CURVE COUNTING IN GENUS ONE: ELLIPTIC SINGULARITIES & RELATIVE GEOMETRY

LUCA BATTISTELLA, NAVID NABIJOU AND DHRUV RANGANATHAN

ABSTRACT. We construct and study the reduced, relative, genus one Gromov–Witten theory of very ample pairs. These invariants form the principal part contribution to relative Gromov–Witten theory in genus one and are relative versions of Zinger's reduced Gromov–Witten invariants. The relative and absolute theories are related by degeneration of the tangency conditions, and the resulting formulas generalize a well-known recursive calculation scheme put forward by Gathmann in genus zero. The geometric input is a desingularization of the principal component of the moduli space of genus one logarithmic stable maps to a very ample pair, using the geometry of elliptic singularities. Our study passes through general techniques for calculating integrals on logarithmic blowups of moduli spaces of stable maps, which may be of independent interest.

CONTENTS

0. INTRODUCTION

The Kontsevich moduli space of stable maps from genus zero curves to projective space exhibits remarkable geometry – it is a smooth orbifold with a "self-similar" normal crossings boundary. These facts have combined to produce a very satisfactory understanding of Gromov–Witten theory in genus zero.

While the higher genus situation is much more difficult, in the last several years, substantial theory has been developed towards a conceptual understanding of the genus one case. This began over a decade ago with the pioneering work of Li, Vakil, and Zinger [Zin09a, VZ08, LZ07, LZ09, Zin08, Zin09b], and was recently reinvigorated involving ideas from logarithmic geometry and singularity theory [BCM18, CM18, HL10, RSW17a, RSW17b, Smy11a, Vis12]. The result is a **reduced** Gromov–Witten theory, which removes certain degenerate contributions from the ordinary theory. While the Atiyah–Bott localization for the reduced invariants was used to great effect by Zinger in his proof of

the BCOV conjectures, two major components of the standard theory – relative invariants and the degeneration formula – have not been developed in this setting. Our results in this paper provide these as new structural tools in reduced genus one Gromov–Witten theory[1].

0.1. **Results**. Our first contribution is a definition and construction of relative reduced Gromov–Witten theory of projective space relative to a hyperplane. The geometric content is the following, see §1 for a precise formulation.

Let $\mathcal{M}^\circ_{1,\alpha}(\mathbb{P}^N|H,d)$ be the moduli space of positive degree maps from smooth curves of genus one to $\mathbb{P}^N$ with dimensionally transverse contact order $\alpha \in \mathbb{Z}^n_{\geq 0}$ along H.

Theorem A. There exists a logarithmically smooth and proper Deligne–Mumford stack $\mathcal{VZ}^\circ_{1,\alpha}(\mathbb{P}^N|H,d)$ having expected dimension, and containing $\mathcal{M}^\circ_{1,\alpha}(\mathbb{P}^N|H,d)$ as a dense open subset. A point in this moduli space parameterizes maps

$$C \to \mathbb{P}^N[s] \to \mathbb{P}^N,$$

to an s-fold expansion of $\mathbb{P}^N$ along H such that the map to the expansion and composite map $C \to \mathbb{P}^N$ each factor nontrivially through an elliptic singularity.

The moduli space above is constructed by identifying two closed conditions that must be satisfied by the space of maps to the pair $(\mathbb{P}^N, H)$. Each condition requires the stable map to factor through an elliptic singularity. The behaviour mixes the geometry in [RSW17a] and [RSW17b]. Together with an analysis of the deformation theory of such maps, we are led to Theorem A.

Our second result explains how to do practical calculations on this space of stable maps.

Theorem B. Given a smooth pair (X, Y) with Y very ample, there is an explicit recursive algorithm to calculate

   (1) the (restricted) reduced genus one Gromov–Witten invariants of Y;
   (2) the reduced genus one relative Gromov–Witten invariants of (X, Y);
   (3) the (restricted) reduced genus one rubber invariants of $\mathbb{P} = \mathbb{P}_Y(N_{Y|X} \oplus \mathcal{O}_Y)$

from the full genus zero and reduced genus one invariants of the ambient variety X.

We pursue a strategy laid out by Vakil and Gathmann, building on work of Caporaso and Harris [CH98, Gat02, Vak00]. The first step is to express the locus of maps where the contact order degenerates in terms of Chern classes of natural vector bundles. The second step is to describe that locus in terms of fibre products of moduli spaces with smaller invariants. Novelties are introduced in both steps. We achieve the first step using techniques that are entirely within tropical and logarithmic geometry, drastically simplifying Gathmann's calculation. For the second step, we contend with the interaction of the relative splitting formula and the geometry of elliptic singularities, which is entirely new (§4.3). We express the factorisation property in terms of tautological classes and combinatorial manipulations, leading to Theorem B.

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[1]Further motivation to consider the genus one case specifically comes simply from the fact that for Calabi–Yau geometries in dimension larger than 3, Gromov–Witten theory vanishes in genera greater than one.

0.2. **Broader contributions**. Beyond genus one, we pass through a number of seemingly useful general techniques. First, we explain how to describe degenerate moduli spaces as fibre products. This is substantially more delicate than the standard situation because of the need to work with elliptic singularities and aligned logarithmic structures. This interaction of the degeneration formula with the geometry of curve singularities is a new phenomenon. The results of [RSW17a] suggest the existence of a reduced higher genus Gromov–Witten theory formed by replacing contracted elliptic components with singularities[2]. The discussion here is likely to carry over to this setting.

Second, a key step in our recursion is a description of the locus of maps with higher than prescribed tangency, which was identified by Gathmann. We realize the locus as the vanishing locus of a section of a line bundle that comes from tropical geometry — from a piecewise linear function on the fan. The systematic understanding of logarithmic line bundles arising in this fashion is likely to play an important rôle in logarithmic enumerative geometry.

We rephrase Gathmann's idea of increasing the tangency requirement within the framework of logarithmic stable maps (where maximal tangency is always assumed) by introducing *fictitious markings*, and identifying the space of maps with higher contact order with a boundary stratum inside that of lower contact order and more (fictitious) markings.

Our analysis leads naturally to the study of a "main component" double ramification cycle for target manifolds in genus one. The virtual geometry of this was studied in recent work of Janda, Pandharipande, Pixton, and Zvonkine [JPPZ18]. We do not know how to calculate a closed formula for this main component contribution (except in the case where the target manifold is a point), but we repurpose E. Katz's topological recursion relations [Kat07] to complete the algorithm. We note that even in genus one, an understanding of the main component contributions of the double ramification cycle for target manifolds would be interesting.

Finally, we note that our results require us to encounter several variants of the space of relative/logarithmic stable maps, with both fixed and expanded target. The conceptual features of one space are often computational bugs, and vice versa, and we believe that the techniques developed in the genus one case here will be used repeatedly in logarithmic Gromov–Witten theory calculations. In particular, we frequently use Kato's perspective on logarithmic blowups as subfunctors that was applied in [RSW17a, RSW17b].

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[2]We have been informed by Jonathan Wise that his Ph.D. candidate Sebastian Bozlee has constructed these higher genus reduced Gromov–Witten invariants. A manuscript is in preparation.

1. Constructions and logarithmic smoothness

1.1. **The absolute case.** Let $\mathfrak{M}_{1,n}$ be the logarithmic algebraic stack of n-pointed prestable curves of genus one. We begin by recalling the logarithmic moduli spaces of genus one curves constructed in [RSW17a, §2 & §4]. Given a logarithmically smooth curve C over a logarithmic geometric point S, let $\sqsubset$ denote its tropicalization, and let $\lambda \in \overline{M}_S$ be the piecewise-linear function giving the distance from each vertex of $\sqsubset$ to the minimal genus one subcurve.

Definition 1.1. A **centrally aligned curve** is a pair $(C/S, \delta)$, where $\delta \in \overline{M}_S$, such that:
  (1) the section δ is comparable to $\lambda(v)$ for all vertices v of $\sqsubset$;
  (2) for any pair of vertices v and w of distance less than δ, the sections $\lambda(v)$ and $\lambda(w)$ are comparable.

In essence, this says that the curve $\sqsubset$ is such that all vertices are either inside, outside, or on the circle of radius δ, and that the collection of vertices inside the circle is totally ordered.

The moduli stack $\mathfrak{M}_{1,n}^{\mathrm{cen}}$ of centrally aligned curves is a logarithmic algebraic stack in the smooth topology, and $\mathfrak{M}_{1,n}^{\mathrm{cen}} \to \mathfrak{M}_{1,n}$ is a logarithmic modification. The main construction of [RSW17a] canonically associates to any centrally aligned family of curves $\mathcal{C}_S$ a partial destabilisation $\widetilde{\mathcal{C}}_S$ and a contraction

$$\widetilde{\mathcal{C}}_S \to \overline{\mathcal{C}}_S,$$

where $\overline{\mathcal{C}}_S$ may contain a Gorenstein elliptic singularity [Smy11a]. The number of branches is equal to the sum of the number of vertices of $\widetilde{\sqsubset}$ which lie on the circle of radius δ. This data uniquely determines the singularity.

The space of stable maps $\widetilde{\mathcal{VZ}}_{1,n}(\mathbb{P}^N, d)$ from the universal nodal curve $\mathcal{C}$ over the stack $\mathfrak{M}_{1,n}^{\mathrm{cen}}$ is proper and algebraic, with projective coarse moduli space. A compatibility condition between the alignment and the map is required, namely that the circle of radius δ passes through at least one component of positive degree. A stable map in this space is said to satisfy the **factorisation condition** if the map $\mathcal{C} \to \mathbb{P}^N$ factors through the contraction $\widetilde{\mathcal{C}} \to \overline{\mathcal{C}}$.

Theorem 1.2 ([RSW17a, Theorem B]). *The substack $\mathcal{VZ}_{1,n}(\mathbb{P}^N, d)$ of $\widetilde{\mathcal{VZ}}_{1,n}(\mathbb{P}^N, d)$ parameterizing maps to $\mathbb{P}^N$ that satisfy the factorisation condition is smooth and proper of the expected dimension.*

The remainder of this section lifts this construction to the category of relative maps to the pair $(\mathbb{P}^N, H)$. The core of the result is identifying the necessary factorisation conditions that remove the obstructions.

1.2. **Relative geometry: compactification.** Fix a hyperplane $H \subseteq \mathbb{P}^N$. Let α be a partition of the degree $d > 0$. Consider the moduli space $\mathcal{M}_{1,\alpha}^{\circ}(\mathbb{P}^N | H, d)$ of maps from smooth elliptic curves $C \to \mathbb{P}^N$ that meet H at finitely many marked points with vanishing orders given by the partition α. This is a smooth, but non-proper Deligne–Mumford stack.

We will first compactify the space described above, and then desingularize it. For the compactification, we begin with Abramovich–Chen–Gross–Siebert's space of logarithmic

stable maps, though we will typically work with various subcategories and variants [AC14, Che14b, GS13, Kim10].

Let $\mathbb{P}^N$ be endowed with the divisorial logarithmic structure induced by H. The moduli space $\overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H, d)$ is a fibred category over logarithmic schemes, whose fibre over (S, M_S) is the groupoid of logarithmically smooth curves of genus one over (S, M_S) equipped with a logarithmic map to $\mathbb{P}^N$ of degree d and contact order α. It is a fundamental fact in the subject that this category is representable by a proper algebraic stack with logarithmic structure, parameterizing minimal objects [Che14b].

1.3. **Relative geometry: expansions**. In order make the obstruction theory more geometric, we expand the target. To elucidate the connection with the static target, consider a logarithmic stable map $[C \to (\mathbb{P}^N, H)]$ over $\mathrm{Spec}\,(\mathbb{N} \to \mathbb{C})$. At the level of tropicalizations, there is a map of fans
$$\mathsf{C} \to \mathbb{R}_{\geq 0}.$$
Choose a subdivision of $\mathbb{R}_{\geq 0}$ whose vertices consist of the images of vertices of C. Pull this subdivision back to C by marking all preimages of the vertices of $\mathbb{R}_{\geq 0}$. Denote the resulting map $\widetilde{\mathsf{C}} \to \widetilde{\mathbb{R}}_{\geq 0}$.

These subdivisions induce logarithmic modifications
$$\widetilde{C} \to \mathbb{P}^N[s],$$
see [AW18]. Here the latter is the s-times iterated deformation to the normal cone of H in $\mathbb{P}^N$. The irreducible components of the target are in bijection with the vertices in $\widetilde{\mathbb{R}}_{\geq 0}$. The curve is modified by adding rational components corresponding to the newly introduced vertices.

The result is a logarithmic stable map to an expanded target $\mathbb{P}^N[s]$, together with a contraction to the main target component $\mathbb{P}^N[s] \to \mathbb{P}^N$. Two maps are considered equivalent if they differ by $\mathbb{G}_{\mathrm{m}}$-scaling the target at higher level.

Globally, Kim constructs a moduli space of logarithmic stable maps to expanded degenerations, which on logarithmic points, reduces to the above construction [Kim10]. Indeed, by the above description, Kim's space is identified with a subcategory of the Abramovich–Chen–Gross–Siebert space, and its dual minimal monoids are cones in a subdivision of the dual minimal cones of the unexpanded space. This process is outlined in complete generality in [Ran19, § 2].

Since we will work mostly with expansions, we will use the notation $\overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d)$ for Kim's space of logarithmic stable maps, with the understanding that this may just as easily be viewed as a logarithmic modification of the Abramovich–Chen–Gross–Siebert space.

1.4. **Relative geometry: alignment**. As in the absolute setting, we consider the space of logarithmic morphisms from the universal curve $\mathcal{C} \to \mathfrak{M}_{1,n}^{\mathrm{cen}}$ to the logarithmic scheme $(\mathbb{P}^N, H)$. Also as in the absolute case, we impose a compatibility condition, which says that the radius must pass through a vertex of positive degree in $\mathbb{P}^N$ and contain only vertices of degree zero in its strict interior. In this way we obtain a logarithmic modification of the Abramovich–Chen–Gross–Siebert space. If we further perform the subdivisions of target and source outlined in the previous section, we obtain a logarithmic modification of Kim's space:
$$\widetilde{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N|H, d) \to \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d)$$

which we refer to as the **moduli space of centrally aligned maps** to $(\mathbb{P}^N, H)$. It parametrises transverse logarithmic maps to the expanded target with an alignment which is compatible with the collapsed map $C \to \mathbb{P}^N$. Because of this compatibility condition, there is a morphism

$$\widetilde{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N | H, d) \to \widetilde{\mathcal{VZ}}_{1,n}(\mathbb{P}^N, d)$$

obtained by forgetting the logarithmic structure on the target and collapsing the map.

