Homework 2:

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2020223

Problem 1:

(a)
$$x_1 = 0$$
$$\sqrt{2}\, x_2 = 0$$
$$\sqrt{3}\ x_3 = 0$$
$$x_1 - x_2 + x_3 = 1$$
$$x_1 + 4x_2 = 2$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \\ 1 & -1 & 1 \\ 1 & 4 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

(b)
$$6x_2 = 4$$
$$-4x_1 + 3x_2 = 1$$
$$x_1 + 8x_2 = 3$$

$$A = \begin{bmatrix} 0 & 6 \\ -4 & 3 \\ 1 & 8 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

(c)
$$\sqrt{2}\,(6x_2 = 4) \Rightarrow 6x_2 = 4$$
$$\sqrt{3}\,(-4x_1 + 3x_2 = 1) \Rightarrow -4x_1 + 3x_2 = 1$$
$$2\,(x_1 + 8x_2 = 3) \Rightarrow x_1 + 8x_2 = 3$$

$$A = \begin{bmatrix} 0 & 6 \\ -4 & 3 \\ 1 & 8 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

(d) We know $(x^T x) = \|x\|_2^2$

we minimise

$$\|x\|_2^2 + \|Bx - d\|_2^2 \quad \text{—— (1)}$$

By Triangle Inequality:

$$\underbrace{\|x + Bx - d\|_2^2}_{\text{Thus the minimum of (1).}} \leq \|x\|_2^2 + \|Bx - d\|_2^2$$

We have, $\|(I+B)x - d\|_2^2$

so

$A = I + B$, $b = d$

(e) $\Delta \in R^{n \times n}$ is a diagonal matrix with positive diagonal elements, thus Δ is a symmetric positive definite matrix.

thus

$$(x^T \Delta x) = \|x\|_2^2$$

so we minimise

$$\|x\|_2^2 + \|Bx - d\|_2^2 \quad \text{—— (1)}$$

By Triangle Inequality:

$$\underbrace{\|x + Bx - d\|_2^2}_{\text{minimization of (1)}} \leq \|x\|_2^2 + \|Bx - d\|_2^2$$

$\Rightarrow$ we have, $\|(I+B)x - d\|_2^2$

so

$A = I + B$, $b = d$.

Problem 2:

Euclidean norm of the minimum residual vector for a system $Ax = b$ is $\|Ax - b\|_2$

In the problem given,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

span(A) is the subspace with vectors of the form $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$

This implies residual vector is $[0\ 0\ 1]^T$ with norm 1.
So solution is given by

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\bar{b}}$$

Clearly $x_1 = 1 = x_2$
so $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$f(x) = \|b - Ax\|^2$$

For the minimum, sufficient condition is

$$\Delta^2 f > 0$$

$$\Delta f = 2A^TAx - 2A^Tb = 0 \quad \text{— (1)}$$

$$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$2A^TAx = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \quad \text{— (3)}$$

$$2A^Tb = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \quad \text{— (4)}$$

$$\begin{bmatrix} 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \end{bmatrix} = 0 \quad (\because (4) \,\&\, (3))$$

so $\Delta f(x) = 0$ is satisfied.

classmate

Date

Page

$$\Delta^2 f = 2A^TA$$

$$2A^TA = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$$

$\forall x \in R^2$

$x^T(2A^TA)x > 0$

$\det(A^TA) > 0$

Thus it is a positive definite matrix

so $\Delta^2 f > 0$

Hence we can say $x = [1 \;\; 1]^T$ is the minimum solution.

Problem 3:

$A \in R^{m \times n}$ $m \geq n$, with LI. columns.

(a) $\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} = K$ (Let)

$I \in R^{m \times m}$

$0_{n \times n} \rightarrow$ null matrix.

$\det(K) = \det(I\,0) - \det(A^TA)$

$= -\det(A^TA)$

Since A is full rank, A^TA is nonsingular.

