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Semistable Modular Compactifications of Moduli Spaces of Genus One Curves

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SEMISTABLE MODULAR COMPACTIFICATIONS OF MODULI SPACES OF GENUS ONE CURVES

by

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The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned disciplines.

Parker, Keli Siqueiros (Ph.D., Mathematics)

Semistable modular compactifications of moduli spaces of genus one curves

Thesis directed by Assistant Professor Jonathan Wise

We clarify the definition of an infinitesimal automorphism of a log smooth curve, and show that logarithmic structure is capable of fixing the underlying infinitesimal automorphisms on a semistable curve. We then introduce the notion of an m-stable partially aligned logarithmic curve and construct a few natural semistable modular compactifications of the moduli space of smooth genus one curves. Finally, we prove that our moduli space of m-stable partially aligned log curves resolves the indeterminacy between the moduli space of Deligne-Mumford stable curves and the moduli space of m-stable curves constructed by Smyth.

Dedication

This is dedicated to my wife Jessica.

Thank you so much for being my everything.

Your love makes all things possible and the world a magical place, let's go exploring!

Te amo siempre.

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1 Introduction

We work over an algebraically closed field of characteristic zero. Throughout, all log structures are fine and saturated unless otherwise specified, and a logarithmic scheme will be given with the explicit notation (X, M_X), while underlying schemes without log structure are indicated by X without further decoration.

Statement of main results

The document that follows contains two main theorems:

Theorem 1.1. *The moduli problem of m-stable partially aligned log curves is representable by a proper, irreducible, logarithmic Deligne-Mumford stack $\mathcal{F}_n^{min}(m)$.*

The universal curve $\pi : \mathcal{U} \to \mathcal{F}_n^{min}(m)$ of this stack comes equipped with natural line bundles. We use these to construct a proper, birational contraction to Smyth m-stable curves, giving us our main result:

Theorem 1.2. *The moduli stack $\mathcal{F}_n^{min}(m)$ resolves the indeterminacy between $\overline{\mathcal{M}}_{1,n}$ and $\overline{\mathcal{M}}_{1,n}(m)$.*

Overview

The Deligne-Mumford moduli space of stable nodal curves [DM69] was the first compactification of the moduli space of smooth curves, beautifully illustrating the heuristic that a compact moduli space should include degenerate objects. Over the years, there has been a great deal of work expanding our knowledge of such compactifications. We will not attempt a review of the extensive literature here, content to instead have our readers recall the pseudostable curves of Schubert [Sch91], and the weighted pointed stable curves of Hassett [Has03] as relevant examples of alternate compactifications of smooth curves. For a more comprehensive (although not complete!) overview of alternate compactifications of $\mathcal{M}_{g,n}$, the birational geometry of $\overline{\mathcal{M}}_{g,n}$, and related work, we point the interested reader to the survey [FS13].

The original moduli space $\overline{\mathcal{M}}_{g,n}$ is a strikingly elegant compactification: it is smooth with a normal crossings boundary, and is stratified by topological type. The combinatorial nature of this stratification is encoded in the notion of the *dual graph* of a nodal curve, which determines the topological type of the curve. We would consider the dual graph a central object in the work that follows, and so we explain the idea here.

For a marked nodal curve C over a separably closed field, the dual graph may be described as a weighted planar graph, allowing loops, consisting of the following:

(1) a set of vertices V corresponding to the irreducible components of C and weighted by their genera;

(2) a set of edges E indexed by the nodes of C, incident to the vertices corresponding to the irreducible component(s) that meet at that node (loops allowed);

(3) a set of half-edges H indexed by the markings, each incident to the vertex corresponding to the marked component.

We are then able to define the stratum $\mathcal{M}_G \subset \overline{\mathcal{M}}_{g,n}$ as the locus of curves with dual graph G. The closure $\overline{\mathcal{M}}_G$ of a stratum has boundary $\partial\overline{\mathcal{M}}_G := \overline{\mathcal{M}}_G \setminus \mathcal{M}_G$ that is further stratified by dual graphs. Specifically, a stratum $\mathcal{M}_{G'}$ is contained in the closure $\overline{\mathcal{M}}_G$ if and only if there is a collection of edges in the graph G' whose contraction produces the graph G.

This stratification indicates that while dual graphs are in general relatively simple combinatorial objects, they nevertheless encode important information about the behavior of curves in families. The process of contracting edges of a dual graph described above corresponds to *smoothing* nodes of the curve: a family of curves $C \to \Delta$ over a discrete valuation ring, whose generic fiber has dual graph G and whose special fiber has dual graph G', witnesses the containment $\mathcal{M}_{G'} \subset \overline{\mathcal{M}}_G$. The nodes of the special fiber corresponding to contracted edges in G' are then *smoothed by the family.*

Considering the stable and semistable reduction theorems, we may expect nodal curves and their dual graphs to be relevant to *any* modular compactification of $\mathcal{M}_{g,n}$. Following Smyth, we consider a modular compactification of $\mathcal{M}_{g,n}$ to be a proper modular stack of smoothable curves. A *stable* compactification is one in which the rational components in the normalization of a geometric fiber have at least three distinguished points, while a semistable compactification allows rational components with at least two. Smyth classified all irreducible *stable* modular compactifications using so called *extremal assignments relative to* $\overline{\mathcal{M}}_{g,n}$, whose definition relies explicitly on dual graphs [Smy13]. We discuss this in section 2.1, but briefly, an extremal assignment $\mathcal{Z}$ is a consistent choice

$$\mathcal{Z}(G) \subset G$$

of a subgraph of each dual graph over $\overline{\mathcal{M}}_{g,n}$, subject to some axioms ensuring this is a well-defined notion. Given an extremal assignment $\mathcal{Z}$, there is then a correponding moduli space $\overline{\mathcal{M}}_{g,n}(\mathcal{Z})$ of $\mathcal{Z}$-stable curves, where the subcurves $C_{\mathcal{Z}} \subset C$ corresponding to $\mathcal{Z}(G) \subset G$ have been replaced by genus $p_a(C_{\mathcal{Z}})$ singularities. Note that if

$$p_a(C_{\mathcal{Z}}) = g \quad \text{and} \quad |\overline{C \setminus C_{\mathcal{Z}}} \cap C_{\mathcal{Z}}| = m,$$

then *any* singularity with genus g and m branches occurs in $\overline{\mathcal{M}}_{g,n}(\mathcal{Z})$ as a replacement for some subcurve defined by $\mathcal{Z}(G)$ (see Definition 2.5).

On the other hand, Smyth has also given us an example of a strictly *semistable* modular compactification in genus one, with $\overline{\mathcal{M}}_{1,\mathcal{A}}(m)$ parametrizing $(m, \mathcal{A})$-stable curves with elliptic $l \leq m$-fold singularities. These singularities are the *unique* Gorenstein singularities of genus one with l branches, see [Smy11] and our discussion in section 2.1. This moduli space is curious for a few reasons. First of all, it is a Deligne-Mumford stack, containing curves whose normalizations have rational components with only two distinguished points, and yet no nontrivial infinitesimal automorphisms. Being semistable in general, $\overline{\mathcal{M}}_{1,\mathcal{A}}(m)$ has no corresponding $\mathcal{Z}$ stability condition relative to $\overline{\mathcal{M}}_{1,n}$. However, for $m = 1$ we can relate it directly to a moduli problem that does, namely a combination of pseudostable curves and weighted pointed stable curves.

Indeed, the pseudostable curves of Schubert may be described as the $\mathcal{Z}^{ps}$-stable curves given by an extremal assignment $\mathcal{Z}^{ps}$ that picks out elliptic tails and replaces them with cusps, the unique genus one singularity with one branch [Smy13, Example 1.10]. Note that Schubert's original paper involves unmarked stable curves of genus $g > 3$, so we are being a bit imprecise in referring to the corresponding moduli problem involving n-marked curves. Similarly, the weighted pointed stable curves of Hassett have a corresponding $\mathcal{Z}^{\mathcal{A}}$-stability condition (see 2.2 below). When $m = 1$, an $(m, \mathcal{A})$-stable curve is precisely a weighted pointed stable curve of genus one for which unmarked elliptic tails have been replaced by cusps. Hence we may interpret $\overline{\mathcal{M}}_{1,\mathcal{A}}(1)$ as a moduli space defined by an extremal assignment $\mathcal{Z} := \mathcal{Z}^{ps} \cup \mathcal{Z}^{\mathcal{A}}$.

However, unlike pseudostable curves, $(m, \mathcal{A})$-stable curves have no analogue in higher genus, with the main obstruction being that in a genus $g > 1$ curve, there may not be a natural choice of distinct elliptic subcurves to replace by m-fold singularities. Furthermore, while there is a *unique* genus one singularity with *one* branch, for $m > 1$ branches, there are several genus one singularities. As $\mathcal{Z}$-stability will include *any* singularity of the correct genus and number of branches, and the $(m,\mathcal{A})$-stable curves contain only the *unique* Gorenstein singularities of genus one, we should not expect that there is a description as some type of analogous $\mathcal{Z}$-stability condition from an extremal assignment relative to a moduli space of semistable nodal curves.

In particular, our interpretation of $(1,\mathcal{A})$-stability as a $\mathcal{Z}$-stability condition does not generalize to higher m since we need some way of contracting specifically to an elliptic m-fold singularity. Once such a contraction is defined, the notion of $\mathcal{A}$-stability may be incorporated without much difficulty. As such, we focus our attention on the case when $\mathcal{A}=(1,1,\ldots,1)$, and follow Smyth to denote the corresponding $(m,\mathcal{A})$-stable curves simply as m-stable, parametrized by an algebraic stack $\overline{\mathcal{M}}_{1,n}(m)$.

Approach

As a preliminary step towards understanding semistable modular compactifications in general, we sought to resolve the indeterminacy between the alternate compactifications given by $\overline{\mathcal{M}}_{1,n}$ and $\overline{\mathcal{M}}_{1,n}(m)$. Thankfully, a result of Hassett and Hyeon gives us an example of what might be done. Briefly, the Hassett-Keel program seeks to give modular interpretations for the *log canonical models* of the moduli space of stable curves. In their paper [HH09], Hassett-Hyeon consider the log canonical models

$$\overline{M}_g(\alpha) := \operatorname{Proj}\Big(\oplus_{n\geq 0} H^0\big(n(K_{\overline{\mathcal{M}}_g}+\alpha\delta)\big)\Big)$$

for $9/11 \leq \alpha \leq 1$, filling in what happens at $9/11$. We review this material further in 2.3, but for the purposes of this discussion, Hassett-Hyeon end up constructing a specific divisorial contraction arising from a morphism of stacks

$$\mathcal{T} : \overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g^{ps}.$$

This is obtained via a contraction on the universal curve $\mathscr{C} \to \overline{\mathcal{M}}_g$ that takes elliptic tails to the cusp singularities of pseudostable curves. A technical achievement of the construction is a precise characterization of the singularities that arise from the contraction. In light of our discussion above on generalizing pseudostable curves to m-stable curves, we sought to follow the Hassett-Hyeon construction and obtain our own contraction from a universal family of nodal curves to m-stable curves. As discussed in section 2.3, one key ingredient (besides a universal family of nodal curves over an appropriate moduli space) is a line bundle on the universal curve that remains locally free under pushforward, and it is this line bundle that (unlike an extremal assignment) allows us to control precisely which singularities are obtained from the contraction. For now let us focus on our approach to finding a universal family, which we now discuss.

To obtain a family of m-stable curves from a nodal family, one would like to replace all elliptic subcurves with $l \leq m$ rational tails by an elliptic l-fold singularity. An obvious requirement is that we are able to identify *which* subcurve is contracted, and our approach admits a pleasingly simple description. Namely, we modify dual graphs by drawing a circle around the unique minimal elliptic component (referred to as the *cycle*), and replace components inside of the circle with an elliptic m-fold singularity.

One should of course ask about what choices are involved when drawing a circle on a dual graph. First of all, every dual graph comes with an intrinsic partial order on vertices given by the length of a path from the cycle. If a path from the cycle to a vertex v contains some other vertex v', then v' is objectively *closer* to the cycle than v, but vertices in disjoint paths are not canonically comparable. So when we draw a circle, this may be interpreted as declaring vertices to be closer or farther from the cycle relative to the fixed length determined by the radius. Thankfully, we do not need the difficulty of fully metrized graphs to do this, so while our ideas have a tropical flavor, we never actually work in this setting. The key ingredient for making all of this precise and useful is a logarithmic structure on our curves.

Logarithmic structures will be introduced fully in section 3. For now, a logarithmic structure on a scheme X is a sheaf of monoids M_X on the étale site of X, with a structure morphism $\alpha : M_X \to \mathcal{O}_X$ that induces an isomorphism $\alpha^{-1}(\mathcal{O}_X^*) \xrightarrow{\alpha} \mathcal{O}_X^*$. An excellent example to have is that of a smooth scheme X with a normal crossings divisor $D \subset X$, with monoid

$$M_X := \{ f \in \mathcal{O}_X : \ f \text{ is invertible outside } D \ \} \xhookrightarrow{\alpha} \mathcal{O}_X.$$

Every nodal curve admits a *canonical* logarithmic structure. In fact, the resulting sheaf of monoids $M_{\overline{\mathcal{M}}_{g,n}}$ on $\overline{\mathcal{M}}_{g,n}$ can be constructed from the dual graph stratification, and we do this later on. It is equivalent to the logarithmic structure induced by the normal crossings boundary $\partial\overline{\mathcal{M}}_{g,n}$.

Given two elements a, b in a monoid where cancellation holds, an element c in the monoid admitting the relation $c + b = a$ has the properties of "$a - b$." This gives us a partial order on elements of a monoid, with $a \geq b$ if $a - b$ is in the monoid. From this perspective, the ordering of vertices in the dual graph as farther or closer from the cycle is given by a global section of a particular sheaf of monoids *on the universal curve* over $\overline{\mathcal{M}}_{1,n}$. This section assigns a *length*, valued in the sheaf of monoids on the *base*, to every vertex. With this in mind, drawing a circle on a dual graph amounts to adjoining an element (the "radius" of the circle) to the canonical monoid on the base, and imposing relations that reflect the relative ordering of vertices with respect this fixed length (see section 4).

We shall see that this construction results in a log algebraic stack of curves that is certainly not separated, given our many available choices of just how we might draw a circle. However, there is a very natural stability condition for any choice of nonnegative integer m, that simply says when we draw our circle, we must do so in such a way that strictly greater than m paths are externally incident to the circle, and less than or equal to m paths are internally incident. The resulting logarithmic curves will be called *m-stable partially aligned log curves*, and the proper stack of such objects will give us our moduli space with a universal curve that we may contract to an m-stable curve of Smyth.

Outline of the document

Section 2 contains a more thorough introduction to the motivating literature mentioned above, and we discuss the extremal assignments and m-stable curves of Smyth, as well as the Hassett-Hyeon divisorial contraction from stable curves to pseudostable curves.

In section 3 we include an overview of logarithmic structure and explain some of its utility and the main points needed to work with log algebraic stacks. A brief construction of the canonical log structure on nodal curves is also given, and serves as a preview to our construction of the minimal log structure for partially aligned log curves later on. In 3.3, we correct an error from Kato regarding the characterization of stable log curves [Kat00, Corollary 1.15]. We expect our analysis of infinitesimal automorphisms of log smooth curves presented here to be of independent interest. We expect the implied (and previously unknown) stack of "log smooth curves without infinitesimal automorphisms" may be useful for a classification of *semistable* modular compactifications following the program of Smyth. Finally, we introduce the tool of a log blowup and explain our modular interpretation of this.

Our main construction and the proof of theorem 1.1 takes up all of section 4. While our moduli spaces require us to work with stacks, the main subtleties come from log theoretic considerations, and we never find the need, for example, to verify Artin's criteria, relying instead on representability of open subfunctors of algebraic stacks.

Finally, the contraction from a universal curve on our moduli space to a Smyth curve occurs in section 5, completing the proof of theorem 1.2.

We would like to note that while we do not explicitly refer to tropical geometry, the results of our perspective are somewhat reminiscent of the "realizability" results in tropical geometry, and we hope that the progress there can provide some intuition as to how to adapt our methods to higher genus situations, see in particular the recent work reported by Ranganathan and Jensen in [JR17]. Via personal communication we have learned that there is a tropical compactification similar to ours, developed independently by Ranganathan. We hope that a combination of perspectives will prove fruitful for further progress on related problems, including classifications of semistable compactifications, application to higher genus, and alternate compactifications of the moduli space of maps of curves.

2 Preliminaries

In this section we compile the required background for the approach that follows. The first sections here focus quite specifically on papers of Smyth and Hassett-Hyeon that serve as a foundation on which we have built our approach.

2.1 Modular compactifications of $\mathcal{M}_{g,n}$

In [Smy13], Smyth provides a classification of all stable modular compactifications of smooth n-marked genus g curves. Here we briefly summarize the definitions and results from this program. This will lead into an overview of Smyth's example in genus one [Smy11] of a modular compactification that allows curves with rational components with only two distinguished points.

It is well-known that there is an algebraic stack $\mathcal{U}_{g,n}$, locally of finite-type over Spec $\mathbf{Z}$, representing the moduli functor on $\mathfrak{Sch}$

$$S \quad \mapsto \quad \mathcal{U}_{g,n}(S) := \left\{ \begin{array}{l} \text{Flat, proper, finitely-presented morphisms } \mathcal{C} \to S \text{ from an algebraic space } \mathcal{C}, \\ \text{together with } n \text{ sections } \{\sigma_i\}, \text{ with reduced, connected, one-dimensional} \\ \text{geometric fibers of arithmetic genus } g \ . \end{array} \right\}$$

We will refer to $\mathcal{U}_{g,n}$ as the moduli stack of all n-marked genus g curves. Smooth curves determine an open substack of this moduli space, and since $\mathcal{M}_{g,n}$ is irreducible, we may consider the unique irreducible component of $\mathcal{U}_{g,n}$ containing smooth curves. This main component will be denoted by $\mathcal{V}_{g,n}$; its points are the *smoothable* curves. Note that $\mathcal{V}_{g,n}$ is highly non-separated.

Smyth suggests classifying families of curves admitting unique limits; it is implied that such compactifications of $\mathcal{M}_{g,n}$ can be given by a moduli functor (for us a loose notion we will refer to as admitting a "modular interpretation"). Hence compactifications $\mathcal{X}$ will come with a universal stack $\mathscr{C} \to \mathcal{X}$, referred to as the universal curve, whose geometric fibers are the curves being parametrized in the moduli problem.

Definition 2.1. A *modular compactification* of $\mathcal{M}_{g,n}$ is a proper open substack $\mathcal{X} \subset \mathcal{V}_{g,n}$. If every rational component of the normalization of any geometric fiber of $\mathscr{C} \to \mathcal{X}$ has at least three distinguished points, then $\mathcal{X}$ is called a *stable* modular compactification. *Semistable* refers to the situation of at least two distinguished points in each rational component.

Recall that for any stable curve with n markings, we have the notion of the *dual graph* of the curve. As described in the introduction, the dual graph of a nodal curve is a planar graph whose vertices correspond to irreducible components of the curve and are weighted by genus, edges correspond to nodes, half edges to marked points, and incidence reflects which component(s) have nodes or markes. Note that for any datum (g, n), there will be a finite number of dual graphs up to isomorphism corresponding to curves of genus g with n marked points.

Smyth has classfied every stable modular compactification by defining a collection of moduli functors that depend on the dual graphs of stable curves, and showing that any modular compactification admits such a description. We present the required definitions below, beginning with the key notion of compatibility with generization.

Suppose $\mathcal{C} \to \Delta$ is a family of curves over a discrete valuation ring with special fiber C_0 and a geometric generic fiber $C_{\overline{\eta}}$. Let G_0 be the dual graph of C_0, $G_{\overline{\eta}}$ the dual graph of $C_{\overline{\eta}}$. Recall from the discussion in the introduction that if C_0 has more nodes than $C_{\overline{\eta}}$, then these additional nodes are *smoothed* by the family, and that the curve $\mathcal{C} \to \Delta$ is witnessing the relationship $\mathcal{M}_{G_0} \subset \overline{\mathcal{M}}_{G_{\overline{\eta}}}$ between the strata of $\overline{\mathcal{M}}_{g,n}$. We could just as well say that $C_{\overline{\eta}}$ is a *geometric generization* of the point $C_0 \in \mathcal{M}_{G_0}$ into the stratum $\mathcal{M}_{G_{\overline{\eta}}}$.

As noted before, in this situation there must be a subset E_0 of edges in G_0 such that the graph $G_{\overline{\eta}}$ may be obtained from G_0 by contracting each edge in E_0 to a single vertex. We call such a contraction of edges a *generization of dual graphs*, and will denote this by

$$G_0 \leftsquigarrow G_{\overline{\eta}}.$$

In particular, under such a generization construction, collections of vertices $v_1, \ldots, v_r$ that are connected by edges in E_0 become identified as a single vertex $v' \in G_{\overline{\eta}}$.

Let $G_1, \ldots G_N$ be an enumeration up to isomorphism of all the dual graphs of n-pointed stable curves of genus g, and let $\mathcal{Z}$ denote a choice of vertices

$$\mathcal{Z}(G_i) \subset V(G_i) \qquad \text{for each } i = 1, \ldots, N.$$

Definition 2.2. (Compatible with generization) We say that $\mathcal{Z}$ *is compatible with generization* if for every generization of dual graphs $G_i \leftsquigarrow G_j$ and any collection of vertices $v_1, \ldots v_r \in G_i$ identified to $v' \in G_j$ by the generization,

$$v' \in \mathcal{Z}(G_j) \text{ if and only if } v_1, \dots, v_r \in \mathcal{Z}(G_i).$$

Definition 2.3 (Extremal assignment relative to $\overline{\mathcal{M}}_{g,n}$)**.** A choice of of vertices $\mathcal{Z}$ is an *extremal assignment over* $\overline{\mathcal{M}}_{g,n}$ if it satisfies the following:

(1) for any dual graph, $\mathcal{Z}(G) \neq V(G)$;

(2) for any dual graph, $\mathcal{Z}(G)$ is invariant under $\mathrm{Aut}(G)$;

(3) $\mathcal{Z}$ is compatible with generization.

The axioms for an extremal assignment allow us to pick out subcurves of stable curves in a universal way. Given an extremal assignment $\mathcal{Z}$ and a stable curve $\mathcal{C} \to S$, we denote by $\mathcal{Z}(\mathcal{C}) \to S$ the subcurve defined on geometric fibers as the union of the irreducible components corresponding to the vertices selected by $\mathcal{Z}$.

Definition 2.4 (Genus and type of a curve singularity)**.** For any curve C over an algebraically closed field k, consider the normalization $\nu : \widetilde{C} \to C$ of C at a singular point p. The *number of branches* at p

$$m(p) := |\nu^{-1}(p)|$$

imposes $m(p) - 1$ conditions for a function on $\widetilde{C}$ to descend to a function on C, namely that the function must agree at every point of $\nu^{-1}(p)$. Depending on the singularity, there may be other conditions for a function to descend, and the overall number of conditions is recorded by the δ-*invariant*

$$\delta(p) := \dim_k(\nu_* \mathcal{O}_{\widetilde{C}} / \mathcal{O}_C).$$

Then the *genus* of a singular point $p \in C$, denoted by $g(p)$, is defined as the number of conditions for a function to descend, beyond agreeing at the points above p in the normalization.

$$g(p) := \delta(p) - m(p) + 1.$$

We refer to the *type* of a singularity $p \in C$ as the pair $(g(p), m(p))$.

Note that there are in general several singularities of any given type, for example,

$$\hat{\mathcal{O}}_{C,p} \simeq k[[x,y]]/(y^2 - yx^2) \qquad \text{and} \qquad \hat{\mathcal{O}}_{C,p} \simeq k[[x,y,z]]/(xz, yz, y^2 - x^3)$$

are nonisomorphic type $(1,2)$ singularities, namely the ordinary tacnode and the cusp with a transverse branch passing through its singular point.