1.5. **Relative geometry: factorisation**. Let $[C \to \mathbb{P}^N[s] \to \mathbb{P}^N]$ be a centrally aligned logarithmic map. Recall that $\mathbb{P}^N[s]$ consists of a union of $\mathbb{P}^N$ with s copies of the projective bundle $\mathbb{P} = \mathbb{P}(\mathcal{O}_H \oplus \mathcal{O}_H(1))$. We refer to these latter components as **the higher levels**. Thus, we will say that a subcurve $D \subseteq C$ maps to **higher level** if the collapsed map

$$C \to \mathbb{P}^N$$

maps D into $H \subset \mathbb{P}^N$.

Let $D_F \subseteq C$ be the maximal genus one subcurve which is mapped to a higher level and contracted by the map $C \to \mathbb{P}^N[s]$. Let δ_F denote the distance from the core to the nearest component of $C \setminus D_F$. Similarly let $D_B \subseteq C$ be the maximal genus one subcurve that is contracted by the collapsed map to $\mathbb{P}^N$ and let δ_B be the associated radius. This coincides with the radius of the underlying map to $\mathbb{P}^N$. Of course, $\delta_F \leq \delta_B$, so (C, δ_F) is centrally aligned also.

The datum (δ_F, δ_B) determines a destabilisation $\widetilde{C}$ of C together with successive contractions $\widetilde{C} \to \overline{C}_F \to \overline{C}_B$. The contractions are constructed directly from [RSW17a, Section 3].

The following condition identifies the locus of smoothable maps.

Definition 1.3. The map $[C \to \mathbb{P}^N[s] \to \mathbb{P}^N]$ **factorises completely** if

(1) the map $C \to \mathbb{P}^N[s]$ factors through $\overline{C}_F$ such that at least one branch of $\overline{C}_F$ has positive degree in the fibre direction;
(2) the collapsed map to $\mathbb{P}^N$ factorises through $\overline{C}_B$ such that at least one branch of $\overline{C}_B$ has positive degree.

In particular, if $[C \to \mathbb{P}^N[s] \to \mathbb{P}^N]$ is a family of centrally aligned maps over S that factorises completely, there is a forgetful moduli map $S \to \mathcal{VZ}_{1,n}(\mathbb{P}^N, d)$, to the principal component of the space of absolute maps. Let

$$\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N | H, d) \subseteq \widetilde{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N | H, d)$$

be the stack of maps from centrally aligned curves to expansions of $\mathbb{P}^N$ that factorise completely.

Theorem 1.4. The stack $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N | H, d)$ is proper and logarithmically smooth over $\operatorname{Spec} \mathbb{C}$.

Proof. We prove the results via the forgetful morphism

$$\nu : \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N | H, d) \to \mathcal{VZ}_{1,n}(\mathbb{P}^N, d),$$

which remembers only the stabilization of the collapsed map. The map is certainly centrally aligned, and we have already argued above that it satisfies the factorisation property for $\mathbb{P}^N$ after composition. The moduli space $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N | H, d)$ is proper. Indeed, the verification of the valuative criterion follows exactly as in Vakil's Lemma [Vak00, Lemma 5.9]. A

direct proof that the factorisation property is a closed condition may be found in [RSW17a, Theorem 4.3].

We come to logarithmic smoothness. If the elliptic curve does not map into higher level, then unobstructedness proceeds exactly as in [RSW17a, Section 4.5].

Assume now that the elliptic curve maps into higher level. Note that the relative logarithmic tangent bundle of the expansion, over the base $\mathbb{P}^N$ is a trivial line bundle of rank 1, as the fibres are toric. Consider an S-family of maps. After replacing the source C by a destabilisation, we have maps

$$
\begin{array}{ccc}
C \longrightarrow \overline{C}_F \longrightarrow \overline{C}_B \\
\downarrow \qquad\qquad \downarrow \\
\mathbb{P}^N[s] \longrightarrow \mathbb{P}^N
\end{array}
$$

factorizing completely. Let f_B and f be the maps from C to the base and total space respectively. Examining the morphism ν, we see that there is a map

$$
\mathrm{Def}(C, f_B) \to \mathrm{Obs}(f/(C, f_B)) = H^1(\overline{C}_F, \mathcal{O}_{\overline{C}_F}),
$$

where the former is the space of deformations of the curve and map to the base, as a factorised centrally aligned map. The latter is the space of obstructions to lifting a map to the base into the total space. The cokernel of this map are the absolute obstructions, which we will show vanishes, proving logarithmic smoothness.

We note the manner in which the group $H^1(\overline{C}_F, \mathcal{O}_{\overline{C}_F})$ functions as the obstruction group for the lifting. It suffices to work near the minimal genus one subcurve D_F of C, since the rest of the curve is rational. This lifting is given by a rational function with prescribed orders of poles given by the slopes of the tropicalization map. That is, if α be the piecewise linear function giving this tropical map, then the lifting is described by a section of the associated bundle $\mathcal{O}_C(-\alpha)$.

Given a strict square-zero extension S' of S, the piecewise linear function α extends uniquely by strictness to any deformation of the curve. The resulting deformation of $\mathcal{O}_C(-\alpha)$ produces an infinitesimal deformation of the trivial bundle in the Picard group, whose class in $H^1(\overline{C}_F, \mathcal{O}_{\overline{C}_F})$, is the obstruction to deformation. We will show that it is possible to choose a deformation of (C, f_B) that cancels out the obstruction to lifting to f, and thus the map to the obstruction space above is surjective.

The line bundle $\mathcal{O}_C(-\alpha)$ is equivalent to the divisor $\sum a_i x_i$ where x_i are the points connecting D_F to the rest of the curve, and a_i are the slopes of α along the edges corresponding to the x_i. Infinitesimally moving the point x_i is a deformation of the curve that is unobstructed by the map f_B, since the map is constant on the interior of the circle of radius δ_B. By deforming the x_i, at least one of which has nonzero a_i, we produce a one-dimensional space of obstructions. The absolute obstructions therefore vanish, so the result follows. $\qquad\square$

The above theorem guarantees that the singularities of the space of totally factorised maps to expansions is logarithmically smooth. In fact, one can say more in this case.

Corollary 1.5. The logarithmically smooth stack $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ has at worst orbifold singularities, i.e. admits a non-representable cover by a smooth Deligne–Mumford stack.

Proof. Since $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ is logarithmically smooth, it will suffice to show that the cones of its tropicalization are simplicial. To see this, consider a logarithmic stable map to an expansion $C \to \mathbb{P}^N[s]$ without a central alignment. The tropical moduli cone obtained as the dual of the minimal base monoid can be identified with $\mathbb{R}^s_{\geq 0}$, see for instance [Che14a, Section 2.2]. The alignment procedure is an iterated barycentric subdivision at the level of tropical moduli spaces, as explained in [RSW17a, Section 4.6], and such subdivisions preserve the property of being simplicial. We conclude from this that the blowup $\widetilde{\mathcal{VZ}}^{\mathrm{exp}}_{1,\alpha}(\mathbb{P}^N|H, d)$ obtained by centrally aligning Kim's spaces has simplicial cones. Finally, the morphism $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d) \to \widetilde{\mathcal{VZ}}^{\mathrm{exp}}_{1,\alpha}(\mathbb{P}^N|H, d)$ is strict, so the cones of $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ are simplicial, as claimed. $\qquad\qquad\square$

1.6. **Rubber variants**. In our implementation of the relative theory, we will need the "rubber" variant of the space of maps $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$, where the entire curve is mapped into higher levels.

Specifically, let $\mathbb{P}$ denote the projective bundle $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O})$ on $\mathbb{P}^{N-1}$. Equip this space with the logarithmic structure coming from the 0 and ∞ sections of the bundle. Consider the moduli space $\mathcal{VZ}^{\leftrightarrow}_{1,\alpha}(\mathbb{P}|H_0 + H_\infty, d)$ of logarithmic maps

$$C \to \mathbb{P}[s] \to \mathbb{P}^{N-1},$$

which factorise completely, and where automorphisms are considered over the space of maps to $\mathbb{P}^{N-1}$. Specifically, two maps that differ by a $\mathbb{G}_m$ translate in the fibre direction are considered equivalent. Total factorisation, similar to the previous case, means that the map to the bundle and the map to the base $\mathbb{P}^{N-1}$ both factorise through possibly different singularities.

Theorem 1.6. *The stack $\mathcal{VZ}^{\leftrightarrow}_{1,\alpha}(\mathbb{P}|H_0 + H_\infty, d)$ is logarithmically smooth.*

The main point in the proof of this statement is that on the space of non-degenerate maps with smooth domains, the map from $\mathcal{VZ}^{\circ}_{1,\alpha}(\mathbb{P}|H_0 + H_\infty, d) \to \mathcal{VZ}^{\leftrightarrow,\circ}_{1,\alpha}(\mathbb{P}|H_0 + H_\infty, d)$ is a $\mathbb{G}_m$-torsor. Correspondingly, the map on compact moduli spaces is a nodal curve fibration. To prove this, it is convenient to work with the logarithmic multiplicative group and its torsors. Recall that the **logarithmic multiplicative group** $\mathbb{G}_{\mathrm{log}}$ is the functor on logarithmic schemes whose value on a logarithmic scheme S is the group of global sections $H^0(S, M^{\mathrm{gp}}_S)$; it is a proper group functor, and it contains $\mathbb{G}_m$ as a subfunctor. This functor is representable only after a logarithmically étale cover. Background may be found in [MW17, RW19].

Each fibre of the bundle $\mathbb{P} \to \mathbb{P}^{N-1}$ admits a natural log étale morphism to $\mathbb{G}_{\mathrm{log}}$. Replacing these fibres by $\mathbb{G}_{\mathrm{log}}$, we obtain a non-representable functor on logarithmic schemes $\mathbb{P}_{\mathrm{log}}$ and a logarithmically étale modification

$$\mathbb{P} \to \mathbb{P}_{\mathrm{log}}.$$

Proof of Theorem 1.6. Consider the stack over logarithmic schemes $\mathcal{VZ}_{1,\alpha}(\mathbb{P}_{\mathrm{log}}, d)$ of stable logarithmic maps that factorise completely. Here, stability for the map coincides with stability for the projection to $\mathbb{P}^{N-1}$. The arguments in the previous section show that maps to $\mathbb{P}$ that factorise completely are unobstructed. Since this space is a logarithmically étale cover of $\mathcal{VZ}_{1,\alpha}(\mathbb{P}_{\mathrm{log}}, d)$, logarithmic smoothness of the latter follows.

The logarithmic multiplicative group $\mathbb{G}_{\log}$ acts on $\mathcal{VZ}_{1,\alpha}(\mathbb{P}_{\log}, d)$ by translation. Tautologically, $\mathcal{VZ}_{1,\alpha}(\mathbb{P}_{\log}, d)$ is a $\mathbb{G}_{\log}$-torsor over the moduli problem of maps up to this $\mathbb{G}_{\log}$ translation. It is representable after a logarithmically étale modification by a family of nodal curves. It follows that the space $\mathcal{VZ}_{1,\alpha}^{\leftrightarrow}(\mathbb{P}_{\log}, d)$ of maps up to $\mathbb{G}_{\log}$-translation that factorise completely is logarithmically smooth.

To conclude the main result, we compare the stacks $\mathcal{VZ}_{1,\alpha}^{\leftrightarrow}(\mathbb{P}_{\log}, d)$ and $\mathcal{VZ}_{1,\alpha}^{\leftrightarrow}(\mathbb{P}|H_0 + H_\infty, d)$. Since a map to $(\mathbb{P}, H_0 + H_\infty)$ gives rise immediately to a map to $\mathbb{P}_{\log}$, there is a morphism

$$\mathcal{VZ}_{1,\alpha}^{\leftrightarrow}(\mathbb{P}|H_0 + H_\infty, d) \to \mathcal{VZ}_{1,\alpha}^{\leftrightarrow}(\mathbb{P}_{\log}, d).$$

An application of the infinitesimal lifting criterion shows that this map is logarithmically étale, so the result follows. □

1.7. **Virtual construction for very ample pairs**. Now let (X, Y) be a smooth pair with Y very ample. The definition given in §1.5 applies in this setting, producing a moduli space $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$ together with a morphism

$$\mathcal{VZ}_{1,\alpha}(X|Y, \beta) \to \overline{\mathcal{M}}_{1,\alpha}^{\log}(X|Y, \beta)$$

obtained by performing a logarithmic modification and then passing to a closed substack. This moduli space will not in general be logarithmically smooth, but we may equip it with a virtual class as follows. The complete linear system $|\mathcal{O}_X(Y)|$ defines an embedding $X \hookrightarrow \mathbb{P}^N$ with $Y = X \cap H$ for H some hyperplane.

Lemma 1.7. The following square is cartesian (in the category of ordinary stacks):

$$\begin{array}{ccc}
\mathcal{VZ}_{1,\alpha}(X|Y, \beta) & \longrightarrow & \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d) \\
\downarrow & \square & \downarrow \\
\mathcal{VZ}_{1,n}(X, \beta) & \xrightarrow{\ i\ } & \mathcal{VZ}_{1,n}(\mathbb{P}^N, d).
\end{array}$$

Proof. It is clear from the modular description that this square is cartesian in the category of fs logarithmic stacks. But the morphism i is strict, which implies that the square is also cartesian in the category of coherent logarithmic stacks. Since fibre products in the latter category are compatible with the functor forgetting logarithmic structures, the claim follows. □

Since $\mathcal{VZ}_{1,n}(\mathbb{P}^N, d)$ is smooth and $\mathcal{VZ}_{1,n}(X, \beta)$ carries a natural virtual class [RSW17a, Theorem 4.4.1] there is a diagonal pull-back morphism [BN17, Appendix C], which we use to define the virtual class on the space of maps to (X, Y):

$$[\mathcal{VZ}_{1,\alpha}(X|Y, \beta)]^{\text{virt}} := i_\Delta^! [\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)].$$

Since $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$ comes equipped with evaluation maps and cotangent line classes, we immediately arrive at a definition of reduced Gromov–Witten invariants for the pair (X, Y). This is the same manner in which Gathmann defines his virtual classes [Gat02].

