GROMOV–WITTEN THEORY OF HYPERSURFACES IN GENUS ONE

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Abstract.

Contents

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1. Gathmann's line bundle via tropical geometry

For each marking x_k we consider the locus in $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$ where x_k belongs to an internal component of the collapsed map. We use the log structure on $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$ to construct a line bundle $\mathcal{L}_k$ together with a section s_k vanishing along this locus. We use the correspondence with tropical geometry to identify $\mathrm{c}_1(\mathcal{L}_k)$ in terms of tautological classes on $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$, and to compute the vanishing order of s_k along the components of its zero set.

The pair $(\mathcal{L}_k, s_k)$ is most naturally constructed on the moduli space $\overline{\mathcal{M}}^{\log}_{1,\alpha}(\mathbb{P}^N|H)$ of non-expanded log maps; the corresponding pair on $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$ will be obtained via pull-back. Consider therefore a moduli point in $\overline{\mathcal{M}}^{\log}_{1,\alpha}(\mathbb{P}^N|H)$ and let $Q^\vee_{\mathbb{R}}$ be the corresponding tropical moduli space (viewed as a rational polyhedral cone). Over this cone we have a universal tropical curve and map

$$\begin{tikzcd} \sqsubset \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{R}_{\geq 0} \\ Q^\vee_{\mathbb{R}} \arrow[u, bend left, "x_k"] & \end{tikzcd}$$

where x_k is the section which for every point $\lambda \in Q^\vee_{\mathbb{R}}$ picks out the vertex of $\sqsubset_\lambda$ containing the leg x_k. The composition $f \circ x_k \colon Q^\vee_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$ defines a piecewise-linear function on $Q^\vee_{\mathbb{R}}$ whose preimage over the open cone $\mathbb{R}_{>0}$ consists of those tropical maps where x_k belongs to an internal component. Dually we obtain an element of the minimal monoid Q at the moduli point, and this globalises to produce a section of the ghost sheaf on $\overline{\mathcal{M}}^{\log}_{1,\alpha}(\mathbb{P}^N|H)$. In the usual way this induces a line bundle and section $(\mathcal{L}_k, s_k)$ on the moduli space, and the tropical description above shows that the zero locus of s_k is (set-theoretically) the locus where x_k belongs to an internal component.

We now calculate $\mathrm{c}_1(\mathcal{L}_k)$. Choose a family of log stable maps over S and let $\mu \in \Gamma(S, \overline{M}_S)$ be the global section of the ghost sheaf constructed in the previous paragraph. This pulls back along π to give a global section $\pi^\flat(\mu) \in \Gamma(C, \overline{M}_C)$. Interpreted as a piecewise-linear function on the tropicalisation $\sqsubset$ with values in $\overline{M}_S$ [CCUW17, Remark 7.3], this assigns μ to every vertex and has slope zero along every edge. By construction, the line bundle associated to this section is $\pi^\star \mathcal{L}_k$. Consider on the other hand the generator $1 \in \mathbb{N} = \Gamma(\mathbb{P}^N, \overline{M})$ with associated line bundle $\mathcal{O}(H)$. The section $f^\flat(1) \in \Gamma(C, \overline{M}_C)$ has associated line bundle $f^\star \mathcal{O}(H)$. If we let v denote the vertex containing x_k, then by construction $f^\flat(1)$ assigns μ to v and has slope α_k along the leg x_k. Thus if we consider the difference $f^\flat(1) - \pi^\flat(\mu)$ then this assigns 0 to v and still has slope α_k along x_k. Thus by [RSW17, Proposition 2.4.1] the corresponding line bundle restricted to C_v is given by

$$\mathcal{O}_{C_v}\left(\alpha_k x_k + \sum_e \mu_e x_e\right)$$

where the sum is over the edges e adjacent to v and distinct from x_k. Thus we see that:

$$\left(f^\star \mathcal{O}(H) \otimes \pi^\star \mathcal{L}_k^{-1}\right)|_{C_v} = \mathcal{O}_{C_v}\left(\alpha_k x_k + \sum_e \mu_e x_e\right).$$

Since x_k factors through C_v we may pull back along x_k to obtain

$$\mathcal{L}_k = x_k^\star \pi^\star \mathcal{L}_k = x_k^\star \mathcal{O}_{C_k}(-\alpha_k x_k) \otimes x_k^\star f^\star \mathcal{O}(H) = x_k^\star \mathcal{O}_{C_k}(-\alpha_k x_k) \otimes \mathrm{ev}_k^\star \mathcal{O}(H)$$

and taking Chern classes gives:

$$\mathrm{c}_1(\mathcal{L}_k) = \alpha_k \psi_k + \mathrm{ev}_k^\star H.$$

This gives the construction of $(\mathcal{L}_k, s_k)$ on $\overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H)$; the construction on $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$ is given by pull-back. Notice in particular that ψ_k should be interpreted as a **collapsed psi class** on $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$, i.e. a psi class coming from the stabilised curve of the collapsed map. All of the psi classes we deal with will be collapsed psi classes.

Remark 1.1. This construction gives the natural logarithmic analogue of Gathmann's line bundle and section [Gat02, Construction 2.1]. A benefit of the logarithmic approach is to make the computation of vanishing orders entirely combinatorial (see §?? below), circumventing the difficult technical calculation given by Gathmann.

2. Quantum Lefschetz recursion

Consider a smooth pair (X, Y) with Y very ample, and let P be the projective bundle $P = \mathbb{P}_Y(\mathrm{N}_{Y|X} \oplus \mathcal{O}_Y)$. We assume that the genus zero and reduced genus one Gromov–Witten theories of X are known. From this starting data, we will apply our recursion formula to compute:

(1) the genus one **reduced restricted absolute Gromov–Witten theory** of Y;
(2) the genus one **reduced relative Gromov–Witten theory** of the pair (X, Y);
(3) a certain slice of the genus one **reduced rubber theory** of P.

2.1. Reduced absolute, relative and special rubber invariants.

To be precise: by a reduced invariant of Y we mean an integral over $\mathcal{VZ}_{1,n}(Y, \beta)$ of products of pullbacks of evaluation and psi classes along morphisms which forget a subset S of the marked points (taking $S = \emptyset$ gives the ordinary evaluation and psi classes). Here the evaluation maps are viewed as mapping into X (hence the adjective "restricted"). Reduced relative invariants of (X, Y) are defined in the same way, except now the forgetful morphism maps into a space of absolute maps:

$$\mathrm{fgt}_S \colon \mathcal{VZ}_{1,\alpha}(X|Y, \beta) \to \mathcal{VZ}_{1,m-\#S}(X, \beta).$$

In particular, all the psi classes which we consider are *collapsed psi classes*, meaning that they are relative cotangent line classes for the corresponding collapsed stable map. Note that, unlike in the absolute case, in the relative case it may well be the case that the entire insertion is pulled back along a single forgetful map fgt_S.

The reduced rubber invariants of P are defined similarly (again using collapsed psi classes). The particular slice we are interested in is the theory of **special reduced rubber invariants**. These are by definition the rubber invariants which involve at least one marked point of negative tangency (meaning nonzero tangency to Y_0), and such that the negative-tangency markings carry only primary insertions. The significance of this class of rubber invariants will become clear during the recursion.

The systems of invariants defined above are equivalent to the classical systems of invariants (which do not use forgetful morphisms) by well-known topological recursion relations.

2.2. Fictitious and true markings.

The recursion procedure is rather delicate. Roughly speaking, we will induct on the degree (meaning $Y \cdot \beta$), number of marked points and total tangency. To get the correct notion of number of markings and total tangency in the relative setting, we introduce the concept of **fictitious markings**. Consider a moduli space $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$ of reduced relative stable maps and a corresponding integrand γ. We let $F \subseteq \{1, \ldots, m\}$ be the maximal subset of marked points such that:

(1) $\alpha_i = 1$ for all $i \in F$;
(2) the entire integrand γ is pulled back along fgt_F.

This subset is uniquely determined, and its elements are referred to as **fictitious markings**. Markings which are not fictitious are referred to as **true**. When inducting on relative invariants we will always be interested in the number of true markings (as opposed to the total number of markings) and the true tangency

$$\sum_{i \notin F} \alpha_i \leq d = Y \cdot \beta$$

as opposed to the total tangency, which is always d. This formalises the idea that relative invariants with non-maximal tangency $t < d$ can be obtained by adding $d - t$ fictitious markings of tangency 1; see [Gat02, Lemma 1.15(i)].

2.3. **Structure of the recursion**. Given the genus-zero Gromov–Witten theory of X, the arguments of [Gat02] give an effective algorithm to reconstruct the genus-zero theories of Y and (X, Y); moreover, the genus-zero rubber theory of P is identical to the genus-zero theory of Y [Gat03]. Thus we may assume that all genus-zero data is known. We assume in addition that we know the genus one reduced theory of X. The structure of the recursion is then as follows:

for $d \geq 0$:
  for $n \geq 0$:
    for $t \geq 0$:
      Step 1: Compute forgetful relative invariants of (X, Y) (degree d, $n+1$ true markings, true tangency t); see below.
    Step 2: Compute absolute invariants of Y (degree d, n markings).
    for $t \geq 0$:
      Step 3: Compute relative invariants of (X, Y) (degree d, n true markings, true tangency t).
  for $n \geq 0$:
    Step 4: Compute special rubber invariants of P (degree d, n relative markings).

Although the loops involving d and n have infinite length, in order to compute any single invariant it is only necessarily to iterate the preceding loops for a finite amount of time. A **forgetful relative invariant** of (X, Y) is by definition a reduced relative invariant with a marked point x_0 such that all of the insertions are pulled back along fgt_{x_0}. In our recursion, we first deal with this special class of relative invariants (with $n+1$ true markings), before computing the absolute invariants (with n markings) and then returning to compute all of the relative invariants (with n true markings). This need to treat separately a particular subclass of the relative invariants is an inescapable feature of the genus one recursion.

The base terms of the recursion all have $d = 0$ and as such are easy to compute: the relative invariants of (X, Y) are nothing but absolute invariants of X, the absolute invariants of Y are given by obstruction bundle integrals over Deligne–Mumford spaces, and the special rubber invariants of P are also given by integrals over Deligne–Mumford spaces, using the formula for the double ramification cycle [Hai13, JPPZ17] in terms of tautological classes. We will now explain how to perform each of the four inductive steps outlined above.

Notation 2.1. Given tuples $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, we say that $\mathbf{a} < \mathbf{b}$ if there exists an $i \in \{1, \ldots, n\}$ such that $a_j = b_j$ for $j < i$ and $a_i < b_i$.

