

RELATIVE STABLE MAPS IN GENUS ONE VIA CENTRAL ALIGNMENTS

LUCA BATTISTELLA, NAVID NABIJOU AND DHURUV RANGANATHAN

ABSTRACT. For a smooth projective variety X and a smooth very ample hypersurface $Y \subseteq X$, we define moduli spaces of relative stable maps to (X, Y) in genus one, as closed substacks of the moduli space of maps from centrally aligned curves, constructed in [RSW17a]. We construct virtual classes for these moduli spaces, which we use to define *reduced relative Gromov–Witten invariants* in genus one.

[GOALS: We prove a recursion formula which allows us to completely determine these invariants in terms of the reduced Gromov–Witten invariants, as defined in [REF]. We also prove a relative version of the Li–Zinger formula, relating our invariants to the usual relative Gromov–Witten invariants. Also say something about quasimaps.]

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1. INTRODUCTION

1.1. Statement of the problem. Contrary to the genus zero case, the moduli space of genus one maps to projective space - with or without markings - is far from smooth; indeed it has various boundary components of different dimensions, representing maps that contract a genus one curve and have all the degree supported on a number of rational tails. The many incarnations of relative moduli spaces also suffer of the same undesirable feature.

Since the work of Vakil–Zinger and Ranganathan–Santos-Parker–Wise, it has been clear that it is possible to identify a desingularisation of the main component by adding the extra data of a contraction of the source curve

$\nu: C \rightarrow \bar{C}$ - where the latter is allowed to acquire a Smyth singularity - and requiring the stable map $f: C \rightarrow \mathbb{P}^N$ to factor through ν .

1.2. Choice of relative space and desingularisation. We focus on the space of logarithmic stable maps to $(\mathbb{P}^N|H)$, following ACGS. We perform a log modification of this space as detailed below. For a log curve $C \rightarrow S = \text{Spec}(k = \bar{k})$, modify the dual graph of C by replacing the minimal genus one subcurve (in case it is a circle of \mathbb{P}^1) by a single vertex of genus one, called the *core* and denoted by \circ , and define a piecewise linear function with values in \mathcal{M}_S on such a graph by setting

$$\lambda(v) = \sum_{q \in [\circ, v]} \rho_q,$$

where the ρ_q are the smoothing parameters of the nodes q separating v from the core. Such a function is related to the log canonical bundle of $C \rightarrow S$. When the map contracts a subcurve of genus one, we endow it with the extra data of a radius $\delta \in \mathcal{M}_S$ subject to the following compatibility condition:

(*) *the circle of radius δ around \circ passes through ≥ 1 vertex of positive f -degree.*

Furthermore, we require all the values of λ to be comparable with δ , and among themselves whenever they are $\leq \delta$. This is called a *centrally aligned* log structure and carries enough information to define a contraction $\nu: C \rightarrow \bar{C}$, possibly after a semistabilisation of (C, f) - in fact even more. The space thus obtained, $\widehat{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N|H, d)$, is a log modification of $\overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d)$.

The main component $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ is then identified by a double factorisation condition:

- (1) If f contracts a genus one subcurve, then f is required to factor through the Smyth singularity $\nu: C \rightarrow \bar{C}$ determined by the contraction radius δ as above.
- (2) If furthermore the core is contracted by the associated tropical map ϕ , let δ_2 be the minimal distance from \circ to a vertex supporting a flag that escapes $\phi^{-1}(\phi(\circ))$; we require f to factor through $\nu_2: C \rightarrow \bar{C}_2$.

The main result is that

Theorem 1.1. $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ is (log) smooth.

1.3. Gathmann-type recursion. There is a forgetful morphism

$$\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d) \rightarrow \mathcal{VZ}_{1,n}(\mathbb{P}^N, d),$$

hitting Gathmann's relative space. We may therefore pullback Gathmann's line bundle and section, cutting out the locus where the k -th marking is tangent to H to order $\alpha_k + 1$. Because α was maximal ($\sum \alpha = d$) by assumption, this means that the curve has to break, and x_k has to lie on an internal component - one which is entirely mapped into H . We identify the zero locus of Gathmann's section explicitly. Here is an interesting remark: the combinatorics of such boundary loci is governed by tropical geometry, and it is otherwise very hard to distillate the interaction between the relative condition and the exceptional loci of the Vakil-Zinger blow-up.

Corollary 1.2. $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ is smooth over its Artin fan, in particular codimension one logarithmic strata can be read off from the latter.

The Artin fan is a tropical gadget. Its local structure is given by subdividing the ACGS minimal monoid according to the alignment. We are only interested in picking its rays. The upshot is that the combinatorics is slightly more involved than in the genus zero case: the alignment may force some teeth of the comb to break.

Theorem 1.3 (Gathmann-type formula, maximal tangency, $(\mathbb{P}^N|H)$ case).

$$(\alpha_k \psi_k + \text{ev}_k^* H)[\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)] = [D_{1,\alpha;k}(\mathbb{P}^N|H, d)],$$

the latter being a sum of broken comb loci indexed by rays of the tropical fan.