2. Stratification and tropicalization

The logarithmic smoothness established in Theorem 1.4 implies that $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$ is an orbifold toroidal embeddeding. Consequently, the irreducible components of its boundary and their intersections form a stratification of the space. This stratification will be important in the sequel.

The tropicalizations of logarithmic maps that factorise completely satisfy a natural combinatorial condition known as well-spacedness. The version we use here is essentially identical to [RSW17b].

Definition 2.1. Let $\sqsubset$ be a tropical curve of genus one and let $\sqsubset_0$ be its minimal subcurve of genus one. A tropical map $F : \sqsubset \to \mathbb{R}_{\geq 0}$ is said to be **well-spaced** if one of the following two conditions are satisfied:

(1) no open neighborhood of $\sqsubset_0$ is contracted to a point in $\mathbb{R}_{>0}$, or
(2) if an open neighborhood of $\sqsubset_0$ is contracted and $t_1, \ldots, t_k$ are the flags whose vertex is mapped to $F(\sqsubset_0)$, but along which F has nonzero slope; then, the minimum of the distances from $\sqsubset_0$ to t_i occurs for at least two indices i.

Proposition 2.2. Let $[C \to \mathbb{P}^N]$ be a logarithmic stable map from a centrally aligned curve to an expansion, that factorises completely. Then the tropicalization $\sqsubset \to \mathbb{R}_{\geq 0}$ is well-spaced.

Proof. The required result is essentially contained in [RSW17b, Section 4], so we explain how to deduce the requisite result from the one in loc. cit. We may focus on a single component of the expansion $\mathbb{P}^N[s]$ that contains the image of a contracted genus one subcurve, as this is the only relevant case. We may thus replace the target with the projective bundle $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O})$ over $\mathbb{P}^{N-1}$ equipped with the divisorial logarithmic structure from the 0 and ∞ sections. Let p be the point to which the genus one subcurve is contracted. Passing to an open neighborhood of p, the map to the bundle is given by a rational function f on an open curve C°, determined by the the subgraph formed by $\sqsubset_0$ and the flags t_i described in the definition above. To describe the tropical map to $\mathbb{R}_{>0}$, we observe that $\sqsubset_0$ is contracted to a fixed point $q \in \mathbb{R}_{>0}$. The flags at t_i correspond to nodes or markings of C. The pole orders of f at these distinguished points determine the slopes of the tropical map. We are now exactly in the situation considered in [RSW17b, Second Paragraph of § 4.6], and Lemma 4.6.1 of loc. cit. guarantees the well-spacedness as required. □

2.1. **The cone complex**.
To understand the stratification via combinatorial data, we will build the stratification from known objects.

Step 1. Let $\Sigma^{\log}$ be the tropical moduli space of genus one tropical stable maps to $\mathbb{R}_{\geq 0}$. This is naturally identified with the tropicalization (in the logarithmic sense) of the Abramovich–Chen–Gross–Siebert space of logarithmic stable maps to the pair $(\mathbb{P}^N, H)$.

Step 2. Given such a tropical stable map, we may subdivide $\mathbb{R}_{\geq 0}$ such that the image of every vertex of $\sqsubset$ is a vertex of $\mathbb{R}_{\geq 0}$. Call this subdivision $\widetilde{\mathbb{R}}_{\geq 0}$. The preimages of vertices of the subdivision form a subdivision of $\sqsubset$. After this procedure, the images of vertices of $\sqsubset$ which lie in $\mathbb{R}_{>0}$ are totally ordered, in a manner extending the partial order obtained from the map to $\mathbb{R}_{\geq 0}$. The combinatorial types of such **image-ordered** maps produce the

cones of a subdivision of $\Sigma^{\log}$ which we denote Σ^{Kim}. As the notation suggests, Σ^{Kim} is the tropicalization of Kim's moduli space of logarithmic maps to $(\mathbb{P}^N, H)$.

Step 3. Given a tropical map $F : \sqsubset \to \mathbb{R}_{\geq 0}$ parameterized by Σ^{Kim}, there is a largest radius δ (possibly equal to 0) such that every vertex strictly contained in the circle of radius δ around the core has degree-marking $d = 0$. Let Σ^{cen} be the subdivision obtained by requiring that the vertices contained within the circle of radius δ around the core of $\sqsubset$ are totally ordered. This involves introducing cones along which certain $\mathbb{Z}$-linear combinations of edge lengths are identified.

(margin note: doesn' picking δ itself imply a subdivision?)

Step 4. Let $\Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$ be the subcomplex of Σ^{cen} consisting of well-spaced tropical maps.

Proposition 2.3. The cone complex $\Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$ is the fan of the toroidal embedding $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$. In particular, the codimension k strata of $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ are in inclusion reversing bijection with the dimension k cones in $\Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$.

Proof. The construction above has been given to mimic the construction of the space $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$. Specifically, the fact that the cone complex Σ^{cen} is the cone complex attached to the logarithmic stack $\widetilde{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N|H, d)$ of centrally aligned maps to expansions, follows immediately from its description as a subcategory of the fibreed category (over logarithmic schemes) of $\mathcal{M}_{1,\alpha}^{\log}(\mathbb{P}^N|H, d)$. To complete the result, we note that $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ has a strict map to $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$, so its cone complex is a subcomplex of Σ^{cen}. The fact that it must be contained in the subcomplex of well-spaced curves is immediate from Proposition 2.2. $\square$

2.2. **Indexing the strata**. The dimension-k cones in $\Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$ can be enumerated as follows. First, the cones in $\Sigma = \Sigma^{\mathrm{Kim}}$ are indexed by **combinatorial types** of tropical maps to $\mathbb{R}_{\geq 0}$. Here a combinatorial type encodes all of the data of a tropical map, except for the edge lengths. To be more precise, a combinatorial type Δ consists of:

(1) a finite graph $\sqsubset$ and topological (not metric) subdivision $\widetilde{\mathbb{R}}_{\geq 0}$ of $\mathbb{R}_{\geq 0}$;
(2) genus, degree and marking assignments on the vertices;
(3) the data of which vertex of $\widetilde{\mathbb{R}}_{\geq 0}$ each vertex of $\sqsubset$ is mapped to;
(4) integral slopes along the edges (both finite and infinite).

The corresponding cone $\sigma \in \Sigma^{\mathrm{Kim}}$ is then given by the resulting moduli space of tropical maps, given by choices of edge lengths on source and target which produce a continuous tropical map. Given $\sigma \in \Sigma^{\mathrm{Kim}}$ we then produce all the cones in $\Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$ mapping to σ by performing Steps 3–4 outlined above. This amounts to taking a particular subdivision of σ. Note that in this process, new cones are created which map into larger-dimensional cones of Σ^{Kim}. The process for enumerating the codimension k strata of $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ is therefore:

(1) fix a combinatorial type Δ;
(2) perform the subdvidision of the resulting cone $\sigma \in \Sigma^{\mathrm{Kim}}$;
(3) identify the dimension k cones of that subdivision.

In §4 we will perform this analysis in the case $k = 1$.

3. Degeneration of contact order

Consider a reduced relative space $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$. For each marking x_k we consider the locus $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H,d) \subseteq \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$ where x_k belongs to an internal component of the collapsed map. In this section, we use the logarithmic structure on $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$ to construct a line bundle $\mathcal{L}_k$ together with a section s_k which vanishes precisely along $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H,d)$. We use the correspondence with tropical geometry to identify $c_1(\mathcal{L}_k)$ in terms of tautological classes on $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$, and to compute the vanishing order of s_k along the components of $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H,d)$. Combined with the relative splitting axioms established in the next section, we obtain a recursion relation inside $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$

The pair $(\mathcal{L}_k, s_k)$ is most naturally constructed on the moduli space $\overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H,d)$ of non-expanded logarithmic maps; the corresponding pair on $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$ will be obtained via pull-back. Consider therefore the tropicalization $\Sigma^{\log}$ of $\overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H,d)$, identified as usual with the moduli space of tropical maps to $\mathbb{R}_{\geq 0}$. We have a universal family

$$
\begin{array}{ccc}
\mathsf{C} & \xrightarrow{\ f\ } & \mathbb{R}_{\geq 0} \\
{\scriptstyle x_k}\!\!\nwarrow\!\!\Big\downarrow{\scriptstyle\pi} & & \\
\Sigma^{\log} & &
\end{array}
$$

where x_k is the section which for every point $\lambda \in \Sigma^{\log}$ picks out the vertex of C_λ containing the leg x_k. The composition $f \circ x_k \colon \Sigma^{\log} \to \mathbb{R}_{\geq 0}$ defines a piecewise-linear function on $\Sigma^{\log}$ whose preimage over the open cone $\mathbb{R}_{>0}$ consists of those tropical maps where x_k belongs to an internal component. This produces a section of the ghost sheaf on $\overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H,d)$, which in the usual way induces a line bundle and section $(\mathcal{L}_k, s_k)$ on the moduli space, and the tropical description above shows that the zero locus of s_k is (set-theoretically) the locus where x_k belongs to an internal component.

We now calculate $c_1(\mathcal{L}_k)$.

Theorem 3.1. $c_1(\mathcal{L}_k) = \alpha_k \psi_k + \mathrm{ev}_k^\star H$.

Proof. Choose a family of logarithmic stable maps over S and let $\mu \in \Gamma(S, \overline{M}_S)$ be the global section of the ghost sheaf constructed in the previous paragraph. This pulls back along π to give a global section $\pi^\flat(\mu) \in \Gamma(C, \overline{M}_C)$. Interpreted as a piecewise-linear function on the tropicalization C with values in $\overline{M}_S$ [CCUW17, Remark 7.3], this assigns μ to every vertex and has slope zero along every edge. By construction, the line bundle associated to this section is $\pi^\star \mathcal{L}_k$. Consider on the other hand the generator $1 \in \mathbb{N} = \Gamma(\mathbb{P}^N, \overline{M}_{\mathbb{P}^N})$ with associated line bundle $\mathcal{O}(H)$. The section $f^\flat(1) \in \Gamma(C, \overline{M}_C)$ has associated line bundle $f^\star \mathcal{O}(H)$. If we let v denote the vertex containing x_k, then by construction $f^\flat(1)$ assigns μ to v and has slope α_k along the leg x_k. Thus if we consider the difference $f^\flat(1) - \pi^\flat(\mu)$ then this assigns 0 to v and still has slope α_k along x_k. Thus by [RSW17a, Proposition 2.4.1] the corresponding line bundle restricted to C_v is given by

$$
\mathcal{O}_{C_v}\!\left(\alpha_k x_k + \sum_e \mu_e x_e \right)
$$

where the sum is over the edges e adjacent to v and distinct from x_k. Thus we see that:

$$\left(f^\star \mathcal{O}(H) \otimes \pi^\star \mathcal{L}_k^{-1}\right)\big|_{C_v} = \mathcal{O}_{C_v}\left(\alpha_k x_k + \sum_e \mu_e x_e\right).$$

Since x_k factors through C_v we may pull back along x_k to obtain

$$\mathcal{L}_k = x_k^\star \pi^\star \mathcal{L}_k = x_k^\star \mathcal{O}_{C_k}(-\alpha_k x_k) \otimes x_k^\star f^\star \mathcal{O}(H) = x_k^\star \mathcal{O}_{C_k}(-\alpha_k x_k) \otimes \mathrm{ev}_k^\star \mathcal{O}(H)$$

and taking Chern classes gives

$$c_1(\mathcal{L}_k) = \alpha_k \psi_k + \mathrm{ev}_k^\star H.$$

as required. □

This gives the construction of $(\mathcal{L}_k, s_k)$ and the calculation of $c_1(\mathcal{L}_k)$ on $\overline{\mathcal{M}}_{1,\alpha}^{\mathrm{log}}(\mathbb{P}^N|H, d)$; the construction on $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ is given by pull-back. The relevant piecewise-linear function is the composition:

$$\Sigma_{1,\alpha}(\mathbb{P}^N|H, d) \to \Sigma^{\mathrm{log}} \to \mathbb{R}_{\geq 0}.$$

Notice in particular that ψ_k should be interpreted as a **collapsed psi class** on $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$, i.e. a psi class coming from the stabilised curve of the collapsed map. In this paper, all psi classes are collapsed unless stated otherwise. We immediately arrive at:

Theorem 3.2. We have the following relation in the Chow ring of $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$:

(1) $$(\alpha_k \psi_k + \mathrm{ev}_k^\star H) \cap [\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)] = \sum_{\mathcal{D}} \lambda_{\mathcal{D}}[\mathcal{D}].$$

The sum is over irreducible components $\mathcal{D}$ of the divisor $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d) \subseteq \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$, and $\lambda_{\mathcal{D}}$ is the vanishing order of s_k along this component.

Remark 3.3. This construction gives the natural logarithmic analogue of Gathmann's line bundle and section [Gat02, Construction 2.1]. A benefit of the logarithmic approach is to make the computation of vanishing orders entirely combinatorial (see §4.2.4 below), circumventing the difficult technical calculation given by Gathmann.

4. SPLITTING THE BOUNDARY

A basic phenomenon is responsible for the nature of the forthcoming analysis. In contrast with nodal singularities, the functions on an elliptic singularity are not simply given as collections of functions on the normalisation subject to topological gluing conditions. As a consequence, in order to compute integrals, it is necessary to give a tautological description of this additional factorisation condition.

In what follows, we provide an explicit description of the terms appearing in the right-hand side of (1). For each term, we provide a combinatorial formula for the vanishing order $\lambda_{\mathcal{D}}$, and a recursive description of the stratum $\mathcal{D}$. in terms of fibre products of moduli spaces with smaller numerical data. The tautological description of the factorisation condition then leads to a recursive structure for calculating integrals. This last step is the heart of the analysis.