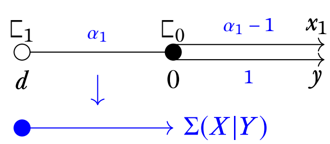
Step 1. Suppose we are given a forgetful relative invariant to compute. That is, we have a relative space $\mathcal{VZ}_{1,\alpha}(X|Y,\beta)$ of degree d, with $n+1$ true markings and true tangency t, and a marking x_0 such that the insertion γ is pulled back along fgt_{x_0}. We assume inductively that every absolute, relative and special rubber invariant with $(d', n') < (d, n)$ is known, and also that every forgetful relative invariant with $(d', n', t') < (d, n+1, t)$ is known. Choose a true marking x_1 with $\alpha_1 \geq 1$ and consider the space:

$$\mathcal{VZ}_{1,(\alpha - e_1)\cup(1)}(X|Y,\beta).$$

Denote the newly-introduced marking by y and consider the integrand $\tilde{\gamma}$ obtained from γ by introducing $\mathrm{fgt}_y^\star$ everywhere. Applying our recursion formula to x_1, we obtain:

$$\left((\alpha_1 - 1)\psi_1 + \mathrm{ev}_1^\star Y\right) \tilde{\gamma} \cap [\mathcal{VZ}_{1,(\alpha-e_1)\cup(1)}(X|Y,\beta)] = \tilde{\gamma} \cap [\mathcal{D}(1)]. \tag{1}$$

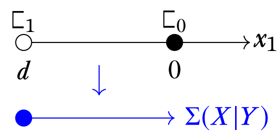
Let us first examine the left-hand side. The class ψ_1 differs from $\mathrm{fgt}_y^\star \psi_1$ by the locus where x_1 and y are contained on a contracted rational bubble. This locus consists of reduced relative stable maps of the form



with all other marked points contained on C_1. This is isomorphic to $\mathcal{VZ}_{1,\alpha}(X|Y,\beta)$ and when we restrict $\tilde{\gamma}$ to this locus we obtain precisely the class γ which we started with. Thus the left-hand side of (1) may be written as

$$(\alpha_1 - 1)I + \mathrm{fgt}_y^\star\left((\alpha_1 - 1)\psi_1 + \mathrm{ev}_1^\star Y\right) \tilde{\gamma} \cap [\mathcal{VZ}_{1,(\alpha-e_1)\cup(1)}(X|Y,\beta)]$$

where I is the invariant we are trying to compute. The second term is a forgetful relative invariant with the same degree and number of true markings (since y is fictitious), and strictly smaller true tangency; hence it is recursively known. We now examine the right-hand side of (1). Recall that all of the genus zero data has already been computed, so we only need to focus on the genus one pieces. First consider the type A loci. The genus one piece has strictly smaller degree (and hence is recursively known) exept in the following situation (with some stable distribution of the remaining markings):



In this situation, C_1 contains at most $n+1$ true markings. If it has $n-1$ or fewer, than it is known recursively. If it has exactly n this means that C_0 contains exactly one true marking (besides x_1, which may be true or fictitious). We claim that in this situation we must have $y \in C_1$, since otherwise we would have a moduli space for C_0 given by $\overline{\mathcal{M}}_{0,k}$ with $k \geq 4$, and applying the projection formula with fgt_y we would conclude that the contribution is zero. Thus we have $y \in C_1$, and so the genus one contribution is a forgetful relative invariant with n true markings, and hence is recursively known.

Finally, if C_1 contains exactly $n+1$ true markings, then the only additional markings on C_0 are fictitious, and by the same argument as in the previous paragraph there can only be one. Thus for each fictitious marked point we obtain a locus isomorphic to $\mathcal{VZ}_{1,\alpha}(X|Y,\beta)$ and $\tilde{\gamma}$ restricts to γ here (from the point of view of computing invariants, the fictitious marked points are indistinguishable, meaning that the contributions are all the same). Thus for each fictitious marked point (of which there is at least one, namely y) we get a contribution of $\alpha_1 I$ to the right-hand side of (1).

The contributions of the type B loci only involve genus zero data and hence are known. The contributions of the type C^0 loci are determined by genus zero data and tautological integrals on Deligne–Mumford space, hence are also known. It remains to consider type C^+ loci. Notice that the rubber integrals which appear are always special rubber invariants, as defined above. If the degree of the genus one piece is less than d then we have a special rubber invariant of strictly smaller degree. The only other possibility is that the entire curve is mapped into the divisor. In this case we may apply the projection formula with fgt_y to identify this with an integral over $\mathcal{VZ}_{1,m}(Y,\beta)$ for some (possibly large) number m of marked points. But by assumption there is another marked point x_0 with all of the insertions pulled back along fgt_{x_0}, so a further application of the projection formula shows that this contribution vanishes. To conclude, we may rearrange (1) to obtain an expression of the form

$$\lambda I = \text{recursively known terms}$$

where λ is an explicit scalar which is always nonzero; we have thus determined I, which completes Step 1.

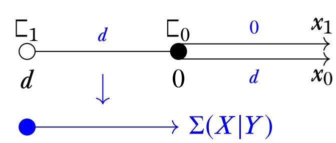
Step 2. Consider now the absolute space $\mathcal{VZ}_{1,n}(Y,\beta)$ with an insertion γ, and suppose inductively that we have computed all forgetful relative invariants with $(d',n') \leq (d,n+1)$, all relative invariants and absolute invariants with $(d',n') < (d,n)$, and all special rubber invariants with $d' < d$. Consider the following moduli space with $n+1$ markings:

$$\mathcal{VZ}_{1,(d,0,\ldots,0)}(X|Y,\beta).$$

Let x_0 denote the relative marking and consider the integrand $\tilde{\gamma}$ obtained from γ by introducing $\mathrm{fgt}_{x_0}^{\star}$ everywhere. Now recurse at x_1 to obtain:

$$\mathrm{ev}_1^{\star} Y \cdot \tilde{\gamma} \cap [\mathcal{VZ}_{1,(d,0,\ldots,0)}(X|Y,\beta)] = \tilde{\gamma} \cap [\mathcal{D}(1)]. \tag{2}$$

The left-hand side is a forgetful relative invariant of degree d and $\leq n+1$ true markings, and so has already been computed. For the right-hand side, let us begin with loci of type A. The genus one contributions from each loci have $(d',n') < (d,n)$ except in the following case



which gives a contribution of

$$d \cdot \gamma \cap [\mathcal{VZ}_{1,(d,\underbrace{0,\ldots,0}_{n-1})}(X|Y,\beta)]$$ [1]

to the right-hand side of (2). Here γ is the insertion we started with; the difference is that we are now considering relative maps to (X,Y) with maximal tangency at x_1, rather than absolute maps to Y. The type B and C^0 loci are recursively determined as in Step 1, and similarly the type C^+ loci are recursively determined except for the locus where the entire curve is mapped into the divisor. On this locus we may apply fgt_{x_0} and identify the contribution with

$$d^2 \cdot \gamma \cap [\mathcal{VZ}_{1,n}(Y,\beta)] = d^2 \cdot I$$

where I is the invariant we are trying to compute. Putting this all together, we obtain

$$I = (-\gamma/d) \cap [\mathcal{VZ}_{1,(d,\underbrace{0,\ldots,0}_{n-1})}(X|Y,\beta)] + \text{recursively known terms} \tag{3}$$

[1](Navid) Fix formatting

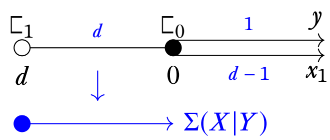
where on the right-hand side there are $n-1$ non-relative markings $x_2, \ldots, x_n$, and a relative marking x_1. We now apply the recursion again to the right-hand side, by considering the space

$$\mathcal{VZ}_{1,(d-1,1,0,\ldots,0)}(X|Y,\beta)$$

where x_1 now has tangency $d-1$ and we have introduced a new marking y with tangency 1. We consider a new insertion, denoted $\tilde{\gamma}$ as usual, by introducing $\mathrm{fgt}_y^\star$ everywhere. Recursing at x_1 we obtain:

$$\big((d-1)\psi_1 + \mathrm{ev}_1^\star Y\big) \cdot (-\tilde{\gamma}/d) \cap [\mathcal{VZ}_{1,(d-1,1,0,\ldots,0)}(X|Y,\beta)] = (-\tilde{\gamma}/d) \cap [\mathcal{D}(1)]. \tag{4}$$

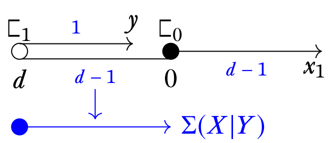
The difference between ψ_1 and $\mathrm{fgt}_y^\star \psi_1$ is given by the locus where x_1 and y belong to a collapsed rational bubble:



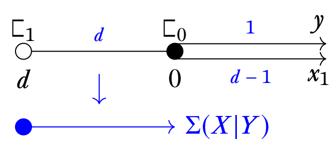
The contribution of this locus to the left-hand side of (4) is:

$$(d-1) \cdot (-\gamma/d) \cap [\mathcal{VZ}_{1,(d,\underbrace{0,\ldots,0}_{n-1})}(X|Y,\beta)].$$

What remains on the left-hand side is a forgetful relative invariant of degree d and $\leq n+1$ true markings (y being the "forgetful" marking), hence is recursively known. On the right-hand side, the type A loci are recursively known except possibly in the following special cases (with some stable distribution of the remaining non-relative markings):



Case 1



Case 2

In Case 1 the contribution from C_1 is a forgetful relative invariant with $\leq n$ true markings, hence is recursively known. In Case 2, we first note that there cannot be any more markings on C_0 (since otherwise we could apply fgt_y and conclude that the contribution vanishes). Thus we obtain a single locus, which contributes precisely

$$d \cdot (-\gamma/d) \cap [\mathcal{VZ}_{1,(d,\underbrace{0,\ldots,0}_{n-1})}(X|Y,\beta)] = -\gamma \cap [\mathcal{VZ}_{1,(d,\underbrace{0,\ldots,0}_{n-1})}(X|Y,\beta)]$$

which gives us the first term on the right-hand side of (4). As usual the type B and C^0 contributions are known recursively, and the only type C^+ contribution not known recursively occurs when the entire curve is mapped into the divisor, in which case we apply fgt_y to calculate the contribution as:

$$(-\gamma/d) \cap [\mathcal{VZ}_{1,n}(Y,\beta)] = -I/d.$$

Substituting into (4) we end up with

$$I(1-d^{-1}) = \text{recursively known terms}$$

which completes the recursion step as long as $d \neq 1$. But since $\mathcal{VZ}_{1,n}(H,1) = \emptyset$ it follows that $\mathcal{VZ}_{1,n}(Y,\beta) = \emptyset$ if $d = Y \cdot \beta = 1$, so we may always assume $d \geq 2$ in the recursion.

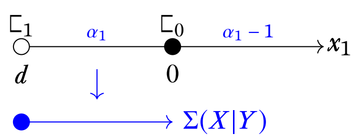
Step 3. Now suppose we are given a relative space $\mathcal{VZ}_{1,\alpha}(X|Y,\beta)$ with an insertion γ, and suppose inductively that we have computed all forgetful relative invariants with $(d',n') \leq (d,n+1)$, all absolute invariants with $(d',n') \leq (d,n)$, all relative invariants with $(d',n',t') < (d,n,t)$ and all special rubber invariants with $d' < d$. Choose a true marking x_1 with $\alpha_1 \geq 1$ and consider the moduli space

$$\mathcal{VZ}_{1,(\alpha-e_1)\cup(1)}(X|Y,\beta)$$

where y is the newly-introduced marking. As usual consider the insertion $\tilde{\gamma}$ obtained from γ by introducing fgt_y everywhere. Recursing at x_1 we obtain:

$$\big((\alpha_1-1)\psi_1 + \mathrm{ev}_1^\star H\big)\,\tilde{\gamma} \cap [\mathcal{VZ}_{1,(\alpha-e_1)\cup(1)}(X|Y,\beta)] = \tilde{\gamma} \cap [\mathcal{D}(1)].$$

The left-hand side is a relative invariant with the same degree and number of true markings, but smaller true tangency: hence it is recursively known. On the right-hand side, the type A contributions are recursively known except for those of the following form



where C_0 contains a single fictitious marking (if it had multiple fictitious markings then the contribution would vanish by projection formula) and all the other markings are on C_1. Thus for each fictitious marking we get a contribution of $\alpha_1 I$ where I is the invariant we are trying to compute. Note that $\alpha_1 \neq 0$, and that this term appears at least once since y is a fictitious marking; so we get a nonzero multiple of I.

