RELATIVE STABLE MAPS IN GENUS ONE VIA CENTRAL ALIGNMENTS

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Abstract. [OLD] For a smooth projective variety X and a smooth very ample hypersurface $Y \subseteq X$, we define moduli spaces of relative stable maps to (X, Y) in genus one, as closed substacks of the moduli space of maps from centrally aligned curves, constructed in [RSW17a]. We construct virtual classes for these moduli spaces, which we use to define *reduced relative Gromov–Witten invariants* in genus one.

[GOALS: We prove a recursion formula which allows us to completely determine these invariants in terms of the reduced Gromov–Witten invariants, as defined in [REF]. We also prove a relative version of the Li–Zinger formula, relating our invariants to the usual relative Gromov–Witten invariants. Also say something about quasimaps.]

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1. Introduction

1.1. **Statement of the problem**. Contrary to the genus zero case, the moduli space of genus one maps to projective space - with or without markings - is far from smooth; indeed it has various boundary components of different dimensions, representing maps that contract a genus one curve and have all the degree supported on a number of rational tails. The many incarnations of relative moduli spaces also suffer of the same undesirable feature.

Since the work of Vakil–Zinger and Ranganathan–Santos-Parker–Wise, it has been clear that it is possible to identify a desingularisation of the main component by adding the extra data of a contraction of the source curve $\nu\colon C \to \bar{C}$ - where the latter is allowed to acquire a Smyth singularity - and requiring the stable map $f\colon C \to \mathbb{P}^N$ to factor through ν.

1.2. **Choice of relative space and desingularisation**. We focus on the space of logarithmic stable maps to $(\mathbb{P}^N|H)$, following ACGS. We perform a log modification of this space as detailed below. For a log curve $C \to S = \operatorname{Spec}(k = \bar{k})$, modify the dual graph of C by replacing the minimal genus one subcurve (in case it is a circle of $\mathbb{P}^1$) by a single vertex of genus one, called the *core* and denoted by $\circ$, and define a piecewise linear function with values in $\overline{\mathcal{M}}_S$ on such a graph by setting

$$\lambda(v) = \sum_{q \in [\circ, v]} \rho_q,$$

where the ρ_q are the smoothing parameters of the nodes q separating v from the core. Such a function is related to the log canonical bundle of $C \to S$. When the map contracts a subcurve of genus one, we endow it with the extra data of a radius $\delta \in \overline{\mathcal{M}}_S$ subject to the following compatibility condition:

(*) *the circle of radius δ around $\circ$ passes through ≥ 1 vertex of positive f-degree.*

Furthermore, we require all the values of λ to be comparable with δ, and among themselves whenever they are $\leq \delta$. This is called a *centrally aligned* log structure and carries enough information to define a contraction $\nu\colon C \to \bar{C}$, possibly after a semistabilisation of (C, f) - in fact even more. The space thus obtained, $\widetilde{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N|H, d)$, is a log modification of $\overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d)$.

The main component $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$ is then identified by a double factorisation condition:

(1) If f contracts a genus one subcurve, then f is required to factor through the Smyth singularity $\nu\colon C \to \bar{C}$ determined by the contraction radius δ as above.
(2) If furthermore the core is contracted by the associated tropical map ϕ, let δ_2 be the minimal distance from $\circ$ to a vertex supporting a flag that escapes $\phi^{-1}(\phi(\circ))$; we require f to factor through $\nu_2\colon C \to \bar{C}_2$.

The main result is that

Theorem 1.1. $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$ is (log) smooth.

1.3. **Gathmann-type recursion.** There is a forgetful morphism

$$\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d) \to \mathcal{VZ}_{1,n}(\mathbb{P}^N,d),$$

hitting Gathmann's relative space. We may therefore pullback Gathmann's line bundle and section, cutting out the locus where the k-th marking is tangent to H to order α_k+1. Because α was maximal ($\sum \alpha = d$) by assumption, this means that the curve has to break, and x_k has to lie on an internal component - one which is entirely mapped into H. We identify the zero locus of Gathmann's section explicitly. Here is an interesting remark: the combinatorics of such boundary loci is governed by tropical geometry, and it is otherwise very hard to districate the interaction between the relative condition and the exceptional loci of the Vakil–Zinger blow-up.

Corollary 1.2. $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$ is smooth over its Artin fan, in particular codimension one logarithmic strata can be read off from the latter.

The Artin fan is a tropical gadget. Its local structure is given by subdividing the ACGS minimal monoid according to the alignment. We are only interested in picking its rays. The upshot is that the combinatorics is slightly more involved than in the genus zero case: the alignment may force some teeth of the comb to break.

Theorem 1.3 (Gathmann-type formula, maximal tangency, $(\mathbb{P}^N|H)$ case)**.**

$$(\alpha_k\psi_k + \mathrm{ev}_k^* H)[\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)] = [D_{1,\alpha;k}(\mathbb{P}^N|H,d)],$$

the latter being a sum of broken comb loci indexed by rays of the tropical fan.

Importantly, the broken comb loci admit a very explicit description in terms of tautological integrals on the underlying boundary of Gathmann's relative space.

Theorem 1.4. Up to a finite cover of the underlying boundary stratum - which is a combinatorially-determined fiber product of moduli spaces of genus zero and one, absolute and relative maps with lower numerical invariants - every component of $D_{\alpha,k}(\mathbb{P}^N|H,d)$ can be described as the transverse intersection of two loci in a projective bundle, where:

- the latter parametrises the possible line bundle isomorphisms imposed by the alignment of the log structure;
- the first locus is a subbundle representing the residual isomorphisms after fixing the ones virtually imposed by tropical continuity;
- the second locus is determined by the factorisation conditions.

The upshot is that we may then push the formula down to the Gathmann's space, so as to obtain multiplicities and tautological classes.

Corollary 1.5.

$$(\alpha_k\psi_k + \mathrm{ev}_k^* H)[\mathcal{VZ}^G_{1,\alpha}(\mathbb{P}^N|H,d)] = [D^G_{1,\alpha;k}(\mathbb{P}^N|H,d)],$$

the latter being expressible as a weighted sum of tautological classes on Gathmann's comb loci.

Once we have this formula, the following extensions are classical:

- A similar formula for raising the tangency holds in the non-maximal tangency case. It can be proven by adding auxiliary markings of contact order 1; forgetting them is then a $(d-\sum\alpha)!:1$ cover because the nice locus is dense inside Gathmann's relative spaces.
- The formula holds more generally for any smooth projective target X relative to a generic hyperplane section $Y = X \cap H \subseteq \mathbb{P}^N$. This follows via virtual pullback.

Finally the recursive structure of the boundary allows us to prove the following

Theorem 1.6 (In-principle quantum Lefschetz)**.** The restricted reduced genus one invariants of Y can be inductively deducted from the full descendant genus zero and one (reduced) Gromov-Witten theory of X.

The proof is more delicate than its genus zero analogue because invariants with the same numerical data appear intertwined in the last steps of the recursion.

2. Desingularisation of the moduli space of log stable maps

The ultimate goal of the paper is to apply Gathmann's techniques to the Vakil-Zinger desingularisation $\mathcal{VZ}_{1,n}(\mathbb{P}^N,d)$ and to obtain a quantum Lefschetz result for reduced invariants under some positivity assumption. The key step is to study the unobstructed case

$(\mathbb{P}^N|H)$. We approach the problem by lifting it to the ACGS space of log stable maps. This allows us to exploit the tools developed in [RSW17a, RSW17b]. We are in an intermediate situation between those two papers, and indeed we get an intermediate answer.

2.1. **The ACGS minimal monoid and central alignments.**

Proposition 2.1. The map $\overline{\mathcal{M}}^{\mathrm{cen}}_{1,\alpha}(\mathbb{P}^N|H,d) \to \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H,d)$ is a log modification. In particular $\overline{\mathcal{M}}^{\mathrm{cen}}_{1,\alpha}(\mathbb{P}^N|H,d)$ is a log algebraic stack.

possibly start pointing out that the log structure is already partially aligned by the map to $(\mathbb{P}^N|H)$

2.2. **Factorisation conditions.**

Proposition 2.2. Factoring through the Smyth curve is a closed condition. In particular $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d) \subseteq_{\mathrm{cl}} \overline{\mathcal{M}}^{\mathrm{cen}}_{1,\alpha}(\mathbb{P}^N|H,d)$ is a log algebraic stack.

2.3. **Log smoothness.**

Theorem 2.3. $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$ is log smooth.

Proof. We reduce to the situation dealt with in [RSW17b] by adding generic extra hyperplanes $H_1,\dots,H_N$.

First, note that, for divisors $D_1 \subseteq D_2$ in X, there is a morphism of log schemes $(X,\mathcal{M}_{D_2}) \to (X,\mathcal{M}_{D_1})$, or equivalently a morphism of log structures $\mathcal{M}_{D_1} \to \mathcal{M}_{D_2}$ over id_X, because functions invertible off D_1 are in particular invertible off D_2 as well, and divisorial log structures are subsheaves of the structure sheaf $\mathcal{M}_D \subseteq \mathcal{O}_X$.

Now fix a point $[(C,f)] \in \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$. Choose hyperplanes $H_1,\dots,H_N$ in such a way that they intersect the image of f transversally, namely $f^{-1}(H_1\cup\ldots\cup H_N)$ is a reduced collection of points $\{q^i_j\}_{\substack{i=1,\dots,N\\ j=1,\dots,d}}$ in the smooth locus of C. This condition will then hold in an open neighbourhood of $[(C,f)]$. Mark C at these points, end endow it with the pullback along f of the divisorial log structure $(\mathbb{P}^N,\Delta)$, where $\Delta = H + \sum_{i=1}^N H_i$. Then

$$f\colon (C,\{p_k\}_{k=1,\dots,n}\{q^i_j\}_{\substack{i=1,\dots,N\\ j=1,\dots,d}}) \to (\mathbb{P}^N,\Delta)$$

is a lift of $[(C,f)]$ to $\overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|\Delta,d)$ (under the forgetful morphism discussed in the previous paragraph).

Looking at the associated tropical map ϕ, observe that:

- new flags have been attached only to vertices of positive degree, and these already have a flag escaping $\phi^{-1}\phi(v)$, because the sum of the incoming slopes is not zero (by modified balancing);

- the image of the new tropical map $\tilde{\phi}$ is entirely contained in the ray of the tropicalisation of $(\mathbb{P}^N, \Delta)$ corresponding to H, with new flags going off to infinity in all the new ray directions from every vertex of positive degree.

In particular, for every quotient N' of the lattice N, the associated tropical map $\tilde{\phi}'$ will either

(1) have image contained in the ray corresponding to H, isomorphically to the original ϕ, so the contraction radius can be seen to coincide with δ_2, or
(2) collapse the entire curve to the zero-cell of the fan, in which case we argue from the previous remarks that the contraction radius is δ.

Hence the lift of $[(C, f)]$ is centrally aligned and satisfies the factorisation property for every subtorus $H < T$, therefore it is well-spaced (see [RSW17b, Definition 3.4.2]) and it belongs to $\mathcal{VZ}_{1,\tilde{\alpha}}(\mathbb{P}^N|\Delta, d)$. Note that the deformation spaces of (C, f) and its lift are isomorphic by construction, as can be checked by the infinitesimal criterion - an infinitesimal deformation of (C, f) brings along a unique deformation of the $\{q_j^i\}$ compatible with the map to $(\mathbb{P}^N, \Delta)$. At the logarithmic level, observe that the ACGS minimal monoid is the same, because no component of C is entirely mapped into any of the newly added hyperplanes; since the global contraction radius δ is the same, the subdivisions corresponding to the alignment procedure do coincide as well. This shows that the forgetful morphism is (log) étale in a neighbourhood of the lift of $[(C, f)]$, hence we may conclude by appealing to [RSW17b, Theorem 3.5.1]. □

Corollary 2.4. $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ is smooth over its Artin fan.

describe the latter as explicitly as possible; comment on the cones that are (possibly?) not there because of the compatibility of the alignment with the log map (this is probably awkward, useless, and superceded by saying "we align subdivide the ACGS minimal dual monoid") and because of smoothability/factorisation (this is probably related to tropical well-spacedness)

3. Description of the logarithmic strata

Let $\mathcal{VZ} = \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ and let $\mathcal{A}$ denote the Artin fan of $\mathcal{VZ}$. Since the map of log stacks

$$\mathcal{VZ} \to \mathcal{A}$$

is smooth, we see that the codimension-k logarithmic strata in $\mathcal{VZ}$ are in bijective correspondence (via pull-back) with the codimension-k logarithmic strata in $\mathcal{A}$. The logarithmic strata in $\mathcal{A}$, on the other hand, have a purely combinatorial description (at least locally) in terms of the associated moduli spaces of tropical maps, coming from the fact that $\mathcal{VZ}$ is a logarithmic blow-up of the usual moduli space of log stable maps. In this section we discuss this circle of ideas, and show how it plays out in a number of examples.