Then means $\det(A^TA) \neq 0$

Since $\det(K) = -\det(A^TA)$

$\det(K) \neq 0$

K is non-singular.

Hence proved.

(b) $\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$

Solving we have:

$I \cdot \hat{x} + A \cdot \hat{y} = b \Rightarrow \hat{x} + A\hat{y} = b$ — ①

$A^T \hat{x} + 0 \cdot \hat{y} = 0 \Rightarrow A^T \hat{x} = 0$ — ②

② : $A^T \hat{x} = 0$

$\rightarrow$ A^T cannot be 0

$\Rightarrow$ $\hat{x} = 0$ — ③

If x is the solution of $Ax = b$ — ④

Using $\hat{x} = b - Ax$ — ⑤

By ④ : $b - Ax = 0$ — ⑥

By ③ & ⑤ : $b - Ax = 0$ as $\hat{x} = 0$ — ⑦

From ⑥ and ⑦ :

$\hat{x} = b - Ax$ holds true.

① & ③ $A\hat{y} = b$

But $Ax = b$

so $\hat{y} = x$ as A has LI columns.

~~as col~~

so

$\hat{x} = b - Ax$ and

$\hat{y} = x$

Hence proved.

Problem 4:

$$[A \quad b] = [q_1 \ldots q_{n+1}][R_1 \ldots R_{n+1}]$$

(a) $\|b\|_2^2 = q_1 R_{1n+1} + q_2 R_{2n+1} + q_3 R_{3n+1} + \cdots q_{n+1} R_{n+1}$

Since $q_i \ \forall i \le n+1$ is orthonormal

$\|q_i\|_2^2 = 1$

$\Rightarrow \|b\|_2^2 = [R_{1n+1}^2 + \ldots R_{n+1n+1}^2]$

so $\|b\|_2^2 = \|R_{n+1}\|_2^2$

(b) Let $\tilde{Q} = [q_1 \ldots q_n]$ and

$\tilde{R} = [R_1 \ldots R_n]$

where $q_i \in Q \ \forall i \le n$ and

$R_i \in R \ \forall i \le n$

To find: $\|Ax\|_2$

($\tilde{Q}$ is an orthogonal matrix)

we have $A^T A \hat{x} = A^T b$

$\Rightarrow (\tilde{Q}\tilde{R})^T (\tilde{Q}\tilde{R})\hat{x} = (\tilde{Q}\tilde{R})^T b$

$\Rightarrow \tilde{R}^T \tilde{Q}^T \tilde{Q}\tilde{R}\hat{x} = \tilde{R}^T \tilde{Q}^T b$

we know $\tilde{Q}^T\tilde{Q} = I$

so $\tilde{R}^T \tilde{R}\hat{x} = \tilde{R}^T \tilde{Q}^T b$

$\Rightarrow \tilde{R}^T(\tilde{R}\hat{x} - \tilde{Q}^T b) = 0$

$\tilde{R}^T$ is non singular and not $\perp$ to $(\tilde{R}\hat{x} - \tilde{Q}^T b)$

so $\hat{R}x - \hat{Q}^T b = 0$

$$\hat{R}x = \hat{Q}^T b \quad \text{--- (1)}$$

From (a) part we have

$\|b\|_2^2 = q_1 R_{1n+1} + \cdots q_{n+1} R_{n+1}$

$$b = \left\| \hat{Q} \begin{bmatrix} R_{1,n+1} \\ \vdots \\ R_{n,n+1} \end{bmatrix} + q_{n+1} R_{n+1,n+1} \right\| \quad \text{--- (2)}$$

(1) & (2) :

$$\hat{R}x = \hat{Q}^T \left[\hat{Q} \begin{bmatrix} R_{1n+1} \\ \vdots \\ R_{nn+1} \end{bmatrix} + q_{n+1} R_{n+1n+1} \right]$$