Remark 2.1. The terminology of *genus* of a singularity is justified by the following observation. Given a smoothable isolated curve singularity $p \in C$, take a family of curves over a smooth one parameter base $\mathcal{C} \to \Delta$ with $\mathcal{C}\big|_0 = C$. Base change and apply stable reduction, and the result will be a nodal curve $\widetilde{\mathcal{C}} \to S$ and a contraction

$$\begin{tikzcd}
\widetilde{\mathcal{C}} \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{C} \arrow[dl] \\
& S &
\end{tikzcd}$$

with exceptional fiber $\phi^{-1}(p)$ of genus equal to the genus of the singularity p.

Definition 2.5 ($\mathcal{Z}$-stability)**.** Given an extremal assignment $\mathcal{Z}$, a smoothable n pointed curve $\mathcal{C} \to S$ is *$\mathcal{Z}$-stable* if for any geometric point $\overline{s} \to S$, there is an n-pointed stable curve $\widetilde{C} \to \overline{s}$ in $\overline{\mathcal{M}}_{g,n}(\overline{s})$ and a morphism

$$\begin{tikzcd}
\widetilde{C} \arrow[rr, "\phi"] \arrow[dr] & & C_{\overline{s}} \arrow[dl] \\
& \overline{s} &
\end{tikzcd}$$

such that

(1) ϕ is surjective with connected fibers;

(2) ϕ is an isomorphism away from $\mathcal{Z}(\widetilde{C})$;

(3) for $Z_1, \dots, Z_k$ the connected components of $\mathcal{Z}(\widetilde{C})$, the image $\phi(Z_i)$ is a point p_i of type $(g(Z_i), |Z_i \cap \overline{\widetilde{C} \setminus Z_i}|)$.

The main result of Smyth's work is a remarkably simple classification of all stable modular compactifications, relying on the following moduli problem.

Definition 2.6. Let $\overline{\mathcal{M}}_{g,n}(\mathcal{Z}) \subset \mathcal{V}_{g,n}$ be the subfunctor of $\mathcal{Z}$-stable curves.

Theorem 2.7. *[Smy13, 1.9]*

(1) If $\mathcal{Z}$ is an extremal assignment over $\overline{\mathcal{M}}_{g,n}$, then $\overline{\mathcal{M}}_{g,n}(\mathcal{Z}) \subset \mathcal{V}_{g,n}$ is a stable modular compactification of $\mathcal{M}_{g,n}$.

(2) If $\mathcal{X} \subset \mathcal{V}_{g,n}$ is a stable modular compactification, then $\mathcal{X} = \overline{\mathcal{M}}_{g,n}(\mathcal{Z})$ for some extremal assignment $\mathcal{Z}$.

In light of this theorem, the important examples of the pseudostable curves of Schubert (naturally extended to pointed curves) and the $\mathcal{A} := (a_1, \dots a_n)$-weighted pointed stable curves of Hasset may be given respectively by the following $\mathcal{Z}$-stability conditions:

$$\mathcal{Z}^{ps}(C) = \{Z \subset C \ : \ p_a(Z) = 1, |Z \cap \overline{C \setminus Z}| = 1, Z \text{ is unmarked } \} \tag{2.1}$$

$$\mathcal{Z}_{\mathcal{A}}(C, \{s_i\}_{i=1}^n) = \{Z \subset C \ : \ p_a(Z) = 0, |Z \cap \overline{C \setminus Z}| = 1, \sum_{s_i \in Z} a_i \leq 1\} \tag{2.2}$$

Many useful examples can also be decribed from the following lemma provided by Smyth.

Lemma 2.8. *[Smy13, 1.13] Let $\mathcal{L}$ be a line bundle on the universal curve $\mathcal{C} \xrightarrow{\pi} \overline{\mathcal{M}}_{g,n}$ such that $\mathcal{L}$ has positive degree on the generic fiber and non-negative degree on every irreducible component of every fiber ($\mathcal{L}$ is π-nef, numerically-nontrivial). Then the assignment*

$$\mathcal{Z}(C) := \{Z \subset C | \deg(\mathcal{L}\big|_Z) = 0\}$$

satisfies the axioms (1)-(3) for an extremal assignment.

It is important to note that a π-nef numerically-nontrivial line bundle $\mathcal{L}$ does not necessarily provide a *specific* contraction yielding the $\mathcal{Z}$-stable curves defined by such an extremal assignment. If $\mathcal{L}$ is π-semiample and $\pi_*\mathcal{L}^n$ happens to be locally free, then certainly $\operatorname{Proj} \oplus_{n\geq 0} \pi_* \mathcal{L}^n$ will yield a flat family. The induced contraction of the universal curve

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & \operatorname{Proj} \oplus_{n\geq 0} \pi_* \mathcal{L}^n \arrow[dl, "\pi'"] \\
& \overline{\mathcal{M}}_{g,n} &
\end{tikzcd}$$

has fibers over $\overline{\mathcal{M}}_{g,n}$ that satisfy the conditions of definition 2.5. In fact, we get a proper, birational map $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}(\mathcal{Z})$, and separatedness of $\mathcal{M}_{g,n}(\mathcal{Z})$ (along with the fact that smooth curves are a dense) tells us that this map is surjective, so in this case the $\mathcal{Z}$ stable curves are determined by the line bundle. However, there is no reason to expect this approach with a π-nef numerically-nontrivial line bundle to yield a flat family in general.

While the classification by $\mathcal{Z}$-stability presented above is quite beautiful in its concise simplicity, Smyth notes that the complete story is not yet known. Indeed, there is no $\mathcal{Z}$ stability condition capable of picking out curves with only nodes, cusps, tacnodes, and planar triple-points. The difficulty compared to psuedostable curves arises because unlike type $(0,2)$ and $(1,1)$ singularities (nodes and cusps), type $(1,2)$ and type $(1,3)$ singularities are not unique. A planar triple point may degenerate to a tacnode with a transverse branch or a cusp with two transverse branches under degeneration of the equation describing the dependency of tangent spaces at the points above the singularity in the normalization.

Thankfully, Smyth also provides us with an example of how one can produce a compact moduli of attaching data producing purely tacnodal or planar triple-point limits. However this construction is semistable, and clearly such compactifications cannot be classified by extremal assignments relative to $\overline{\mathcal{M}}_{g,n}$. We give a construction of a log stack $\mathcal{F}_n^{min}(m)$ (Theorem 1.1) that provides a semistable (but still Deligne-Mumford!) analog of $\overline{\mathcal{M}}_{1,n}$, and one might expect that a $\mathcal{Z}$-stable relative to $\mathcal{F}_n^{min}(m)$ condition might interpret the Smyth space. Such an interpretation of this example (and so the classification of semistable compactifications in general) would require a careful check that a corresponding semistable $\mathcal{Z}$-stability condition remains separated.

We give one last example of an extremal assignment before moving on:

$$\mathcal{Z}(C) := \{Z \subset C | p_a(Z) = 1, |Z \cap \overline{C \setminus Z}| = l, l \leq m, Z \text{ is unmarked}\} \tag{2.3}$$

2.2 m-stable curves

We provide a brief overview of Smyth's moduli space of n pointed m-stable curves of arithmetic genus one [Smy11].

Recall that a curve is *Gorenstein* if its dualizing sheaf is invertible, and this occurs precisely if it only has isolated curve singularities whose local rings are Gorenstein.

Proposition 1. *[Smy11, Prop. A.2] If $p \in C$ is singular, genus zero, and Gorenstein, then p is an ordinary node.*

Hence the node is the unique genus zero Gorenstein singularity, in particular genus zero points with more than two branches are not Gorenstien. In genus one, however, there is a unique Gorenstein singularity for each number of branches $m \geq 1$.

Definition 2.9. An *elliptic m-fold point* is a point p of a curve C satisfying one of the following:

$$\hat{\mathcal{O}}_{C,p} \simeq \begin{cases} k[[x,y]]/(y^2-x^3) & m=1 \text{ (ordinary cusp)} \\ k[[x,y]]/(y^2-yx^2) & m=2 \text{ (ordinary tacnode)} \\ k[[x,y]]/(x^2y-xy^2) & m=3 \text{ (planar triple-point)} \\ k[[x_1,\dots,x_{m-1}]]/I_m & m\geq 4 \text{ (cone over } m \text{ general points in } \mathbb{A}^{m-2}) \end{cases}$$

$$I_m := \big(x_h(x_i-x_j) : i,j,h \in \{1,\dots,m-1\} \text{ distinct }\big).$$

Proposition 2. *[Smy11, Prop. A.3] If $p \in C$ is a Gorenstein curve singularity of type $(1,m)$, then p is an elliptic m-fold point.*

This implies that if $(C, p_1, \dots p_n)$ is an n-pointed Gorenstein curve of arithmetic genus 1, then C has only nodes and elliptic m-fold points. However, the moduli problem of n-pointed Gorenstein curves of arithmetic genus one is not separated. For instance, a family could degenerate to either a tacnode or an elliptic bridge. There is, however, a natural stability condition from Smyth for each $m \geq 1$ that destabilizes, for example, elliptic bridges when $m \geq 2$.

Definition 2.10. Fix a positive integer m, and let $(C, p_1, \dots, p_n)$ be an n-pointed Gorenstein curve of arithmetic genus one, and let Σ denote the support of the divisor $p_1 + \dots + p_n$. Then C is *m-stable* if:

(1) the singularities of C are nodes or elliptic l-fold points, $l \leq m$;

(2) the minimal elliptic subcurve $Z \subset C$ satisfies

$$|Z \cap \overline{C \setminus Z}| + |Z \cap \Sigma| > m\,;$$

(3) over any singular point $p \in C$, there is at least one rational branch $\widetilde{B}'$ in the normalization of C with at least three distinguished points;

(4) any rational component $\widetilde{B}$ of the normalization of C satisfies the condition that either

  (a) $\widetilde{B}$ has at least three distinguished points, or

  (b) $\widetilde{B}$ has two distinguished points, and one of them lies above an elliptic l-fold point.

Theorem 2.11 ([Smy11, Theorem 3.8]). *The moduli functor $\overline{\mathcal{M}}_{1,n}(m)$ of m-stable curves is represented by a proper irreducible Deligne-Mumford stack over* $\operatorname{Spec}\mathbb{Z}[1/6]$.

Non-trivial automorphisms in characteristic 2 and 3 for cusps, and characteristic 2 for tacnodes leads to the restriction that 6 be invertible. Of course, we work exclusively over characteristic 0.

2.3 Divisorial contractions

As mentioned in the introduction, work on the birational geometry of the moduli space of genus g curves provides us with much insight into alternate modular compactifications. We particularly focused on the work of Hassett-Hyeon in [HH09] as inspiration for our own approach. We recall the relevant results of that paper here, from our perspective. This also serves to introduce our technique for section 5.

Let $\overline{\mathcal{M}}_g$ denote the moduli stack of stable genus g curves, $K_{\overline{\mathcal{M}}_g}$ the canonical divisor, and $\delta := \sum \delta_i$ the sum of boundary divisors δ_i. Recall that the boundary divisors in this setting are given by the codimension one strata indexed by dual graphs with two vertices connected by one edge. Since there are no marks and we are in genus g, there is such a stratum δ_i for $i = 0, 1, \ldots, \lfloor g/2 \rfloor$, with δ_0 corresponding to the locus of irreducible nodal genus g curves, and δ_i corresponding to curves with one component of genus i and another of genus $g - i$. In [HH09], Hassett and Hyeon investigate the *log canonical model*

$$\overline{M}_g(\alpha) := \operatorname{Proj}\left(\oplus_{n\geq 0} \Gamma(n(K_{\overline{\mathcal{M}}_g} + \alpha\delta))\right)$$

for the critical value $\alpha = 9/11$. In this situation, $K_{\overline{\mathcal{M}}_g} + (9/11)\delta$ is not ample, but rather is ample away from a unique extremal ray - the locus of elliptic tails. The model then is given by a *divisorial contraction*

$$\overline{M}_g \to \overline{M}_g(9/11).$$

A natural goal with the log minimal model program is to obtain modular interpretations of these log canonical models, and this situation does not disappoint. Indeed, Hassett and Hyeon proceed to produce a morphism of algebraic stacks from stable genus g curves to the psuedostable curves of Schubert.

$$\mathcal{T} : \overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g^{ps}$$

This in turn then allows them to relate the coarse moduli spaces, showing that the induced

$$T : \overline{M}_g \to \overline{M}_g^{ps} \simeq \overline{M}_g(9/11)$$

is equivalent to the divisorial contraction.

We are primarily intrigued by the construction of the morphism $\mathcal{T} : \overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g^{ps}$ as an example of what might be done for the m-stable curves of Smyth. In section 5, we will construct an analogous contraction to the moduli space $\overline{\mathcal{M}}_{1,n}(m)$ of m-stable curves, building the appropriate (log) stack to contract *from* in section 4. In our language, the construction of Hassett and Hyeon can essentially be given by building a line bundle $L := \omega_\pi(E)$, where E is the divisor of elliptic tails in the universal curve $\pi : \mathcal{C} \to \overline{\mathcal{M}}_g$. Then L is π-nef numerically nontrivial and so defines psuedostable curves as an extremal assigment, but even more specifically the pushforward $\pi_* L^n$ is also locally free. Hassett and Hyeon achieve this technical detail with an analysis of limiting linear series, showing that the limit over a discrete valuation ring is independent of the DVR itself. Then local freeness follows from an application of flattening and blowing up. Armed with local freeness and taking Proj of the section ring, the resulting contraction ϕ on the universal curve produces a flat family of pseudostable curves over $\overline{\mathcal{M}}_g$. Then the map $\overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g^{ps}$ is the classifying morphism induced by the following diagram.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "\phi"] \arrow[dr] & & \operatorname{Proj}\left(\oplus_{n\geq 0} \pi_* L^n\right) \arrow[dl] \\
& \overline{\mathcal{M}}_g &
\end{tikzcd}$$

In section 5, we follow nearly the same construction. Note that a major difference between elliptic tails and the loci replaced by m-fold singularities in $\overline{\mathcal{M}}_{1,n}(m)$ is that elliptic tails already form a divisor, whereas elliptic bridges, for instance, do not. Furthermore, there is a unique one-branched singularity of genus one (the cusp), so the local freeness of $\pi_* L$ was not needed to achieve the existence of $\overline{\mathcal{M}}^{ps}$ from an extremal assignment $\mathcal{Z}$-stability condition. This will not be true for m-fold curves. We will not deviate from the approach to resolving the indeterminacy in the birational map

$$\overline{\mathcal{M}}_{1,n} \dashrightarrow \overline{\mathcal{M}}_{1,n}(m)$$

with some kind of blowup. The novel tool will be using logarithmic structure to perform the blowup for us, see section 3.4.

3 Moduli of log smooth curves

In this section we recall the needed background on logarithmic structures, logarithmic algebraic stacks, and develop the tools we will need for our own constructions in the sections that follow. Our analysis of infinitesimal automorphisms of log smooth curves corrects a minor error that has persisted in the literature, and may be of independent interest for its utility in moduli problems involving semistable rational curves.

3.1 Preliminaries on log structures

Here we briefly summarize the relevant definitions regarding logarithmics structure, referring to [Ogu06] for additional detail. To begin, for us a *monoid* will always be a commutative monoid (a semigroup) with a unit, which we will write additively.

Any monoid has an associated group

$$M^{gp} := \{(a-b) : (a-b) \sim (c-d) \text{ if there is an } s \in M \text{ such that } s+a+d = s+b+c\},$$

and will be called called *integral* if $M \hookrightarrow M^{gp}$ is injective. Note that for an integral monoid the cancellation property holds, so we may write $(a-b) = (c-d)$ if and only if $a+d = c+b$ in M^{gp}.

M is *saturated* if it is integral and for any $a \in M^{gp}$ such that $na \in M$ for some integer $n > 0$, then $a \in M$. For any integral monoid M, we denote by M^{sat} the smallest saturated submonoid of M^{gp} containing M, and refer to M^{sat} as the *saturation* of M.

A monoid will be called *fine* if it is integral and coherent (finitely generated). We will be working with *fs* log structures, which means that monoids will be fine and saturated. A monoid whose only invertible element is the identity is called *sharp*.

Let X be a scheme, and consider $\mathcal{O}_X$ as a sheaf of monoids under multiplication. A *pre-log structure* on X is a sheaf of monoids M_X on the étale site $X_{\text{ét}}$, together with a morphism of sheaves of monoids $\exp : M_X \to \mathcal{O}_X$, refered to as the *structure morphism*. We will assume the étale topology throughout, and so do not include the decoration ét in our notation.

A pre-log structure is a *log structure* if the restriction of exp to the invertible elements of the monoid, $\exp^{-1}(\mathcal{O}_X^*) \xrightarrow{\exp} \mathcal{O}_X^*$, is an isomorphism. Given a log structure, the reverse inclusion will be denoted

$\log : \mathcal{O}_X^* \to M_X$. A *log scheme* is a pair (X, M_X), with X a scheme and M_X a log structure on X.

The *characteristic monoid* of a log scheme is the quotient sheaf $\overline{M}_X := M_X/\mathcal{O}_X^*$, and we note this will always be *sharp* (the only element in $\overline{M}_X^*$ is 0). We will use the overline notation in general to denote the quotient of a monoid by its group of units, $\overline{P} := P/P^*$.

Given a pre-log structure $\exp : P \to \mathcal{O}_X$ on a scheme X, the *associated log structure* $P^a := P \oplus_{\exp^{-1}(\mathcal{O}_X^*)} \mathcal{O}_X^*$ is constructed as the pushout.

$$\begin{tikzcd}
0 \arrow[r] & \exp^{-1}(\mathcal{O}_X^*) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & P \arrow[r] \arrow[d] & \overline{P} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_X^* \arrow[r] & P^a \arrow[r] & \overline{P} \arrow[r] & 0
\end{tikzcd} \tag{3.1}$$

Given a scheme X, we can talk about the log structures over X, with morphisms $N \to M$ defined as morphisms on sheaves of monoids over $\mathcal{O}_X$. For $f : X \to S$ a morphism of schemes, and M_S a log structure on S, the pullback of M_S *as a log structure* is defined to be the log structure associated to

$$f^{-1}(M_S) \to f^{-1}(\mathcal{O}_S) \to \mathcal{O}_X.$$

Note that the construction of the associated log structure 3.1 gives us the fact that $f^{-1}(\overline{M_S}) = \overline{f^* M_S}$.

Finally we are prepared to define a morphism of log schemes $(f, f^\flat) : (X, M_X) \to (S, M_S)$ as a pair of morphisms, one on the underlying schemes $f : X \to S$ and a morphism of log structures $f^\flat : f^*(M_S) \to M_X$. A morphism is *strict* if the induced map on log structures is an isomorphism.

3.1.1 Charts

Briefly, a chart of a log scheme (X, M_X) is a morphism from a constant sheaf of monoids P to M_X so that the associated log structure of the composition $P \to M_X \to \mathcal{O}_X$ is M_X. Given a monoid P, we will denote the sheaf of constant monoids on X induced by P as P_X. Note that charts always exist étale locally.

A log structure M_X on X is called *fine* (resp. fs) if (X, M_X) admits a chart $P \to M_X$ from a fine (resp. fs) monoid P.

We may also think of a chart on the level of the characteristic sheaf as a morphism from a sheaf of constant monoids $\overline{P} \to \overline{M}_X$ such that when we pull back along the quotient map for $\overline{M}_X$, the extension $P := M_X \times_{\overline{M}_X} \overline{P}$ is a chart in the previous sense. This notion is quite useful, so we explain a bit more in detail. Let $\overline{P} \to \overline{M}_X$ be any morphism from a sharp sheaf of constant monoids, and consider the diagram below obtained by pullback.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X^* \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \overline{P} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X^* \arrow[r] & M_X \arrow[r] & \overline{M}_X \arrow[r] & 0
\end{tikzcd}$$

We may pushout the top row along the automorphism of $\mathcal{O}_X^*$ induced by the structure morphism $\exp : M_X \to \mathcal{O}_X$, thereby obtaining the associated log structure of the extension P.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X^* \arrow[r] & P^a \arrow[r] & \overline{P} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_X^* \arrow[r] \arrow[u, "\exp"] \arrow[d, equal] \arrow[ur, phantom, "\urcorner", very near end] & P \arrow[r] \arrow[u] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \overline{P} \arrow[r] \arrow[u, equal] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X^* \arrow[r] & M_X \arrow[r] & \overline{M}_X \arrow[r] & 0
\end{tikzcd} \tag{3.2}$$

From the universal property of P^a as a pushout, there is a natural morphism $\beta : P^a \to M_X$, and we say that $\overline{P} \to \overline{M}_X$ is a *characteristic chart* if β is an isomorphism. One immediate advantage of using a characteristic chart is that $\overline{P} \to \overline{M}_X$ is necessarily an isomorphism, whereas a chart $P \to M_X$ in the usual sense is not.

Another very nice consequence of this perspective is that from any $\overline{P} \to \overline{M}_X$, we obtain a canonical log structure on X and a morphism of log structures $P^a \to M_X$ using the construction above, so we have a natural way to build alternate log structures on a log scheme. This is precisely the property we exploit to describe the canonical log structure of a log smooth curve.

We also have a natural notion of a chart of a map of fine log schemes. Given $\pi : (X, M_X) \to (S, M_S)$, a chart is a pair of monoids P and Q, charts $P_X \to M_X$ and $Q_S \to M_S$, and a commutative diagram:

$$\begin{tikzcd}
Q_X \arrow[r] \arrow[d] & P_X \arrow[d] \\
\pi^*(M_S) \arrow[r] & M_X
\end{tikzcd} .$$

3.1.2 Log smooth and integral morphisms

An *integral* morphism of integral monoids is a morphism $h : M_S \to M_X$ with the additional property that for any $a_1, a_2 \in M_S$, $b_1, b_2 \in M_X$, with $h(a_1) + b_1 = h(a_2) + b_2$, there exist $a_3, a_4 \in M_S$ and $b_3 \in M_X$ so that $b_1 = h(a_3) + b_3, b_2 = h(a_4) + b_3$, and $a_1 + a_3 = a_2 + a_4$. In other words, we may rewrite

$$h(a_1) + b_1 = h(a_2) + b_2 \qquad \text{as} \qquad h(a_1) + h(a_3) + b_3 = h(a_2) + h(a_4) + b_3,$$

and recall that the cancellation property holds for integral monoids.

Definition 3.1. An *integral* morphism of log schemes $\pi : (X, M_X) \to (S, M_S)$ is given by the property that at any point $x \in X$, the induced $h : \overline{M}_{\pi(x)} \to \overline{M}_x$ is an integral morphism of integral monoids.

A strict closed immersion of log schemes is a morphism $i : (T, M_T) \to (S, M_S)$ with underlying closed immersion of schemes and $i^*(M_S) \to M_X$ an isomorphism. A *log square zero extension* is a strict closed immersion $(T', M'_T) \to (T, M_T)$ with underlying map of schemes $T' \to T$ defined in by a square zero ideal in T. Then the analogous definitions of smooth and étale for a log map $(X, M_X) \to (S, M_S)$ whose underlying morphism is locally finitely presented are given in [Kat89] by the following infinitesimal lifting properties.

Definition 3.2. For any commutative diagram of solid arrows below with vertical arrow on the left a log square zero extension, $(X, M_X) \to (S, M_S)$ is *log smooth* if it is locally finitely presented and there exists a dashed arrow as in the diagram below. If there is a unique such arrow then the map is *log étale.*

$$\begin{tikzcd}
(T', M_{T'}) \arrow[r] \arrow[d] & (X, M_X) \arrow[d] \\
(T, M_T) \arrow[r] \arrow[ur, dashed] & (S, M_S)
\end{tikzcd}$$

Remark 3.1. It is foundational [Kat89, Cor. 4.5] that for a log smooth and integral morphism $(X, M_X) \to (S, M_S)$, the underlying map $X \to S$ is flat.

Definition 3.3. [ACG$^+$13, 3.3] For $(X, M_X) \to (Y, M_Y)$ a morphism of fine log schemes and an $\mathcal{O}_X$-module I, a *log derivation of* (X, M_X) *over* (Y, M_Y) *to* I is a pair (∂, D) where ∂ is a derivation of X over Y to I in the usual sense, and $D : M_X \to I$ is an additive map such that

(1) $D(ab) = D(a) + D(b)$ for $a, b \in M_X$;

(2) $\exp(a)D(a) = \partial(\exp(a))$, for $a \in M_X$ (and $\exp : M_X \to \mathcal{O}_X$ the structure morphism);

(3) $D(a) = 0$, for $a \in f^{-1}M_Y$.