Importantly, the broken comb loci admit a very explicit description in terms of tautological integrals on the underlying boundary of Gathmann's relative space.

Theorem 1.4. Up to a finite cover of the underlying boundary stratum - which is a combinatorially-determined fiber product of moduli spaces of genus zero and one, absolute and relative maps with lower numerical invariants - every component of $D_{\alpha,k}(\mathbb{P}^N|H, d)$ can be described as the transverse intersection of two loci in a projective bundle, where:

- the latter parametrises the possible line bundle isomorphisms imposed by the alignment of the log structure;
- the first locus is a subbundle representing the residual isomorphisms after fixing the ones virtually imposed by tropical continuity;
- the second locus is determined by the factorisation conditions.

The upshot is that we may then push the formula down to the Gathmann's space, so as to obtain multiplicities and tautological classes.

Corollary 1.5.

$$(\alpha_k \psi_k + \text{ev}_k^* H)[\mathcal{VZ}_{1,\alpha}^G(\mathbb{P}^N|H, d)] = [D_{1,\alpha;k}^G(\mathbb{P}^N|H, d)],$$

the latter being expressible as a weighted sum of tautological classes on Gathmann's comb loci.

Once we have this formula, the following extensions are classical:

- A similar formula for raising the tangency holds in the non-maximal tangency case. It can be proven by adding auxiliary markings of contact order 1; forgetting them is then a $(d - \sum \alpha)! : 1$ cover because the nice locus is dense inside Gathmann's relative spaces.
- The formula holds more generally for any smooth projective target X relative to a generic hyperplane section $Y = X \cap H \subseteq \mathbb{P}^N$. This follows via virtual pullback.

Finally the recursive structure of the boundary allows us to prove the following

Theorem 1.6 (In-principle quantum Lefschetz). The restricted reduced genus one invariants of Y can be inductively deduced from the full descendant genus zero and one (reduced) Gromov-Witten theory of X .

The proof is more delicate than its genus zero analogue because invariants with the same numerical data appear intertwined in the last steps of the recursion.

2. DESINGULARISATION OF THE MODULI SPACE OF LOG STABLE MAPS

The ultimate goal of the paper is to apply Gathmann's techniques to the Vakil-Zinger desingularisation $\mathcal{VZ}_{1,n}(\mathbb{P}^N, d)$ and to obtain a quantum Lefschetz result for reduced invariants under some positivity assumption. The key step is to study the unobstructed case $(\mathbb{P}^N|H)$. We approach the problem by lifting it to the ACGS space of log stable maps. This allows us to exploit the tools developed in [RSW17a, RSW17b]. We are in an intermediate situation between those two papers, and indeed we get an intermediate answer.

2.1. The ACGS minimal monoid and central alignments.

Proposition 2.1. The map $\overline{\mathcal{M}}_{1,\alpha}^{\text{cen}}(\mathbb{P}^N|H, d) \rightarrow \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d)$ is a log modification. In particular $\overline{\mathcal{M}}_{1,\alpha}^{\text{cen}}(\mathbb{P}^N|H, d)$ is a log algebraic stack.

possibly start pointing out that the log structure is already partially aligned by the map to $(\mathbb{P}^N|H)$

2.2. Factorisation conditions.

Proposition 2.2. Factoring through the Smyth curve is a closed condition. In particular $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d) \subseteq_{\text{cl}} \overline{\mathcal{M}}_{1,\alpha}^{\text{cen}}(\mathbb{P}^N|H, d)$ is a log algebraic stack.

2.3. Log smoothness.

Theorem 2.3. $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ is log smooth.

Proof. We reduce to the situation dealt with in [RSW17b] by adding generic extra hyperplanes H_1, \dots, H_N .

First, note that, for divisors $D_1 \subseteq D_2$ in X , there is a morphism of log schemes $(X, \mathcal{M}_{D_2}) \rightarrow (X, \mathcal{M}_{D_1})$, or equivalently a morphism of log structures $\mathcal{M}_{D_1} \rightarrow \mathcal{M}_{D_2}$ over id_X , because functions invertible off D_1 are in particular invertible off D_2 as well, and divisorial log structures are subsheaves of the structure sheaf $\mathcal{M}_D \subseteq \mathcal{O}_X$.

Now fix a point $[(C, f)] \in \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$. Choose hyperplanes H_1, \dots, H_N in such a way that they intersect the image of f transversally, namely $f^{-1}(H_1 \cup \dots \cup H_N)$ is a reduced collection of points $\{q_j^i\}_{\substack{i=1,\dots,N \\ j=1,\dots,d}}$ in the smooth locus of C . This condition will then hold in an open neighbourhood of $[(C, f)]$. Mark C at these points, endow it with the pullback along f of the divisorial log structure (\mathbb{P}^N, Δ) , where $\Delta = H + \sum_{i=1}^N H_i$. Then

$$f : (C, \{p_k\}_{k=1,\dots,n} \{q_j^i\}_{\substack{i=1,\dots,N \\ j=1,\dots,d}}) \rightarrow (\mathbb{P}^N, \Delta)$$

is a lift of $[(C, f)]$ to $\overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|\Delta, d)$ (under the forgetful morphism discussed in the previous paragraph).