As usual the type B and C^0 contributions are recursively known. The type C^+ contributions are determined by lower-degree special rubber invariants, except for when the whole curve is mapped into Y; however in this case we may apply fgt_y to identify the contribution with an absolute invariant of Y with degree d and n markings, which is also recursively known. Thus we have determined the invariant I.

Step 4. Finally, suppose we are given a special reduced rubber invariant to compute. So we have a reduced rubber space $\mathcal{VZ}_{1,\alpha}(P|Y_0 + Y_\infty, \beta)_{\mathbb{G}_m}$ together with an insertion γ satisfying the conditions outlined above. We first note that if there exists a relative marking with no insertion, we may forget this marking and identify the rubber invariant with a recursively-known absolute invariant (up to some boundary terms which can also be computed recursively). This approach will suffice for the majority of computations.

[TO BE FINISHED]

3. Recursion procedure for reduced rubber integrals

[2] In the recursion above, certain terms appear which are integrals over moduli spaces of reduced log stable maps to rubber (as defined in §??):

$$\mathcal{VZ}_{1,\alpha}(P|H_0+H_\infty)_{\mathbb{G}_{\mathrm{m}}}.$$

In this section we provide a recursive algorithm for computing such integrals. We have three main techniques at our disposal: direct comparison with the double ramification cycle, forgetting of a marking with no insertion, and rubber recursion. Each technique has its own domain of applicability, and these domains have non-trivial intersection. For convenience we denote the **horizontal degree** of the rubber map by:

$$d := \sum_{i=1}^{n} \alpha_i.$$

Recall that the α_i may be positive or negative (recording tangencies with respect to H_∞ and H_0, respectively) and that $d \geq 0$ always. First suppose that $d = 0$. Then the usual rubber moduli space decomposes as

$$\overline{\mathcal{M}}^{\log}_{1,\alpha}(P|H_0+H_\infty)_{\mathbb{G}_{\mathrm{m}}} = \overline{\mathcal{M}}^{\log}_{1,\alpha}(\mathbb{P}^1|0+\infty)_{\mathbb{G}_{\mathrm{m}}} \times H$$

and the first term on the right-hand side is birational to the double ramification locus in $\overline{\mathcal{M}}_{1,n}$. This locus has a main component (the closure of the locus of smooth curves) and one other "exceptional" component, consisting of split curves where all the markings belong to the rational tail. The corresponding reduced rubber space is birational to the main component, and thus we can identify its class by subtracting off the class of the exceptional component. This gives

$$\pi_\star[\mathcal{VZ}_{1,\alpha}(P|H_0+H_\infty)_{\mathbb{G}_{\mathrm{m}}}] = \left([\overline{\mathcal{M}}^{\alpha}_{1,n}] - [D]\right) \times [H] \in \mathrm{A}_\star(\overline{\mathcal{M}}_{1,n} \times H) = \mathrm{A}_\star(\overline{\mathcal{M}}_{1,n}(H,0))$$

where $\pi\colon \mathcal{VZ}_{1,\alpha}(P|H_0+H_\infty)_{\mathbb{G}_{\mathrm{m}}} \to \overline{\mathcal{M}}_{1,n}(H,0)$ is the map which forgets the log structure and collapses the map, D is the boundary locus of split curves described above, and $[\overline{\mathcal{M}}^{\alpha}_{1,n}]$ is the classical double ramification cycle, which has an explicit formula in terms of tautological classes [Hai13, JPPZ17]. These observations allow us to compute the integral over the rubber space as a tautological integral on the moduli space of curves.

Now suppose that $d > 0$. If there is a marking x_k which carries no insertion and has $\alpha_k \neq 0$, then consider the morphism which forgets x_k and collapses the map:

$$\mathcal{VZ}_{1,\alpha}(P|H_0+H_\infty)_{\mathbb{G}_{\mathrm{m}}} \xrightarrow{\pi} \mathcal{VZ}_{1,n-1}(H,d).$$

[2] (Navid) To be superseded by general recursion section

The moduli spaces above are log smooth and of the same dimension. Over the nice locus in $\mathcal{VZ}_{1,n-1}(H,d)$, π has degree α_k^2, arising from a choice of α_k-torsion point on the source elliptic curve. Thus we have:

$$\pi_\star[\mathcal{VZ}_{1,\alpha}(P|H_0+H_\infty)_{\mathbb{G}_\mathrm{m}}] = \alpha_k^2 \cdot [\mathcal{VZ}_{1,n-1}(H,d)].$$

Since all the insertions are pulled back from the right-hand side (up to modifying psi classes by some boundary strata which we can then also compute recursively) we can perform the integral on $\mathcal{VZ}_{1,n-1}(H,d)$ inductively. Notice that, when we apply our recursion **??** to compute a reduced invariant of H, the only non-trivial reduced rubber integrals which appear will have smaller degree than the one we started with, so the calculation is not circular.

It remains to deal with the case where all of the relative markings carry insertions. The idea is to reduce to the case above by introducing an additional marking. Note first that every reduced rubber integral which appears in our recursion must either have a marking of negative tangency (i.e. tangent to H_0), or a marking with no insertions: indeed, the only boundary loci with no negative-tangency markings have the entire curve mapped to higher level, in which case (since we are assuming $d>0$, so we are not in the C_0 case) the curve is generically irreducible, and on this curve we have at least one fictitious marking (of positive tangency 1) with no insertion.

Therefore we may assume that there is at least one point of negative tangency, corresponding to a node at which we will glue. Notice that in our recursion these points only ever carry evaluation classes. Consider the following rubber space of dimension one higher, obtained by introducing an additional non-relative marking:

$$\mathcal{VZ}_{1,\alpha\cup(0)}(P|H_0+H_\infty)_{\mathbb{G}_\mathrm{m}}. \tag{5}$$

Given a moduli point in this space, consider the piecewise-linear function on the associated tropical moduli space which sends a tropical map to the sum of the target edge lengths. This defines a line bundle and section on the space (5) which vanishes precisely on the locus where the target splits; moreover the vanishing orders may be read off from the tropical data as in §??. It is not hard to see [Kat07] that the first Chern class of the line bundle is $\Psi_0+\Psi_\infty$, a sum of target psi classes which may be identified with evaluation and psi classes on the source curve as in [Gat03, Construction 5.1.17].

Now choose a negative-tangency point x_1 which carries only evaluation classes, and let x_0 denote the new non-relative marking we have introduced. If γ was the class we were originally trying to integrate, consider the class $\tilde{\gamma}$ on (5) obtained from γ by replacing $\mathrm{ev}_1^\star$

by $\mathrm{ev}_0^\star$. We obtain a recursion formula:

$$(\Psi_0 + \Psi_\infty) \cdot \tilde{\gamma} \cap [\mathcal{VZ}_{1,\alpha\cup(0)}(P|H_0 + H_\infty)_{\mathbb{G}_\mathrm{m}}] = \sum_\Gamma m_\Gamma \cdot \tilde{\gamma} \cap [\overline{\mathcal{M}}_\Gamma].$$

The right-hand side is a sum over irreducible components $\overline{\mathcal{M}}_\Gamma$ of the locus where the target splits, with multiplicites m_Γ calculated from the tropical data. Each of these components can be written as a fibre product of (reduced) rubber spaces. One of these components is the locus where x_0 and x_1 belong to a trivial rational bubble:

[FIGURE]

The corresponding moduli space is $\mathcal{VZ}_{1,\alpha}(P|H_0 + H_\infty)_{\mathbb{G}_\mathrm{m}}$ and we have $\mathrm{ev}_0 = \mathrm{ev}_1$ here, so the contribution of this locus is precisely the original rubber integral we were trying to calculate. The remaining terms on the right-hand side can be written either as a product of rubber integrals of smaller degree (which we can compute recursively), or rubber integrals of the same degree but with no insertion at x_1 (which we can compute by forgetting x_1). Finally, the left-hand side also has no insertion at x_1 and so can be computed by the same method. Here we are assuming that x_1 is not the only negative-tangency marking, so that Ψ_0 can be interpreted as an insertion on a marking different from x_1.

If x_1 is in fact the only negative-tangency marking, then we can modify the above argument by considering the following rubber space of dimension two larger:

$$\mathcal{VZ}_{1,\alpha\cup(0,0)}(P|H_0 + H_\infty)_{\mathbb{G}_\mathrm{m}} \tag{6}$$

Let us denote the newly-introduced non-relative markings by x_0 and x_{n+1}. We now let $\tilde{\gamma}$ denote the class obtained from γ by replacing $\mathrm{ev}_1^\star$ by $\mathrm{ev}_0^\star$ and then multiplying by ψ_{n+1}. Following [Kat07][3], there is a line bundle with a section cutting out the locus where x_{n+1} does not belong to the leftmost component of the target. The first Chern class of this line bundle is $\Psi_\infty + \mathrm{ev}_{n+1}^\star H$, which we may multiply with $\tilde{\gamma}$ and integrate over (6) to obtain a rubber integral with no insertion at x_1 (which we may then compute by forgetting x_1). On the right-hand side we obtain the integral of $\tilde{\gamma}$ over the loci where x_{n+1} does not belong to the leftmost component of the target (this is just a union of some of the irreducible components of the split locus described above). One such locus is the following [FIGURE] and integrating $\tilde{\gamma}$ over this locus gives

$$\gamma \cdot \psi_{n+1} \cap [\mathcal{VZ}_{1,\alpha\cup(0)}(P|H_0 + H_\infty)_{\mathbb{G}_\mathrm{m}}]$$

which we may identify with a (nonzero) multiple of our original rubber integral via the rubber dilaton equation (see[?]; the same proof carries over to the reduced setting[4]). The

[3] (Navid) Tropica interpretation of this? Vanishing multiplicities?

[4] (Navid) Think about this a bit

remaining loci may be computed recursively exactly as before, and thus we obtain a complete recursion procedure.

Remark 3.1. This idea — of finding a section of a line bundle cutting out the locus of split curves, and using this to derive recursion formulae for rubber integrals — has been pursued in the non-reduced setting in [Kat07].

4. Description of the logarithmic strata

Let $\mathcal{VZ} = \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$ and let $\mathcal{A}$ denote the Artin fan of $\mathcal{VZ}$. Since the map of log stacks

$$\mathcal{VZ} \to \mathcal{A}$$

is smooth, we see that the codimension-k logarithmic strata in $\mathcal{VZ}$ are in bijective correspondence (via pull-back) with the codimension-k logarithmic strata in $\mathcal{A}$. The logarithmic strata in $\mathcal{A}$, on the other hand, have a purely combinatorial description (at least locally) in terms of the associated moduli spaces of tropical maps, coming from the fact that $\mathcal{VZ}$ is a logarithmic blow-up of the usual moduli space of log stable maps. In this section we discuss this circle of ideas, and show how it plays out in a number of examples.