Note that for any log square zero extension $(T', M'_T) \to (T, M_T)$ with underlying map of schemes $T' \to T$ defined by a square zero ideal I, and a commutative diagram as below

$$\begin{tikzcd}
(T', M_{T'}) \arrow[r] \arrow[d] & (X, M_X) \arrow[d] \\
(T, M_T) \arrow[r] \arrow[ur, dashed, shift left, "{(g,g^\flat)}"] \arrow[ur, dashed, shift right, "{(f,f^\flat)}"'] & (Y, M_Y)
\end{tikzcd} \quad ,$$

the difference of the two dashed arrows yields a pair $(f - g, f^\flat - g^\flat)$ that is a log derivation of (X, M_X) over (Y, M_Y) to I.

We may define the *sheaf of log derivations of* (X, M_X) *over* (Y, M_Y) *to* I, denoted $\mathrm{Der}_Y(X, I)$ when the log structures are clear from context, as the sheaf of germs of pairs (∂, D). The *sheaf of logarithmic differentials* is a universal $\mathcal{O}_X$-module $\Omega^1_{X/Y}$ representing the functor $\mathrm{Der}_Y(X, \cdot)$ on $\mathcal{O}_X$-modules. It has a description as

$$\Omega^1_{X/Y} = \big(\Omega_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M_X^{gp})\big)/K,$$

where K is the $\mathcal{O}_X$-module generated by local sections of the form:

(1) $(d\exp(a), 0) - (0, \exp(a) \otimes a)$, for $a \in M_X$;

(2) $(0, 1 \otimes a)$ for $a \in \mathrm{Im}(f^{-1}(M_Y) \to M_X)$.

3.1.3 Definition of a log curve

The following definitions and characterizations are given in [Kat00], specifically definition 1.2, theorem 1.3, and the construction in section 2.

Definition 3.4. A *log curve* is a log smooth and integral morphism to an fs log scheme $\pi : (X, M_X) \to (S, M_S)$ such that every geometric fiber of the underlying map $X \to S$ is a reduced and connected curve.

Let $f : (X, M_X) \to (S, M_S)$ be a log curve. The *relative characteristic* is $\overline{M}_{X/S} := M_X/\mathrm{im}(f^*\mathcal{M}_S \to \mathcal{M}_X)$.

Proposition 3. *[Kat00, Theorem 1.3] Any log curve over a separably closed field has at worst nodal singularities, and if there are ℓ nodes r_i, then there exist smooth disjoint points $s_1, \ldots, s_n$ of X such that*

$$\overline{M}_{X/S} = \mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_\ell} \oplus \mathbb{N}_{s_1} \oplus \cdots \oplus \mathbb{N}_{s_n}.$$

(These are skyscraper sheaves at the respective points). Note that a log curve determines an underlying marked nodal curve!

Definition 3.5. A log curve is said to have *type* (g, n) if the underlying curve is of genus g and there are n sections s_i as above, corresponding to unlabeled marked points. A log curve is *stable* if the underlying curve is a stable nodal curve (see further discussion of stability in the next section).

Any nodal curve $X \to S$ with ℓ nodes may be equipped with a *canonical* log structure of the form

$$M_X^{min} = M_1 \oplus_{\mathcal{O}_X^*} \cdots \oplus_{\mathcal{O}_X^*} M_\ell \oplus_{\mathcal{O}_X^*} \mathcal{N}$$

$$M_S^{min} = \mathcal{L}_1 \oplus_{\mathcal{O}_X^*} \cdots \oplus_{\mathcal{O}_X^*} \mathcal{L}_\ell$$

that induces a log curve $(X, M_X^{can}) \to (S, M_S^{can})$ satisfying the universal property of a final object in the category of log structures making $X \to S$ into a log curve. See section 3.2.1 for further description of this canonical log structure. Note that what we are calling *canonical* is referred to as *basic* in [Kat00], and has been termed *minimal* or *special* by other authors as well. We note that there may be many log structures over an underlying scheme that deserve the term "minimal" in the sense that they are given by *some* universal property. The point of using alternate terminology here is that the canonical log structure is minimal among *all* possible log structures.

3.1.4 Log structures on stacks

We denote the categories of schemes and fs log schemes by $\mathfrak{Sch}$ and $\mathfrak{LogSch}^{\mathrm{fs}}$, respectively. There is a forgetful functor $\mathfrak{LogSch}^{\mathrm{fs}} \to \mathfrak{Sch}$ taking an fs log scheme to its underlying scheme.

Lemma 3.6. *[Kat00, Lemma 4.1] The moduli problem of stable log curves of type (g, n) is representable by an algebraic stack $\mathcal{LM}_{g,n}$ over $\mathfrak{LogSch}^{\mathrm{fs}}$.*

Definition 3.7. An *fs logarithmic structure* on a fibered category $\mathcal{S}$ over $\mathfrak{Sch}$ is a cartesian section of $\mathfrak{LogSch}^{\mathrm{fs}} \to \mathfrak{Sch}$ over $\mathcal{S}$. This ammounts to a commutative triangle

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "M_{\mathcal{S}}"] \arrow[d] & \mathfrak{LogSch}^{\mathrm{fs}} \arrow[dl] \\
\mathfrak{Sch} &
\end{tikzcd}$$

that takes morphisms in $\mathcal{S}$ to strict morphisms of log schemes in $\mathfrak{LogSch}^{\mathrm{fs}}$. An fs *log algebraic stack* is a pair $(\mathcal{S}, M_{\mathcal{S}})$ such that $\mathcal{S}$ is an algebraic stack over $\mathfrak{Sch}$ and $M_{\mathcal{S}} : \mathcal{S} \to \mathfrak{LogSch}^{fs}$ is a logarithmic structure on $\mathcal{S}$. A logarithmic Deligne-Mumford stack is a log algebraic stack such that the underlying algebraic stack is Deligne-Mumford algebraic.

Lemma 3.8. *[Kat00, Lemma 4.4] The canonical log structure on a nodal curve defines an fs log structure on $\overline{\mathcal{M}}_{g,n}$, equivalent to the log structure determined by the normal crossings boundary.*

We denote the resulting fs log algebraic stack (a logarithmic Deligne-Mumford stack) by $(\overline{\mathcal{M}}_{g,n}, M_{\overline{\mathcal{M}}_{g,n}})$.

An fs log algebraic stack $(\mathcal{S}, M_{\mathcal{S}})$ may naturally be considered a stack over $\mathfrak{LogSch}^{fs}$ simply by taking the associated category of maps of log stacks $(\mathcal{X}, M_{\mathcal{X}}) \to (\mathcal{S}, M_{\mathcal{S}})$, with morphisms defined as morphisms over $(\mathcal{S}, M_{\mathcal{S}})$. We should think of this as the stack of log schemes over $(\mathcal{S}, M_{\mathcal{S}})$, and will refer explicitly to the *associated stack* of $(\mathcal{S}, M_{\mathcal{S}})$ over $\mathfrak{LogSch}^{\mathrm{fs}}$.

Definition 3.9. We say that an algebraic stack $\mathcal{F}$ over $\mathfrak{LogSch}^{\mathrm{fs}}$ is *represented* by a log algebraic stack $(\mathcal{S}, M_{\mathcal{S}})$ if $\mathcal{F}$ is isomorphic to the associated stack of $(\mathcal{S}, M_{\mathcal{S}})$ over $\mathfrak{LogSch}^{\mathrm{fs}}$.

Finally, for any algebraic stack $\mathcal{F}$ over $\mathfrak{LogSch}^{\mathrm{fs}}$, we denote by $\mathbf{Log}(\mathcal{F})$ the naturally associated log algebraic stack (so an algebraic stack over $\mathfrak{Sch}$ equipped with a log structure) given by the forgetful functor $\mathfrak{LogSch}^{\mathrm{fs}} \to \mathfrak{Sch}$. Specifically, for a scheme S, $\mathbf{Log}(\mathcal{F})(S)$ is the category of pairs (M_S, ξ), where M_S is a log structure on S, ξ is an object of $\mathcal{F}(S, M_S)$, and arrows are isomorphisms of pairs. The result is a category $\mathbf{Log}(\mathcal{F})$ whose objects $\xi : (S, M_S) \to \mathcal{F}$ are the same as the objects of $\mathcal{F}$, but morphisms

$$\begin{tikzcd}
(S, M_S) \arrow[r] \arrow[d] & (S', M_S') \arrow[dl] \\
\mathcal{F} &
\end{tikzcd}$$

are required to be strict.

We identify the substack of $\mathbf{Log}(\mathcal{LM}_{g,n})$ consisting of stable log curves equipped with the canonical log structure with $(\overline{\mathcal{M}}_{g,n}, M_{\overline{\mathcal{M}}_{g,n}})$.

Theorem 3.10. *[Kat00, Theorem 4.5] The logarithmic Deligne-Mumford stack $(\overline{\mathcal{M}}_{g,n}, M_{\overline{\mathcal{M}}_{g,n}})$ represents $\mathcal{LM}_{g,n}$.*

3.2 Stable log smooth curves

Log geometry is particularly well-suited to naturally compactifying moduli problems, due to the fact that log smooth morphisms include particular types of degenerations (see [Gil16, Definition 4.1.1]), of course we will be interested specifically in nodal degenerations. Heuristically, compactifying a moduli problem in a meaningful way amounts to a natural choice for degenerate objects to include, our main example being the compactification of smooth curves of genus g with n markings by stable marked nodal curves, $\mathcal{M}_{g,n} \hookrightarrow \overline{\mathcal{M}}_{g,n}$. The representability of the moduli space of stable log smooth curves of type (g, n) by stable n-marked nodal curves of genus g equipped with the canonical log structure is a compelling argument for the utility of log structures in describing these problems. We use this section to highlight this example, showing in our language that Kato's moduli space $\mathcal{LM}_{g,n}$ of stable log smooth curves of type (g, n) is representable by the log algebraic stack $(\overline{\mathcal{M}}_{g,n}, M_{\overline{\mathcal{M}}_{g,n}})$ of stable curves equipped with the canonical log structure.

The stratification of $\overline{\mathcal{M}}_{g,n}$ by topological type exhibits the combinatorial nature of the boundary $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$, and we will see that log algebraic stacks are similarly stratified, with the minimal log structure on our logarithmic curves admitting a description depending solely on a modified version of the dual graph of underlying geometric fibers. Tied to these statifications is the important intuition that log structure is determined by its behavior on stalks. We state the following well known facts regarding log structures and charts, which will allow for us to frequently reduce our discussions to sufficiently small étale neighborhoods of geometric points and relate the combinatorics of the logarithmic structures directly to the combinatorics of dual graphs.

Lemma 3.11. *[GS13, Definition 2.11] Let $\overline{M}$ be a constructible sheaf on a quasi-compact quasi-separated scheme S and $A \subset |S|$ a set of geometric points. Define the subset $U_A \subset |S|$ to be the collection of points $\overline{x} \in |S|$ for which there is a finite sequence $\overline{s}_1 \in A$, $\overline{t}_1, \ldots, \overline{t}_{r-1}, \overline{s}_r, \overline{x} \in |S|$ of* geometric *generizations and specializations*

$$\begin{tikzcd}
& \overline{t}_1 \arrow[dl, rightsquigarrow] \arrow[dr, rightsquigarrow] & & \overline{t}_2 \arrow[dl, rightsquigarrow] \arrow[dr, rightsquigarrow] & & \cdots \arrow[dl, rightsquigarrow] \arrow[dr, rightsquigarrow] & & \overline{x} \arrow[dl, rightsquigarrow] \\
\overline{s}_1 & & \overline{s}_2 & & \overline{s}_3 & & \overline{s}_r &
\end{tikzcd}$$

such that for any specialization $\overline{t}_i \rightsquigarrow \overline{s}_{i+1}$, the associated map $\overline{M}_{\overline{s}_{i+1}} \to \overline{M}_{\overline{t}_i}$ is an isomorphism. Then U_A is open.

Proof. Since $\overline{M}$ is constructible, U_A is a constructible set [Sta16, Lemma 50.67.2]. It is closed under generization by definition. $\square$

Note that this simple statement does not a priori have anything to do with log structures. To make the importance clear, we will use the example of minimal log structure in the case of stable log curves, following the ideas presented by Chen in [Che14] for using the dual graphs of the underlying curves to pick out the minimal log structure. We summarize a few results that will be used here and in the construction of minimal log structure for our curves later on, and make the connection between lemma 3.11 and fine log structure explicit.

Lemma 3.12. *[Ols03, 2.1] Let M be a fine log structure on a scheme S with $\overline{M}^{gp}$ torsion free (as is automatically the case for fs log structures), and let $\overline{s}$ be a geometric point. Then there is an étale neighborhood $f : S' \to S$ of $\overline{s}$ and a chart $P \to f^*M$ for which the natural map $P \to f^{-1}\overline{M}_{\overline{s}}$ is bijective.*

Lemma 3.13. *[Wis16, Lemma B.3] Let $\varphi : M \to M'$ be a morphism of coherent log structures on a scheme S. Then the locus in S on which φ is an isomorphism is open.*

The essential idea is that if φ is an isomorphism at a geometric point, then in a neighborhood of that point, we have charts given by the stalks of the respective characteristic sheaves. Isomorphic charts give rise to isomorphic associated logarithmic structure, and the result follows.

Lemma 3.14. *[Ols03, 3.5] Let S be a quasi-compact scheme and M a fine log structure on S. Then the sheaf $\overline{M}^{gp}$ is a constructible sheaf of abelian groups. Furthermore, for $\overline{t} \to S$ a generization of a geometric point $\overline{s} \to S$, the specialization map $\overline{M}^{gp}_{\overline{s}} \to \overline{M}^{gp}_{\overline{t}}$ is surjective.*

Remark 3.2. Suppose we have a geometric point $\overline{s} \to S$ of a log scheme (S, M_S). By Lemma 3.14, $\overline{M}^{gp}_S$ is constructible and specialization maps associated to generizations are surjective. We may apply lemma 3.11 with $A := \{\overline{s}\}$, and we then have a neighborhood U such that any geometric fiber of $\overline{M}^{gp}_S|_U$ admits a surjection from $\overline{M}^{gp}_{S,\overline{s}}$. In particular, there is a neighborhood U of $\overline{s}$ on which the natural map $\overline{M}^{gp}_S|_U \to \overline{M}^{gp}_{S,\overline{s}}$

is an isomorphism. That is, the germs of the characteristic sheaf may be *simultaneously* extended to an open neighborhood on which there are no other sections.

This is also reflected by the behavior of dual graphs. Specifically, given a subcurve $C \to S$ and a dual graph G of a geometric fiber $C_{\overline{s}} \to \overline{s}$, there is a neighborhood U of $\overline{s}$ on which every vertex of G is associated with a unique vertex in every dual graph over a geometric fiber in U. This corresponds to an irreducible component of C over all of S. We will need this observation later on.

Definition 3.15. For a geometric point $\overline{s} \to S$ of a log scheme S we say that a neighborhood U on which we have $\overline{M}_S^{gp}\big|_U \xrightarrow{\sim} \overline{M}_{S,\overline{s}}^{gp}$ is an *atomic* neighborhood. This notion will be used in the following ways:

(1) to refer to a sufficiently local neighborhood of a geometric point of a log scheme in general, including log curves;

(2) to refer to a sufficiently local neighborhood of a geometric point of a log scheme parametrizing a log curve so as to identify "smoothing parameter sections";

(3) to refer to a sufficiently local neighborhood of a geometric point $\overline{s}$ of a *scheme* parametrizing a *stable* curve, such that the dual graph of any geometric fiber over the neighborhood admits a surjection from the dual graph of the fiber over $\overline{s}$. This is the same as (2) above in the case that the log curve is equipped with the canonical log structure.

We hope that our use of "atomic" in non-log situations is not too confusing, as the notions agree when they overlap for log curves equipped with the canonical log structure. The existence of atomic neighborhoods for dual graphs can be made precise by associating dual graphs with a constructible sheaf of partially ordered sets.

Remark 3.3. The existence of atomic neighborhoods implies that constructing a new log structure on a log scheme (X, M_X) is particularly simple in a neighborhood of a geometric point. Namely, let $\overline{x}$ be a geometric point, and take any map $\overline{P} \to \overline{M}_{X,\overline{x}}$ of monoids. Then in an atomic neighborhood U of $\overline{x}$, we have a map from a constant sheaf of monoids $\overline{P}_U \to \overline{M}_U$, and this gives rise to an associated log structure P_U^a over U by the construction of (3.2). We will take advantage of precisely such a construction, describing a canonical morphism of monoids over any geometric point that is compatible with generization. It is essential that the constructed log structure maps to $\overline{M}_X$, as then lemma 3.13 is very useful to note that the locus on which the constructed log structure is isomorphic to M_X is open. This argument will be the recurring theme for the constructions of the canonical and minimal log structures below.

3.2.1 Canonical log structure on stable curves

It is well known that the canonical log structure equips $\overline{M}_{g,n}$ with the log structure associated to the boundary divisor of nodal curves. We recall here this canonical log structure, introducing the useful approach of associating monoids to the dual graphs of the underlying curves.

Local description of log structures

Let $f : (C, M_C) \to (S, M_S)$ be a log smooth curve of type (g, n), $(C_{\overline{s}}, M_{C,\overline{s}}) \to (\overline{s}, M_{S,\overline{s}})$ a geometric fiber, and x a point in the fiber. Possibly by shrinking to an étale neighborhood of $\overline{s}$, we will assume we have an S itself as atomic neighborhood, so $\overline{M}_S \to \overline{M}_{S,\overline{s}}$ is a bijection. We will let $Q := \overline{M}_{S,\overline{s}}$. There are three cases to consider:

(i) For x any smooth point of the fiber, there is a neighborhood $U \subset C$ on which $M_C\big|_U \to M_S\big|_{f(U)}$ is strict and $U \to f(U)$ is smooth.

(ii) If x is a marked point, we have a neighborhood U on which the log structure is described by the chart

$$\begin{tikzcd}
\mathbf{N} \arrow[r] & (Q \oplus \mathbf{N})_U \arrow[r] & M_U \\
0 \arrow[u] \arrow[r] \arrow[ur, phantom, "\urcorner", very near end] & Q_{f(U)} \arrow[u] \arrow[r] & M_S \arrow[u]
\end{tikzcd}$$

(iii) For x a nodal point, we have a neighborhood U on which the chart of f takes the form

$$\begin{tikzcd}
\mathbf{N}^2 \arrow[r] & Q_U \arrow[r] & M_U \\
\mathbf{N} \arrow[u, "\Delta"] \arrow[r] \arrow[ur, phantom, "\urcorner", very near end] & Q_{f(U)} \arrow[u] \arrow[r] & M_S \arrow[u]
\end{tikzcd}$$

Recall that $\overline{M}_S$ is sharp, and that the irreducible elements of a sharp fs monoid will be a finite generating set. Furthermore, from the local description above, for every node in $C_{\overline{s}}$, we get a well defined element in

Q from the generator of $\mathbf{N}$, which we will refer to as the *smoothing parameter* δ of the node. The elements from the generators of $\mathbf{N}^2$ we will typically be denoted by $\log x$ and $\log y$, with $\delta \mapsto \log x + \log y$.

We are able to use this map from nodes to irreducible elements to give a simple description of the canonical log structure. For our description, we recall that the dual graph G of $C_{\overline{s}}$ consists of the following data:

(1) a set of vertices V corresponding to the irreducible components of $C_{\overline{s}}$ and weighted by their genera;

(2) a set of edges E indexed by the nodes of $C_{\overline{s}}$, incident to the vertices corresponding to the irreducible component(s) that meet at that node (loops allowed);

(3) a set of half-edges H indexed by the markings, each incident to the vertex corresponding to the marked component.

We associate a monoid $\overline{M}(G) = \mathbf{N}^E$ to the dual graph, freely generated by irreducible elements corresponding to the edges of G. For e an edge, we have $\delta_e \in \overline{M}_{S,\overline{s}}$ the smoothing parameter of the corresponding node, and hence we have a canonical map of monoids.

$$\overline{M}(G) \to \overline{M}_S \tag{3.3}$$

Now we define a log curve of type (g, n) equipped with the canonical log structure to be a log curve of type (g, n) for which the map 3.3 is an isomorphism. Note that $\overline{M}(G)$ and the map above exist as defined over S an atomic neighborhood of a geometric point, and that we have such a map in a neighborhood of *every* geometric fiber of $\mathbf{Log}(\mathcal{LM}_{g,n})$. In particular we have a new log structure on $\mathbf{Log}(\mathcal{LM}_{g,n})$ and a map of log structures $M^{can} \to M_{\mathbf{Log}(\mathcal{LM}_{g,n})}$ (as in remark 3.3). From Lemma 3.13, we immediately have

Proposition 4. *The locus of log smooth curves equipped with the canonical log structure is open in* $\mathbf{Log}(\mathcal{LM}_{g,n})$*, and hence is algebraic.*

To show representability of $\mathcal{LM}_{g,n}$ by this locus of canonical log curves, we need to verify Gillam's criteria, see [Wis16, Proposition B.1] or [Gil12].

Proposition 5 (Universal Property of canonical log structure)**.** *Let* $(X, M_X) \to (S, M_S)$ *be a log smooth curve of type* (g, n)*. There is a log smooth curve of type* (g, n) *equipped with the canonical log structure,* *unique up to unique isomorphism,* $(X, M_X^{can}) \to (S, M_S^{can})$ *and a cartesian diagram of log schemes as below.*

$$\begin{tikzcd}
(X, M_X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & (X, M_X^{can}) \arrow[d] \\
(S, M_S) \arrow[r] & (S, M_S^{can})
\end{tikzcd}$$

Furthermore, the pull back of $(X, M_X^{can}) \to (S, M_S^{can})$ *along any strict morphism* $(S', M_{S'}) \to (S, M_S^{can})$ *is a canonical log curve* $(X_{S'}, M_{X_{S'}}^{can}) \to (S', M_{S'} = M_{S'}^{can})$.

This is given by the construction contained in [Kat00] and is Proposition 2.3 and Theorem 2.6 there. We only include the details of the minimal log structure on S below, as this strategy is used again later on.

Proof. The statement is local on S, so we may assume that we are in an atomic neighborhood for some geometric point $\overline{s}$ yielding an isomorphism $\overline{M}_S \xrightarrow{\sim} \overline{M}_{S,\overline{s}}$. Then the desired canonical log structure M_S^{can} is the log structure associated to the map of sharp constant sheaves of monoids $\overline{M}_G \to \overline{M}_{S,\overline{s}} \cong \overline{M}_S$. ☐

We now briefly summarize the relationships between the various algebraic stacks over $\mathfrak{Sch}$ and $\mathfrak{LogSch}^{\text{fs}}$ implicit in the constructions above. As previously mentioned, the Deligne-Mumford stack of stable curves $\overline{\mathcal{M}}_{g,n}$ may naturally be considered an fs log algebraic stack in the sense of Olsson (that is, an algebraic stack over $\mathfrak{Sch}$, equipped with an fs log structure). The log structure $M_{\overline{\mathcal{M}}_{g,n}}$ is given by the canonical log structure induced by the boundary divisor of nodal curves.

Such an fs log algebraic stack may naturally be viewed as a stack over $\mathfrak{LogSch}^{\text{fs}}$, and the universal property of the canonical log structure shows that the algebraic stack $\mathcal{LM}_{g,n}$ of stable log curves of type (g, n) is the associated stack of $(\overline{\mathcal{M}}_{g,n}, M_{\overline{\mathcal{M}}_{g,n}})$ over $\mathfrak{LogSch}^{\text{fs}}$. We hence may say that $(\overline{\mathcal{M}}_{g,n}, M_{\overline{\mathcal{M}}_{g,n}})$ *represents* $\mathcal{LM}_{g,n}$ [Kat00, Definition 3.5].