Looking at the associated tropical map ϕ , observe that:

- new flags have been attached only to vertices of positive degree, and these already have a flag escaping $\phi^{-1}\phi(v)$, because the sum of the incoming slopes is not zero (by modified balancing);
- the image of the new tropical map $\tilde{\phi}$ is entirely contained in the ray of the tropicalisation of (\mathbb{P}^N, Δ) corresponding to H , with new flags going off to infinity in all the new ray directions from every vertex of positive degree.

In particular, for every quotient N' of the lattice N , the associated tropical map $\tilde{\phi}'$ will either

- (1) have image contained in the ray corresponding to H , isomorphically to the original ϕ , so the contraction radius can be seen to coincide with δ_2 , or
- (2) collapse the entire curve to the zero-cell of the fan, in which case we argue from the previous remarks that the contraction radius is δ .

Hence the lift of $[(C, f)]$ is centrally aligned and satisfies the factorisation property for every subtorus $H < T$, therefore it is well-spaced (see [RSW17b, Definition 3.4.2]) and it belongs to $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|\Delta, d)$. Note that the deformation spaces of (C, f) and its lift are isomorphic by construction, as can be checked by the infinitesimal criterion - an infinitesimal deformation of (C, f) brings along a unique deformation of the $\{q_j^i\}$ compatible with the map to (\mathbb{P}^N, Δ) . At the logarithmic level, observe that the ACGS minimal monoid is the same, because no component of C is entirely mapped into any of the newly added hyperplanes; since the global contraction radius δ is the same, the subdivisions corresponding to the alignment procedure do coincide as well. This shows that the forgetful morphism is (log) étale in a neighbourhood of the lift of $[(C, f)]$, hence we may conclude by appealing to [RSW17b, Theorem 3.5.1]. \square

Corollary 2.4. $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ is smooth over its Artin fan.

describe the latter as explicitly as possible; comment on the cones that are (possibly?) not there because of the compatibility of the alignment with the log map (this is probably awkward, useless, and superceded by saying “we align subdivide the ACGS minimal dual monoid”) and because of smoothability/factorisation (this is probably related to tropical well-spacedness)

2.4. A desingularisation of punctured log maps. Describe an analogous desingularisation for punctured log maps. The degree zero case seems more delicate and complicated, even though it should just be a substack of a moduli space of marked curves after all (the log structure is possibly trickier).

Definition 2.5. A (z, g, n) -punctured log map is a punctured log map where the markings with negative contact order to H have an extra label, either z

(the tropical map sends them to zero), or g (stands for gluing), or n (stands for nihil, nothing...)

We would like to extend the minimal monoid in order to account for some of the punctures, i.e. take the sum of $p^* \mathcal{M}_{C^\circ}$ (for every such puncture p) fibered over the log structure of the base, which seems to have the same effect as topping those edges with a uertex (which doesn't carry any other information). This looks to me like sprouting a \mathbb{P}^1 at those puncture. Now, the point of the (z, g, n) -labelling is that: we only extend the log structure at the z and g punctures; uertices labelled with a z are sent under the tropical map to 0, so continuity applies to them (therefore the extension of the log structure is fake because that edge length is determined by the position of the adjacent vertex); while uertices labelled with an n are free to roam about. Finally, we perform a radial alignment (and we take rays). This distinction comes about when you split your centrally aligned map according to the strict interior of the δ -circle and the outside; the intersection of a tree with the δ -circle can either be a vertex over 0, or another vertex (corresponding to a degeneration of the genus zero map outside), or not a vertex.

3. INDEXING THE LOGARITHMIC STRATA

Let $\mathcal{VZ} = \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ and let \mathcal{A} denote the Artin fan of \mathcal{VZ} . Since the map of log stacks

$$\mathcal{VZ} \rightarrow \mathcal{A}$$

is smooth, we see that the codimension- k logarithmic strata in \mathcal{VZ} are in bijective correspondence (via pull-back) with the codimension- k logarithmic strata in \mathcal{A} . The logarithmic strata in \mathcal{A} , on the other hand, have a purely combinatorial description (at least locally) in terms of the associated moduli spaces of tropical maps, coming from the fact that \mathcal{VZ} is a logarithmic blow-up of the usual moduli space of log stable maps. In this section we discuss this circle of ideas, and show how it plays out in a number of examples.