4.1. **Blowing up the Artin fan**. Let $\mathcal{M} = \overline{\mathcal{M}}^{\log}_{1,\alpha}(\mathbb{P}^N|H,d)$ denote the Abramovich–Chen–Gross–Siebert moduli space of log stable maps. Recall that $\mathcal{VZ}$ is obtained as a closed substack of a log modification

$$\widetilde{\mathcal{VZ}} \to \mathcal{M}.$$

Since the map $\mathcal{VZ} \hookrightarrow \widetilde{\mathcal{VZ}}$ is strict, the Artin fan $\mathcal{A}$ of $\mathcal{VZ}$ is locally isomorphic to the Artin fan of $\widetilde{\mathcal{VZ}}$ (with the latter being, in general, larger)[5]. Since the following discussion is entirely local, we will ignore the distinction between the two, and pretend as though $\mathcal{A}$ is the Artin fan of $\widetilde{\mathcal{VZ}}$. There is then a commuting square:

$$\begin{array}{ccc} \widetilde{\mathcal{VZ}} & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}_{\mathcal{M}}. \end{array}$$

Note that neither of the vertical maps are smooth, since the moduli spaces on the top row are not log smooth. The construction of $\widetilde{\mathcal{VZ}} \to \mathcal{M}$ as the log modification obtained by imposing an alignment condition gives us a combinatorial description of the map $\mathcal{A} \to \mathcal{A}_{\mathcal{M}}$ which we will use to study $\mathcal{A}$. Recall that $\mathcal{A}_{\mathcal{M}}$ is locally isomorphic to the stack quotient

$$\left[\operatorname{Spec} \Bbbk[Q] / \operatorname{Spec} \Bbbk[Q^{\mathrm{gp}}]\right]$$

where Q is a monoid giving a[6] local chart for $\mathcal{M}$, which we may take to be the minimal monoid of [GS13, §1.5]. The real dual $Q^\vee_{\mathbb{R}} = \operatorname{Hom}(Q, \mathbb{R}_{\geq 0})$ of this monoid can be viewed as a moduli space of tropical maps; see [GS13, Remark 1.12]. We call this *the tropical moduli space*; contained in the tropical moduli space are the edge lengths of the associated tropical curve (corresponding to the smoothing parameters of the nodes of the logarithmic curve). Since the alignment condition amounts to imposing a partial ordering amongst certain

[5] (Navid) Make this more precise

[6] (Navid) Neat?

sums of these edge lengths, it produces a polyhedral decomposition of the tropical moduli space, into chambers where different partial order relations hold. If we only consider the integral points, this produces a polyhedral decomposition of the cone $Q^\vee = \operatorname{Hom}(Q, \mathbb{N})$. Dualising, we obtain a toric blow-up

$$Z \to \operatorname{Spec} \Bbbk[Q]$$

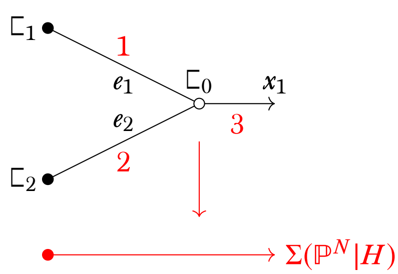
which, since it is equivariant, descends to a morphism of the associated zero-dimensional stacks:

$$[Z/T_Z] \to \left[\operatorname{Spec} \Bbbk[Q] / \operatorname{Spec} \Bbbk[Q^{\mathrm{gp}}]\right].$$

This gives a local description of the map $\mathcal{A} \to \mathcal{A}_{\mathcal{M}}$ and in particular of the Artin fan $\mathcal{A}$. By the orbit-cone correspondence, the codimension-k strata of $\mathcal{A}$ which intersect this open locus correspond to the k-dimensional cones in the polyhedral decomposition of $Q^\vee$ described above. These can be understood entirely in terms of tropical combinatorics. This is best explained through a number of examples, which we now present.

4.2. **Examples of logarithmic strata.** In these examples we will proceed as follows: we will start by fixing an element of the ordinary moduli space $\mathcal{M}$ of log stable maps. We will then compute the associated tropical moduli space, giving a description of $\mathcal{A}_{\mathcal{M}}$ local to our chosen element. We will then describe the necessary polyhedral subdivison, and thus give a description of $\mathcal{A}$ in a neighbourhood of the preimage in $\mathcal{VZ}$ of our chosen element of $\mathcal{M}$. Finally we will use this to describe the logarithmic strata which intersect this neighbourhood.

Example 4.1. Consider an element of $\mathcal{M}$ whose associated tropical map has the following combinatorial type:



Here each edge (corresponding to a node of the curve) has a length e_i and an expansion factor u_i (indicated in red). The moduli space of such tropical maps is generated by the edge lengths e_1 and e_2

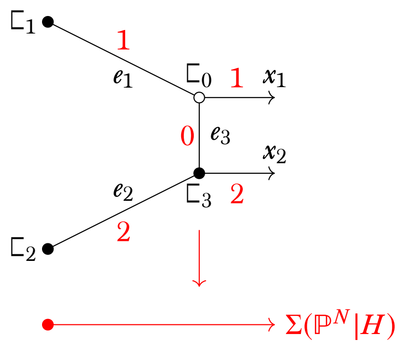
$$(\mathbb{R}_{\geq 0})_{e_1} \times (\mathbb{R}_{\geq 0})_{e_2}$$

subject to the continuity condition $e_1 = 2e_2$. So the tropical moduli space is simply $\mathbb{R}_{\geq 0}$ generated by e_2. Note that for any $e_2 > 0$ (i.e. on the interior of the cone) we have:

$$\lambda(\sqsubset_2) = e_2 = e_1/2 < e_1 = \lambda(\sqsubset_1).$$

Thus, any logarithmic map with combinatorial type given by the above picture is automatically aligned. This means that the map $\mathcal{VZ} \to \mathcal{M}$ is an isomorphism in a neighbourhood of our chosen element (there is no blowing up necessary). Indeed, the tropical moduli space is $\mathbb{R}_{\geq 0}$ and this cone does not admit a polyhedral subdivision (there is no non-trivial toric blow-up of $\mathbb{A}^1$).

Example 4.2. Consider now an element of $\mathcal{M}$ whose associated tropical map has the corresponding combinatorial type:



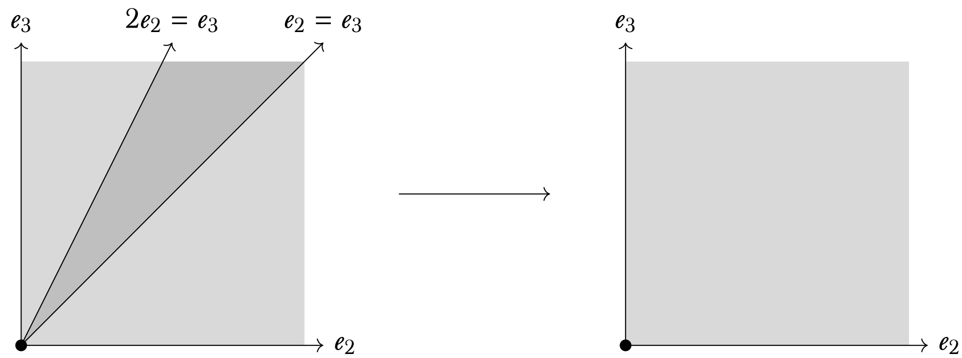
As before, expansion factors are indicated in red and the edge lengths are e_1, e_2, e_3. The tropical moduli space is $\mathbb{R}^3_{\geq 0}$ generated by these three lengths, subject to the continuity condition $e_1 = 2e_2$. Thus the moduli space is isomorphic to $\mathbb{R}^2_{\geq 0}$ generated by e_2 and e_3. In order to have an alignment, at least one of $\lambda(\sqsubset_1)$ and $\lambda(\sqsubset_2)$ must be equal to the radius δ. Note that

$$\lambda(\sqsubset_1) = e_1 = 2e_2$$
$$\lambda(\sqsubset_2) = e_2 + e_3$$

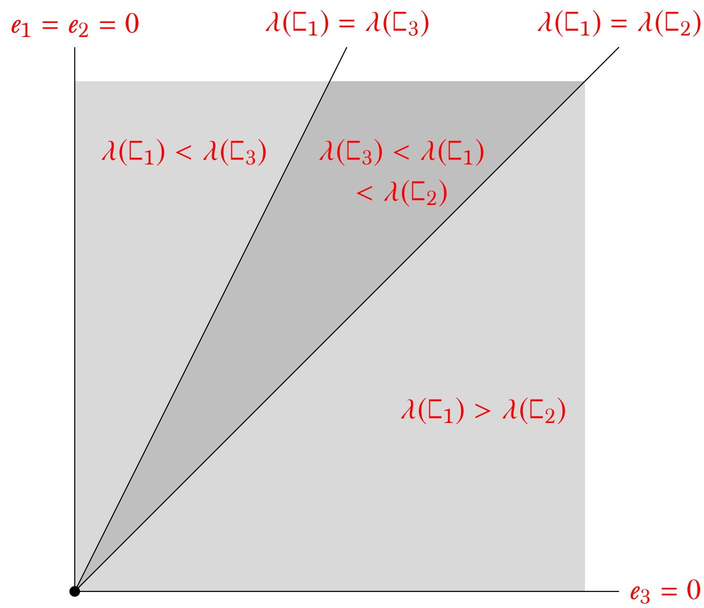
and without more information we cannot say which of these is larger. The subdivision of $(\mathbb{R}_{\geq 0})^2$ is obtained by dividing the cone into regions where different order relations hold amongst the distances $\lambda(\sqsubset_1), \lambda(\sqsubset_2), \lambda(\sqsubset_3)$. The walls of this subdivison correspond to where some of these distances are equal. Note that in this setting we always have $\lambda(\sqsubset_2) > \lambda(\sqsubset_3)$ (at least, as long as we remain in the interior of the cone). The remaining possibilities are:

$$\lambda(\sqsubset_1) = \lambda(\sqsubset_3) \qquad (\Leftrightarrow e_1 = e_3 \Leftrightarrow 2e_2 = e_3)$$
$$\lambda(\sqsubset_1) = \lambda(\sqsubset_2) \qquad (\Leftrightarrow e_1 = e_2 + e_3 \Leftrightarrow e_2 = e_3).$$

Thus, the subdivision of the tropical moduli space $(\mathbb{R}^2_{\geq 0})_{e_2 e_3}$ is given by:



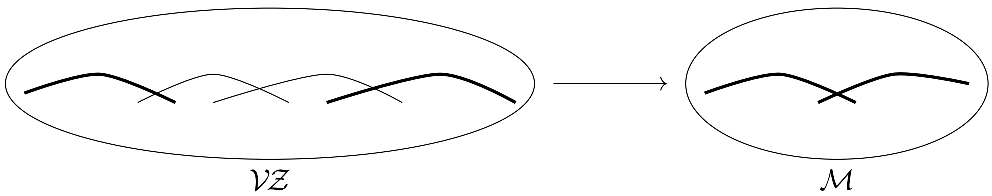
The cones of the subdivison index the logarithmic strata in a neighbourhood of the preimage in $\mathcal{VZ}$ of our chosen element of $\mathcal{M}$. These can be described as follows:



There are four codimension–1 logarithmic strata of $\mathcal{VZ}$ intersecting our chosen neighbourhood, corresponding to the rays in the above picture. Two of these – those corresponding to the rays labeled $\{e_1 = e_2 = 0\}$ and $\{e_3 = 0\}$ – are the proper transforms of codimension–1 strata in $\mathcal{M}$. These strata consist of log stable maps where some of the tropical edge lengths are equal to zero, meaning that the corresponding nodes have been smoothed. Notice that although the curve has three nodes, there are only two such strata: the nodes q_1 and q_2 cannot be smoothed independently because of the relation $e_1 = 2e_2$.