Remark 4.1. We remark that for the reduced relative theory of $(\mathbb{P}^N, H)$ with only primary fields, an analysis of the factorisation condition can be avoided by dimension considerations. This leads to the main results in [Vak00]. In order to add descendant fields, study with more general pairs, or indeed work cycle theoretically, an analysis of these tautological classes is crucial.

4.1. **Irreducible components of the degenerate locus.** We explain the tropical geometric procedure to identify irreducible components of $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H,d)$.

Lemma 4.2. Every irreducible component of $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H,d)$ is a codimension 1 logarithmic stratum.

Proof. The locus where the logarithmic structure is trivial coincides precisely with the locus where the source curve is smooth, and consequently not mapped inside H. Since $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$ is toroidal this locus is open and dense. By definition $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H,d)$ is contained in the complement of this locus and can be expressed as a union of strata of positive codimension. Since $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H,d)$ has codmension 1, it must therefore be equal to a union of logarithmic divisors. □

In §2.2 we discussed a procedure for enumerating the logarithmic strata of $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$, using the cones of the tropicalization $\Sigma_{1,\alpha}(\mathbb{P}^N|H,d)$. Every divisorial logarithmic stratum is obtained as follows:

(1) choose a combinatorial type Δ of a tropical map;
(2) subdivide the corresponding tropical moduli space σ to align the type;
(3) choose a ray in this subdivision.

This process contains some redundancies: upon choosing a ray in the tropical moduli space, some of the edge lengths or vertex positions may get set to 0. This induces a generization of the intial combinatorial type Δ, given by contracting the corresponding edges of the dual graph and moving the corresponding vertices from $\mathbb{R}_{>0}$ to 0. When we speak of the combinatorial type of a stratum in $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$, we will always mean this generisation. This is independent of the choice of Δ, and in fact we can and will always choose Δ to coincide with the generization.

The logarithmic divisors contained in $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H,d)$ are precisely those whose combinatorial types map the vertex of the dual graph containing x_k into the interior $\mathbb{R}_{>0} \subseteq \mathbb{R}_{\geq 0}$. Thus via the above procedure, we enumerate the irreducible components of $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H,d)$ in a combinatorial manner, and the combinatorial type Δ allows us to describe the general element of such a component. The remainder of this section gives a recursive description of the components, and a procedure to compute integrals over them in a recursive manner.

4.2. **Recursive description of the divisors: types I,II and III.** Choose an irreducible component $\mathcal{D} \subseteq \mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H,d)$. By Lemma 4.2 this is a logarithmic divisor, and hence may be written as:

$$\mathcal{D} = \widetilde{\mathcal{D}} \cap \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$$

for a unique logarithmic stratum $\widetilde{\mathcal{D}} \subseteq \widetilde{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N|H,d)$. Since

$$\widetilde{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N|H,d) \to \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H,d)$$

is a logarithmic modification, the divisor $\widetilde{\mathcal{D}}$ is either exceptional or the strict transform of a logarithmic divisor. These two cases correspond, respectively, to when the core is

assigned zero degree or nonzero degree by the combinatorial type. To begin with we will focus on the latter case. Suppose therefore that $\widetilde{\mathcal{D}}$ is the proper transform of a logarithmic divisor:

$$\mathcal{E} \subseteq \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H,d).$$

The birational map $\widetilde{\mathcal{D}} \to \mathcal{E}$ induces a morphism $\mathcal{D} \to \mathcal{E}$. We now show how to interpret this morphism as a desingularisation of the principal component. Since $\mathcal{E}$ admits a recursive description in terms of relative and rubber moduli spaces, this will allow us to compute integrals over $\mathcal{D}$.

Lemma 4.3. Let $\mathcal{D} \subseteq \mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H,d)$ be an irreducible component which contributes nontrivially to the Gromov–Witten invariant, and let Δ be the corresponding combinatorial type. Suppose that Δ assigns positive degree to the core. Then Δ takes one of the following forms:
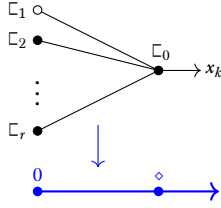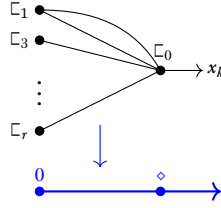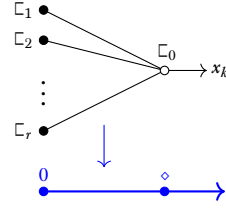

FIGURE 1. I

FIGURE 2. II

FIGURE 3. III

The degrees of vertices, expansions factors, and remaining markings are distributed arbitrarily, subject to the following constraints:

(1) the vertices $\sqsubset_1, \ldots, \sqsubset_r$ have positive degree;
(2) the core has positive degree;
(3) every vertex is stable;
(4) the balancing condition is satisfied.

In [Vak00] these cases are referred to, respectively, as types A, B and C^+.

Proof. Let $\sigma \in \Sigma_{1,\alpha}(\mathbb{P}^N|H,d)$ be the cone corresponding to the $\mathcal{E} \subseteq \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H,d)$. By the discussion above $\mathcal{D}$ corresponds to a ray in the subdivision of σ given by imposing the central alignment condition. Since the elliptic core is assigned positive degree, both radii are equal to 0 and the subdivision is trivial. We conclude that $\sigma = \mathbb{R}_{\geq 0}$. Since there must be at least one vertex mapped to higher level, the tropical target $\widetilde{\mathbb{R}}_{\geq 0}$ is obtained from $\mathbb{R}_{\geq 0}$ by subdividing at a single point $\diamond \in \mathbb{R}_{>0}$.

In order for the cone σ to be 1-dimensional, the dual graph Γ must be bipartite, with vertices over 0 and $\diamond$. The cases I,II and III enumerated above cover situations where there is a single vertex mapped to $\diamond$. If there is more than one such vertex, then the contribution to the Gromov–Witten invariant vanishes. To see this, we consider the stratum

$$\mathcal{E}^{\log} \subseteq \overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H,d)$$

to which $\mathcal{E}$ maps under the collapsing morphism, and examine the composition $\mathcal{D} \to \mathcal{E} \to \mathcal{E}^{\log}$. If we let $\mathcal{F}^{\log}$ denote the intersection of $\mathcal{E}^{\log}$ with the main component of the moduli space, then we obtain a factorisation $\mathcal{D} \to \mathcal{F}^{\log} \hookrightarrow \mathcal{E}^{\log}$. Since the moduli space is

generically unobstructed along $\mathcal{F}^{\log}$, the codimension of $\mathcal{F}^{\log}$ is given by the dimension of the associated cone in the tropicalization $\Sigma^{\log}$. If there is more than one vertex mapped to higher level, then this cone has dimension ≥ 2. Therefore the map $\mathcal{D} \to \mathcal{F}^{\log}$ has positive-dimensional fibres, and since all insertions are pulled back from the latter space the contribution vanishes by the projection formula. $\qquad\square$

Remark 4.4. The difference in dimensions between $\mathcal{D}$ and $\mathcal{F}^{\log}$ (or equivalently, the difference in virtual dimensions between $\mathcal{E}$ and $\mathcal{E}^{\log}$) may be interpreted as the difference in dimensions between moduli spaces of *disconnected* rubber and their images under the collapsing morphisms.

We now investigate the three types I,II and III separately, giving a recursive description of the boundary divisor in each case. For the remainder of this subsection, therefore, we fix a one-dimensional cone $\tau \in \Sigma_{1,\alpha}(\mathbb{P}^N|H,d)$, let $\mathcal{D} \subseteq \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$ be the associated logarithmic divisor, and assume that $\mathcal{D}$ is contained in $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H,d)$ and is of type I, II or III. We let $\mathcal{E} \subseteq \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H,d)$ be the logarithmic stratum into which $\mathcal{D}$ is mapped; this is indexed by a cone $\epsilon \in \Sigma^{\mathrm{Kim}}$ corresponding to a combinatorial type Δ, and ϵ is one-dimensional since we are restricting to the type I,II,III cases.

4.2.1. *Type I.* Suppose Δ is of type I. Then $\mathcal{E}$ admits a finite and surjective *splitting morphism* onto the fibre product:

$$\mathcal{E} \to \left(\overline{\mathcal{M}}_{1,\alpha^{(1)}\cup(m_1)}(\mathbb{P}^N|H,d_1) \times \prod_{i=2}^{r} \overline{\mathcal{M}}_{0,\alpha^{(i)}\cup(m_i)}(\mathbb{P}^N|H,d_i)\right) \times_{H^r} \overleftrightarrow{\overline{\mathcal{M}}}_{0,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(\mathbb{P}|H_0+H_\infty,d_0).$$

Lemma 4.5. The divisor $\mathcal{D}$ admits a natural splitting morphism

$$\mathcal{D} \xrightarrow{\rho} \left(\mathcal{VZ}_{1,\alpha^{(1)}\cup(m_1)}(\mathbb{P}^N|H,d_1) \times \prod_{i=2}^{r} \overline{\mathcal{M}}_{0,\alpha^{(i)}\cup(m_i)}(\mathbb{P}^N|H,d_i)\right) \times_{H^r} \overleftrightarrow{\overline{\mathcal{M}}}_{0,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(\mathbb{P}|H_0+H_\infty,d_0)$$

such that the map $\mathcal{D} \to \mathcal{E}$ covers the map on fibre products obtained by desingularising the main component of the factor corresponding to $\llcorner_1$.

Proof. We first consider the statement of the lemma in the category of fine but not necessarily saturated logarithmic schemes, so that a relative stable map to an expansion admits a unique logarithmic lifting. The fibre product description preceding the lemma is an isomorphism in this category [AMW14, Lemma 4.2.2].

The map $\mathcal{D} \to \mathcal{E}$ is an isomorphism away from the exceptional centers, and the latter is contained in the locus where the core is contracted. Given an element of $\mathcal{D}$ we can split it along the nodes $q_1,\ldots,q_r$. It is then clear that $\llcorner_1$ is aligned. We claim that $\llcorner$ satisfies the factorisation property if and only if $\llcorner_1$ does. This immediately implies the lemma.

On $\llcorner$ there are two contraction radii δ_F, δ_B. Let $\delta \in \{\delta_F, \delta_B\}$. An examination of the stratification by combinatorial type of $\mathcal{D}$ shows that $\lambda(\llcorner') > \delta$ for any component $\llcorner'$ of $\llcorner_0$. This means that $\llcorner_0, \llcorner_2, \llcorner_3, \ldots, \llcorner_r$ lie outside both contraction radii, and so the aligned curve $\llcorner$ satisfies the factorisation condition if and only if $\llcorner_1$ does. $\qquad\square$

Lemma 4.5 provides a means to calculate integrals over the class $\lambda_{\mathcal{D}}[\mathcal{D}]$ appearing on the right-hand side of Theorem 3.2, provided that we can calculate $\lambda_{\mathcal{D}}$ and the degree of the splitting morphism. Simple closed formulae for these are given in §4.2.4 below.

4.2.2. *Type II.* Now suppose Δ is of type II. Note that in this case, it is impossible for the core to be contracted. Hence $\mathcal{E}$ is disjoint from the blown-up locus, and the map $\mathcal{D} \to \mathcal{E}$ is an isomorphism. We thus obtain the following description, entirely in terms of genus zero data:

Lemma 4.6. $\mathcal{D}$ admits a finite and surjective splitting morphism:

$$\mathcal{D} \xrightarrow{\rho} \left(\overline{\mathcal{M}}_{0,\alpha^{(1)} \cup (m_1,m_2)}(\mathbb{P}^N|H, d_1) \times \prod_{i=3}^{r} \overline{\mathcal{M}}_{0,\alpha^{(i)} \cup (m_i)}(\mathbb{P}^N|H, d_i) \right) \times_{H^r} \overline{\mathcal{M}}_{0,\alpha^{(0)} \cup (-m_1,\ldots,-m_r)}^{\leftrightarrow}(\mathbb{P}|H_0+H_\infty, d_0).$$

4.2.3. *Type III.* Finally suppose that Δ is of type III. As before we have a finite and surjective morphism:

$$\mathcal{E} \to \left(\prod_{i=1}^{r} \overline{\mathcal{M}}_{0,\alpha^{(i)} \cup (m_i)}(\mathbb{P}^N|H, d_i) \right) \times_{H^r} \overline{\mathcal{M}}_{1,\alpha^{(0)} \cup (-m_1,\ldots,-m_r)}^{\leftrightarrow}(\mathbb{P}|H_0 + H_\infty, d_0).$$

The same arguments as in §4.2.1 then apply to give:

Lemma 4.7. $\mathcal{D}$ admits a finite splitting morphism

$$\mathcal{D} \xrightarrow{\rho} \left(\prod_{i=1}^{r} \overline{\mathcal{M}}_{0,\alpha^{(i)} \cup (m_i)}(\mathbb{P}^N|H, d_i) \right) \times_{H^r} \mathcal{VZ}_{1,\alpha^{(0)} \cup (-m_1,\ldots,-m_r)}^{\leftrightarrow}(\mathbb{P}|H_0 + H_\infty, d_0)$$

such that $\mathcal{D} \to \mathcal{E}$ corresponds to the obvious map on fibre products.

Here the first factor in the fibre product is the logarithmic blow-up of the moduli space of rubber maps constructed in §1.6. We will compute integrals over such spaces as part of the recursion (see §5).

4.2.4. *Splitting degree and vanishing order.* In each of the subsections 4.2.1–4.2.3 above, we obtained a finite splitting morphism ρ from $\mathcal{D}$ to a fibre product of moduli spaces with smaller numerical data. Here we describe the degree of ρ and calculate the vanishing order $\lambda_{\mathcal{D}}$ of the section s_k constructed in §3. This gives a complete recursive description of the terms appearing on the right-hand side of Theorem 3.2 which are of type I, II or III.

