

CURVE COUNTING IN GENUS ONE: ELLIPTIC SINGULARITIES AND RELATIVE GEOMETRY

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ABSTRACT.

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INTRODUCTION

Gromov–Witten theory in genus zero has been studied intensely using a large assortment of techniques, many of which rely crucially on the remarkable geometry of the Kontsevich space of genus zero maps to projective space. On the other hand, the higher genus spaces have uncontrollable singularities, and an entirely different set of techniques must be appealed to. Indeed, much less is known in higher genus, though the last few years have seen remarkable progress.

0.1. The transition point in genus one. The singularities of the space of maps in genus one are delicate, but understandable. Vakil and Zinger exhibited this by giving an explicit blowup algorithmic that desingularized the space of maps to projective space, giving rise to “reduced” Gromov–Witten invariants, which are closer to enumerative counts [VZ08]. Certain tools, such as localization and quantum Lefschetz, can thus be extended to genus one. However, there is as yet no degeneration formula for the Vakil–Zinger invariants.

Recently, a more conceptual and geometric understanding of Vakil and Zinger’s construction has come out of tropical geometry and logarithmic Gromov–Witten theory.

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Specifically, Ranganathan, Santos-Parker, and Wise give a modular interpretation of the construction and a deformation theoretic proof its smoothness by using the geometry of elliptic singularities [RSW17a, RSW17b]. In this paper we seek an understanding of Vakil—Zinger invariants in relative geometries and the degeneration formula.

0.2. Results. Our first contribution is a definition and construction of relative reduced Gromov–Witten theory of projective space relative to a hyperplane. The geometric content is the following.

Theorem A. There exists a logarithmically smooth and proper Deligne–Mumford stack giving a modular compactification for the space of genus one maps to projective space with fixed contact order with a hyperplane.

With regards to the degeneration formula, we pursue the strategy laid out by Vakil and Gathmann, building on work of Caporaso and Harris. We reinterpret and extend their methods using logarithmic geometry and tropical techniques. Our main result is the following.

Theorem B. Given a smooth pair (X, Y) with Y very ample, there is an explicit recursive algorithm to calculate

- (1) the reduced genus one Gromov–Witten invariants of Y ;
- (2) the reduced genus one relative Gromov–Witten invariants of (X, Y) ;
- (3) the reduced genus one rubber invariants of $P = \mathbb{P}_Y(N_{Y|X} \oplus \mathcal{O}_Y)$

from the invariants of the ambient variety X .

0.3. Broader contributions. Beyond genus one, we pass through a number of seemingly useful general techniques. First, we explain how to describe degenerate moduli spaces as fiber products. This is substantially more delicate than the standard situation because of the need to work with elliptic singularities and aligned logarithmic structures. This interaction of the degeneration formula with the geometry of curve singularities is a new phenomenon. The results of [RSW17a] suggest the existence of a reduced higher genus Gromov–Witten theory formed by replacing contracted elliptic components with singularities¹. The discussion here is likely to carry over to this setting.

Second, a key step in our recursion is a description of the locus of maps with higher than prescribed tangency, which was identified by Gathmann. We realize the locus as the vanishing locus of a section of a line bundle that comes from tropical geometry — from a piecewise linear function on the fan. The systematic understanding of logarithmic line bundles arising in this fashion is likely to play an important in logarithmic enumerative geometry.

Our analysis leads naturally to a study of a “main component” double ramification cycle for target manifolds in genus one. The virtual geometry of this was studied in recent work of Janda, Pandharipande, Pixton, and Zvonkine [JPPZ18]. We do not know how to calculate an expression for this main component contribution, but we repurpose E. Katz’s topological recursion relations [Kat07] to complete the algorithm. We note that even in genus one, an understanding of the main component contributions of the double ramification cycle for target manifolds would be interesting.

¹We have been informed by Jonathan Wise that his Ph.D. candidate Sebastian Bozlee has constructed these higher genus reduced Gromov–Witten invariants. A manuscript is in preparation.

Finally, we note that our results require us to encounter several variants of the space of relative/logarithmic stable maps, with both fixed and expanded target. The conceptual features of one space are often computational bugs, and vice versa, and we believe that the techniques developed in the genus one case here will be used repeatedly in logarithmic Gromov–Witten theory calculations. In particular, we use frequently use Kato’s perspective on logarithmic blowups as subfunctors that was applied in [RSW17a, RSW17b].

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Part 1. The moduli space and its deformation theory

1. CONSTRUCTION AND LOGARITHMIC SMOOTHNESS

1.1. The absolute story. Let $\mathfrak{M}_{1,n}$ be the logarithmic algebraic stack of genus 1 n -pointed prestable curves. Let \sqsubset be a n -pointed genus 1 tropical curve. Let C be a logarithmic curve over S , let \sqsubset denote its tropicalization, and let λ be the section of $\overline{M}_{C/S}$ giving the distance from the circuit.

We begin by recalling the logarithmic moduli spaces of genus 1 curves constructed in [RSW17a, Sections 2 & 4]. Given a family of tropical curves over a base σ , a central alignment is a piecewise-linearly varying choice of radius δ_s for $s \in \sigma$, together with a consistent ordering of the vertices that lie inside the circle of radius s around the circuit components. Precisely, let T be a geometric point with logarithmic structure.

Definition 1.1. A **central alignment** of C/T is an element $\delta \in \overline{M}_S$ such that

- (1) the section δ is comparable to $\lambda(v)$ for all vertices v of \sqsubset ,
- (2) for any pair of vertices v and w at distance less than δ , the sections $\lambda(v)$ and $\lambda(w)$ are comparable.

The moduli stack $\mathfrak{M}_{1,n}^{\text{cen}}$ is a logarithmic algebraic stack in the smooth topology. The main construction of [RSW17a] canonically associates to any radially aligned family of curves \mathcal{C}_S a partial destabilization $\widetilde{\mathcal{C}}_S$ and a contraction

$$\widetilde{\mathcal{C}}_S \rightarrow \overline{\mathcal{C}}_S,$$

where $\overline{\mathcal{C}}_S$ is a Gorenstein elliptic singular curve. The number of branches is equal to the number of excident edges at the circle of radius δ . This data uniquely determines the singularity.

The space of stable maps $\overline{\mathcal{M}}_{1,n}^{\text{cen}}(\mathbb{P}^N, d)$ from the universal nodal curve \mathcal{C} over the stack $\mathfrak{M}_{1,n}^{\text{cen}}$ is a proper and algebraic, with projective coarse moduli. The **factorization condition** that the map $\mathcal{C} \rightarrow \mathbb{P}^N$ factors through the contraction $\widetilde{\mathcal{C}} \rightarrow \overline{\mathcal{C}}$ is a closed condition.

Theorem 1.2 ([RSW17a, Theorem B]). The substack $\mathcal{V}\mathcal{Z}_{1,n}(\mathbb{P}^N, d)$ of $\overline{\mathcal{M}}_{1,n}^{\text{cen}}(\mathbb{P}^N, d)$ parameterizing maps to \mathbb{P}^N that satisfy the factorization property is smooth and proper of the expected dimension.

1.2. Relative geometry: compactification. Fix $H \subset \mathbb{P}^N$ be a hyperplane. Let α be a partition of the degree d . Consider the moduli space $\mathcal{M}_{1,\alpha}^{\circ}(\mathbb{P}^N|H)$ of maps from smooth elliptic curves $C \rightarrow \mathbb{P}^N$ that meet H at finitely many marked points with vanishing orders given by the partition α . This is a smooth non-compact Deligne–Mumford stack.

We will first compactify the space described above, and then desingularize it. For the compactification, we begin with Abramovich–Chen–Gross–Siebert’s space of logarithmic stable maps, though we will typically work with various subcategories and variants.

The moduli space $\overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H)$ is a fibred category over logarithmic schemes, whose fiber over (S, M_S) is the groupoid of logarithmic curves of genus 1 over (S, M_S) equipped with a map to \mathbb{P}^N of degree d and contact order α . It is a fundamental fact in the subject that this category is representable by a proper algebraic stack with logarithmic structure.

There is a representable finite logarithmic morphism to the Kontsevich space, forgetting the logarithmic structure on the target:

$$\overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H) \rightarrow \overline{\mathcal{M}}_{1,n}(\mathbb{P}^N, d).$$

The space of centrally aligned maps is a logarithmic modification

$$\overline{\mathcal{M}}_{1,n}^{\text{cen}}(\mathbb{P}^N, d) \rightarrow \overline{\mathcal{M}}_{1,n}(\mathbb{P}^N, d)$$

The fiber product leads to a fourth moduli space of **centrally aligned logarithmic maps** to (\mathbb{P}^n, H) , which we denote $\widetilde{\mathcal{V}}_{1,\alpha}(\mathbb{P}^N|H)$.

1.3. Relative geometry: expansions. Our next task is to pick out a non-singular principal component in $\widetilde{\mathcal{V}}_{1,\alpha}(\mathbb{P}^N|H)$. The principal component of this space, consisting of the closure of the space of maps from nonsingular curves, maps into the principal component of $\overline{\mathcal{M}}_{1,n}^{\text{cen}}(\mathbb{P}^N, d)$. Indeed, smoothable logarithmic maps are, in particular, smoothable as ordinary maps. However, an additional condition is required to isolate the principal component of the space of logarithmic maps.

In order make the obstruction theory more geometric, we expand the target. To elucidate the connection with the static target, consider a logarithmic stable map $[C \rightarrow (\mathbb{P}|H)]$ over $\text{Spec}(\mathbb{N} \rightarrow \mathbb{C})$. At the level of tropicalizations, we have a map of fans

$$\square \rightarrow \mathbb{R}_{\geq 0}.$$

Choose a subdivision of $\mathbb{R}_{\geq 0}$ whose vertices consist of the images of vertices of \square . Pull this subdivision back to \square by marking all preimages of the vertices of $\mathbb{R}_{\geq 0}$. Denote the resulting map $\widetilde{\square} \rightarrow \widetilde{\mathbb{R}}_{\geq 0}$.

These subdivisions induce logarithmic modifications

$$\widetilde{C} \rightarrow \mathbb{P}^N[s],$$

see [?]. Here the latter is the s -times iterated deformation to the normal cone of H in \mathbb{P}^N . The number of components is equal to the number of vertices in $\widetilde{\mathbb{R}}_{\geq 0}$. The curve is modified by adding rational components corresponding to the newly introduced vertices.

The result is a logarithmic stable map to an expanded target $\mathbb{P}^N[s]$, together with a contraction to the main target component $\mathbb{P}^N[s] \rightarrow \mathbb{P}^N$. More generally, this construction is easily modified for any family of logarithmic maps, such that the vertices of \square are totally ordered in the family, see [] for details.

Globally, Kim constructs a moduli space of logarithmic stable maps to expanded degenerations, which on logarithmic points, reduces to the above construction [Kim10]. Indeed, by above description, Kim's space is identified with a subcategory of the Abramovich–Chen–Gross–Siebert space, and its minimal objects are identified with a logarithmic modification of the unexpanded space.

1.4. Relative geometry: factorization. Let $[C \rightarrow \mathbb{P}^N[s] \rightarrow \mathbb{P}^N]$ be a logarithmic map from a centrally aligned curve to an expansion. Recall that $\mathbb{P}^N[s]$ consists of a union of \mathbb{P}^N with the projective bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$. We refer to these latter components as **the higher levels**. Thus, we will say that subcurve $D \subset C$ maps to **higher level** if the collapsed map

$$D \rightarrow \mathbb{P}^N$$

is contained in $H \subset \mathbb{P}^N$.

Let $D_1 \subset C$ be the maximal genus 1 subcurve that is mapped to higher level contracted by the map $C \rightarrow \mathbb{P}^N[s]$ in the fiber direction. Let δ_1 be the associated radius from the circuit to the nearest non-contract component. Let D_2 be the maximal genus 1 subcurve that is contracted by the collapsed map to \mathbb{P}^N , and let δ_2 be the associated radius. This coincides with the radius of the underlying map to \mathbb{P}^N . Of course, $\delta_1 \leq \delta_2$.

The datum (δ_1, δ_2) determines a destabilization \tilde{C} of C together with successive contractions $\tilde{C} \rightarrow \bar{C}_1 \rightarrow \bar{C}_2$. Keeping this definition, we come to the key definition.

Definition 1.3. The map $[C \rightarrow \mathbb{P}^N[s] \rightarrow \mathbb{P}^N]$ **factorizes completely** if

- the map $C \rightarrow \mathbb{P}^N[s]$ factors through \bar{C}_1 such that at least one branch of \bar{C}_1 has positive degree in the fiber direction.
- the collapsed map to \mathbb{P}^N factorizes through \bar{C}_2 such that at least one branch of \bar{C}_2 has positive degree.

In particular, if $[C \rightarrow \mathbb{P}^N[s] \rightarrow \mathbb{P}^N]$ is a family of centrally aligned maps over S that factorizes completely, there is a forgetful moduli map $S \rightarrow \mathcal{VZ}_{1,n}(\mathbb{P}^N, d)$, to the principal component of the space of absolute maps.

Let $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$ be stack of maps from centrally aligned curves to expansions of \mathbb{P}^N that factorize completely.

Theorem 1.4. The stack $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$ is proper and logarithmically non-singular over $\text{Spec } \mathbb{C}$.

Proof. We prove the results via the forgetful morphism

$$\nu : \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H) \rightarrow \mathcal{VZ}_{1,n}(\mathbb{P}^N, d),$$

which remembers only the stabilization of the collapsed map. The map is certainly centrally aligned, and we have already argued above that it satisfies the factorization property for \mathbb{P}^N after composition. The morphism ν is in fact proper: the verification for the valuative criterion is the essential content of Vakil's Lemma [?, Lemma 5.9], and a detailed proof may be found in [RSW17a, Theorem 4.3].

We come to logarithmic smoothness. Assume first that the elliptic curve maps into higher level. Note that the relative logarithmic tangent bundle of the expansion, over the base \mathbb{P}^N is a trivial line bundle of rank 1, as the fibers are toric. Consider an S -family of maps. After replacing the source C by a destabilization, we have maps

$$\begin{array}{ccccc}
C & \longrightarrow & \overline{C}_1 & \longrightarrow & \overline{C}_2 \\
& & \downarrow & & \downarrow \\
& & \mathbb{P}^N[s] & \longrightarrow & \mathbb{P}^N
\end{array}$$

factorizing completely. Let f_B and f_T be the maps from C to the base and total space respectively. Examining the morphism ν , we see that there is a map

$$\mathrm{Def}(C, f_B) \rightarrow \mathrm{Obs}(f_T/(C, f_B)) = H^1(\overline{C}_1, \mathcal{O}_{\overline{C}_1}),$$

where the former is the space of deformations of the curve and map to the base, as a factorized centrally aligned map. The latter is the space of obstructions to lifting a map to the base into the total space. The cokernel of this map are the absolute obstructions, which we will show vanishes, proving logarithmic smoothness.