3.1. **Blowing up the Artin fan**. Let $\mathcal{M} = \overline{\mathcal{M}}^{\log}_{1,\alpha}(\mathbb{P}^N|H, d)$ denote the Abramovich–Chen–Gross–Siebert moduli space of log stable maps. Recall that $\mathcal{VZ}$ is obtained as a closed substack of a log modification

$$\widetilde{\mathcal{VZ}} \to \mathcal{M}.$$

Since the map $\mathcal{VZ} \hookrightarrow \widetilde{\mathcal{VZ}}$ is strict, the Artin fan $\mathcal{A}$ of $\mathcal{VZ}$ is locally isomorphic to the Artin fan of $\widetilde{\mathcal{VZ}}$ (with the latter being, in general, larger)[1]. Since the following discussion is entirely local, we will ignore the distinction between the two, and pretend as though $\mathcal{A}$ is the Artin fan of $\widetilde{\mathcal{VZ}}$. There is then a commuting square:

$$\begin{tikzcd} \widetilde{\mathcal{VZ}} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ \mathcal{A} \arrow[r] & \mathcal{A}_{\mathcal{M}}. \end{tikzcd}$$

Note that neither of the vertical maps are smooth, since the moduli spaces on the top row are not log smooth. The construction of $\widetilde{\mathcal{VZ}} \to \mathcal{M}$ as the log modification obtained by imposing an alignment condition gives us a combinatorial description of the map $\mathcal{A} \to \mathcal{A}_{\mathcal{M}}$ which we will use to study $\mathcal{A}$. Recall that $\mathcal{A}_{\mathcal{M}}$ is locally isomorphic to the stack quotient

$$\left[\operatorname{Spec} \Bbbk[Q] / \operatorname{Spec} \Bbbk[Q^{\mathrm{gp}}]\right]$$

where Q is a monoid giving a[2] local chart for $\mathcal{M}$, which we may take to be the minimal monoid of [GS13, §1.5]. The real dual $Q^\vee_{\mathbb{R}} = \operatorname{Hom}(Q, \mathbb{R}_{\geq 0})$ of this monoid can be viewed as a moduli space of tropical maps; see [GS13, Remark 1.12]. We call this *the tropical moduli space*; contained in the tropical moduli space are the edge lengths of the associated tropical curve (corresponding to the smoothing parameters of the nodes of the logarithmic curve). Since the alignment condition amounts to imposing a partial ordering amongst certain

[1](Navid) Make this more precise

[2](Navid) Neat?

sums of these edge lengths, it produces a polyhedral decomposition of the tropical moduli space, into chambers where different partial order relations hold. If we only consider the integral points, this produces a polyhedral decomposition of the cone $Q^\vee = \operatorname{Hom}(Q, \mathbb{N})$. Dualising, we obtain a toric blow-up

$$Z \to \operatorname{Spec} \Bbbk[Q]$$

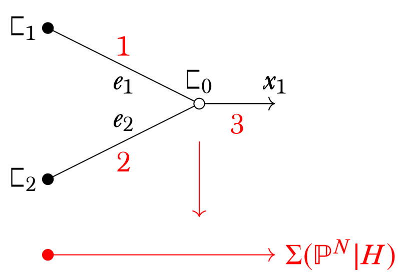
which, since it is equivariant, descends to a morphism of the associated zero-dimensional stacks:

$$[Z/T_Z] \to \left[\operatorname{Spec} \Bbbk[Q] / \operatorname{Spec} \Bbbk[Q^{\mathrm{gp}}]\right].$$

This gives a local description of the map $\mathcal{A} \to \mathcal{A}_\mathcal{M}$ and in particular of the Artin fan $\mathcal{A}$. By the orbit-cone correspondence, the codimension-k strata of $\mathcal{A}$ which intersect this open locus correspond to the k-dimensional cones in the polyhedral decomposition of $Q^\vee$ described above. These can be understood entirely in terms of tropical combinatorics. This is best explained through a number of examples, which we now present.

3.2. **Examples of logarithmic strata.** In these examples we will proceed as follows: we will start by fixing an element of the ordinary moduli space $\mathcal{M}$ of log stable maps. We will then compute the associated tropical moduli space, giving a description of $\mathcal{A}_\mathcal{M}$ local to our chosen element. We will then describe the necessary polyhedral subdivison, and thus give a description of $\mathcal{A}$ in a neighbourhood of the preimage in $\mathcal{VZ}$ of our chosen element of $\mathcal{M}$. Finally we will use this to describe the logarithmic strata which intersect this neighbourhood.

Example 3.1. Consider an element of $\mathcal{M}$ whose associated tropical map has the following combinatorial type:



Here each edge (corresponding to a node of the curve) has a length e_i and an expansion factor u_i (indicated in red). The moduli space of such tropical maps is generated by the edge lengths e_1 and e_2

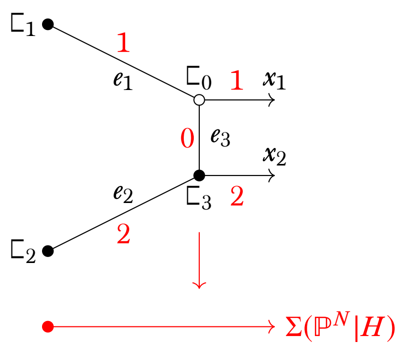
$$(\mathbb{R}_{\geq 0})_{e_1} \times (\mathbb{R}_{\geq 0})_{e_2}$$

subject to the continuity condition $e_1 = 2e_2$. So the tropical moduli space is simply $\mathbb{R}_{\geq 0}$ generated by e_2. Note that for any $e_2 > 0$ (i.e. on the interior of the cone) we have:

$$\lambda(C_2) = e_2 = e_1/2 < e_1 = \lambda(C_1).$$

Thus, any logarithmic map with combinatorial type given by the above picture is automatically aligned. This means that the map $\mathcal{VZ} \to \mathcal{M}$ is an isomorphism in a neighbourhood of our chosen element (there is no blowing up necessary). Indeed, the tropical moduli space is $\mathbb{R}_{\geq 0}$ and this cone does not admit a polyhedral subdivision (there is no non-trivial toric blow-up of $\mathbb{A}^1$).

Example 3.2. Consider now an element of $\mathcal{M}$ whose associated tropical map has the corresponding combinatorial type:



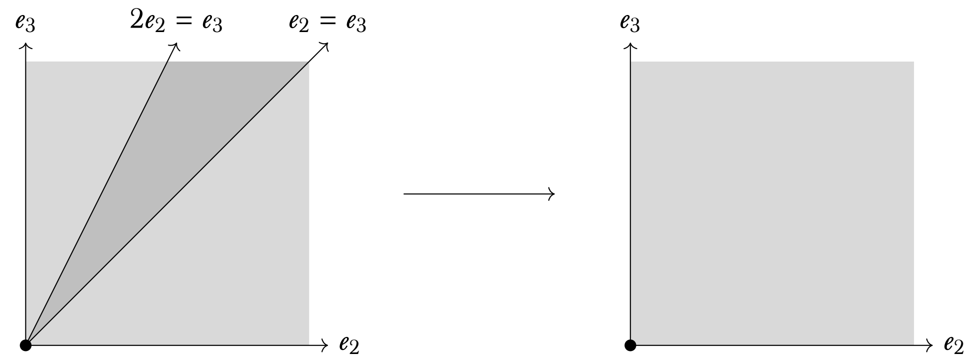
As before, expansion factors are indicated in red and the edge lengths are e_1, e_2, e_3. The tropical moduli space is $\mathbb{R}^3_{\geq 0}$ generated by these three lengths, subject to the continuity condition $e_1 = 2e_2$. Thus the moduli space is isomorphic to $\mathbb{R}^2_{\geq 0}$ generated by e_2 and e_3. In order to have an alignment, at least one of $\lambda(C_1)$ and $\lambda(C_2)$ must be equal to the radius δ. Note that

$$\lambda(C_1) = e_1 = 2e_2$$
$$\lambda(C_2) = e_2 + e_3$$

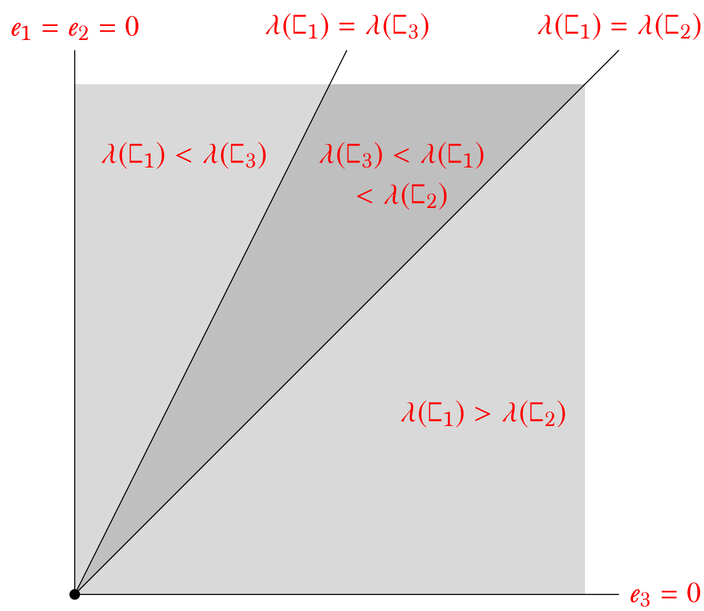
and without more information we cannot say which of these is larger. The subdivision of $(\mathbb{R}_{\geq 0})^2$ is obtained by dividing the cone into regions where different order relations hold amongst the distances $\lambda(C_1), \lambda(C_2), \lambda(C_3)$. The walls of this subdivison correspond to where some of these distances are equal. Note that in this setting we always have $\lambda(C_2) > \lambda(C_3)$ (at least, as long as we remain in the interior of the cone). The remaining possibilities are:

$$\lambda(C_1) = \lambda(C_3) \qquad (\Leftrightarrow e_1 = e_3 \Leftrightarrow 2e_2 = e_3)$$
$$\lambda(C_1) = \lambda(C_2) \qquad (\Leftrightarrow e_1 = e_2 + e_3 \Leftrightarrow e_2 = e_3).$$

Thus, the subdivision of the tropical moduli space $(\mathbb{R}^2_{\geq 0})_{e_2 e_3}$ is given by:



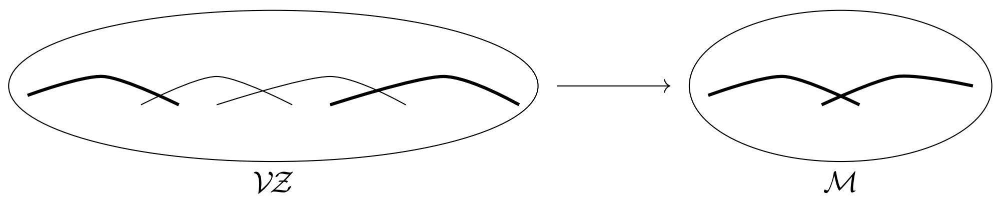
The cones of the subdivison index the logarithmic strata in a neighbourhood of the preimage in $\mathcal{VZ}$ of our chosen element of $\mathcal{M}$. These can be described as follows:



There are four codimension–1 logarithmic strata of $\mathcal{VZ}$ intersecting our chosen neighbourhood, corresponding to the rays in the above picture. Two of these – those corresponding to the rays labeled $\{e_1 = e_2 = 0\}$ and $\{e_3 = 0\}$ – are the proper transforms of codimension–1 strata in $\mathcal{M}$. These strata consist of log stable maps where some of the tropical edge lengths are equal to zero, meaning that the corresponding nodes have been smoothed. Notice that although the curve has three nodes, there are only two such strata: the nodes q_1 and q_2 cannot be smoothed independently because of the relation $e_1 = 2e_2$.

The remaining two codimension–1 strata in $\mathcal{VZ}$ – corresponding to the interior rays in the above picture – consist of log stable maps where some of the vertex distances become equal. Here none of the nodes are smoothed. From the construction of the subdivision, we see that both these strata map onto a codimension–2 stratum of $\mathcal{M}$ (namely, the locus in which all of the nodes persist); this coheres with the fact that they should be thought of as exceptional loci of the blow-up. The extra dimension of moduli comes from the choice of alignment.