$\hat{Q}^T \hat{Q} = I$ so

$$\hat{R}x = \begin{bmatrix} R_{1n+1} \\ \vdots \\ R_{nn+1} \end{bmatrix} + \hat{Q}^T q_{n+1} R_{n+1n+1}$$

$$\forall ij \quad q_i \cdot q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$\hat{Q}^T q_{n+1} = 0$ since $q_{n+1} q_i = 0 \quad \forall i \leq n$

so

$$\hat{R}x = \begin{bmatrix} R_{1n+1} \\ \vdots \\ R_{nn+1} \end{bmatrix}, \quad Ax = \hat{Q}\hat{R}x = \hat{Q} \begin{bmatrix} R_{1n+1} \\ \vdots \\ R_{nn+1} \end{bmatrix}$$

$$\|Ax\|_2 = \|\hat{Q}\hat{R}x\|_2$$

Hence $\|Ax\|_2$ can be calculated using the last column of $\underline{R}$.

(c) $\| b - Ax \|_2$

From (b) part we have

$$b = \tilde{Q} \begin{bmatrix} R_{1\,n+1} \\ \vdots \\ R_{n\,n+1} \end{bmatrix} + q_{n+1} R_{n+1\,n+1}$$

and $$Ax = \tilde{Q}\tilde{R}x = \tilde{Q} \begin{bmatrix} R_{1\,n+1} \\ \vdots \\ R_{n\,n+1} \end{bmatrix}$$

$$b - Ax = \tilde{Q} \begin{bmatrix} R_{1\,n+1} \\ \vdots \\ R_{n\,n+1} \end{bmatrix} + q_{n+1} R_{n+1\,n+1} - \tilde{Q} \begin{bmatrix} R_{1\,n+1} \\ \vdots \\ R_{n\,n+1} \end{bmatrix}$$

$$= q_{n+1} R_{n+1\,n+1}$$

$$\| b - Ax \|_2 = \| q_{n+1} R_{n+1\,n+1} \|$$

$\downarrow$

this is solved using the last column of R.

Problem 5:

(a) $$y = \frac{e^{\alpha t+\beta}}{1+e^{\alpha t+\beta}}$$

$$y\left(1+e^{\alpha t+\beta}\right) = e^{\alpha t+\beta}$$

$$e^{\alpha t+\beta} = \frac{y}{1-y}$$

Taking log both sides

$$\alpha t+\beta = \log\left(\frac{y}{1-y}\right)$$

let $\log\left(\frac{y}{1-y}\right) = z$

so $\alpha t + \beta = z$

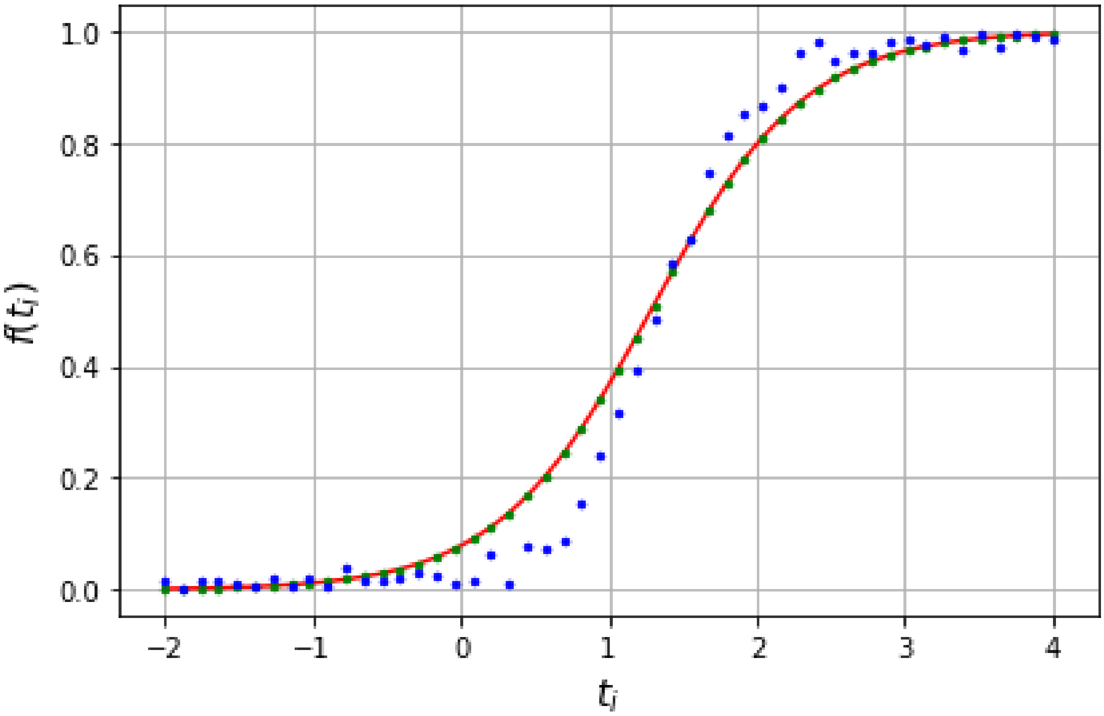
This is a linear least squares problem.