The remaining two codimension–1 strata in $\mathcal{VZ}$ – corresponding to the interior rays in the above picture – consist of log stable maps where some of the vertex distances become equal. Here none of the nodes are smoothed. From the construction of the subdivision, we see that both these strata map onto a codimension–2 stratum of $\mathcal{M}$ (namely, the locus in which all of the nodes persist); this coheres with the fact that they should be thought of as exceptional loci of the blow-up. The extra dimension of moduli comes from the choice of alignment.

Finally, there are three codimension–2 strata, corresponding to different *strict* orderings of the vertex distances. Note that the divisorial strata corresponding to $\{e_1 = e_2 = 0\}$ and $\{e_3 = 0\}$, which intersected in $\mathcal{M}$, no longer intersect in $\mathcal{VZ}$, since we have blown up. The picture is something like:



4.3. **Global logarithmic strata**. In the previous subsections we used tropical geometry to identify the logarithmic strata of $\mathcal{VZ} = \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ local to the preimage of a point in $\mathcal{M}$. In fact, the whole discussion carries over if we replace a point in $\mathcal{M}$ by a locally closed logarithmic stratum: the key observation is simply that the tropical moduli space does not vary if we move inside a locally closed logarithmic stratum, so the subdivision process makes sense over that whole stratum. This fact will allow us to describe the logarithmic strata of $\mathcal{VZ}$ globally.

4.3.1. *Logarithmic strata of* $\mathcal{M}$. The locally closed logarithmic strata of $\mathcal{M}$ consist of loci where the combinatorial type of the associated tropical curve is constant[7]. This is because the combinatorial type determines the minimal monoid Q, which coincides with the stalk of the ghost sheaf on $\mathcal{M}$.

If we have two strata $\mathcal{S}_1$ and $\mathcal{S}_2$ corresponding to combinatorial types Δ_1 and Δ_2, then $\mathcal{S}_2$ is contained in the closure of $\mathcal{S}_1$ if and only if the combinatorial type Δ_1 is obtained from Δ_2 by a process of generisation: namely, by contracting some edges (i.e. smoothing some nodes) and moving some of the vertices from the interior $\mathbb{R}_{>0} \subseteq \mathbb{R}_{\geq 0}$ of the tropicalisation $\Sigma(\mathbb{P}^N|H)$ to the vertex $0 \in \mathbb{R}_{\geq 0}$ (i.e. moving some components of the curve outside H). This allows us to completely describe the dual intersection complex of the logarithmic strata of $\mathcal{M}$.

Note that we are *not* able to easily read off the codimension of a logarithmic stratum from the combinatorial data: the codimension of the associated stratum in the Artin fan $\mathcal{A}_{\mathcal{M}}$ is given by the dimension of the tropical moduli space, but the map $\mathcal{M} \to \mathcal{A}_{\mathcal{M}}$ is not

[7](Navid) Reference for this?

smooth (since $\mathcal{M}$ is not log smooth) so we are not able to say anything about the locus in $\mathcal{M}$.

4.3.2. *Logarithmic strata of* $\mathcal{VZ}$. Now let us pick a stratum $\mathcal{S} \subseteq \mathcal{M}$ indexed by a combinatorial type Δ and with associated monoid Q. We may choose a sufficiently small open neighbourhood of $\mathcal{S}$ which only intersects logarithmic strata $\mathcal{S}'$ which contain $\mathcal{S}$ in their closures. The previous discussion then shows that if we pick any point in this open neighbourhood, the combinatorial type of the associated tropical curve is obtained from Δ by contracting some edges and specialising some vertices. Thus we see that the associated map on tropical moduli spaces is injective, and so the generisation map on the level of ghost sheaves is surjective. This allows us to produce a chart on the open neighbourhood of $\mathcal{S}$ with monoid given by Q.

The discussion in the previous subsections then applies *mutatis mutandis* to this open set, giving a description of the logarithmic strata of $\mathcal{VZ}$ which intersect an open neighbourhood of the preimage of $\mathcal{S}$. Since log strata must map to log strata, this gives a procedure for enumerating all of the locally closed logarithmic strata of $\mathcal{VZ}$, namely:

(1) Enumerate the locally closed logarithmic strata of $\mathcal{M}$ by enumerating all possible combinatorial types of tropical curves with the given numerical data. The dual intersection complex of these strata is specified by generisation of combinatorial types, as described earlier.

(2) For each locally closed stratum $\mathcal{S} \subseteq \mathcal{M}$ identify the tropical moduli space $Q_{\mathbb{R}}^{\vee}$ associated to the given combinatorial type, and perform the subdivision specified by the alignment condition. The arguments of the previous subsections carry over to give a description of the logarithmic strata of $\mathcal{VZ}$ which map to a neighbourhood of $\mathcal{S}$. The dual intersection complex of these strata is specified by the combinatorics of each subdivision together with the dual intersection complex of the strata of $\mathcal{M}$.

We now illustrate this is in some examples:

Example 4.3. Example of low-degree computation of logarithmic strata. (Note that for $\mathcal{M}$ we can't read off the codimension from the dimension of the tropical moduli space because $\mathcal{M}$ is not log smooth.)[8]

[8](Navid) To be done.

5. Splitting axiom and recursion formula for $(\mathbb{P}^N|H)$

5.1. **Recursive description of the divisors: types A, B and C^+.** The basic idea is as follows. Choose an irreducible component $\mathcal{D} \subseteq \mathcal{D}(k)$. As discussed above, we can obtain this by choosing an "initial" combinatorial type together with a ray of the resulting subdivision of the tropical moduli space. Without loss of generality we may assume that this "initial" combinatorial type coincides with the "true" combinatorial type of $\mathcal{D}$ obtained by performing the edge contractions specified by the choice of ray.

Since $\mathcal{VZ} \to \mathcal{M}$ is a log modification, the divisor $\mathcal{D}$ is either exceptional or the proper transform of a logarithmic divisor on $\mathcal{M}$. In this subsection, we will focus on the latter case. Then if we let $\mathcal{D}^\circ_{\mathcal{M}} \subseteq \mathcal{M}$ denote the locally closed stratum corresponding to our choice of combinatorial type, and $\mathcal{D}_{\mathcal{M}} = \overline{\mathcal{D}^\circ_{\mathcal{M}}}$ denote the corresponding logarithmic divisor in $\mathcal{M}$, then our divisor $\mathcal{D} \subseteq \mathcal{VZ}$ is the proper transform of $\mathcal{D}_{\mathcal{M}}$, which is obtained by taking the preimage $\mathcal{D}^\circ$ of $\mathcal{D}^\circ_{\mathcal{M}}$ (since $\mathcal{D}^\circ_{\mathcal{M}}$ is disjoint from the blown-up locus) and then taking its closure in $\mathcal{VZ}$. This is, of course, different from taking the preimage of $\mathcal{D}_{\mathcal{M}}$, since in general $\mathcal{D}_{\mathcal{M}}$ will intersect the blown-up locus. By continuity we have an induced map:

$$\mathcal{D} \to \mathcal{D}_{\mathcal{M}}.$$

The remainder of this section will be dedicated to showing how the map $\mathcal{D} \to \mathcal{D}_{\mathcal{M}}$ may be interpreted as a blow-up. The reason this is useful is that $\mathcal{D}_{\mathcal{M}}$ itself has a recursive description in terms of moduli spaces of punctured maps (see §??), which in our setting we are able to compute integrals over.

For the remainder of this subsection, we will consider only the situation where the circuit is assigned a positive degree by the combinatorial type; the other, more complicated case, will be taken up in §??.

Lemma 5.1. Let $\mathcal{D} \subseteq \mathcal{D}(k)$ be an irreducible component and let Δ be the corresponding combinatorial type. Suppose that Δ assigns positive degree to the circuit. Then Δ takes one of the following forms:

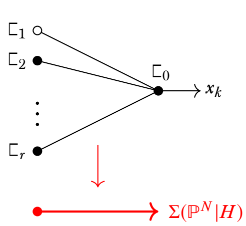


Figure 1. $\mathcal{Y}_A$

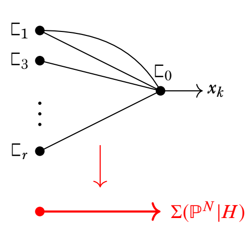


Figure 2. $\mathcal{Y}_B$

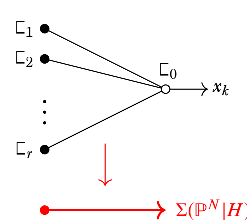


Figure 3. $\mathcal{Y}_C^+$

The terminology is due to Vakil, and the picture in this case is very similar to [Vak00] (as we will see later, when the circuit is contracted the picture becomes *very* different). Note that in these pictures we have omitted the marked points (apart from x_k), the degree of each vertex, and the expansion factors of the edges. These combinatorial data can be distributed arbitrarily, as long as:

(1) the vertices $\sqsubset_1, \ldots, \sqsubset_r$ have positive degree (and $\sqsubset_0$ has positive degree in the $\mathcal{Y}_C^+$ case);
(2) every vertex is stable;
(3) the balancing condition is satisfied.

Proof. Recall that we have $\mathcal{D} \subseteq \mathcal{VZ}$ obtained by a choice of combinatorial type and ray in the subdivison of the tropical moduli space. In this situation, since the circuit has positive degree no alignment is necessary (both radii are zero), and hence the subdivision is trivial. Following our procedure, we choose a ray of the tropical moduli space $Q_{\mathbb{R}}^{\vee}$. Since this is generated by edge lengths, and we are assuming that the initial and generised combinatorial types coincide, we conclude that the tropical moduli space must be $\mathbb{R}_{\geq 0}$. Since we are assuming that $\mathcal{D} \subseteq \mathcal{D}(k)$, we know that the vertex of the tropicalisation which contains the flag x_k must map into the interior $\mathbb{R}_{>0} \subseteq \mathbb{R}_{\geq 0}$.

We claim that this is the only vertex of the tropical curve mapped into the interior; if there were more, the positions of their images in $\mathbb{R}_{>0}$ would be independent, and thus the tropical moduli space would have dimension ≥ 2, a contradiction.[9]

Thus there is a single vertex $\sqsubset_0$ mapped into the interior $\mathbb{R}_{>0} \subseteq \mathbb{R}_{\geq 0}$ and a number of vertices $\sqsubset_1, \ldots, \sqsubset_r$ mapped onto the vertex $0 \in \mathbb{R}_{\geq 0}$. No pair of $\sqsubset_1, \ldots, \sqsubset_r$ can be connected by an edge, since this would introduce an extra edge length to the tropical moduli space, which then once again would have dimension ≥ 2. Each of $\sqsubset_1, \ldots, \sqsubset_r$ must have positive degree by the tropical balancing condition and must be connected to $\sqsubset_0$ (since the tropical curve must be connected).

Thus, we see that the combinatorial type takes the form of a bipartite graph, with $\sqsubset_0$ on the right-hand side and $\sqsubset_1, \ldots, \sqsubset_r$ on the left-hand side. Note that the continuity of the tropical map identifies all of the edge lengths, up to weights given by the expansion factors (i.e. tangency orders) at the edges; thus the tropical moduli space is isomorphic to $\mathbb{R}_{\geq 0}$ and so we have a divisor as expected.

We distinguish three cases, depending on the position of the circuit, giving the three forms presented in the statement of the lemma. □

We now now investigate the three types A, B, C^+ separately, giving a recursive description of the boundary locus in each case.