In turn, the log algebraic stack $(\overline{\mathcal{M}}_{g,n}, M_{\overline{\mathcal{M}}_{g,n}})$ may be recovered as the substack of $\mathbf{Log}(\mathcal{LM}_{g,n})$ cut out by the canonical log structure [ACG+13]. Here we have used the notation $\mathbf{Log}(-)$ to denote the natural operation of taking an algebraic stack over $\mathfrak{LogSch}^{\text{fs}}$ to the associated category fibered over $\mathfrak{Sch}$ via the forgetful map. This just means that the objects of $\mathbf{Log}(\mathcal{LM}_{g,n})$ are the same (families of log curves of type (g, n)), but morphisms are required to be strict: morphisms in $\mathbf{Log}(\mathcal{LM}_{g,n})$ must be cartesian over $\mathfrak{Sch}$.

There is slight variation in the use of these functors. We have used the typesetting found in [Wis16] and [Gil12] for $\mathbf{Log}(-)$. Olsson's $\mathcal{L}og_{(\cdot)}$ [Ols03] (also used by Chen), and the $\mathrm{LOG}_{(\cdot)}$ of the handbook

[ACG+13] are the *composite* functors taking a log algebraic stack $(\mathcal{X}, M_{\mathcal{X}})$ over $\mathfrak{Sch}$ to $\mathbf{Log}(-)$ of the stack over $\mathfrak{LogSch}^{\text{fs}}$ that $(\mathcal{X}, M_{\mathcal{X}})$ *represents*. The careful reader will note that these other works are over fine log schemes, while we have been working with fs log schemes, but the distinction is not critical to the discussion here.

It should be noted that the meaning of *stable* in reference to a log curve of type (g, n) is that the underlying curve is stable in the Deligne-Mumford sense. Note this is *not* the notion of stability in [Kat00, Definition 1.12], and we discuss this mild distinction (which has proved harmless to the literature) and further implications in the next section. It will also be useful to have some notation for the algebraic stack over $\mathfrak{LogSch}^{\text{fs}}$of pre-stable log smooth curves of genus g with n markings, and of the corresponding log algebraic stack over $\mathfrak{Sch}$ of pre-stable log smooth curves equipped with canonical log structure, we will denote these as $\mathfrak{M}^{pre}_{g,n}$ and $\mathfrak{M}^{can}_{g,n} \subset \mathbf{Log}(\mathfrak{M}^{pre}_{g,n})$, respectively (c.f. [Che14, Appendix B], where our $\mathbf{Log}(\mathfrak{M}^{pre}_{g,n})$ is denoted by $\mathfrak{M}^{pre}_{g,n}$). Given $\mathfrak{M}^{pre}_{g,n}$, it is not hard to construct $\mathfrak{M}^{can}_{g,n}$ following as a template the example above (note stability was not used anywhere in the construction as it is just an assumption on the underlying curves).

3.3 Infinitesimal automorphisms of log smooth curves

Typically, a *stability condition* on elements of a moduli problem is an open condition on a larger stack meant to cut out a "nice" collection, for our purposes this will always be to ensure finite automorphisms of objects, which for nodal curves is typically described as a consequence of the nonexistence of infinitesimal automorphisms. We will see that there are potentially multiple notions of what an *infinitesimal automorphism* should be for a log curve, and will devote this section to this investigation. In general, we will be discussing various natural stability conditions, but we fix the terminology of *pre-stable*, *semistable* and *stable* curves by the following characterizations on the fibers. All curves will be connected and reduced.

(1) pre-stable - has only nodal singularities

(2) semi-stable - pre-stable and rational components in the normalization of a geometric fiber have at least two distinguished points

(3) stable - pre-stable and rational components in the normalization of a geometric fiber have at least three distinguished points

Note that these conditions are combinatorial, and that only an underlying curve that is stable in this sense will have no infinitesimal automorphisms. The stability conditions we discuss below will have a combinatorial presentation as well, with logarithmic structure playing a bookkeeping role. Note as well that any log smooth curve is necessarily pre-stable, and in this section all curves should be assumed to be such, with additional stability conditions specified explicitly.

Suppose we have a log scheme (S, M_S), and an automorphism of the log structure $\varphi \in \mathrm{Aut}(M_S)$. Denote by $\overline{\varphi}$ the corresponding isomorphism $\overline{M}_S \to \overline{M}_S$ on the characteristic sheaf. Then from lemma 3.14, we know the set of points on which $\overline{\varphi} = \overline{\mathrm{id}}$ is open (constuctible and stable under generization). In particular, if a geometric stalk over $\overline{s}$ is fixed by an automorphism of the log structure, then the induced automorphism of the characteristic sheaf is the identity in a neighborhood of the geometric point. One implication is that unlike automorphisms of the underlying curves, triviality of automorphisms of log structure on the level of the characteristic sheaf is an open condition. The other implication will be that "infinitesimal automorphisms" of log smooth curves with the canonical log structure are induced by infinitesimal automorphisms of the underlying curve, so log smooth curves with stable underlying curves have finite automorphism groups. However, the log structure *itself* can in fact rigidify some of the underlying automorphisms, so in general there is a much larger class of log curves with finite automorphisms, containing stable curves equipped with the canonical log structure.

Lemma 3.16. *For (S, M_S) a log scheme, the sequence below is exact.*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(\overline{M}, \mathcal{O}_S^*) \arrow[r] & \mathrm{Aut}(M_S) \arrow[r] & \mathrm{Aut}(\overline{M}_S) \end{tikzcd}$$

Proof. It is enough to work étale locally, so we assume we are on an atomic neighborhood and have a chart $f : \overline{M}_S \to M_S$. Consider the diagram below, which specifies an isomorphism $\varphi : M_S \to M_S$ fixing $\overline{M}_S$.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_S^* \arrow[r] \arrow[d] & M_S \arrow[r] \arrow[d, "\wr"] & \overline{M}_S \arrow[r] \arrow[l, bend right] \arrow[d, equal] \arrow[dll, dashed] & 0 \\ 0 \arrow[r] & \mathcal{O}_S^* \arrow[r] & M_S \arrow[r] & \overline{M}_S \arrow[r] & 0 \end{tikzcd}$$

Let $f(a) \in M_S$ be the image of a generator of $\overline{M}_S$. Then we may consider $f(a)$ and $\varphi \circ f(a)$ as elements of M_S^{gp}, and the difference $f(a) - \varphi \circ f(a)$ goes to zero in $\overline{M}_S^{gp}$. Hence $f(a) - \varphi \circ f(a)$ corresponds to a unique unit $\lambda_a \in \mathcal{O}_S^*$, and we have an element of $\mathrm{Hom}(\overline{M}_S, \mathcal{O}_S^*)$ given by $f - \varphi \circ f$.

On the other hand, for a morphism $h : \overline{M} \to \mathcal{O}_S^*$, let λ be the composition $M_S \to \overline{M}_S \xrightarrow{h} \mathcal{O}_S^*$. Then we get a unique automorphism of M_S, fixing $\overline{M}_S$, by $\alpha \mapsto \alpha + \lambda(\alpha)$ for any $\alpha \in M_S$. $\square$

3.3.1 Automorphisms of smooth liftings

Let $(X, M_X) \to (S, M_S)$ be a log smooth curve, and fix $i : (S, M_S) \to (S', M_{S'})$ a strict closed immersion given by a nilpotent ideal $\mathcal{I}$ in $\mathcal{O}_{S'}$, which we will assume satisfies $\mathcal{I}^2 = (0)$. A *smooth lifting* of (X, M_X) over i is a completion of the following diagram of solid arrows to a cartesian square.

$$\begin{tikzcd}
(X, M_X) \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & (X', M_{X'}) \arrow[d, dashed] \\
(S, M_S) \arrow[r, "i"] & (S', M_S)
\end{tikzcd}$$

Lemma 3.17 ([Kat89, Prop 3.14])**.** *By standard results of deformation theory, there is a canonical isomorphism giving a description of the automorphisms of a smooth lifting over i,*

$$\mathrm{Aut}_i(X', M'_X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{I}\mathcal{O}_{X'}),$$

where $\Omega^1_{X/S}$ denotes the sheaf of log differentials.

For the sake of this discussion, we sketch a proof. Consider the commutative diagrams induced by an automorphism α of a smooth lifting over i.

$$\begin{tikzcd}
 & & (X', M_{X'}) \arrow[dl] \\
(X, M_X) \arrow[r] \arrow[d] \arrow[urr] \arrow[dr, phantom, very near start, "\ulcorner"] & (X', M_{X'}) \arrow[d] \arrow[ur, "\alpha"] & \\
(S, M_S) \arrow[r, "i"] & (S', M_{S'}) &
\end{tikzcd}
\qquad
\begin{tikzcd}
(X, M_X) \arrow[r] \arrow[d] & (X', M_{X'}) \arrow[d] \\
(X', M_{X'}) \arrow[r] \arrow[ur, shift left, "\mathrm{id}"] \arrow[ur, shift right, "\alpha"'] & (S', M_{S'})
\end{tikzcd}$$

Then the difference $\mathrm{id} - \alpha$ provides a log derivation (definition 3.3) of $(X', M_{X'})$ over $(S', M_{S'})$ to $\mathcal{I}\mathcal{O}_{X'}$. Noting that the square on the left is cartesian, $\Omega^1_{X/S}$ represents log derivations of X' over S' to $\mathcal{I}\mathcal{O}_{X'}$ as well [Kat89, 1.7], and the identification of these automorphisms with $\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{I}\mathcal{O}_{X'})$ follows.

Corollary 3.17.1. *Vanishing of $\mathrm{H}^0(X, \Omega^1_{X/S}{}^{\vee})$ implies that, fixing an infinitesimal extension $i : (S, M_S) \to (S', M_{S'})$, any smooth lift $(X', M_{X'})$ over i has only the trivial automorphism.*

The definition (1.12) of a stable log curve of type (g, n) in [Kat00] is given as a log smooth curve with vanishing $\mathrm{H}^0(X, {\Omega^1_{X/S}}^\vee)$. While this does restrict the curve to have no automorphisms of smooth liftings over a *logarithmic* infinitesimal extension, interpreting these automorphisms as the infinitesimal automorphsims of a log curve leads to the conclusion of [Kat00, Corollary 1.15] that the underlying curve of a stable log curve is in fact a stable curve in the classical sense, which we will see fails to be true.

Most literature avoids any technicality from this by simply defining a stable log curve to be a log smooth curve whose underlying curve is in fact stable, equivalently for which the *underlying* curve has no non-trivial infinitesimal automorphisms, or equivalently (by [Kat00, Propostion 1.13], which tells us that $\Omega^1_{X/S} \cong \omega_{X/S}(s_1 + \cdots + s_n)$) for which $\Omega^1_{X/S}$ is *ample*. The confusion that remains would appear to be in choosing a notion of an infinitesimal automorphism of a log smooth curve, and the automorphisms of smooth liftings described above may be a natural choice when working on $\mathfrak{M}^{pre}_{g,n}$ over $\mathfrak{LogSch}^{\mathrm{fs}}$. However, this description provides some pathological behavior we would rather avoid, and is in fact not the natural stability condition from the perspective of the log algebraic stack of curves equipped with canonical log structure.

Lemma 3.18. *Any log smooth curve $(X, M_X) \to (S, M_S)$ equipped with the canonical log structure whose underlying curve is connected and semistable, with at least one stable irreducible component, is stable in the sense of vanishing $\mathrm{H}^0(X, {\Omega^1_{X/S}}^\vee)$.*

Proof. The identification $\Omega^1_{X/S} \cong \omega_{X/S}(s_1 + \cdots + s_n)$ tells us that certainly there are global sections when the underlying curve is genus zero with two marked points. Indeed, on any component C that is a smooth rational curve with two special points, $\omega_{C/S}(p + q)$ is trivial. However, this triviality also implies that a section of $\Omega^{1\vee}_{X/S}$ which vanishes on a component attached to such a semistable rational component must vanish on the entire semistable component as well, so stability of a component in the rest of the curve is enough to kill global sections. □

For a bit more insight on the role of the logarithmic structure, we give an alternate proof here. The point is essentially that by fixing the logarithmic structure of the infinitesimal extension $(S', M_{S'})$, we fix some of the log structure of any smooth lift, namely the parameters of nodes and marked points, and this in turn prevents the usual action of $\mathbb{G}_m$ on the underlying semistable curve from lifting to the log curve.

We proceed by induction on the number of strictly semistable components of X. Since X is connected, each such component will be a rational curve with a node connecting it to the rest of the curve and one other special point. First we note that if X contains no strictly semistable components then obviously the statement holds as in this case $\Omega^1_{X/S}$ is ample.

For the inductive step, we assume that $(X, M_X) \to (S, M_S)$ is a log smooth curve over a geometric point, $(S, M_S) \to (S', M'_S)$ is an exact closed immersion given by a nilpotent ideal $\mathcal{I}$ in $\mathcal{O}_{S'}$, and that X contains a rational component (C, p, q), with $p \in C \cap \overline{X \setminus C}$ a node. Assume that we have a smooth lifting $(X', M_{X'})$ and an automorphism φ. Take an atomic neighborhood of p, and let $\overline{M}_{X,p} \to M_X\big|_U$ be the corresponding chart. From the local description of the log structure, we have irreducible elements of the monoid $\log x$ and $\log y$, for x a local parameter of p on C, y a local parameter of p on $\overline{X \setminus C}$, and these generate $\overline{M}_{X,p}$. Let $\delta \in \overline{M}_S$ be the smoothing parameter of the node, so $\delta = \log x + \log y$. The automorphism and the chart induce a diagram

$$\begin{tikzcd}
 & & & \overline{M}_{X,p} \arrow[dll, dashed] \arrow[dl] \arrow[ddl] \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_X^* \arrow[r] \arrow[d] & M_X \arrow[r] \arrow[d] & \overline{M}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X^* \arrow[r] & M_X \arrow[r] & \overline{M}_X \arrow[r] & 0
\end{tikzcd} \tag{3.4}$$

where the dashed arrow is the difference of the two maps to M_X. Hence the automorphism must be of the form $\log x \mapsto \log x + \log \lambda_x$, and hence δ is sent to $\delta + \log \lambda_x$. However, by the inductive assumption, $\log y$ and the log structure on S' are fixed, so we must have that $\lambda_x = 1$ on S', that is that the automorphism is trivial.

This suggests that in defining an infinitesimal automorphism of a log smooth curve, we might let the log structure on the base vary to get a stability condition that is closer to Deligne-Mumford stability. Indeed, instead of working on $\mathfrak{M}^{pre}_{g,n}$, we pass to the associated log algebraic stack $\mathbf{Log}(\mathfrak{M}^{pre}_{g,n})$, and give our notion of an infinitesimal automorphism there, as an automorphism of a lift over an infinitesimal extension of *schemes* rather than an infinitesimal extension of log schemes. The result is a natural stability condition that gives underlying curves that are stable in the classical sense when we restrict to the substack cut out by the canonical log structure.

3.3.2 Automorphisms of log smooth curves

We make a few simplifying remarks. We have seen in lemma 3.16 that for a log scheme (S, M_S), automorphisms of the log structure fixing the characteristic sheaf are given by elements of $\mathrm{Hom}(\overline{M}_S, \mathcal{O}_S^*)$. This type of information is extremely useful, as it provides us with information about automorphisms in general. Since an automorphism of a log scheme is uniquely determined by the induced automorphisms of the underlying scheme and the log structure, we have an injection

$$\mathrm{Aut}(S, M_S) \to \mathrm{Aut}(S) \times \mathrm{Aut}(M_S).$$

We would like to have similar characterizations of automorphisms of log smooth curves. We give a definition of an automorphism of a log smooth curve, following [Che14, Def. B.3.1].

Definition 3.19 (Automorphism of a log smooth curve)**.**

An *automorphism of a log smooth curve* $(X, M_X) \to (S, M_S)$ is a commutative diagram

$$\begin{tikzcd}
(X, M_X) \arrow[r] \arrow[d] & (X, M_X) \arrow[d] \\
(S, M_S) \arrow[r] & (S, M_S)
\end{tikzcd}$$

such that the horizontal arrows are isomorphisms of log schemes, the underlying map $S \to S$ is the identity, and the underlying morphism $X \to X$ is an isomorphism.

Lemma 3.20. *Let $\xi^{can} := (X, M_X^{can}) \to (S, M_S^{can})$ be the associated log smooth curve with the canonical log structure. Then*

$$\mathrm{Aut}(\xi^{can}) \cong \mathrm{Aut}(X).$$

Proof. We only provide a brief sketch of this fact, likely well known but to our knowledge not explicitly appearing in the literature. Recall that $\mathfrak{M}_{g,n}^{pre}$, the stack over $\mathfrak{LogSch}^{fs}$ of pre-stable fs log smooth curves is represented by the log algebraic stack $\mathfrak{M}_{g,n}^{can} \subset \mathbf{Log}(\mathfrak{M}_{g,n}^{pre})$, which is an algebraic stack over $\mathfrak{Sch}$. The proof of this implies that there is an equivalence of algebraic stacks over schemes $\mathfrak{M}_{g,n}^{can} \to \mathfrak{M}_{g,n}$ via the forgetful morphism to the stack of pre-stable n-pointed curves of genus g. Hence an isomorphism of automorphism groups is an immediate consequence of the equivalence of categories. □

Lemma 3.21. *An automorphism of a log smooth curve $\xi := (X, M_X) \to (S, M_S)$ is uniquely determined by the induced automorphism of X and the induced automorphism of M_S. That is, the canonical map given by restriction,*

$$\mathrm{Aut}(\xi) \to \mathrm{Aut}(X) \times \mathrm{Aut}(M_S),$$

is injective.

Proof. We will show that the kernel is trivial. Let $\rho \in \mathrm{Aut}(\xi)$, and assume that ρ restricts to the identity on (S, M_S) and on X. Hence by lemma 3.20, the canonical log structure is also fixed. Furthermore, since the induced automorphism of M_X lies over the identity of M_S, it factors through the relative characteristic sheaf $\overline{M}_{X/S}$. However, the relative characteristic is fixed since the canonical log structure is fixed by ρ. Therefore, ρ is the trivial automorphism. $\square$

Remark 3.4. Consider a rational curve equipped with the log structure associated to the divisor of a single point. Considering the point to be ∞, the log structure is fixed by the natural $\mathbb{G}_a$ action, and so log smooth curves containing rational components with only one distinguished point will always have infinite automorphisms. However, if we have a semistable curve X with r rational components with only two distinguished points each, then $\mathrm{Aut}(X) \cong \mathbb{G}_m^r$, and this *does* induce automorphisms of the log structure.

Let $\log x \in \overline{M}_X$ be a local parameter of a node in some component, and assume we have an automorphism of M_X fixing the characteristic sheaf, that is assume we have $\varphi \in \mathrm{Hom}(\overline{M}_X, \mathcal{O}_X^*)$. This induces an element $\varphi' \in \mathrm{Hom}(\overline{M}_X^{can}, \mathcal{O}_X^*)$ via composition with the canonical morphism $\overline{M}_X^{can} \to \overline{M}_X$. From lemma 3.20, this corresponds to an automorphism of X. On the other hand, let $\lambda \in \mathcal{O}_X^*$ be a unit giving an automorphism of X from a $\mathbb{G}_m$ action on a rational component with two distinguished points. Then from the local description of the log structure, the associated automorphism of $\overline{M}_X^{can}$ is given by $\varphi' \in \mathrm{Hom}(\overline{M}_X, \mathcal{O}_X^*)$ sending the local parameter $\log x$ of a node in the rational component to λ.

Definition 3.22. Let $(X, M_X) \to (S, M_S)$ be a log curve over a geometric point $S = \mathrm{Spec}\, k$. An *infinitesimal automorphism* of such a log curve is an automorphism of a lift $(X', M_{X'}) \to (S', M_{S'})$ of an infinitesimal extension $S \to S'$ by a nilpotent ideal $\mathcal{I}$, that is a commutative diagram as below. It suffices to consider the trivial extension $S' = \operatorname{Spec} k[\epsilon]/\epsilon^2$.

$$\begin{tikzcd}
 & & (X', \mathcal{M}_{X'}) \arrow[d] \\
(X, M_X) \arrow[r] \arrow[urr] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & (X', M_{X'}) \arrow[ur, "\sim"'] \arrow[d] & (S', \mathcal{M}_{S'}) \arrow[ddl] \\
(S, M_S) \arrow[r] \arrow[urr] \arrow[d] & (S', M_{S'}) \arrow[ur, "\sim"'] \arrow[d] & \\
S \arrow[r] & S' &
\end{tikzcd}$$

Recall from the discussion at the beginning of this section that given an automorphisms of log structure, triviality on the characteristic sheaf is an open condition. Then the fact that an infinitesimal automorphism fixes the central fiber immediately gives the following.

Lemma 3.23. *Let the notation be as above. An automorphism of $(X', M_{X'}) \to (S', M_{S'})$ necessarily fixes the characteristic sheaves $\overline{M}_{S'}$ and $\overline{M}_{X'}$.*

Remark 3.5. From lemma 3.20, the infinitesimal automorphisms of a log curve with the canonical log structure are the same as the infinitesimal automorphisms of the underlying curve. Hence we have that vanishing of infinitesimal automorphisms for a log smooth curve with the canonical log structure is equivalent to the underlying curve being Deligne-Mumford stable.

This gives us the expected class of curves when we consider the canonical log structure, so $(\overline{\mathcal{M}}_{g,n}, M_{\overline{M}_{g,n}})$ can be defined as a log algebraic stack as the substack of $\mathfrak{M}^{can}_{g,n}$ of log smooth curves cut out by the stability condition of no infinitesimal automorphisms. However it is very interesting to ask what other curves in $\mathbf{Log}(\mathfrak{M}^{pre}_{g,n})$ are stabilized by this notion of infinitesimal automorphism. We first define the notion of length between two components of a log curve, and the notion of a stable sum in $\overline{M}_S$.

Definition 3.24. Given two irreducible components D, Q of a log smooth curve and a path in the dual graph connecting $v(D), v(Q)$, the *length* of the path is the unique sum $d \in \overline{M}_S$ of smoothing parameters indexed by the edges of the path.

If a length d corresponds to a path between two *stable* components, we will say that d is stable. Furthermore, while there is no element $\delta/2 \in \overline{M}_S$ for an irreducible element δ, we may take an equality such as $\gamma = \frac{2}{3}\delta$ to mean that $3\gamma = 2\delta$ in $\overline{M}_S$, that is they are equal in $\mathbb{Q} \otimes \overline{M}_S^{gp}$.

Definition 3.25. Let $\Sigma \in \overline{M}_S$. We say that Σ is a *stable sum* if there is a finite collection of stable lengths $\{d_i\}$ and $\{q_i\} \in \mathbb{Q}$ with $\Sigma = \sum q_i d_i$ in $\mathbb{Q} \otimes \overline{M}_S^{gp}$.

Lemma 3.26. *Let $\varphi \in \operatorname{Hom}(\overline{M}_S, 1 + \epsilon\mathcal{O}_S)$ be induced by an infinitesimal automorphism, and let δ be a stable sum. Then $\varphi(\delta) = 1$.*

Proof. Since stable components have no underlying infinitesimal automorphisms, the smoothing parameters of nodes joining stable components are fixed by any infinitesimal automorphism. ☐

We use this in the following characterization of log smooth curves with finite automorphisms.

Proposition 6. *Let $(X, M_X) \to (S, M_S)$ be a log smooth curve. It has trivial infinitesimal automorphism group if and only if the following hold:*

(1) the underlying curve is semistable;

(2) the natural map $M_S^{can,gp} \to M_S^{gp}$ is surjective.

(3) every strictly semistable rational component has a node with a smoothing parameter δ in $\overline{M}_S$ that is a stable sum.

Proof. Let $(X', M_{X'}) \to (S', M'_S)$ be the trivial square zero extension, and assume that X is semistable. The topological space of the underlying curve of the extension is the same as $X \to S$, and the log structures are isomorphic since $(S, M_S) \to (S', M_{S'})$ is strict. Let $(\phi, \varphi) \in \operatorname{Aut}(X) \times \operatorname{Aut}(M_S)$ be an infinitesimal automorphism. Since X is semistable, $\operatorname{Aut}(X) \cong \mathbb{G}_m^r$, and ϕ determines a tuple of units $(\lambda_1, \ldots, \lambda_r)$, with $\lambda_i = 1$ on the central fiber. Hence $\lambda_i \in 1 + \epsilon\mathcal{O}_X$, and $\varphi \in \operatorname{Hom}(\overline{M}_S, 1 + \epsilon\mathcal{O}_S)$.