Finally, there are three codimension–2 strata, corresponding to different *strict* orderings of the vertex distances. Note that the divisorial strata corresponding to $\{e_1 = e_2 = 0\}$ and $\{e_3 = 0\}$, which intersected in $\mathcal{M}$, no longer intersect in $\mathcal{VZ}$, since we have blown up. The picture is something like:



3.3. **Global logarithmic strata**. In the previous subsections we used tropical geometry to identify the logarithmic strata of $\mathcal{VZ} = \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ local to the preimage of a point in $\mathcal{M}$. In fact, the whole discussion carries over if we replace a point in $\mathcal{M}$ by a locally closed logarithmic stratum: the key observation is simply that the tropical moduli space does not vary if we move inside a locally closed logarithmic stratum, so the subdivision process makes sense over that whole stratum. This fact will allow us to describe the logarithmic strata of $\mathcal{VZ}$ globally.

3.3.1. *Logarithmic strata of* $\mathcal{M}$. The locally closed logarithmic strata of $\mathcal{M}$ consist of loci where the combinatorial type of the associated tropical curve is constant[3]. This is because the combinatorial type determines the minimal monoid Q, which coincides with the stalk of the ghost sheaf on $\mathcal{M}$.

If we have two strata $\mathcal{S}_1$ and $\mathcal{S}_2$ corresponding to combinatorial types Δ_1 and Δ_2, then $\mathcal{S}_2$ is contained in the closure of $\mathcal{S}_1$ if and only if the combinatorial type Δ_1 is obtained from Δ_2 by a process of generisation: namely, by contracting some edges (i.e. smoothing some nodes) and moving some of the vertices from the interior $\mathbb{R}_{>0} \subseteq \mathbb{R}_{\geq 0}$ of the tropicalisation $\Sigma(\mathbb{P}^N|H)$ to the vertex $0 \in \mathbb{R}_{\geq 0}$ (i.e. moving some components of the curve outside H). This allows us to completely describe the dual intersection complex of the logarithmic strata of $\mathcal{M}$.

Note that we are *not* able to easily read off the codimension of a logarithmic stratum from the combinatorial data: the codimension of the associated stratum in the Artin fan $\mathcal{A}_{\mathcal{M}}$ is given by the dimension of the tropical moduli space, but the map $\mathcal{M} \to \mathcal{A}_{\mathcal{M}}$ is not

[3] (Navid) Reference for this?

smooth (since $\mathcal{M}$ is not log smooth) so we are not able to say anything about the locus in $\mathcal{M}$.

3.3.2. *Logarithmic strata of $\mathcal{VZ}$.* Now let us pick a stratum $\mathcal{S} \subseteq \mathcal{M}$ indexed by a combinatorial type Δ and with associated monoid Q. We may choose a sufficiently small open neighbourhood of $\mathcal{S}$ which only intersects logarithmic strata $\mathcal{S}'$ which contain $\mathcal{S}$ in their closures. The previous discussion then shows that if we pick any point in this open neighbourhood, the combinatorial type of the associated tropical curve is obtained from Δ by contracting some edges and specialising some vertices. Thus we see that the associated map on tropical moduli spaces is injective, and so the generisation map on the level of ghost sheaves is surjective. This allows us to produce a chart on the open neighbourhood of $\mathcal{S}$ with monoid given by Q.

The discussion in the previous subsections then applies *mutatis mutandis* to this open set, giving a description of the logarithmic strata of $\mathcal{VZ}$ which intersect an open neighbourhood of the preimage of $\mathcal{S}$. Since log strata must map to log strata, this gives a procedure for enumerating all of the locally closed logarithmic strata of $\mathcal{VZ}$, namely:

(1) Enumerate the locally closed logarithmic strata of $\mathcal{M}$ by enumerating all possible combinatorial types of tropical curves with the given numerical data. The dual intersection complex of these strata is specified by generisation of combinatorial types, as described earlier.

(2) For each locally closed stratum $\mathcal{S} \subseteq \mathcal{M}$ identify the tropical moduli space $Q^\vee_{\mathbb{R}}$ associated to the given combinatorial type, and perform the subdivision specified by the alignment condition. The arguments of the previous subsections carry over to give a description of the logarithmic strata of $\mathcal{VZ}$ which map to a neighbourhood of $\mathcal{S}$. The dual intersection complex of these strata is specified by the combinatorics of each subdivision together with the dual intersection complex of the strata of $\mathcal{M}$.

We now illustrate this is in some examples:

Example 3.3. Example of low-degree computation of logarithmic strata. (Note that for $\mathcal{M}$ we can't read off the codimension from the dimension of the tropical moduli space because $\mathcal{M}$ is not log smooth.)[4]

[4](Navid) To be done.

4. Splitting axiom and recursion formula for $(\mathbb{P}^N|H)$

In the previous section we explained how to determine the logarithmic strata of $\mathcal{VZ}$. We now focus our attention on certain logarithmic divisors, providing a recursive description for these loci: that is, a description in terms of fibre products of smaller moduli spaces. This is the technically most difficult part of the paper. The recursive description bears some similarities with the recursive description of the boundary of the absolute space $\mathcal{VZ}_{1,n}(\mathbb{P}^N, d)$ given in [VZ08]; in keeping with the theme of this paper, however, we will see that log geometry provides a convenient framework for dealing with the combinatorics.

The end result is that we obtain an explicit formula for tautological integrals over these boundary loci. This produces a Gathmann recursion for relative invariants in genus one.

The plan for this section is as follows [5]

4.1. **Recursion setup**. Consider an element of $\mathcal{VZ}$ and choose a marked point $x_k \in C$. We note that by the existence of a log morphism, either f has contact order α_k to H at the marking x_k, or else the irreducible component of C containing x_k is mapped entirely inside H (we say "x_k belongs to an internal component" in this case).

By pulling back the equation defining H along f and then taking its α_k-th derivative at x_k (which makes sense since all the lower-order derivatives vanish by assumption) we obtain a section of a jet bundle on $\mathcal{VZ}$ whose vanishing locus coincides (set-theoretically) with the locus where x_k belongs to an internal component. As in [Gat02, Construction 2.1] there is an exact sequence of jet bundles

$$0 \to x_k^* \omega_{\mathcal{C}/\mathcal{VZ}}^{\otimes \alpha_k} \otimes \mathrm{ev}_k^* \mathcal{O}_{\mathbb{P}^N}(H) \to x_k^* \mathcal{J}^{\alpha_k}(f^* \mathcal{O}_{\mathbb{P}^N}(H)) \to x_k^* \mathcal{J}^{\alpha_k - 1}(f^* \mathcal{O}_{\mathbb{P}^N}(H)) \to 0$$

and thus we obtain a section of the line bundle $x_k^* \omega_{\mathcal{C}/\mathcal{VZ}}^{\otimes \alpha_k} \otimes \mathrm{ev}_k^* \mathcal{O}_{\mathbb{P}^N}(H)$ with vanishing locus equal to the locus where x_k belongs to an internal component. Writing this locus as

$$\mathcal{D}(k) = \mathcal{D}^k_{1,\alpha}(\mathbb{P}^N|H, d) \subseteq \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d) = \mathcal{VZ}$$

we obtain:

Lemma 4.1. $(\alpha_k \psi_k + \mathrm{ev}_k^* H) \cap [\mathcal{VZ}] = [\mathcal{D}(k)]$.[6]

The remainder of this section will be devoted to providing an explicit recursive description of the class $[\mathcal{D}(k)]$ in terms of tautological classes on spaces of maps with smaller numerical invariants. As we will see, this provides an algorithm for computing reduced relative invariants, as well as a quantum Lefschetz algorithm for reduced (absolute) invariants in genus one.

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[5](Navid) Finish this

[6](Navid) Should we work out the vanishing orders of the section at this point?

4.2. **Identifying the irreducible components of $\mathcal{D}(k)$.** Here we explain the strategy for identifying the irreducible components of $\mathcal{D}(k)$ via tropical geometry.

Lemma 4.2. Every irreducible component of $\mathcal{D}(k)$ is a codimension-1 logarithmic stratum.

Proof. This is not hard to see: the locus where the log structure is trivial coincides precisely with the locus where the source curve is smooth and not mapped inside H, and since $\mathcal{VZ}$ is log smooth this locus is open and dense. By definition $\mathcal{D}(k)$ is disjoint from this locus, hence is contained in the complement of this locus, which (by the very definition of logarithmic strata) can be written as a union of logarithmic strata of positive codimension. Since $\mathcal{D}(k)$ has codmension 1, it must therefore be equal to a union of logarithmic divisors. □

In the previous section we discussed at length a procedure for describing the logarithmic strata of $\mathcal{VZ}$. This shows that every divisorial logarithmic stratum is obtained via a three-step process:

(1) choose the combinatorial type of a tropical map;
(2) subdivide the corresponding tropical moduli space;
(3) choose a ray in this subdivision.

Notice that this process contains some redundancies: once we choose a ray of the tropical moduli space, we obtain a generisation of the intial combinatorial type, given by contracting some edges of the dual graph.

Example 4.3.

Whenever we consider a logarithmic stratum in $\mathcal{VZ}$, by its combinatorial type we will always mean the generisation of the "initial" combinatorial type induced by our choice of cone in the polyhedral subdivision. This does not depend on the choice of initial combinatorial type.

Now, the logarithmic divisors which are contained in $\mathcal{D}(k)$ are precisely those whose corresponding combinatorial type has the vertex of the dual graph containing x_k mapped into the interior $\mathbb{R}_{>0} \subseteq \mathbb{R}_{\geq 0}$.

Thus, via the above procedure, we are able to enumerate the irreducible components of $\mathcal{D}(k)$ in a combinatorial manner. The combinatorial type and choice of ray allows us to describe the general element of such a component. What we are still lacking is a recursive description of the components, which we need in order to compute integrals over them. This is the subject of the remainder of this section.

4.3. **Splitting and gluing punctured logarithmic maps.** We still start by considering the usual moduli space $\mathcal{M}$ of log stable maps. In [ACGS], a recursive description of the logarithmic strata of $\mathcal{M}$ is given, in terms of moduli spaces of punctured maps. As discussed above, the logarithmic strata $\mathcal{S}$ of $\mathcal{M}$ are indexed by combinatorial types of tropical curves. Given such a combinatorial type Δ, each vertex v of the associated graph defines a moduli space of punctured stable maps:

$$\overline{\mathcal{M}}_v = \overline{\mathcal{M}}^{\text{punct}}_{g(v),\alpha(v)}(Z_v, d(v)).$$

Here $g(v)$ and $d(v)$ are the genus and degree assigned to the vertex, while Z_v is the logarithmic stratum of the target corresponding to the tropical stratum into which the vertex is mapped by the combinatorial type (in our setting, $Z_v = \mathbb{P}^N$ or H). The length of the vector $\alpha(v)$ is given by the number of flags adjacent to v, and the entries (which represent tangencies, and may be positive or negative) are given by the expansion factors along those flags. If all of the tangencies are positive, then this is nothing but an ordinary moduli space of log stable maps.

The corresponding logarithmic stratum is then obtained as a follows. For each vertex v and each adjacent flag $E \ni v$ we define $\widetilde{\mathcal{M}}^E_v$ to be the stack $\overline{\mathcal{M}}_v$ equipped with the log structure

$$p^\star_{v,E}\mathcal{M}_{\mathcal{C}^\circ_v}$$

where $p_{v,E}\colon \overline{\mathcal{M}}_v \to \mathcal{C}^\circ_v$ is the marking section corresponding to E. This definition makes $p_{v,E}$ into a strict logarithmic map, and composing with the universal map produces a logarithmic evaluation morphism $\mathrm{ev}_{v,E}\colon \widetilde{\mathcal{M}}^E_v \to Z_E$, where $Z_E \subseteq Z_v$ is the logarithmic stratum into which the marking corresponding to E is mapped.

If $E_1, \ldots, E_k$ are the flags adjacent to v then we define

$$\widetilde{\mathcal{M}}_v = \widetilde{\mathcal{M}}^{E_1}_v \times_{\overline{\mathcal{M}}_v} \ldots \times_{\overline{\mathcal{M}}_v} \widetilde{\mathcal{M}}^{E_k}_v$$

where the fibre product is taken in the fs log category. With this definition, all the evaluation maps $\mathrm{ev}_{v,E}\colon \widetilde{\mathcal{M}}_v \to Z_E$ are naturally logarithmic morphisms. Then the corresponding logarithmic stratum is obtained by taking the following fibre product (again, in the fs log category):

$$\begin{tikzcd} \mathcal{S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \prod_v \widetilde{\mathcal{M}}_v \arrow[d] \\ \prod_E Z_E \arrow[r] & \prod_E Z_E \times Z_E. \end{tikzcd}$$

In the above discussion, there were two steps at which we needed to take fs fibre products: first, when constructing each $\widetilde{\mathcal{M}}_v$ and second, when construction $\mathcal{S}$. Typically the fs fibre product does not agree with the usual fibre product of stacks; rather, one must take the fibre product in the category of coherent log stacks (which *does* agree with the usual fibre product) and then integralise and saturate the result.