[9](Luca) Say something about rigid tropical curves as in ACGS and KLR

5.1.1. *Type A.* Let Δ be a combinatorial type of type A and let $\mathcal{D} \subseteq \mathcal{VZ}$ be the corresponding logarithmic divisor (note there is no choice of ray here). The corresponding locus $\mathcal{D}_{\mathcal{M}} \subseteq \mathcal{M}$ is given (up to a finite cover) by[10]:

$$\mathcal{D}_{\mathcal{M}} = \overline{\mathcal{M}}^{\text{punct}}_{0,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H,d_0) \times_{H^r} \left(\overline{\mathcal{M}}^{\log}_{1,\alpha^{(1)}\cup(m_1)}(\mathbb{P}^N|H,d_1) \times \prod_{i=2}^{r} \overline{\mathcal{M}}^{\log}_{0,\alpha^{(i)}\cup(m_i)}(\mathbb{P}^N|H,d_i) \right).$$

Lemma 5.2. We have the following description of $\mathcal{D}$

$$\mathcal{D} = \overline{\mathcal{M}}^{\text{punct}}_{0,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H,d_0) \times_{H^r} \left(\mathcal{VZ}_{1,\alpha^{(1)}\cup(m_1)}(\mathbb{P}^N|H,d_1) \times \prod_{i=2}^{r} \overline{\mathcal{M}}^{\log}_{0,\alpha^{(i)}\cup(m_i)}(\mathbb{P}^N|H,d_i) \right) \tag{7}$$

i.e. the map $\mathcal{D} \to \mathcal{D}_{\mathcal{M}}$ is given by blowing up one of the factors of the fibre product.

Note that there is a birational map which forgets the log structures

$$\overline{\mathcal{M}}^{\text{punct}}_{0,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H,d_0) \to \overline{\mathcal{M}}_{0,n_0+r}(H,d_0)$$

and all of our insertions are pulled back from the latter space. Therefore the integrals over the punctured space are determined by the genus zero Gromov–Witten theory of $H \cong \mathbb{P}^{N-1}$.

Proof. Note first of all that the map $\mathcal{D} \to \mathcal{D}_{\mathcal{M}}$ is an isomorphism away from the blown-up locus, which is contained inside the locus where the circuit is contracted. Given an element of $\mathcal{D}$ we can split it along the nodes $q_1,\ldots,q_r$ as in [ACGS]. It is then clear that $\sqsubset_1$ is aligned. We claim that $\sqsubset$ satisfies the factorisation property if and only if $\sqsubset_1$ does. This is enough to show (7).

On $\sqsubset$ there is an associated contraction radius δ passing through a non-contracted vertex, such that the strict interior only contains contracted vertices.

Lemma 5.3. $\lambda(\sqsubset') > \delta$ for any component $\sqsubset'$ of $\sqsubset_0$.

Assuming this holds, we see as a consequence that $\sqsubset_0, \sqsubset_2, \ldots, \sqsubset_r$ must lie outside the contraction radius. Consequently the aligned curve $\sqsubset$ satisfies the factorisation condition if and only if $\sqsubset_1$ does, so (7) holds.

Proof. The basic point is that $\mathcal{D}$ consists of the union of the locally closed logarithmic strata adjacent to the locally closed stratum where all the $\sqsubset_i$ are irreducible. If we look at one of these boundary strata, the tropical moduli space contains a ray σ corresponding to the stratum where all the $\sqsubset_i$ are irreducible; this amounts to setting all edge lengths other than $e_1,\ldots,e_r$ to zero. If $f_1,\ldots,f_l$ are some number of these additional edge lengths (corresponding to internal nodes in degenerations of the $\sqsubset_i$) then all of the cones of the

[10] (Navid) Include short appendix on integralisation/saturation when gluing punctured maps in our setting?

subdivision adjacent to σ will have $f_1 + \ldots + f_l < e_j$ for all $j \in \{1, \ldots, r\}$, since $f_1 = \ldots = f_l = 0$ and $e_j \neq 0$ on σ. In particular, if $\sqsubset_1$ is degenerate and if f denotes the minimal distance from the circuit to a non-contracted vertex of $\sqsubset_1$ (which certainly exists since $\sqsubset_1$ has positive degree) then $f < e_1 \leq \lambda(\sqsubset')$. Thus $\delta = f$ and $\delta < \lambda(\sqsubset')$ as claimed.[11] □

Notice we did not specify whether δ was the relative or absolute radius. The point is that it does not matter; the proof goes through the same in either case. Thus we see that $\sqsubset$ satisfies the (double) factorisation condition if and only if $\sqsubset_1$ does. □

5.1.2. *Type B.* Now let $\mathcal{D} \subseteq \mathcal{VZ}$ be a component of $\mathcal{D}(k)$ with combinatorial type Δ of type B. In this case, it is impossible for the circuit to be contracted. Thus, $\mathcal{D}_{\mathcal{M}} \subseteq \mathcal{M}$ is disjoint from the blown-up locus, and the map

$$\mathcal{D} \to \mathcal{D}_{\mathcal{M}}$$

is an isomorphism. Thus we obtain:

$$\mathcal{D} = \overline{\mathcal{M}}^{\text{punct}}_{0,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H, d_0) \times_{H^r} \left(\overline{\mathcal{M}}^{\log}_{0,\alpha^{(1)}\cup(m_1,m_2)}(\mathbb{P}^N|H, d_1) \times \prod_{i=3}^{r} \overline{\mathcal{M}}^{\log}_{0,\alpha^{(i)}\cup(m_i)}(\mathbb{P}^N|H, d_i) \right).$$

As before, the integrals over the punctured space are determined by the Gromov–Witten theory of H.

5.1.3. *Type C^+.* Finally let $\mathcal{D} \subseteq \mathcal{VZ}$ be a component of $\mathcal{D}(k)$ with combinatorial type Δ of type C^+. The corresponding locus in $\mathcal{M}$ can be written as:

$$\mathcal{D}_{\mathcal{M}} = \overline{\mathcal{M}}^{\text{punct}}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H, d_0) \times_{H^r} \left(\prod_{i=1}^{r} \overline{\mathcal{M}}^{\log}_{0,\alpha^{(i)}\cup(m_i)}(\mathbb{P}^N|H, d_i) \right).$$

Then the same argument as in Lemma 5.3 shows that given any point in $\mathcal{D}$ the interior of the radius can only contain components of C_0. Thus the alignment and factorisation condition apply exclusively to C_0, which shows that:

$$\mathcal{D} = \mathcal{VZ}^{\text{punct}}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H, d_0) \times_{H^r} \left(\prod_{i=1}^{r} \overline{\mathcal{M}}^{\log}_{0,\alpha^{(i)}\cup(m_i)}(\mathbb{P}^N|H, d_i) \right).$$

Here the first factor

$$\mathcal{VZ}^{\text{punct}}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H, d_0)$$

is the logarithmic blow-up of the moduli space of punctured maps, obtained by imposing an alignment and factorisation condition. To be more precise: its elements consist of punctured maps to H which are aligned in the sense of Definition **??**, and satisfy the

[11] (Navid) Do an example?

(double) factorisation condition (simply replace $\mathbb{P}^N$ by H everywhere in that definition). As in §??, this construction produces a closed substack of a logarithmic modification

$$\mathcal{VZ}^{\text{punct}}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H,d_0) \subseteq \widetilde{\mathcal{VZ}}^{\text{punct}}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H,d_0) \xrightarrow{\psi} \overline{\mathcal{M}}^{\text{punct}}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H,d_0)$$

Lemma 5.4. The logarithmic modification ψ restricts to a birational map:

$$\mathcal{VZ}^{\text{punct}}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H,d) \to \overline{\mathcal{M}}^{\text{punct},\circ}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H,d).$$

Note that the latter space (the main component of the moduli space of punctured maps) is birational to the main component of the double ramification locus. In Lemma 5.5 below we explain how to compute integrals over this.cd

Proof. If it is true, we might be able to prove it via deformation theory. For this it might be useful to notice that there is a morphism $\overline{\mathcal{M}}^{\text{punct}}_{g,\alpha}(H,d) \to \overline{\mathcal{M}}^{\text{punct}}_{g,\alpha}(\mathbb{P}^N,d)$ (because $H \subseteq \mathbb{P}^N$ is strict), which is probably a closed immersion, and the loci with smooth source curve are isomorphic under this map. It seems plausible that the main omponent is log smooth over $B\mathbb{G}_{\mathrm{m}} \subseteq [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ (or the standard log point). □

Integrals over the main component of the double ramification locus can be computed using the following lemma.

Lemma 5.5. The morphism which forgets a marking

$$\mathrm{fgt}_i \colon \overline{\mathcal{M}}^{\alpha,\circ}_{1,n}(H,d) \to \overline{\mathcal{M}}^{\circ}_{1,n-1}(H,d)$$

is generically finite, of degree α_i^2 (except in one special case, described in the proof).

Proof. Let us consider the case $d = 0$ first. We may assume the source curve is smooth elliptic E. The map

$$\phi\colon E \to \mathrm{Pic}^0(E), \qquad x \mapsto \mathcal{O}_E\left(\alpha_i x + \sum_{j=1,\ldots,\hat{i}\ldots,n} \alpha_j p_j\right)$$

is an isogeny of degree α_i^2. The locus of $(C, p_1, \ldots, \hat{p}_i, \ldots, p_n)$ such that the kernel of ϕ contains one of the points $\{p_1, \ldots, \hat{p}_i, \ldots, p_n\}$ is itself a double ramification locus inside $\overline{\mathcal{M}}_{1,n}$, hence non-generic - with one exception: namely, when $n = 2$, $x = p_1$ is always a solution, but the curve lying above such point bubbles off a $\mathbb{P}^1$. To see that it does not belong to the closure of the nice locus, notice that the rational function trivialising $\mathcal{O}_C(\alpha_1 p_1 - \alpha_2 p_2)$ should descend to the cusp, thus having a ramification point at the node; yet its ramification profile is determined by Riemann-Hurwitz, and it is entirely supported on p_1 and p_2.

For a different proof: notice that we should obtain the class of the main component from the full double ramification cycle by subtracting the boundary class $[D_{1,\emptyset|0,\{1,\ldots,n\}}]$.