We recall how the group $H^1(\overline{C}_1, \mathcal{O}_{\overline{C}_1})$ functions as the obstruction group for the lifting. It suffices to work near the minimal genus 1 subcurve D_1 of C_1 , since the rest of the curve is rational. This lifting is given by a rational function with prescribed orders of poles given by the slopes of the tropicalization map. That is, if α be the piecewise linear function giving this tropical map, then the lifting is described by a section of the associated bundle $\mathcal{O}_{C_1}(-\alpha)$.

Given a strict square-zero extension S' of S , the piecewise linear function α extends uniquely by strictness to any deformation of the curve. The resulting deformation of $\mathcal{O}_{C_1}(-\alpha)$ produces an infinitesimal deformation of the trivial bundle in the Picard group, whose class in $H^1(\overline{C}_1, \mathcal{O}_{\overline{C}_1})$, is the obstruction to deformation. We will show that it is possible to choose a deformation of (C, f_B) that cancels out the obstruction to lifting to f_T , and thus the map to the obstruction space above is surjective.

The line bundle $\mathcal{O}_{C_1}(-\alpha)$ is equivalent to the divisor $\sum a_i x_i$ where x_i are the points connecting D_1 to the rest of the curve, and a_i are the slopes of α along the edges corresponding to the x_i . Infinitesimally moving the point x_i is a deformation of the curve that is unobstructed by the map f_B , since the map is constant on the interiors of the circuits. By deforming the x_i , at least one of which is nonzero, we produce a one-dimensional space of obstructions. The absolute obstructions therefore vanish, since they are the cokernel of a surjective map, so the result follows. \square

The above theorem guarantees that the singularities of the space of totally factorized maps to expansions is logarithmically smooth. In fact, one can say more in this case.

Corollary 1.5. The logarithmically smooth stack $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$ has at worst orbifold singularities, i.e. admits a non-representable cover by a smooth Deligne–Mumford stack.

Proof. Since $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$ is logarithmically smooth, it will suffice to show that the cones of its tropicalization are simplicial. To see this, consider a logarithmic stable map to an expansion $C \rightarrow \mathbb{P}^N[s]$ without a central alignment. The tropical moduli cone obtained as the dual of the minimal base monoid can be identified with $\mathbb{R}_{\geq 0}^s$, see for instance [Che14, Section 2.2]. The alignment procedure is an iterated barycentric subdivision at the level of tropical moduli spaces, as explained in [RSW17a, Section 4.6], and such subdivisions preserve the property of being simplicial. We conclude from this that the

blowup $\widetilde{\mathcal{VZ}}_{1,\alpha}^{\text{exp}}(\mathbb{P}^N|H)$ obtained by centrally aligning Kim's spaces has simplicial cones. Finally, the morphism $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H) \rightarrow \widetilde{\mathcal{VZ}}_{1,\alpha}^{\text{exp}}(\mathbb{P}^N|H)$ is strict, so the cones of $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$ are simplicial, as claimed. \square

2. STRATIFICATION

The logarithmic smoothness established in Theorem 1.4 implies that $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ is an orbifold toroidal embedding. Consequently, the irreducible components of its boundary and their intersections form a stratification of the space. This stratification will be important in the sequel.

The tropicalizations of logarithmic maps that factorize completely satisfy a natural combinatorial condition known as well-spacedness. The version we use here is essentially identical to [RSW17b].

Definition 2.1. Let \sqsubset be a tropical curve of genus 1 and let \sqsubset_0 be its minimal subcurve of genus 1. A tropical map $F : \sqsubset \rightarrow \mathbb{R}_{\geq 0}$ is said to be **well-spaced** if one of the following two conditions are satisfied:

- (1) no open neighborhood of \sqsubset_0 is contracted to a point in $\mathbb{R}_{>0}$, or
- (2) if an open neighborhood of \sqsubset_0 is contracted and t_1, \dots, t_k are the vertex-edge direction flags whose vertex is mapped to $F(\sqsubset_0)$, but along which F has nonzero slope; then, the minimum of the distances from \sqsubset_0 to t_i occurs for at least two indices i .

Proposition 2.2. Let $[C \rightarrow \mathbb{P}^N]$ be a logarithmic stable map from a centrally aligned curve to an expansion, that factorizes completely. Then the tropicalization $\sqsubset \rightarrow \mathbb{R}_{\geq 0}$ is well-spaced.

Proof. The required result is essentially contained in [RSW17b, Section 4], so we explain how to deduce the requisite result from the one in loc. cit. We may focus on a single component of the expansion $\mathbb{P}^N[s]$ that contains the image of a contracted genus 1 subcurve, as this is the only relevant case. We may thus replace the target with the projective bundle $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O})$ over \mathbb{P}^{N-1} equipped with the divisorial logarithmic structure from the 0 and ∞ sections. Let p be the point to which the genus 1 subcurve is contracted. Passing to an open neighborhood of p , the map to the bundle is given by a rational function f on an open curve C° , determined by the subgraph formed by \sqsubset_0 and the flags t_i described in the definition above. To describe the tropical map to $\mathbb{R}_{>0}$, we observe that \sqsubset_0 is contracted to a fixed point $q \in \mathbb{R}_{>0}$. The flags at the vertex bases at t_i correspond to nodes or markings of C . The pole orders of f at these distinguished points determine the slopes of the tropical map. We are now exactly in the situation considered in [RSW17b, Second Paragraph of Section 4.6], and Lemma 4.6.1 of loc. cit. guarantees the well-spacedness as required. \square

2.1. The cone complex. To understand the stratification via combinatorial data, we will build the stratification from known objects.

Step 1. Let Σ^{\log} be the tropical moduli space of genus 1 tropical stable maps to $\mathbb{R}_{\geq 0}$. This is naturally identified with the tropicalization (in the logarithmic sense) of the Abramovich–Chen–Gross–Siebert space of logarithmic stable maps to the pair (\mathbb{P}^N, H) .

Step 2. Given such a tropical stable map, we may subdivide $\mathbb{R}_{\geq 0}$ such that the image of every vertex of \square is a vertex of $\mathbb{R}_{\geq 0}$. Call this subdivision $\widetilde{\mathbb{R}}_{\geq 0}$. The preimages of vertices of the subdivision form a subdivision of \square . After this procedure, the images of vertices of \square which lie in $\mathbb{R}_{>0}$ are totally ordered, in a manner extending the partial order obtained from the map to $\mathbb{R}_{\geq 0}$. The combinatorial types of such **image-ordered** maps produce the cones of a subdivision of Σ^{\log} which we denote Σ^{Kim} .

Step 3. Given a tropical map $F : \square \rightarrow \mathbb{R}_{\geq 0}$ parameterized by Σ^{Kim} , there is a largest radius δ (possibly equal to 0) such that every vertex strictly contained in the circle of radius δ around the core has degree-marking $d = 0$. Let Σ^{cen} be the subdivision obtained by requiring that the vertices contained within the circle of radius δ around the core of \square are totally ordered. This involves introducing cones along which certain \mathbb{Z} -linear combinations of edge lengths are identified.

Step 4. Let $\Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$ be the subcomplex of Σ^{cen} obtained by passing to the subcomplex of Σ^{cen} that consists of well-spaced tropical maps.

Proposition 2.3. The cone complex $\Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$ is the fan of the toroidal embedding $\mathcal{V}\mathcal{Z}_{1,\alpha}(\mathbb{P}^N|H)$. In particular, the codimension k strata of $\mathcal{V}\mathcal{Z}_{1,\alpha}(\mathbb{P}^N|H)$ are in inclusion reversing bijection with the dimension k cones in $\Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$.

Proof. The construction above has been given to mimic the construction of the space $\mathcal{V}\mathcal{Z}_{1,\alpha}(\mathbb{P}^N|H)$. Specifically, the fact that cone complex Σ^{cen} is the cone complex attached to the logarithmic stack $\widetilde{\mathcal{V}\mathcal{Z}_{1,\alpha}(\mathbb{P}^N|H)}$ of centrally aligned maps to expansions, follows immediately from its description as a subcategory of the fibered category (over logarithmic schemes) of $\mathcal{M}_{1,\alpha}^{\log}(\mathbb{P}^N|H)$. To complete the result, we note that $\mathcal{V}\mathcal{Z}_{1,\alpha}(\mathbb{P}^N|H)$ has a strict map to $\mathcal{V}\mathcal{Z}_{1,\alpha}(\mathbb{P}^N|H)$, so its cone complex is a subcomplex of Σ^{cen} . The fact that it must be contained in the subcomplex of well-spaced curves is immediate from Proposition 2.2.1 \square

2.2. Indexing the strata. The dimension- k cones in $\Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$ can be enumerated as follows. First, the cones in Σ^{\log} are indexed by **combinatorial types** of tropical maps to $\mathbb{R}_{\geq 0}$. Here a combinatorial type encodes all of the data of a tropical map, except for the edge lengths and precise vertex positions. To be more precise, a combinatorial type Δ consists of:

- (1) a finite graph;
- (2) genus, degree and marking assignments on the vertices
- (3) the data of which stratum of $\mathbb{R}_{\geq 0}$ each vertex is mapped to;
- (4) integral slopes along the edges (both finite and infinite).

The corresponding cone $\sigma \in \Sigma^{\log}$ is then given by the resulting moduli space of tropical maps, given by choices of edge lengths and vertex positions which produce a continuous tropical map. Given $\sigma \in \Sigma^{\log}$ we then produce all the cones in $\Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$ mapping to σ by performing Steps 2–4 outlined above. This amounts to taking a particular subdivision

of σ . Note that in this process, new cones are created which map into larger-dimensional cones of Σ^{\log} . The process for enumerating the codimension k strata of $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ is therefore:

- (1) fix a combinatorial type Δ ;
- (2) perform the subdivision of the resulting cone $\sigma \in \Sigma^{\log}$;
- (3) identify the dimension k cones of that subdivision.

In §?? we will perform this analysis in the case $k = 1$.

3. RUBBER VARIANTS AND VIRTUAL CONSTRUCTION

3.1. Rubber variants. In our implementation of the relative theory, we will need the naturally “rubber” variant of the space of maps $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$, where the entire curve is mapped into higher levels.

Specifically, let \mathbb{P} denote the projective bundle $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O})$ on \mathbb{P}^{N-1} . Equip this space with the logarithmic structure coming from the 0 and ∞ sections of the bundle. Consider the moduli space $\mathcal{VZ}_{1,\alpha}(\mathbb{P}/\mathbb{G}_m)$ of logarithmic maps

$$C \rightarrow \mathbb{P}[s] \rightarrow \mathbb{P}^{N-1},$$

which factorize completely, and where automorphisms are considered over the space of maps to \mathbb{P}^{N-1} . Specifically, two maps that differ by a \mathbb{G}_m translate in the fiber direction are considered equivalent. Total factorization, similar to the previous case, means that the map to the bundle and the map to the base \mathbb{P}^{N-1} both factorize through possibly different singularities.

Theorem 3.1. The stack $\mathcal{VZ}_{1,\alpha}(\mathbb{P}/\mathbb{G}_m)$ is logarithmically smooth.

It is convenient to work with the logarithmic multiplicative group and its torsors. Recall that the *logarithmic multiplicative group* \mathbb{G}_{\log} is the functor on logarithmic schemes whose value on a logarithmic scheme S is the group of global sections $H^0(S, M_S^{\text{gp}})$. This functor is representable only after a logarithmically étale cover.

The bundle $\mathbb{P} \rightarrow \mathbb{P}^{N-1}$ gives rise to a \mathbb{G}_m -torsors by deleting the zero and infinity sections. Replacing these fibers by their \mathbb{G}_{\log} compactifications, we obtain a non-representable functor on logarithmic schemes \mathbb{P}_{\log} and a logarithmically étale modification

$$\mathbb{P} \rightarrow \mathbb{P}_{\log}.$$

Add reference to Wise–R ’19 and Marcus–Wise ’17 for the multiplicative group.

Proof. Consider the stack over logarithmic schemes $\mathcal{VZ}_{1,\alpha}(\mathbb{P}_{\log})$ of stable logarithmic maps that factorize completely. Here, stability for the map coincides with stability for the projection to \mathbb{P}^{N-1} . The arguments in the previous section show that maps to \mathbb{P} that factorize completely are unobstructed. Since this space is a logarithmically étale cover of $\mathcal{VZ}_{1,\alpha}(\mathbb{P}_{\log})$, logarithmic smoothness of the latter follows.

The logarithmic multiplicative group \mathbb{G}_{\log} acts on $\mathcal{VZ}_{1,\alpha}(\mathbb{P}_{\log})$ by translation. Tautologically, $\mathcal{VZ}_{1,\alpha}(\mathbb{P}_{\log})$ is a \mathbb{G}_{\log} -torsor over the moduli problem of maps up to this \mathbb{G}_{\log} translation. It is representable after a logarithmically étale modification by a family of nodal curves. It follows that the space $\mathcal{VZ}_{1,\alpha}(\mathbb{P}_{\log}/\mathbb{G}_{\log})$ of maps up to translation that factorize completely is logarithmically smooth.

To conclude the main result, we compare the stacks $\mathcal{VZ}_{1,\alpha}(\mathbb{P}_{\log}/\mathbb{G}_{\log})$ and $\mathcal{VZ}_{1,\alpha}(\mathbb{P}/\mathbb{G}_m)$. Since a map to \mathbb{P} gives rise immediately to a map to \mathbb{P}_{\log} , there is a morphism

$$\mathcal{VZ}_{1,\alpha}(\mathbb{P}/\mathbb{G}_m) \rightarrow \mathcal{VZ}_{1,\alpha}(\mathbb{P}_{\log}/\mathbb{G}_{\log}).$$

A straightforward application of the infinitesimal lifting criterion shows that this map is logarithmically étale, so the result follows. \square

The main point in the proof above is that on the space of non-degenerate maps with smooth domains, the map from $\mathcal{VZ}_{1,\alpha}^{\circ}(\mathbb{P}) \rightarrow \mathcal{VZ}_{1,\alpha}^{\circ}(\mathbb{P}/\mathbb{G}_m)$ is a \mathbb{G}_m torsor, and correspondingly on the compact spaces it is a \mathbb{G}_{\log} -torsor. The logarithmically étale modification guarantee representability, and turn this into a nodal curve fibration.

3.2. Virtual construction for very ample pairs. Now let (X, Y) be a smooth pair with Y very ample. The definition given in §1.4 applies in this setting as well, producing a moduli space $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$ together with a morphism

$$\mathcal{VZ}_{1,\alpha}(X|Y, \beta) \rightarrow \overline{\mathcal{M}}_{1,\alpha}^{\log}(X|Y, \beta)$$

obtained by performing a logarithmic modification and then passing to a closed substack. This moduli space will not in general be logarithmically smooth, but we may equip it with a virtual class as follows. The complete linear system $|\mathcal{O}_X(Y)|$ defines an embedding $X \hookrightarrow \mathbb{P}^N$ with $Y = X \cap H$ for H some hyperplane.