We will now argue that in our situation, when we take fibre products of coherent log stacks, the result is automatically integral. This means that the integralisation process is redundant, and so the fs fibre product is obtained from the ordinary fibre product by a process of saturation. This amounts to taking a finite cover of the underlying stack (whose degree we will compute).

The following lemma holds for arbitrary moduli spaces of punctured maps.

Lemma 4.4. Given a combinatorial type Δ indexing a stratum of $\mathcal{M}$ and a vertex v of the associated graph, consider the corresponding moduli space of punctured stable maps $\overline{\mathcal{M}}_v$. As in the above discussion, define for each adjacent flag $E \ni v$ the log stack $\widetilde{\mathcal{M}}_v^E$ by pulling back the log structure of the punctured curve along the marking section. Consider the fibre product *in the category of coherent log stacks*:

$$\widetilde{\mathcal{M}}_v^{\text{not-fs}} = \widetilde{\mathcal{M}}_v^{E_1} \times_{\overline{\mathcal{M}}_v} \ldots \times_{\overline{\mathcal{M}}_v} \widetilde{\mathcal{M}}_v^{E_k}.$$

Then $\widetilde{\mathcal{M}}_v^{\text{not-fs}}$ is integral (and consequently, the map $\widetilde{\mathcal{M}}_v \to \widetilde{\mathcal{M}}_v^{\text{not-fs}}$ is finite).

Proof. Locally we have a chart for $\overline{\mathcal{M}}_v$ given by Q (the minimal monoid as constructed, for instance, in [GS13, Construction 1.16]) and charts for $\widetilde{\mathcal{M}}_v^{E_i}$ given by Q_{E_i} where $Q_{E_i} = Q \oplus \mathbb{N}$ if E_i corresponds to an ordinary marked point, and Q_{E_i} is the submonoid of $Q \oplus \mathbb{Z}$ generated by $Q \oplus \mathbb{N}$ and $(\overline{\varphi}_{\eta_i}(1), u_{E_i})$ if E_i corresponds to a puncturing (so that $u_{E_i} \in \mathbb{Z}_{<0}$). In either case, we have $Q_{E_i} \subseteq Q \oplus \mathbb{Z} \subseteq Q^{\text{gp}} \oplus \mathbb{Z}$ (since Q is integral) and so Q_{E_i} is integral (recall that a monoid is integral if and only if it admits an injective homomorphism into a group). In this notation, $\widetilde{\mathcal{M}}_v^{\text{not-fs}}$ admits a chart given by the monoid:

$$P = Q_{E_1} \oplus_Q Q_{E_2} \oplus_Q \ldots \oplus_Q Q_{E_k}.$$

If we consider the map $D\colon Q^{r-1} \to Q^r$ given by the $(r-1) \times r$ matrix

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 1 & 0 & \ldots & 0 \\ & & & \vdots & & \\ 0 & 0 & \ldots & 0 & 1 & 1 \end{pmatrix}$$

then the monoid P can be expressed as a cokernel

$$Q^{r-1} \to \oplus_{i=1}^r Q_{E_i} \to P$$

where the first map is obtained by composing D with the natural inclusion $Q^r \to \oplus_{i=1}^r Q_{E_i}$. We may now form a diagram

$$\begin{tikzcd}
Q^{r-1} \arrow[r] \arrow[d, "\mathrm{Id}"] & \oplus_{i=1}^r Q_{E_i} \arrow[r] \arrow[d, hook] & P \arrow[d] \\
Q^{r-1} \arrow[r] & Q^r \oplus \mathbb{Z}^r \arrow[r] & S \oplus \mathbb{Z}^r
\end{tikzcd}$$

where both rows are cokernel diagrams and the map $P \to S \oplus \mathbb{Z}^r$ is induced by the universal property. Here S is the cokernel of D and is given by:

$$S = \begin{cases} Q & \text{if } r = 1 \\ Q^{\mathrm{gp}} & \text{if } r \geq 2 \end{cases}.$$

In any case, S is integral and the map $P \to S \oplus \mathbb{Z}^r$ is injective because the middle vertical maps in injective. Hence we conclude that P is integral, as required. □

[7]The previous lemma holds for arbitrary moduli spaces of punctured maps. The next lemma, on the other hand, is special to our setting.

Lemma 4.5. Consider the fibre product in the category of coherent log stacks:

$$\begin{tikzcd}
\mathcal{S}^{\text{not-fs}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \prod_v \widetilde{\mathcal{M}}_v \arrow[d] \\
\prod_E Z_E \arrow[r] & \prod_E Z_E \times Z_E.
\end{tikzcd}$$

Then $\mathcal{S}^{\text{not-fs}}$ is integral.

Proof. □

This gives a recursion description of the logarithmic strata in $\mathcal{M}$. We will now[8] use this to obtain a similar (though somewhat more complicated) description of the strata in $\mathcal{VZ}$.

4.4. **Punctured maps and double ramification loci.** [9]

We will now present some arguments and constructions aimed at demystifying the moduli spaces of punctured stable maps. In our context, we only ever encounter these spaces when the associated logarithmic stratum Z_v of the target is the divisor H. Consider therefore a moduli space of punctured stable maps

$$\overline{\mathcal{M}}^{\text{punct}}_{g,\alpha}(H, d) \tag{1}$$

where $g \in \{0, 1\}$ and α is a vector of tangency orders. The target H is given the log structure pulled back from $\mathbb{P}^N$, and consequently is not log smooth. As such, the natural obstruction theory on the above moduli space will not, in general, be perfect. This is

[7](Navid) Is this a good point to calculate the degree of $\widetilde{\mathcal{M}}_v \to \widetilde{\mathcal{M}}_v^{\text{not-fs}}$?

[8](Navid) Really, after the next section, which maybe we should move to an appendix

[9](Navid) Move this to an appendix?

not really a problem for us, since we are mainly interested in studying the non-virtual geometry of our moduli spaces. Indeed, we will see that the space (1) is birational to the moduli space of ordinary stable maps to H satisfying the condition:

$$f^*[H] - \sum_{i=1}^{n} \alpha_i x_i = 0 \in \mathrm{A}_0(C). \tag{2}$$

This space forms a closed substack

$$\overline{\mathcal{M}}^{\alpha}_{g,n}(H,d) \subseteq \overline{\mathcal{M}}_{g,n}(H,d)$$

and is referred to as the *double ramification locus.* These moduli spaces have been the focus of intense study over the years (see, for instance, [Gat03] [JPPZ18]). The double ramification locus is a closed substack of $\overline{\mathcal{M}}_{g,n}(H,d)$ of virtual codimension g (though, as with $\overline{\mathcal{M}}_{g,n}(H,d)$ itself, it typically has excess dimension).

Proposition 4.6. Suppose that $g \in \{0,1\}$ and consider the morphism forgetting all log structures:

$$\overline{\mathcal{M}}^{\mathrm{punct}}_{g,\alpha}(H,d) \to \overline{\mathcal{M}}_{g,n}(H,d).$$

This factors through the double ramification locus, and is in fact birational onto its image.

Proof. The fact that the forgetful morphism factors through the double ramification locus is an immediate consequence of the balancing condition satisfied by punctured stable maps, interpreted as a condition in $\mathrm{Pic}(C)$ (the more familiar balancing condition is obtained from this one by restricting to a component of C and then taking degrees). Thus indeed we have a map:

$$\varphi\colon \overline{\mathcal{M}}^{\mathrm{punct}}_{g,\alpha}(H,d) \to \overline{\mathcal{M}}^{\alpha}_{g,n}(H,d).$$

To show that φ is birational, we will choose an open dense locus of the domain and show that φ induces an isomorphism on geometric points when restricted to this locus. The locus we take is the preimage of the locus inside $\overline{\mathcal{M}}^{\alpha}_{g,n}(H,d)$ consisting of curves of compact type (i.e. those whose dual graph has genus zero); essentially we are excluding stable maps whose source curve contains a cycle of $\mathbb{P}^1$s. The fact that this is open dense in $\overline{\mathcal{M}}^{\alpha}_{g,n}(H,d)$ is not hard to see: when $g=1$ (the only case in which non-compact type curves can arise), the double ramification locus contains many components, but on every such component the maximal genus one subcurve of the source will be generically smooth, which shows again that the locus of stable maps not of compact type is closed inside $\overline{\mathcal{M}}^{\alpha}_{g,n}(H,d)$. Once we show that φ is bijective on geometric points when restricted to the preimage of this locus, it follows (since φ is a finite morphism) that the preimage of this locus is dense in $\overline{\mathcal{M}}^{\mathrm{punct}}_{g,\alpha}(H,d)$, and hence that φ is birational.

So, let us start with a compact type stable map which belongs to the double ramification locus:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & H \\ S = \operatorname{Spec} \Bbbk & \end{tikzcd}$$

We will show that there exists a unique lift of this to a punctured stable map. Since C has compact type, there is a unique choice of associated combinatorial type Δ. This determines a minimal monoid Q such that any punctured map enhancing the given stable map has base log structure given by:

$$\begin{aligned} Q \oplus \Bbbk^\star &\to \Bbbk \\ (q, \lambda) &\mapsto \begin{cases} \lambda & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

If we let r denote the number of nodes of C then by construction there is a natural map $\mathbb{N}^r \to Q$ and by the universal property of minimality our punctured curve must be obtained by puncturing a log smooth curve pulled back along a morphism of log schemes:

$$(\operatorname{Spec} \Bbbk, Q \oplus \Bbbk^\star) \to (\operatorname{Spec} \Bbbk, \mathbb{N}^r \oplus \Bbbk^\star).$$

This log morphism must enhance the given map $\mathbb{N}^r \to Q$ on the level of ghost sheaves. The moduli for such a log map is $\mathbb{G}_{\mathrm{m}}^r$, but all these choices are canceled out by automorphisms of $(\operatorname{Spec} \Bbbk, Q \oplus \Bbbk^\star)$[10]. Thus we obtain, by pulling back the minimal log smooth curve over $(\operatorname{Spec} \Bbbk, \mathbb{N}^r \oplus \Bbbk^\star)$, a unique log smooth curve:

$$(C, \mathcal{M}_C) \to (\operatorname{Spec} \Bbbk, Q \oplus \Bbbk^\star).$$

We must now construct the puncturing $\mathcal{M}_{C^\circ}$. Let $\mathcal{P}_C$ denote the divisorial log structure on C with respect to the punctures. It is easy to see that there is a unique log structure $\mathcal{M}$ on C such that

$$\mathcal{M}_C = \mathcal{M} \oplus_{\mathcal{O}_C^\star} \mathcal{P}_C$$

where the pushout is taken in the category of monoid sheaves. We consider the sheaf:

$$\mathcal{N}_C = \mathcal{M} \oplus_{\mathcal{O}_C^\star} \mathcal{P}_C^{\mathrm{gp}}.$$

By definition the puncturing $\mathcal{M}_{C^\circ}$ should be an intermediate sheaf:

$$\mathcal{M}_C \subseteq \mathcal{M}_{C^\circ} \subseteq \mathcal{N}_C.$$

[10](Navid) Here we are using the fact that Q is in some sense bigger than $\mathbb{N}^r$.

We will now construct this, and argue that our construction gives the only puncturing possible. First we note that it is sufficient to define $\mathcal{M}_{C^\circ}$ on a base for the topology on C. We choose a base such that for each base open set U:

(1) U is connected;
(2) U contains at most one of the punctures of C;
(3) if U contains a puncture $p \in C$, then U does not contain any of the nodes or markings of C, and $\mathcal{O}_C(p)|_U$ is trivial.

Note that these properties are stable under taking subsets, so to obtain such a base we simply choose an open cover with this property (which is certainly possible) and then take the collection of all open subsets of our chosen cover.

