RELATIVE STABLE MAPS IN GENUS ONE VIA CENTRAL ALIGNMENTS

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ABSTRACT. For a smooth projective variety X and a smooth very ample hypersurface $Y \subseteq X$, we define moduli spaces of relative stable maps to (X, Y) in genus one, as closed substacks of the moduli space of maps from centrally aligned curves, constructed in [RSW17]. We construct virtual classes for these moduli spaces, which we use to define *reduced relative Gromov–Witten invariants* in genus one.

[GOALS: We prove a recursion formula which allows us to completely determine these invariants in terms of the reduced Gromov–Witten invariants, as defined in [REF]. We also prove a relative version of the Li–Zinger formula, relating our invariants to the usual relative Gromov–Witten invariants. Also say something about quasimaps.]

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1. THE SPACE OF RELATIVE CENTRALLY ALIGNED MAPS

Recall [RSW17] that the moduli space of maps from centrally aligned curves is obtained by considering the Cartesian diagram

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$$\begin{tikzcd} \widetilde{\mathcal{VZ}}^{\mathrm{ctr}}_{1,n}(X,\beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_{1,n}(X,\beta) \arrow[d] \\ \mathfrak{M}^{\mathrm{ctr}}_{1,n} \arrow[r] & \mathfrak{M}^{\dagger}_{1,n} \end{tikzcd}$$

so that objects of $\widetilde{\mathcal{VZ}}^{\mathrm{ctr}}_{1,n}(X,\beta)$ consist of

(1) a centrally aligned curve (C, M_C, δ);
(2) a stable map $f\colon C \to X$;

subject to the condition that the subcurve $C_0 \subseteq C$, consisting of those components C_v for which $\lambda(v) < \delta$, coincides with the maximal connected genus one subcurve contracted by f. They then define

$$\mathcal{VZ}^{\mathrm{ctr}}_{1,n}(X,\beta) \subseteq \widetilde{\mathcal{VZ}}^{\mathrm{ctr}}_{1,n}(X,\beta)$$

to be the closed substack consisting of maps satisfying the *factorisation condition*, namely that the map $f\colon C \to X$ factors through the associated contraction to a Smyth curve, i.e. there exists a map $\bar{f}$ making the following square commute:

$$\begin{tikzcd} \widetilde{C} \arrow[r] \arrow[d] & \overline{C} \arrow[d, dotted, "\bar{f}"'] \\ C \arrow[r, "f"] & X \end{tikzcd}$$

One should think of the factorisation condition as identifying the main component of the moduli space.

1.1. **Definition of the relative space.** We now give a definition of the relative space inside the moduli space of centrally aligned stable maps.

Definition 1.1. Let X be a smooth projective variety with a smooth divisor Y. Fix a vector of tangency conditions $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\Sigma_{i=1}^n \alpha_i = Y \cdot \beta$. Then the *centrally aligned relative space*

$$\mathcal{VZ}^{\mathrm{ctr}}_{1,\alpha}(X|Y,\beta)$$

is defined to be the closed substack of $\mathcal{VZ}^{\mathrm{ctr}}_{1,n}(X,\beta)$ satisfying the following conditions:

(1) *Gathmann's relative condition:* for each connected component Z of $f^{-1}(Y)$, either:
   (a) Z is a single point of C, in which case if $Z = x_i$ is a marking then we require that the multiplicity of f at x_i along Y is at least α_i (if Z is not a marking then there is no condition);
   (b) Z is a subcurve $Z \subseteq C$, in which case we require that

$$f^*[Y]|_Z - \sum_{x_i \in Z} \alpha_i x_i$$

   is an effective class in $\mathrm{A}_0(Z)$.

(2) *Compatibility condition:* this applies when the minimal subcurve of genus one (the *circuit* or *core*) is contracted to a point in Y. In this case we require that the following diagram of solid arrows may be completed to a commutative diagram:

$$\begin{tikzcd} Q_{\min}^{\text{prestable}} \arrow[r] \arrow[d] & Q_{\min}^{\text{log.map}} \\ Q_{\min}^{\text{cen.align}} \arrow[ur, dashed] & \end{tikzcd}$$

Remark 1.2 (Explaining Gathmann). Gathmann's condition is promptly extended to the case that $\Sigma_{i=1}^{n}\alpha_i \leq Y \cdot \beta$. Question: what is the correct generalisation of the compatibility condition? Here we try to make the condition for a curve component of $f^{-1}(Y)$ into an easily verifiable criterion: if Z is a smooth elliptic curve, recall that every line bundle of positive degree on it is effective, hence the relative condition can be rephrased as the following numerical criterion:

$$f_*[Z]\cdot[Y]+\sum_{j=1}^{r} m^{(j)} \geq \sum_{x_i\in Z}\alpha_i$$

together with the additional requirement that, when the above is an equality, there is an isomorphism of line bundles:

$$(f|_Z)^*\mathcal{O}_X(Y)\otimes\mathcal{O}_Z\left(\sum_{j=1}^{r} m^{(j)}y_j\right)=\mathcal{O}_Z\left(\sum_{x_i\in Z}\alpha_i x_i\right) \tag{1}$$

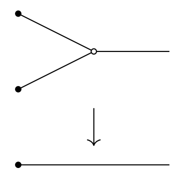
If Z is reducible, the correct extension is not to impose the same condition componentwise; rather, we should ask the numerical condition for the total degree, and, if Z is obtained by gluing a smooth elliptic curve E with a forest of rational trees T_i at roots q_i^E, then the line bundle equality (1) should be required in $\mathrm{Pic}(E)$ after counting each q_i^E towards the left hand side with multiplicity

$$f_*[T_i]\cdot[Y]+\sum_{\text{external components attached to } T_i} m^{(j)}-\sum_{x_j\in T_i}\alpha_j.$$