$\Rightarrow$ $\alpha t_i + \beta - z_i = 0 \quad \forall i = 1 \ldots 50.$

Problem 5:
a) Alpha is -2.46632514
Beta is 1.91883939

b) Residual for the first part (using solve) is 6.980329177231976
Residual for the second part (using lstsq) is 6.980329177231976

Graph:



Green dots: using np.linalg.lstsq
Red line: using np.linalg.solve
Blue line: Original data

Time taken by using np.linalg.solve is lesser than when we use np.linalg.lstsq.
When run over 100000 loops, np.linalg.solve took 10.62985460400023 seconds while np.linalg.lstsq took 13.495780264000132 seconds.

Problem 6:

Problem 6:

(a) $A = \begin{bmatrix} 1 & 1 \\ 10^{-k} & 0 \\ 0 & 10^{-k} \end{bmatrix}$, $b = \begin{bmatrix} -10^{-k} \\ 1+10^{-k} \\ 1-10^{-k} \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 10^{-k} & 0 \\ 1 & 0 & 10^{-k} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 10^{-k} & 0 \\ 0 & 10^{-k} \end{bmatrix}$$

$$= \begin{bmatrix} 1+10^{-2k} & 1 \\ 1 & 1+10^{-2k} \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 10^{-k} & 0 \\ 1 & 0 & 10^{-k} \end{bmatrix} \begin{bmatrix} -10^{-k} \\ 1+10^{-k} \\ 1-10^{-k} \end{bmatrix}$$

$$= \begin{bmatrix} 10^{-2k} \\ -10^{-2k} \end{bmatrix}$$

so if $x = [x_1 \quad x_2]^T$

Normal eqn:

$$\begin{bmatrix} 1+10^{-2k} & 1 \\ 1 & 1+10^{-2k} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10^{-2k} \\ -10^{-2k} \end{bmatrix}$$

b) for k= 6 x is

[[1.]
 [-1.]]

for k= 7 x is
[[1.]
 [-1.]]

for k= 8 x is
[[1.00000001]
 [-1.00000001]]

for k= 9 x is
[[1.00000005]
 [-1.00000005]]

for k= 10 x is
[[1.00000052]
 [-1.00000052]]

for k= 11 x is
[[1.00002329]
 [-1.00002329]]

for k= 12 x is
[[0.99991338]
 [-0.99991338]]

for k= 13 x is
[[0.99936385]
 [-0.99936385]]

for k= 14 x is
[[1.01270964]
 [-1.01270964]]

for k= 15 x is
[[0.86355085]
 [-0.86355085]]