The latter pushes forward to 0 under fgt_i, unless $n = 2$. Therefore we may apply $\mathrm{fgt}_{i,*}$ to the Hain-Pixton formula:

$$DR_1(A) = \frac{1}{2}\left(\sum_{i=1}^{n} a_i^2 \psi_i - \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I| \geq 2}} a_I^2 [D_{1,I^c|0,I}] - \frac{1}{12}\delta_0\right),$$

where $a_I = \sum_{i \in I} a_i$ and $\delta_0 = \mathrm{glue}_*([\overline{\mathcal{M}}_{0,n+2}])$. From $\psi_j = \mathrm{fgt}_i^\star \psi_j + [D_{i,j}]$ for $i \neq j$, and the dilaton equation, we see that $\mathrm{fgt}_{i,*}\psi_j = 1$ and $\mathrm{fgt}_{i,*}\psi_i = n - 1$. On the other hand, the only surviving boundary classes are $[D_{i,j}]$, and they push down to 1. Hence the formula pushes down to $\frac{1}{2}\left(\sum_{j \neq i} \alpha_j^2 + (n-1)\alpha_i^2 - \sum_{j \neq i}(\alpha_j + \alpha_i)^2\right) = -\alpha_i \left(\sum_{j \neq i} \alpha_j\right) = \alpha_i^2$. □

Definition 5.6. A combinatorial type Δ of a tropical curve consists of the following data:

(1) G a finite graph, with a set $V(G)$ of vertices, a set $F(G)$ of flags and an involution $\iota\colon F(G) \to F(G)$ with $\iota^2 = \mathrm{Id}$. We let $L(G)$ denote the fixed set of ι, and think of its elements as infinite legs; we let $E(G)$ denote the set of equivalence classes of elements not fixed by ι, and think of them as finite edges connecting two (possibly equal) vertices of G;
(2) a genus assignment $g\colon V(G) \to \mathbb{Z}_{\geq 0}$;
(3) a degree assignment $\deg\colon V(G) \to \mathbb{Z}_{\geq 0}$;
(4) a weight assignment $u\colon F(G) \to \mathbb{Z}$ such that for every edge $e \in E(G)$ we have $u(f_1) = -u(f_2)$ where $e = [f_1] = [f_2]$, and $u(l) \geq 0$ for any $l \in L(G)$;

[12] Review here the process of gluing for punctured maps. In particular the basic monoid should be modified in order to make the relevant evaluations log morphisms. Claim: evaluations are strict. Consequence: the fiber product in the category of log stacks is fine. We have then to apply saturation. This is a finite morphism of degree... (I think it could be $\frac{\prod m^{(i)}}{lcm(m^{(i)})}$).

Lemma 5.7 (Virtual pushforward). The following hold.

- $\mathrm{fgt}_*[\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d)] = [\overline{\mathcal{M}}^{\mathrm{G}}_{0,\alpha}(\mathbb{P}^N|H,d)]$ (follows from [Gat03, AMW14]).
- $\mathrm{fgt}_*[\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)]$ computes the reduced relative invariants by definition.
- $\mathrm{fgt}_*[\overline{\mathcal{M}}_{0,\mu}(H,d_0)^{\sim}] = [\overline{\mathcal{M}}_{0,|\mu|}(H,d_0)]$ (follows from [Gat03] and... comparison of punctured with rubber invariants).
- $\mathrm{fgt}_*[\mathcal{VZ}_{1,\mu}(H,d_0)^{\sim}]$ here we should need a variation on Pixton's DRC formula; hopefully it's enough to avoid the graphs that tropical well-spacedness discards.

5.2. **Recursive description of the divisors: type C_0.** Consider now a combinatorial type Δ such that the circuit is contracted into the divisor. The corresponding stratum $\mathcal{D}_{\mathcal{M}} \subseteq \mathcal{M}$ is contained in the blown-up locus. Our task is to identify the logarithmic divisors in the preimage of $\mathcal{D}_{\mathcal{M}}$ under the map $\mathcal{VZ} \to \mathcal{M}$. As discussed, these are indexed by choices of ray in the subdivision of the tropical moduli space. Suppose we have fixed such data.

Claim 1. The teeth of the comb can break at most once, along the circuit.

Proof. If not there would be more parameters in the tropical moduli space. □

Proposition 5.8. Let Δ be a combinatorial type as above. Then:

(1) the circuit is a single vertex, and is mapped to the interior $\mathbb{R}_{>0} \subseteq \mathbb{R}_{\geq 0}$;
(2) the degree of the circuit is zero;

[12] (Luca) to be made homogeneous with what comes earlier

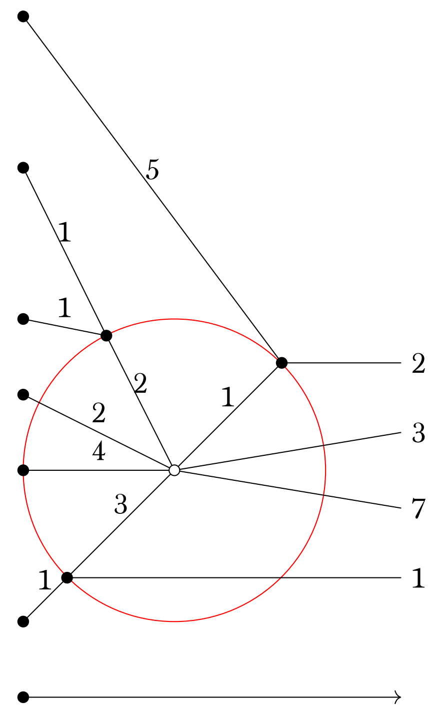


FIGURE 4. The degrees can be figured out from the balancing equation.

(3) there is a vertex which is adjacent to the circuit and mapped to the interior $0 \in \mathbb{R}_{\geq 0}$;
(4) let $e \in E(\sqsubset)$ be an edge which is adjacent to the circuit, and such that $u(e)$ is maximal among the edges adjacent to the circuit. Then the other vertex v contained in e is mapped to $0 \in \mathbb{R}_{\geq 0}$, and v lies on the (absolute) radius;
(5) if $v \in V(\sqsubset)$ and $\lambda(v) > \delta$ then $f(v) = 0$.

On the other hand, when there is a contracted elliptic subcurve - and it will be contracted into the hyperplane, because otherwise it wouldn't be generic, by density of the nice locus in $\mathcal{VZ}_{1,\alpha'}(\mathbb{P}^N|H, d')$ - the picture becomes more complicated due to the alignment. The combs may break. We label these loci $\mathcal{Y}_C^0$.

In the following we describe the rays of the tropical moduli space.

Lemma 5.9. A one-parameter tropical map ϕ to $\mathbb{R}_{\geq 0}$ is a decorated tree (with expansion factors - *contact orders* - along edges and legs, and degrees on vertices, satisfying the balancing condition) with a circle (of radius δ) around the root (sometimes called the *core* and denoted by $\circ$) satisfying:

(1) the circle of radius δ passes through at least one vertex of $\phi^{-1}(0)$ - which necessarily has positive degree - call m its contact order with H;

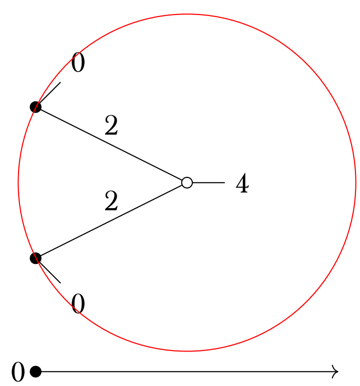
(2) teeth may break only when they intersect the circle of radius δ; in particular, $\circ$ is the only vertex contained in its strict interior, and every edge heading out from the circle goes directly to a vertex of $\phi^{-1}(0)$;

(3) every tooth that starts with contact order m goes directly to a vertex of $\mathrm{circle}(\circ, \delta) \cap \phi^{-1}(0)$, and every other tooth starts with contact order $< m$ (possibly negative).

Proof. Otherwise there would be more than one parameters. □

Remark 5.10. The core being contracted in the fiber of the tropical map is not a phenomenon that we should worry about in codimension one. Indeed, assume that the core is contracted in the fiber along a ray. Then all the edges departing from the core have expansion factor 0; call the corresponding coordinates $U = \{u_i\}_{i\in I}$. Call the remaining coordinates $E = \{e_j\}_{j\in J}$. Note that tropical continuity involves only E. Alignments on the other hand assume the form $\lambda(v) = \lambda(v')$, where $\lambda(v) = \sum_{i\in I(v)} u_i \sum_{j\in J(v)} e_j$. Pick the shortest elements of U; then these can be shortened to zero without affecting the rest (by hypothesis, alignments can only identify them among themselves). This shows that we could not have started with a ray.

Example 5.11. We look at the following example in some detail. The ambient space is



$\mathcal{VZ}_{1,(4,0,0)}(\mathbb{P}^N|H,4)$, of dimension $4N+3$. The underlying moduli space is $X = \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H,2)\times_H \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H,2) \times \mathcal{VZ}_{1,(-2,-2,4)}$ of dimension $5N+1$. Consider the fiber product:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H,2) \times \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H,2) \arrow[d] \\ H \arrow[r] & H \times H \end{tikzcd}$$

At the level of ghost sheaves, $\overline{\mathcal{M}}_F = \mathbb{N} \oplus_{\mathbb{N}^2} \overline{\mathcal{M}}_1^{\mathrm{enl}} \oplus \overline{\mathcal{M}}_2^{\mathrm{enl}}$, where the map $\mathbb{N}^2 \to \mathbb{N}$ is the sum, and the map $\mathbb{N}^2 \to \overline{\mathcal{M}}_1^{\mathrm{enl}} \oplus \overline{\mathcal{M}}_2^{\mathrm{enl}}$ generically is multiplication by 2, so $\overline{\mathcal{M}}_F = \mathbb{N}^2/(2e = 2f)$ generically. Saturation gives a finite cover $G \to F$ with $\overline{\mathcal{M}}_G = \mathbb{N}_{e=f}$ generically. Lifting this to actual log structures, what we are doing (again generically) is taking a square root of the isomorphism $T_{R_1,q_1}^{\otimes 2} \simeq T_{R_2,q_2}^{\otimes 2}$, which is obtained passing through $N_{H/\mathbb{P}^N,f(Z)}$ via $\mathrm{d} f_{|R_i,q_i}$. This breaks when $f_{|R_i}$ is not tangent to H of order exactly 2 at q_i, for either i; but by the maximality assumption this happens precisely along Gathmann's comb loci Δ_i. So in fact, rather than with $T_{R_i,q_i}^{\otimes 2}$, we should be working with $T_{R_i,q_i}^{\otimes 2}(-\Delta_i)$: but this is exactly $\mathrm{ev}_i^\star(-H)$ by Gathmann's genus zero formula, and the isomorphism $\mathrm{ev}_1^\star(H) = \mathrm{ev}_2^\star(H)$ holds on all of F.

On the other hand, generically on $\mathcal{VZ}_{1,(-2,-2,4)}$ we have $T_{q_1}Z \simeq T_{q_2}Z$ by exploiting the group structure on the elliptic curve. This breaks when either (but not both) is on a rational tail. Yet we have $T_{q_1}Z(\Delta_{1\in P}) \simeq T_{q_2}Z(\Delta_{2\in P})$ by Vakil-Zinger's construction of a universal ψ-class (i.e. by comparing both with $\pi_*\omega(\Delta)$; notice that our further blow-up has the only effect of twisting *all* the relevant line bundles by $\Delta_{1,2\notin P}$).

Now, the fiber of the Vakil-Zinger blow-up over X can be described as follows. Generically it looks like

$$\mathbb{P}(T_{q_1}R_1 \otimes T_{q_1}Z \oplus T_{q_2}R_2 \otimes T_{q_2}Z)$$

but this has to be modified along the boundary:

- this has to do with the fact that the normal bundle of the strict transform is the pullback of the normal bundle twisted by the intersection with the exceptional divisor (so it relates with previous steps of the blow-up);
- it is not globally a $\mathbb{P}^1$-bundle (so it relates with further stages of the blow-up; it also has to do with a choice of compactification for the moduli space of attachments);
- it has the effect of replacing $T_{q_i}Z$ with Vakil-Zinger's universal $\mathbb{T}$, so that this can be factored out of the projective bundle, and in fact we are left with a projective bundle $\mathbb{P} = \mathbb{P}(T_{q_1}R_1 \oplus T_{q_2}R_2)$ over F, and its open part $\mathrm{Iso}(T_{q_1}R_1 \oplus T_{q_2}R_2)$ represents the attachment data for a contraction to a tacnode $R_1 \sqcup_q R_2 \to \bar{C}$.