Remark 1.3 (Explaining compatibility). First I wish to define the first line $Q_{\min}^{\text{prestable}} \to Q_{\min}^{\text{log.map}}$; when looking at a geometric point $\mathrm{Spec}(k=\bar{k})$ it is well-known that

$$Q_{\min}^{\text{prestable}}=\prod_{q \text{ node}}\mathbb{N}_q.$$

Let me assume that all the nodes are internal, i.e. map to Y, for all the action happens there, and also that there is only one internal connected component Z, which has arithmetic genus one. Then I define $Q_{\min}^{\text{log.map}}$ as the saturation of



the quotient of

$$\prod_{q \text{ node}} \mathbb{N}_q \times \prod_{S \text{ internal irr. compo.}} \mathbb{N}_S \tag{2}$$

by the relation generated in the groupification of the latter by

$$\langle \mathrm{Rel_q = (u_q, \ldots, 1, \ldots, -1, \ldots), \;\; q \text{ internal node}} \rangle$$

where 1 is in position S_q^+ (adjacent to q and further from the circuit), -1 in position S_q^- (adjacent to q and closer to the circuit), and u_q in position q is defined as to include the tropical balancing condition (I don't know whether this is standard):

$$u_q = f_*[T_q] \cdot [Y] + \sum_{R_i \text{external compo. attached to } T_q} m^{(i)} - \sum_{x_j \in T_q} \alpha_j,$$

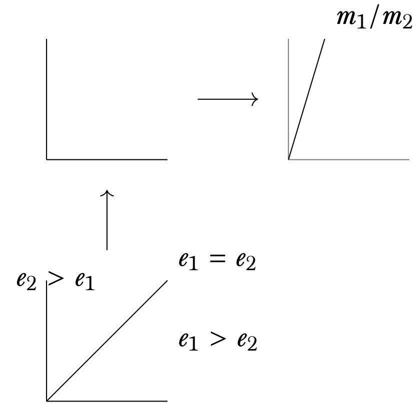
where T_q is the tree of internal components that q separates from the circuit. Notice that the nodes separating Z from the external components have no 1, so the corresponding relations look like

$$(u_q, \ldots, -1, \ldots).$$

The map $Q_{\min}^{\text{prestable}} \to Q_{\min}^{\text{log.map}}$ is induced by the inclusion of the node factors in (2). If we call e_q the image in $Q_{\min}^{\text{log.map}}$ of the q-th basis vector, then we can verify that the following equations hold: chosen any internal component S_i and any two external component R_i^1, R_i^2 such that their path to the circuit goes through S_i, then

$$\sum_{q \in [R_i^1, S_i]} u_q e_q = \sum_{q \in [R_i^2, S_i]} u_q e_q. \tag{3}$$

Question: is this monoid factorisation condition a closed one?

Remark 1.4. Here is a tidy example: start with an elliptic curve E contracted to Y, with two external components R_1, R_2 attached to it and a single marking $x \in E$. The tropical picture would be: The dual picture of the compatibility condition would be: 

which is compatible with the fact that $Q^{\text{prestable}}_{\text{min}} \to Q^{\text{log.map}}_{\text{min}}$ is given by: $e_1 \mapsto m_2, e_2 \mapsto m_1$.

2. The relative space is equal to the closure of the nice locus for $(\mathbb{P}^N, H)$

In general we do not know very much about the space we have just defined. The aim of this section is to show that, in the case where $X = \mathbb{P}^N$ and $Y = H$ is a hyperplane, the space is proper and irreducible of the expected dimension. As such it has a fundamental class, which we can use to define reduced relative Gromov–Witten invariants in genus one.

Remark 2.1. In the case of a general pair (X, Y) we will see that the moduli space is still proper, but is not in general irreducible or even equidimensional. Nevertheless, we can equip it with a virtual class by "pulling back" from the case of $(\mathbb{P}^N, H)$; for details, see §[REF].

The strategy is as follows. We define (Definition 2.2) an open subspace

$$\mathcal{VZ}^{\text{ctr}}_{1,\alpha}(\mathbb{P}^N|H, d)^\circ \subseteq \mathcal{VZ}^{\text{ctr}}_{1,\alpha}(\mathbb{P}^N|H, d)$$

called the *nice locus*, on which the source curve and the map take a particularly simple form. Because of this simplicity, it is easy to show that the nice locus is irreducible (Lemma 2.3). We then prove that $\mathcal{VZ}^{\text{ctr}}_{1,\alpha}(\mathbb{P}^N|H, d)$ is equal to the closure of the nice locus inside $\mathcal{VZ}^{\text{ctr}}_{1,n}(\mathbb{P}^N, d)$. Thus it is proper, since it is a closed subspace of a proper space, and it is irreducible, since it is the closure of an irreducible space.

Definition 2.2. The *nice locus* is defined as the open substack

$$\mathcal{VZ}^{\text{ctr}}_{1,\alpha}(\mathbb{P}^N|H, d)^\circ \subseteq \mathcal{VZ}^{\text{ctr}}_{1,\alpha}(\mathbb{P}^N|H, d)$$

of centrally aligned maps satisfying the following two conditions:

(1) the source curve C is irreducible;
(2) f does not map C inside H.

Lemma 2.3. The nice locus $\mathcal{VZ}^{\text{ctr}}_{1,\alpha}(\mathbb{P}^N|H, d)^\circ$ is irreducible.