Lemma 4.8. The degree of ρ is given by:

$$\frac{\prod_{i=1}^{r} m_i}{\mathrm{lcm}(m_1, \ldots, m_r)}.$$

Proof. The calculation is well known, and is recorded for instance in [Che14a, Section 7.9], [ACGS17, Section 5.3]. $\qquad\qquad\square$

Lemma 4.9. The section s_k vanishes along the divisor $\mathcal{D}$ with order given by

$$\lambda_{\mathcal{D}} = \mathrm{lcm}(m_1, \ldots, m_r).$$

Proof. Let $\tau \in \Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$ be the cone corresponding to $\mathcal{D}$ and let

$$\varphi \colon \Sigma_{1,\alpha}(\mathbb{P}^N|H, d) \to \Sigma^{\log} \to \mathbb{R}_{\geq 0}$$

be the piecewise-linear function constructed in §3. It follows from the tropical description that $\lambda_{\mathcal{D}}$ is equal to the index of the map of integral cones $\tau \to \mathbb{R}_{\geq 0}$ obtained by restricting

φ. Observe that $\tau \subseteq \mathbb{R}^r_{\geq 0}$ with integral generator:

$$w = \left(\frac{\mathrm{lcm}(m_1, \ldots, m_r)}{m_1}, \ldots, \frac{\mathrm{lcm}(m_1, \ldots, m_r)}{m_r} \right).$$

The map $\tau \to \mathbb{R}_{\geq 0}$ is given by projecting onto the ith factor and then multiplying by m_i. Note that by piecewise linear continuity of the tropical map, this is independent of i. We conclude that the index is equal to $\mathrm{lcm}(m_1, \ldots, m_r)$ as claimed. $\square$

We summarize that the contribution of each term $\lambda_{\mathcal{D}}[\mathcal{D}]$ is given by integrating over the appropriate fibre product and then multiplying the result by $\lambda_{\mathcal{D}} \cdot \deg \rho = \prod_{i=1}^r m_i$.

4.3. **Recursive description of the divisors: type** †.

The treatment of the type I, II and III cases above is a fairly straightforward extension of the ideas of Vakil and Gathmann (albeit updated with logarithmic and tropical technology, which streamlines the presentation and simplifies the proofs). The remaining type † case, however, is entirely new. In this section we provide a recursive description of these boundary divisors. This constitutes the technical heart of the paper.

4.3.1. *Possible combinatorial types.* A boundary divisor $\mathcal{D} \subseteq \mathcal{D}_{1,\alpha,k}(\mathbb{P}^N | H, d)$ is said to have **type** † if the core is assigned degree zero by the combinatorial type. Divisors of this type occur as principal components in compactified torus bundles over strata in the space of relative maps. As a result, their combinatorial types exhibit very degenerate behaviour. Precisely, over the generic point of such a divisor, the target may expand multiple times, and the source tropical curve need not be bipartite with a single interior vertex. We stress once again that such degenerate contributions are at the heart of the reduced theory and cannot be removed. We begin with the following lemma:

Lemma 4.10. Let $\mathcal{D}$ be a locus of type † with combinatorial type Δ which contributes nontrivially to the Gromov–Witten invariant. For every nonzero vertex $\diamond$ of the tropical target, the fibre of the tropical map over $\diamond$ contains exactly one stable vertex, which either is the core, or lies on the circle. Moreover, the maximal contact order at the core is $m > 1$.

Proof. Stability guarantees that at least one stable vertex lies over each vertex of the target. If there is more than one stable vertex, as in the proof of Lemma 4.3, the corresponding locus $\mathcal{F}$ in the moduli space of unexpanded maps has high codimension. As both insertions and the factorisation property for the collapsed map to $\mathbb{P}^N$ are pulled back from the moduli space of maps without expansions, the contribution vanishes by the projection formula.

For each vertex of the target there must always be at least one vertex in the preimage which lies on the circle. This is implied by the fact that the cone τ corresponding to $\mathcal{D}$ is a ray. If the vertex on the circle is strictly semistable – so that in particular the collapsed map f_B is constant along the corresponding component – it can be seen that the factorisation condition is unaffected by $\mathbb{G}_m$-scaling the coordinate corresponding to that vertex in the moduli space of attaching data, so the map $\mathcal{D} \to \mathcal{E}$ has generic fibre of positive dimension and the contribution vanishes.

In particular, notice that vertices lying on the circle and tropically mapped to $0 \in \mathbb{R}_{\geq 0}$ all have the same contact order at the core, because by the previous paragraph we may assume that only strictly semistable vertices are encountered on the paths from these

vertices to the core. For the same reason, this contact order m is maximal among those of the edges adjacent to the core.

Finally, in case the maximal contact order is $m = 1$, the image tangent vectors must be linearly dependent, in which case we can deduce as above that factorisation will be satisfied indepedently of the choice of alignment. □

For the remainder of this section we use the following notation for the combinatorial type Δ; we let L_0 denotes the vertex of dual graph corresponding to the core, $\mathsf{L}_1, \ldots, \mathsf{L}_r$ the other stable vertices (genus zero and lying on the circle, by the previous lemma) and $q_1, \ldots, q_r$ the corresponding splitting nodes.

4.3.2. *Recursive description of $\mathcal{E}$.* Recall that we have maps

$$\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d) \hookrightarrow \widetilde{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N|H, d) \to \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d).$$

The first morphism is a closed embedding cut out by the two factorisation properties. The second morphism is a logarithmic modification. Let $\tau \in \Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$ be the ray corresponding to $\mathcal{D} \subseteq \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$, and to the virtual divisor $\widetilde{\mathcal{D}} \subseteq \widetilde{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N|H, d)$.

Let $\sigma \in \Sigma^{\mathrm{Kim}}$ be the minimal cone containing the image of τ. The dimension of this cone could be large – its dimension is equal to the the number of levels of the target expansion in the type Δ. The cone σ determines a stratum $\mathcal{E} \subseteq \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d)$, and the morphism above restricts to

$$\mathcal{D} \to \mathcal{E}.$$

Splitting in the ordinary geometry. The locus $\mathcal{E}$ has the following description. Let Λ_0 for the collection of stable vertices of the combinatorial type which are mapped to $0 \in \mathbb{R}_{\geq 0}$, and $\Lambda_{>0}$ for the stable vertices which are mapped to $\mathbb{R}_{>0}$. Each vertex v has associated discrete data Γ_v. The vertices over 0 determine moduli spaces of ordinary relative stable maps, while the higher level vertices determine maps to rubber. Then $\mathcal{E}$ admits a finite and surjective splitting morphism

$$(2) \qquad \mathcal{E} \xrightarrow{\rho} \left(\prod_{v \in \Lambda_0} \overline{\mathcal{M}}_{\Gamma_v}(\mathbb{P}^N|H) \times \prod_{v \in \Lambda_{>0}} \overline{\mathcal{M}}^{\leftrightarrow}_{\Gamma_v}(\mathbb{P}|H_0 + H_\infty) \right) \times_{H^{2\epsilon}} H^\epsilon$$

where ϵ is the number of edges of the stabilised dual graph, and the fibre product is taken over the appropriate evaluation maps. Splitting the diagonal of $H^{2\epsilon}$ we see that tautological integrals over $\mathcal{E}$ are completely determined in terms of integrals over moduli spaces with "smaller" combinatorial data.

Exportation to the ordinary geometry. Observe that we have a virtual divisor $\widetilde{\mathcal{D}} \subseteq \widetilde{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N|H, d)$ such that

$$\mathcal{D} = \widetilde{\mathcal{D}} \cap \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d).$$

We wish to use the splitting of $\mathcal{E}$ in the ordinary geometry to recursively compute integrals in the reduced geometry. Noting that $\mathcal{D}$ is cut out of the excess dimensional space $\widetilde{\mathcal{D}}$ by the factorisation properties, we make the reduction in two steps:

(1) in §4.3.3 we explain how to construct $\widetilde{\mathcal{D}}$ from $\mathcal{E}$;
(2) in §4.3.4 we explain how to express the class of $\mathcal{D} \subseteq \widetilde{\mathcal{D}}$ in terms of tautological classes.

To compute integrals over $\mathcal{D}$ we simply push down the appropriate classes from $\widetilde{\mathcal{D}}$ to $\mathcal{E}$ and then integrate over $\mathcal{E}$ using the ordinary splitting axiom.

Remark 4.11 (Splitting degree and vanishing order, revisited). Similarly to §4.2.4 above, we can calculate the degree of ρ and the vanishing order $\lambda_{\mathcal{D}}$ of the section s_k along $\mathcal{D}$. The degree of ρ is given exactly as in Lemma 4.8, where now we obtain one such factor for each bounded edge of the tropical target. The vanishing order $\lambda_{\mathcal{D}}$ is slightly more delicate, but may be computed as the index of an explicit morphism of one-dimensional lattices, as explained in the proof of Lemma 4.9.

4.3.3. *Compactified bundles over strata: $\widetilde{\mathcal{D}}$ from $\mathcal{E}$.* Recall that the map $\widetilde{\mathcal{D}} \to \mathcal{E}$ is obtained by restricting the logarithmic blow-up

$$\widetilde{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N | H, d) \to \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N | H, d)$$

to the locus $\widetilde{\mathcal{D}}$. Toroidal geometry gives a construction of $\widetilde{\mathcal{D}}$ as a compactification of a torus bundle over a locally closed stratum in $\overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N | H, d)$.

Logarithmic structures on strata. Let $W \subseteq X$ be an irreducible stratum in a toroidal embedding X. The stratum itself is toroidal, and hence logarithmically smooth. If X and W are merely logarithmic and not toroidal, there is nonetheless a strict map

$$X \to \mathcal{A}_X,$$

where $\mathcal{A}_X$ is the Artin fan of X, which is a toroidal Artin stack. If $W \subseteq X$ maps to a stratum $\mathcal{A}_W$ under this map, the pullback of the logarithmic structure from $\mathcal{A}_W$ to W endows the stratum with a logarithmic structure. Note that the inclusion of a stratum $W \hookrightarrow X$ is *not* a logarithmic morphism.

Equipped with the strata logarithmic structures, the morphism

$$\widetilde{\mathcal{D}} \to \mathcal{E}$$

is a logarithmic morphism. The generic fibre of this morphism is a toric variety.

Flattening the map to $\mathcal{E}$. The morphism $\widetilde{\mathcal{D}} \to \mathcal{E}$ is not flat. In order to push integrals forward, it will be convenient to flatten the map by blowing up $\mathcal{E}$. This can be done in a universal manner, following [AK00, Mol16].

The combinatorics of the flattening can be described explicitly. Consider the map on tropicalizations

$$\widetilde{\mathcal{D}}^{\mathrm{trop}} \to \mathcal{E}^{\mathrm{trop}}.$$

The tropical curves on the left inherit a central alignment from the moduli space $\widetilde{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N | H, d)$, which orders a certain subset of the edge lengths. The polyhedral criterion for equidimensionality is that every cone of the source maps surjectively onto a cone of the target. There is a universal logarithmic modification $\widetilde{\mathcal{E}} \to \mathcal{E}$ such that $\widetilde{\mathcal{D}} \to \widetilde{\mathcal{E}}$ is flat, constructed by taking the minimal subdivison of $\mathcal{E}^{\mathrm{trop}}$ such that cones map surjectively to cones.

This subdivision is best described using the formalism of extended tropicalizations []. The subdividing cones are obtained from cones in the star of τ under the limit $e_1 = \cdots = e_r = \infty$. The cones in the star of τ are given by (in)equalities relating distances of pairs of vertices, and there are three cases to consider:

   (1) both vertices are components of the subgraph $\mathsf{L} \setminus \mathsf{L}_0$;
   (2) both vertices are components of the subgraph L_0;

(3) one vertex is a component of Γ_0 and one is a component of $\Gamma \setminus \Gamma_0$.

As we pass to the limit, the first relations impose a central alignment condition on the genus zero moduli, the second relations impose a radial alignment condition on the genus one moduli, and the third relations disappear entirely. We are therefore led to the following moduli theoretic description of the logarithmic modification $\widetilde{\mathcal{E}}$.

Keeping in mind the fibre product description of $\mathcal{E}$, construct a modification $\widetilde{\mathcal{E}}$ from $\mathcal{E}$ as follows:

(1) Given a logarithmic map of type Δ with splitting nodes $q_1, \ldots, q_r$, remove the genus one piece Γ_0 and formally glue together the r vertices of the tropicalization which carry the splitting nodes, and declare the resulting vertex to be the root of the resulting tree. Impose the central alignment condition, where the special root takes on the rôle previously occupied by the core.

(2) Replace the fibre product factor in $\mathcal{E}$ associated to the vertex Γ_0

$$\overline{\mathcal{M}}^{\leftrightarrow}_{\Gamma_{\Gamma_0}}(\mathbb{P}|H_0 + H_\infty)$$

with the radially aligned rubber space:

$$\mathcal{VZ}^{\leftrightarrow}_{\Gamma_{\Gamma_0}}(\mathbb{P}|H_0 + H_\infty).$$

(Here we are imposing fibre-direction factorisation in order to isolate the main component.)

This procedure induces a logarithmic modification $\widetilde{\mathcal{E}} \to \mathcal{E}$ which admits a modular description as a fibre product of genus zero centrally aligned maps coming from the vertices $\Gamma_1, \ldots, \Gamma_r$, and genus one aligned rubber maps satisfying fibre-direction factorisation from the vertex Γ_0.

By construction, we have a map $\widetilde{\mathcal{D}} \to \widetilde{\mathcal{E}}$ which is a compactification of a torus bundle over $\widetilde{\mathcal{E}}$. The strata of $\widetilde{\mathcal{D}}$ can be read off from its tropicalization, and in particular we have the information of how (polynomials in) the classes of boundary strata push forward to $\widetilde{\mathcal{E}}$.

We have produced a factorization

$$\widetilde{\mathcal{D}} \to \widetilde{\mathcal{E}} \to \mathcal{E},$$

where the first map is flat and the second map is a logarithmic modification. $\widetilde{\mathcal{D}}$ is a compactification of a torus bundle over $\widetilde{\mathcal{E}}$ whose strata can be read off from the tropicalization; in particular, we have the information of how (polynomials in) the classes of boundary strata push forward to $\widetilde{\mathcal{E}}$. In fact, we can say more and describe the generic fibre of $\widetilde{\mathcal{D}} \to \widetilde{\mathcal{E}}$:

Lemma 4.12. Generically over $\mathcal{E}$, $\widetilde{\mathcal{D}} \to \widetilde{\mathcal{E}}$ is the projectivisation of a tautological vector bundle.