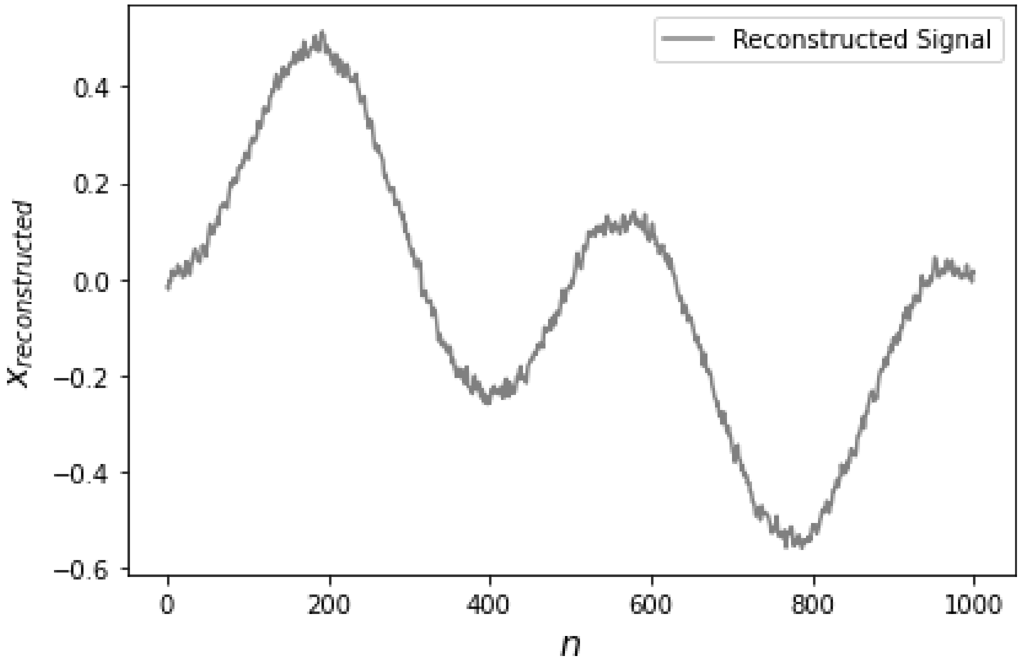
c) for k= 6 x is
[[0.99991111]
 [-0.99991111]]

for k= 7 x is
[[1.00079992]
 [-1.00079992]]

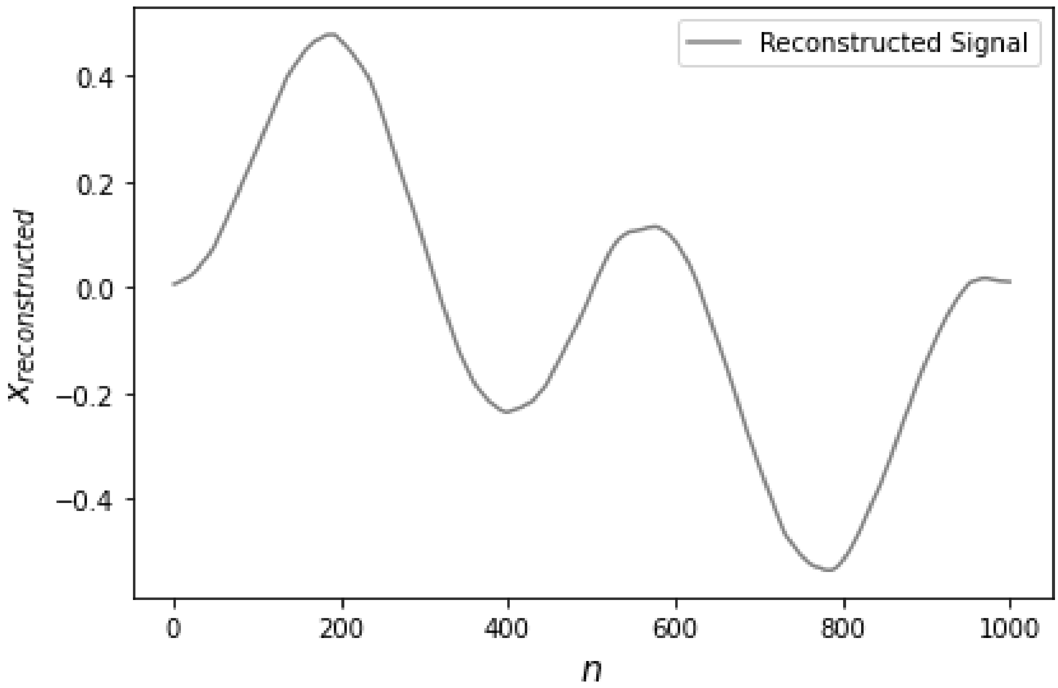
After k=7, A^TA includes numbers like 1+10^2k which contains 10^2k which is such a small number that due to the limitations of the precision it is ignored.
The epsilon machine is 10^-16 which is 10^-2k for k=8 thus 1+10^2k is rounded off to 1 for k=8 onwards and the matrix becomes a singular matrix with each element equal to 1.
Hence, we get an error when we use np.linalg.solve. The QR factorisation method does not give an error as the A matrix does not have 10^16 in any case, the round off error does not happen and it remains decomposable into matrices Q and R which are then used to solve the equation.