Proof. By definition the contraction radius δ is compatible with the map, i.e. the subcurve $C_0 \subseteq C$ where $\lambda < \delta$ is equal to the maximal connected genus one subcurve contracted by f. Hence when the source curve is irreducible we must have $\delta = 0$, and we see that the central alignment on C is uniquely and trivially determined. Thus to specify a point in the nice locus we only need to specify the source curve C (as a scheme) and the map f. A parametrisation can be given from the vector bundle:

I'm sure there's a simpler way to say this - Navid

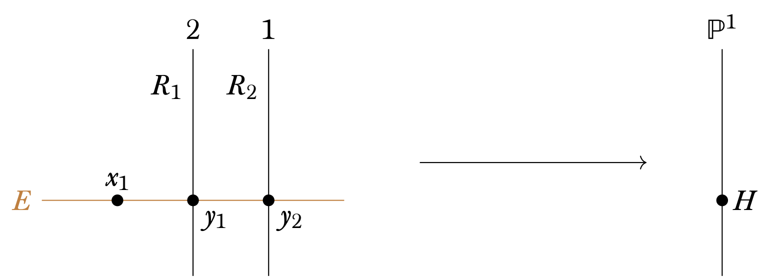
$$\mathrm{Vb}\left(\pi_*\mathcal{O}_\mathcal{E}(\sum_{j=n+1}^{n+\delta}\sigma_j) \oplus \pi_*\mathcal{O}_\mathcal{E}(\sum_{j=1}^{n+\delta}\sigma_j)^{\oplus r}\right) \quad \text{on} \quad \mathcal{M}_{1,n+\delta}$$

where $\pi\colon \mathcal{E} \to \mathcal{M}$ is the universal curve and $\delta = d - \sum \alpha_i$. □

2.1. **Justifying the novel condition**. Remember that we are trying to impose conditions which determine the closure of the nice locus inside the moduli space of centrally aligned maps. Here is one example where we show by a dimensional computation that the novel condition must be included if we hope to determine the closure of the nice locus.

Example 2.4. Consider $\overline{\mathcal{M}}_{1,(3)}(\mathbb{P}^1|H,3)$. This moduli space has virtual dimension $6+1-3=4$. Here is a parametrisation of the nice locus: choose an object $(E,p) \in \mathcal{M}_{1,1}$, and let s_0 be the natural section $s_0\colon \mathcal{O}_E \hookrightarrow \mathcal{O}_E(3p)$ and s_1 any other section of $\mathcal{O}_E(3p)$ not vanishing at p (notice that $\mathrm{h}^0(E,\mathcal{O}_E(3p)) = 3$). Then $(E,p,[\lambda s_0, s_1])$ gives a well-defined element of the nice locus for $\lambda \neq 0$.

Consider now the following weighted graph for a map in the boundary



where the brown line represents a contracted genus 1 curve. Now (E,x_1,y_1,y_2) is a point of $\mathcal{M}_{1,3}$ subject to the divisorial condition $3x_1-2y_1-y_2 = 0 \in \mathrm{A}_0(E)$; furthermore we have to choose the second branch point of the $2:1$ map from R_1 to $\mathbb{P}^1$. This already makes up for a 3-dimensional moduli space of degenerate relative maps corresponding to such a graph. The minimal log structure for this curve has a chart from $\mathbb{N}^2$, with generators e_1 and e_2 corresponding to the smoothing parameters of the two nodes. If we allowed e_1 and e_2 to be identified in the characteristic sheaf, then we would get an extra $\mathbb{G}_\mathrm{m}$ of choices for the log structure, so in total a 4-dimensional moduli space. Thus if we don't impose the novel condition then we get a whole other component of the relative space, which of course for dimensional reasons cannot be contained in the closure of the nice locus. □

Remark 2.5. For the non-expert reader, it might be convenient to recall that the extra data of a centrally aligned log structure on the curves gives a modular interpretation to the iterated blow-up procedure of Vakil and Zinger. In particular the novel condition on the log structure can be translated into a closed condition on the exceptional loci of these blow-ups.

Example 2.6. In this example we will give further moral justification for the novel condition, using as motivation the expanded degenerations approach to relative stable maps. This is not strictly necessary for what we want to do (since our spaces don't involve expanded degenerations), but it helps to explain where this condition comes from.

So, let us pretend that we have a good definition of the "centrally aligned Li space" of maps from centrally aligned curves to expanded degenerations:

$$\mathcal{VZ}^{\mathrm{ctr,Li}}_{1,\alpha}(X|D,\beta)$$

Objects of this moduli space should consist of maps $C \to X_l$ where $X_l \to X$ is an expansion (of some length l) of the pair (X, D), together with a central alignment on C which is compatible with this map. The factorisation property should say that the following diagram commutes:

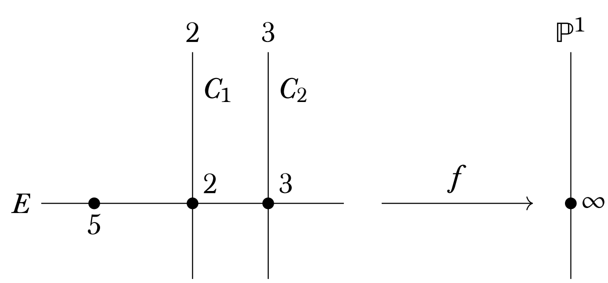
$$\begin{tikzcd}
& \tilde{C} \arrow[dl, "\nu"'] \arrow[dr, "\tau"] & \\
C \arrow[d, "f"'] & & \overline{C} \arrow[d, dotted, "\overline{f}"] \\
X_l \arrow[dr, "\pi"] & & X_l \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

Morally speaking, the projection map

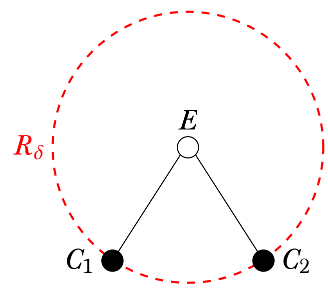
$$\pi_* \colon \mathcal{VZ}^{\mathrm{ctr,Li}}_{1,\alpha}(X|D,\beta) \to \mathcal{VZ}^{\mathrm{ctr}}_{1,n}(X,\beta)$$

should have as its image our space $\mathcal{VZ}^{\mathrm{ctr}}_{1,\alpha}(X|D,\beta)$. We will see that everything in the image of this space must satisfy the novel condition, thus providing further justification for us imposing it.