Lemma 3.2. The following square is cartesian (in the category of ordinary stacks):

$$\begin{array}{ccc} \mathcal{VZ}_{1,\alpha}(X|Y, \beta) & \longrightarrow & \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d) \\ \downarrow & \square & \downarrow \\ \mathcal{VZ}_{1,n}(X, \beta) & \xrightarrow{i} & \mathcal{VZ}_{1,n}(\mathbb{P}^N, d). \end{array}$$

Proof. It is clear from the modular description that this square is cartesian in the category of fs log stacks. But the morphism i is strict, which implies that the square is also cartesian in the category of coherent log stacks. Since fibre products in the latter category are compatible with the functor forgetting log structures, the claim follows. \square

Since $\mathcal{VZ}_{1,n}(\mathbb{P}^N, d)$ is smooth and $\mathcal{VZ}_{1,n}(X, \beta)$ carries a natural virtual class [RSW17a, Theorem 4.4.1] there is a diagonal pull-back morphism [Gat02] [BN17, Appendix C], which we use to define the virtual class on the space of maps to (X, Y) :

$$[\mathcal{VZ}_{1,\alpha}(X|Y, \beta)]^{\text{virt}} := i_{\Delta}^! [\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)].$$

Since $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$ comes equipped with evaluation maps and cotangent line classes, we immediately arrive at a definition of reduced Gromov–Witten invariants for the pair (X, Y) .

Part 2. The recursion formula and applications

² In this part we prove recursion relations on the moduli space of reduced stable maps (Theorem 4.1). These are inspired by the constructions of Vakil and Gathmann, but go

²(Navid) From here on, when I write a relative space without any decoration, I mean a Kim space. We should establish this convention from the start

beyond these in several key respects, giving a full treatment of descendant invariants for very ample pairs. We apply the recursions to produce a quantum Lefschetz algorithm in genus one, reconstructing the absolute theory of Y and the relative theory of (X, Y) from the absolute theory of X . Along the way we prove a splitting axiom for the reduced relative theory; the form of this splitting axiom is nontrivial, insofar as it involves the introduction of additional tautological classes.

In §§4–5 we prove the recursion formula for (\mathbb{P}^N, H) . In §6 we indicate how to obtain from this the recursion formula for an arbitrary very ample pair (X, Y) . Finally in §7 we describe the quantum Lefschetz algorithm.

4. GATHMANN'S LINE BUNDLE VIA TROPICAL GEOMETRY

Consider a reduced relative space $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$. For each marking x_k we consider the locus $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d) \subseteq \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ where x_k belongs to an internal component of the collapsed map. In this section, we use the log structure on $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ to construct a line bundle \mathcal{L}_k together with a section s_k which vanishes precisely along $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$. We use the correspondence with tropical geometry to identify $c_1(\mathcal{L}_k)$ in terms of tautological classes on $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$, and to compute the vanishing order of s_k along the components of $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$. Combined with the relative splitting axioms established in the next section, we obtain a recursion relation inside $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$.

The pair (\mathcal{L}_k, s_k) is most naturally constructed on the moduli space $\overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H, d)$ of non-expanded log maps; the corresponding pair on $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ will be obtained via pull-back. Consider therefore the tropicalisation Σ^{\log} of $\overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H, d)$, identified as usual with the moduli space of tropical maps to $\mathbb{R}_{\geq 0}$. We have a universal family

$$\begin{array}{ccc} \square & \xrightarrow{f} & \mathbb{R}_{\geq 0} \\ \nwarrow x_k & \downarrow \pi & \\ & \Sigma^{\log} & \end{array}$$

where x_k is the section which for every point $\lambda \in \Sigma^{\log}$ picks out the vertex of \square_λ containing the leg x_k . The composition $f \circ x_k: \Sigma^{\log} \rightarrow \mathbb{R}_{\geq 0}$ defines a piecewise-linear function on Σ^{\log} whose preimage over the open cone $\mathbb{R}_{>0}$ consists of those tropical maps where x_k belongs to an internal component. This produces a section of the ghost sheaf on $\overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H, d)$, which in the usual way induces a line bundle and section (\mathcal{L}_k, s_k) on the moduli space, and the tropical description above shows that the zero locus of s_k is (set-theoretically) the locus where x_k belongs to an internal component.

We now calculate $c_1(\mathcal{L}_k)$. Choose a family of log stable maps over S and let $\mu \in \Gamma(S, \overline{\mathcal{M}}_S)$ be the global section of the ghost sheaf constructed in the previous paragraph. This pulls back along π to give a global section $\pi^b(\mu) \in \Gamma(C, \overline{\mathcal{M}}_C)$. Interpreted as a piecewise-linear function on the tropicalisation \square with values in $\overline{\mathcal{M}}_S$ [CCUW17, Remark 7.3], this assigns μ to every vertex and has slope zero along every edge. By construction, the line bundle associated to this section is $\pi^*\mathcal{L}_k$. Consider on the other hand the generator $1 \in \mathbb{N} = \Gamma(\mathbb{P}^N, \overline{\mathcal{M}}_{\mathbb{P}^N})$ with associated line bundle $\mathcal{O}(H)$. The section $f^b(1) \in \Gamma(C, \overline{\mathcal{M}}_C)$ has associated line bundle $f^*\mathcal{O}(H)$. If we let v denote the vertex containing x_k , then by construction $f^b(1)$ assigns μ to v and has slope α_k along the leg x_k . Thus if we consider

the difference $f^b(1) - \pi^b(\mu)$ then this assigns 0 to v and still has slope α_k along x_k . Thus by [RSW17a, Proposition 2.4.1] the corresponding line bundle restricted to C_v is given by

$$\mathcal{O}_{C_v} \left(\alpha_k x_k + \sum_e \mu_e x_e \right)$$

where the sum is over the edges e adjacent to v and distinct from x_k . Thus we see that:

$$(f^* \mathcal{O}(H) \otimes \pi^* \mathcal{L}_k^{-1})|_{C_v} = \mathcal{O}_{C_v} \left(\alpha_k x_k + \sum_e \mu_e x_e \right).$$

Since x_k factors through C_v we may pull back along x_k to obtain

$$\mathcal{L}_k = x_k^* \pi^* \mathcal{L}_k = x_k^* \mathcal{O}_{C_k}(-\alpha_k x_k) \otimes x_k^* f^* \mathcal{O}(H) = x_k^* \mathcal{O}_{C_k}(-\alpha_k x_k) \otimes \text{ev}_k^* \mathcal{O}(H)$$

and taking Chern classes gives:

$$c_1(\mathcal{L}_k) = \alpha_k \psi_k + \text{ev}_k^* H.$$

This gives the construction of (\mathcal{L}_k, s_k) on $\overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H, d)$; the construction on $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ is given by pull-back. The relevant piecewise-linear function is the composition:

$$\Sigma_{1,\alpha}(\mathbb{P}^N|H, d) \rightarrow \Sigma^{\log} \rightarrow \mathbb{R}_{\geq 0}.$$

Notice in particular that ψ_k should be interpreted as a **collapsed psi class** on $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$, i.e. a psi class coming from the stabilised curve of the collapsed map. In this paper, all psi classes are collapsed unless stated otherwise. We immediately arrive at:

Theorem 4.1. We have the following relation in the Chow ring of $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$:

$$(1) \quad (\alpha_k \psi_k + \text{ev}_k^* H) \cap [\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)] = \sum_{\mathcal{D}} \lambda_{\mathcal{D}} [\mathcal{D}].$$

The sum is over irreducible components \mathcal{D} of the divisor $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d) \subseteq \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$, and $\lambda_{\mathcal{D}}$ is the vanishing order of s_k along this component.

Remark 4.2. This construction gives the natural logarithmic analogue of Gathmann's line bundle and section [Gat02, Construction 2.1]. A benefit of the logarithmic approach is to make the computation of vanishing orders entirely combinatorial (see §?? below), circumventing the difficult technical calculation given by Gathmann.

5. RECURSIVE DESCRIPTION OF BOUNDARY STRATA: THE RELATIVE SPLITTING AXIOM

In this section we provide an explicit description of the terms appearing on the right-hand side of (1) above. For each such term, we provide a combinatorial method to calculate the vanishing order $\lambda_{\mathcal{D}}$, and a recursive description of the stratum \mathcal{D} in terms of fibre products of moduli spaces with smaller numerical data. The form which this recursive description takes is non-obvious, in the sense that additional tautological classes appear, which must be calculated. This constitutes the most technically difficult part of the paper.

5.1. Identifying the irreducible components of $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$. Here we explain the strategy for identifying the irreducible components of $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$ via tropical geometry.

Lemma 5.1. Every irreducible component of $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$ is a codimension 1 log stratum.

Proof. This is not hard to see: the locus where the log structure is trivial coincides precisely with the locus where the source curve is smooth and not mapped inside H , and since $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ is log smooth this locus is open and dense [Niz06]. By definition $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$ is contained in the complement of this locus, which (by the very definition of logarithmic strata) can be written as a union of log strata of positive codimension. Since $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$ has codimension 1, it must therefore be equal to a union of log divisors. \square

In §2.2 we discussed a procedure for enumerating the log strata of $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$, using the cones of the tropicalisation $\Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$. To recap: every divisorial logarithmic stratum is obtained via a three-step process:

- (1) choose a combinatorial type Δ of a tropical map;
- (2) subdivide the corresponding tropical moduli space σ ;
- (3) choose a ray in this subdivision.

Notice that this process contains some redundancies: upon choosing a ray in the tropical moduli space, some of the edge lengths or vertex positions may get set to 0. This induces a generisation of the initial combinatorial type Δ , given by contracting the corresponding edges of the dual graph and moving the corresponding vertices from $\mathbb{R}_{>0}$ to 0. By the combinatorial type of a stratum in $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ we will always mean this generisation. This is independent of the choice of Δ , and in fact we can (and will) always choose Δ to coincide with the generisation.

The log divisors contained in $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$ are precisely those whose combinatorial types map the vertex of the dual graph containing x_k into the interior $\mathbb{R}_{>0} \subseteq \mathbb{R}_{\geq 0}$. Thus via the above procedure, we are able to enumerate the irreducible components of $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$ in a combinatorial manner, and the combinatorial type Δ allows us to describe the general element of such a component. What we are still lacking is a recursive description of the components, which we need in order to compute integrals over them. This is the subject of the remainder of this section.

5.2. Recursive description of the divisors: types A, B and C^+ . Choose an irreducible component $\mathcal{D} \subseteq \mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$. By Lemma 5.1 this is a log divisor, and hence may be written as:

$$\mathcal{D} = \widetilde{\mathcal{D}} \cap \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$$

for a unique log stratum $\widetilde{\mathcal{D}} \subseteq \widetilde{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N|H, d)$. Since

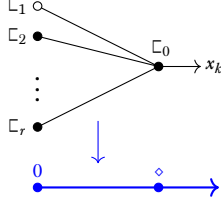
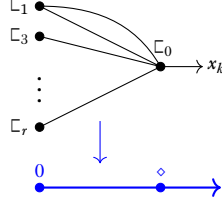
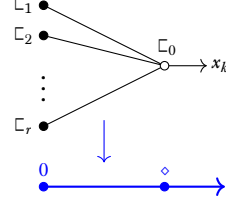
$$\widetilde{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N|H, d) \rightarrow \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d)$$

is a log modification, $\widetilde{\mathcal{D}}$ is either exceptional or the proper transform of a log divisor. These cases correspond, respectively, to when the circuit is assigned zero degree or nonzero degree by the combinatorial type. To begin with we will focus on the latter case. Suppose therefore that $\widetilde{\mathcal{D}}$ is the proper transform of a log divisor:

$$\mathcal{E} \subseteq \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d).$$

The birational map $\tilde{\mathcal{D}} \rightarrow \mathcal{E}$ induces a morphism $\mathcal{D} \rightarrow \mathcal{E}$. In the remainder of this subsection, we will show how to interpret this morphism as a “desingularisation of the main component”. Since \mathcal{E} admits a recursive description in terms of relative and rubber moduli spaces, this will allow us to compute integrals over \mathcal{D} .

Lemma 5.2. Let $\mathcal{D} \subseteq \mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H,d)$ be an irreducible component which contributes nontrivially to the recursion, and let Δ be the corresponding combinatorial type. Suppose that Δ assigns positive degree to the circuit. Then Δ takes one of the following forms:

FIGURE 1. A FIGURE 2. B FIGURE 3. C^+

The terminology is due to Vakil, and the pictures in this case are very similar to [Vak00] (we will see later that when the circuit is assigned zero degree the picture becomes *very* different). Note that in these pictures we have deliberately omitted the marked points (except for x_k), the degree of each vertex, and the expansion factors of the edges. The omitted combinatorial data can be distributed arbitrarily, as long as:

- (1) the vertices C_1, \dots, C_r have positive degree;
- (2) the ellipse³ has positive degree;
- (3) every vertex is stable;
- (4) the balancing condition is satisfied.

Proof. Let $\sigma \in \Sigma$ be the cone corresponding to the stratum $\mathcal{E} \subseteq \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H,d)$. By the discussion above \mathcal{D} corresponds to a ray in the subdivision of σ given by imposing the central alignment condition. But since the ellipse is assigned positive degree, both radii are equal to 0 and the subdivision is trivial. We conclude that $\sigma = \mathbb{R}_{\geq 0}$ (recall we are assuming that the initial and generised combinatorial types coincide). Since there must be at least one vertex mapped to higher level, we conclude that the tropical target $\tilde{\mathbb{R}}_{\geq 0}$ is obtained from $\mathbb{R}_{\geq 0}$ by subdividing at a single point $\diamond \in \mathbb{R}_{>0}$.

In order to have $\sigma = \mathbb{R}_{\geq 0}$, the dual graph Γ must be a bipartite graph with vertices over 0 and \diamond . The cases A, B and C^+ enumerated above cover situations where there is a single vertex mapped to \diamond . We claim that if there is more than one such vertex, then the contribution to the recursion vanishes. To see this, we consider the stratum

$$\mathcal{E}^{\log} \subseteq \overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H,d)$$

to which \mathcal{E} maps under the collapsing morphism, and examine the composition $\mathcal{D} \rightarrow \mathcal{E} \rightarrow \mathcal{E}^{\log}$. If we let \mathcal{F}^{\log} denote the intersection of \mathcal{E}^{\log} with the main component of the moduli space, then we obtain a factorisation $\mathcal{D} \rightarrow \mathcal{F}^{\log} \hookrightarrow \mathcal{E}^{\log}$. Since the moduli space is generically unobstructed along \mathcal{F}^{\log} , the codimension of \mathcal{F}^{\log} is given by the dimension of the associated cone in the tropicalisation Σ^{\log} . If there is more than one vertex mapped

³(Navid) Didn't we decide this was our new terminology for the circuit?

to higher level, then this cone has dimension ≥ 2 . Therefore the map $\mathcal{D} \rightarrow \mathcal{F}^{\log}$ has positive-dimensional fibres, and since all insertions are pulled back from the latter space the contribution vanishes by the projection formula. \square

Remark 5.3. The difference in dimensions between \mathcal{D} and \mathcal{F}^{\log} (or equivalently, the difference in virtual dimensions between \mathcal{E} and \mathcal{E}^{\log}) may be interpreted as the difference in dimensions between moduli spaces of *disconnected* rubber and their images under the collapsing morphisms (see ?? below).