In fact, let x, y be the local coordinates of a node, with x the coordinate of a semistable component on which λ_i acts. Then we have a smoothing parameter of the node $\delta = \log x + \log y$, and ϕ induces $\log x \mapsto \log x + \log \lambda_i$, $\delta \mapsto \delta + \log \lambda_i$ as an automorphism of ξ^{can}. However, by assumption there is some stable sum Σ with $\delta - \Sigma$ in the kernel of

$$M_S^{can,gp} \to M_S^{gp},$$

so $\phi(\delta - \Sigma) = \delta + \log \lambda_i - \Sigma$. By assumption, ϕ preserves this kernel, so $\lambda_i = 1$, and we see that in general ϕ must be trivial. Finally, φ is trivial as well since $\operatorname{coker}(M_S^{can,gp} \to M_S^{gp}) = 0$ by assumption.

For the other direction, recall that if the underlying curve is not semistable, we always have a nontrivial $\mathbb{G}_a$ action. Similarly, if assumption (2) is not met, then

$$\operatorname{Aut}\big(\operatorname{coker}(M_S^{can,gp} \to M_S^{gp})\big)$$

is nontrivial and does not depend on automorphisms of the underlying curve. Finally, for a semistable X with $\operatorname{Aut}(X) \cong \mathbb{G}_m^r$, a nontrivial element of $\operatorname{Aut}(X)$ will lift to a nontrivial infinitesimal automorphism unless it does not preserve $\ker(M_S^{can,gp} \to M_S^{gp})$. That is, if $\lambda_i \neq 1$ in $\phi = (\lambda_1, \ldots, \lambda_r)$, and there is some element γ in the kernel with $\gamma \mapsto \gamma + \log \lambda_i$, then we note γ must be of the form $\delta - \Sigma$ for some smoothing parameter $\delta \mapsto \delta + \log \lambda_i$ and Σ fixed by ϕ. Then δ is necessarily a smoothing parameter for a node of a semistable component, and to prevent every automorphism from lifting, Σ must be a stable sum. $\square$

Intuitively, what we have shown is that if we have a log smooth curve with a semistable component, there will be no infinitesimal automorphisms if a node of the semistable component is smoothed identically with a node of a stable component. While all examples come down to smoothing parameters that are stable sums, there are two key types of strictly semistable components that occur with the above notion of log curves without infintesimal automorphisms, obtained over the usual stable curves by "taking roots" and "adding order."

All we mean by this is that if there is a stable component with a node whose smoothing parameter is a, then any element $b \in \overline{M}_S$ such that there is an integer n with $nb = a$ is also a stable sum. Hence a rational component with smoothing parameter b would be stable, courtesy of its smoothing parameter being a "root" of a stable sum.

On the other hand, given any stable smoothing parameters a and b, any element $c \in \overline{M}_S$ such that $b + c = a$ is also a stable sum, interpreted as "$a - b$." Recall that we have $\overline{M}_S$ as a partially ordered set given by the relation $a \geq b$ if the element $a - b$, naturally in $\overline{M}_S^{gp}$, is in $\overline{M}_S$. Hence a rational component with stable smoothing parameter c would witness the partial ordering of the other smoothing parameters a and b.

3.4 A modular interpretation of log blowups

We begin this section with the preliminaries of a log blowup defined by a log ideal (due to Kazuya Kato), using [Kat99] as a reference.

Definition 3.27. An *ideal* of a monoid M will be a subset $P \subseteq M$ such that $a + b \in P$ for *any* $a \in M$, $b \in P$. Let $\pi : M \to \overline{M} := M/M^*$ be the usual projection. There is a one-to-one correspondence between ideals of M and ideals of $\overline{M}$, given by

$$P \mapsto \overline{P} := \pi(P)\overline{M}.$$

A *log-ideal* of a log scheme (X, M_X) is a sheaf of ideals of M_X. For a morphism $(X, M_X) \to (S, M_S)$ of log schemes, and P a log ideal of (S, M_S), the log ideal generated by the image of $f^{-1}P$ is denoted $f^{\bullet}P$.

Note that for a log ideal P of (S, M_S) and a composition

$$\begin{tikzcd}(X, M_X) \arrow[r, "g"] & (T, M_T) \arrow[r, "f"] & (S, M_S),\end{tikzcd}$$

$$(g \circ f)^{\bullet}P = g^{\bullet}f^{\bullet}P.$$

Definition 3.28. A log ideal P of a locally noetherian fine log scheme (S, M_S) is *coherent* if for any $s, t \in S$ with $\overline{t} \rightsquigarrow \overline{s}$, the specialization map $P_{\overline{s}} \to P_{\overline{t}}$ is surjective. A coherent log ideal is *locally principal* if it is étale locally generated by a single global section.

Definition 3.29. Let (S, M_S) be a locally noetherian fs log scheme and P a coherent log ideal of (S, M_S). The *log blow-up*

$$\rho_P : (B_P(S), M_{B_P}) \to (S, M_S)$$

of (S, M_S) *along* P is given by the universal property that

(1) $\rho_P : (B_P(S), M_{B_P}) \to (S, M_S)$ is a morphism of fs log schemes and the log ideal $\rho_P^{\bullet}P$ of $(B_P(S), M_{B_P})$ is locally principal;

(2) for any morphism $f : (T, M_T) \to (S, M_S)$ of locally noetherian fs log schemes such that $f^{\bullet}P$ is locally principal, f factors uniquely through $\rho_P : (B_P(S), M_{B_P}) \to (S, M_S)$.

The construction of $(B_P(S), M_{B_P})$ is given étale locally by charts and then glued.

Construction of the blowup

For $K \subset P$ an ideal of an fs monoid, let $I(K)$ be the ideal of $\mathbf{Z}[P]$ generated by K, and consider

$$\operatorname{Proj} \oplus_n nI(K),$$

which comes with a natural map to $\operatorname{Spec} \mathbf{Z}[P]$, and a covering by open affines

$$\operatorname{Proj} \oplus_n nI(K) = \bigcup_{p \in K} \operatorname{Spec} \mathbf{Z}[P[K-p]].$$

Here $P[K-p]$ denotes the smallest fs submonoid of P^{gp} that contains P and $k-p$ for all $k \in K$, and the log structures $P[K-p] \to \mathbf{Z}[P[K-p]]$ glue to give a fine log structure on $\operatorname{Proj} \oplus_n nI(K)$. Then we define $(B_K(\operatorname{Spec} \mathbf{Z}[P]), M_{B_K})$ as the saturation of the log scheme just described.

For the general situation, we take an étale covering of our log scheme (S, M_S) such that each element of the cover admits a chart on which the log ideal K is generated by global sections. On such an open log scheme (U, M_U) with chart $(U, M_U) \to \operatorname{Spec} \mathbf{Z}[P]$, define

$$(B_K(U), M_{B_K}) := (U, M_U) \times_{\operatorname{Spec} \mathbf{Z}[P]} (B_K(\operatorname{Spec} \mathbf{Z}[P]), M_{B_K})$$

which then glue to give $(B_P(S), M_{B_P})$.

Note that the log blowup is log étale and proper.

Lemma 3.30 ([Kat99, 3.10])**.** *Let K and K' be coherent log ideals of (X, M_X). Then there are canonical isomorphisms*

$$B_{K \cdot K'}(X, M_X) \cong B_{\rho_K^\bullet K'} B_K(X, M_X) \cong B_{\rho_K^\bullet K'} B_{K'}(X, M_X)$$

over (X, M_X), where $K \cdot K'$ is the log ideal generated as the product of K and K'.

Example 3.31. The two notions of log blowup and usual blowup do overlap. Let (S, M_S) be an fs log scheme with log structure determined by a normal crossings divisor $D = \cup D_i$, for D_i the individual divisors. Assume the D_i all intersect at one locus, and let I be the ideal of the intersection. It will be generated by sections s_i according to the local functions defining the D_i. Let K_I be the corresponding log ideal of (S, M_S), and note that the ideal generated by the image of K_I under the structure morphism is I. Then $(B_{K_I}(S), M_{B_{K_I}})$ is the log scheme associated to the usual blowup, equipped with the log structure associated to the proper transform of the normal crossings divisor and the exceptional divisor.

To see this, it suffices to consider a local example since both constructions are local, and we simplify to the following situation. Let (S, M_S) be a log smooth scheme with log structure given by an underlying normal crossings divisor with two components D_1 and D_2. Restrict to a sufficiently local neighborhood U of the intersection on which the log ideal K of the chart $P \to M_U$ is generated by global sections (and such that the divisors are given by a single equation, obviously this can be done simultaneously). Let $Bl_I U$ be the usual blowup along the ideal generated by the local sections u_1, u_2 cutting out D_1 and D_2, and let $K = \langle p, q \rangle$ be the corresponding ideal of the global chart. Then the underlying scheme of the log blowup is given by

$$U \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Proj} \oplus_n nK.$$

The map $K \to I$ given by $\exp p = u_1$, $\exp q = u_2$ induces a map on graded rings, so we get a natural map

$$Bl_I U \to U \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Proj} \oplus_n nK.$$

Since $D_1 \cap D_2$ pulls back to a cartier divisor in $U \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Proj} \oplus_n nK$, the map is an isomorphism.

Definition 3.32. [ACMW14] A *logarithmic modification* of logarithmically smooth Artin stacks is a proper, birational, logarithmically étale morphism.

There is an alternate definition of log modification due to $F.Kato$ that we give here, proposing to call it a *weak* log modificiation. It is equivalent for log smooth schemes.

Definition 3.33. [Kat99, 3.14] A morphism $\rho : (\widetilde{X}, M_{\widetilde{X}}) \to (X, M_X)$ of locally noetherian fs log schemes is called a *weak log modification* if for any point $x \in X$ there exists an étale neighborhood $U \to X$ of x, and a log blowup $(Z, M_Z) \to (\widetilde{X}, M_{\widetilde{X}})$ such that the composition $(Z, M_Z) \to (\widetilde{X}_U, M_{\widetilde{X_U}}) \to (X, M_X)$ is also a log blowup.

Lemma 3.34. *[Kat99, 3.15] Weak log modifications are universally surjective and stable under base change and composition. They are also log étale.*

We would like to introduce an alternate perspective on the log blowup in the situation that the ideal along which we are blowing up corresponds to an ideal of the characteristic sheaf generated by two sections. Recall that any monoid M comes equipped with a natural partial order $>$ on M^{gp} defined by $a > b$ if and only if the element $a - b \in M^{gp}$ is contained in M.

This notion is related to the log blowup in the following way: given an ideal of the characteristic $\overline{M}$ generated by elements a and b, we have an affine cover of the blowup by charts on which we adjoin elements $a-b$ and $b-a$. The open set along which they glue is the locus on which these elements are a unit, that is $a-b=0$ in $\overline{M}$. Hence the result is that a and b are ordered with respect to one another, with the specific ordering locally defined. In general, the universal property of the log blowup is equivalent to locally choosing a minimal element (the local principal generator of the ideal). To obtain additional order beyond a minimal element, one can simply perform additional blowups.

Example 3.35. Consider $\mathbb{A}^3$ as the log smooth scheme induced by the toric structure, and let e_1, e_2, e_3 be the generators of the characteristic monoid $\mathbb{N}^3$. The blowup along the ideal I generated by the e_i glues together three affine opens on which one of the e_i is minimal.

Now consider the blowup along the ideal J generated by e_1+e_2, e_2+e_3, and e_1+e_3, and to illustrate our point, consider the patch on which e_1+e_2 is minimal. Then we have elements in the monoid

$$(e_1+e_3)-(e_1+e_2)$$

$$(e_2+e_3)-(e_1+e_2)$$

But this in turn implies that both e_3-e_2 and e_3-e_1 are in the monoid, that is that e_3 is *maximal.* Then it is not hard to check that if we blowup along the ideal $I \cdot J$, we in fact have a modification resulting in the generators being totally ordered with respect to one another.

Lemma 3.36. *For a log scheme (S, M_S) and a finite collection $\mathcal{K}$ of sections of $\overline{M}_S$, there is a log scheme $Ord_{\mathcal{K}}(S) \to (S, M_S)$ defined by the universal property that for any map of log schemes $(T, M_T) \to (S, M_S)$ that sends the elements of $\mathcal{K}$ to relatively ordered elements of $\overline{M}_T$, the map factors through $Ord_{\mathcal{K}}(S)$.*

Proof. It follows from the discussion above that $Ord_{\mathcal{K}}(S)$ is given by a sequential log blowup of (S, M_S) along log ideals generated by sums of elements of $\mathcal{K}$. ☐

Lemma 3.37. *If A is a finite collection of sections of $\overline{M_S}$, for (S, M_S) a log smooth scheme, then $Ord_A(S) \to S$ is proper, surjective, and logarithmically étale.*

Proof. As $Ord_A(S) \to S$ is a log blowup, it is a weak log modification. Proper follows from the fact that it is also a log modification. ☐

Lemma 3.38. *Let $\alpha, \beta \in \overline{M}_S$, and consider the log blowup $Ord_{\alpha,\beta}(S) \to (S, M_S)$. Then the loci where $\alpha > \beta$ and $\beta > \alpha$ are disjoint.*

Proof. Indeed, by the construction above, the locus where $\alpha > \beta$ is given by the complement of the open set along which the affine opens glue in the chart containing $\alpha - \beta$. Hence $\alpha > \beta$ and $\beta > \alpha$ determine closed loci contained in distinct open affine patches and disjoint from the intersection.. ☐

4 Partially aligned log curves

As we saw in section 1, dual graphs of log smooth curves provide us with a very concrete description of canonical log structure on nodal curves. From here on, we explore the genus one case, working over $\overline{\mathcal{M}}_{1,n}$. We begin with the fact that the dual graph of a stable arithmetic genus one curve with n markings is a rooted tree when the unique cycle is contracted to a single vertex.

Let $X \to S$ be a stable curve of arithmetic genus one with n markings, $\overline{s} \to S$ a geometric point, and assume that S is an atomic neighborhood of $\overline{s}$. The dual graph of the fiber over $\overline{s}$ contains a unique minimal genus one subgraph which may take the form of either a finite cycle of genus zero vertices, or a single vertex labelled as genus 1. Furthermore, as S is an atomic neighborhood, this minimal genus one subgraph is compatible with generization, so we may refer to it in families.

Definition 4.1. The *cycle* of a genus one nodal curve $X \to S$ will be the unique minimal genus one subgraph of its dual graph and (by slight abuse), the unique minimal genus one subcurve $Z \subset X \to S$ it corresponds to.

Now each connected component of $\overline{X \setminus Z}$ is genus zero, and the corresponding subgraph of the dual graph consists of a collection of trees whose roots are the unique vertices that connect to the cycle. In this sense we think of the dual graph of $X \to S$ as a rooted tree, where the root may be a cycle of genus zero vertices or a single genus one vertex. For any graph there is automatically a natural partial order on paths in the dual graph by inclusion, but having a unique cycle allows us to put an orientation on paths in a consistent manner. This orientation then gives rise to a partial order on vertices consistent with the ordering on paths from the cycle. The seemingly canonical choice is to consider the vertices of the cycle as the set of minimal elements.

Definition 4.2. The *canonical* ordering on the vertices of a genus one dual graph is defined by $v_1 < v_2$ if the unique path in the dual graph from the root to v_2 contains v_1.

In particular, vertices in different branches of the tree are not canonically ordered! For each marking i, we take the i-th *path to infinity* to be the path in the dual graph that connects a single vertex of the cycle to the half edge of the marking, oriented to radiate outward from the cycle.

Of course, the picture we have in mind is tropical, but at this point our dual graphs are abstract cartoons with no *true* meaning of distance beyond saying that for vertices $v_1 < v_2$, v_2 is "farther" away from the cycle. This discussion might be made precise by interpreting the dual graph stratification of $\overline{\mathcal{M}}_{1,n}$ as a *sheaf of partially ordered finite sets*, together with a collection of n subsheafs of *totally ordered finite sets* in the sense of [ACFW13], one for each marking. While there is no doubt an interpretation of our work in terms of ordered configurations of line bundles and aligned log structures, we make no attempt at such descriptions here.

For example, in figure 1, the cycle is the unique vertex of genus one (indicated by an open dot), all other vertices are rational (closed dots), and the 6-th path to infinity is highlighed.

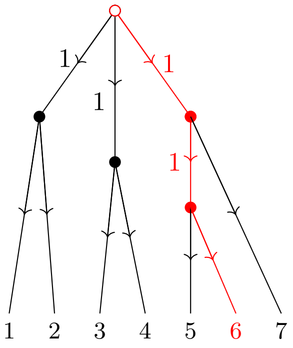

Figure 1: Marked tree of a genus one log curve with the 6-th path to infinity indicated

While our discussion above was canonical for a nodal genus one curve, we would like to extend this notion to stable log smooth curves of genus one. We follow the construction given by Chen in [Che14] of a *marked graph.*

Definition 4.3. A *weighted graph* consists of the data:

(1) an underlying connected graph. We denote the set of vertices by V and the set of edges by E;

(2) a subset $V_0 \subset V$ called the *set of nondegenerate vertices*;

(3) each edge $l \in E$ is assigned a nonnegative integer weight c_l, the *contact order* of l.

An *orientation* on a weighted graph G is an orientation on the underlying graph, with some edges allowed to be *non-oriented.* If an edge l is oriented from vertex v_1 to vertex v_2, we write $v_1 < v_2$, and note that an orientation on a weighted graph induces a partial order on V in this way.

An orientation on G is *compatible* if

(1) for every edge $l \in E$, the contact order $c_l = 0$ if and only if l is nonoriented;

(2) the nondegenerate vertices are minimal with respect to the partial order induced by the orientation.

A weighted graph G with a compatible orientation is called a *marked* graph. Let G_0 denote the subgraph consisting of nondegenerate vertices and nonoriented edges. If G_0 is connected, we say G is a marked *tree*, which will be our setting here in genus one.

4.0.1 Marked tree of a genus one stable log smooth curve

Now assume we have $(X, M_X) \to (S, M_S)$ a stable log smooth curve of type $(1, n)$. The *marked tree* of $(X, M_X) \to (S, M_S)$ will be defined to be the dual graph of the underlying curve, together with the canonical orientation and contact orders given by a section λ of $\overline{M}_X$ that we construct below. We may specify this canonical section λ of the characteristic monoid $\overline{M}_X$ by identifying a consistent collection of sections of $\overline{M}_S$ according to the following rules:

(1) $\lambda(v) = 0$ for each vertex of the cycle, implying λ has zero contact order on internal edges of the cycle;

(2) λ has contact order 1 for each edge of the graph that is not in the cycle, including the marked half edges.

These conditions imply that given two adjacent vertices $v_1 < v_2$, joined by an edge l with smoothing parameter e_l, we have $\lambda(v_2) = \lambda(v_1) + e_l$. The result is a marked tree (see figure 1), with the vertices of the cycle as the nondegenerate vertices. We will typically not indicate the contact orders or the orientation when specifying the associated marked tree of a log smooth curve of genus one, as they are canonical.

We will refer to λ as the *length* section, and think of this as defining a piecewise linear function on the dual graph valued in $\overline{M}_S$ that measures the length of a path from the cycle. In the next section we will discuss the line bundles naturally arising as lifts of global sections of the characteristic sheaf on the curve, and the lift of λ will naturally yield ω_π, the relative dualizing sheaf of the underlying nodal curve. In figure 2, we have labelled the edges by their smoothing parameters, and so the vertex marked by $\{5, 6\}$ has length

$c + d$. In general, $\overline{M}_S$ is a partially ordered set under the relation $\alpha > \beta$ if $\alpha - \beta \in \overline{M}_S$, but need not be totally ordered. In the example above, $c + d > c$ since $d = (c + d) - c$, but a and c, for example, are not neccesarily ordered with respect to one another, and in the canonical log structure are certainly not (recall the canonical log structure is locally free).

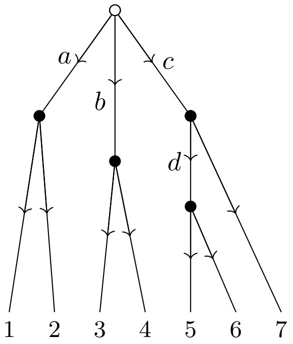

Figure 2: Marked tree of a genus one log curve, further labelled by smoothing parameters

Definition 4.4. We will say that a section δ of $\overline{M}_S$ is *ordered in a path to infinity* if for each vertex v in the path, we have $\lambda(v) \geq \delta$ or $\delta \geq \lambda(v)$. A *partially aligned log curve* is a log curve of genus one with the additional data of a section δ of the characteristic sheaf on the base, satisfying the condition that the restriction of δ to a geometric fiber is ordered in each path to infinity.

4.1 The stack of partially aligned log curves

We will arrive at the stack of partially aligned log curves as a subfunctor of the moduli problem for stable log curves and a section of the characteristic monoid on the base.

Definition 4.5. Let $\mathcal{E}_n$ denote the functor on $\mathfrak{LogSch}^{\text{fs}}$ given by

$$(S, M_S) \quad \mapsto \quad \mathcal{E}_n(S, M_S) := \left\{ \begin{matrix} \text{Families of stable log curves of genus one with } n \text{ marked points } (X, M_X) \to \\ (S, M_S), \text{ along with a section } \delta \text{ of } \overline{M}_S \end{matrix} \right\}.$$

Let $\mathcal{F}_n$ denote the subfunctor of partially aligned log curves.

Lemma 4.6. *$\mathcal{E}_n$ is representable by an algebraic stack over $\mathfrak{LogSch}^{\text{fs}}$, étale over $\mathcal{LM}_{1,n}$ (and hence locally of finite presentation).*

Proof. Since étale sheaves are representable by étale algebraic spaces [Mil80, Theorem 1.5], the characteristic sheaf $\overline{M}_{\mathcal{LM}_{1,n}}$ is representable by an algebraic space $\mathcal{E}_n$ relative to $\mathcal{LM}_{1,n}$. Hence $\mathcal{E}_n$ is representable by an algebraic stack over $\mathfrak{LogSch}^{\text{fs}}$, étale over $\mathcal{LM}_{1,n}$ [Sta16, Tag 05UM]. In fact, $\mathcal{E}_n \cong \overline{\mathcal{M}}_{1,n} \times [\mathbb{A}^1/\mathbb{G}_m]$. ☐

Proposition 7. *$\mathcal{F}_n$ is representable by an algebraic stack over $\mathfrak{LogSch}^{\text{fs}}$ and locally of finite presentation, and the inclusion $\mathcal{F}_n \to \mathcal{E}_n$ is a log blowup.*

Proof. The description of the subfunctor as a log blowup also implies it is proper, log étale (hence locally of finite presentation), and birational over $\mathcal{E}_n$ (however we shall see in section 4.5 that properness over $\overline{M}_{1,n}$ will require additional open conditions). Let (S, M_S) be a log scheme and consider the commutative diagram below, with bottom arrow giving $((X, M_X) \to (S, M_S), \delta)$ and $\mathcal{F}_S := (S, M_S) \times_{\mathcal{E}_n} \mathcal{F}_n$. We will show directly that $\mathcal{F}_S$ is a log blowup of S, giving the result.

$$\begin{tikzcd}
\mathcal{F}_S \arrow[r] \arrow[d] & \mathcal{F}_n \arrow[d] \\
(S, M_S) \arrow[r] & \mathcal{E}_n
\end{tikzcd}$$

As blowing up is a local construction based on charts, we may assume that S is an atomic neighborhood of a geometric point $\overline{s} \in S$. We take P to be the stalk of the characteristic sheaf at $\overline{s}$. Now suppose that δ is not ordered in some path to infinity in the dual graph of X. Then there is a vertex v in the dual graph such that δ is not comparable to $\lambda(v)$. We may then replace P by $P[\lambda(v) - \delta]$ or $P[\delta - \lambda(v)]$. The resulting $\operatorname{Spec} \mathbf{Z}[P[\lambda(v) - \delta]]$ and $\operatorname{Spec} \mathbf{Z}[P[\delta - \lambda(v)]]$ glue along the strata giving $\delta = \lambda(v)$ in the characteristic sheaf, and provide characteristic charts for the log-blowup $\widetilde{S} \to S$ along the ideal generated by δ and $\lambda(v)$. This is none other than $Ord_{\delta,\lambda(v)}(S)$. Continuing in this way until δ is ordered in every path to infinity results in a log-blowup $B \to \operatorname{Spec}(P)$, whose pullback $S_B \to S$ is the log blowup of S along each of the ideals generated by δ and $\lambda(v)$, where v goes over the vertices of the dual graph of $X_{\overline{s}}$ not comparable to δ. From lemma 3.30, the order in which the blowup is done does not matter.