We proceed by contradiction. Consider a genus one map $C \to \mathbb{P}^1$ of the following form

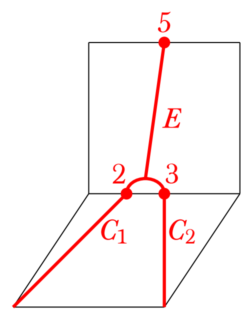


and equip the source curve C with a central alignment which identifies the lengths of the components C_1 and C_2:

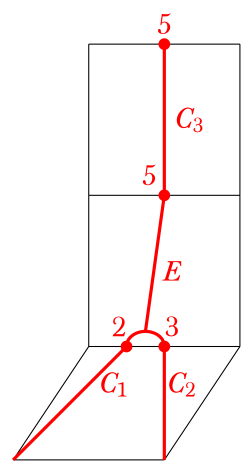


This defines an "object" of $\mathcal{VZ}^{\mathrm{ctr}}_{1,(5)}(\mathbb{P}^1|\infty, 5)$, satisfying all the necessary conditions *except for the novel condition.* We will show that this cannot belong to the image of π_*.

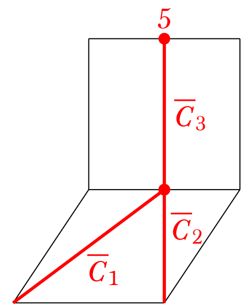
First we have to identify what the possible lifts of this object to the centrally aligned Li space are. The map $C \to X_1$ takes the following form:



(Here we use Navid's convention for drawing maps to expanded degenerations, so that the box at level 1 represents a *single fibre* of the expanded degeneration, i.e. E is mapped into a fibre and has zero horizontal degree.) Now, $\widetilde{C}$ is obtained from C by bubbling a rational component C_3 at the single marking, and so the map $\widetilde{C} \to X_2$ is given by:



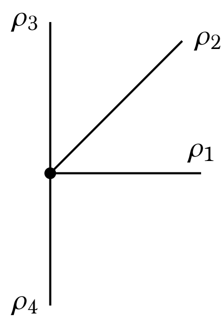
The map $\widetilde{C} \to \overline{C}$ contracts the elliptic component E, and so the map $\overline{C} \to X_1$ *should* take the form:



We will now see that, in this example, the map drawn above cannot actually exist, and that this occurs precisely because we have violated the novel condition.

We start by choosing a one-parameter smoothing $\mathcal{C} \to \mathbb{A}^1$ of the centrally aligned curve C. This induces one-parameter smoothings $\widetilde{\mathcal{C}}$ of $\widetilde{C}$ and $\overline{\mathcal{C}}$ of $\overline{C}$. Since the factorisation condition must also hold in families, we must have a map $\overline{\mathcal{C}} \to \mathfrak{X}$, whose general fibre maps to the space $X_0 = X$ (with no expansion) and whose general fibre maps to $X_1 = X \sqcup_D Y$.

We thus have a map $\overline{\mathcal{C}} \to \mathfrak{X}$, where $\mathfrak{X}$ is the total space of the expanded degeneration. This is obtained as the toric blow-up of $\mathbb{P}^1 \times \mathbb{A}^1$ at the point $(\infty, 0)$. Thus it is also a smooth toric variety, with fan given by:



There is a map $\mathfrak{X} \to \mathbb{A}^1$ whose general fibre is isomorphic to X and whose central fibre is isomorphic to X_1. We see from the fan above that $\mathrm{Pic}(\mathfrak{X})$ is generated by $\mathcal{O}_{\mathfrak{X}}(D_{\rho_i})$ for $i = 1, \ldots, 4$, subject to the relations:

$$\mathcal{O}_{\mathfrak{X}}(D_{\rho_1}) \otimes \mathcal{O}_{\mathfrak{X}}(D_{\rho_2}) \cong \mathcal{O}_{\mathfrak{X}}$$
$$\mathcal{O}_{\mathfrak{X}}(D_{\rho_2}) \otimes \mathcal{O}_{\mathfrak{X}}(D_{\rho_3}) \cong \mathcal{O}_{\mathfrak{X}}(D_{\rho_4})$$

Choosing as $\{\mathcal{O}_{\mathfrak{X}}(D_{\rho_1}), \mathcal{O}_{\mathfrak{X}}(D_{\rho_3})\}$ our minimal set of generators, we see that a map $\overline{\mathcal{C}} \to \mathfrak{X}$ is given by the data of line bundes L_1 and L_3 on $\overline{\mathcal{C}}$ together with sections:

$$\begin{aligned}
s_1 &\in \mathrm{H}^0(\overline{\mathcal{C}}, L_1)\\
s_2 &\in \mathrm{H}^0(\overline{\mathcal{C}}, L_1^{-1})\\
s_3 &\in \mathrm{H}^0(\overline{\mathcal{C}}, L_3)\\
s_4 &\in \mathrm{H}^0(\overline{\mathcal{C}}, L_1^{-1} \otimes L_3)
\end{aligned}$$

The divisors D_{ρ_1} and D_{ρ_2} are, respectively, the level-0 and level-1 pieces of the central fibre of $\mathfrak{X}$. Therefore we must have

$$\begin{aligned}
L_1 &\cong \mathcal{O}_{\overline{\mathcal{C}}}(a_1\overline{C}_1 + a_2\overline{C}_2)\\
L_1^{-1} &\cong \mathcal{O}_{\overline{\mathcal{C}}}(a_3\overline{C}_3)
\end{aligned}$$

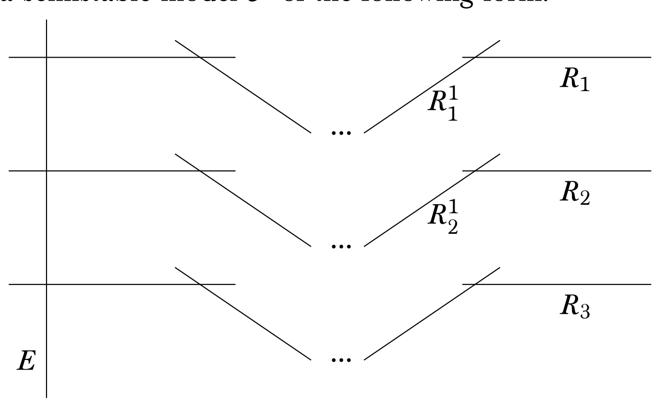
for some positive integers a_1, a_2 and a_3. Here we are using the fact that each $\overline{C}_i$ is a $\mathbb{Q}$-Cartier divisor on $\overline{\mathcal{C}}$, so that (after possibly multipltying by some positive integer) the corresponding line bundle makes sense. Furthermore, the intersection multiplicites of $\overline{C}_1$ and $\overline{C}_2$ with the level-1 piece mean that we must have:

$$\begin{aligned}
\deg(L_1^{-1}|_{\overline{C}_1}) &= 2\\
\deg(L_1^{-1}|_{\overline{C}_2}) &= 3
\end{aligned}$$