Now let U be such an open set in our base. If U does not contain a puncture then $\mathcal{M}_C|_U = \mathcal{N}_C|_U$ and so we set

$$\mathcal{M}_{C^\circ}(U) = \mathcal{M}_C(U).$$

If U contains a puncture p then by assumption $U \setminus \{p\}$ contains only smooth unmarked points, and hence we have a splitting

$$\mathcal{N}_C|_U = (\pi^\star \mathcal{M}_S|_U) \oplus_{\mathcal{O}_U^\star} \left(\mathcal{P}_p^{\mathrm{gp}}|_U\right)$$

where $\mathcal{P}_p$ is the divisorial log structure with respect to p. Since $\mathcal{M}_S = Q \oplus \Bbbk^\star$ it follows that $\pi^* \mathcal{M}_S|_U = \underline{Q} \oplus \mathcal{O}_U^\star$. Thus we see that

$$\mathcal{N}_C|_U = \underline{Q} \oplus \mathcal{P}_p^{\mathrm{gp}}|_U$$

and similarly:

$$\mathcal{M}_C|_U = \underline{Q} \oplus \mathcal{P}_p|_U.$$

We therefore have (since U is connected):

$$\mathcal{N}_C(U) = Q \oplus \mathcal{P}_p^{\mathrm{gp}}(U)$$

Recall we are assuming that $\mathcal{O}_C(p)|_U$ is trivial. Choosing a trivialisation, we obtain a function $s \in \mathcal{O}_U(U)$ such that $p = \{s = 0\}$ on U. Then $\mathcal{P}_p(U)$ consists of functions of the form $\lambda \cdot s^n$ for $n \in \mathbb{N}$ and $\lambda \in \mathcal{O}_U^\star(U)$. Groupifying, we obtain:

$$\mathcal{P}_p^{\mathrm{gp}}(U) = \left\{\lambda \cdot s^k \mid \lambda \in \mathcal{O}_U^\star(U), k \in \mathbb{Z}\right\}.$$

(Indeed, $\mathcal{P}_p^{\mathrm{gp}}$ can be described as the sheaf of rational functions which are regular and invertible outside of p.) Recall that p has attached to it a tangency order $\alpha_p \in \mathbb{Z}_{<0}$. If we let η denote the generic point of the component of C containing p, then we also have a pullback map $\overline{\varphi}_\eta \colon \mathbb{N} \to Q$ which is sharp (i.e. it is not the zero map). We define $\mathcal{M}_{C^\circ}(U) \subseteq \mathcal{N}_C(U)$ to be the submonoid generated by $Q \oplus \mathcal{P}_p(U)$ and the element:

$$(\overline{\varphi}_\eta(1), s^{\alpha_p}).$$

This clearly satisfies the conditions to be a puncturing, and by the prestability condition it is the only possibility for $\mathcal{M}_{C^\circ}$. We thus have a punctured curve:

$$(C, \mathcal{M}_{C^\circ}) \to (C, \mathcal{M}_C) \to (\operatorname{Spec} \Bbbk, Q \oplus \Bbbk^\star).$$

It remains to construct the log enhancement of the map $f\colon C \to H$. The map $f^\flat\colon f^\star\mathcal{M}_H \to \mathcal{M}_{C^\circ}$ is already determined on the level of ghost sheaves. To enhance this to a map on the level of log structures entails choosing an isomorphism of line bundles:

$$f^*\mathcal{O}_H(1) \cong \mathcal{O}_C\left(\overline{f}^\flat(1)\right).$$

The component-wise description of the right-hand side in [RSW17a] (together with the fact that C is compact type, so its Picard group is simply the product of the Picard groups of its components) shows that such an isomorphism exists if and only if the original stable map is in the double ramification locus. There is a $\mathbb{G}_{\mathrm{m}}$ worth of choices for such a map, but these are canceled out by the automorphisms of $(C, \mathcal{M}_{C^\circ})$ induced by pulling back along (strict) automorphisms of $(\operatorname{Spec} \Bbbk, Q \oplus \Bbbk^\star)$. This completes the proof of unique punctured lifting. □

Thus, the spaces of punctured maps we encounter are always birational to double ramification loci. We now briefly discuss the geometry of these loci. In the genus zero case, the double ramification locus is equal to the entire space

$$\overline{\mathcal{M}}^{\alpha}_{0,n}(H, d) = \overline{\mathcal{M}}_{0,n}(H, d)$$

which is smooth. Thus tautological integrals over $\overline{\mathcal{M}}^{\alpha}_{0,n}(H, d)$ are determined by the genus-zero Gromov–Witten theory of H.

The genus one case is more delicate: there is a "main component" of the double ramification locus, obtained as the closure of the locus where the source curve is smooth. This forms a divisor inside the main component of the moduli space of stable maps:

$$\overline{\mathcal{M}}^{\alpha,\circ}_{1,n}(H, d) \subseteq \overline{\mathcal{M}}^{\circ}_{1,n}(H, d).$$

However, we know that $\overline{\mathcal{M}}_{1,n}(H, d)$ contains many other components besides $\overline{\mathcal{M}}^{\circ}_{1,n}(H, d)$, some of which have excess dimension. Certainly the double ramification locus intersects these other components, and sometimes it can do so in quite dramatic fashion. For instance, consider the locus where the source curve splits into a contracted elliptic component and a rational component containing all the markings. This locus forms an irreducible component of the moduli space, isomorphic to

$$\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,n+1}(H, d)$$

which has excess dimension $N-2$. This entire component belongs to the double ramification locus, since the condition (2) becomes entirely numerical here. Thus, we see that the double ramification locus can be quite badly-behaved in genus one. Fortunately, we will only be interested in the main component of this locus (which is birational to the main component of the moduli space of punctured maps); see §?? below.

4.5. **Recursive description of the divisors: types A, B and C^+.** The basic idea is as follows. Choose an irreducible component $\mathcal{D} \subseteq \mathcal{D}(k)$. As discussed above, we can obtain this by choosing an "initial" combinatorial type together with a ray of the resulting subdivision of the tropical moduli space. Without loss of generality we may assume that this "initial" combinatorial type coincides with the "true" combinatorial type of $\mathcal{D}$ obtained by performing the edge contractions specified by the choice of ray.

Since $\mathcal{VZ} \to \mathcal{M}$ is a log modification, the divisor $\mathcal{D}$ is either exceptional or the proper transform of a logarithmic divisor on $\mathcal{M}$. In this subsection, we will focus on the latter case. Then if we let $\mathcal{D}^\circ_{\mathcal{M}} \subseteq \mathcal{M}$ denote the locally closed stratum corresponding to our choice of combinatorial type, and $\mathcal{D}_{\mathcal{M}} = \overline{\mathcal{D}^\circ_{\mathcal{M}}}$ denote the corresponding logarithmic divisor in $\mathcal{M}$, then our divisor $\mathcal{D} \subseteq \mathcal{VZ}$ is the proper transform of $\mathcal{D}_{\mathcal{M}}$, which is obtained by taking the preimage $\mathcal{D}^\circ$ of $\mathcal{D}^\circ_{\mathcal{M}}$ (since $\mathcal{D}^\circ_{\mathcal{M}}$ is disjoint from the blown-up locus) and then taking its closure in $\mathcal{VZ}$. This is, of course, different from taking the preimage of $\mathcal{D}_{\mathcal{M}}$, since in general $\mathcal{D}_{\mathcal{M}}$ will intersect the blown-up locus. By continuity we have an induced map:

$$\mathcal{D} \to \mathcal{D}_{\mathcal{M}}.$$

The remainder of this section will be dedicated to showing how the map $\mathcal{D} \to \mathcal{D}_{\mathcal{M}}$ may be interpreted as a blow-up. The reason this is useful is that $\mathcal{D}_{\mathcal{M}}$ itself has a recursive description in terms of moduli spaces of punctured maps (see §4.3), which in our setting we are able to compute integrals over.

For the remainder of this subsection, we will consider only the situation where the circuit is assigned a positive degree by the combinatorial type; the other, more complicated case, will be taken up in §??.

Lemma 4.7. Let $\mathcal{D} \subseteq \mathcal{D}(k)$ be an irreducible component and let Δ be the corresponding combinatorial type. Suppose that Δ assigns positive degree to the circuit. Then Δ takes one of the following forms:

The terminology is due to Vakil, and the picture in this case is very similar to [Vak00] (as we will see later, when the circuit is contracted the picture becomes *very* different). Note that in these pictures we have omitted the marked points (apart from x_k), the degree of each vertex, and the expansion factors of the edges. These combinatorial data can be distributed arbitrarily, as long as:

(1) the vertices $\sqsubset_1, \ldots, \sqsubset_r$ have positive degree (and $\sqsubset_0$ has positive degree in the $\mathcal{Y}_C^+$ case);

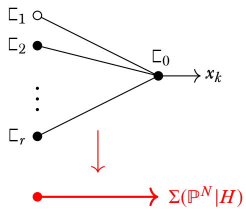


FIGURE 1. $\mathcal{Y}_A$

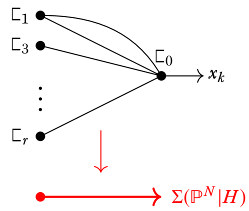


FIGURE 2. $\mathcal{Y}_B$

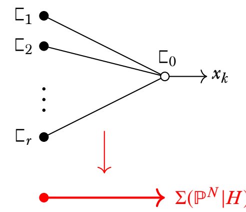


FIGURE 3. $\mathcal{Y}_C^+$

(2) every vertex is stable;
(3) the balancing condition is satisfied.

Proof. Recall that we have $\mathcal{D} \subseteq \mathcal{VZ}$ obtained by a choice of combinatorial type and ray in the subdivison of the tropical moduli space. In this situation, since the circuit has positive degree no alignment is necessary (both radii are zero), and hence the subdivision is trivial. Following our procedure, we choose a ray of the tropical moduli space $Q_{\mathbb{R}}^{\vee}$. Since this is generated by edge lengths, and we are assuming that the initial and generised combinatorial types coincide, we conclude that the tropical moduli space must be $\mathbb{R}_{\geq 0}$. Since we are assuming that $\mathcal{D} \subseteq \mathcal{D}(k)$, we know that the vertex of the tropicalisation which contains the flag x_k must map into the interior $\mathbb{R}_{>0} \subseteq \mathbb{R}_{\geq 0}$.

We claim that this is the only vertex of the tropical curve mapped into the interior; if there were more, the positions of their images in $\mathbb{R}_{>0}$ would be independent, and thus the tropical moduli space would have dimension ≥ 2, a contradiction.[11]

Thus there is a single vertex $\sqsubset_0$ mapped into the interior $\mathbb{R}_{>0} \subseteq \mathbb{R}_{\geq 0}$ and a number of vertices $\sqsubset_1, \ldots, \sqsubset_r$ mapped onto the vertex $0 \in \mathbb{R}_{\geq 0}$. No pair of $\sqsubset_1, \ldots, \sqsubset_r$ can be connected by an edge, since this would introduce an extra edge length to the tropical moduli space, which then once again would have dimension ≥ 2. Each of $\sqsubset_1, \ldots, \sqsubset_r$ must have positive degree by the tropical balancing condition and must be connected to $\sqsubset_0$ (since the tropical curve must be connected).

Thus, we see that the combinatorial type takes the form of a bipartite graph, with $\sqsubset_0$ on the right-hand side and $\sqsubset_1, \ldots, \sqsubset_r$ on the left-hand side. Note that the continuity of the tropical map identifies all of the edge lengths, up to weights given by the expansion factors (i.e. tangency orders) at the edges; thus the tropical moduli space is isomorphic to $\mathbb{R}_{\geq 0}$ and so we have a divisor as expected.

We distinguish three cases, depending on the position of the circuit, giving the three forms presented in the statement of the lemma. □

[11] (Luca) Say something about rigid tropical curves as in ACGS and KLR

We now now investigate the three types A, B, C^+ separately, giving a recursive description of the boundary locus in each case.

4.5.1. *Type A*. Let Δ be a combinatorial type of type A and let $\mathcal{D} \subseteq \mathcal{VZ}$ be the corresponding logarithmic divisor (note there is no choice of ray here). The corresponding locus $\mathcal{D}_{\mathcal{M}} \subseteq \mathcal{M}$ is given (up to a finite cover) by[12]:

$$\mathcal{D}_{\mathcal{M}} = \overline{\mathcal{M}}^{\mathrm{punct}}_{0,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H,d_0) \times_{H^r} \left(\overline{\mathcal{M}}^{\log}_{1,\alpha^{(1)}\cup(m_1)}(\mathbb{P}^N|H,d_1) \times \prod_{i=2}^{r} \overline{\mathcal{M}}^{\log}_{0,\alpha^{(i)}\cup(m_i)}(\mathbb{P}^N|H,d_i) \right).$$

Lemma 4.8. We have the following description of $\mathcal{D}$

$$\mathcal{D} = \overline{\mathcal{M}}^{\mathrm{punct}}_{0,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H,d_0) \times_{H^r} \left(\mathcal{VZ}_{1,\alpha^{(1)}\cup(m_1)}(\mathbb{P}^N|H,d_1) \times \prod_{i=2}^{r} \overline{\mathcal{M}}^{\log}_{0,\alpha^{(i)}\cup(m_i)}(\mathbb{P}^N|H,d_i) \right) \tag{3}$$

i.e. the map $\mathcal{D} \to \mathcal{D}_{\mathcal{M}}$ is given by blowing up one of the factors of the fibre product.