We now investigate the three types A, B, C^+ separately, giving a recursive description of the boundary divisor in each case. For the remainder of this subsection, therefore, we fix a one-dimensional cone $\tau \in \Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$, let $\mathcal{D} \subseteq \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ be the associated log divisor, and assume that \mathcal{D} is contained in $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$ and is of type A, B or C^+ . We let $\mathcal{E} \subseteq \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d)$ be the log stratum into which \mathcal{D} is mapped; this is indexed by a cone $\epsilon \in \Sigma$ corresponding to a combinatorial type Δ , and ϵ is one-dimensional since we are restricting to the type A, B, C^+ cases.

5.2.1. Type A. Suppose Δ is of type A . Then \mathcal{E} admits a finite splitting morphism onto the fibre product:

$$\mathcal{E} \rightarrow \left(\overline{\mathcal{M}}_{1,\alpha^{(1)} \cup (m_1)}(\mathbb{P}^N|H, d_1) \times \prod_{i=2}^r \overline{\mathcal{M}}_{0,\alpha^{(i)} \cup (m_i)}(\mathbb{P}^N|H, d_i) \right) \times_{H^r} \overline{\mathcal{M}}_{0,\alpha^{(0)} \cup (-m_1, \dots, -m_r)}^{\leftrightarrow}(P|H_0+H_\infty, d_0).$$

Lemma 5.4. \mathcal{D} admits a natural splitting morphism

$$\mathcal{D} \xrightarrow{\rho} \left(\mathcal{VZ}_{1,\alpha^{(1)} \cup (m_1)}(\mathbb{P}^N|H, d_1) \times \prod_{i=2}^r \overline{\mathcal{M}}_{0,\alpha^{(i)} \cup (m_i)}(\mathbb{P}^N|H, d_i) \right) \times_{H^r} \overline{\mathcal{M}}_{0,\alpha^{(0)} \cup (-m_1, \dots, -m_r)}^{\leftrightarrow}(P|H_0+H_\infty, d_0)$$

such that the map $\mathcal{D} \rightarrow \mathcal{E}$ covers the map on fibre products obtained by desingularising the main component of the factor corresponding to \square_1 .

Proof. The choices involved in gluing log maps to expansions (which accounts for the degrees of the splitting morphisms above) is not affected by the introduction of alignments, so for the purposes of this proof we may pretend as if these splitting morphisms are the identity maps.

Note that the map $\mathcal{D} \rightarrow \mathcal{E}$ is an isomorphism away from the blown-up locus, which is contained inside the locus where the circuit is contracted. Given an element of \mathcal{D} we can split it along the nodes q_1, \dots, q_r as in []. It is then clear that \square_1 is aligned. We claim that \square satisfies the factorisation property if and only if \square_1 does. This immediately implies the lemma.

On \square there are associated contraction radii δ_1, δ_2 . Let $\delta \in \{\delta_1, \delta_2\}$. By Lemma 5.5 below, we have $\lambda(\square') > \delta$ for any component \square' of \square_0 . This means that $\square_0, \square_2, \square_3, \dots, \square_r$ lie outside both contraction radii, and so the aligned curve \square satisfies the factorisation condition if and only if \square_1 does. \square

Lemma 5.5. $\lambda(\square') > \delta$ for any component \square' of \square_0 (with notation as in the proof above).

Proof. The basic point is that \mathcal{D} is obtained as the disjoint union of the locally closed logarithmic strata contained in the closure of the stratum where all of the \square_i are irreducible. If we look at one of these boundary strata, then by definition the associated cone

$\sigma \in \Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$ contains a ray τ corresponding to the stratum where all the \square_i are irreducible; this amounts to setting all edge lengths other than e_1, \dots, e_r to zero. If f_1, \dots, f_l are some collection of additional edge lengths in σ (corresponding to internal nodes in degenerations of the \square_i) then since σ is adjacent to τ we must have (by construction of the subdivision) $f_1 + \dots + f_l < e_j$ for all $j \in \{1, \dots, r\}$, since $f_1 = \dots = f_l = 0$ and $e_j \neq 0$ on τ . In particular, if \square_1 is degenerate and if δ denotes the minimal distance from the circuit to a non-contracted vertex of \square_1 (which certainly exists since \square_1 has positive degree) then $\delta < e_1 \leq \lambda(\square')$, as claimed. \square

Lemma 5.4 provides a means to calculate integrals over the class $\lambda_{\mathcal{D}}[\mathcal{D}]$ appearing on the right-hand side of Theorem 4.1, provided that we can calculate $\lambda_{\mathcal{D}}$ and the degree of the splitting morphism. Simple closed formulae for these are given in §5.2.4 below.

5.2.2. Type B. Now suppose Δ is of type B. Note that in this case, it is impossible for the circuit to be contracted. Hence \mathcal{E} is disjoint from the blown-up locus, and the map $\mathcal{D} \rightarrow \mathcal{E}$ is an isomorphism. We thus obtain the following description, entirely in terms of genus zero data:

Lemma 5.6. \mathcal{D} admits a finite splitting morphism:

$$\mathcal{D} \xrightarrow{\rho} \left(\overline{\mathcal{M}}_{0,\alpha^{(1)} \cup (m_1, m_2)}(\mathbb{P}^N|H, d_1) \times \prod_{i=3}^r \overline{\mathcal{M}}_{0,\alpha^{(i)} \cup (m_i)}(\mathbb{P}^N|H, d_i) \right) \times_{H^r} \overline{\mathcal{M}}_{0,\alpha^{(0)} \cup (-m_1, \dots, -m_r)}^{\leftrightarrow}(P|H_0 + H_{\infty}, d_0).$$

5.2.3. Type C⁺. Finally suppose that Δ is of type C⁺. As before we have a finite morphism:

$$\mathcal{E} \rightarrow \left(\prod_{i=1}^r \overline{\mathcal{M}}_{0,\alpha^{(i)} \cup (m_i)}(\mathbb{P}^N|H, d_i) \right) \times_{H^r} \overline{\mathcal{M}}_{1,\alpha^{(0)} \cup (-m_1, \dots, -m_r)}^{\leftrightarrow}(P|H_0 + H_{\infty}, d_0).$$

The same arguments as in §5.2.1 then apply to give:

Lemma 5.7. \mathcal{D} admits a finite splitting morphism

$$\mathcal{D} \xrightarrow{\rho} \left(\prod_{i=1}^r \overline{\mathcal{M}}_{0,\alpha^{(i)} \cup (m_i)}(\mathbb{P}^N|H, d_i) \right) \times_{H^r} \mathcal{VZ}_{1,\alpha^{(0)} \cup (-m_1, \dots, -m_r)}^{\leftrightarrow}(P|H_0 + H_{\infty}, d_0)$$

such that $\mathcal{D} \rightarrow \mathcal{E}$ corresponds to the obvious map on fibre products.

Here the first factor in the fibre product is the logarithmic blow-up of the moduli space of rubber maps constructed in §7. We will compute integrals over such spaces as part of the recursion (see §7).

5.2.4. Splitting degree and vanishing order. In each of the subsections 5.2.1–5.2.3 above, we obtained a finite splitting morphism ρ from \mathcal{D} to a fibre product of moduli spaces with smaller numerical data. Here we describe the degree of ρ and calculate the vanishing order $\lambda_{\mathcal{D}}$ of the section s_k constructed in §4. This gives a complete recursive description of the terms appearing on the right-hand side of Theorem 4.1 which are of type A, B or C⁺.

Lemma 5.8. The degree of ρ is given by:

$$\frac{\prod_{i=1}^r m_i}{\text{lcm}(m_1, \dots, m_r)}.$$

Proof. This calculation is by now standard, and is given by examining the base order of the corresponding family of targets. See, for instance [Che14], [ACGS17]. \square

Lemma 5.9. $\lambda_{\mathcal{D}} = \text{lcm}(m_1, \dots, m_r)$.

Proof. Let $\tau \in \Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$ be the cone corresponding to \mathcal{D} and let

$$\varphi: \Sigma_{1,\alpha}(\mathbb{P}^N|H, d) \rightarrow \Sigma^{\log} \rightarrow \mathbb{R}_{\geq 0}$$

be the piecewise-linear function constructed in §4. It follows from the tropical description that $\lambda_{\mathcal{D}}$ is equal to the index of the map of integral cones $\tau \rightarrow \mathbb{R}_{\geq 0}$ obtained by restricting φ . By tropical continuity we have $\tau \subseteq \mathbb{R}_{\geq 0}^r$ with integral generator:

$$w = \left(\frac{\text{lcm}(m_1, \dots, m_r)}{m_1}, \dots, \frac{\text{lcm}(m_1, \dots, m_r)}{m_r} \right).$$

The map $\tau \rightarrow \mathbb{R}_{\geq 0}$ is given by projecting onto the i th factor and then multiplying by m_i (this is independent of i by continuity of the tropical map). Thus we conclude that the index is equal to $\text{lcm}(m_1, \dots, m_r)$ as claimed. \square

In summary, then: the contribution of each term $\lambda_{\mathcal{D}}[\mathcal{D}]$ is given by integrating over the appropriate fibre product and then multiplying the result by $\lambda_{\mathcal{D}} \cdot \deg \rho = \prod_{i=1}^r m_i$.

5.3. Recursive description of the divisors: type C^0 . In this case, the core has (horizontal) degree 0 and is mapped into H . We start by describing the relevant combinatorial types of rays in $\widetilde{\mathcal{V}}_{1,\alpha}(\mathbb{P}^N|H, d)$.

Lemma 5.10. In what follows it is understood that we insert trivial bubbles whenever the curve or target need to be subdivided in order for the image and preimage of every polyhedral subcomplex to be a union of polyhedral subcomplexes.

- (1) The core is represented by a single vertex of genus one and degree zero, tropically mapped to $\mathbb{R}_{>0}$.
- (2) The vertices adjacent to the core with maximal expansion factor are tropically mapped to 0.
- (3) For every point of the target different from 0, the fiber of the tropical map contains at most one stable vertex, which either is the core or lies on the circle.
- (4) x_k is supported by a vertex tropically mapped to $\mathbb{R}_{>0}$.

Proof. By assumption, the core is mapped to $\mathbb{R}_{>0}$, hence the distance of its image from 0 is one parameter ℓ ; since we are describing a ray, all other parameters have to be related to ℓ , and the only way it can be so is via the alignment. If the fiber over $c > 0$ contains two stable vertices, because both the insertions and the factorisation condition are pulled back from the moduli space without expansions, it can be seen as in Lemma 5.2 that the collapsing morphism $\mathcal{D} \rightarrow \mathcal{F}$ has positive-dimensional fibers (disconnected rubber), so the corresponding combinatorial type does not contribute to the invariants. We conclude that every stable vertex over $\mathbb{R}_{>0}$ different from the core has to lie on the circle. \square

See Figure ?? . We observe that the corresponding stratum \mathcal{E} of $\overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d)$ has logarithmic codimension equal to the number of non-core vertices mapping to $\mathbb{R}_{>0}$: indeed, the combinatorial type of \mathcal{E} can be read off by forgetting the circle, and the logarithmic

codimension in $\overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d)$ is accounted for by the number of subdivisions of the target. The discrepancy in dimension is to be expected, as the loci of type C^0 are contained in the exceptional locus of the blow-up.

The geometry of $\widetilde{\mathcal{D}} \rightarrow \mathcal{E}$ is encoded by the combinatorics of the alignment. The reader should bear the following toric construction in mind: suppose we subdivide a fan, and we want to describe the relation between the varieties corresponding respectively to a cone ρ of the subdivision, and the original cone σ containing it. The fan of X_τ is obtained by projecting the star of τ to the quotient $N_\tau = N_{\mathbb{R}}/\langle\tau\rangle_{\mathbb{R}}$ of the ambient vector space by the linear span of the cone⁴. Notice that the dimension of the vector spaces N_ρ and N_σ (or the corresponding toric varieties) is off by the difference c between the dimension of the cones. Furthermore, we may need to subdivide the fan of X_σ in order for cones of Σ_ρ to be mapped surjectively to cones of Σ_σ . The geometric interpretation of this is that X_τ is obtained as an appropriate compactification of a toric bundle of dimension c over X_σ ; observe that the fiber dimension may not be constant (but it can be made so by blowing up X_σ , see [?, Lemma 4.1]). It is a fundamental insight due to Tevelev [?] that a good compactification of the toric bundle is suggested by the tropicalisation of the locus we are interested in: here, “good” means that the intersection of the interesting locus with the boundary is dimensionally transverse. In the case at hand, the interesting locus is that of maps satisfying the factorisation property (i.e. smoothable maps), and the tropicalisation has been worked out in [RSW17b] to consist of well-spaced maps.

We proceed to describe \mathcal{D} step-by-step, building on \mathcal{E} .

5.3.1. *The fiber product.* We start from the finite splitting morphism

$$(2) \quad \rho: \mathcal{E} \rightarrow \left(\prod_{i=1}^k \overline{\mathcal{M}}_{0,\alpha^{(i)} \cup (m)}(\mathbb{P}^N|H, d_i) \times \prod_{i=k+1}^s \overline{\mathcal{M}}_{0,\alpha^{(i)} \cup (m_i)}(\mathbb{P}^N|H, d_i) \right) \times_{H^s} \\ \prod_{j=k+1}^r \overline{\mathcal{M}}_{0,\alpha^{(j)} \cup (-m_i)_{i \in I_j \subseteq \{k+1, \dots, s\}} \cup (\mu_j)}(P|H_0 + H_\infty, d_j) \times_{H^{r-k}} \\ \overline{\mathcal{M}}_{1,\alpha^{(0)} \cup (-m, \dots, -m, -\mu_{k+1}, \dots, -\mu_r) \cup (-m_i)_{i \in \{k+1, \dots, s\} \setminus (\cup I_j)}}(P|H_0 + H_\infty, 0)$$

The degree of ρ can be computed as in Lemma 5.8 above. From the logarithmic point of view, ρ is a saturation map; generically it should be thought of as follows. On the fiber product, the tangent vectors of the components on level 0 (corresponding to the factors in the first line above) are identified with the target tangent line up to finite powers; the saturation is taking roots of these isomorphisms (it is interesting to notice that it fails precisely on Gathmann’s boundary comb loci).

5.3.2. *The generic alignment.* In the first line of the product above, we have grouped the k factors adjacent to the core with maximal expansion factor m . It is fundamental to realise that, generically on \mathcal{E} , the vertices corresponding to these factors are (equidistant and) already aligned with respect to the core. We can construct a modification $\widetilde{\mathcal{E}}$ of \mathcal{E} by:

⁴We find it useful to encode this within the formalism of extended tropicalisations [?].