Problem 7:

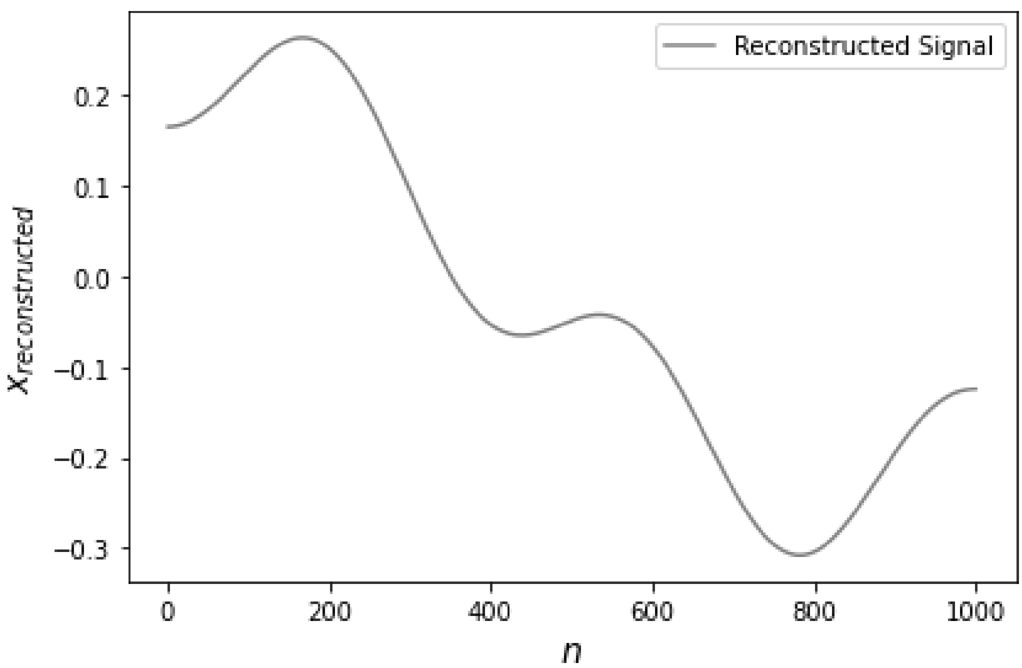
For lamba=1:



For lamba=100:



For lambda=10000:



As lambda increases the “noise” is decreased and the curve becomes smoother and more definite as the solution x becomes more accurate.