Proof. Let us assume that the curve does not degenerate, so that the dual graph is as prescribed by Δ. In particular there are components $\Gamma_1, \ldots, \Gamma_r$ generically lying on the circle of radius δ. The moduli space of attaching data for the elliptic singularity is the complement of the coordinate hyperplanes in

$$\mathbb{P}\left(\bigoplus_{i=1}^{r} T_{q_i} C_i\right).$$

FIGURE 4. Moving vertices off the circle creates new tropical parameters.

On the boundary, the circle is allowed to miss some of the L_i – but not all of them! – cutting the edge between them and the core (correspondingly, there is a destabilisation of the curve, and an elliptic singularity containing strictly semistable branches). Tropically, pushing some of the vertices off the circle creates new finite-length edges with coordinates $\ell_i \in \mathbb{R}_{\geq 0}$ (see Figure 4). Since not all of them can be strictly positive, this represents the fan of a projective space.

A slight modification of the argument above is required to account for the fact that all vertices that lie on the circle and are mapped to $0 \in \mathbb{R}_{\geq 0}$ by the tropical map are generically aligned on $\mathcal{E}$: indeed, by Lemma 4.10, they all have the same contact order m, therefore their $T_{q_i} C_i^{\otimes m}$ can be identified via the target tangent line on the fibre product (2), and, after saturation, the $T_{q_i} C_i$ themselves are canonically isomorphic on $\mathcal{E}$. Thus, only one summand should be taken to represent all of them in the projective bundle above. □

4.3.4. $\mathcal{D}$ *from* $\widetilde{\mathcal{D}}$*: first way.* On $\widetilde{\mathcal{D}}$ there is a tautological line bundle arising from the modular description of $\widetilde{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N | H, d)$, namely $\mathcal{O}(\delta)$.

Lemma 4.13. The fibre of $\mathcal{O}(\delta)$ over a moduli point is naturally isomorphic to

$$T_p C' \otimes T C_0$$

where C' is any component of C which lies on the circle, p is the adjacent node which points towards the core, and $T C_0$ is the universal tangent line bundle on the moduli space for L_0.

Proof. Choose a component C' of the curve which lies on the circle. Then $\mathcal{O}(\delta)$ is equal to the tensor product of tangent line bundles along the path of edges connecting C' to the core. However, this tensor product of line bundles is telescoping: any vertex which appears in the middle of the path will have an incoming and outgoing node, and since the vertex corresponds to a smooth rational curve these tangent spaces are naturally dual to each other [VZ08, §2.2]. Thus these factors cancel, and the result follows. □

The content of Lemma 4.13 is that the choice of point in the fibre of $\widetilde{\mathcal{D}} \to \widetilde{\mathcal{E}}$ provides a natural identification between the inward-pointing tangent spaces on the circle. This identification amounts to a specific linear dependence relation between the tangent vectors, which gives a point in the moduli space of attaching data for the corresponding elliptic singularity [Smy11b, §2.2]. We may therefore express the factorisation condition in terms of the following natural morphism of vector bundles on $\widetilde{\mathcal{D}}$:

$$\Sigma \, \mathrm{d}f \colon \mathcal{O}(\delta) \otimes T C_0^\vee \to \mathrm{ev}_q^\star T H.$$

Here $\Sigma \, \mathrm{d}f$ is defined by summing the images of the tangent vectors at the inward-pointing nodes on the circle, using the natural identification of each of these tangent spaces with $\mathcal{O}(\delta) \otimes T C_0^\vee$ established above. By Lemma 4.10 we have that it takes values in $T H$, rather than $T \mathbb{P}^N$. This gives a section of the vector bundle

$$V = (\mathrm{ev}_q^\star T H) \otimes (T C_0) \otimes \mathcal{O}(-\delta)$$

which is transverse and whose vanishing locus coincides with $\mathcal{D}$. Thus we obtain:

Proposition 4.14. $[\mathcal{D}] = \mathrm{e}(V) \cap [\widetilde{\mathcal{D}}]$.

We claim that the class $\mathrm{e}(V)$ is tautological and computable. The only part which needs justification is $\mathcal{O}(\delta)$. We claim that this can be obtained by pulling back a line bundle from $\mathcal{E}$ and twisting by exceptional divisors.

Lemma 4.15. For $i \in \{1, \ldots, r\}$ consider the line bundle

$$T_i = T_{q_i} C_i \otimes T_{q_i} C_0$$

on $\mathcal{E}$, where q_i is the splitting node. Then $\mathcal{O}(\delta)$ is isomorphic to the pull-back of T_i to $\widetilde{\mathcal{D}}$ twisted by all exceptional divisors.

We illustrate this lemma in the following example, which brings out the interplay between logarithmic moduli and blow-ups.

Example 4.16. Consider the locus $\mathcal{E}$ given by σ below, and the exceptional divisor $\widetilde{\mathcal{D}}_\rho \subseteq \widetilde{\mathcal{D}}$ given by ρ:

On the cone ρ we have the following tropical continuity equations

$$e := e_1 = e_2, \qquad f := f_1 = f_2 = 2f_3, \qquad g := g_1 = g_2 = 2g_3$$

together with the equation imposed by the choice of alignment:

$$e_1 + f_1 = e_2 + f_3 + g_3 \qquad (\Longleftrightarrow f = g).$$

From now on we use the same symbol to denote an edge length and the corresponding node on $\widetilde{\mathcal{D}}_\rho$. On this locus we have line bundles pulled back from $\mathcal{E}$
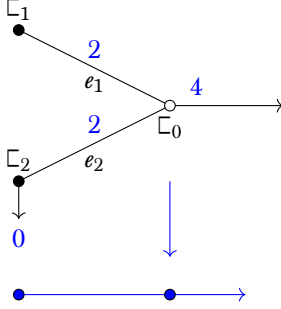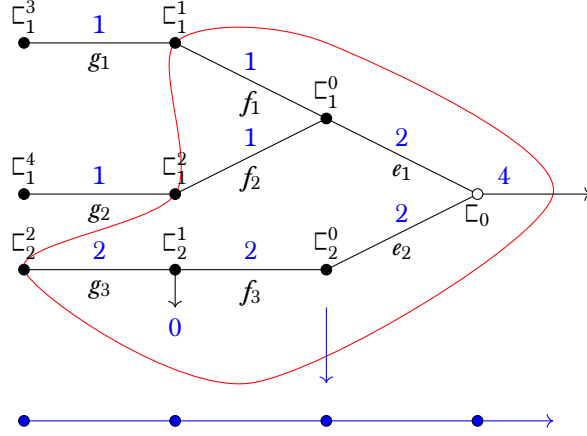
$$T_1 = T_{e_1} C_1^0 \otimes T_{e_1} C_0$$
$$T_2 = T_{e_2} C_2^0 \otimes T_{e_2} C_0$$

as well as the logarithmic line bundle

$$\mathcal{O}(\delta) = T_{f_1} C_1^1 \otimes T_{e_1} C_0 = T_{f_2} C_1^2 \otimes T_{e_1} C_0 = T_{g_3} C_2^2 \otimes T_{e_2} C_0$$

where the identifications follow from Lemma 4.13. We claim that:

(3) $$\mathcal{O}(\delta) = T_1 \otimes \mathcal{O}(\widetilde{\mathcal{D}}_\rho) = T_2 \otimes \mathcal{O}(\widetilde{\mathcal{D}}_\rho).$$

FIGURE 5. σ

FIGURE 6. ρ

Notice that $\mathcal{O}(\widetilde{\mathcal{D}}_\rho) = \mathcal{O}(f) = \mathcal{O}(g)$. Thus we have

$$
\begin{aligned}
T_1 \otimes \mathcal{O}(\widetilde{\mathcal{D}}_\rho) &= \mathcal{O}(f_1) \otimes T_{e_1} C_1^0 \otimes T_{e_1} C_0 \\
&= T_{f_1} C_1^1 \otimes T_{f_1} C_1^0 \otimes T_{e_1} C_1^0 \otimes T_{e_1} C_0 \\
&= T_{f_1} C_1^1 \otimes T_{e_1} C_0 \qquad (= \mathcal{O}(\delta))
\end{aligned}
$$

where in the last line we have used the same telescoping trick used in the proof of Lemma 4.13, namely that $T_{f_1} C_1^0 \otimes T_{e_1} C_1^0 = \mathcal{O}$. On the other hand, for T_2 we may write $f = f/2 + g/2 = f_3 + g_3$ in order to obtain:

$$
T_2 \otimes \mathcal{O}(\widetilde{\mathcal{D}}_\rho) = \mathcal{O}(g_3) \otimes \mathcal{O}(f_3) \otimes T_{e_2} C_2^0 \otimes T_{e_2} C_0 = T_{g_3} C_2^2 \otimes T_{e_2} C_0 = \mathcal{O}(\delta)
$$

where we have used the same telescoping trick. Thus we have proven (3). This observation generalises, giving a precise relation between the intrinsically-defined line bundle $\mathcal{O}(\delta)$ on $\widetilde{\mathcal{D}}$ (which we use to impose the factorisation condition) and the the line bundles T_i pulled back from $\mathcal{E}$. This allows us to express the Chern classes of $\mathcal{O}(\delta)$ (and hence V) in terms of pull-backs of tautological classes from $\mathcal{E}$ and boundary strata on $\widetilde{\mathcal{D}}$, which is sufficient in order to compute integrals.

4.3.5. $\mathcal{D}$ *from* $\widetilde{\mathcal{D}}$: *second way.* To establish a more direct connection with the geometry of the fibre product (2), recall that $\widetilde{\mathcal{E}}$ is built as an explicit modification of a finite cover of (2). For $i \in \{1, \dots, r\}$ (with the convention established at the end of Lemma 4.12), it makes sense on $\widetilde{\mathcal{E}}$ to consider the line bundle $\mathcal{O}(\delta_i)$ associated to a shortest path from the core to the circle in the L_i-direction. Construct

$$
\mathcal{P} = \mathbb{P}_{\widetilde{\mathcal{E}}} \left(\bigoplus_{i=1}^r \mathcal{O}(\delta_i) \right).
$$

Since $\mathsf{L}_1, \dots, \mathsf{L}_r$ are always ordered on $\widetilde{\mathcal{D}}$ (they may lie on the circle, be the first component off the circle, or degenerate and lie in the strict interior), there is a morphism $\widetilde{\mathcal{D}} \to \mathcal{P}$

lifting $\widetilde{\mathcal{D}} \to \widetilde{\mathcal{E}}$. Consider the map of vector bundles on $\mathcal{P}$:

$$\mathcal{O}_{\mathcal{P}}(-1) \to p^* \left(\bigoplus_{i=1}^{r} \mathcal{O}(\delta_i) \right) \xrightarrow{\sum \mathrm{d}f} \mathrm{ev}_q^* TH.$$

The composite, regarded as a section s of $W = \mathcal{O}_{\mathcal{P}}(1) \otimes \mathrm{ev}_q^*(TH)$, vanishes precisely where f factors through the elliptic singularity determined by the point in the fibre of p. Generically on $\widetilde{\mathcal{E}}$, this is exactly what we are after (compare with Lemma 4.12). On the boundary, though, it may happen that a simultaneous degeneration in the base and the fibre makes this factorisation trivial, in the sense that the map $\bar{f}$ is constant on all the branches of the elliptic singularity.

Lemma 4.17. For every $\Lambda_B \subseteq \{\mathsf{C}_1, \ldots, \mathsf{C}_r\}$ and $\Lambda_F = \{\mathsf{C}_1, \ldots, \mathsf{C}_r\} \setminus \Lambda_B$, s vanishes along the loci where the factors corresponding to Λ_B in $\widetilde{\mathcal{E}}$ degenerate so that $f_i \colon C_i \to \mathbb{P}^N$ is constant in a neighbourhood of q_i, and the fibre coordinates $(\lambda_j)_{j \in \Lambda_F}$ along $p \colon \mathcal{P} \to \widetilde{\mathcal{E}}$ are set to be 0.

Call these loci Ξ_Λ for $\Lambda = (\Lambda_B, \Lambda_F)$ (compare with [VZ08, §3.2] for an analogous phenomenon in the non-relative case); clearly, they may be in excess with respect to the expected codimension of $V(s)$, that is $N - 1$. These loci are logarithmic substrata; therefore, there exists a logarithmic blow-up that principalizes them. We claim which $\widetilde{\mathcal{D}}$ is (possibly a refinement of) such a blow-up.

Lemma 4.18. The pullback of Ξ_Λ is principal in $\widetilde{\mathcal{D}}$.

Proof. We analyse the ideal of Ξ_Λ: it is generated by

(1) equations on $\widetilde{\mathcal{E}}$ cutting the locus where C_i degenerates so that f_i is constant on a neighbourhood of q_i for $i \in \Lambda_B$: these correspond to the tropical functions $\delta_i - \lambda(q_i)$ where λ is the distance function to the core;

(2) equations on the fibre of p for the linear subspace corresponding to $i \in \Lambda_F$: tropically, these correspond to the ℓ_i discussed in Lemma 4.12.

By construction, on $\widetilde{\mathcal{D}}$ it is always possible to tell which of these tropical lengths is the shortest (the circle we have drawn on Ξ_Λ is degenerate – it passes through no vertex of positive horizontal degree – therefore we need to decide for a strictly larger circle in order to lift to $\widetilde{D}$), which corresponds to principalising the ideal (see Figure 7). $\qquad \square$

Finally, on $\widetilde{\mathcal{D}}$ (or, in fact, on an intermediate blow-up $\widetilde{\mathcal{P}} \to \mathcal{P}$) there are exceptional divisors $\widetilde{\Xi}_\Lambda$ corresponding to Ξ_Λ in $\mathcal{P}$. Since the Ξ_Λ exhaust all the excess-dimensional components of the vanishing locus of $s \in H^0(\mathcal{P}, W)$, the latter induces a section $\tilde{s} \in H^0(\widetilde{\mathcal{D}}, W(-\Sigma_\Lambda \widetilde{\Xi}_\Lambda))$ by pullback and twisting, whose vanishing locus is dimensionally transverse to the boundary, and whose open part coincides with the generic point of $\mathcal{D}$, therefore $V(\tilde{s}) = \mathcal{D}$.

