Coherent Ising Machines Cheatsheet

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Coherent Ising Machines

Coherent Ising Machines are physical hardware solvers designed to find the ground state of the Ising Hamiltonian [1]. They consist of a set of coupled degenerate optical parametric oscillators (DOPOs) that can efficiently solve combinatorial optimization problems (COPs) in terms of both time and energy [2]. Figure 1 illustrates the procedure for solving these types of problems using Ising machines.

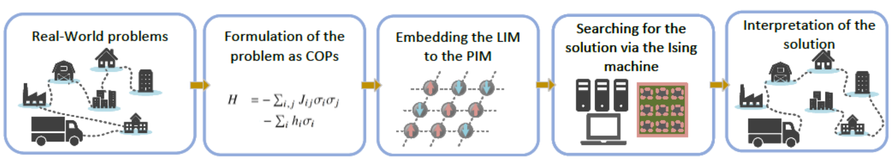

Figure 1: Procedure of solving a COP problem [1].

In this tutorial, I aim to explore the dynamics of coherent Ising machines by solving their governing differential equations and demonstrating how they can obtain solutions for certain types of COPs. The code is divided into three sections, which I will describe in the following section.

Code Structure

Part One: Dynamics of a Single DOPO

In the first part, I describe the dynamical equation of a single degenerate parametric oscillator and explain how to map Ising spins to it. The mean-field description of the in-phase quadrature has the following dynamical equation:

$$\frac{dx}{dt} = (p-1)x - \beta x^3 \tag{1}$$

where x denotes the amplitude of each DOPO, px represents the normalized pump gain, $-x$ is the normalized linear loss, and $-\beta x^3$ is the saturation gain.

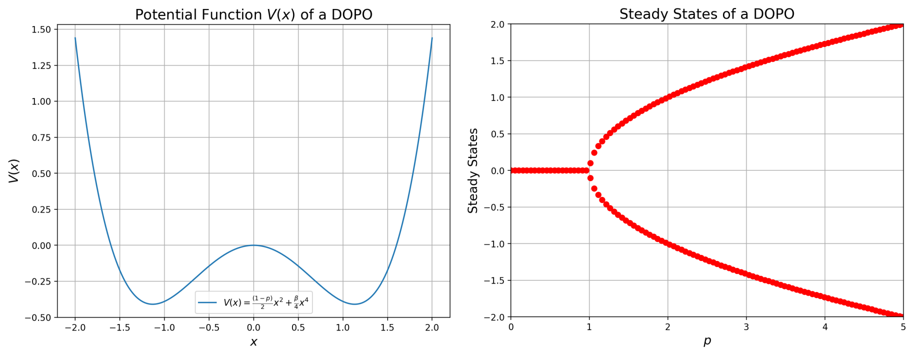

Figure 2: Bistable behavior in the potential and steady states of a single DOPO.

We can assign the potential

$$V(x) = \frac{1-p}{2}x^2 + \frac{\beta}{4}x^4 \tag{2}$$

to this equation. This potential has two global minima above a certain threshold for p, leading to bistable behavior. This bistability provides the basis for defining the concept of a spin for a single DOPO.

Part Two: Finding Ising Model Parameters

The Ising model Hamiltonian is:

$$H = -\sum_{<i,j>} J_{ij}\sigma_i\sigma_j - \sum_{1\leq i\leq N} h_i\sigma_i \tag{3}$$

where J_{ij} is the adjacency matrix representing the interaction strength between spins, and h_i is the external magnetic field vector. In the second part, I use the Python `NetworkX` package to generate different graphs and their adjacency matrices as inputs to the coherent Ising machine (CIM) to solve the max-cut problem on these graphs. Then, I explain how to map the number partitioning problem and the traveling salesman problem to the Ising model and derive the corresponding h and J parameters for these problems.