3.1. Blowing up the Artin fan. Let $\mathcal{M} = \overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H, d)$ denote the Abramovich–Chen–Gross–Siebert moduli space of log stable maps. Recall that \mathcal{VZ} is obtained as a closed substack of a log modification

$$\widetilde{\mathcal{VZ}} \rightarrow \mathcal{M}.$$

Since the map $\mathcal{VZ} \hookrightarrow \widetilde{\mathcal{VZ}}$ is strict, the Artin fan \mathcal{A} of \mathcal{VZ} is locally isomorphic to the Artin fan of $\widetilde{\mathcal{VZ}}$ (with the latter being, in general, larger). Since the following discussion is entirely local, we will ignore the distinction between the two, and pretend as though \mathcal{A} is the Artin fan of $\widetilde{\mathcal{VZ}}$. There is then a commuting square:

$$\begin{array}{ccc} \widetilde{\mathcal{VZ}} & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}_{\mathcal{M}}. \end{array}$$

Note that neither of the vertical maps are smooth, since the moduli spaces on the top row are not log smooth. The construction of $\widetilde{\mathcal{VZ}} \rightarrow \mathcal{M}$ as the log modification obtained by imposing an alignment condition gives us a combinatorial description of the map $\mathcal{A} \rightarrow \mathcal{A}_{\mathcal{M}}$ which we will use to study \mathcal{A} . Recall that $\mathcal{A}_{\mathcal{M}}$ is locally isomorphic to the stack quotient

$$[\mathrm{Spec} \mathbb{k}[Q] / \mathrm{Spec} \mathbb{k}[Q^{\mathrm{gp}}]]$$

where Q is a monoid giving a¹ local chart for \mathcal{M} , which we may take to be the minimal monoid of [GS13, §1.5]. The real dual $Q_{\mathbb{R}}^{\vee} = \mathrm{Hom}(Q, \mathbb{R}_{\geq 0})$ of this monoid can be viewed as a moduli space of tropical maps; see [GS13, Remark 1.12]. We call this *the tropical moduli space*; contained in the tropical moduli space are the edge lengths of the associated tropical curve (corresponding to the smoothing parameters of the nodes of the logarithmic curve). Since the alignment condition amounts to imposing a partial ordering amongst certain sums of these edge lengths, it produces a polyhedral decomposition of

¹(Navid) Neat?

the tropical moduli space into chambers where different partial order relations hold. If we only consider the integral points, this produces a polyhedral decomposition of the cone $Q^\vee = \text{Hom}(Q, \mathbb{N})$. Dualising, we obtain a toric blow-up

$$Z \rightarrow \text{Spec } \mathbb{k}[Q]$$

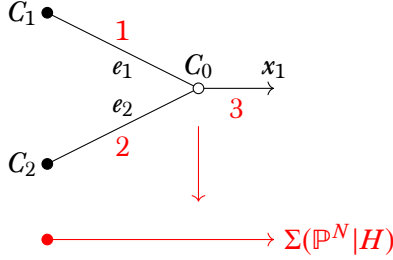
which, since it is equivariant, descends to a morphism of the associated zero-dimensional stacks:

$$[Z/T_Z] \rightarrow [\text{Spec } \mathbb{k}[Q]/\text{Spec } \mathbb{k}[Q^{\text{gp}}]].$$

This gives a local description of the map $\mathcal{A} \rightarrow \mathcal{A}_{\mathcal{M}}$ and in particular of the Artin fan \mathcal{A} . By the orbit-cone correspondence, the codimension- k strata of \mathcal{A} which intersect this open locus correspond to the k -dimensional cones in the polyhedral decomposition of Q^\vee described above. These can be understood entirely in terms of tropical combinatorics. This is best explained through a number of examples, which we now present.

3.2. Examples of logarithmic strata. In these examples we will proceed as follows: we will start by fixing an element of the ordinary moduli space \mathcal{M} of log stable maps. We will then compute the associated tropical moduli space, giving a description of $\mathcal{A}_{\mathcal{M}}$ local to our chosen element. We will then describe the necessary polyhedral subdivision, and thus give a description of \mathcal{A} in a neighbourhood of the preimage in \mathcal{VZ} of our chosen element of \mathcal{M} . Finally we will use this to describe the logarithmic strata which intersect this neighbourhood.

Example 3.1. Consider an element of \mathcal{M} whose associated tropical map has the following combinatorial type:



Here each edge (corresponding to a node of the curve) has a length e_i and an expansion factor u_i (indicated in red). The moduli space of such tropical maps is generated by the edge lengths e_1 and e_2

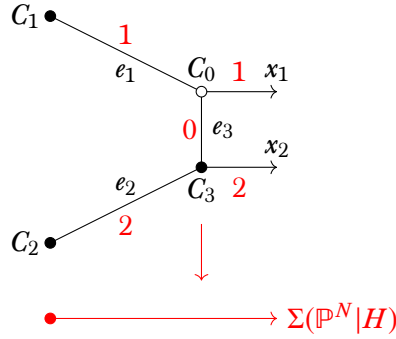
$$(\mathbb{R}_{\geq 0})_{e_1} \times (\mathbb{R}_{\geq 0})_{e_2}$$

subject to the continuity condition $e_1 = 2e_2$. So the tropical moduli space is simply $\mathbb{R}_{\geq 0}$ generated by e_2 . Note that for any $e_2 > 0$ (i.e. on the interior of the cone) we have:

$$\lambda(C_2) = e_2 = e_1/2 < e_1 = \lambda(C_1).$$

Thus, any logarithmic map with combinatorial type given by the above picture is automatically aligned. This means that the map $\mathcal{VZ} \rightarrow \mathcal{M}$ is an isomorphism in a neighbourhood of our chosen element (there is no blowing up necessary). Indeed, the tropical moduli space is $\mathbb{R}_{\geq 0}$ and this cone does not admit a polyhedral subdivision (there is no non-trivial toric blow-up of \mathbb{A}^1).