- (1) radially aligning the core, and imposing the double factorisation condition, thus effectively replacing the factor

$$\overline{\mathcal{M}}_{1,\alpha^{(0)} \cup (-m, \dots, -m, -\mu_{k+1}, \dots, -\mu_r) \cup (-m_i)_{i \in \{k+1, \dots, s\} \setminus (\cup I_j)}}^{\leftrightarrow} (P|H_0 + H_\infty, 0)$$

by $\mathcal{VZ}_{1,\alpha^{(0)} \cup (-m, \dots, -m, -\mu_{k+1}, \dots, -\mu_r) \cup (-m_i)_{i \in \{k+1, \dots, s\} \setminus (\cup I_j)}}^{\leftrightarrow} (P|H_0 + H_\infty, 0)$ in the product above;

in particular, there is a universal cotangent line bundle at the markings \mathbb{L}_E , obtained by suitably twisting any marking cotangent line (they are all isomorphic over a smooth elliptic curve) by the appropriate bubbling loci (see [VZ08, §2.3]);

- (2) aligning the first k factors with respect to the core (this blows up some diagonal subloci of the product, so it is not enough to align the individual factors).

See Figure ?? below. It is obvious that $\tilde{\mathcal{D}} \rightarrow \mathcal{E}$ factors through $\tilde{\mathcal{E}}$. On $\tilde{\mathcal{E}}$, we have a tropical line bundle $\mathcal{O}(\delta)$ representing the minimum distance from the core to the tropical preimage of 0; generically, this line bundle coincides with the normal bundle of the locus where the node between the core and one (any) of the first k vertices persists, but on the boundary this needs to be twisted by further smoothing parameters according to the (partial) alignment on $\tilde{\mathcal{E}}$. We can obtain a line bundle \mathbb{T}_R accounting only for the behaviour of the rational side of the fiber product by twisting $\mathcal{O}(\delta)$ with \mathbb{L}_E .

5.3.3. *The projective bundle.* Generically on $\tilde{\mathcal{E}}$, the compactified torus bundle $\tilde{\mathcal{D}} \rightarrow \tilde{\mathcal{E}}$ coincides with the following projective bundle:

$$\mathcal{P} = \mathbb{P}_{\tilde{\mathcal{E}}}(\mathbb{T}_R \oplus \bigoplus_{j=k+1}^r \mathbb{T}_{q_j} R_j),$$

where $\mathbb{T}_{q_j} R_j$ is the tangent line at the gluing marking closest to the core on the j -th factor of the product in the second line of (2). The open stratum in the fiber consists of moduli of attachments (i.e. moduli for the contraction to the Smyth's singularity); the compactification is by adding strictly semistable Gorenstein singularities as in [?, §§2.2-2.3]. In terms of alignment, a coordinate hyperplane corresponds to pushing the corresponding vertex off the circle, and destabilising the corresponding edge to the core. Tropically, the fibers can be analysed as follows: pushing some of the vertices off the circle creates new finite length edges with coordinates $\ell_j \in \mathbb{R}_{\geq 0}$ (see Figure ??); not all of them can be strictly positive, because the circle needs to pass through at least one vertex of positive horizontal degree. This is patently the fan of a projective space.

5.3.4. *Factorisation.* Assume $m \geq 2$. Over \mathcal{P} we have a tautological sequence:

$$\mathcal{O}_F(-1) \hookrightarrow p^*(F) \xrightarrow{+df} \text{ev}_E^*(TH),$$

where $p: \mathcal{P} = \mathbb{P}(F) \rightarrow \tilde{\mathcal{E}}$, and $+df$ should be interpreted as follows: k copies of \mathbb{T}_R are taken to account - via the preferred isomorphisms - for all factors of the first product in (2); then, the differential of the universal map to \mathbb{P}^N is applied to each tangent vector (possibly including some of the contracted rational bubbles in the boundary), and the sum of these contributions is taken in $T_{f(E)}\mathbb{P}^N$ - it actually lies in $T_{f(E)}H$ because all the normal contributions are trivial by assumption. The composition is a section s of $G = \mathcal{O}_F(1) \otimes \text{ev}_E^*(TH)$, and it vanishes precisely where f factors through the Smyth's

singularity determined by the point in the fiber of p . Generically on $\tilde{\mathcal{E}}$, this is exactly what we are after.

5.3.5. Dimensional transversality. In general, the vanishing locus of s corresponds to maps $f: C \rightarrow \mathbb{P}^N$ and diagrams $C \leftarrow \tilde{C} \rightarrow \bar{C}$ - with \bar{C} the elliptic r -fold point determined by the position in the fiber of $p: \mathcal{P} \rightarrow \tilde{\mathcal{E}}$ - such that f factors through \bar{C} . When the curve is allowed to degenerate (so, on the boundary of $\tilde{\mathcal{E}}$), this is not always what we want: indeed, due to the combined action of degeneration (in the base direction) and destabilisation (in the fiber direction), it may very well be that f is constant along every component of the elliptic singularity, so that it factors trivially; in other words, the vanishing locus of s is not dimensionally transverse to the boundary of \mathcal{P} .

Lemma 5.11. For every $K \subseteq \{1, \dots, r\}$ and $K^c = \{1, \dots, r\} \setminus K$, s vanishes along the loci where the factors corresponding to K in $\tilde{\mathcal{E}}$ degenerate so that $f_k: R_k \rightarrow \mathbb{P}^N$ is constant in a neighbourhood of q_k , and the fiber coordinates $(\lambda_h)_{h \in K^c}$ along $p: \mathcal{P} \rightarrow \tilde{\mathcal{E}}$ are set to be 0.

Call these loci $\Delta_i, i \in I$. See [VZ08, §3.2] for a treatment of the analogous situation in the non-relative case; the study of the normal bundles is also dealt with there.

It is clear that these loci are logarithmic substrata; therefore, there exists a logarithmic blow-up that principalizes them. We claim that $\tilde{\mathcal{D}}$ is (possibly a refinement of) this blow-up. This follows from the tropical discussion in Section 5.3.3, and the fact that the degenerate circle that we are drawing on the Δ_i falls in the strict interior of the circle of radius δ on $\tilde{\mathcal{D}}$ (because the former only passes through vertices of horizontal degree 0), thus all the distances from the vertices to the former circle are ordered. See Figure ?? above.

Finally, on $\tilde{\mathcal{D}}$ there are exceptional divisors $\tilde{\Delta}_i$ corresponding to Δ_i in \mathcal{P} . By the preceding analysis, $s \in \Gamma(\mathcal{P}, G)$ induces a section $\tilde{s} \in \Gamma(\tilde{\mathcal{D}}, G(-\sum_i \tilde{\Delta}_i))$ by pullback, whose vanishing locus is dimensionally transverse to the boundary, and whose open part coincides with the generic point of \mathcal{D} , therefore $V(\tilde{s}) = \mathcal{D}$.

5.4. Tautological contributions. Expressing the pushforward of the Chern classes of $\mathcal{O}_F(1)$ and $\mathcal{O}(\tilde{\Delta}_i)$ in terms of tautological classes on the fiber product underlying \mathcal{E} , get:

Corollary 5.12. After pushforward to \mathcal{E} , the contribution of \mathcal{D} to the invariants can be computed in terms of tautological integrals on moduli spaces of maps with lower numerics.

Lemma 5.13. The logarithmic modification ψ restricts to a birational map:

$$\mathcal{VZ}_{1,\alpha^{(0)} \cup (-m_1, \dots, -m_r)}^{\text{punct}}(H, d) \rightarrow \overline{\mathcal{M}}_{1,\alpha^{(0)} \cup (-m_1, \dots, -m_r)}^{\text{punct}, \circ}(H, d).$$

Note that the latter space (the main component of the moduli space of punctured maps) is birational to the main component of the double ramification locus. In Lemma 5.14 below we explain how to compute integrals over this cd

Proof. If it is true, we might be able to prove it via deformation theory. For this it might be useful to notice that there is a morphism $\overline{\mathcal{M}}_{g,\alpha}^{\text{punct}}(H, d) \rightarrow \overline{\mathcal{M}}_{g,\alpha}^{\text{punct}}(\mathbb{P}^N, d)$ (because $H \subseteq \mathbb{P}^N$ is strict), which is probably a closed immersion, and the loci with smooth source curve are isomorphic under this map. It seems plausible that the main component is log smooth over $B\mathbb{G}_m \subseteq [\mathbb{A}^1/\mathbb{G}_m]$ (or the standard log point). \square

Integrals over the main component of the double ramification locus can be computed using the following lemma.

Lemma 5.14. The morphism which forgets a marking

$$\text{fgt}_i: \overline{\mathcal{M}}_{1,n}^{\alpha, \circ}(H, d) \rightarrow \overline{\mathcal{M}}_{1,n-1}^{\circ}(H, d)$$

is generically finite, of degree α_i^2 (except in one special case, described in the proof).

Proof. Let us consider the case $d = 0$ first. We may assume the source curve is smooth elliptic E . The map

$$\phi: E \rightarrow \text{Pic}^0(E), \quad x \mapsto \mathcal{O}_E \left(\alpha_i x + \sum_{j=1, \dots, i, \dots, n} \alpha_j p_j \right)$$

is an isogeny of degree α_i^2 . The locus of $(C, p_1, \dots, \hat{p}_i, \dots, p_n)$ such that the kernel of ϕ contains one of the points $\{p_1, \dots, \hat{p}_i, \dots, p_n\}$ is itself a double ramification locus inside $\overline{\mathcal{M}}_{1,n}$, hence non-generic - with one exception: namely, when $n = 2$, $x = p_1$ is always a solution, but the curve lying above such point bubbles off a \mathbb{P}^1 . To see that it does not belong to the closure of the nice locus, notice that the rational function trivialising $\mathcal{O}_C(\alpha_1 p_1 - \alpha_2 p_2)$ should descend to the cusp, thus having a ramification point at the node; yet its ramification profile is determined by Riemann-Hurwitz, and it is entirely supported on p_1 and p_2 .

For a different proof: notice that we should obtain the class of the main component from the full double ramification cycle by subtracting the boundary class $[D_{1, \emptyset | 0, \{1, \dots, n\}}]$. The latter pushes forward to 0 under fgt_i , unless $n = 2$. Therefore we may apply $\text{fgt}_{i,*}$ to the Hain-Pixton formula:

$$DR_1(A) = \frac{1}{2} \left(\sum_{i=1}^n a_i^2 \psi_i - \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I| \geq 2}} a_I^2 [D_{1, I^c | 0, I}] - \frac{1}{12} \delta_0 \right),$$

where $a_I = \sum_{i \in I} a_i$ and $\delta_0 = \text{glue}_*([\overline{\mathcal{M}}_{0,n+2}])$. From $\psi_j = \text{fgt}_i^* \psi_j + [D_{i,j}]$ for $i \neq j$, and the dilaton equation, we see that $\text{fgt}_{i,*} \psi_j = 1$ and $\text{fgt}_{i,*} \psi_i = n - 1$. On the other hand, the only surviving boundary classes are $[D_{i,j}]$, and they push down to 1. Hence the formula pushes down to $\frac{1}{2} \left(\sum_{j \neq i} \alpha_j^2 + (n - 1) \alpha_i^2 - \sum_{j \neq i} (\alpha_j + \alpha_i)^2 \right) = -\alpha_i \left(\sum_{j \neq i} \alpha_j \right) = \alpha_i^2$. \square

Definition 5.15. A combinatorial type Δ of a tropical curve consists of the following data:

- (1) G a finite graph, with a set $V(G)$ of vertices, a set $F(G)$ of flags and an involution $\iota: F(G) \rightarrow F(G)$ with $\iota^2 = \text{Id}$. We let $L(G)$ denote the fixed set of ι , and think of its elements as infinite legs; we let $E(G)$ denote the set of equivalence classes of elements not fixed by ι , and think of them as finite edges connecting two (possibly equal) vertices of G ;
- (2) a genus assignment $g: V(G) \rightarrow \mathbb{Z}_{\geq 0}$;
- (3) a degree assignment $\deg: V(G) \rightarrow \mathbb{Z}_{\geq 0}$;
- (4) a weight assignment $u: F(G) \rightarrow \mathbb{Z}$ such that for every edge $e \in E(G)$ we have $u(f_1) = -u(f_2)$ where $e = [f_1] = [f_2]$, and $u(l) \geq 0$ for any $l \in L(G)$;

⁵ Review here the process of gluing for punctured maps. In particular the basic monoid should be modified in order to make the relevant evaluations log morphisms. Claim: evaluations are strict. Consequence: the fiber product in the category of log stacks is fine. We have then to apply saturation. This is a finite morphism of degree... (I think it could be $\prod \frac{m^{(i)}}{lc(m^{(i)})}$).

Lemma 5.16 (Virtual pushforward). The following hold.

- $\text{fgt}_*[\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H, d)] = [\overline{\mathcal{M}}_{0,\alpha}^G(\mathbb{P}^N|H, d)]$ (follows from [Gat03, AMW14]).
- $\text{fgt}_*[\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)]$ computes the reduced relative invariants by definition.
- $\text{fgt}_*[\overline{\mathcal{M}}_{0,\mu}(H, d_0)^\sim] = [\overline{\mathcal{M}}_{0,|\mu|}(H, d_0)]$ (follows from [Gat03] and... comparison of punctured with rubber invariants).
- $\text{fgt}_*[\mathcal{VZ}_{1,\mu}(H, d_0)^\sim]$ here we should need a variation on Pixton's DRC formula; hopefully it's enough to avoid the graphs that tropical well-spacedness discards.

5.5. Recursive description of the divisors: type C_0 . Consider now a combinatorial type Δ such that the circuit is contracted into the divisor. The corresponding stratum $\mathcal{D}_{\mathcal{M}} \subseteq \mathcal{M}$ is contained in the blown-up locus. Our task is to identify the logarithmic divisors in the preimage of $\mathcal{D}_{\mathcal{M}}$ under the map $\mathcal{VZ} \rightarrow \mathcal{M}$. As discussed, these are indexed by choices of ray in the subdivision of the tropical moduli space. Suppose we have fixed such data.

Claim 1. The teeth of the comb can break at most once, along the circuit.

Proof. If not there would be more parameters in the tropical moduli space. \square

Proposition 5.17. Let Δ be a combinatorial type as above. Then:

- (1) the circuit is a single vertex, and is mapped to the interior $\mathbb{R}_{>0} \subseteq \mathbb{R}_{\geq 0}$;
- (2) the degree of the circuit is zero;
- (3) there is a vertex which is adjacent to the circuit and mapped to the interior $0 \in \mathbb{R}_{\geq 0}$;
- (4) let $e \in E(\square)$ be an edge which is adjacent to the circuit, and such that $u(e)$ is maximal among the edges adjacent to the circuit. Then the other vertex v contained in e is mapped to $0 \in \mathbb{R}_{\geq 0}$, and v lies on the (absolute) radius;
- (5) if $v \in V(\square)$ and $\lambda(v) > \delta$ then $f(v) = 0$.