Note that there is a birational map which forgets the log structures

$$\overline{\mathcal{M}}^{\mathrm{punct}}_{0,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H,d_0) \to \overline{\mathcal{M}}_{0,n_0+r}(H,d_0)$$

and all of our insertions are pulled back from the latter space. Therefore the integrals over the punctured space are determined by the genus zero Gromov–Witten theory of $H \cong \mathbb{P}^{N-1}$.

Proof. Note first of all that the map $\mathcal{D} \to \mathcal{D}_{\mathcal{M}}$ is an isomorphism away from the blown-up locus, which is contained inside the locus where the circuit is contracted. Given an element of $\mathcal{D}$ we can split it along the nodes $q_1, \ldots, q_r$ as in [ACGS]. It is then clear that C_1 is aligned. We claim that C satisfies the factorisation property if and only if C_1 does. This is enough to show (3).

On C there is an associated contraction radius δ passing through a non-contracted vertex, such that the strict interior only contains contracted vertices.

Lemma 4.9. $\lambda(\mathsf{C}') > \delta$ for any component C' of C_0.

Assuming this holds, we see as a consequence that $\mathsf{C}_0, \mathsf{C}_2, \ldots, \mathsf{C}_r$ must lie outside the contraction radius. Consequently the aligned curve C satisfies the factorisation condition if and only if C_1 does, so (3) holds.

Proof. The basic point is that $\mathcal{D}$ consists of the union of the locally closed logarithmic strata adjacent to the locally closed stratum where all the C_i are irreducible. If we look at one of these boundary strata, the tropical moduli space contains a ray σ corresponding to the stratum where all the C_i are irreducible; this amounts to setting all edge lengths

[12] (Navid) Include short appendix on integralisation/saturation when gluing punctured maps in our setting?

other than $e_1, \ldots, e_r$ to zero. If $f_1, \ldots, f_l$ are some number of these additional edge lengths (corresponding to internal nodes in degenerations of the $\sqsubset_i$) then all of the cones of the subdivision adjacent to σ will have $f_1 + \ldots + f_l < e_j$ for all $j \in \{1, \ldots, r\}$, since $f_1 = \ldots = f_l = 0$ and $e_j \neq 0$ on σ. In particular, if $\sqsubset_1$ is degenerate and if f denotes the minimal distance from the circuit to a non-contracted vertex of $\sqsubset_1$ (which certainly exists since $\sqsubset_1$ has positive degree) then $f < e_1 \leq \lambda(\sqsubset')$. Thus $\delta = f$ and $\delta < \lambda(\sqsubset')$ as claimed.[13] □

Notice we did not specify whether δ was the relative or absolute radius. The point is that it does not matter; the proof goes through the same in either case. Thus we see that $\sqsubset$ satisfies the (double) factorisation condition if and only if $\sqsubset_1$ does. □

4.5.2. *Type B.* Now let $\mathcal{D} \subseteq \mathcal{VZ}$ be a component of $\mathcal{D}(k)$ with combinatorial type Δ of type B. In this case, it is impossible for the circuit to be contracted. Thus, $\mathcal{D}_{\mathcal{M}} \subseteq \mathcal{M}$ is disjoint from the blown-up locus, and the map

$$\mathcal{D} \to \mathcal{D}_{\mathcal{M}}$$

is an isomorphism. Thus we obtain:

$$\mathcal{D} = \overline{\mathcal{M}}^{\text{punct}}_{0,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H, d_0) \times_{H^r} \left(\overline{\mathcal{M}}^{\log}_{0,\alpha^{(1)}\cup(m_1,m_2)}(\mathbb{P}^N|H, d_1) \times \prod_{i=3}^{r} \overline{\mathcal{M}}^{\log}_{0,\alpha^{(i)}\cup(m_i)}(\mathbb{P}^N|H, d_i) \right).$$

As before, the integrals over the punctured space are determined by the Gromov–Witten theory of H.

4.5.3. *Type C^+.* Finally let $\mathcal{D} \subseteq \mathcal{VZ}$ be a component of $\mathcal{D}(k)$ with combinatorial type Δ of type C^+. The corresponding locus in $\mathcal{M}$ can be written as:

$$\mathcal{D}_{\mathcal{M}} = \overline{\mathcal{M}}^{\text{punct}}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H, d_0) \times_{H^r} \left(\prod_{i=1}^{r} \overline{\mathcal{M}}^{\log}_{0,\alpha^{(i)}\cup(m_i)}(\mathbb{P}^N|H, d_i) \right).$$

Then the same argument as in Lemma 4.9 shows that given any point in $\mathcal{D}$ the interior of the radius can only contain components of C_0. Thus the alignment and factorisation condition apply exclusively to C_0, which shows that:

$$\mathcal{D} = \mathcal{VZ}^{\text{punct}}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H, d_0) \times_{H^r} \left(\prod_{i=1}^{r} \overline{\mathcal{M}}^{\log}_{0,\alpha^{(i)}\cup(m_i)}(\mathbb{P}^N|H, d_i) \right).$$

Here the first factor

$$\mathcal{VZ}^{\text{punct}}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H, d_0)$$

is the logarithmic blow-up of the moduli space of punctured maps, obtained by imposing an alignment and factorisation condition. To be more precise: its elements consist of punctured maps to H which are aligned in the sense of Definition **??**, and satisfy the

[13] (Navid) Do an example?

(double) factorisation condition (simply replace $\mathbb{P}^N$ by H everywhere in that definition). As in §??, this construction produces a closed substack of a logarithmic modification

$$\mathcal{VZ}^{\mathrm{punct}}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H,d_0) \subseteq \widetilde{\mathcal{VZ}}^{\mathrm{punct}}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H,d_0) \xrightarrow{\psi} \overline{\mathcal{M}}^{\mathrm{punct}}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H,d_0)$$

Lemma 4.10. The logarithmic modification ψ restricts to a birational map:

$$\mathcal{VZ}^{\mathrm{punct}}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H,d) \to \overline{\mathcal{M}}^{\mathrm{punct},\circ}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H,d).$$

Note that the latter space (the main component of the moduli space of punctured maps) is birational to the main component of the double ramification locus. In Lemma 4.11 below we explain how to compute integrals over this.cd

Proof. If it is true, we might be able to prove it via deformation theory. For this it might be useful to notice that there is a morphism $\overline{\mathcal{M}}^{\mathrm{punct}}_{g,\alpha}(H,d) \to \overline{\mathcal{M}}^{\mathrm{punct}}_{g,\alpha}(\mathbb{P}^N,d)$ (because $H \subseteq \mathbb{P}^N$ is strict), which is probably a closed immersion, and the loci with smooth source curve are isomorphic under this map. It seems plausible that the main omponent is log smooth over $B\mathbb{G}_{\mathrm{m}} \subseteq [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ (or the standard log point). □

Integrals over the main component of the double ramification locus can be computed using the following lemma.

Lemma 4.11. The morphism which forgets a marking

$$\mathrm{fgt}_i : \overline{\mathcal{M}}^{\alpha,\circ}_{1,n}(H,d) \to \overline{\mathcal{M}}^{\circ}_{1,n-1}(H,d)$$

is generically finite, of degree α_i^2 (except in one special case, described in the proof).

Proof. Let us consider the case $d = 0$ first. We may assume the source curve is smooth elliptic E. The map

$$\phi\colon E \to \mathrm{Pic}^0(E), \qquad x \mapsto \mathcal{O}_E\left(\alpha_i x + \sum_{j=1,\ldots,\hat{i}\ldots,n} \alpha_j p_j\right)$$

is an isogeny of degree α_i^2. The locus of $(C,p_1,\ldots,\hat{p}_i,\ldots,p_n)$ such that the kernel of ϕ contains one of the points $\{p_1,\ldots,\hat{p}_i,\ldots,p_n\}$ is itself a double ramification locus inside $\overline{\mathcal{M}}_{1,n}$, hence non-generic - with one exception: namely, when $n = 2$, $x = p_1$ is always a solution, but the curve lying above such point bubbles off a $\mathbb{P}^1$. To see that it does not belong to the closure of the nice locus, notice that the rational function trivialising $\mathcal{O}_C(\alpha_1 p_1 - \alpha_2 p_2)$ should descend to the cusp, thus having a ramification point at the node; yet its ramification profile is determined by Riemann-Hurwitz, and it is entirely supported on p_1 and p_2.

For a different proof: notice that we should obtain the class of the main component from the full double ramification cycle by subtracting the boundary class $[D_{1,\emptyset|0,\{1,\ldots,n\}}]$.

The latter pushes forward to 0 under fgt_i, unless $n = 2$. Therefore we may apply $\mathrm{fgt}_{i,*}$ to the Hain-Pixton formula:

$$DR_1(A) = \frac{1}{2}\left(\sum_{i=1}^{n} a_i^2 \psi_i - \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I| \geq 2}} a_I^2 [D_{1,I^c|0,I}] - \frac{1}{12}\delta_0\right),$$

where $a_I = \sum_{i \in I} a_i$ and $\delta_0 = \mathrm{glue}_*([\overline{\mathcal{M}}_{0,n+2}])$. From $\psi_j = \mathrm{fgt}_i^* \psi_j + [D_{i,j}]$ for $i \neq j$, and the dilaton equation, we see that $\mathrm{fgt}_{i,*}\psi_j = 1$ and $\mathrm{fgt}_{i,*}\psi_i = n - 1$. On the other hand, the only surviving boundary classes are $[D_{i,j}]$, and they push down to 1. Hence the formula pushes down to $\frac{1}{2}\left(\sum_{j \neq i} \alpha_j^2 + (n-1)\alpha_i^2 - \sum_{j \neq i}(\alpha_j + \alpha_i)^2\right) = -\alpha_i \left(\sum_{j \neq i} \alpha_j\right) = \alpha_i^2$. □

[14] Review here the process of gluing for punctured maps. In particular the basic monoid should be modified in order to make the relevant evaluations log morphisms. Claim: evaluations are strict. Consequence: the fiber product in the category of log stacks is fine. We have then to apply saturation. This is a finite morphism of degree... (I think it could be $\frac{\prod m^{(i)}}{lcm(m^{(i)})}$).

Lemma 4.12 (Virtual pushforward). The following hold.

- $\mathrm{fgt}_*[\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d)] = [\overline{\mathcal{M}}^{\mathrm{G}}_{0,\alpha}(\mathbb{P}^N|H,d)]$ (follows from [Gat03, AMW14]).
- $\mathrm{fgt}_*[\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)]$ computes the reduced relative invariants by definition.
- $\mathrm{fgt}_*[\overline{\mathcal{M}}_{0,\mu}(H,d_0)^{\sim}] = [\overline{\mathcal{M}}_{0,|\mu|}(H,d_0)]$ (follows from [Gat03] and... comparison of punctured with rubber invariants).
- $\mathrm{fgt}_*[\mathcal{VZ}_{1,\mu}(H,d_0)^{\sim}]$ here we should need a variation on Pixton's DRC formula; hopefully it's enough to avoid the graphs that tropical well-spacedness discards.

4.6. **Recursive description of the divisors: type** C_0. On the other hand, when there is a contracted elliptic subcurve - and it will be contracted into the hyperplane, because otherwise it wouldn't be generic, by density of the nice locus in $\mathcal{VZ}_{1,\alpha'}(\mathbb{P}^N|H,d')$ - the picture becomes more complicated due to the alignment. The combs may break. We label these loci $\mathcal{Y}^0_C$.

In the following we describe the rays of the tropical moduli space.

Lemma 4.13. A one-parameter tropical map ϕ to $\mathbb{R}_{\geq 0}$ is a decorated tree (with expansion factors - *contact orders* - along edges and legs, and degrees on vertices, satisfying the balancing condition) with a circle (of radius δ) around the root (sometimes called the *core* and denoted by $\circ$) satisfying:

(1) the circle of radius δ passes through at least one vertex of $\phi^{-1}(0)$ - which necessarily has positive degree - call m its contact order with H;
(2) teeth may break only when they intersect the circle of radius δ; in particular, $\circ$ is the only vertex contained in its strict interior, and every edge heading out from the circle goes directly to a vertex of $\phi^{-1}(0)$;
(3) every tooth that starts with contact order m goes directly to a vertex of $\mathrm{circle}(\circ,\delta)\cap\phi^{-1}(0)$, and every other tooth starts with contact order $< m$ (possibly negative).