Example 3.2. Consider now an element of \mathcal{M} whose associated tropical map has the corresponding combinatorial type:



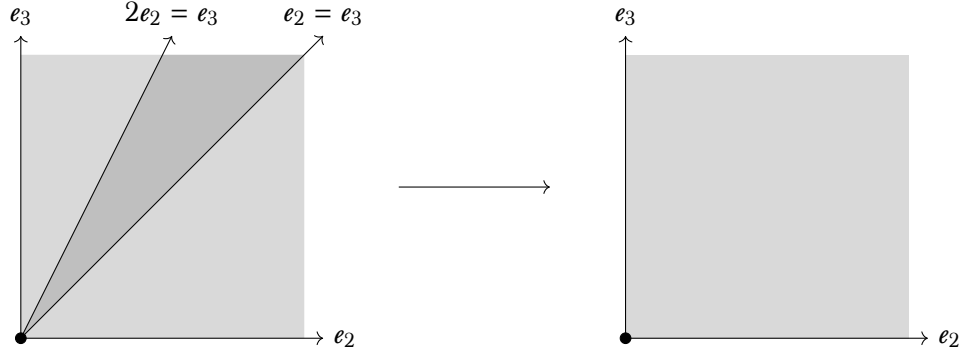
As before, expansion factors are indicated in red and the edge lengths are e_1, e_2, e_3 . The tropical moduli space is $\mathbb{R}_{\geq 0}^3$ generated by these three lengths, subject to the continuity condition $e_1 = 2e_2$. Thus the moduli space is isomorphic to $\mathbb{R}_{\geq 0}^2$ generated by e_2 and e_3 . In order to have an alignment, at least one of $\lambda(C_1)$ and $\lambda(C_2)$ must be equal to the radius δ . Note that

$$\begin{aligned}\lambda(C_1) &= e_1 = 2e_2 \\ \lambda(C_2) &= e_2 + e_3\end{aligned}$$

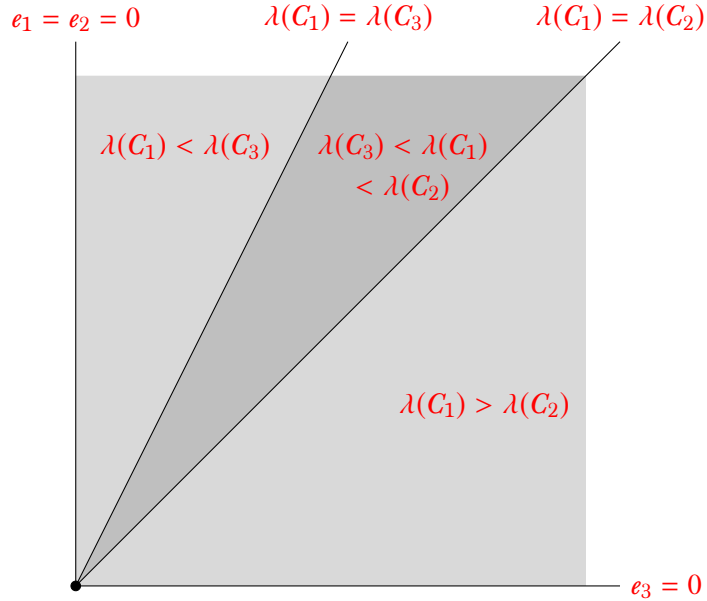
and without more information we cannot say which of these is larger. The subdivision of $(\mathbb{R}_{\geq 0})^2$ is obtained by dividing the cone into regions where different order relations hold amongst the distances $\lambda(C_1), \lambda(C_2), \lambda(C_3)$. The walls of this subdivision correspond to where some of these distances are equal. Note that in this setting we always have $\lambda(C_2) > \lambda(C_3)$ (at least, as long as we remain in the interior of the cone). The remaining possibilities are:

$$\begin{aligned}\lambda(C_1) &= \lambda(C_3) & (\Leftrightarrow e_1 = e_3 \Leftrightarrow 2e_2 = e_3) \\ \lambda(C_1) &= \lambda(C_2) & (\Leftrightarrow e_1 = e_2 + e_3 \Leftrightarrow e_2 = e_3).\end{aligned}$$

Thus, the subdivision of the tropical moduli space $(\mathbb{R}_{\geq 0}^2)_{e_2 e_3}$ is given by:



The cones of the subdivision index the logarithmic strata in a neighbourhood of the preimage in \mathcal{VZ} of our chosen element of \mathcal{M} . These can be described as follows:

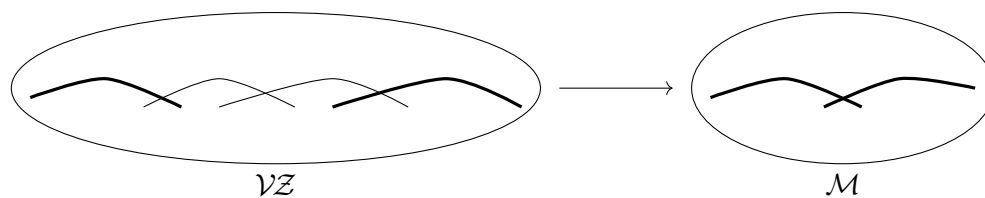


There are four codimension–1 logarithmic strata of \mathcal{VZ} intersecting our chosen neighbourhood, corresponding to the rays in the above picture. Two of these – those corresponding to the rays labeled $\{e_1 = e_2 = 0\}$ and $\{e_3 = 0\}$ – are the proper transforms of codimension–1 strata in \mathcal{M} . These strata consist of log stable maps where some of the tropical edge lengths are equal to zero, meaning that the corresponding nodes have been smoothed. Notice that although the curve has three nodes, there are only two such strata: the nodes q_1 and q_2 cannot be smoothed independently because of the relation $e_1 = 2e_2$.

The remaining two codimension–1 strata in \mathcal{VZ} – corresponding to the interior rays in the above picture – consist of log stable maps where some of the vertex distances become equal. Here none of the nodes are smoothed. From the construction of the subdivision, we see that both these strata map

onto a codimension-2 stratum of \mathcal{M} (namely, the locus in which all of the nodes persist); this coheres with the fact that they should be thought of as exceptional loci of the blow-up. The extra dimension of moduli comes from the choice of alignment.

Finally, there are three codimension-2 strata, corresponding to different *strict* orderings of the vertex distances. Note that the divisorial strata corresponding to $\{e_1 = e_2 = 0\}$ and $\{e_3 = 0\}$, which intersected in \mathcal{M} , no longer intersect in \mathcal{VZ} , since we have blown up. The picture is something like:



4. RECURSION FORMULA FOR $(\mathbb{P}^N|H)$

In the previous section we used tropical geometry to identify the logarithmic strata of $\mathcal{VZ} = \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$. We now focus our attention on the logarithmic divisors, providing a recursive description for these loci: that is, a description in terms of fibre products of smaller moduli spaces. This is the technically most difficult part of the paper. The recursive description bears some similarities with the recursive description of the boundary of the absolute space $\mathcal{VZ}_{1,n}(\mathbb{P}^N, d)$ given in [VZ08]; in keeping with the theme of this paper, however, we will see that the logarithmic structures provide a convenient framework for dealing with the combinatorics.

The end result is that we obtain an explicit formula for tautological integrals over these boundary loci. This produces a Gathmann recursion formula for relative invariants in genus one.

4.1. Recursion setup. Consider an element of \mathcal{VZ} and choose a marked point $x_k \in C$. We note that by the existence of a log morphism, either f has contact order α_k to H at the marking x_k , or else the irreducible component of C containing x_k is mapped entirely inside H (we say “ x_k belongs to an internal component” in this case).

By pulling back the equation defining H along f and then taking its α_k -th derivative at x_k (which makes sense since all the lower-order derivatives vanish by assumption) we obtain a section of a jet bundle on \mathcal{VZ} whose vanishing locus coincides (set-theoretically) with the locus where x_k belongs to an internal component. As in [Gat02, Construction 2.1] there is an exact sequence of jet bundles

$$0 \rightarrow x_k^* \omega_{C/\mathcal{VZ}}^{\otimes \alpha_k} \otimes \text{ev}_k^* \mathcal{O}_{\mathbb{P}^N}(H) \rightarrow x_k^* \mathcal{J}^{\alpha_k}(f^* \mathcal{O}_{\mathbb{P}^N}(H)) \rightarrow x_k^* \mathcal{J}^{\alpha_k-1}(f^* \mathcal{O}_{\mathbb{P}^N}(H)) \rightarrow 0$$

and thus we obtain a section of the line bundle $x_k^* \omega_{C/\mathcal{VZ}}^{\otimes \alpha_k} \otimes \text{ev}_k^* \mathcal{O}_{\mathbb{P}^N}(H)$ with vanishing locus equal to the locus where x_k belongs to an internal component. Writing this as

$$\mathcal{D}^k = \mathcal{D}_{1,\alpha}^k(\mathbb{P}^N|H, d) \subseteq \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d) = \mathcal{VZ}$$

we obtain:

Lemma 4.1. $(\alpha_k \psi_k + \text{ev}_k^* H) \cap [\mathcal{VZ}] = [\mathcal{D}^k]$.²

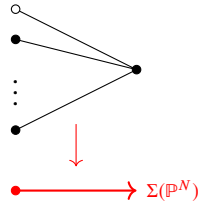
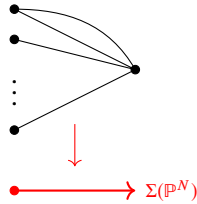
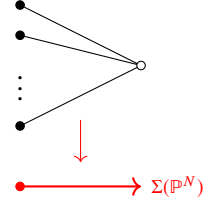
The remainder of this section will be devoted to providing an explicit recursive description of $[\mathcal{D}^k]$ in terms of tautological classes on spaces of maps with smaller numerical invariants. As we will see, this provides an algorithm for computing reduced relative invariants, as well as a quantum Lefschetz algorithm for reduced (absolute) invariants in genus one.