Expressing the pushforward of the Chern classes of $\mathcal{O}_W(1)$ and $\mathcal{O}(\widetilde{\Xi}_\Lambda)$ in terms of tautological classes on the fibre product underlying $\mathcal{E}$, we get:

Corollary 4.19. After pushforward to $\mathcal{E}$, the contribution of $\mathcal{D}$ to the invariants can be computed in terms of tautological integrals on moduli spaces of maps with lower numerics.
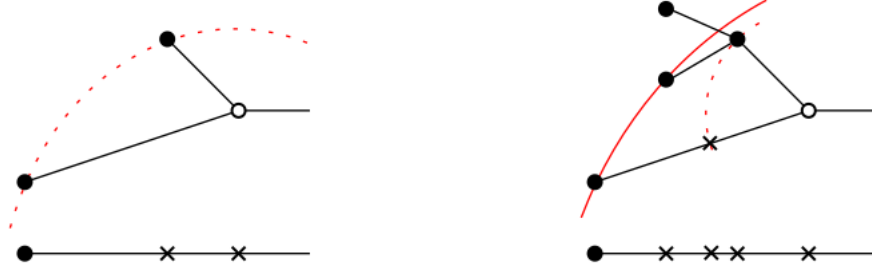
FIGURE 7. On the left, a generic point of $\mathcal{P}$. On the right, a point of Ξ. The dotted circle is the one we draw on $\mathcal{P}$. The continuous circle represents one of the possible choices of δ enlarging the dotted one (a stratum of $\widetilde{\mathcal{D}}$).

4.4. **Recursion for general** (X, Y). Now let (X, Y) be a general smooth very ample pair. In §1.7 we studied the moduli space $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$ of centrally aligned maps to (X, Y) and equipped it with a virtual fundamental class by diagonal pull-back from $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$.

This construction of the virtual fundamental class means that once we have established a recursion formula for $(\mathbb{P}^N, H)$, it immediately pulls back to give a recursion formula for (X, Y). To be more precise, Theorem 3.2 holds *mutatis mutandis*, with the same values of $\lambda_{\mathcal{D}}$. The computation of the splitting orders is exactly the same, and the recursive description of the boundary strata pulls back to give an entirely analogous description on $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$.

5. QUANTUM LEFSCHETZ ALGORITHM

Consider a smooth pair (X, Y) with Y very ample, and let $\mathbb{P}$ be the projective bundle $\mathbb{P} = \mathbb{P}_Y(\mathrm{N}_{Y|X} \oplus \mathcal{O}_Y)$. We assume that the genus zero and reduced genus one Gromov–Witten theories of X are known. From this starting data, we will apply our recursion formula to compute:

(1) the genus one **reduced restricted absolute Gromov–Witten theory** of Y;
(2) the genus one **reduced relative Gromov–Witten theory** of the pair (X, Y);
(3) the genus one **reduced rubber theory** of $\mathbb{P}$.

5.1. **Reduced absolute, relative and rubber invariants**. To be precise: by a reduced invariant of Y we mean an integral over $\mathcal{VZ}_{1,n}(Y, \beta)$ of products of pullbacks of evaluation and psi classes along morphisms which forget a subset S of the marked points (taking $S = \emptyset$ gives the ordinary evaluation and psi classes). Here the evaluation maps are viewed as mapping into X (hence the adjective "restricted"). Reduced relative invariants of (X, Y) are defined in the same way, except now the forgetful morphism maps into a space of absolute maps:

$$\mathrm{fgt}_S \colon \mathcal{VZ}_{1,\alpha}(X|Y, \beta) \to \mathcal{VZ}_{1,m-\#S}(X, \beta).$$

In particular, all the psi classes which we consider are **collapsed psi classes**, meaning that they are relative cotangent line classes for the corresponding collapsed stable map. Note that, unlike in the absolute case, in the relative case it may well be the case that the entire insertion is pulled back along a single forgetful map fgt_S. The reduced rubber invariants of $\mathbb{P}$ are defined similarly (again using collapsed psi classes).

The systems of invariants defined above are equivalent to the classical systems of invariants (which do not use forgetful morphisms) by well-known topological recursion relations.

5.2. **Fictitious and true markings**. The recursion procedure is rather delicate. Roughly speaking, we will induct on the degree (meaning $Y \cdot \beta$), number of marked points and total tangency. To get the correct notion of number of markings and total tangency in the relative setting, we introduce the concept of **fictitious markings**. Consider a moduli space $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$ of reduced relative stable maps and a corresponding integrand γ. We let $F \subseteq \{1, \ldots, m\}$ be the maximal subset of marked points such that:

(1) $\alpha_i = 1$ for all $i \in F$;
(2) the entire integrand γ is pulled back along fgt_F.

This subset is uniquely determined, and its elements are referred to as **fictitious markings**. Markings which are not fictitious are referred to as **true**. When inducting on relative invariants we will always be interested in the number of true markings (as opposed to the total number of markings) and the true tangency

$$\sum_{i \notin F} \alpha_i \leq d = Y \cdot \beta$$

as opposed to the total tangency, which is always d. This formalises the idea that relative invariants with non-maximal tangency $t < d$ can be obtained by adding $d - t$ fictitious markings of tangency 1; see [Gat02, Lemma 1.15(i)].

5.3. **Structure of the recursion**. Given the genus-zero Gromov–Witten theory of X, the arguments of [Gat02] give an effective algorithm to reconstruct the genus-zero theories of Y and (X, Y); moreover, the genus-zero rubber theory of $\mathbb{P}$ is identical to the genus-zero theory of Y [Gat03]. Thus we may assume that all genus-zero data is known. We assume in addition that we know the genus one reduced theory of X. The structure of the recursion is then as follows:

---

for $d \geq 0$:
    for $n \geq 0$:
        for $t \geq 0$:
            Step 1: Compute forgetful relative invariants of (X, Y) (degree d, $n + 1$ true markings, true tangency t); see below.

        Step 2: Compute absolute invariants of Y (degree d, n markings).
        for $t \geq 0$:
            Step 3: Compute relative invariants of (X, Y) (degree d, n true markings, true tangency t).
    for $n \geq 0$:
        for $m \geq 0$:
            Step 4: Compute rubber invariants of $\mathbb{P}$ (degree d, n relative markings, m non-relative markings).

Although the loops involving d, n and m have infinite length, in order to compute any single invariant it is only necessarily to iterate the preceding loops for a finite amount of time. A **forgetful relative invariant** of (X, Y) is by definition a reduced relative invariant with a marked point x_0 such that all of the insertions are pulled back along fgt_{x_0}. In our recursion, we first deal with this special class of relative invariants (with $n + 1$ true markings), before computing the absolute invariants (with n markings) and then returning to compute all of the relative invariants (with n true markings). This need to treat separately a particular subclass of the relative invariants is an inescapable feature of the genus one recursion.

The base terms of the recursion all have $d = 0$ and as such are easy to compute: the relative invariants of (X, Y) are nothing but absolute invariants of X and the absolute invariants of Y are given by obstruction bundle integrals over Deligne–Mumford spaces. The rubber invariants of $\mathbb{P}$ in degree zero are given by integrals over the main component of the double ramification cycle; this can likewise be computed as an integral over Deligne–Mumford space, by correcting the formula for the ordinary double ramification cycle [Hai13, JPPZ17] by the obvious contribution from the boundary irreducible component (the observation which makes this easy to do in genus one is that the rubber moduli space is pure-dimensional, so that the virtual class is simply the sum of the fundamental classes of the irreducible components). We will now explain how to perform each of the four inductive steps outlined above.

Notation 5.1. Given tuples $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, we say that $\mathbf{a} < \mathbf{b}$ if there exists an $i \in \{1, \ldots, n\}$ such that $a_j = b_j$ for $j < i$ and $a_i < b_i$.

Step 1. Suppose we are given a forgetful relative invariant to compute. That is, we have a relative space $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$ of degree d, with $n+1$ true markings and true tangency t, and a marking x_0 such that the insertion γ is pulled back along fgt_{x_0}. We assume inductively that every absolute, relative and rubber invariant with $(d', n') < (d, n)$ is known, and also that every forgetful relative invariant with $(d', n', t') < (d, n + 1, t)$ is known. Choose a true marking x_1 with $\alpha_1 \geq 1$ and consider the space:

$$\mathcal{VZ}_{1,(\alpha - e_1) \cup (1)}(X|Y, \beta).$$

Denote the newly-introduced marking by y and consider the integrand $\tilde{\gamma}$ obtained from γ by introducing $\mathrm{fgt}_y^\star$ everywhere. Applying our recursion formula to x_1, we obtain:
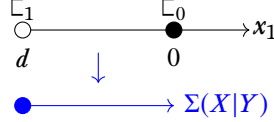
$$(4) \qquad \left((\alpha_1 - 1)\psi_1 + \mathrm{ev}_1^\star Y\right) \tilde{\gamma} \cap [\mathcal{VZ}_{1,(\alpha - e_1) \cup (1)}(X|Y, \beta)] = \tilde{\gamma} \cap [\mathcal{D}(1)].$$

Let us first examine the left-hand side. The class ψ_1 differs from $\mathrm{fgt}_y^\star \psi_1$ by the locus where x_1 and y are contained on a contracted rational bubble. This locus consists of reduced relative stable maps of the form

with all other marked points contained on $\llcorner_1$. This is isomorphic to $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$ and when we restrict $\tilde{\gamma}$ to this locus we obtain precisely the class γ which we started with. Thus the left-hand side of (4) may be written as

$$(\alpha_1 - 1)I + \mathrm{fgt}_y^\star \left((\alpha_1 - 1)\psi_1 + \mathrm{ev}_1^\star Y\right) \tilde{\gamma} \cap [\mathcal{VZ}_{1,(\alpha - e_1) \cup (1)}(X|Y, \beta)]$$

where I is the invariant we are trying to compute. The second term is a forgetful relative invariant with the same degree and number of true markings (since y is fictitious), and strictly smaller true tangency; hence it is recursively known. We now examine the right-hand side of (4). Recall that all of the genus zero data has already been computed, so we only need to focus on the genus one pieces. First consider the type I loci. The genus one piece has strictly smaller degree (and hence is recursively known) exept in the following situation (with some stable distribution of the remaining markings):

$$
\begin{array}{ccc}
\overset{\llcorner_1}{\underset{d}{\circ}} & \overset{\llcorner_0}{\underset{0}{\bullet}} & \!\!\!\!\to x_1 \\[2pt]
& \downarrow & \\[2pt]
\bullet & \longrightarrow & \Sigma(X|Y)
\end{array}
$$

In this situation, C_1 contains at most $n+1$ true markings. If it has $n-1$ or fewer, than it is known recursively. If it has exactly n this means that C_0 contains exactly one true marking (besides x_1, which may be true or fictitious). We claim that in this situation we must have $y \in C_1$, since otherwise we would have a moduli space for C_0 given by $\overline{\mathcal{M}}_{0,k}$ with $k \geq 4$, and applying the projection formula with fgt_y we would conclude that the contribution is zero. Thus we have $y \in C_1$, and so the genus one contribution is a forgetful relative invariant with n true markings, and hence is recursively known.

Finally, if C_1 contains exactly $n+1$ true markings, then the only additional markings on C_0 are fictitious, and by the same argument as in the previous paragraph there can only be one. Thus for each fictitious marked point we obtain a locus isomorphic to $\mathcal{VZ}_{1,\alpha}(X|Y,\beta)$ and $\tilde\gamma$ restricts to γ here (from the point of view of computing invariants, the fictitious marked points are indistinguishable, meaning that the contributions are all the same). Thus for each fictitious marked point (of which there is at least one, namely y) we get a contribution of $\alpha_1 I$ to the right-hand side of (4).

The contributions of the type II loci only involve genus zero data and hence are known. The contributions of the type † loci are determined by genus zero data and tautological integrals on Deligne–Mumford space, hence are also known. It remains to consider type III loci. If the degree of the genus one piece is less than d then we have a rubber invariant of strictly smaller degree. The only other possibility is that the entire curve is mapped into the divisor. In this case we may apply the projection formula with fgt_y to identify this with an integral over $\mathcal{VZ}_{1,m}(Y,\beta)$ for some (possibly large) number m of marked points. But by assumption there is another marked point x_0 with all of the insertions pulled back along fgt_{x_0}, so a further application of the projection formula shows that this contribution vanishes. To conclude, we may rearrange (4) to obtain an expression of the form

$$\lambda I = \text{recursively known terms}$$

where λ is an explicit scalar which is always nonzero; we have thus determined I, which completes Step 1.