Proof. Otherwise there would be more than one parameters. □

Remark 4.14. The core being contracted in the fiber of the tropical map is not a phenomenon that we should worry about in codimension one. Indeed, assume that the core is contracted in the fiber along a ray. Then all the edges departing from the core have

[14] (Luca) to be made homogeneous with what comes earlier

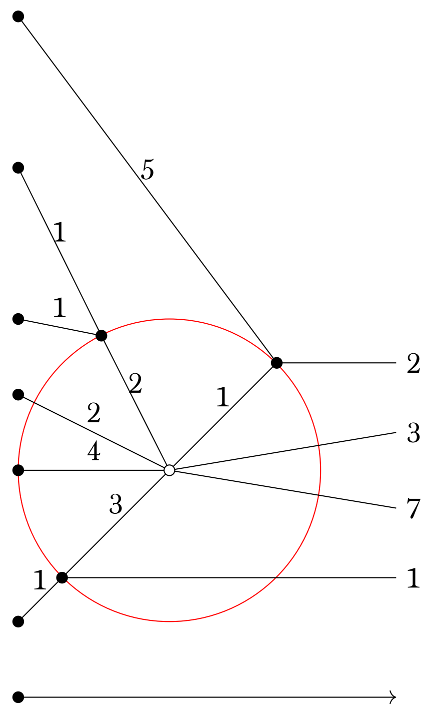
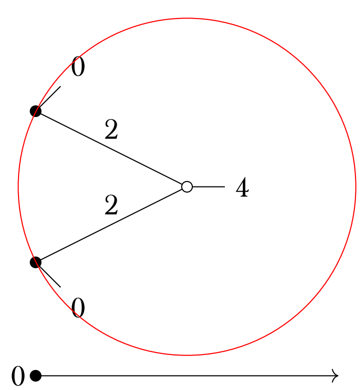


Figure 4. The degrees can be figured out from the balancing equation.

expansion factor 0; call the corresponding coordinates $U = \{u_i\}_{i\in I}$. Call the remaining coordinates $E = \{e_j\}_{j\in J}$. Note that tropical continuity involves only E. Alignments on the other hand assume the form $\lambda(v) = \lambda(v')$, where $\lambda(v) = \sum_{i\in I(v)} u_i \sum_{j\in J(v)} e_j$. Pick the shortest elements of U; then these can be shortened to zero without affecting the rest (by hypothesis, alignments can only identify them among themselves). This shows that we could not have started with a ray.

Example 4.15. We look at the following example in some detail. The ambient space is



$\mathcal{VZ}_{1,(4,0,0)}(\mathbb{P}^N|H,4)$, of dimension $4N+3$. The underlying moduli space is $X = \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H,2) \times_H \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H,2) \times \mathcal{VZ}_{1,(-2,-2,4)}$ of dimension $5N+1$. Consider the fiber product:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H,2) \times \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H,2) \arrow[d] \\ H \arrow[r] & H \times H \end{tikzcd}$$

At the level of ghost sheaves, $\overline{\mathcal{M}}_F = \mathbb{N} \oplus_{\mathbb{N}^2} \overline{\mathcal{M}}_1^{\mathrm{enl}} \oplus \overline{\mathcal{M}}_2^{\mathrm{enl}}$, where the map $\mathbb{N}^2 \to \mathbb{N}$ is the sum, and the map $\mathbb{N}^2 \to \overline{\mathcal{M}}_1^{\mathrm{enl}} \oplus \overline{\mathcal{M}}_2^{\mathrm{enl}}$ generically is multiplication by 2, so $\overline{\mathcal{M}}_F = \mathbb{N}^2/(2e = 2f)$ generically. Saturation gives a finite cover $G \to F$ with $\overline{\mathcal{M}}_G = \mathbb{N}_{e=f}$ generically. Lifting this to actual log structures, what we are doing (again generically) is taking a square root of the isomorphism $T^{\otimes 2}_{R_1,q_1} \simeq T^{\otimes 2}_{R_2,q_2}$, which is obtained passing through $N_{H/\mathbb{P}^N,f(Z)}$ via $\mathrm{d} f_{|R_i,q_i}$. This breaks when $f_{|R_i}$ is not tangent to H of order exactly 2 at q_i, for either i; but by the maximality assumption this happens precisely along Gathmann's comb loci Δ_i. So in fact, rather than with $T^{\otimes 2}_{R_i,q_i}$, we should be working with $T^{\otimes 2}_{R_i,q_i}(-\Delta_i)$: but this is exactly $\mathrm{ev}_i^*(-H)$ by Gathmann's genus zero formula, and the isomorphism $\mathrm{ev}_1^*(H) = \mathrm{ev}_2^*(H)$ holds on all of F.

On the other hand, generically on $\mathcal{VZ}_{1,(-2,-2,4)}$ we have $T_{q_1}Z \simeq T_{q_2}Z$ by exploiting the group structure on the elliptic curve. This breaks when either (but not both) is on a rational tail. Yet we have $T_{q_1}Z(\Delta_{1\in P}) \simeq T_{q_2}Z(\Delta_{2\in P})$ by Vakil-Zinger's construction of a universal ψ-class (i.e. by comparing both with $\pi_*\omega(\Delta)$; notice that our further blow-up has the only effect of twisting *all* the relevant line bundles by $\Delta_{1,2\notin P}$).

Now, the fiber of the Vakil-Zinger blow-up over X can be described as follows. Generically it looks like

$$\mathbb{P}(T_{q_1}R_1 \otimes T_{q_1}Z \oplus T_{q_2}R_2 \otimes T_{q_2}Z)$$

but this has to be modified along the boundary:

- this has to do with the fact that the normal bundle of the strict transform is the pullback of the normal bundle twisted by the intersection with the exceptional divisor (so it relates with previous steps of the blow-up);
- it is not globally a $\mathbb{P}^1$-bundle (so it relates with further stages of the blow-up; it also has to do with a choice of compactification for the moduli space of attachments);
- it has the effect of replacing $T_{q_i}Z$ with Vakil-Zinger's universal $\mathbb{T}$, so that this can be factored out of the projective bundle, and in fact we are left with a projective bundle $\mathbb{P} = \mathbb{P}(T_{q_1}R_1 \oplus T_{q_2}R_2)$ over F, and its open part $\mathrm{Iso}(T_{q_1}R_1 \oplus T_{q_2}R_2)$ represents the attachment data for a contraction to a tacnode $R_1 \sqcup_q R_2 \to \bar{C}$.

On $\mathbb{P}$ there is a natural vector bundle map

$$s\colon \mathcal{O}_{\mathbb{P}}(-1) \hookrightarrow p^*(T_{q_1}R_1 \oplus T_{q_2}R_2) \xrightarrow{+\mathrm{d}f} \mathrm{ev}_q^* T\mathbb{P}^N$$

that vanishes along the locus where f descends to $\bar{C}$. In general, it is not a transversal section:

- we should replace $T\mathbb{P}^N$ by TH as long as all the $m^{(i)}$ are ≥ 2;
- Vakil and Zinger construct a blow-up of $\mathbb{P}$ along the vanishing loci of s of low codimension, and twist s by the exceptional divisors, so that it becomes a transverse section $\tilde{s}$.

On the other hand, the finite cover $G \to F$ factors through $\mathbb{P}$, because the two vertices are already aligned on G. We claim that the boundary locus of $\mathcal{VZ}_{1,(4,0,0)}(\mathbb{P}^N|H,4)$ corresponding to the combinatorial type of the tropical map above is the transverse intersection

$$(G \cap V(\tilde{s}) \subseteq \mathbb{P}) \times \mathcal{VZ}_{1,(-2,-2,4)}.$$

This has the expected dimension (codimension $N-1$ with respect to X). To compute its class, we can pull $\mathbb{P}$ back to G, and then notice that $G \hookrightarrow \mathbb{P}_G$ is the inclusion of a (trivial) subbundle.

Lemma 4.16. The class of $\mathbb{P}(\mathcal{F}) \subseteq \mathbb{P}(\mathcal{E})$ is $c_{\mathrm{top}}(\mathcal{O}_{\mathcal{E}}(1) \otimes p^*(\mathcal{E}/\mathcal{F}))$.

See [EH16, Prop. 9.13]. It is a good time to remember that $\mathcal{E}$ was in fact $(\bigoplus_{i=1}^r TR_{i,q_i}) \otimes \mathbb{T}$. By writing c for $c_1(\mathcal{O}_{\mathcal{E}}(1))$, ψ_i for $c_1(T^*R_{i,q_i})$, ψ_Z for Vakil-Zinger's universal psi class, and H for $\mathrm{ev}_q^* H$, we need to compute

$$\begin{aligned} &p_*\left((c-\psi_1-\psi_2-2\psi_Z)[(1+c+H)^N(1+c)^{-1}]_{N-1}\right) = \\ &p_*\left((c-\psi_1-\psi_2-2\psi_Z)(\sum_{k=0}^{N-1}\binom{N}{1+k}c^k H^{N-1-k})\right) = \\ &\sum_{k=0}^{N-1}\binom{N}{k}H^{N-1-k}\left(s_k(\mathcal{E}) - s_{k-1}(\mathcal{E})(\psi_1+\psi_2+\psi_Z)\right) \end{aligned}$$

We now generalise this picture. Recall that the map $C \to \bar{C}$ is given by a (generic) line in the sum of the tangent spaces to the rational tails at the nodes that join them to the contracted curve of genus one. This is equivalent to an alignment, and it is parametrised by an open subset of a projective bundle over the moduli space for the tails corresponding to vertices of the dual graph lying on the circle of radius δ. Yet, notice that those vertices lying in $\phi^{-1}(0)$ are already aligned among themselves. This is why we find it convenient to distinguish among four groups of vertices:

(1) the core;

(2) vertices on $\phi^{-1}(0) \cap \operatorname{circle}(\circ, \delta)$;
(3) vertices on $\operatorname{circle}(\circ, \delta) \setminus \phi^{-1}(0)$;
(4) vertices on $\phi^{-1}(0) \setminus \operatorname{circle}(\circ, \delta)$.

We shall first argue that gluing of log maps can be performed separately for the exterior and interior of the circle (the analogous classical picture is that, since the core is contracted, this moduli space is the product of a genus one curve, and a fiber product of genus zero maps under evaluation morphisms).

Lemma 4.17. $\mathcal{VZ}^{\text{punct}}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(H,0) \simeq \mathcal{VZ}^{\text{punct}}_{1,\alpha^{(0)}\cup(-m_1,\ldots,-m_r)}(\operatorname{Spec}(k\oplus\mathbb{N})) \times H$.[15]

Let us now deal with vertices of type 2.

Lemma 4.18. Consider the following fiber product in the category of fs log stacks:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \prod_{i=1}^r \overline{\mathcal{M}}_{0,\alpha^{(i)}\cup\{m\}}(\mathbb{P}^N|H,d_i) \arrow[d, "\mathrm{ev}_{q_i}"] \\ H \arrow[r] & H^r \end{tikzcd}$$

On F there is a canonical isomorphism $\mathbb{L}_{q_i} \cong \mathbb{L}_{q_j}$[16], the latter being the cotangent line bundles at the gluing markings on two different components i and j. Furthermore, $\underline{F}$ is a finite cover of degree m^{r-1} of the fiber product of the underlying stacks.

Proof. Recall that each $\overline{\mathcal{M}}_{0,\alpha^{(i)}\cup\{m\}}(\mathbb{P}^N|H,d_i)$ is endowed with the log structure induced by pulling back along q_i the divisorial log structure of the universal curve at the image of q_i itself. Thus ev_{q_i} is made into a log morphism to H with its induced DF(1) log structure. We claim that the subtext of such morphism is Gathmann's formula; namely, the log morphism to H corresponds to an isomorphism betwee $\mathrm{ev}_{q_i}^*\,\mathcal{O}_H(-H)$ on one side, and $\mathcal{I}_{D_i}\otimes\mathbb{L}_{q_i}^m$ on the other, where $\mathcal{I}_{D_i}$ is the ideal sheaf of the union of the comb loci D_i in $\overline{\mathcal{M}}_{0,\alpha^{(i)}\cup\{m\}}(\mathbb{P}^N|H,d_i)$.