²(Navid) Should we work out the vanishing orders of the section at this point?

4.2. Recursive description of the divisors: types A, B and C^+ . We begin by noting that every irreducible component of \mathcal{D}^k is a codimension-1 logarithmic stratum. This is obvious: the locus where the log structure is trivial coincides precisely with the nice locus, where the source curve is smooth and not mapped inside H . By definition \mathcal{D}^k is disjoint from this locus, and since it has codimension 1 it must be equal to a union of logarithmic strata.

As discussed at length in the previous section, the divisorial logarithmic strata of \mathcal{VZ} correspond to the rays of the polyhedral subdivision of the tropical moduli space which produces the Artin fan \mathcal{A} . The logarithmic strata contained in \mathcal{D}^k correspond precisely to the rays of the tropical moduli space where the vertex of the tropicalisation containing x_k is mapped into the interior of $\mathbb{R}_{\geq 0}$ by the tropical map.

We proceed as in the previous section, fixing an element of \mathcal{M} and considering the resulting subdivision of the tropical moduli space. In fact, it makes sense to choose a locally closed substack of \mathcal{M} where the combinatorial type of the tropical curve (i.e. the ghost sheaf) is constant. Then as long as the minimal subcurve of genus one is not contracted, the picture is classical (i.e. the subdivision is trivial, and $\mathcal{VZ} = \mathcal{M}$ locally). In this case there can only be one vertex of the tropical curve mapped into the interior of $\mathbb{R}_{\geq 0}$; if there were more, the positions of their images in $\mathbb{R}_{\geq 0}$ would be independent and thus the corresponding maximal cone in the tropical moduli space would be at least 2 dimensional, producing a locus in \mathcal{VZ} of larger codimension.³ We adopt the terminology of Vakil, distinguishing three cases as below:


 FIGURE 1. \mathcal{Y}_A

 FIGURE 2. \mathcal{Y}_B

 FIGURE 3. \mathcal{Y}_C^+

Notice that the continuity of the tropical map identifies all of the edge lengths, up to weights given by the expansion factors (i.e. tangency orders) at the edges; thus the tropical moduli space is isomorphic to $\mathbb{R}_{\geq 0}$ and so we have a divisor (with no subdivision required).

These divisors live inside an open neighbourhood of the preimage in \mathcal{VZ} of our chosen locus in \mathcal{M} , and are thus locally closed substacks of \mathcal{VZ} . To obtain the desired components of \mathcal{D}^k , we must take the closure of these locally closed substacks. We will analyse this procedure on a case-by-case basis, and see once again that the tropical picture gives a complete and recursive description of the combinatorics. As usual, this will be done by comparing loci in \mathcal{VZ} with loci in \mathcal{M} . To be more precise: we will take the closure of our locally closed

³(Luca) Say something about rigid tropical curves as in ACGS and KLR

substack in \mathcal{M} and relate it to the closure of our locally closed substack in \mathcal{VZ} . We will see that the induced map on closures is another blow-up. The closed locus in \mathcal{M} is understood by [ACGS] as a fibre product of moduli spaces of punctured maps.⁴

4.2.1. *Type A.* By definition the locus \mathcal{Y}_A contains those elements of \mathcal{VZ} which appear in the closure of the locus of aligned maps whose associated tropical curve has the same combinatorial type as Figure 1⁵. We will now argue that such an element restricts to give an aligned map on the piece of the curve corresponding to C_1 , and that conversely given an aligned map corresponding to C_1 and ordinary punctured maps corresponding to C_0, C_2, \dots, C_r , the glued map is aligned. This shows that \mathcal{Y}_A is equal, up to an étale quotient, to the fibre product:

$$\overline{\mathcal{M}}_{0,\alpha^{(0)} \cup (-m_1, \dots, -m_r)}(\mathbb{P}^N | H, d_0) \times_{H^r} \left(\mathcal{VZ}_{1,\alpha^{(1)} \cup (m_1)}(\mathbb{P}^N | H, d_1) \times \prod_{i=2}^r \overline{\mathcal{M}}_{0,\alpha^{(i)} \cup (m_i)}(\mathbb{P}^N | H, d_i) \right).$$

Put differently, the map from the closed locus in \mathcal{VZ} to the closed locus in \mathcal{M} is given by blowing up one of the factors of the fibre product

$$\overline{\mathcal{M}}_{0,\alpha^{(0)} \cup (-m_1, \dots, -m_r)}(\mathbb{P}^N | H, d_0) \times_{H^r} \prod_{i=1}^r \overline{\mathcal{M}}_{0,\alpha^{(i)} \cup (m_i)}(\mathbb{P}^N | H, d_i).$$

and leaving the rest alone.

Let us start by considering possible limiting points of our original locus in \mathcal{M} . These correspond to further degenerations of the (combinatorial type of the) tropical map.

⁴(Navid) Introduce notation for these locally closed substacks, because this shit is getting out of hand.