We thus obtain:

$$\begin{aligned}
a_3(\overline{C}_1 \cdot \overline{C}_3) &= 2\\
a_3(\overline{C}_2 \cdot \overline{C}_3) &= 3
\end{aligned}$$

However, we claim that in fact $\overline{C}_1 \cdot \overline{C}_3 = \overline{C}_2 \cdot \overline{C}_3$, so that the above equations cannot hold. To calculate these intersection numbers, we pass to the semistable model of $\overline{\mathcal{C}}$. This is obtained by performing further blow-ups to $\widetilde{\mathcal{C}}$, resulting in a semistable model $\widetilde{\mathcal{C}}^{\mathrm{ss}}$of the following form:



Here the component labeled E is smooth elliptic and all the other components are smooth rational. The projection $\pi : \widetilde{\mathcal{C}}^{\mathrm{ss}} \to \overline{\mathcal{C}}$ sends each R_i to $\overline{C}_i$ and contracts all the other components. The semistable model has the advantage that the total space is regular, so the intersections are precisely those that you "see" in the picture. In particular, any two adjacent components have intersection 1, and the self-intersection of any component is equal to minus the number of adjacent components.

Now, $\overline{C}_i \cdot \overline{C}_j = (\pi^*\overline{C}_i) \cdot (\pi^*\overline{C}_j)$, so we want to show that:

$$(\pi^*\overline{C}_1) \cdot (\pi^*\overline{C}_3) = (\pi^*\overline{C}_2) \cdot (\pi^*\overline{C}_3)$$

For each i, $\pi^*\overline{C}_i$ is equal to R_i plus a non-negative linear combination of the components in $\widetilde{\mathcal{C}}^{\mathrm{ss}}$ contracted by π. By the projection formula, any component contracted by π will multiply to zero with a class pulled back along π. Thus

$$(\pi^*\overline{C}_1) \cdot (\pi^*\overline{C}_3) = R_1 \cdot (\pi^*\overline{C}_3) = a_3(R_1^1)$$

the coefficient of R_1^1 in $\pi^*\overline{C}_3$. Similarly

$$(\pi^*\overline{C}_2) \cdot (\pi^*\overline{C}_3) = a_3(R_2^1)$$

the coefficient of R_2^1 in $\pi^*\overline{C}_3$. So we need to show that:

$$a_3(R_1^1) = a_3(R_1^2)$$

Now, we have

$$0 = R_1^1 \cdot (\pi^*\overline{C}_3) = a_3(R_1^2) - 2a_3(R_1^1)$$

because $(R_1^1)^2 = -2$. So $a_3(R_1^1) = a_3(R_1^2)/2$. Similarly:

$$0 = R_1^2 \cdot (\pi^*\overline{C}_3) = a_3(R_1^1) - 2a_3(R_1^2) + a_3(R_1^3) = -3a_3(R_1^1) + a_3(R_1^3)$$

So $a_3(R_1^1) = a_3(R_1^3)/3$. Continuing in this way, we find that

$$a_3(R_1^1) = a_3(E)/l_1$$

where l_1 is the length of the rational tail connecting E to R_1. Similarly $a_3(R_1^2) = a_3(E)/l_2$. But by Smyth's balancing condition for the semistable model [Smy11, Proposition 2.11], we must have $l_1 = l_2$. This concludes the proof that $\overline{C}_1 \cdot \overline{C}_3 = \overline{C}_2 \cdot \overline{C}_3$, which shows the necessity of the novel condition in this example.

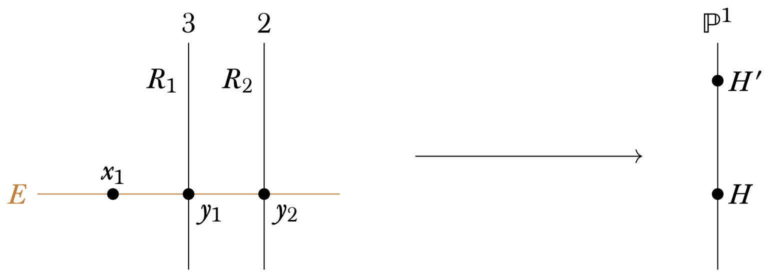
Notice that here we used the fact that the rational tails $\overline{C}_1$ and $\overline{C}_2$ are adjacent to the same irreducible component (namely E) of the core. □

2.2. **Proof that the relative space equals the closure of the nice locus.** We now want to show that the relative space in the centrally aligned setting is equal to the closure of the nice locus; irreducibility then follows immediately. We start with one direction:

Lemma 2.7. The closure of the nice locus is contained in the relative space.

Proof. We address the relative conditions one at a time. Notice that they are all obviously satisfied on the nice locus. Also, Gathmann's relative condition is obviously closed; see [Vak00, Proposition 4.9].

It remains to show that the novel condition is satisfied on the closure of the nice locus. Here is no proof but rather some heuristics. Consider for example a map similar to the one above



and assume that it is in the closure of the nice locus; I want to argue that the two smoothing parameters cannot be identified. By the previous points I may assume that the factorisation property and Gathmann's relative conditions hold. I have a diagram:

$$\begin{tikzcd} \mathcal{C}^{\mathrm{ss}} \arrow[r] \arrow[dr] & \widetilde{\mathcal{C}} \arrow[r] \arrow[d] & \overline{\mathcal{C}} \arrow[d, "\bar{f}"] \\ & \mathcal{C} \arrow[r, "f"] & X \end{tikzcd}$$

where I have included the semistable model of $\mathcal{C}$ (and $\widetilde{\mathcal{C}}$), thought of as a curve marked with $f^{-1}(H')$ as well, where $H \neq H' \in \mathbb{P}^1$.