We note that this construction also does not in fact depend on the choice of $s \in S$, since the vertices of the dual graphs admit a local definition compatible with the corresponding sections $\lambda(v)$ and δ.

It remains to show that S_B represents $\mathcal{F}_S$, from which it follows that $\mathcal{F}_n$ is algebraic. Since $S_B \to \mathcal{E}_n$ evidently gives rise to a partially aligned log curve, we have the dashed arrow in the commutative diagram below, so there is a canonical map $S_B \to F_S$.

$$\begin{tikzcd}
B \arrow[d] & S_B \arrow[l] \arrow[r, dashed] \arrow[d] \arrow[dl, phantom, "\urcorner" very near start] & \mathcal{F}_n \arrow[d] \\
\operatorname{Spec}(P) & S \arrow[l] \arrow[r] & \mathcal{E}_n
\end{tikzcd}$$

Now for any $\xi \in \mathcal{F}_S(T)$, ξ factors through S_B by the construction, and this completes the proof. $\square$

It is interesting to consider related moduli problems that impose additional order on the log structure. We will see below that the "naive" proper moduli space that arises from partially aligned log curves will not in fact be smooth. However, if we ask that in addition to having δ ordered in every path to infinity, the vertices inside the circle of radius δ should also be ordered, then the space will be smooth, and there is a natural sequence of birational moduli problems corresponding to allowing δ to "increase in size." We will call these curves *centrally aligned* log curves. Finally *ordered* log curves will be those on which the order on vertices imposed by the length section is a total order. When centrally aligned log curves are equipped with a "maximum" δ, the resulting order on vertices will also be total, relating these two notions. We summarize these definitions here.

Definition 4.7. A *centrally aligned* log curve is a log curve of genus one with the additional data of a section δ of the characteristic sheaf on the base, satisfying the condition that the restriction of δ to a geometric fiber is ordered in each path to infinity, as are the vertices of the dual graph at length less than δ. We let $\mathcal{F}_n^\circ$ denote the subfunctor of $\mathcal{F}_n$ parametrizing centrally aligned log curves.

Definition 4.8. An *ordered* log curve is a log curve of genus one such that the partial order on vertices induced by the length section is a total order. We let $\mathcal{LM}_{1,n}^{ord}$ denote the subfunctor of $\mathcal{LM}_{1,n}$ parametrizing ordered log curves. Note that the definition of an ordered log curve does not involve a section of the characteristic sheaf, unlike our previous two notions. An ordered log curve together with a section of the characteristic sheaf on the base that is ordered in every path to infinity will be called a *totally aligned* log curve, and we will let $\mathcal{F}_n^{tot}$ denote the subfunctor of $\mathcal{F}_n^\circ$ parametrizing totally aligned log curves.

As these subfunctors are determined by imposing natural conditions for additional order (i.e. blowing up), they are representable by a similar argument. In particular, $\mathcal{LM}_{1,n}^{ord}$ is a log blowup of $\mathcal{LM}_{1,n}$ itself.

Corollary 4.8.1. *Each of $\mathcal{LM}_{1,n}^{ord}$, $\mathcal{F}_n^{tot}$ and $\mathcal{F}_n^{\circ}$ is representable by an algebraic stack over $\mathfrak{LogSch}^{\mathrm{fs}}$, and the inclusions $\mathcal{LM}_{1,n}^{ord} \to \mathcal{LM}_{1,n}$ and $\mathcal{F}_n^{tot} \to \mathcal{F}_n^{\circ} \to \mathcal{F}_n \to \mathcal{E}_n$ are log blowups.*

4.2 Minimal log structure

We begin by extending the definition of the marked tree of a genus one stable log smooth curve to partially aligned curves.

Partially aligned marked tree of a partially aligned log curve

For any partially aligned log curve $(X, M_X) \to (S, M_S), \delta$ over a geometric point, we have an associated marked tree G from the marked tree of the underlying stable log smooth curve of genus one. We then obtain a *partially aligned marked tree* by the additional data of a circle on the marked tree, that intersects each path to infinity according to the ordering of δ in that path.

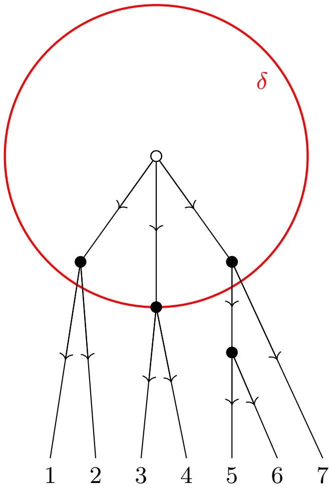

Figure 3: A partially aligned marked tree. The vertices are ordered with respect to δ and the canonical ordering only.

Definition 4.9 (The minimal monoid of a partially aligned marked tree)**.** To any partially aligned marked tree G, we associate a monoid $\overline{M}_{part}(G)$. For each edge l in the graph we have a generator e_l, and we also have a generator δ corresponding to the radius of the circle. Let F be the free monoid on the generators $\{e_l\}, \delta$, and for each vertex v, denote by $\lambda(v)$ the sum $\sum e_l$ in F, taken over all edges in the path from the cycle to v. Let

$$Ord := \{\lambda(v) - \delta : v \text{ not inside the circle }\} \cup \{\delta - \lambda(v') : v' \text{ not outside the circle }\} \subset F^{gp}$$

record the data of the relative ordering of δ in every path to infinity. Then we define

$$\overline{M}_{part}(G) := \frac{F[Ord]}{F[Ord]^*}.$$

Note that if v is *on* the circle, then $\lambda(v) - \delta$ and $\delta - \lambda(v)$ are both in Ord, that is are both a *unit* in $F[Ord]$. Hence $\delta = \lambda(v)$ in $\overline{M}_{part}(G)$.

Proposition 8. *For $((X, M_X) \to (S, M_S), \delta)$ a partially aligned log curve, with S an atomic neighborhood of a geometric point $\overline{s}$, there is a canonical morphism of monoids*

$$\phi : \overline{M}_{part}(G) \to \overline{M}_{S,s} \cong \overline{M}_S,$$

where G is the partially aligned marked tree constructed as above.

Proof. We have a diagram of solid arrows below

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & \overline{M}_{S,s} \\
\overline{M}_{part}(G) \arrow[ur, dashed, "\phi"] &
\end{tikzcd}$$

where the horizontal arrow is given by sending e_l to the corresponding smoothing parameter in $\overline{M}_{S,s}$, and δ to δ. However, since the relations in $\overline{M}_{part}(G)$ are imposed from relations in $\overline{M}_{S,s}$, the map factors uniquely through the dashed arrow ϕ. □

Note that the construction of $\overline{M}_{part}(G)$ and $\phi : \overline{M}_{part}(G) \to \overline{M}_S$ is compatible with generization, and hence defines a log structure and a map on any partially aligned log curve.

Definition 4.10. Consider a partially aligned log curve $((X, M_X) \to (S, M_S), \delta)$. We will say that this has been equipped with the *minimal log structure* if the canonical map

$$\phi : \overline{M}_{part}(G) \to \overline{M}_S$$

is an isomorphism. As $\overline{M}_{part}(G)$ is constant in an atomic neighborhood, this amounts to ϕ locally defining a chart.

Lemma 4.11. *The subfunctor $\mathcal{F}_n^{min} \subset \mathbf{Log}(\mathcal{F}_n)$ of partially aligned log curves with minimal log structure is open, and hence representable by an algebraic stack over $\mathfrak{Sch}$.*

Proof. By the construction above, for any partially aligned log curve over S with partially aligned marked tree G, we obtain a morphism $\phi : \overline{M}_{part}(G) \to \overline{M}_S$ of fs log structures on S. The locus in S where φ is an isomorphism is an open subset of S by lemma 3.13. □

Lemma 4.12. *Let notation be as in proposition 8 above, and let $\overline{M}_{can}(G) \to \overline{M}_S$ denote the corresponding map for the canonical log structure as constructed in the proof of Proposition 5 in section 1. Then this map factors as*

$$\overline{M}_{can}(G) \to \overline{M}_{part}(G) \to \overline{M}_S.$$

Proof. Note that $\overline{M}_{part}(G)$ contains generators $\{e_l\}$ for each edge l in the underlying dual graph, and $\overline{M}_S$ admits the relations given in the construction $\overline{M}_{part}(G)$. Let $\overline{e}_l$ denote the generators of $\overline{M}(G)$, e_l' the corresponding smoothing parameter in $\overline{M}_S$. Then the factorization is given by

$$\overline{e}_l \mapsto e_l \mapsto e_l'$$

□

Proposition 9 (Universal Property of mimimal log structure)**.** *Let $\big((X, M_X) \to (S, M_S), \delta\big)$ be a partially aligned log curve. Then there is a partially aligned log curve equipped with the minimal log structure, unique up to unique isomorphism, $\big((X, M_X^{min}) \to (S, M_S^{min}), \delta^{min}\big)$, fitting in a cartesian diagram of log schemes*

$$\begin{tikzcd}
(X, M_X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & (X, M_X^{min}) \arrow[d] \\
(S, M_S) \arrow[r] & (S, M_S^{min})
\end{tikzcd}$$

such that $\overline{M}_S^{min} \to \overline{M}_S$ sends δ^{min} to δ. Furthermore, the pullback along any strict morphism $(S', M_{S'}) \to (S, M_S^{min})$ is another partially aligned log curve with minimal log structure.

Proof. We first construct the log structures M_S^{min}, M_X^{min}, and then show that the resulting log curve satisfies the universal properties. We construct the minimal log structure (S, M_S^{min}) by following the argument for (S, M_S^{can}) in the case of canonical log curves (see Proposition 5).

The statement is local on S, so we may assume that we are in an atomic neighborhood for some geometric point $\overline{s}$ and we have a chart $\overline{M}_{S,\overline{s}} \cong \overline{M}_S \to M_S$. Then the desired minimal log structure is given on S by the log structure associated to $\overline{M}_{part}(G_{\overline{s}}) \to \overline{M}_S$. Note δ^{min} in $\overline{M}_S^{min}$ is given by δ in $\overline{M}_{part}(G_{\overline{s}})$. As in the previous argument, this does not depend on the choice of chart.

It remains to construct (X, M_X^{min}). We note that we do have the unique cartesian diagram below, where the bottom arrow is the identity on S and factors through (S, M_S^{min}) by lemma 4.12.

$$\begin{tikzcd}
(X, M_X) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & (X, M_X^{can}) \arrow[d] \\
(S, M_S) \arrow[r] & (S, M_S^{can})
\end{tikzcd}$$

This demonstrates the utility of the canonical log structure for other notions of minimality, as then (X, M_X^{min}) is defined uniquely up to unique isomorphism by the pullback:

$$\begin{tikzcd}
(X, M_X) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & (X, M_X^{min}) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & (X, M_X^{can}) \arrow[d] \\
(S, M_S) \arrow[r] & (S, M_S^{min}) \arrow[r] & (S, M_S^{can})
\end{tikzcd}$$

Let ξ^{min} denote the partially aligned log curve $\big((X, M_X^{min}) \to (S, M_S^{min}), \delta^{min}\big)$, and assume there are partially aligned log curves ξ, ξ' fitting in the diagram of solid arrows below, lying over the identity of the underlying curve $X \to S$.

$$\begin{tikzcd}
\xi \arrow[r] \arrow[d] & \xi^{min} \\
\xi' \arrow[ur, dashed] &
\end{tikzcd}$$

The canonical log structure on the underlying curve ensures that we may complete the diagram uniquely.

$$\begin{tikzcd}
\xi \arrow[r] \arrow[d] & \xi^{min} \arrow[r] & \xi^{can} \\
\xi' \arrow[ur, dashed] \arrow[urr] & &
\end{tikzcd}$$

Now $\xi \to \xi'$ is strict and the maps on the underlying curve are the identity, so all associated partially aligned marked trees are identical. In particular, if $\xi' = \big((X, M'_X) \to (S, M'_S), \delta'\big)$, then we have a natural map $(S, M'_S) \to (S, M_S^{min})$ that takes δ' to δ (see the proof of proposition 8). Then the dashed arrow above is completed uniquely via the universal property of (X, M_X^{min}) as a fiber product, and ξ^{min} satisifies the universal property of a minimal object.

It remains to show that the pullback of such a minimal partially aligned log curve along a morphism of schemes is again minimal. Let $\big((X, M_X^{min}) \to (S, M_S^{min}), \delta^{min}\big)$ be a minimal partially aligned log curve, and assume we have a map of curves $X'/S' \to X/S$. Denote by η the log curve obtained by pulling back the log structure. Then for any diagram of solid arrows below over the identity of X'/S', the completion with a dashed arrow is equivalent to a map from η'' to $\big((X, M_X^{min}) \to (S, M_S^{min}), \delta^{min}\big)$.

$$\begin{tikzcd}
\eta' \arrow[r] \arrow[d] & \eta \\
\eta'' \arrow[ur, dashed] &
\end{tikzcd}$$

However, such a map is guaranteed (after considering direct images) by the universal property of $\big((X, M_X^{min}) \to (S, M_S^{min}), \delta^{min}\big)$. □

The minimal log structures for ordered, centrally aligned and totally aligned log curves are similarly constructed from their associated *ordered*, *centrally aligned* and *totally aligned* marked trees, following exactly the same construction as above with the additional ordering data on the vertices of the marked trees and ordering relations in $M(G)$.

Lemma 4.13. *The subfunctors $\mathcal{LM}_{1,n}^{ord,min} \subset \mathbf{Log}(\mathcal{LM}_{1,n}^{ord})$, $\mathcal{F}_n^{\circ,min} \subset \mathbf{Log}(\mathcal{F}_n^{\circ})$ and $\mathcal{F}_n^{tot,min} \subset \mathbf{Log}(\mathcal{F}_n^{tot})$ are open, and hence algebraic.*

Hence $\mathcal{LM}_{1,n}^{ord,min}$, $\mathcal{F}_n^{min}$, $\mathcal{F}_n^{\circ,min}$, and $\mathcal{F}_n^{tot,min}$ *represent* $\mathcal{LM}_{1,n}^{ord}$, $\mathcal{F}_n$, $\mathcal{F}_n^{\circ}$, and $\mathcal{F}_n^{tot}$ as log algebraic stacks.

4.3 Stratification by partially aligned marked trees

Let $\xi \in \mathcal{F}_n(S)$ be a partially aligned log curve $((X, M_X) \to (S, M_S), \delta)$ with $\xi_{\bar{s}}$ a geometric fiber. We represent the relative ordering of $\delta_{\bar{s}}$ as a circle on the partially aligned marked tree, and by construction this is compatible with generization. It will be useful to have a characterization of what can happen to the partially aligned marked tree under such a generization. The main point is that the ordering of δ in each path to infinity forces additional order on vertices relative to one another (those at length less than or equal to δ are ordered relative to those at length greater than or equal to δ) and all of the ordering must be compatible with generization. We will say that a vertex v is *adjacent* to δ if $\lambda(v) \neq \delta$ and there is no vertex v' with $\lambda(v') \neq \delta$ and in the path from v to length δ.

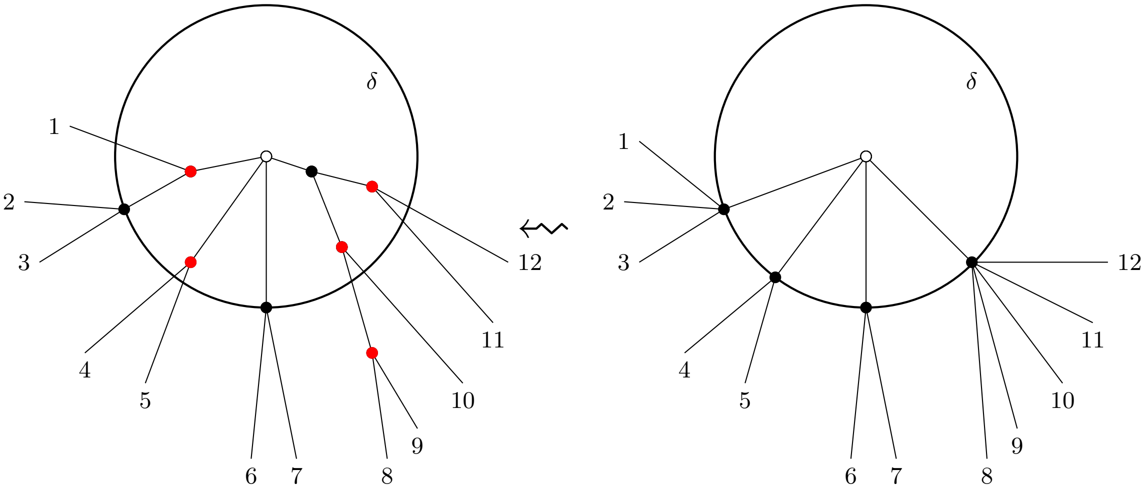

Figure 4: Vertices adjacent to δ are in red, and a possible generization

Lemma 4.14. *Let $\bar{t} \rightsquigarrow \bar{s}$ be a generization in S, and consider $\xi_t \mapsto \xi_s$ the specialization map. Then the partially aligned marked tree of ξ_t is related to the partially aligned marked tree of ξ_s by some combination of the following possibilities:*

(1) two vertices are contracted to a single vertex along an edge that is not incident to δ;

(2) a vertex v adjacent to δ is contracted along the unique edge to δ.

Proof. We recall from lemma 3.14 that we have a surjection $\overline{M}_{S,\bar{s}} \to \overline{M}_{S,\bar{t}}$. The relevant elements in the kernel consist of smoothing parameters and differences $\delta - \lambda(v)$ or $\lambda(v') - \delta$. □

Remark 4.1. Situation (1) in the lemma corresponds to the smoothing of a node, and the smoothing parameter of this node must be an element of the kernel of the specialization map on the level of the characteristic. Since the edge is not incident to δ, the *relative* ordering of the contracted vertices with the rest of the dual graph is *not affected beyond the canonical ordering*. Specifically, if v_1 is adjacent to and less than both v_2 and v_2', and the edge connecting v_1 and v_2' is contracted under the generization, then vertices v_2 and v_2' go from being incomparable to having their images canonically ordered with $\mathrm{Im}(v_2') < \mathrm{Im}(v_2)$

The result of $\delta - \lambda(v) \mapsto 0$ is that v is at length δ in the generization, this is the situation of (2) from the lemma. Prior to $\delta - \lambda(v) \mapsto 0$, every outgoing path from v crossed δ at some point, after the generization each of these paths begins at length δ. The relative order of vertices can only be minimally affected. First if v' is another vertex that is not canonically ordered with v and $\lambda(v') = \delta$, then the generization takes the order $v' > v$ to $v' = v$. Otherwise, if $\lambda(v') < \delta$ and is not canonically ordered with v, then the generization imposes the order $v > v'$. The situation of $\lambda(v') - \delta \mapsto 0$ is handled similarly.

Theorem 4.15. *The algebraic stack $\mathcal{F}_n^{min}$ is stratified by partially aligned marked trees. This stratification is a refinement of the stratification of $\overline{\mathcal{M}}_{1,n}$ by dual graphs. Furthermore, the codimension of a stratum is given by $e+1-d$, where e is the number of edges of the dual graph and d the number of vertices at length δ.*

Proof. Our argument is that we have a stratification from the log structure that is a refinement of the stratification of $\overline{\mathcal{M}}_{1,n}$, and that this is further refined by a disjoint union of strata.

Strata

We need to show that given a partially aligned marked tree G, the set $\mathcal{F}_G$ of minimal partially aligned log curves with associated partially aligned marked tree isomorphic to G is locally closed in $\mathcal{F}_n^{min}$, and that allowing G to range over all possible partially aligned marked trees provides a covering $\bigsqcup \mathcal{F}_G = \mathcal{F}_n^{min}$. Clearly we will have a covering, and note that fixing an n, there are finitely many possible partially aligned marked trees up to isomorphism. To see that $\mathcal{F}_G$ is locally closed, first consider the fact that $\overline{M}^{gp}_{\mathcal{F}_n^{min}}$ is constructible, hence we have a stratification by locally closed sets on which $\overline{M}^{gp}_{\mathcal{F}_n^{min}}$ is constant. Note that if $\overline{M}^{gp}_{\mathcal{F}_n^{min}}$ is constant, then smoothing parameters, and hence the underlying dual graphs, are fixed. So this preliminary stratification is a refinement of the stratification of $\overline{M}_{1,n}$ by dual graphs.

However, the partially aligned marked trees of geometric points in such a strata may not be constant. For a simple example, there are multiple partially aligned log curves over a canonical stable log curve of genus one with a two pointed rational branch. Let the smoothing parameter of the unique node be given by α, and consider the partially aligned log curves with $\delta - \alpha \in \overline{M}_S$ versus $\alpha - \delta \in \overline{M}_S$. In each case $\overline{M}_S^{gp}$ is isomorphic to the free abelian group on two generators, while the partially aligned marked trees are quite different.

Now we note that given a locally closed subset $U \subset \mathcal{X}$ of a topological space, with U a finite disjoint union of sets $\bigcup U_i = U$ whose closures are disjoint in $\mathcal{X}$, each U_i is locally closed as well.

In this case of partially aligned marked trees giving rise to minimal monoids with isomorphic associated groups, if the partially aligned marked trees are not isomorphic, it must be that the relative ordering of vertices is different between them. Such partially aligned marked trees in fact give strata whose closures are disjoint by lemma 3.38, and so the first part of the theorem follows from lemma 3.38. It is clear, from the definition of partially aligned marked trees as modifications of the dual graphs of the underlying curves that the stratification is compatible with that of $\overline{M}_{1,n}$.

Codimension

Since the stratifications are a refinement, the codimension of the preimage of a stratum of $\overline{\mathcal{M}}_{1,n}$ indexed by a dual graph Γ is given by the number e of edges in Γ. The additional data of δ allows for an additional dimension of smoothing ($\mathcal{F}_n^{min}$ has relative dimension one) given by an automorphism since $\delta \in \mathrm{coker}(M_S^{can,gp} \to M_S^{gp})$. However, every vertex *at* length δ corresponds to an identification on the smoothings that are allowed, cutting down the codimension by one for each such identification. Hence the codimension is $e+1-d$ for d the number of conditions forced on the smoothings of nodes by δ. In particular, the partially aligned marked tree to the right in figure 4 identifies a codimension one stratum. If there is precisely *one* vertex at length δ, then the corresponding stratum has the same codimension as the stratum it maps to in $\overline{\mathcal{M}}_{1,n}$. This concludes the proof to the theorem. $\square$

It may seem reasonably clear that $\mathcal{F}_n$ has no hope of being proper. Indeed the fiber over an underlying stable curve with specified dual graph contains multiple possibilities for a section δ ordered in every path to infinity. However, the possibilities are finite, combinatorially tractable, and stratified as above. Using this, we find the following a natural route to a proper moduli problem, and we will use the stratification once more in our proof of properness of the modified moduli problem.

4.4 m-stability

In this section we will define two natural functions on the geometric points of $\mathcal{F}_n$ (and by obvious extension on $\mathcal{F}_n^{tot}$ and $\mathcal{F}_n^{\circ}$) that will serve both to quantize the choices of δ and eliminate infinitesimal automorphisms of the partially aligned log curve. In section 3, we will construct two particular line bundles E and L on the universal curve, and interpret these functions as the degrees of E and L restricted to a subcurve determined by δ. The relationship $L := \omega_\pi \otimes E$ allows us to interpret the (overall) degree 0 line bundle E as a *tool* for redistributing the degree of the canonical sheaf. This redistribution constructs a line bundle that is degree zero on an elliptic subcurve determined by δ.

Let $(X, M_X) \to (S, M_S), \delta$ be a partially aligned log curve, and $\overline{s}$ a geometric point of S. We will rely on two functions on the geometric fibers of S.

Definition 4.16. We define $\eta(\overline{s})$ to be the unique integer that gives the number of internal edges incident to the circle δ (this is just the number of times a circle drawn at length δ crosses the dual graph).