On the other hand, when there is a contracted elliptic subcurve - and it will be contracted into the hyperplane, because otherwise it wouldn't be generic, by density of the nice locus

⁵(Luca) to be made homogeneous with what comes earlier

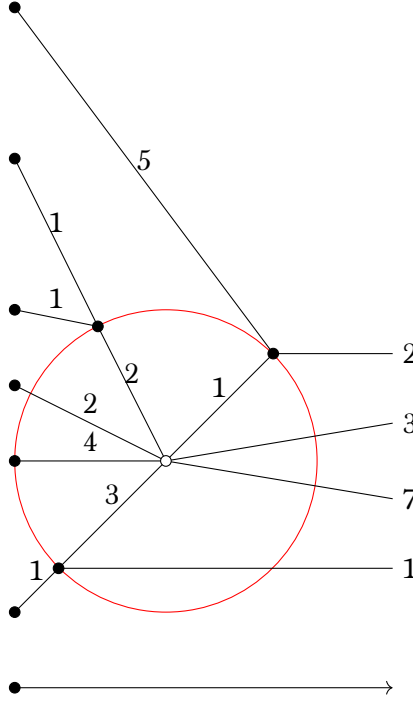


FIGURE 4. The degrees can be figured out from the balancing equation.

in $\mathcal{VZ}_{1,\alpha'}(\mathbb{P}^N|H, d')$ - the picture becomes more complicated due to the alignment. The combs may break. We label these loci \mathcal{Y}_C^0 .

In the following we describe the rays of the tropical moduli space.

Lemma 5.18. A one-parameter tropical map ϕ to $\mathbb{R}_{\geq 0}$ is a decorated tree (with expansion factors - *contact orders* - along edges and legs, and degrees on vertices, satisfying the balancing condition) with a circle (of radius δ) around the root (sometimes called the *core* and denoted by \circ) satisfying:

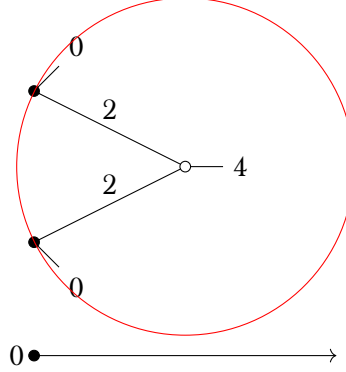
- (1) the circle of radius δ passes through at least one vertex of $\phi^{-1}(0)$ - which necessarily has positive degree - call m its contact order with H ;
- (2) teeth may break only when they intersect the circle of radius δ ; in particular, \circ is the only vertex contained in its strict interior, and every edge heading out from the circle goes directly to a vertex of $\phi^{-1}(0)$;
- (3) every tooth that starts with contact order m goes directly to a vertex of $\text{circle}(\circ, \delta) \cap \phi^{-1}(0)$, and every other tooth starts with contact order $< m$ (possibly negative).

Proof. Otherwise there would be more than one parameters. \square

Remark 5.19. The core being contracted in the fiber of the tropical map is not a phenomenon that we should worry about in codimension one. Indeed, assume that the core is contracted in the fiber along a ray. Then all the edges departing from the core have expansion factor 0; call the corresponding coordinates $U = \{u_i\}_{i \in I}$. Call the remaining coordinates $E = \{e_j\}_{j \in J}$. Note that tropical continuity involves only E . Alignments on the other hand assume the form $\lambda(v) = \lambda(v')$, where $\lambda(v) = \sum_{i \in I(v)} u_i \sum_{j \in J(v)} e_j$. Pick the

shortest elements of U ; then these can be shortened to zero without affecting the rest (by hypothesis, alignments can only identify them among themselves). This shows that we could not have started with a ray.

Example 5.20. We look at the following example in some detail. The ambient space is



$\mathcal{VZ}_{1,(4,0,0)}(\mathbb{P}^N|H, 4)$, of dimension $4N+3$. The underlying moduli space is $X = \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H, 2) \times_H \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H, 2) \times \mathcal{VZ}_{1,(-2,-2,4)}$ of dimension $5N+1$. Consider the fiber product:

$$\begin{array}{ccc} F & \longrightarrow & \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H, 2) \times \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H, 2) \\ \downarrow & & \downarrow \\ H & \longrightarrow & H \times H \end{array}$$

At the level of ghost sheaves, $\overline{\mathcal{M}}_F = \mathbb{N} \oplus_{\mathbb{N}^2} \overline{\mathcal{M}}_1^{\text{enl}} \oplus \overline{\mathcal{M}}_2^{\text{enl}}$, where the map $\mathbb{N}^2 \rightarrow \mathbb{N}$ is the sum, and the map $\mathbb{N}^2 \rightarrow \overline{\mathcal{M}}_1^{\text{enl}} \oplus \overline{\mathcal{M}}_2^{\text{enl}}$ generically is multiplication by 2, so $\overline{\mathcal{M}}_F = \mathbb{N}^2/(2e = 2f)$ generically. Saturation gives a finite cover $G \rightarrow F$ with $\overline{\mathcal{M}}_G = \mathbb{N}_{e=f}$ generically. Lifting this to actual log structures, what we are doing (again generically) is taking a square root of the isomorphism $T_{R_1, q_1}^{\otimes 2} \simeq T_{R_2, q_2}^{\otimes 2}$, which is obtained passing through $N_{H/\mathbb{P}^N, f(Z)}$ via $d f|_{R_i, q_i}$. This breaks when $f|_{R_i}$ is not tangent to H of order exactly 2 at q_i , for either i ; but by the maximality assumption this happens precisely along Gathmann's comb loci Δ_i . So in fact, rather than with $T_{R_i, q_i}^{\otimes 2}$, we should be working with $T_{R_i, q_i}^{\otimes 2}(-\Delta_i)$; but this is exactly $\text{ev}_i^*(-H)$ by Gathmann's genus zero formula, and the isomorphism $\text{ev}_1^*(H) = \text{ev}_2^*(H)$ holds on all of F .

On the other hand, generically on $\mathcal{VZ}_{1,(-2,-2,4)}$ we have $T_{q_1}Z \simeq T_{q_2}Z$ by exploiting the group structure on the elliptic curve. This breaks when either (but not both) is on a rational tail. Yet we have $T_{q_1}Z(\Delta_{1 \in P}) \simeq T_{q_2}Z(\Delta_{2 \in P})$ by Vakil-Zinger's construction of a universal ψ -class (i.e. by comparing both with $\pi_*\omega(\Delta)$; notice that our further blow-up has the only effect of twisting *all* the relevant line bundles by $\Delta_{1,2 \notin P}$).

Now, the fiber of the Vakil-Zinger blow-up over X can be described as follows. Generically it looks like

$$\mathbb{P}(T_{q_1}R_1 \otimes T_{q_1}Z \oplus T_{q_2}R_2 \otimes T_{q_2}Z)$$

but this has to be modified along the boundary:

- this has to do with the fact that the normal bundle of the strict transform is the pullback of the normal bundle twisted by the intersection with the exceptional divisor (so it relates with previous steps of the blow-up);
- it is not globally a \mathbb{P}^1 -bundle (so it relates with further stages of the blow-up; it also has to do with a choice of compactification for the moduli space of attachments);
- it has the effect of replacing $T_{q_i}Z$ with Vakil-Zinger's universal \mathbb{T} , so that this can be factored out of the projective bundle, and in fact we are left with a projective bundle $\mathbb{P} = \mathbb{P}(T_{q_1}R_1 \oplus T_{q_2}R_2)$ over F , and its open part $\text{Iso}(T_{q_1}R_1 \oplus T_{q_2}R_2)$ represents the attachment data for a contraction to a tacnode $R_1 \sqcup_q R_2 \rightarrow \bar{C}$.

On \mathbb{P} there is a natural vector bundle map

$$s: \mathcal{O}_{\mathbb{P}}(-1) \hookrightarrow p^*(T_{q_1}R_1 \oplus T_{q_2}R_2) \xrightarrow{+df} \text{ev}_q^* T\mathbb{P}^N$$

that vanishes along the locus where f descends to \bar{C} . In general, it is not a transversal section:

- we should replace $T\mathbb{P}^N$ by TH as long as all the $m^{(i)}$ are ≥ 2 ;
- Vakil and Zinger construct a blow-up of \mathbb{P} along the vanishing loci of s of low codimension, and twist s by the exceptional divisors, so that it becomes a transverse section \tilde{s} .

On the other hand, the finite cover $G \rightarrow F$ factors through \mathbb{P} , because the two vertices are already aligned on G . We claim that the boundary locus of $\mathcal{VZ}_{1,(4,0,0)}(\mathbb{P}^N|H, 4)$ corresponding to the combinatorial type of the tropical map above is the transverse intersection

$$(G \cap V(\tilde{s}) \subseteq \mathbb{P}) \times \mathcal{VZ}_{1,(-2,-2,4)}.$$

This has the expected dimension (codimension $N - 1$ with respect to X). To compute its class, we can pull \mathbb{P} back to G , and then notice that $G \hookrightarrow \mathbb{P}_G$ is the inclusion of a (trivial) subbundle.

Lemma 5.21. The class of $\mathbb{P}(\mathcal{F}) \subseteq \mathbb{P}(\mathcal{E})$ is $c_{\text{top}}(\mathcal{O}_{\mathcal{E}}(1) \otimes p^*(\mathcal{E}/\mathcal{F}))$.

See [EH16, Prop. 9.13]. It is a good time to remember that \mathcal{E} was in fact $(\bigoplus_{i=1}^r TR_{i,q_i}) \otimes \mathbb{T}$. By writing c for $c_1(\mathcal{O}_{\mathcal{E}}(1))$, ψ_i for $c_1(T^*R_{i,q_i})$, ψ_Z for Vakil-Zinger's universal psi class, and H for $\text{ev}_q^* H$, we need to compute

$$\begin{aligned} p_* \left((c - \psi_1 - \psi_2 - 2\psi_Z) [(1 + c + H)^N (1 + c)^{-1}]_{N-1} \right) &= \\ p_* \left((c - \psi_1 - \psi_2 - 2\psi_Z) \left(\sum_{k=0}^{N-1} \binom{N}{1+k} c^k H^{N-1-k} \right) \right) &= \\ \sum_{k=0}^{N-1} \binom{N}{k} H^{N-1-k} (s_k(\mathcal{E}) - s_{k-1}(\mathcal{E})(\psi_1 + \psi_2 + \psi_Z)) & \end{aligned}$$

We now generalise this picture. Recall that the map $C \rightarrow \bar{C}$ is given by a (generic) line in the sum of the tangent spaces to the rational tails at the nodes that join them to the contracted curve of genus one. This is equivalent to an alignment, and it is parametrised by an open subset of a projective bundle over the moduli space for the tails corresponding to vertices of the dual graph lying on the circle of radius δ . Yet, notice that those vertices

lying in $\phi^{-1}(0)$ are already aligned among themselves. This is why we find it convenient to distinguish among four groups of vertices:

- (1) the core;
- (2) vertices on $\phi^{-1}(0) \cap \text{circle}(\circ, \delta)$;
- (3) vertices on $\text{circle}(\circ, \delta) \setminus \phi^{-1}(0)$;
- (4) vertices on $\phi^{-1}(0) \setminus \text{circle}(\circ, \delta)$.

We shall first argue that gluing of log maps can be performed separately for the exterior and interior of the circle (the analogous classical picture is that, since the core is contracted, this moduli space is the product of a genus one curve, and a fiber product of genus zero maps under evaluation morphisms).

Lemma 5.22. $\mathcal{VZ}_{1, \alpha^{(0)} \cup (-m_1, \dots, -m_r)}^{\text{punct}}(H, 0) \simeq \mathcal{VZ}_{1, \alpha^{(0)} \cup (-m_1, \dots, -m_r)}^{\text{punct}}(\text{Spec}(k \oplus \mathbb{N})) \times H$.⁶

Let us now deal with vertices of type 2.

Lemma 5.23. Consider the following fiber product in the category of fs log stacks:

$$\begin{array}{ccc} F & \longrightarrow & \prod_{i=1}^r \overline{\mathcal{M}}_{0, \alpha^{(i)} \cup \{m\}}(\mathbb{P}^N | H, d_i) \\ \downarrow & & \downarrow \text{ev}_{q_i} \\ H & \longrightarrow & H^r \end{array}$$

On F there is a canonical isomorphism $\mathbb{L}_{q_i} \cong \mathbb{L}_{q_j}$ ⁷, the latter being the cotangent line bundles at the gluing markings on two different components i and j . Furthermore, \underline{F} is a finite cover of degree m^{r-1} of the fiber product of the underlying stacks.

Proof. Recall that each $\overline{\mathcal{M}}_{0, \alpha^{(i)} \cup \{m\}}(\mathbb{P}^N | H, d_i)$ is endowed with the log structure induced by pulling back along q_i the divisorial log structure of the universal curve at the image of q_i itself. Thus ev_{q_i} is made into a log morphism to H with its induced DF(1) log structure. We claim that the subtext of such morphism is Gathmann's formula; namely, the log morphism to H corresponds to an isomorphism between $\text{ev}_{q_i}^* \mathcal{O}_H(-H)$ on one side, and $\mathcal{I}_{D_i} \otimes \mathbb{L}_{q_i}^m$ on the other, where \mathcal{I}_{D_i} is the ideal sheaf of the union of the comb loci D_i in $\overline{\mathcal{M}}_{0, \alpha^{(i)} \cup \{m\}}(\mathbb{P}^N | H, d_i)$.

On the fiber product there is a canonical isomorphism between $\text{ev}_{q_i}^* H$ and $\text{ev}_{q_j}^* H$.

Say something about integrality.

Saturation is a local operation, as much as computing the degree, hence we can concentrate on the dense open locus where all the curves we are gluing are smooth, and they are not mapped entirely into H . There the minimal log structure on $\overline{\mathcal{M}}_{0, \alpha^{(i)} \cup \{m\}}(\mathbb{P}^N | H, d_i)$ is trivial, therefore the isomorphism between $\text{ev}_{q_i}^* H$ and $\text{ev}_{q_j}^* H$ translates into an isomorphism $\mathbb{L}_{q_i}^m \cong \mathbb{L}_{q_j}^m$. The saturation F is obtained by taking an m -th root of this isomorphism. \square

Let us denote by \mathbb{L}_F the universal cotangent line at q .

⁶(Luca) check log structure, could be fibered over std log point

⁷(Luca) check how it degenerates

The projective bundle we are seeking has base

$$\mathcal{X} = \left(F \times \prod_{i=1}^s \overline{\mathcal{M}}_{0, \tilde{\alpha}^{(i)} \cup \{m_i\}}^{\text{punct}}(H, \tilde{d}_i) \right) \times_{H^{s+1}} H$$

(with $m_i < m$) and it is

$$\mathbb{P} = \text{Proj}_{\mathcal{X}} \left(\mathbb{T}_F \oplus \bigoplus_{i=1}^s \mathbb{T}_{\tilde{q}_i} \right).$$

We are interested in the vanishing locus of the section

$$s: \mathcal{O}_{\mathbb{P}}(-1) \hookrightarrow p^* \left(\mathbb{T}_F \oplus \bigoplus_{i=1}^s \mathbb{T}_{\tilde{q}_i} \right) \xrightarrow{+df_q} f^*(T \mathbb{P}^N)_q,$$

because it represents the geometric condition that $f: C \rightarrow \mathbb{P}^N$ factors through the normalisation map $C \rightarrow \bar{C}$ prescribed by the given point of \mathbb{P} . As is, s is not transverse to the zero section. First of all, unless $m = 1$, $T \mathbb{P}^N$ can be replaced by TH in the definition of s above, because the projection of all $df_i(T_{q_i} R_i)$ to N_{H/\mathbb{P}^N} is zero. The case $m = 1$ has to be dealt with separately and it turns out that, once s is made transverse, the dimension of its zero locus is smaller than the expected dimension, hence the corresponding combinatorial types are in fact irrelevant.