On $\mathbb{P}$ there is a natural vector bundle map

$$s\colon \mathcal{O}_{\mathbb{P}}(-1) \hookrightarrow p^\star(T_{q_1}R_1 \oplus T_{q_2}R_2) \xrightarrow{+\mathrm{d} f} \mathrm{ev}_q^\star T\mathbb{P}^N$$

that vanishes along the locus where f descends to $\bar{C}$. In general, it is not a transversal section:

- we should replace $T\mathbb{P}^N$ by TH as long as all the $m^{(i)}$ are ≥ 2;
- Vakil and Zinger construct a blow-up of $\mathbb{P}$ along the vanishing loci of s of low codimension, and twist s by the exceptional divisors, so that it becomes a transverse section $\tilde{s}$.

On the other hand, the finite cover $G \to F$ factors through $\mathbb{P}$, because the two vertices are already aligned on G. We claim that the boundary locus of $\mathcal{VZ}_{1,(4,0,0)}(\mathbb{P}^N|H,4)$ corresponding to the combinatorial type of the tropical map above is the transverse intersection

$$(G \cap V(\tilde{s}) \subseteq \mathbb{P}) \times \mathcal{VZ}_{1,(-2,-2,4)}.$$

This has the expected dimension (codimension $N-1$ with respect to X). To compute its class, we can pull $\mathbb{P}$ back to G, and then notice that $G \hookrightarrow \mathbb{P}_G$ is the inclusion of a (trivial) subbundle.

Lemma 5.12. The class of $\mathbb{P}(\mathcal{F}) \subseteq \mathbb{P}(\mathcal{E})$ is $c_{\text{top}}(\mathcal{O}_{\mathcal{E}}(1) \otimes p^\star(\mathcal{E}/\mathcal{F}))$.

See [EH16, Prop. 9.13]. It is a good time to remember that $\mathcal{E}$ was in fact $(\bigoplus_{i=1}^r TR_{i,q_i}) \otimes \mathbb{T}$. By writing c for $c_1(\mathcal{O}_{\mathcal{E}}(1))$, ψ_i for $c_1(T^\star R_{i,q_i})$, ψ_Z for Vakil-Zinger's universal psi class, and H for $\mathrm{ev}_q^\star H$, we need to compute

$$\begin{aligned} p_*\left((c-\psi_1-\psi_2-2\psi_Z)[(1+c+H)^N(1+c)^{-1}]_{N-1}\right) &= \\ p_*\left((c-\psi_1-\psi_2-2\psi_Z)(\sum_{k=0}^{N-1}\binom{N}{1+k}c^kH^{N-1-k})\right) &= \\ \sum_{k=0}^{N-1}\binom{N}{k}H^{N-1-k}\big(s_k(\mathcal{E})-s_{k-1}(\mathcal{E})(\psi_1+\psi_2+\psi_Z)\big) & \end{aligned}$$

We now generalise this picture. Recall that the map $C \to \bar{C}$ is given by a (generic) line in the sum of the tangent spaces to the rational tails at the nodes that join them to the contracted curve of genus one. This is equivalent to an alignment, and it is parametrised by an open subset of a projective bundle over the moduli space for the tails corresponding to vertices of the dual graph lying on the circle of radius δ. Yet, notice that those vertices lying in $\phi^{-1}(0)$ are already aligned among themselves. This is why we find it convenient to distinguish among four groups of vertices:

1. the core;
2. vertices on $\phi^{-1}(0) \cap \mathrm{circle}(\circ,\delta)$;
3. vertices on $\mathrm{circle}(\circ,\delta) \setminus \phi^{-1}(0)$;
4. vertices on $\phi^{-1}(0) \setminus \mathrm{circle}(\circ,\delta)$.

We shall first argue that gluing of log maps can be performed separately for the exterior and interior of the circle (the analogous classical picture is that, since the core is contracted, this moduli space is the product of a genus one curve, and a fiber product of genus zero maps under evaluation morphisms).

Lemma 5.13. $\mathcal{VZ}^{\text{punct}}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H,0) \simeq \mathcal{VZ}^{\text{punct}}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(\mathrm{Spec}(k \oplus \mathbb{N})) \times H$.[13]

[13] (Luca) check log structure, could be fibered over std log point

Let us now deal with vertices of type 2.

Lemma 5.14. Consider the following fiber product in the category of fs log stacks:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \prod_{i=1}^r \overline{\mathcal{M}}_{0,\alpha^{(i)}\cup\{m\}}(\mathbb{P}^N|H,d_i) \arrow[d, "\mathrm{ev}_{q_i}"] \\ H \arrow[r] & H^r \end{tikzcd}$$

On F there is a canonical isomorphism $\mathbb{L}_{q_i} \cong \mathbb{L}_{q_j}$[14], the latter being the cotangent line bundles at the gluing markings on two different components i and j. Furthermore, $\underline{F}$ is a finite cover of degree m^{r-1} of the fiber product of the underlying stacks.

Proof. Recall that each $\overline{\mathcal{M}}_{0,\alpha^{(i)}\cup\{m\}}(\mathbb{P}^N|H,d_i)$ is endowed with the log structure induced by pulling back along q_i the divisorial log structure of the universal curve at the image of q_i itself. Thus ev_{q_i} is made into a log morphism to H with its induced DF(1) log structure. We claim that the subtext of such morphism is Gathmann's formula; namely, the log morphism to H corresponds to an isomorphism betwee $\mathrm{ev}_{q_i}^\star \mathcal{O}_H(-H)$ on one side, and $\mathcal{I}_{D_i} \otimes \mathbb{L}_{q_i}^m$ on the other, where $\mathcal{I}_{D_i}$ is the ideal sheaf of the union of the comb loci D_i in $\overline{\mathcal{M}}_{0,\alpha^{(i)}\cup\{m\}}(\mathbb{P}^N|H,d_i)$.

On the fiber product there is a canonical isomorphism between $\mathrm{ev}_{q_i}^\star H$ and $\mathrm{ev}_{q_j}^\star H$.

Say something about integrality.

Saturation is a local operation, as much as computing the degree, hence we can concentrate on the dense open locus where all the curves we are gluing are smooth, and they are not mapped entirely into H. There the minimal log structure on $\overline{\mathcal{M}}_{0,\alpha^{(i)}\cup\{m\}}(\mathbb{P}^N|H,d_i)$ is trivial, therefore the isomorphism between $\mathrm{ev}_{q_i}^\star H$ and $\mathrm{ev}_{q_j}^\star H$ translates into an isomorphism $\mathbb{L}_{q_i}^m \cong \mathbb{L}_{q_j}^m$. The saturation F is obtained by taking an m-th root of this isomorphism. □

Let us denote by $\mathbb{L}_F$ the universal cotangent line at q.

The projective bundle we are seeking has base

$$\mathcal{X} = \left(F \times \prod_{i=1}^s \overline{\mathcal{M}}^{\text{punct}}_{0,\tilde{\alpha}^{(i)}\cup\{m_i\}}(H,\tilde{d}_i) \right) \times_{H^{s+1}} H$$

(with $m_i < m$) and it is

$$\mathbb{P} = \mathrm{Proj}_{\mathcal{X}} \left(\mathbb{T}_F \oplus \bigoplus_{i=1}^s \mathbb{T}_{\tilde{q}_i} \right).$$

We are interested in the vanishing locus of the section

$$s\colon \mathcal{O}_{\mathbb{P}}(-1) \hookrightarrow p^\star \left(\mathbb{T}_F \oplus \bigoplus_{i=1}^s \mathbb{T}_{\tilde{q}_i} \right) \xrightarrow{+df_q} f^\star(\mathrm{T}\,\mathbb{P}^N)_q,$$

[14] (Luca) check how it degenerates

because it represents the geometric condition that $f\colon C \to \mathbb{P}^N$ factors through the normalisation map $C \to \bar{C}$ prescribed by the given point of $\mathbb{P}$. As is, s is not transverse to the zero section. First of all, unless $m = 1$, $\mathrm{T}\,\mathbb{P}^N$ can be replaced by $\mathrm{T}\,H$ in the definition of s above, because the projection of all $\mathrm{d}\,f_i(\mathrm{T}_{q_i} R_i)$ to $N_{H/\mathbb{P}^N}$ is zero. The case $m = 1$ has to be dealt with separately and it turns out that, once s is made transverse, the dimension of its zero locus is smaller than the expected dimension, hence the corresponding combinatorial types are in fact irrelevant.

The procedure to make s transverse is the same as described in [VZ08, §3], namely we need to blow up inside $\mathbb{P}$ the projective subbundle $\mathbb{P}(\mathcal{E})$ of $\mathbb{P}_{|\mathcal{X}_\sigma}$, where $\mathcal{X}_\sigma$ is the closed substack of $\mathcal{X}$ where some of the tails degenerate so that the corresponding gluing marking lies on a component contracted by f, and $\mathcal{E}$ is the sum of the tangent line bundles at such subset of the gluing markings. Because $\overline{\mathcal{M}}^{\mathrm{punct}}_{0,\tilde{\alpha}^{(i)}\cup\{m_i\}}(H,\tilde{d}_i)$ is isomorphic to $\overline{\mathcal{M}}_{0,\tilde{n}_i}(H,d_i)$, the construction of Vakil and Zinger in the section "A blowup of a moduli space of genus-zero maps" of their paper carries through unchanged. The result of the blow-up is to replace $T_{q_i}R_i$ with $T_{q_i}R_i \otimes \bigoplus_{j=1}^k T_{\tilde{q}_j}R_i \otimes T_{\tilde{q}_j}S_{ij}$. We interpret this in terms of alignments.[15]

We claim that $\tilde{s}$ obtained from s by twisting by the exceptional divisors of the blowup is a transverse section. We also claim that comb loci of type $\mathcal{Y}^0_c$ can be described as a fibered product of

- moduli of genus one punctured maps to the standard log point, radially aligned and satisfying factorisation;
- $V(\tilde{s})$;
- moduli of genus zero maps relative to $(\mathbb{P}^N|H)$, corresponding to vertices of type 4 above.

Finally, we claim that integrals of psi and evaluation classes over these loci can be translated into tautological integrals, i.e. descendant Gromov-Witten invariants, whose numerics is governed by the combinatorial type of the tropical map.

6. Recursion formula in general

Now let $(X|Y)$ be a smooth pair with Y very ample. The complete linear system $|\mathcal{O}_X(Y)|$ defines an embedding $X \hookrightarrow \mathbb{P}^N$ with $Y = X \cap H$ for H some hyperplane.

Lemma 6.1. The following square is cartesian (in the category of ordinary stacks):

$$\begin{tikzcd} \mathcal{VZ}_{1,\alpha}(X|Y,\beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d) \arrow[d] \\ \mathcal{VZ}_{1,n}(X,\beta) \arrow[r, "i"] & \mathcal{VZ}_{1,n}(\mathbb{P}^N,d). \end{tikzcd}$$

[15](Luca) this has to be written properly

Since $\mathcal{VZ}_{1,n}(\mathbb{P}^N, d)$ is smooth and $\mathcal{VZ}_{1,n}(X, \beta)$ carries a natural virtual class, there is a diagonal pull-back morphism which we use to define the virtual class on the space of maps to $(X|Y)$:

$$[\mathcal{VZ}_{1,\alpha}(X|Y, \beta)]^{\mathrm{virt}} := i_\Delta^![\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)].$$

The recursion formula in $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ immediately pulls back along i to give a recursion formula in $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$. [16]

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[16] (Navid) Is it clear how to compute integrals over the pulled back classes?

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