16 -1.20642700e-16]
 [ 0.00000000e+00  2.13307448e+00  1.34800587e-16]
 [ 0.00000000e+00  0.00000000e+00  5.78933386e-01]]
```