The procedure to make s transverse is the same as described in [VZ08, §3], namely we need to blow up inside \mathbb{P} the projective subbundle $\mathbb{P}(\mathcal{E})$ of $\mathbb{P}|_{\mathcal{X}_\sigma}$, where \mathcal{X}_σ is the closed substack of \mathcal{X} where some of the tails degenerate so that the corresponding gluing marking lies on a component contracted by f , and \mathcal{E} is the sum of the tangent line bundles at such subset of the gluing markings. Because $\overline{\mathcal{M}}_{0, \tilde{\alpha}^{(i)} \cup \{m_i\}}^{\text{punct}}(H, \tilde{d}_i)$ is isomorphic to $\overline{\mathcal{M}}_{0, \tilde{n}_i}(H, d_i)$, the construction of Vakil and Zinger in the section “A blowup of a moduli space of genus-zero maps” of their paper carries through unchanged. The result of the blow-up is to replace $T_{q_i} R_i$ with $T_{q_i} R_i \otimes \bigoplus_{j=1}^k T_{\tilde{q}_j} R_i \otimes T_{\tilde{q}_j} S_{ij}$. We interpret this in terms of alignments.⁸

We claim that \tilde{s} obtained from s by twisting by the exceptional divisors of the blowup is a transverse section. We also claim that comb loci of type \mathcal{Y}_c^0 can be described as a fibered product of

- moduli of genus one punctured maps to the standard log point, radially aligned and satisfying factorisation;
- $V(\tilde{s})$;
- moduli of genus zero maps relative to $(\mathbb{P}^N|H)$, corresponding to vertices of type 4 above.

Finally, we claim that integrals of psi and evaluation classes over these loci can be translated into tautological integrals, i.e. descendant Gromov–Witten invariants, whose numerics is governed by the combinatorial type of the tropical map.

⁸(Luca) this has to be written properly

6. RECURSION FOR GENERAL (X, Y)

7. QUANTUM LEFSCHETZ ALGORITHM

Consider a smooth pair (X, Y) with Y very ample, and let P be the projective bundle $P = \mathbb{P}_Y(N_{Y|X} \oplus \mathcal{O}_Y)$. We assume that the genus zero and reduced genus one Gromov–Witten theories of X are known. From this starting data, we will apply our recursion formula to compute:

- (1) the genus one **reduced restricted absolute Gromov–Witten theory** of Y ;
- (2) the genus one **reduced relative Gromov–Witten theory** of the pair (X, Y) ;
- (3) the genus one **reduced rubber theory** of P .

7.1. Reduced absolute, relative and rubber invariants. To be precise: by a reduced invariant of Y we mean an integral over $\mathcal{VZ}_{1,n}(Y, \beta)$ of products of pullbacks of evaluation and psi classes along morphisms which forget a subset S of the marked points (taking $S = \emptyset$ gives the ordinary evaluation and psi classes). Here the evaluation maps are viewed as mapping into X (hence the adjective “restricted”). Reduced relative invariants of (X, Y) are defined in the same way, except now the forgetful morphism maps into a space of absolute maps:

$$\text{fgt}_S: \mathcal{VZ}_{1,\alpha}(X|Y, \beta) \rightarrow \mathcal{VZ}_{1,m-\#S}(X, \beta).$$

In particular, all the psi classes which we consider are *collapsed psi classes*, meaning that they are relative cotangent line classes for the corresponding collapsed stable map. Note that, unlike in the absolute case, in the relative case it may well be the case that the entire insertion is pulled back along a single forgetful map fgt_S . The reduced rubber invariants of P are defined similarly (again using collapsed psi classes).

The systems of invariants defined above are equivalent to the classical systems of invariants (which do not use forgetful morphisms) by well-known topological recursion relations.

7.2. Fictitious and true markings. The recursion procedure is rather delicate. Roughly speaking, we will induct on the degree (meaning $Y \cdot \beta$), number of marked points and total tangency. To get the correct notion of number of markings and total tangency in the relative setting, we introduce the concept of **fictitious markings**. Consider a moduli space $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$ of reduced relative stable maps and a corresponding integrand γ . We let $F \subseteq \{1, \dots, m\}$ be the maximal subset of marked points such that:

- (1) $\alpha_i = 1$ for all $i \in F$;
- (2) the entire integrand γ is pulled back along fgt_F .

This subset is uniquely determined, and its elements are referred to as **fictitious markings**. Markings which are not fictitious are referred to as **true**. When inducting on relative invariants we will always be interested in the number of true markings (as opposed to the total number of markings) and the true tangency

$$\sum_{i \notin F} \alpha_i \leq d = Y \cdot \beta$$

as opposed to the total tangency, which is always d . This formalises the idea that relative invariants with non-maximal tangency $t < d$ can be obtained by adding $d - t$ fictitious markings of tangency 1; see [Gat02, Lemma 1.15(i)].

7.3. Structure of the recursion. Given the genus-zero Gromov–Witten theory of X , the arguments of [Gat02] give an effective algorithm to reconstruct the genus-zero theories of Y and (X, Y) ; moreover, the genus-zero rubber theory of P is identical to the genus-zero theory of Y [Gat03]. Thus we may assume that all genus-zero data is known. We assume in addition that we know the genus one reduced theory of X . The structure of the recursion is then as follows:

```

for  $d \geq 0$  :
  for  $n \geq 0$  :
    for  $t \geq 0$  :
      | Step 1: Compute forgetful relative invariants of  $(X, Y)$  (degree  $d$ ,  $n + 1$ 
      | true markings, true tangency  $t$ ); see below.
    Step 2: Compute absolute invariants of  $Y$  (degree  $d$ ,  $n$  markings).
    for  $t \geq 0$  :
      | Step 3: Compute relative invariants of  $(X, Y)$  (degree  $d$ ,  $n$  true
      | markings, true tangency  $t$ ).
    for  $n \geq 0$  :
      for  $m \geq 0$  :
        | Step 4: Compute rubber invariants of  $P$  (degree  $d$ ,  $n$  relative markings,
        |  $m$  non-relative markings).
    
```

Although the loops involving d , n and m have infinite length, in order to compute any single invariant it is only necessary to iterate the preceding loops for a finite amount of time. A **forgetful relative invariant** of (X, Y) is by definition a reduced relative invariant with a marked point x_0 such that all of the insertions are pulled back along fgt_{x_0} . In our recursion, we first deal with this special class of relative invariants (with $n + 1$ true markings), before computing the absolute invariants (with n markings) and then returning to compute all of the relative invariants (with n true markings). This need to treat separately a particular subclass of the relative invariants is an inescapable feature of the genus one recursion.

The base terms of the recursion all have $d = 0$ and as such are easy to compute: the relative invariants of (X, Y) are nothing but absolute invariants of X , the absolute invariants of Y are given by obstruction bundle integrals over Deligne–Mumford spaces, and the rubber invariants of P are also given by integrals over Deligne–Mumford spaces, using the formula for the double ramification cycle [Hai13, JPPZ17] in terms of tautological classes. We will now explain how to perform each of the four inductive steps outlined above.

Notation 7.1. Given tuples $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$, we say that $\mathbf{a} < \mathbf{b}$ if there exists an $i \in \{1, \dots, n\}$ such that $a_j = b_j$ for $j < i$ and $a_i < b_i$.

Step 1. Suppose we are given a forgetful relative invariant to compute. That is, we have a relative space $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$ of degree d , with $n + 1$ true markings and true tangency t , and a marking x_0 such that the insertion γ is pulled back along fgt_{x_0} . We assume inductively that every absolute, relative and rubber invariant with $(d', n') < (d, n)$ is known, and also

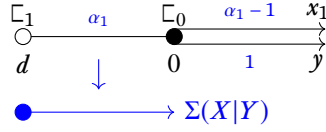
that every forgetful relative invariant with $(d', n', t') < (d, n + 1, t)$ is known. Choose a true marking x_1 with $\alpha_1 \geq 1$ and consider the space:

$$\mathcal{VZ}_{1,(\alpha-e_1)\cup(1)}(X|Y, \beta).$$

Denote the newly-introduced marking by y and consider the integrand $\tilde{\gamma}$ obtained from γ by introducing fgt_y^* everywhere. Applying our recursion formula to x_1 , we obtain:

$$(3) \quad ((\alpha_1 - 1)\psi_1 + \text{ev}_1^* Y) \tilde{\gamma} \cap [\mathcal{VZ}_{1,(\alpha-e_1)\cup(1)}(X|Y, \beta)] = \tilde{\gamma} \cap [\mathcal{D}(1)].$$

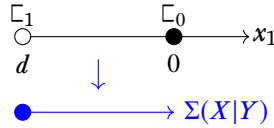
Let us first examine the left-hand side. The class ψ_1 differs from $\text{fgt}_y^* \psi_1$ by the locus where x_1 and y are contained on a contracted rational bubble. This locus consists of reduced relative stable maps of the form



with all other marked points contained on C_1 . This is isomorphic to $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$ and when we restrict $\tilde{\gamma}$ to this locus we obtain precisely the class γ which we started with. Thus the left-hand side of (3) may be written as

$$(\alpha_1 - 1)I + \text{fgt}_y^* ((\alpha_1 - 1)\psi_1 + \text{ev}_1^* Y) \tilde{\gamma} \cap [\mathcal{VZ}_{1,(\alpha-e_1)\cup(1)}(X|Y, \beta)]$$

where I is the invariant we are trying to compute. The second term is a forgetful relative invariant with the same degree and number of true markings (since y is fictitious), and strictly smaller true tangency; hence it is recursively known. We now examine the right-hand side of (3). Recall that all of the genus zero data has already been computed, so we only need to focus on the genus one pieces. First consider the type A loci. The genus one piece has strictly smaller degree (and hence is recursively known) except in the following situation (with some stable distribution of the remaining markings):



In this situation, C_1 contains at most $n + 1$ true markings. If it has $n - 1$ or fewer, then it is known recursively. If it has exactly n this means that C_0 contains exactly one true marking (besides x_1 , which may be true or fictitious). We claim that in this situation we must have $y \in C_1$, since otherwise we would have a moduli space for C_0 given by $\overline{\mathcal{M}}_{0,k}$ with $k \geq 4$, and applying the projection formula with fgt_y we would conclude that the contribution is zero. Thus we have $y \in C_1$, and so the genus one contribution is a forgetful relative invariant with n true markings, and hence is recursively known.

Finally, if C_1 contains exactly $n + 1$ true markings, then the only additional markings on C_0 are fictitious, and by the same argument as in the previous paragraph there can only be one. Thus for each fictitious marked point we obtain a locus isomorphic to $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$ and $\tilde{\gamma}$ restricts to γ here (from the point of view of computing invariants, the fictitious marked points are indistinguishable, meaning that the contributions are all the same).

Thus for each fictitious marked point (of which there is at least one, namely y) we get a contribution of $\alpha_1 I$ to the right-hand side of (3).

The contributions of the type B loci only involve genus zero data and hence are known. The contributions of the type C^0 loci are determined by genus zero data and tautological integrals on Deligne–Mumford space, hence are also known. It remains to consider type C^+ loci. If the degree of the genus one piece is less than d then we have a rubber invariant of strictly smaller degree. The only other possibility is that the entire curve is mapped into the divisor. In this case we may apply the projection formula with fgt_y to identify this with an integral over $\mathcal{VZ}_{1,m}(Y, \beta)$ for some (possibly large) number m of marked points. But by assumption there is another marked point x_0 with all of the insertions pulled back along fgt_{x_0} , so a further application of the projection formula shows that this contribution vanishes. To conclude, we may rearrange (3) to obtain an expression of the form

$$\lambda I = \text{recursively known terms}$$

where λ is an explicit scalar which is always nonzero; we have thus determined I , which completes Step 1.

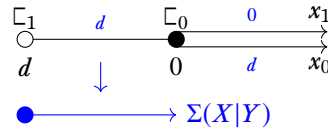
Step 2. Consider now the absolute space $\mathcal{VZ}_{1,n}(Y, \beta)$ with an insertion γ , and suppose inductively that we have computed all forgetful relative invariants with $(d', n') \leq (d, n+1)$, all relative invariants and absolute invariants with $(d', n') < (d, n)$, and all special rubber invariants with $d' < d$. Consider the following moduli space with $n+1$ markings:

$$\mathcal{VZ}_{1,(d,0,\dots,0)}(X|Y, \beta).$$

Let x_0 denote the relative marking and consider the integrand $\tilde{\gamma}$ obtained from γ by introducing $\text{fgt}_{x_0}^*$ everywhere. Now recurse at x_1 to obtain:

$$(4) \quad \text{ev}_1^* Y \cdot \tilde{\gamma} \cap [\mathcal{VZ}_{1,(d,0,\dots,0)}(X|Y, \beta)] = \tilde{\gamma} \cap [\mathcal{D}(1)].$$

The left-hand side is a forgetful relative invariant of degree d and $\leq n+1$ true markings, and so has already been computed. For the right-hand side, let us begin with loci of type A . The genus one contributions from each loci have $(d', n') < (d, n)$ except in the following case



which gives a contribution of

$$d \cdot \gamma \cap [\mathcal{VZ}_{1,(d,0,\dots,0)}(X|Y, \beta)]^9$$

to the right-hand side of (4). Here γ is the insertion we started with; the difference is that we are now considering relative maps to (X, Y) with maximal tangency at x_1 , rather than absolute maps to Y . The type B and C^0 loci are recursively determined as in Step 1, and similarly the type C^+ loci are recursively determined except for the locus where the

⁹(Navid) Fix formatting

entire curve is mapped into the divisor. On this locus we may apply fgt_{x_0} and identify the contribution with

$$d^2 \cdot \gamma \cap [\mathcal{VZ}_{1,n}(Y, \beta)] = d^2 \cdot I$$

where I is the invariant we are trying to compute. Putting this all together, we obtain

$$(5) \quad I = (-\gamma/d) \cap [\mathcal{VZ}_{1,(d,0,\underbrace{\dots,0}_{n-1})}(X|Y, \beta)] + \text{recursively known terms}$$

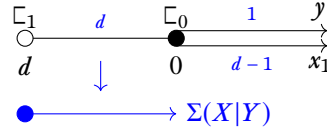
where on the right-hand side there are $n-1$ non-relative markings x_2, \dots, x_n , and a relative marking x_1 . We now apply the recursion again to the right-hand side, by considering the space

$$\mathcal{VZ}_{1,(d-1,1,0,\dots,0)}(X|Y, \beta)$$

where x_1 now has tangency $d-1$ and we have introduced a new marking y with tangency 1. We consider a new insertion, denoted $\tilde{\gamma}$ as usual, by introducing fgt_y^* everywhere. Recursing at x_1 we obtain:

$$(6) \quad ((d-1)\psi_1 + \text{ev}_1^* Y) \cdot (-\tilde{\gamma}/d) \cap [\mathcal{VZ}_{1,(d-1,1,0,\dots,0)}(X|Y, \beta)] = (-\tilde{\gamma}/d) \cap [\mathcal{D}(1)].$$