On the fiber product there is a canonical isomorphism between $\mathrm{ev}_{q_i}^*\,H$ and $\mathrm{ev}_{q_j}^*\,H$.

Say something about integrality.

Saturation is a local operation, as much as computing the degree, hence we can concentrate on the dense open locus where all the curves we are gluing are smooth, and they are not mapped entirely into H. There the minimal log structure on $\overline{\mathcal{M}}_{0,\alpha^{(i)}\cup\{m\}}(\mathbb{P}^N|H,d_i)$ is trivial, therefore the isomorphism between $\mathrm{ev}_{q_i}^*\,H$ and $\mathrm{ev}_{q_j}^*\,H$ translates into an isomorphism $\mathbb{L}_{q_i}^m \cong \mathbb{L}_{q_j}^m$. The saturation F is obtained by taking an m-th root of this isomorphism. □

[15] (Luca) check log structure, could be fibered over std log point

[16] (Luca) check how it degenerates

Let us denote by $\mathbb{L}_F$ the universal cotangent line at q.

The projective bundle we are seeking has base

$$\mathcal{X} = \left(F \times \prod_{i=1}^{s} \overline{\mathcal{M}}^{\text{punct}}_{0,\tilde{\alpha}^{(i)}\cup\{m_i\}}(H, \tilde{d}_i) \right) \times_{H^{s+1}} H$$

(with $m_i < m$) and it is

$$\mathbb{P} = \operatorname{Proj}_{\mathcal{X}} \left(\mathbb{T}_F \oplus \bigoplus_{i=1}^{s} \mathbb{T}_{\tilde{q}_i} \right).$$

We are interested in the vanishing locus of the section

$$s\colon \mathcal{O}_{\mathbb{P}}(-1) \hookrightarrow p^* \left(\mathbb{T}_F \oplus \bigoplus_{i=1}^{s} \mathbb{T}_{\tilde{q}_i} \right) \xrightarrow{+df_q} f^*(\mathrm{T}\,\mathbb{P}^N)_q,$$

because it represents the geometric condition that $f\colon C \to \mathbb{P}^N$ factors through the normalisation map $C \to \bar{C}$ prescribed by the given point of $\mathbb{P}$. As is, s is not transverse to the zero section. First of all, unless $m = 1$, $\mathrm{T}\,\mathbb{P}^N$ can be replaced by $\mathrm{T}\,H$ in the definition of s above, because the projection of all $\mathrm{d}\,f_i(\mathrm{T}_{q_i}\,R_i)$ to $N_{H/\mathbb{P}^N}$ is zero. The case $m = 1$ has to be dealt with separately and it turns out that, once s is made transverse, the dimension of its zero locus is smaller than the expected dimension, hence the corresponding combinatorial types are in fact irrelevant.

The procedure to make s transverse is the same as described in [VZ08, §3], namely we need to blow up inside $\mathbb{P}$ the projective subbundle $\mathbb{P}(\mathcal{E})$ of $\mathbb{P}_{|\mathcal{X}_\sigma}$, where $\mathcal{X}_\sigma$ is the closed substack of $\mathcal{X}$ where some of the tails degenerate so that the corresponding gluing marking lies on a component contracted by f, and $\mathcal{E}$ is the sum of the tangent line bundles at such subset of the gluing markings. Because $\overline{\mathcal{M}}^{\text{punct}}_{0,\tilde{\alpha}^{(i)}\cup\{m_i\}}(H, \tilde{d}_i)$ is isomorphic to $\overline{\mathcal{M}}_{0,\tilde{n}_i}(H, d_i)$, the construction of Vakil and Zinger in the section "A blowup of a moduli space of genus-zero maps" of their paper carries through unchanged. The result of the blow-up is to replace $T_{q_i}R_i$ with $T_{q_i}R_i \otimes \bigoplus_{j=1}^{k} T_{\tilde{q}_j}R_i \otimes T_{\tilde{q}_j}S_{ij}$. We interpret this in terms of alignments.[17]

We claim that $\tilde{s}$ obtained from s by twisting by the exceptional divisors of the blowup is a transverse section. We also claim that comb loci of type $\mathcal{Y}^0_c$ can be described as a fibered product of

- moduli of genus one punctured maps to the standard log point, radially aligned and satisfying factorisation;
- $V(\tilde{s})$;
- moduli of genus zero maps relative to $(\mathbb{P}^N|H)$, corresponding to vertices of type 4 above.

[17] (Luca) this has to be written properly

Finally, we claim that integrals of psi and evaluation classes over these loci can be translated into tautological integrals, i.e. descendant Gromov-Witten invariants, whose numerics is governed by the combinatorial type of the tropical map.

5. Recursion formula in general

Now let $(X|Y)$ be a smooth pair with Y very ample. The complete linear system $|\mathcal{O}_X(Y)|$ defines an embedding $X \hookrightarrow \mathbb{P}^N$ with $Y = X \cap H$ for H some hyperplane.

Lemma 5.1. The following square is cartesian (in the category of ordinary stacks):

$$\begin{tikzcd} \mathcal{VZ}_{1,\alpha}(X|Y,\beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d) \arrow[d] \\ \mathcal{VZ}_{1,n}(X,\beta) \arrow[r, "i"] & \mathcal{VZ}_{1,n}(\mathbb{P}^N,d). \end{tikzcd}$$

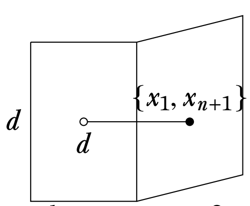
Since $\mathcal{VZ}_{1,n}(\mathbb{P}^N,d)$ is smooth and $\mathcal{VZ}_{1,n}(X,\beta)$ carries a natural virtual class, there is a diagonal pull-back morphism which we use to define the virtual class on the space of maps to $(X|Y)$:

$$[\mathcal{VZ}_{1,\alpha}(X|Y,\beta)]^{\mathrm{virt}} := i_\Delta^![\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)].$$

The recursion formula in $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$ immediately pulls back along i to give a recursion formula in $\mathcal{VZ}_{1,\alpha}(X|Y,\beta)$. [18]

6. Quantum Lefschetz in genus one

Assume [19] we want to compute $\langle \tau_{h_1}H^{k_1},\ldots,\tau_{h_1}H^{k_1}\rangle_{1,n,d}^{\mathbb{P}^{N-1},\mathrm{red}}$ by writing it as $\frac{1}{d^2}(\psi_1^{\{n+1\}})^{h_1}H^{k_1}\cdots(\psi_n^{\{n+1\}})^{h_n}H^{k_n}\cdot$ $[\mathcal{VZ}(0,\ldots,0,d)\mathbb{P}^N|Hd\tilde{\ }]$. Let me start from the recursion step $H_1\cdot[\mathcal{VZ}(0,\ldots,0,d)\mathbb{P}^N|Hd] =$ $\ldots$:

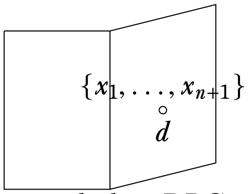
$\mathcal{Y}_a$:: 

contributes $\frac{1}{d}\psi_1^{h_1}H^{k_1}\cdots\psi_n^{h_n}H^{k_n}\cdot[\mathcal{VZ}(d,\ldots,0)\mathbb{P}^N|Hd]$; if this is the curve configuration, while there is any other marking at level one, then fgt_{n+1} has positive dimensional fibers; otherwise there must be another component of positive horizontal degree (these contributions are computed inductively on d);

$\mathcal{Y}_b$:: is an entirely rational story;

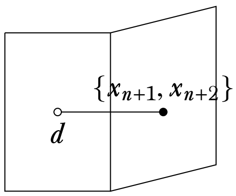
[18] (Navid) Is it clear how to compute integrals over the pulled back classes?

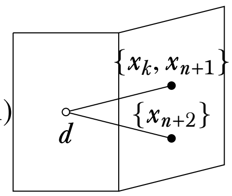
[19] (Luca) copied and pasted from my thesis

$\mathcal{Y}_c$:: 

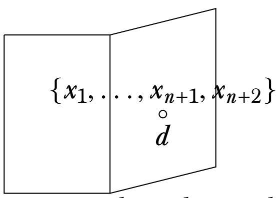
is the term we are meant to compute, while all other terms will include a DRC integral of lower degree.

I shall look one step further into the recursion for computing $[\mathcal{VZ}(0,\ldots,0,d)\mathbb{P}^N|Hd]$: in order to do so consider the formula for $((d-1)\psi_{n+1}+H_{n+1})\cdot[\mathcal{VZ}(0,\ldots,0,d-1,1)\mathbb{P}^N|Hd]$. This produces:

$\mathcal{Y}_a$:: d 

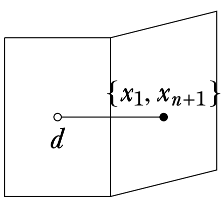
this is the one we are after; while $(d-1)$ 

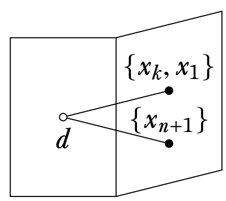
gives a lower tangency condition $[\mathcal{VZ}(0,\ldots,d-1,\ldots,0)\mathbb{P}^N|Hd]$; and all other terms have lower degree.

$\mathcal{Y}_c$:: 

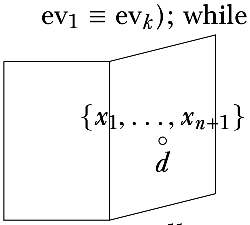
has positive dimensional fibers for $\mathrm{fgt}_{n+1,n+2}$, and all other terms have lower degree.

Finally, let me look one step further into the reduction of $[\mathcal{VZ}(d,\ldots,0)\mathbb{P}^N|Hd]$: we have $((d-1)\psi_1^{\{n+1\}}+H_1)\cdot[\mathcal{VZ}(d-1,\ldots,0,1)\mathbb{P}^N|Hd] = \ldots$:

$\mathcal{Y}_a$:: d 

is the one we care about; $(d-1)$ 

has a lower number of markings (indeed $\psi_1^{\{n+1\}}$ and $\psi_k^{\{n+1\}}$ restrict to 0 on this locus, and $\mathrm{ev}_1 \equiv \mathrm{ev}_k$); while all other terms have lower degree.

$\mathcal{Y}_c$:: 

contributes $\frac{1}{d}\psi_1^{h_1}H^{k_1}\cdots\psi_n^{h_n}H^{k_n}\cdot[\mathcal{VZ}(0,\ldots,0,d)\mathbb{P}^N|Hd^{\sim}]$ so *there is no cancellation*, while all other terms have lower degree.

To sum up, it is possible to determine the reduced genus one restricted invariants of H in terms of those of $\mathbb{P}^N$ and the genus zero theory, by induction on the degree, number of markings, and total tangency order.

References

[ACGS] D. Abramovich, Q. Chen, M. Gross, and B. Siebert. Punctured logarithmic maps. In preparation.

[AMW14] Dan Abramovich, Steffen Marcus, and Jonathan Wise. Comparison theorems for Gromov-Witten invariants of smooth pairs and of degenerations. *Ann. Inst. Fourier (Grenoble)*, 64(4):1611–1667, 2014.

[EH16] David Eisenbud and Joe Harris. *3264 and all that—a second course in algebraic geometry*. Cambridge University Press, Cambridge, 2016.

[Gat02] Andreas Gathmann. Absolute and relative Gromov-Witten invariants of very ample hypersurfaces. *Duke Math. J.*, 115(2):171–203, 2002.

[Gat03] Andreas Gathmann. Gromov-Witten invariants of hypersurfaces, 2003. Habilitation thesis.

[GS13] Mark Gross and Bernd Siebert. Logarithmic Gromov-Witten invariants. *J. Amer. Math. Soc.*, 26(2):451–510, 2013.

[JPPZ18] F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine. Double ramification cycles with target varieties. *arXiv e-prints*, page arXiv:1812.10136, December 2018.

[RSW17a] D. Ranganathan, K. Santos-Parker, and J. Wise. Moduli of stable maps in genus one and logarithmic geometry I. *ArXiv e-prints*, August 2017.

[RSW17b] D. Ranganathan, K. Santos-Parker, and J. Wise. Moduli of stable maps in genus one and logarithmic geometry II. *ArXiv e-prints*, September 2017.

[Vak00] Ravi Vakil. The enumerative geometry of rational and elliptic curves in projective space. *J. Reine Angew. Math.*, 529:101–153, 2000.

[VZ08] Ravi Vakil and Aleksey Zinger. A desingularization of the main component of the moduli space of genus-one stable maps into $\mathbb{P}^n$. *Geom. Topol.*, 12(1):1–95, 2008.

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