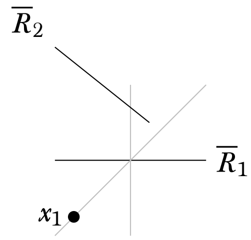
Notice that $\mathcal{C}$ is a normal surface with at worst singular points at the nodes of $C = \mathcal{C}_0$ (the central fibre $C = E + R_1 + R_2$ is Cartier and a variety is smooth at any smooth point of a Cartier divisor) and the singularities are of type A_{n_i}, $i = 1, 2$ (from the deformation theory of nodal curves).

I assume maximal multiplicity $\sum \alpha = d$, i.e. I am looking at the moduli space $\overline{\mathcal{M}}_{1,(5)}(\mathbb{P}^1|H, 5)$. Hence the line bundle and s_0 are determined as $f^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_{\mathcal{C}}(5x_1) \otimes \mathcal{O}_{\mathcal{C}}(\beta E)$ for some (positive rational) β and s_0 is the natural inclusion of $\mathcal{O}_{\mathcal{C}}$ (up to $\mathbb{G}_{\mathrm{m}}$).

The map is totally ramified at y_i by Gathmann's condition. We conclude:

$$\frac{\beta}{n_i + 1} = (\beta E) \cdot R_i = m^{(i)}$$

Hence, having fixed the multiplicities, the two possible singularities of $\mathcal{C}$ determine each other. In our example we may pick $n_1 = 1$, $n_2 = 2$, $\beta = 6$. But knowing the singularity determines the semistable model: in our case y_1 is replaced by a (-2)-curve and y_2 by a chain of two (-2)-curves. Now we know from the work of Smyth [Smy11, Proposition 2.12] that the exceptional locus of $\mathcal{C}^{\mathrm{ss}} \to \overline{\mathcal{C}}$ is balanced, therefore we may deduce that $\bar{f}$ is constant on the branch of the genus 1 singularity to which R_2 is joined, so $\overline{\mathcal{C}}$ looks like this:



where a gray line is contracted by the map. On the other hand if $\delta = \lambda(v_1) = \lambda(v_2)$ then the prescription of [RSW17, Proposition 3.6.1] implies that $\tilde{C} \to \overline{C}$ looks like:

which is a contradiction. □

It thus remains to show that, given a relative radially aligned map, we can smooth it to one in the nice locus. This is done by considering different cases locally, then gluing.

What do we exactly mean by this? How does gluing work in the centrally aligned setting?

Case 1: **non-contracted genus one internal component**. Assume that the curve takes the form

$$C = C_0 \cup C_1 \cup \ldots \cup C_k$$

where all the C_i are smooth, C_0 has genus one, all the other C_i have genus zero, and for $i \in \{1, \ldots, k\}$, C_i intersects C_0 at a single node (denoted q_i) and does not intersect any other components.

Suppose furthermore that C_0 is a non-contracted *internal component*, meaning that it is mapped into H via f, and that $C_1,\ldots,C_k$ are *external components*, meaning that they are not mapped into H via f. The picture is:

[FIGURE]

Suppose that this is a relative stable map. This means that [BLAH]. We claim that it can be smoothed to a relative stable map in the nice locus. The construction depends on choosing an appropriate smoothing of the curve C, so that the map also smooths.

We start with $W = C_0 \times \mathbb{A}^1_t$ (where t denotes a fixed co-ordinate on the affine line). This is a smooth surface, fibred over $\mathbb{A}^1_t$, with fibre equal to the elliptic curve C_0. Consider the points $q_1, \ldots, q_k$ on C_0. We will perform a series of weighted blow-ups at the points $(q_i, 0) \in W$, in order to obtain a surface whose general fibre is smooth (in fact, isomorphic to C_0) and whose central fibre is isomorphic to C.

Fix $i \in \{1, \ldots, k\}$ and let m_i be the multiplicity of f with H at $q_i \in C_i$. We define:

$$l = \mathrm{lcm}(m_1, \ldots, m_k) \qquad r_i = l/m_i$$

We now blow-up the surface W at the points $(q_i, 0)$ with weight r_i in the horizontal direction and weight 1 in the vertical direction: if x_i is a local co-ordinate for the fibre around q_i, this means that we blow-up in the ideal (x_i, t^{r_i}).

The result is a fibred surface $W' \to \mathbb{A}^1_t$ with general fibre equal to C_0 and central fibre $W'_0 \cong C$. The total space of W' is no longer smooth (its singular points are [BLAH]), but this is not a problem since the projection to $\mathbb{A}^1_t$ is still flat. The central fibre is a linearly trivial Cartier divisor:

$$W'_0 = C_0 + C_1 + \ldots + C_k = 0 \in \operatorname{Pic} W'$$

For $i \in \{1, \ldots, k\}$ we have that $r_i C_i$ is Cartier, although the same is not necessarily true of C_i. Furthermore, since

$$lC_0 = -\sum_{i=1}^{k} lC_i = -\sum_{i=1}^{k} m_i(r_i C_i)$$

in $A_1(W')$, it follows that lC_0 is Cartier. Finally, a local computation shows that

$$r_i C_i \cdot C_0 = 1$$

for $i \in \{1, \ldots, k\}$. Now, let $x_1, \ldots, x_n$ denote the marked points of C. These are smooth points of the central fibre W'_0, and hence can be extended to Cartier divisors $\tilde{x}_1, \ldots, \tilde{x}_n$ on W'. Consider the line bundle:

$$\tilde{L} = \mathcal{O}_{W'}(lC_0 + \Sigma_{j=1}^{n} \alpha_j \tilde{x}_j)$$

on W'. We claim that this gives a smoothing of the line bundle $L = f^*\mathcal{O}(1)$ on C, i.e. that $\tilde{L}|_{W'_0} = L$. We show this by first restricting $\tilde{L}$ to each of the components C_i of $W'_0 \cong C$. For $i \in \{1, \ldots, k\}$, we have

$$\begin{aligned} \tilde{L}|_{C_i} &= \mathcal{O}_{C_i}\left((lC_0 \cdot C_i)q_i + \sum_{x_j \in C_i} \alpha_j x_j\right) = \mathcal{O}_{C_i}\left((l/r_i)q_i + \sum_{x_j \in C_i} \alpha_j x_j\right) \\ &= \mathcal{O}_{C_i}\left(m_i q_i + \sum_{x_j \in C_i} \alpha_j x_j\right) = L|_{C_i} \end{aligned}$$

while for $i = 0$ we have:

$$\tilde{L}|_{C_0} = \mathcal{O}_{C_0}\left(-\sum_{i=1}^{k}(lC_i \cdot C_0)q_i + \sum_{x_j \in C_0} \alpha_j x_j\right) = \mathcal{O}_{C_0}\left(-\sum_{i=1}^{k} m_i q_i + \sum_{x_j \in C_0} \alpha_j x_j\right) = L|_{C_0}$$

Finally the fact that $\tilde{L}|_{W'_0} = L$ follows from the fact that the dual intersection graph of C has genus zero.

Now, $\tilde{L}$ comes with a unique section whose restriction to $W'_0 \cong C$ is s_0. After we extend the sections $s_1, \ldots, s_N$, it is clear that the resulting stable map is in the nice locus (i.e. that it is not mapped into H).

In order to extend the sections $s_1, \ldots, s_N$, we simply check that they are unobstructed. The space containing the obstructions to extending the sections is [Wan12, Theorem 3.1]:

$$\mathrm{H}^1(C, L)$$

By taking the normalisation exact sequence for C, tensoring with L and passing to cohomology, we obtain an exact sequence:

$$0 \to \mathrm{H}^0(C, L) \to \bigoplus_{i=0}^{k} \mathrm{H}^0(C_i, L) \xrightarrow{\theta} \bigoplus_{i=1}^{k} L_{q_i} \to$$
$$\to \mathrm{H}^1(C, L) \to \bigoplus_{i=0}^{k} \mathrm{H}^1(C_i, L) \to 0$$

Now, each of $C_1, \ldots, C_k$ is isomorphic to $\mathbb{P}^1$ and $L|_{C_i}$ has non-negative degree; hence the map θ is surjective. Thus the map

$$\mathrm{H}^1(C, L) \to \bigoplus_{i=0}^{k} \mathrm{H}^1(C_i, L)$$

is an isomorphism. But $\mathrm{H}^1(C_i, L) = 0$ for $i \in \{1, \ldots, k\}$ since $C_i \cong \mathbb{P}^1$ and $L|_{C_i}$ has non-negative degree; also we have by Serre duality

$$\mathrm{H}^1(C_0, L) \cong \mathrm{H}^0(C_0, L^\vee \otimes \omega_{C_0}) = \mathrm{H}^0(C_0, L^\vee) = 0$$

where the penultimate equality holds because $\mathrm{g}(C_0) = 1$ and the last equality holds because $L|_{C_0}$ has *strictly* positive degree (here we are using the fact that $f|_{C_0}$ is non-constant).

To conclude, we have a family $\tilde{C} = W'$ of nodal curves and a map from this family to $\mathbb{P}^N$

$$\begin{array}{ccc} \tilde{C} & \xrightarrow{\tilde{f}} & \mathbb{P}^N \\ \downarrow{\scriptstyle \pi} & & \\ \mathbb{A}^1_t & & \end{array}$$

such that when we restrict to $0 \in \mathbb{A}^1_t$ we recover the map $f : C \to \mathbb{P}^N$ and such that the general fibre is an element of the nice locus.

Address the case that C_0 is not irreducible, and that $C_i,\ i \neq 0$ are chains of $\mathbb{P}^1$'s.

This should follow from gluing, whatever that means.

Case 2: contracted genus one internal component. This is similar to the previous case, but slightly more delicate. The smoothing W', the line bundle $\tilde{L}$ and the section s_0 are constructed as before. Now, however, $\mathrm{H}^1(C, L) \neq 0$, so we need to work harder to show that we can extend the other sections.

Before extending the sections let us first extend the centrally aligned log structure. On the base $\mathbb{A}^1_t$ the chart sends $e_i \in \mathbb{N}^r$ to t^{r_i}. By the novel condition, if $e_i = e_j$ in the minimal centrally aligned structure that we start with on C, then $r_i = r_j$, so the chart that we have just defined factors indeed through the sharpening of the submonoid of $\mathbb{Z}^r$ generated by $\mathbb{N}^r$ and the differences $\lambda(v)-\lambda(w)$. We declare the contraction radius δ to be e_i for those i's corresponding to branches of the genus 1 singularity on which $\bar{f}$ is non-constant.

actually this sounds like bs and I probably have δ already?

The central alignment provides us with a diagram:

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r] \arrow[d] & \overline{\mathcal{C}} \\ \mathcal{C} & \end{tikzcd}$$

We can now extend the sections $\bar{s}_1, \ldots, \bar{s}_N$ onto $\overline{\mathcal{C}}$ because $\mathrm{H}^1(\overline{\mathcal{C}}, \bar{L}) = 0$, so we get a map $\bar{F}\colon \overline{\mathcal{C}} \to \mathbb{P}^N$ deforming $\bar{f}$. Finally we claim that $F\colon \mathcal{C} \to \mathbb{P}^N$ is the stable model of $\bar{F}$.

3. Novel Condition 2.0

This is done with $(\mathbb{P}^1, H)$ in mind; in general it will need to be modified to account for internal components of positive degree.