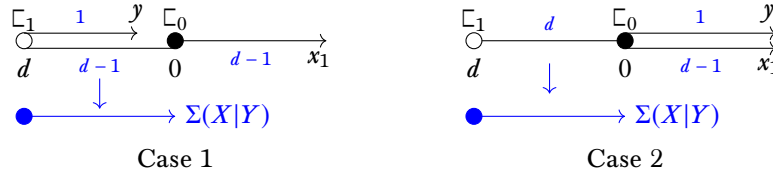
The difference between ψ_1 and $\text{fgt}_y^* \psi_1$ is given by the locus where x_1 and y belong to a collapsed rational bubble:



The contribution of this locus to the left-hand side of (6) is:

$$(d-1) \cdot (-\gamma/d) \cap [\mathcal{VZ}_{1,(d,0,\underbrace{\dots,0}_{n-1})}(X|Y, \beta)].$$

What remains on the left-hand side is a forgetful relative invariant of degree d and $\leq n+1$ true markings (y being the “forgetful” marking), hence is recursively known. On the right-hand side, the type A loci are recursively known except possibly in the following special cases (with some stable distribution of the remaining non-relative markings):



In Case 1 the contribution from C_1 is a forgetful relative invariant with $\leq n$ true markings, hence is recursively known. In Case 2, we first note that there cannot be any more markings on C_0 (since otherwise we could apply fgt_y and conclude that the contribution vanishes). Thus we obtain a single locus, which contributes precisely

$$d \cdot (-\gamma/d) \cap [\mathcal{VZ}_{1,(d,0,\underbrace{\dots,0}_{n-1})}(X|Y, \beta)] = -\gamma \cap [\mathcal{VZ}_{1,(d,0,\underbrace{\dots,0}_{n-1})}(X|Y, \beta)]$$

which gives us the first term on the right-hand side of (6). As usual the type B and C^0 contributions are known recursively, and the only type C^+ contribution not known

recursively occurs when the entire curve is mapped into the divisor, in which case we apply fgt_y to calculate the contribution as:

$$(-\gamma/d) \cap [\mathcal{VZ}_{1,n}(Y, \beta)] = -I/d.$$

Substituting into (6) we end up with

$$I(1 - d^{-1}) = \text{recursively known terms}$$

which completes the recursion step as long as $d \neq 1$. But since $\mathcal{VZ}_{1,n}(H, 1) = \emptyset$ it follows that $\mathcal{VZ}_{1,n}(Y, \beta) = \emptyset$ if $d = Y \cdot \beta = 1$, so we may always assume $d \geq 2$ in the recursion.

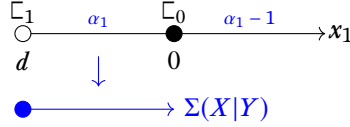
Step 3. Now suppose we are given a relative space $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$ with an insertion γ , and suppose inductively that we have computed all forgetful relative invariants with $(d', n') \leq (d, n+1)$, all absolute invariants with $(d', n') \leq (d, n)$, all relative invariants with $(d', n', t') < (d, n, t)$ and all rubber invariants with $d' < d$. Choose a true marking x_1 with $\alpha_1 \geq 1$ and consider the moduli space

$$\mathcal{VZ}_{1,(\alpha-\epsilon_1)\cup(1)}(X|Y, \beta)$$

where y is the newly-introduced marking. As usual consider the insertion $\tilde{\gamma}$ obtained from γ by introducing fgt_y everywhere. Recursing at x_1 we obtain:

$$((\alpha_1 - 1)\psi_1 + \text{ev}_1^* H) \tilde{\gamma} \cap [\mathcal{VZ}_{1,(\alpha-\epsilon_1)\cup(1)}(X|Y, \beta)] = \tilde{\gamma} \cap [\mathcal{D}(1)].$$

The left-hand side is a relative invariant with the same degree and number of true markings, but smaller true tangency: hence it is recursively known. On the right-hand side, the type A contributions are recursively known except for those of the following form



where C_0 contains a single fictitious marking (if it had multiple fictitious markings then the contribution would vanish by projection formula) and all the other markings are on C_1 . Thus for each fictitious marking we get a contribution of $\alpha_1 I$ where I is the invariant we are trying to compute. Note that $\alpha_1 \neq 0$, and that this term appears at least once since y is a fictitious marking; so we get a nonzero multiple of I .

As usual the type B and C^0 contributions are recursively known. The type C^+ contributions are determined by lower-degree rubber invariants, except for when the whole curve is mapped into Y ; however in this case we may apply fgt_y to identify the contribution with an absolute invariant of Y with degree d and n markings, which is also recursively known. Thus we have determined the invariant I .

Step 4. Finally, consider a rubber space $\mathcal{VZ}_{1,\alpha}(P|Y_0 + Y_\infty, \beta)_{\mathbb{G}_m}$ with insertion γ . Suppose inductively that we have computed all absolute and relative invariants with $d' \leq d$ and all rubber invariants with $(d', n', m') < (d, n, m)$.

We first note the following important reduction: if there exists a relative marking x_k such that all insertions are pulled back along fgt_k , then we may apply the projection formula, together with the fact that

$$(\text{fgt}_k)_* [\mathcal{VZ}_{1,\alpha}(P|Y_0 + Y_\infty, \beta)_{\mathbb{G}_m}] = \alpha_k^2 \cdot [\mathcal{VZ}_{1,n+m-1}(Y, \beta)]$$

to identify the rubber invariant with a multiple of a reduced invariant of Y , which has the same degree and hence is known recursively.

We will deal with the general case by reducing to the one above. Consider the moduli space

$$\mathcal{VZ}_{1,\alpha \cup (0)}(P|Y_0 + Y_\infty, \beta)_{\mathbb{G}_m}$$

obtained by introducing a marked point y with no tangency. Let x_1 be a positive-tangency marking (such a marking always exists since $d \geq 2$) and let $\tilde{\gamma}$ be the insertion obtained from γ by replacing ev_1 and ψ_1 by ev_y and ψ_y , and then introducing fgt_1^\star everywhere.

We will make use of a recursion formula for rubber spaces analogous to the recursion formula for relative spaces used in Steps 1–3. Following [Kat07], there is a line bundle L_y^{bot} on $\mathcal{VZ}_{1,\alpha \cup (0)}(P|Y_0 + Y_\infty, \beta)_{\mathbb{G}_m}$, together with a section s_y^{bot} whose vanishing locus consists of the locus $\mathcal{D}(y)$ of rubber maps where y is not mapped into the bottom level of the expanded target. As in §??, we can give a logarithmic interpretation of this: it corresponds to the piecewise-linear function on the tropical moduli space which associates, to every rubber tropical map, the distance between $\varphi(\mathbb{C}_y)$ and the leftmost vertex of the tropical target, where \mathbb{C}_y is the vertex of the source curve containing the flag corresponding to y . Using this tropical description, we may easily calculate the vanishing orders of s_y^{bot} along the various components of $\mathcal{D}(y)$, and show that $c_1(L_y^{\text{bot}}) = \Psi_0 - \text{ev}_y^\star Y$ (see also [Kat05], where similar results in the non-reduced setting are obtained, using different methods). From this, we obtain a rubber recursion formula

$$(7) \quad (\Psi_0 - \text{ev}_y^\star Y) \tilde{\gamma} \cap [\mathcal{VZ}_{1,\alpha \cup (0)}(P|Y_0 + Y_\infty, \beta)_{\mathbb{G}_m}] = \tilde{\gamma} \cap [\mathcal{D}(y)]$$

where the fundamental class $[\mathcal{D}(y)]$ is weighted by vanishing orders on the components. We will first show that the left-hand side is recursively known. By construction the class $\tilde{\gamma}$ is pulled back via fgt_1^\star , and the same is true for $\text{ev}_y^\star Y$. It remains to examine Ψ_0 . If there exists a negative-tangency marking x_2 , then we have [Gat03, Construction 5.1.17]

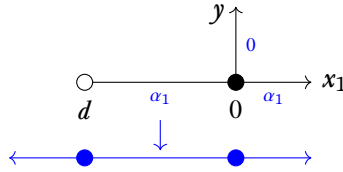
$$\Psi_0 = -\alpha_2 \hat{\psi}_2 - \text{ev}_2^\star Y$$

where $\hat{\psi}_2$ is a *non-collapsed* psi class. (If there are no negative-tangency markings, then the construction given in [MP06, §1.5.2] shows that $\Psi_0 = 0$.) Now, $\hat{\psi}_2 - \psi_2$ is given by the loci where x_2 belongs to a trivial bubble. This entails a splitting of the curve into pieces, each of which contributes a rubber integral. Typically each of these pieces will have $(g', d', n', m') < (1, d, n, m)$ and hence be recursively known. The one exception is when all of the genus and degree is concentrated on the top level of the expansion, with the bottom level containing only a single non-relative marking in addition to all the negative-tangency markings. But in this case the contribution is a rubber invariant where all of the insertions are pulled back via fgt_1^\star , and hence we may apply the projection formula to identify this with an absolute invariant of Y which is recursively known. We conclude that, up to recursively known terms, we may replace $\hat{\psi}_2$ by ψ_2 in the left-hand side of (7). If we now compare ψ_2 with $\text{fgt}_1^\star \psi_2$ we see that the difference is given by the locus where x_1 and x_2 belong to a collapsed rational piece. The contribution of this locus consists of rubber invariant with strictly fewer relative markings, and hence is recursively known. So up to recursively-known terms (7) becomes

$$\text{fgt}_1^\star(-\alpha_2 \psi_2 - \text{ev}_y^\star Y) \tilde{\gamma} \cap [\mathcal{VZ}_{1,\alpha \cup (0)}(P|Y_0 + Y_\infty, \beta)_{\mathbb{G}_m}] = \tilde{\gamma} \cap [\mathcal{D}(y)]$$

and now the left-hand side is also recursively known, by the projection formula. Let us now examine the right-hand side. The components of $\mathcal{D}(y)$ are indexed by splittings of the curve, and certainly the contributions are recursively known unless there is a piece of the curve carrying all of the genus and degree, so we may restrict to examining these contributions.

Let us denote the piece of the curve carrying all of the genus and degree by $C' \subseteq C$. If C' is mapped to top level, then either it contains x_1 , in which case we apply fgt_1 to compute the contribution, or it does not contain x_1 , in which case it has fewer than n relative markings and is known recursively. If on the other hand C' is not mapped to top level, then generically it is mapped to the bottom level (since generically the circuit is not contracted on this locus, so the desingularisation process does nothing). One possible contribution is given by the following locus



which contributes a nonzero multiple of the invariant I we are trying to compute. For the other possibilities, first note that, unless every component at the top level contains a single positive-tangency marking and a single node (together with possibly some tangency-zero markings), then C' has fewer than n relative markings and hence the contribution is known recursively. On the other hand, if any non-relative marking other than y is mapped to the top level, C' has $\leq n$ relative markings and less than m non-relative markings, and hence again the contribution is known recursively. The only remaining possibilities are when x_1 is replaced by another positive-tangency marking x_k in the above picture; but then the contribution can be calculated by applying the projection formula to fgt_1 . We therefore conclude that the only contribution to the right-hand side of (7) which is not known recursively is a nonzero multiple of the invariant we were trying to compute. This completes the recursion step.

REFERENCES

- [ACGS17] D. Abramovich, Q. Chen, M. Gross, and B. Siebert. Decomposition of degenerate Gromov-Witten invariants. *arXiv e-prints*, page arXiv:1709.09864, Sep 2017.
- [AMW14] D. Abramovich, S. Marcus, and J. Wise. Comparison theorems for Gromov-Witten invariants of smooth pairs and of degenerations. *Ann. Inst. Fourier (Grenoble)*, 64(4):1611–1667, 2014.
- [BN17] L. Battistella and N. Nabijou. A Lefschetz-type Theorem for Quasimap Invariants via Relative Quasimaps. *arXiv e-prints*, page arXiv:1710.11158, Oct 2017.
- [CCUW17] R. Cavalieri, M. Chan, M. Ulirsch, and J. Wise. A moduli stack of tropical curves. *arXiv e-prints*, page arXiv:1704.03806, April 2017.
- [Che14] Q. Chen. The degeneration formula for logarithmic expanded degenerations. *J. Algebraic Geom.*, 23(2):341–392, 2014.
- [EH16] D. Eisenbud and J. Harris. *3264 and all that—a second course in algebraic geometry*. Cambridge University Press, Cambridge, 2016.
- [Gat02] A. Gathmann. Absolute and relative Gromov-Witten invariants of very ample hypersurfaces. *Duke Math. J.*, 115(2):171–203, 2002.
- [Gat03] A. Gathmann. Gromov-Witten invariants of hypersurfaces, 2003. Habilitation thesis.

- [Hai13] R. Hain. Normal functions and the geometry of moduli spaces of curves. In *Handbook of moduli. Vol. I*, volume 24 of *Adv. Lect. Math. (ALM)*, pages 527–578. Int. Press, Somerville, MA, 2013.
- [JPPZ17] F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine. Double ramification cycles on the moduli spaces of curves. *Publ. Math. Inst. Hautes Études Sci.*, 125:221–266, 2017.
- [JPPZ18] F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine. Double ramification cycles with target varieties. *arXiv e-prints*, page arXiv:1812.10136, December 2018.
- [Kat05] E. Katz. Line-Bundles on Stacks of Relative Maps. *arXiv Mathematics e-prints*, page math/0507322, Jul 2005.
- [Kat07] E. Katz. An algebraic formulation of symplectic field theory. *J. Symplectic Geom.*, 5(4):385–437, 2007.
- [Kim10] B Kim. Logarithmic stable maps. In *New developments in algebraic geometry, integrable systems and mirror symmetry (RIMS, Kyoto, 2008)*, volume 59 of *Adv. Stud. Pure Math.*, pages 167–200. Math. Soc. Japan, Tokyo, 2010.
- [MP06] D. Maulik and R. Pandharipande. A topological view of Gromov-Witten theory. *Topology*, 45(5):887–918, 2006.
- [Niz06] W. Niziol. Toric singularities: log-blow-ups and global resolutions. *J. Algebraic Geom.*, 15(1):1–29, 2006.
- [RSW17a] D. Ranganathan, K. Santos-Parker, and J. Wise. Moduli of stable maps in genus one and logarithmic geometry I. *ArXiv e-prints*, August 2017.
- [RSW17b] D. Ranganathan, K. Santos-Parker, and J. Wise. Moduli of stable maps in genus one and logarithmic geometry II. *ArXiv e-prints*, September 2017.
- [Vak00] R. Vakil. The enumerative geometry of rational and elliptic curves in projective space. *J. Reine Angew. Math.*, 529:101–153, 2000.
- [VZ08] R. Vakil and A. Zinger. A desingularization of the main component of the moduli space of genus-one stable maps into \mathbb{P}^n . *Geom. Topol.*, 12(1):1–95, 2008.

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