Step 2. Consider now the absolute space $\mathcal{VZ}_{1,n}(Y,\beta)$ with an insertion γ, and suppose inductively that we have computed all forgetful relative invariants with $(d',n') \leq (d,n+1)$, all relative invariants and absolute invariants with $(d',n') < (d,n)$, and all special rubber invariants with $d' < d$. Consider the following moduli space with $n+1$ markings:

$$\mathcal{VZ}_{1,(d,0,\dots,0)}(X|Y,\beta).$$

Let x_0 denote the relative marking and consider the integrand $\tilde{\gamma}$ obtained from γ by introducing $\mathrm{fgt}^{\star}_{x_0}$ everywhere. Now recurse at x_1 to obtain:

$$(5) \qquad \mathrm{ev}^{\star}_1 Y \cdot \tilde{\gamma} \cap [\mathcal{VZ}_{1,(d,0,\ldots,0)}(X|Y,\beta)] = \tilde{\gamma} \cap [\mathcal{D}(1)].$$

The left-hand side is a forgetful relative invariant of degree d and $\leq n+1$ true markings, and so has already been computed. For the right-hand side, let us begin with loci of type I. The genus one contributions from each loci have $(d',n') < (d,n)$ except in the following case
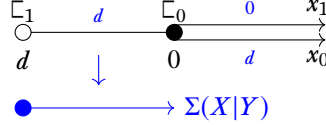
which gives a contribution of

$$d \cdot \gamma \cap [\mathcal{VZ}_{1,(d,\underbrace{0,\ldots,0}_{n-1})}(X|Y,\beta)]^3$$

to the right-hand side of (5). Here γ is the insertion we started with; the difference is that we are now considering relative maps to (X,Y) with maximal tangency at x_1, rather than absolute maps to Y. The type II and $\dagger$ loci are recursively determined as in Step 1, and similarly the type III loci are recursively determined except for the locus where the entire curve is mapped into the divisor. On this locus we may apply fgt_{x_0} and identify the contribution with

$$d^2 \cdot \gamma \cap [\mathcal{VZ}_{1,n}(Y,\beta)] = d^2 \cdot I$$

where I is the invariant we are trying to compute. Putting this all together, we obtain

$$(6) \qquad I = (-\gamma/d) \cap [\mathcal{VZ}_{1,(d,\underbrace{0,\ldots,0}_{n-1})}(X|Y,\beta)] + \text{recursively known terms}$$

where on the right-hand side there are $n-1$ non-relative markings $x_2,\ldots,x_n$, and a relative marking x_1. We now apply the recursion again to the right-hand side, by considering the space

$$\mathcal{VZ}_{1,(d-1,1,0,\ldots,0)}(X|Y,\beta)$$

where x_1 now has tangency $d-1$ and we have introduced a new marking y with tangency 1. We consider a new insertion, denoted $\tilde{\gamma}$ as usual, by introducing $\mathrm{fgt}^{\star}_y$ everywhere. Recursing at x_1 we obtain:

$$(7) \qquad ((d-1)\psi_1 + \mathrm{ev}^{\star}_1 Y) \cdot (-\tilde{\gamma}/d) \cap [\mathcal{VZ}_{1,(d-1,1,0,\ldots,0)}(X|Y,\beta)] = (-\tilde{\gamma}/d) \cap [\mathcal{D}(1)].$$

The difference between ψ_1 and $\mathrm{fgt}^{\star}_y \psi_1$ is given by the locus where x_1 and y belong to a collapsed rational bubble:
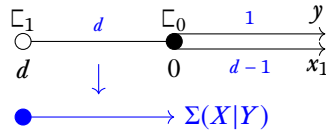
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The contribution of this locus to the left-hand side of (7) is:

$$(d-1) \cdot (-\gamma/d) \cap [\mathcal{VZ}_{1,(d,\underbrace{0,\ldots,0}_{n-1})}(X|Y,\beta)].$$

What remains on the left-hand side is a forgetful relative invariant of degree d and $\leq n+1$ true markings (y being the "forgetful" marking), hence is recursively known. On the right-hand side, the type I loci are recursively known except possibly in the following special cases (with some stable distribution of the remaining non-relative markings):

Case 1   Case 2
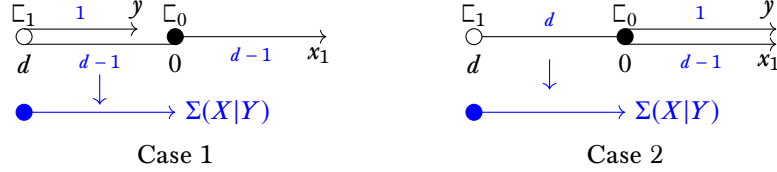
In Case 1 the contribution from C_1 is a forgetful relative invariant with $\leq n$ true markings, hence is recursively known. In Case 2, we first note that there cannot be any more markings on C_0 (since otherwise we could apply fgt_y and conclude that the contribution vanishes). Thus we obtain a single locus, which contributes precisely

$$d \cdot (-\gamma/d) \cap [\mathcal{VZ}_{1,(d,\underbrace{0,\ldots,0}_{n-1})}(X|Y,\beta)] = -\gamma \cap [\mathcal{VZ}_{1,(d,\underbrace{0,\ldots,0}_{n-1})}(X|Y,\beta)]$$

which gives us the first term on the right-hand side of (7). As usual the type II and † contributions are known recursively, and the only type III contribution not known recursively occurs when the entire curve is mapped into the divisor, in which case we apply fgt_y to calculate the contribution as:

$$(-\gamma/d) \cap [\mathcal{VZ}_{1,n}(Y,\beta)] = -I/d.$$

Substituting into (7) we end up with

$$I(1 - d^{-1}) = \text{recursively known terms}$$

which completes the recursion step as long as $d \neq 1$. But since $\mathcal{VZ}_{1,n}(H,1) = \emptyset$ it follows that $\mathcal{VZ}_{1,n}(Y,\beta) = \emptyset$ if $d = Y \cdot \beta = 1$, so we may always assume $d \geq 2$ in the recursion.

Step 3. Now suppose we are given a relative space $\mathcal{VZ}_{1,\alpha}(X|Y,\beta)$ with an insertion γ, and suppose inductively that we have computed all forgetful relative invariants with $(d',n') \leq (d,n+1)$, all absolute invariants with $(d',n') \leq (d,n)$, all relative invariants with $(d',n',t') < (d,n,t)$ and all rubber invariants with $d' < d$. Choose a true marking x_1 with $\alpha_1 \geq 1$ and consider the moduli space

$$\mathcal{VZ}_{1,(\alpha-e_1)\cup(1)}(X|Y,\beta)$$

where y is the newly-introduced marking. As usual consider the insertion $\tilde{\gamma}$ obtained from γ by introducing fgt_y everywhere. Recursing at x_1 we obtain:

$$\left((\alpha_1 - 1)\psi_1 + \mathrm{ev}_1^\star H\right)\tilde{\gamma} \cap [\mathcal{VZ}_{1,(\alpha-e_1)\cup(1)}(X|Y,\beta)] = \tilde{\gamma} \cap [\mathcal{D}(1)].$$

The left-hand side is a relative invariant with the same degree and number of true markings, but smaller true tangency: hence it is recursively known. On the right-hand side, the type I contributions are recursively known except for those of the following form
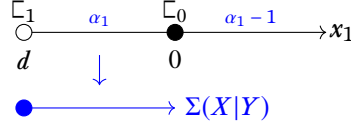
where C_0 contains a single fictitious marking (if it had multiple fictitious markings then the contribution would vanish by projection formula) and all the other markings are on C_1. Thus for each fictitious marking we get a contribution of $\alpha_1 I$ where I is the invariant we are trying to compute. Note that $\alpha_1 \neq 0$, and that this term appears at least once since y is a fictitious marking; so we get a nonzero multiple of I.

As usual the type II and $\dagger$ contributions are recursively known. The type III contributions are determined by lower-degree rubber invariants, except for when the whole curve is mapped into Y; however in this case we may apply fgt_y to identify the contribution with an absolute invariant of Y with degree d and n markings, which is also recursively known. Thus we have determined the invariant I.

Step 4. Finally, consider a rubber space $\mathcal{VZ}^{\leftrightarrow}_{1,\alpha}(\mathbb{P}|Y_0 + Y_\infty, \beta)$ with insertion γ. Suppose inductively that we have computed all absolute and relative invariants with $d' \leq d$ and all rubber invariants with $(d', n', m') < (d, n, m)$.

We first note the following important reduction: if there exists a relative marking x_k such that all insertions are pulled back along fgt_k, then we may apply the projection formula, together with the fact that

$$(\mathrm{fgt}_k)_\star [\mathcal{VZ}^{\leftrightarrow}_{1,\alpha}(\mathbb{P}|Y_0 + Y_\infty, \beta)] = \alpha_k^2 \cdot [\mathcal{VZ}_{1,n+m-1}(Y, \beta)]$$

to identify the rubber invariant with a multiple of a reduced invariant of Y, which has the same degree and hence is known recursively.

We will deal with the general case by reducing to the one above. Consider the moduli space

$$\mathcal{VZ}^{\leftrightarrow}_{1,\alpha\cup(0)}(\mathbb{P}|Y_0 + Y_\infty, \beta)$$

obtained by introducing a marked point y with no tangency. Let x_1 be a positive-tangency marking (such a marking always exists since $d \geq 2$) and let $\tilde{\gamma}$ be the insertion obtained from γ by replacing ev_1 and ψ_1 by ev_y and ψ_y, and then introducing $\mathrm{fgt}_1^\star$ everywhere.

We will make use of a recursion formula for rubber spaces analogous to the recursion formula for relative spaces used in Steps 1–3. Following [Kat07], there is a line bundle $L_y^{\notin\mathrm{bot}}$ on $\mathcal{VZ}^{\leftrightarrow}_{1,\alpha\cup(0)}(\mathbb{P}|Y_0 + Y_\infty, \beta)$, together with a section $s_y^{\notin\mathrm{bot}}$ whose vanishing locus consists of the locus $\mathcal{D}(y)$ of rubber maps where y is not mapped into the bottom level of the expanded target. As in §3, we can give a logarithmic interpretation of this: it corresponds to the piecewise-linear function on the tropical moduli space which associates, to every rubber tropical map, the distance between $\varphi(\mathsf{L}_y)$ and the leftmost vertex of the tropical target, where L_y is the vertex of the source curve containing the flag corresponding to y. Using this tropical description, we may easily calculate the vanshing orders of $s_y^{\notin\mathrm{bot}}$ along the various components of $\mathcal{D}(y)$, and show that $\mathrm{c}_1(L_y^{\notin\mathrm{bot}}) = \Psi_0 - \mathrm{ev}_y^\star Y$ (see also [Kat05], where similar results in the non-reduced setting are obtained, using different methods). From this, we obtain a rubber recursion formula

$$(8) \qquad (\Psi_0 - \mathrm{ev}_y^\star Y)\tilde{\gamma} \cap [\mathcal{VZ}^{\leftrightarrow}_{1,\alpha\cup(0)}(\mathbb{P}|Y_0 + Y_\infty, \beta)] = \tilde{\gamma} \cap [\mathcal{D}(y)]$$

where the fundamental class $[\mathcal{D}(y)]$ is weighted by vanishing orders on the components. We will first show that the left-hand side is recursively known. By construction the class $\tilde{\gamma}$ is pulled back via $\mathrm{fgt}_1^\star$, and the same is true for $\mathrm{ev}_y^\star Y$. It remains to examine Ψ_0. If there exists a negative-tangency marking x_2, then we have [Gat03, Construction 5.1.17]

$$\Psi_0 = -\alpha_2 \hat{\psi}_2 - \mathrm{ev}_2^\star Y$$

where $\hat{\psi}_2$ is a **non-collapsed** psi class. (If there are no negative-tangency markings, then the construction given in [MP06, §1.5.2] shows that $\Psi_0 = 0$.) Now, $\hat{\psi}_2 - \psi_2$ is given by the loci where x_2 belongs to a trivial bubble. This entails a splitting of the curve into pieces, each of which contributes a rubber integral. Typically each of these pieces will have $(g', d', n', m') < (1, d, n, m)$ and hence be recursively known. The one exception is when all of the genus and degree is concentrated on the top level of the expansion, with the bottom level containing only a single non-relative marking in addition to all the negative-tangency markings. But in this case the contribution is a rubber invariant where all of the insertions are pulled back via $\mathrm{fgt}_1^\star$, and hence we may apply the projection formula to identify this with an absolute invariant of Y which is recursively known. We conclude that, up to recursively knwon terms, we may replace $\hat{\psi}_2$ by ψ_2 in the left-hand side of (8). If we now compare ψ_2 with $\mathrm{fgt}_1^\star \psi_2$ we see that the difference is given by the locus where x_1 and x_2 belong to a collapsed rational piece. The contribution of this locus consists of rubber invariant with strictly fewer relative markings, and hence is recursively known. So up to recursively-known terms (8) becomes

$$\mathrm{fgt}_1^\star(-\alpha_2 \psi_2 - \mathrm{ev}_y^\star Y)\tilde{\gamma} \cap [\mathcal{VZ}_{1,\alpha\cup(0)}^{\leftrightarrow}(\mathbb{P}|Y_0 + Y_\infty, \beta)] = \tilde{\gamma} \cap [\mathcal{D}(y)]$$

and now the left-hand side is also recursively known, by the projection formula. Let us now examine the right-hand side. The components of $\mathcal{D}(y)$ are indexed by splittings of the curve, and certainly the contributions are recursively known unless there is a piece of the curve carrying all of the genus and degree, so we may restrict to examining these contributions.

Let us denote the piece of the curve carrying all of the genus and degree by $C' \subseteq C$. If C' is mapped to top level, then either it contains x_1, in which case we apply fgt_1 to compute the contribution, or it does not contains x_1, in which case it has fewer than n relative markings and is known recursively. If on the other hand C' is not mapped to top level, then generically it is mapped to the bottom level (since generically the core is not contracted on this locus, so the desingularisation process does nothing). One possible contribution is given by the following locus
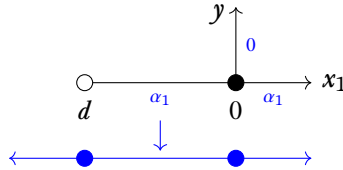
which contributes a nonzero multiple of the invariant I we are trying to compute. For the other possibilities, first note that, unless every component at the top level contains a single positive-tangency marking and a single node (together with possibly some tangency-zero markings), then C' has fewer than n relative markings and hence the contribution is known recursively. On the other hand, if any non-relative marking other than y is mapped

to the top level, C' has $\leq n$ relative markings and less than m non-relative markings, and hence again the contribution is known recursively. The only remaining possibilities are when x_1 is replaced by another positive-tangency marking x_k in the above picture; but then the contribution can be calculated by applying the projection formula to fgt_1. We therefore conclude that the only contribution to the right-hand side of (8) which is not known recursively is a nonzero multiple of the invariant we were trying to compute. This completes the recursion step.

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Luca Battistella
Max-Planck-Institut für Mathematik, Bonn
battistella@mpim-bonn.mpg.de

Navid Nabijou
School of Mathematics and Statistics, University of Glasgow
navid.nabijou@glasgow.ac.uk

Dhruv Ranganathan
Department of Pure Mathematics and Mathematical Statistics, University of Cambridge
dr508@cam.ac.uk