Novel condition 2.0: if v_1, v_2 are two vertices belonging to R_δ, then the corresponding tangencies $m^{(j_1)}$ and $m^{(j_2)}$ to H must be equal *if the corresponding rational tails are attached to the same irreducible component of* $\sqsubset_0$. Here R_δ is the set of vertices $v \in \sqsubset$ with $\lambda(v) = \delta$ and $\sqsubset_0$ is the set of vertices with $\lambda(v) < \delta$.

Sketch of "sufficient" direction: Denote by $S_0 = E, S_1, \ldots, S_k$ the irreducible components of Z and assume E is smooth elliptic (or the core). Every S_i for $i \geq 0$ comes with a bunch of special points: divide them into $B^-(i) =$ the singleton representing the only node separating S_i from S_0 (this is not defined for $i = 0$); $B^+(i) =$ the set of remaining nodes; and $A(i) =$ the set of markings on S_i. Start with the trivial family $E \times \mathbb{A}^1$, where E is marked with the points of $B(0) = B^+(0)$ and $A(0)$. Now perform a weighted blow-up at the points $\{q \in B(0)\} \times \{0 \in \mathbb{A}^1\}$ of weight r_q; call each resulting rational tail S_i correspondingly to the original picture. On S_i we may mark points $B^+(i)$ and $A(i)$ in such a way that the resulting point of $\overline{\mathcal{M}}_{0,1+|B^+(i)|+|A(i)|}$ corresponds to the one we started with. Now blow up this family in all the $B^+(i) \times \{0\}$ with a weight. Keep going until you have recreated C. Call R_j the first external components, namely those corresponding to the vertices with $\lambda(v) \geq \delta$, $\nexists v'$ with $\lambda(v) > \lambda(v') \geq \delta$.

We define the line bundle $\mathcal{L} = \mathcal{O}_\mathcal{C}(\sum \beta_i S_i)(\sum \alpha_i x_i)$; let's look at the equations that it gives us, starting from the outside.

(1) $m^{(j)} = \mathcal{L}_{|R_j} = \frac{\beta^-(j)}{r_{q^-(j)}}$;
(2) on every $S_i, i \geq 1$ we can prove inductively that:

$$0 = \mathcal{L}_{|S_i} = \frac{\beta^-(i) - \beta_i}{r_{q^-(i)}} + \sum_{x_h \in A(i)} \alpha_i + \sum_{y_h \in B^+(i)} M^{(h)},$$

where I have denoted by $M^{(h)}$ the sum of all the contributions (both α_l and $m^{(l)}$) coming from the rational tree attached to the corresponding point of $B^+(i)$;

(3) it turns out that $\mathcal{O}_{S_0} \simeq \mathcal{L}_{|S_0}$ is precisely Gathmann's condition (in the numeric form if S_0 is a circle of rational curves).

Now notice that $\lambda(v_j) = \lambda(v'_j)$ may happen only at the first external components; in this case $r_{q^-(j)} = r_{q^-(j')}$, and from the first equation we see that $m^{(j)} = m^{(j')}$ if they are attached to the same component ($\beta^-(j) = \beta^-(j')$). The second equation can be made to hold for every S_i by appropriately choosing $\beta^-(i), r_{q^-(i)}$. The last equation is automatically satisfied.

is this true? if not, probably there are contributions $M^{(h)}$ and $M^{(h')}$ to equate

Extending sections is the usual business of exploiting the fatorisation through the elliptic m-fold.

Sketch of "necessary" direction: pick a smoothing $\mathcal{C}$ of (C, f). Suppose that $\lambda(v) = \lambda(v')$ and the corresponding R, R' are attached to the same S_i, i.e. they share the same the tree separating them from C_0. Notice that R and R' map to two branches of the Smyth singularity. Looking at the semistable model of $\mathcal{C}$, it follows that (the strict transforms of) R and R' still share the same tree there. Then by writing $(f^{ss})^*\mathcal{O}_{\mathbb{P}}(1) = \mathcal{O}_{\mathcal{C}^{ss}}(\sum \beta_i S_i)(\sum \alpha_i x_i)$ and intersecting with R, R', we see that they must have the same contact order to H.

Appendix A. Basic facts about centrally aligned curves

Proposition A.1 ([RSW17, Proposition 4.6.2.2]). The morphism $\mathfrak{M}^{\mathrm{ctr}}_{1,n} \to \mathfrak{M}^{\dagger}_{1,n}$ is a log-modification.

Explanation: this is a local statement so I can probably reduce to an atomic neighbourhood S of a point p. S and the curve over it are endowed with the minimal log structure; let $P = \overline{\mathcal{M}}_p$ determine a chart for this log structure. Observe that the subcurve $\sqsubset_0$ of the tropicalisation $\sqsubset$ of C_p determines a subset MinPos of the set of vertices, namely those adjacent to $\sqsubset_0$. Perform the following log-blowups: consider the set of primitive values of the function $\lambda\colon \sqsubset \to P$, and blow up the ideal that they generate; now locally the set of values of λ is principal with generator p: blow up the ideal generated by $\{\lambda(v) - p\} \setminus \{-p\}$. Keep going until $\lambda(v_i)$ is reached for some $v_i \in$ MinPos; at this point stop and declare the contraction radius $\delta := \lambda(v_i)$. Finish by adjoining $\lambda(v) - \delta$ for all the vertices untouched to this stage. This shows that the choice of δ is not an extra degree of freedom.

do I sound like a physicist?

References

[RSW17] D. Ranganathan, K. Santos-Parker, and J. Wise. Moduli of stable maps in genus one and logarithmic geometry I. *ArXiv e-prints*, August 2017.

[Smy11] David Ishii Smyth. Modular compactifications of the space of pointed elliptic curves I. *Compos. Math.*, 147(3):877–913, 2011.

[Vak00] Ravi Vakil. The enumerative geometry of rational and elliptic curves in projective space. *J. Reine Angew. Math.*, 529:101–153, 2000.

[Wan12] Jie Wang. Deformations of pairs (X, L) when X is singular. *Proc. Amer. Math. Soc.*, 140(9):2953–2966, 2012.

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