Definition 4.17. We define $\tau(\overline{s})$ to be the unique integer that gives the number of external edges incident to the circle δ (this can be thought of as the number of times a circle slightly larger than δ crosses the dual graph).

Example 4.18. If we let δ be indicated by the circle in figure 5, then we have $\eta = 7$ and $\tau = 11$.

It is important to note that for any partially aligned log curve, $\eta \leq \tau$ since the underlying curve is a stable tree. That is, as we move further out from the cycle, more branching will occur, and in particular, for every node that δ passes through there can only be one internal edge incident at that node, but there must be at least two external edges incident at that node.

Proposition 10. *Let $\xi \in \mathcal{F}_n(S)$ be a partially aligned log curve. The function η is upper semicontinuous on the geometric points of S, and the function τ is lower semicontinuous on the geometric points of S.*

Proof. As η and τ each take integer values, it suffices to show that in an atomic neighborhood of a geometric point $\overline{s}$, the value of η is maximized at $\overline{s}$, and the value of τ is minimized at $\overline{s}$. Hence we briefly analyze the possible modifications of the dual graph of a partially aligned log curve in an atomic neighborhood. Recall

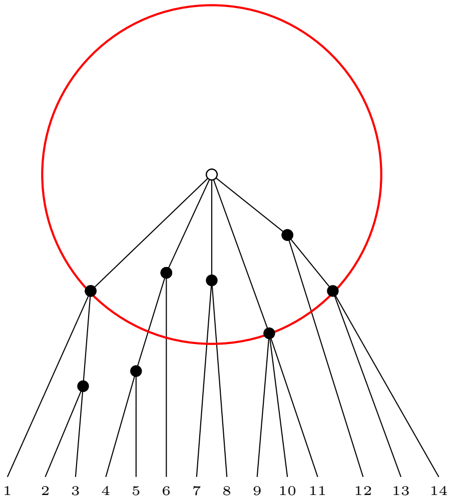

Figure 5: A 7-stable partially aligned marked tree

that in an atomic neighborhood, the ordering of δ with respect to any vertex is compatible with generization. Hence we make use of lemma 4.14 and consider the possibilities.

Clearly each of η and τ is unaffected by a generization that keeps the relative ordering of vertices the same, as in this case internally and externally incident edges remain so. The remaining case of (2) involves taking adjacent vertices to vertices at length δ, possibly by contracting some edges. We recognize that an adjacent vertex may be at length less than δ or at length greater than delta. When an adjecent vertex v is of length *less* than δ, the paths extending from v each contribute to η, and this contribution becomes 1 after $\delta - \lambda(v) \mapsto 0$. On the other hand, since v is adjacent, each path extending from v is also externally incident to δ, and remains so under the generization, so τ is unaffected. For an adjacent vertex v' that is at length *greater* than δ, the contribution to both η and τ is one, from the unique path connecting the cycle and v'. Under a generization $\lambda(v') - \delta \mapsto 0$, there is no effect on η as this unique path remains the only edge internally incident to δ. However, every edge extending from v' (of which there are at least two by stability) will be externally incident to δ in the generization. Hence η is unneffected while τ increases.

Therefore, in an atomic neighborhood of $\overline{s}$, η cannot increase from its value at $\overline{s}$ while τ cannot decrease, and the claim holds. ☐

Definition 4.19. A partially aligned log curve is called *m-stable* if $\eta \leq m < \tau$ for every geometric fiber.

We will refer to such curves as partially aligned m-stable log curves. (Note $m < n$ the number of markings is required).

Corollary 4.19.1. *Let $\mathcal{F}_n(m)$ be the substack of $\mathcal{F}_n$ parametrizing partially aligned m-stable log curves. $\mathcal{F}_n(m)$ is an open substack, hence it is algebraic.*

Proof. This follows immediately from the upper and lower semicontinous functions η and τ. $\square$

Corollary 4.19.2. *Let $\mathcal{F}_n^{\circ}(m)$ and $\mathcal{F}_n^{tot}(m)$ be the substacks of $\mathcal{F}_n^{\circ}$ and $\mathcal{F}_n^{tot}$ parametrizing centrally and totally aligned m-stable log curves, respectively. Each of these is an open substack, hence they are algebraic.*

Corollary 4.19.3. *The definitions of η and τ are compatible with the minimal log structure, and $\mathcal{F}_n^{min}(m)$, $\mathcal{F}_n^{\circ,min}(m)$, and $\mathcal{F}_n^{tot,min}(m)$ are log algebraic stacks representing $\mathcal{F}_n(m)$, $\mathcal{F}_n^{\circ}(m)$, and $\mathcal{F}_n^{tot}(m)$.*

Infinitesimal automorphisms and the semistable model

In general, the objects of each of $\mathcal{F}_n^{tot}(m)$, $\mathcal{F}_n^{\circ}(m)$, and $\mathcal{F}_n(m)$ may have infinitesimal automorphisms. However we may consider the corresponding algebraic stack cut out by each of the minimal log structures decribed above, yielding the log algebraic stacks $\mathcal{F}_n^{tot,min}(m)$, $\mathcal{F}_n^{\circ,min}(m)$, and $\mathcal{F}_n^{min}(m)$. As δ is a stable sum in each, and the underlying curves are stable, these curves have no infinitesimal automorphisms when equipped with the "minimal" m-fold log structure by proposition 6.

We hope the reader will agree that our construction of these spaces over stable curves $\overline{M}_{1,n}$ has been pleasingly natural, but we now wish to modify the universal curves so as to take advantage of our observation in section 1 that log structure is capable of stabilizing rational components with only two special points. The idea is that we should be able to modify our curves such that not only is δ ordered in every path to infinity, by there is a rational component at length δ in every path to infinity.

Construction of the semistable model

Let $\big((X, M_X) \to (S, M_S), \delta\big)$ be an m-stable, partially aligned log curve, and we will assume that (S, M_S) is an versal atomic neighborhood of a geometric point $\overline{s}$. Fix a path to infinity, and assume that in the partially aligned marked tree of the fiber over $\overline{s}$, δ crossed the through an edge instead of a vertex. Let $v_0 < v_1$ be the vertices connected by the edge.

This corresponds to a node of the curve, and we let $\log x$, $\log y$ be the sections of $\overline{M}_X$ corresponding to the node ($a := \log x + \log y$ is identified with a smoothing parameter in the base) and assume that x is the local parameter for v_0. We already have $\delta - \lambda(v_0)$ and $\lambda(v_1) - \delta$ in the log structure. Then the universal property of $Ord_{\lambda(v_0)+\log x,\delta}(X)$ gives a modification over (S, M_S).

$$\begin{tikzcd}
(X', M_{X'}) \arrow[r] \arrow[dr] & (X, M_X) \arrow[d] \\
 & (S, M_S)
\end{tikzcd}$$

Clearly this construction glues, is universal, and can be done for every path to infinity without a component of length δ. Let $(\widetilde{X}, M_{\widetilde{X}}) \to (S, M_S)$ denote the corresponding universal replacement which has rational components at length δ in every path to infinity. The fiber of $(X', M_{X'}) \to (X, M_X)$ over a node is a $\mathbb{P}^1$ bundle, and we call $\big((\widetilde{X}, M_{\widetilde{X}}) \to (S, M_S), \delta\big)$ the semistable model of $\big((X, M_X) \to (S, M_S), \delta\big)$.

Note that the universal property also gives us a minimal log structure on the semistable model (it is the same minimal log structure on the base). Finally, since δ is a stable sum, such curves equipped with their minimal log structure are without infinitesimal automorphisms. Hence we have described a modification of the universal log curve

$$\begin{tikzcd}
(\widetilde{\mathscr{C}}, M_{\widetilde{\mathscr{C}}}) \arrow[r] \arrow[dr] & (\mathscr{C}, M_{\mathscr{C}}) \arrow[d] \\
 & (\mathcal{F}_n^{min}(m), M_{\mathcal{F}_n^{min}(m)}).
\end{tikzcd}$$

From this point on we will take the universal curves of each of our moduli functors to be the corresponding semistable models, that is we assume that curves have a rational component (possibly semistable!) at length δ in every path to infinity.

4.5 Properness

Note that each of the functors defined above lies over $\left(\overline{\mathcal{M}}_{1,n}, M_{\overline{\mathcal{M}}_{1,n}}\right)$, with the map forgetting δ and non-canonical ordering corresponding to inclusions of the log structure $M_{\overline{\mathcal{M}}_{1,n}}$ into the various extensions. Recall that a log blowup is a proper modification, and as such $\mathcal{LM}_{1,n}^{ord,min} \to \left(\overline{\mathcal{M}}_{1,n}, M_{\overline{\mathcal{M}}_{1,n}}\right)$ is proper.

We devote this section to the proof of the following proposition.

Proposition 11. *The natural map* $\mathcal{F}_n^{min}(m) \to \left(\overline{\mathcal{M}}_{1,n}, M_{\overline{\mathcal{M}}_{1,n}}\right)$ *forgetting* δ *and non-canonical ordering is a proper morphism of log algebraic stacks.*

Proof. Note that we have three proofs of properness, as we assert that the $\mathcal{F}_n^{min}(m)$ is actually a subdivision (hence is proper), it is dominated by a log blowup (hence is proper), and there is a valuative criterion, which is the path we follow here. To begin, consider the diagram below relating our various moduli spaces to one another.

$$\begin{tikzcd}
\mathcal{F}_n^{tot,min}(m) \arrow[r] \arrow[d, "\wr"'] & \mathcal{F}_n^{\circ,min}(m) \arrow[r] & \mathcal{F}_n^{min}(m) \arrow[d] \\
\mathcal{LM}_{1,n}^{ord,min} \arrow[rr] & & (\overline{\mathcal{M}}_{1,n}, M_{\overline{\mathcal{M}}_{1,n}})
\end{tikzcd}$$

The left vertical arrow is an isomorphism given by the fact that for fixed m, any ordered log curve has a unique section of the characteristic on the base satisfying the conditions of m-stability. The horizontal arrows are all explicit log blowups. In particular, $\mathcal{F}_n^{tot,min}(m)$ is proper over $(\overline{\mathcal{M}}_{1,n}, M_{\overline{\mathcal{M}}_{1,n}})$, and each of $\mathcal{F}_n^{\circ,min}(m)$ and $\mathcal{F}_n^{min}(m)$ is a log modification (definition 3.32). In fact, this is expressed by $\mathcal{F}_n^{tot,min}(m)$ as a cover by a global log blowup. In particular this implies that every functor in the diagram above is universally surjective and log étale over $(\overline{\mathcal{M}}_{1,n}, M_{\overline{\mathcal{M}}_{1,n}})$, in particular locally of finite presentation. We are really done at this point since log modifications are proper, but we may proceed by showing the valuative criterion (existence and uniqueness of limits) and boundedness. We learned about the technique for the valuative criterion from Jonathan Wise, and find it to be the most natural valuative criterion argument (of course there are others related to the alternate proofs of properness).

Existence and uniqueness of limits

Let R be a DVR with residue field K and suppose we have the commutative diagram of solid arrows below.

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\xi"] \arrow[d] & \mathcal{F}_n^{min}(m) \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \overline{\mathcal{M}}_{1,n}
\end{tikzcd}$$

Let $(\operatorname{Spec} K, M_K^{min})$ denote the underlying fs log scheme of the partially aligned log curve induced by ξ, let $(\operatorname{Spec} K, M_K) \to (\operatorname{Spec} K, M_K^{can})$ be the map to the canonical log structure, and $(\operatorname{Spec} K, M_C^{can}) \to (\operatorname{Spec} R, M_R^{can})$ the inclusion of the generic point.

We need to find some log scheme $(\operatorname{Spec} R, M)$ that extends the relative ordering of the log structure in M_R^{can} so that vertices that map to relatively ordered vertices over $\operatorname{Spec} K$ are relatively ordered by M. We would also like $(\operatorname{Spec} K, M_K) \to (\operatorname{Spec} R, M_R^{can})$ to factor through $(\operatorname{Spec} R, M)$. Since the underlying curve is stable, and ξ satisfies m-stability, such an extension of order ensures that there is a unique $\delta \in \overline{M}$ satisfying the condition of m-stability. This ensures that the map $(\operatorname{Spec} R, M) \to (\operatorname{Spec} R, M_R^{can})$ factors through our moduli problem, and the universal property of the minimal log structure provides the dashed arrow in the diagram above.

There is a natural such object, essentially the maximum log structure we may put on $\operatorname{Spec} R$ so that $(\operatorname{Spec} K, M_K) \to (\operatorname{Spec} R, M_R^{can})$ factors through. We construct this as

$$M_R^{max} := M_K^{min} \times_K R \to R,$$

and note that this object is *not* a fine log structure. However, the forgetful map to $(\operatorname{Spec} R, M_R^{can})$ ensures that it maps to many fine log schemes, and factors uniquely through a minimal partially aligned log curve.

Suppose α, β are sections of $\overline{M}_R^{max}$ such that $\alpha - \beta$ is in the kernel of the specialization map $\overline{M}_R^{max} \to \overline{M}_K^{min}$. Lifting α and β to M_R^{max} and letting $\tilde{\alpha}$, $\tilde{\beta}$ be their images in M_K^{min}, we see that there is a unit λ in K with

$$\tilde{\alpha} = \tilde{\beta} + \log(\lambda).$$

Furthermore, since R is a DVR, either λ or λ^{-1} is in R. In other words, any vertices in the dual graph over the closed point whose lengths are identified over $(\operatorname{Spec} K, M_K)$ are ordered in $(\operatorname{Spec} R, M_R^{max})$. Let δ be the unique section of $\overline{M}_R^{max}$ satisfying the m-stable condition, so the forgetful map to $(\operatorname{Spec} R, M_R^{can})$ factors uniquely through a minimal m-stable partially aligned log curve over $\operatorname{Spec} R$ by virtue of the universal property of the minimal log structure. Hence we have existence and uniqueness of the desired lift.

Boundedness

Let $(X, M_X) \to (S, M_S)$ be a minimal stable log curve with S quasicompact. Since we can freely pass to constructible étale covers, we assume we have constant log structure and constant dual graph. Then the claim follows from the universal property of the minimal log structure, the fact there are finitely many partially aligned marked trees over any dual graph of a stable curve, and that the corresponding choice of a log structure is finite type. ☐

Figure 6 depicts how the valuative criterion works in practice by considering a general example. Let us assume that we have a partially aligned log curve such that the associated partially aligned marked tree has at least two vertices at length $\delta > 0$. In the figure, we have only depicted the portion of the graph specified by these assumptions, and hence we may represent our curve by the graph labelled C, with a horizontal line to indicate that both vertices are at length δ.

Suppose we have a specialization in which one of the two nodes splits. The fact that a node was able to split and remain stable implies that there are at least three marks on one node, which we indicate by c, d, e. The remaining node must have at least two marks by stability, and we have labelled these as a and b. In general there are then at least three values of m which could have given rise to such a curve, that is the fiber of partially aligned log curves over the underlying stable curve in $\overline{\mathcal{M}}_{1,n}$ has at least three points. We denote these three possible stability conditions by m, $m+1$, and $m+2$.

In the figure, we depict the curve itself as lying in the intersection of two boundary components, with the specialization described above lying in the intersection with a third component. The corresponding intersections of underlying stable curves are depicted by the lower triangle in the figure. The edges of the triangle should be thought of as having codimension one higher than the verices, and the interior as codimension one higher than the edges. The specialization itself is indicated by a small arrow, moving the curve from the generic intersection of two boundary components to the triple intersection.

Now, putting order on our log curves is a proper modification (it is a subdivision), so the specialization of stable curves lifts uniquely to the ordered log curves. What remains is a choice of δ. However, of the possible m, $m+1$, and $m+2$ stability conditions, C has a fixed one. In each ordered curve there is only one possible choice for δ (indicated by a dashed line) for each of these stability conditions. Hinted at here is the much simpler proof of properness given by the fact that $F_n^{min}(m)$ is a subdivision.

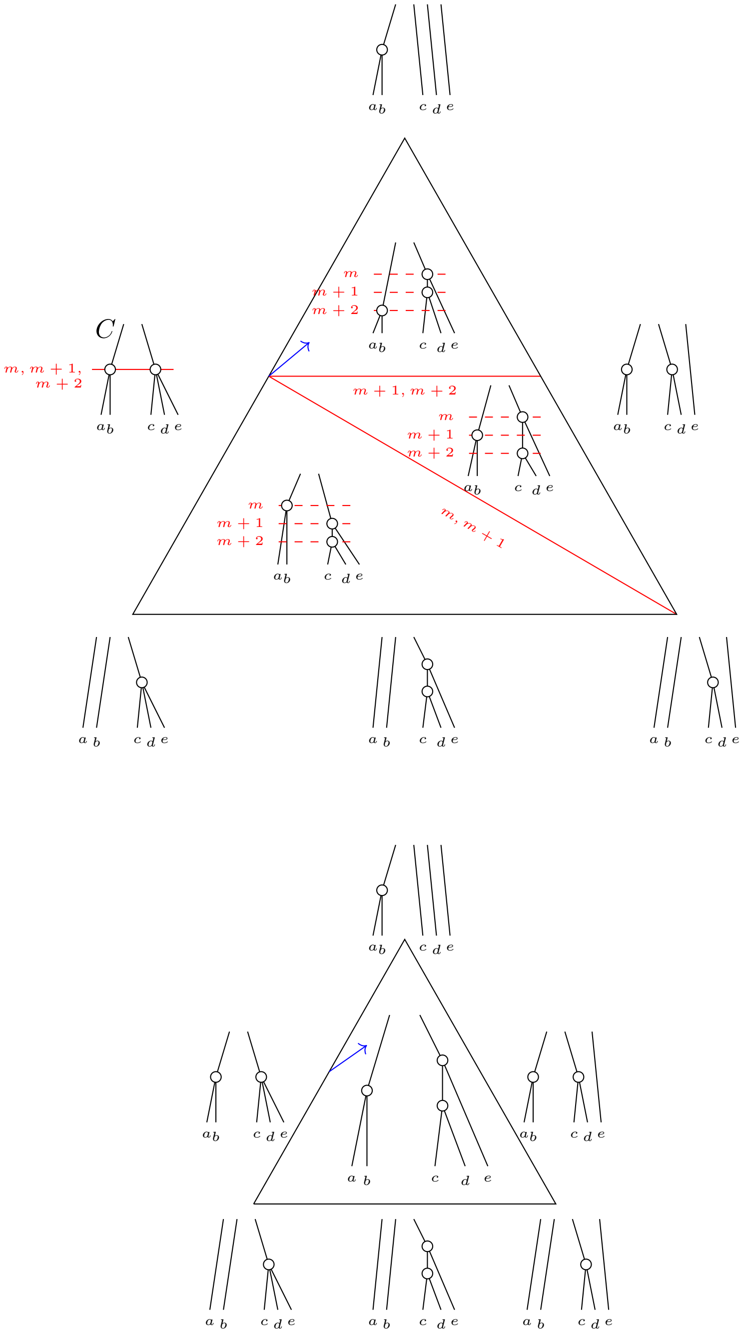

Figure 6: Valuative Criterion of properness

Representability of the moduli problem

We recall the statement here:

Theorem (1.1)**.** *The moduli problem of* m*-stable partially aligned log curves is representable by a proper, irreducible, logarithmic Deligne-Mumford algebraic stack* $\mathcal{F}_n^{min}(m)$*.*

Proof. Representability by an algebraic stack follows from openness of m-stability (Corollary 4.19.1) and the openness (Lemma 4.11) and universality (Proposition 9) of the minimal log structure. Properness is completed in the first part of Section 4.5 above. The fact that $\mathcal{F}_n^{min}(m)$ is irreducible follows from the fact that the locus of smooth curves $\mathcal{M}_{1,n}$ is dense since this is the locus of trivial log structures, which is dense since our objects are log smooth. The fibered category is a Deligne-Mumford stack since finite automorphisms implies formally unramified diagonal. □

5 Contraction to m-stable curves

In the previous section, the length section λ was developed as a canonical section of the characteristic sheaf on the universal curve, and this allowed us to define what it means for a section δ of the characteristic sheaf on the base to be ordered in every path to infinity. As we have noted previously, there are in general many options for defining proper modular compactifications, and it seems reasonable that one could use any of our stacks to define an analogous notion of an extremal assignment with

$$G \supset Z(G) := \{v \in G | \lambda(v) < \delta\}.$$

However, we are interested in resolving the indeterminacy of the birational map $\overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,n}(m)$, that is we would like to realize a universal contraction of the universal curve over $\mathcal{F}_n^{min}(m)$ to an m-stable curve in the sense of Smyth. Such curves contain only the *unique* Gorenstein singularities of type $(1, l)$, and so the usual machinery of extremal assignments is not enough as it does specify the singularities beyond genus and branch type.

The necessary ingredient, as suggested by the work presented in [HH09], is a π-semiample line bundle on the universal curve, *that remains locally free when pushed forward*, so that we may achieve a contraction to a flat family of curves. It turns out this will be enough to ensure the dualizing sheaf of the fibers is invertible as well, so we can say that the resulting contracted curves exhibit Gorenstein singularities. In fact, once such a line bundle is found, the singularity type is completely determined by Smyth's contraction lemma [Smy11, lemma 2.12].

Note that the modification of the universal curve of $\mathcal{F}_n^{min}(m)$ to include semistable components is essential for the line bundle constructions we will carry out here, it would be optional for the hinted Z-stability moduli space suggested above. Briefly, lifts of a section of the characteristic sheaf on any log scheme X form a torsor under $\mathcal{O}_X^*$. In the particular case of λ, we will see that $Lift(\lambda) \simeq \omega_\pi$, the relative dualizing sheaf of $\mathcal{U} \to \mathcal{F}_n^{min}(m)$. In turn, δ will similarly determine a line bundle $\mathcal{O}(E)$ on the universal curve, and $L := Lift(\lambda + \tilde{\delta}) = Lift(\lambda) \otimes Lift(\tilde{\delta}) = \omega_\pi(E)$ will be an essential object introduced below. The work we do here will be to prove the following theorem.

Theorem 5.1. *There is a proper, birational, surjective morphism* $\phi : \mathcal{F}_n^{min}(m) \to \overline{\mathcal{M}}_{1,n}(m)$.

The proof will rely on a canonical construction of an m-stable curve $X' \to S$ associated to any m-stable partially aligned log curve $\big((X, M_X) \to (S, M_S), \delta\big)$, determining an assignment

$$\Big((X, M_X) \to (S, M_S), \delta\Big) \quad \mapsto \quad \Big(X' \to S\Big)$$

that is compatible with isomorphisms and base extension (satisfies the conditions for descent).

A key ingredient for this construction is

Proposition 12. *$\pi_* L^k$ is locally free for all k.*

We will prove this in section 5.2 below by describing the boundary divisor $\Delta_E \subset \mathcal{F}_n^{min}(m)$ parametrizing m-stable partially aligned log curves with nontrivial δ. The proposition follows from the fact that $\pi_* L^k$ is locally free away from Δ_E, and lemma 5.8 below, which tells us that it is locally free of the same rank on Δ_E as well.

5.1 Line bundles from sections of the characteristic sheaf

Let $\big((X, M_X) \to (S, M_S), \delta\big)$ be an m-stable partially aligned log curve, $\overline{s} \in S$ a geometric point, and assume that S is an atomic neighborhood of $\overline{s}$. Recall that on an atomic neighborhood, a section of the characteristic sheaf $\overline{M}_X$ is determined by a collection of sections of $\overline{M}_S$ for each vertex of the dual graph of the underlying curve such that on adjacent vertices, sections differ by an integer multiple of the smoothing parameter of the corresponding node. There is also a choice of integer for each marked half edge of the dual graph.

The length section λ is defined as the canonical section of the characteristic sheaf that is zero on the vertices of the cycle and has contact order one at each marked point and node external to the cycle. We will associate to δ a section of the characteristic sheaf of the curve. Note δ is a section of the characteristic sheaf *on the base*. Associating a section on the curve amounts to assigning vertices that will take on the value δ, along with a choice of orientation, and contact orders for each edge.