Part Three: Dynamics of Coherent Ising Machines

In the third part, I demonstrate various models for the dynamics of CIM and attempt to solve their dynamical equations to obtain results for specific values of J and h. Table 1 provides an overview of the different models used in the code. For detailed examination, please refer to the code. After introducing all the models, I will benchmark their performance for a particular problem to identify which model achieves the solution more effectively. However, I will not

Model	Equations
Meanfield	$\frac{dx_i}{dt} = (p-1)x_i - x_i^3 + \epsilon \sum_{i \neq j}^{N} J_{ij} x_j$ where p represents the pump gain, ϵ is the coupling constant, and J_{ij} is the adjacency matrix of the model [3].
AHC	$\frac{dx_i(t)}{dt} = (p-1)x_i(t) - x_i(t)^3 - e_i(t) \sum_j J_{ij} x_j(t)$ $\frac{de_i(t)}{dt} = -\beta e_i(t) \left(x_i(t)^2 - \alpha \right)$ where $e_i(t)$ is the error signal that dynamically modulates the coupling constant to converge to the correct result, and α and β are fine-tuning parameters for optimizing the system's behavior [4]. Additional magnetic field term: $\frac{h_i}{N} \sum_k \lvert x_k \rvert$
CAC	$f_i(t) = e_i(t) \sum_j J_{ij} x_j(t)$ $\frac{dx_i(t)}{dt} = (p-1)x_i(t) - x_i(t)^3 - f_i(t)$ $\frac{de_i(t)}{dt} = -\beta e_i(t) \left(f_i(t)^2 - \alpha \right)$ Parameters similar to those used in the AHC [4].
C-NUMBER	$\frac{dc_i}{dt} = \left(p - 1 - \left(c_i^2 + s_i^2 \right) \right) c_i + \eta \sum_{j=1}^{N} J_{ij} c_j + g^2 \sqrt{\left(c_i^2 + s_i^2 \right) + \frac{1}{2}} W_c$ $\frac{ds_i}{dt} = \left(-p - 1 - \left(c_i^2 + s_i^2 \right) \right) s_i + g^2 \sqrt{\left(c_i^2 + s_i^2 \right) + \frac{1}{2}} W_s$ where c_i and s_i are normalized complex numbers, η is the coupling constant, W_c and W_s are Gaussian white noise terms, and g^2 is the saturation parameter [3].
Discrete CIM	$c_i(\mathrm{t}+1) = \sqrt{\gamma G_i(t)} c_i(t) + \sqrt{G_i(t)} \eta \sum_{j=1}^{N} J_{ij} c_j(t) + f_i$ where γ indicates the total loss, $G_i(t)$ is the nonlinear gain, and f_i is the noise term [3].
Quantum	DOPO Hamiltonian: $\hat{H}_{\text{dopo}} := \frac{\mathrm{i}}{2} \left(p \hat{a}^{\dagger 2} - p^* \hat{a}^2 \right)$ Jump Operators: $\hat{L}_{\text{xtal}} := -\sqrt{\eta} \hat{a}^2$, $\hat{L}_{\text{loss}} := \sqrt{2\gamma} \hat{a}$, $\hat{L}_{\text{out}} := \sqrt{2\kappa} \hat{a}$ Lindblad Equation: $\partial_t \hat{\rho}(t) = -\mathrm{i} \left[\hat{H}_{\text{dopo}}, \hat{\rho}(t) \right] + \left(\mathcal{D} \left[\hat{L}_{\text{loss}} \right] + \mathcal{D} \left[\hat{L}_{\text{xtal}} \right] + \mathcal{D} \left[\hat{L}_{\text{out}} \right] \right) \hat{\rho}(t)$ where $\mathcal{D}[\hat{L}] \hat{\rho} = \hat{L} \hat{\rho} \hat{L}^\dagger - \frac{1}{2} \left\{ \hat{L}^\dagger \hat{L}, \hat{\rho} \right\}$ [5, 6]

Table 1: Dynamical equations for different CIM models.

benchmark the quantum model, as running it for large-scale problems requires an enormous amount of memory.
In Figure 3, the flowchart for Part 3, outlining the process of solving the dynamical equations of CIM, is presented. The key parameters for the solver, along with their respective solution methods, are summarized in Table 2.

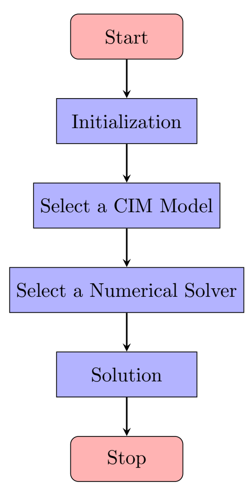

Figure 3: Different steps of the solver

Initialization Parameters	**CIM Models**	**Solvers**
Pump function	Mean-field	Euler
J and h matrix	AHC	RK4
Model parameters	CAC	RK45
	c-number	RK23
	Discrete model	DOP853
	Quantum model	mesolve
		smesolve

Table 2: CIM Initialization Parameters, Models, and Solvers

References

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