Recall that at the end of the previous section we detailed how to modify the universal curve of $\mathcal{F}_n(m)$, etc. so as to have rational components at length δ in every path to infinity. As a result we may specify a canonical section $\tilde{\delta}$ of $\overline{M}_X$ as follows:

(1) $\tilde{\delta}(v) = 0$ for every vertex v with $\lambda(v) \geq \delta$;

(2) $\tilde{\delta}$ has zero contact order at each marked point;

(3) $\tilde{\delta}(v) = \delta$ for every vertex v of the cycle.

These conditions *imply* that the contact order of $\tilde{\delta}$ on each edge of a path from the cycle to the circle at length δ is one, and the orientation is towards the cycle, while on every other edge the contact order is 0.

Since the involved contact orders are zero or one, it is useful to think of λ and $\tilde{\delta}$ as orientations of the dual graphs of the underlying curves. For example, figure 7 gives the corresponding orientations on the partially aligned log curve given by the sections λ and δ.

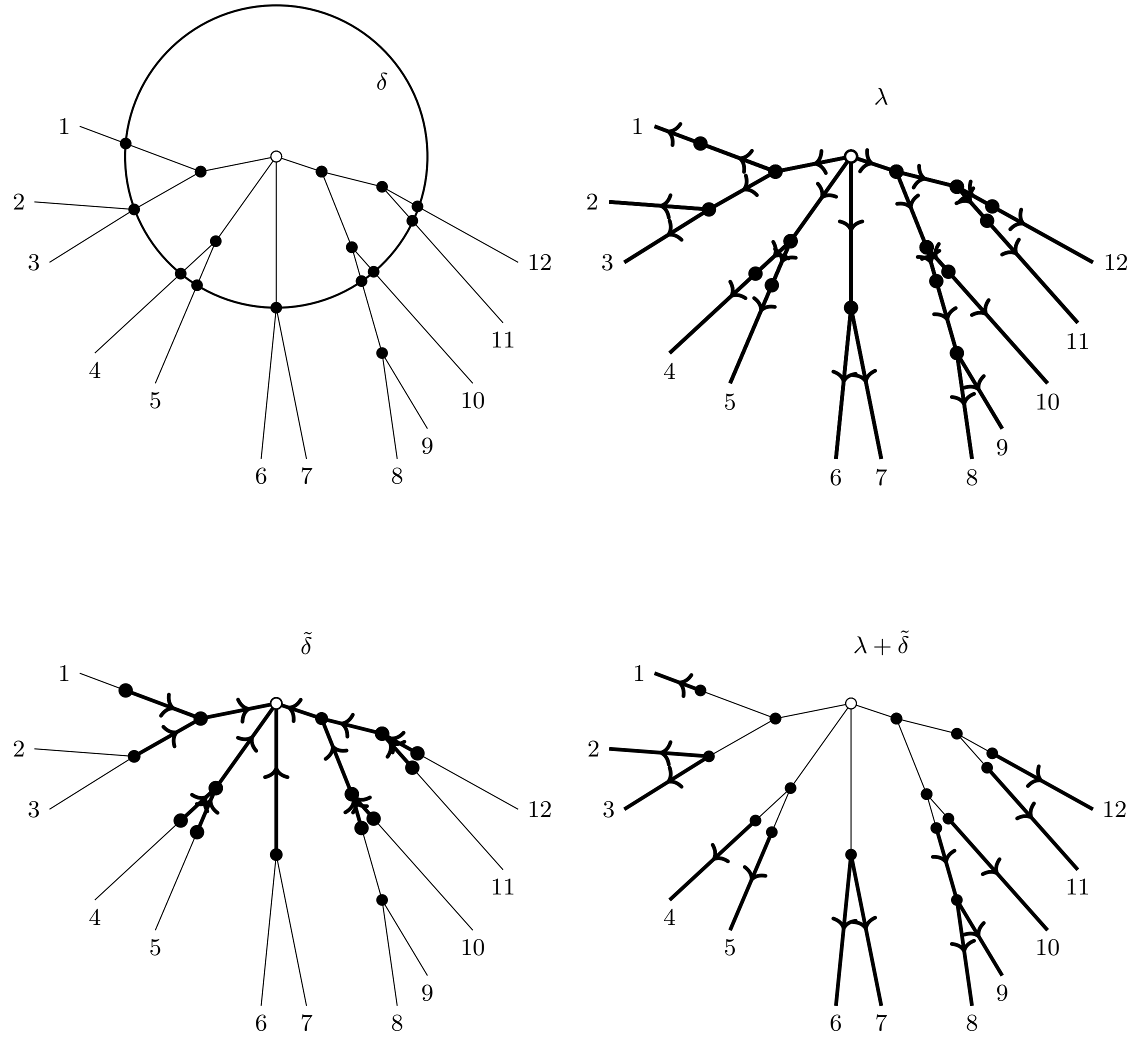

Figure 7: The diagramatic representations of λ and $\tilde{\delta}$ make degree calculations of the line bundles of lifts particularly simple.

Definition 5.2. Let γ be a section of $\overline{M}_X$ for $(X, M_X) \to (S, M_S)$ a log smooth curve. Define the line bundle $Lift(\gamma)$ on S to be the $\mathcal{O}_X^*$-torsor determined by the sheaf of lifts of the section γ to M_X as in the diagram below

$$\begin{tikzcd}
 & & & \mathbb{N} \arrow[dl, dashed] \arrow[d, "\gamma"] & \\
0 \arrow[r] & \mathcal{O}_X^* \arrow[r] & M_X \arrow[r] & \overline{M}_X \arrow[r] & 0.
\end{tikzcd}$$

We define the line bundles $Lift(\lambda)$ and $Lift(\tilde{\delta})$ as given by this construction.

Lemma 5.3. *Let γ be a global section of $\overline{M}_X$ such that on components v_1, v_2 intersecting at a node with smoothing parameter δ, γ takes on values satisfying $\gamma(v_2) = \gamma(v_1) + \delta$. Then in particular the line bundle defined by $Lift(\gamma)$ restricts locally to $\mathcal{O}_{v_1}(p)$ and $\mathcal{O}_{v_2}(-q)$ for p, q the node in each component.*

Proof. Recall the local description of the log structure near the node with $\delta = \log x + \log y$ for local parameters, and the fact that $\gamma(v_2) = \gamma(v_1) + \delta$. The generator on v_2 differs from the generator on v_1 by a factor of xy. In particular, consistent lifts are generated by sections of the form $x(1)$, $y^{-1}(xy)$, that is required a zero on v_1 at the node, and allowed a pole on v_2 at the node. $\square$

Remark 5.1. The above lemma tells us that in the particularly simple case that a global section of the characteristic sheaf has contact order one on each oriented edge (and necessarily contact order zero on non-oriented edges), the degree of the line bundle is determined by the orientation on the underlying dual graph. Namely, the degree on a component corresponding to a vertex v is equal to the number of incident edges oriented away from v minus the number of incident edges oriented towards v.

Lemma 5.4. *There is an isomorphism of line bundles $Lift(\lambda) \simeq \omega_\pi(\Sigma)$ on the universal curve over $\mathcal{F}_n^{min}(m)$, for Σ the divisor of markings.*

Proof. Since the dual graph of the underlying curve is a rooted tree, and from the above lemma computing degrees on components, $Lift(\lambda)$ on the minimal elliptic subcurve is given as $\mathcal{O}(p_1 + \cdots + p_r)$ for $\{p_i\}$ the nodes of the elliptic component. For rational components, $Lift(\lambda)$ restricts to $\mathcal{O}(n-2)$ for n the number of nodes. This is essentially due to the fact that the dual graph is a tree, there is a single incoming edge of the dual graph contributing -1 to the degree, and $n-1$ outgoing edges of the dual graph contributing positive degree. Hence we obtain an isomorphism with $\omega_\pi(\Sigma)$. $\square$

Definition 5.5. Let E be the locus in the universal curve consisting of components on which $Lift(\tilde{\delta})$ has negative degree (components of length strictly less than δ). E is an effective cartier divisor on the universal curve, and $Lift(\tilde{\delta}) \simeq \mathcal{O}(E)$. There is a corresponding boundary divisor on the base given as the locus of curves with $\delta > 0$, we will denote this by Δ_E.

We now wish to define a line bundle on the universal curve over $\mathcal{F}_n^{min}(m)$ that will have positive degree on all components of length at least δ, and degree 0 on components of length less than δ. Thanks to our preliminary work, such a line bundle is not difficult to describe, it will simply be $L := Lift(\lambda) \otimes Lift(\tilde{\delta})$.

Lemma 5.6. *We have the following isomorphisms:*

$$L \cong \omega_\pi(E + \Sigma) \cong Lift(\lambda + \tilde{\delta}).$$

Lemma 5.7. *The line bundle L is trivial on the divisor of components of length strictly less than δ:*

$$L\big|_E \simeq \mathcal{O}_E$$

Proof. On E, the section $\lambda + \tilde{\delta}$ is trivial, hence $Lift(\lambda + \tilde{\delta})$ is trivial. ☐

Corollary 5.7.1. $R^1\pi_* L^k\big|_{\Delta_E}$ *is a line bundle on Δ_E.*

5.2 Cohomology and base change with a boundary

We will now prove the key lemma required for proposition 12. The point will be that although $R^1\pi_*L^k$ is nontrivial, it itself is a line bundle when restricted to the *divisor* Δ_E. This line bundle serves to obstruct the deformation of local sections on E to outside the fiber, preserving the local freeness of π_*L^k despite a nonzero $h^1(E, L^k\big|_E)$. Hence we shall see that although the cohomology of L^k does not commute with base change in general, it does commute with "base change with a boundary" (see lemma 5.8 below). The argument here was suggested to us by Jonathan Wise, and is an alternate to the flattening an blowing up strategy in [HH09], although the idea is essentially the same.

As π_*L^k is certainly locally free away from Δ_E by stability, so we work locally near Δ_E. Hence we assume we have S a normal scheme, $\pi : (X, M_X) \to (S, M_S)$ an m-stable partially aligned log curve, and define ∂S to be the pullback of Δ_E along the classifying morphism $S \to \mathcal{F}_n^{min}(m)$ (we also assume S is versal).

Lemma 5.8. *Let T be the spectrum of a DVR with closed point t, generic point t', and assume we have $f : T \to S$ a map taking t to δS and t' to $S \setminus \delta S$. Then*

$$f^*\pi_*L^k = \pi_*f^*L^k.$$

Proof. By cohomology and base change, we can find K^0, K^1 finitely generated and locally free fitting into an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \pi_*L^k \arrow[r] & K^0 \arrow[r] & K^1 \arrow[r] & R^1\pi_*L^k \arrow[r] & 0 \end{tikzcd} \tag{5.1}$$

such that the sequence below is exact as well.

$$\begin{tikzcd} 0 \arrow[r] & \pi_*f^*L^k \arrow[r] & f^*K^0 \arrow[r] & f^*K^1 \arrow[r] & R^1\pi_*f^*L^k \arrow[r] & 0 \end{tikzcd}$$

Our strategy will be to show that the sequence

$$\begin{tikzcd} 0 \arrow[r] & f^*\pi_*L^k \arrow[r] & f^*K^0 \arrow[r] & f^*K^1 \arrow[r] & f^*R^1\pi_*L^k \arrow[r] & 0 \end{tikzcd}$$

is exact.

We now pull back 5.1 in a derived manner, yielding a spectral sequence $L_pf^*R^q\pi_*L^k$ converging to 0.

$$\begin{tikzcd} 0 \arrow[r] & L_2f^*\pi_*L^k \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & L_2f^*R^1\pi_*L^k \arrow[r] & 0 \\ 0 \arrow[r] & L_1f^*\pi_*L^k \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & L_1f^*R^1\pi_*L^k \arrow[r] & 0 \\ 0 \arrow[r] & f^*\pi_*L^k \arrow[r] & f^*K^0 \arrow[r] & f^*K^1 \arrow[r] & f^*R^1\pi_*L^k \arrow[r] & 0 \end{tikzcd} \tag{E1}$$

$$\begin{tikzcd} L_2f^*\pi_*L^k & 0 & 0 & L_2f^*R^1\pi_*L^k \\ L_1f^*\pi_*L^k \arrow[urr] & 0 \arrow[urr] & 0 & L_1f^*R^1\pi_*L^k \\ \ker(f^*\pi_*L^k \to \pi_*f^*L^k) \arrow[urr] & \operatorname{coker}(f^*\pi_*L^k \to \pi_*f^*L^k) \arrow[urr, "\sim"] & 0 & \end{tikzcd} \tag{E2}$$

$$\begin{tikzcd}
 & & & L_3 f^* R^1 \pi_* L^k \\
L_2 f^* \pi_* L^k & 0 & 0 & L_2 f^* R^1 \pi_* L^k \\
L_1 f^* \pi_* L^k \arrow[uurrr, "\sim"'] & 0 & 0 & 0 \\
\ker(f^* \pi_* L^k \to \pi_* f^* L^k) \arrow[uurrr, "\sim"'] & 0 & 0 & 0
\end{tikzcd} \tag{E3}$$

Hence obstructions to commuting with the base change come from $L_1 f^* R^1 \pi_* L^k$ and $L_2 f^* R^1 \pi_* L^k$. However, $R^1 \pi_* L^k$ is a line bundle on ∂S, and furthermore ∂S is a divisor on S, given locally by a single equation $s = 0$. Looking locally, we have that $R^1 \pi_* L^k \cong \mathcal{O}_{\delta S}$ and since S is integral there is an exact sequence (a projective resolution)

$$0 \longrightarrow \mathcal{O}_S \xrightarrow{s} \mathcal{O}_S \longrightarrow \mathcal{O}_{\delta S} \longrightarrow 0.$$

Now we apply f^* to get

$$0 \longrightarrow L_1 f^* \mathcal{O}_{\delta S} \longrightarrow f^* \mathcal{O}_S \xrightarrow{f^* s} f^* \mathcal{O}_S \longrightarrow f^* \mathcal{O}_{\delta S} \longrightarrow 0$$

and that $L_2 f^* R^1 \pi_* L^k = 0$. By assumption, $f(t') \notin \delta S$, so $f^* s \neq 0$. Since T is a DVR (so integral), this implies that $f^* s$ is injective, so $L_1 f^* \mathcal{O}_{\delta S} = 0$. Thus $L_1 f^* R^1 \pi_* L^k = 0$ and $L_2 f^* R^1 \pi_* L^k = 0$, and

$$f^* \pi_* L^k = \pi_* f^* L^k.$$

$\square$

(proof of proposition 12)

Proof. Since $\pi_* L^k$ is locally free away from Δ_E and locally free of the same rank on Δ_E, $\pi_* L^k$ is locally free. $\square$

5.3 Divisorial Contraction to m-stable curves

We are now prepared to complete our contraction of partially aligned log curves to the m-stable curves of Smyth. Our argument is essentially that of Smyth's contraction lemma [Smy11, Lemma 2.12], the only difference being that we are able to glue our contractions up to the universal curve $\pi : \mathcal{C} \to \mathcal{F}_n^{min}(m)$.

Let $f : (X, M_X) \to (S, M_S), \delta$ be an m-stable partially aligned log curve over a normal locally noetherian base such that the classifying map $\xi : S \to \mathcal{F}_n^{min}(m)$ is faithfully flat. We assume that ξ has smooth generic fiber. We briefly compile the pertinent information that we have on ξ:

(1) a line bundle $L := \omega_{X/S}(E + \Sigma)$ on X defined as the pullback of $\omega_\pi(E + \Sigma)$ along the classifying map $(X, M_X) \to \mathscr{C}$ to the universal curve, where Σ denotes the divisor of marked points;

(2) δ_S a cartier divisor on S given as the pullback of Δ_E along $\xi : S \to \mathcal{F}_n^{min}(m)$;

(3) $f_* L^k$ is locally free by lemma 5.8.

Proposition 13. *Given the above situation, L is f-semiample and we have a diagram*

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "f"'] & & X' := \mathrm{Proj}(\oplus_{n \geq 0} f_* L^k) \arrow[dl, "f'"] \\ & S & \end{tikzcd}$$

with ϕ proper, birational, and with the exceptional locus equal to E. Furthermore,

(1) $f' : X' \to S$ is flat and projective with reduced fibers;

(2) $\phi\big|_{\overline{\mathrm{X} \setminus E}\big|_s} : \overline{\mathrm{X} \setminus E}\big|_s \to \mathrm{X}'\big|_s$ is the normalization of $\mathrm{X}'\big|_s$ at $\phi(E\big|_s)$ for each fiber;

(3) $\phi(E\big|_s)$ is an elliptic m-fold point in each fiber X'_s, and $X' \to S$ together with the image of Σ is an m-stable curve in the sense of Smyth.

Proof. Since $f_* L^n$ is locally free by proposition 12, $X' \to S$ is flat. For the remainder, we may assume that S is a DVR and apply Smyth's contraction lemma [Smy11]. It is not in general necessary for us to restrict to a DVR, however, and we include a proof below for completeness.

To begin, L being f-semiample is equivalent to having the adjunction map

$$f^* f_* L^k \to L^k$$

be surjective for n sufficiently large. Note that we have that L is ample on generic fibers, and over δS

$$L\big|_E \simeq \mathcal{O}_E \ ;$$

$$L\big|_{\overline{X_{\delta S} \setminus E}} \text{ is ample.}$$

Since we just need the surjection to hold for $n \gg 0$, and L is generically ample, we simplify the problem to finding a section $s \in f_*L\big|_{\delta S}$ for each point of $X_{\delta S}$ that does not vanish.

Pushing forward the tensored ideal sheaf sequence

$$\begin{tikzcd} 0 \arrow[r] & L^n(-E) \arrow[r] & L^n \arrow[r] & \mathcal{O}_E \arrow[r] & 0 \end{tikzcd}$$

yields the sequence below.

$$\begin{tikzcd} 0 \arrow[r] & f_*L^n(-E) \arrow[r] & f_*L^n \arrow[r] & f_*\mathcal{O}_E \arrow[r] & R^1f_*L^n(-E) \end{tikzcd}$$

Note that $f_*\mathcal{O}_E\big|_{\delta S}$ is a line bundle on ∂S, so showing that $f_*L^n \to f_*\mathcal{O}_E$ is surjective gives the required sections on the points of E. Of course, $L^n(-E)$ is the dualizing sheaf for $n = 1$, and we may observe vanishing R^1 for $n \gg 0$ in general by considering the trivial higher cohomology of the tensored ideal sheaf sequence of E in $X_{\delta S}$:

$$\begin{tikzcd} 0 \arrow[r] & L^n \otimes I_{E/X_{\delta S}} \arrow[r] & L^n(-E)\big|_{X_{\delta S}} \arrow[r] & \mathcal{O}_E(-E) \arrow[r] & 0. \end{tikzcd}$$

The vanishing of $R^1f_*L^n(-E)$ further implies that $f_*L^n(-E)\big|_{\delta S} = H^0(X_{\delta S}, L^n \otimes I_{E/X_{\delta S}})$ since $\mathcal{O}_E(-E)$ has no global sections. Finally, the ampleness of $L\big|_{\overline{X_{\delta S}\setminus E}}$ and the fact that $f_*L^n\big|_{\delta S}$ contains $f_*L^n(-E)\big|_{\delta S}$ gives us all the sections we need.

From f-semiampleness, we now have that ϕ is proper, birational, and with exceptional locus equal to E.

Note that S is irreducible and normal. X is regular in codimension one (R1) since $X \to S$ has smooth generic fiber and has isolated singularities in fibers. Now $X \to S$ is flat, with fibers X_s satisfying (S2) (since reduced, one dimensional implies Cohen-Macaulay (Sk for all $k > 0$)) and S also satisfying S2, so X is S2 [Gro65, Corollary 6.4.2] and by Serre's criterion is normal.

From flatness, the fiber $X'_{\delta S}$ is a Cartier divisor in X', and so has no embedded points. It is also connected since $X_{\delta S}$ is connected, and reduced as the birational image of $\overline{X_{\delta S} \setminus E}$. In fact, since $\overline{X_{\delta S} \setminus E}$ is smooth along all the points of $E \cap \overline{X_{\delta S} \setminus E}$, we have that for each fiber, $\phi\big|_{\overline{\mathrm{X}\setminus E}\big|_s} : \overline{\mathrm{X} \setminus E}\big|_s \to \mathrm{X}'\big|_s$ is the normalization of $\mathrm{X}'\big|_s$ at $\phi(E\big|_s)$.

The same argument for normality above applies to X', so X' is normal. Furthermore, ϕ is equal to its own Stein factorization since a finite birational morphism of normal algebraic spaces is an isomorphism [Knu71, Lemma V.4.7], so in particular $\phi_*\mathcal{O}_X = \mathcal{O}'_X$.

Since Σ is disjoint from the locus on which the line bundle fails to have positive degree, its image is a Cartier divisor as well. To see that fibers $E\big|_s$ are contracted to a genus one singularity in each fiber, note that $\delta = \chi(X_{\delta S}\big|_s, \mathcal{O}_{\overline{X_{\delta S}\setminus E}\big|_s}) - \chi(X'_{\delta S}\big|_s, \mathcal{O}_{X'_{\delta S}\big|_s})$ by definition since $\overline{X_{\delta S}\setminus E}\big|_s$ is the normalization. Then since we have that $X'_{\delta S}\big|_s$ and $X_{\delta S}\big|_s$ occur in flat families with the same generic fiber, $\delta = \chi(X_{\delta S}\big|_s, \mathcal{O}_{\overline{X_{\delta S}\setminus E}\big|_s}) - \chi(X_{\delta S}\big|_s, \mathcal{O}_{X_{\delta S}\big|_s})$. Finally, by additivity of the Euler characteristic and the fact that $I_{\overline{X_{\delta S}\setminus E}}\big|_s$ is supported on $E\big|_s$,

$$\delta = -\chi(X_{\delta S}\big|_s, I_{\overline{X_{\delta S}\setminus E}\big|_s}) = -\chi(E\big|_s, I_{\overline{X_{\delta S}\setminus E}}\big|_{E\big|_s}) = -\chi(E\big|_s, \mathcal{O}_{E\big|_s}(-E \cap \overline{X_{\delta S}\setminus E}\big|_s)) = p_a(E\big|_s) + m - 1$$

It remains to show the image of $E\big|_s$ is a gorenstein curve singularity in each fiber.

Reduced fibers imply Cohen-Macaulay fibers, so since any flat, projective, finitely presented morphism $X \to S$ whose geometric fibers are Cohen-Macaulay admits a relative dualizing sheaf [Kle80, Theorem 21] whose formation commutes with base change [Kle80, Proposition 9], and the relative dualizing sheaf is S2 [KM98, corollary 5.69] together with [Gro65, proposition 6.4.1], it will suffice to show that $\omega_{X'/S}$ is isomorphic to a line bundle in codimension one (since on a reduced scheme of finite type over a field S2 sheaves isomorphic in codimension one are isomorphic [FvdGL11, lemma 5.1.1]). But this is true, take

$$\mathcal{O}_{X'}(1)(-\Sigma)\big|_{X'\setminus\phi(E)} \cong \omega_{X'/S}\big|_{X'\setminus\phi(E)}.$$

Note $\phi(E)$ is the exceptional image and it is codimension 2, so this is an isomorphism in codimension one by definition. So we have shown that the relative dualizing sheaf on X', which commutes with base extension, is isomorphic to a line bundle $\mathcal{O}_{X'}(1)(-\Sigma)$. In particular the fibers are Gorenstein curves.

The fact that the fibers are stable in the sense of Smyth is immediate from our stability condition, so we have proved (3). ☐

Proof of theorem 5.1

Now that we have developed the machinery contracting an m-stable partially aligned log curve to an m-stable curve in the sense of Smyth, we finish the proof of Theorem 5.1.

Proof. Since our line bundles are pulled back from $\omega_\pi(E + \Sigma)$ on the universal curve, the construction of contractions of m-stable partially aligned minimal log curves to m stable curves of Smyth are compatible with isomorphisms and base extension, so we have a well-defined map $\mathcal{F}_n^{min}(m) \to \overline{M}_{1,n}(m)$. Together with the forgetful morphism to stable curves, we have achieved the commutative diagram below.

$$\begin{tikzcd}
\mathcal{F}_n^{min}(m) \arrow[d] \arrow[dr] & \\
\overline{\mathcal{M}}_{1,n} \arrow[r, dashed] & \overline{\mathcal{M}}_{1,n}(m)
\end{tikzcd}$$

$\square$

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