⁵(Navid) Bad LaTeX reference

Now we may use the recursive description of the boundary in \mathcal{M} using punctured maps [ACGS] to obtain a description of these loci:

(A)

$$\mathcal{Y}_a = \overline{\mathcal{M}}_{0, \{-m^{(1)}, \dots, -m^{(r)}\} \cup \alpha^{(0)}}(H, d_0)^\sim \times_{H^r} \left(\mathcal{VZ}_{1, \alpha^{(1)} \cup \{m^{(1)}\}}(\mathbb{P}^N | H, d_1) \times \prod_{i=2}^r \overline{\mathcal{M}}_{0, \alpha^{(i)} \cup \{m^{(i)}\}}(\mathbb{P}^N | H, d_i) \right)$$

(B)

$$\mathcal{Y}_b = \overline{\mathcal{M}}_{0, \{-m^{(1)}, \dots, -m^{(r)}\} \cup \alpha^{(0)}}(H, d_0)^\sim \times_{H^r} \left(\overline{\mathcal{M}}_{0, \alpha^{(1)} \cup \{m^{(1)}, m^{(2)}\}}(\mathbb{P}^N | H, d_1) \times \prod_{i=3}^r \overline{\mathcal{M}}_{0, \alpha^{(i)} \cup \{m^{(i)}\}}(\mathbb{P}^N | H, d_i) \right)$$

(C)

$$\mathcal{Y}_c^+ = \mathcal{VZ}_{1, \{-m^{(1)}, \dots, -m^{(r)}\} \cup \alpha^{(0)}}(H, d_0)^\sim \times_{H^r} \prod_{i=1}^r \overline{\mathcal{M}}_{0, \alpha^{(i)} \cup \{m^{(i)}\}}(\mathbb{P}^N | H, d_i)$$

where $\overline{\mathcal{M}}_{0, \mu}(H, d)^\sim$ denotes the space of genus zero punctured maps, while $\mathcal{VZ}_{0, \mu}(H, d)$ denotes the desingularisation of the main component of the space of punctured maps discussed in Section ???. The numerical data run over all splitting A of the tangency conditions (resp. B of the degree) in $r + 1$ parts, such that:

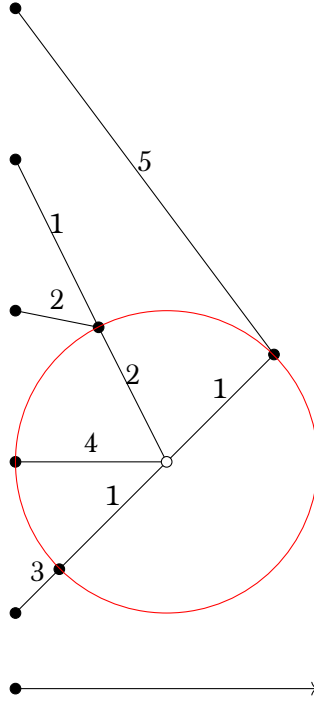
- (1) $d_0 + \sum_{i=1}^r m^{(i)} = \sum \alpha^{(0)}$;
- (2) $d_i = m^{(i)} + \sum \alpha^{(i)}$;
- (3) $d_0 > 0$ in case c .

Review here the process of gluing for punctured maps. In particular the basic monoid should be modified in order to make the relevant evaluations log morphisms. Claim: evaluations are strict. Consequence: the fiber product in the category of log stacks is fine. We have then to apply saturation. This is a finite morphism of degree... (I think it could be $\frac{\prod m^{(i)}}{\text{lcm}(m^{(i)})}$).

Lemma 4.2 (Virtual pushforward). The following hold.

- $\text{fgt}_*[\overline{\mathcal{M}}_{0, \alpha}(\mathbb{P}^N | H, d)] = [\overline{\mathcal{M}}_{0, \alpha}^G(\mathbb{P}^N | H, d)]$ (follows from [Gat03, AMW14]).
- $\text{fgt}_*[\mathcal{VZ}_{1, \alpha}(\mathbb{P}^N | H, d)]$ computes the reduced relative invariants by definition.
- $\text{fgt}_*[\overline{\mathcal{M}}_{0, \mu}(H, d_0)^\sim] = [\overline{\mathcal{M}}_{0, |\mu|}(H, d_0)]$ (follows from [Gat03] and... comparison of punctured with rubber invariants).
- $\text{fgt}_*[\mathcal{VZ}_{1, \mu}(H, d_0)^\sim]$ here we should need a variation on Pixton's DRC formula; hopefully it's enough to avoid the graphs that tropical well-spacedness discards.

4.3. Recursive description of the divisors: type C_0 . On the other hand, when there is a contracted elliptic subcurve - and it will be contracted into the hyperplane, because otherwise it wouldn't be generic, by density of the nice locus in $\mathcal{VZ}_{1, \alpha'}(\mathbb{P}^N | H, d')$ - the picture becomes more complicated due to the alignment. The combs may break (multiple times) and they may point in opposite directions. We label these loci \mathcal{Y}_c^0 .



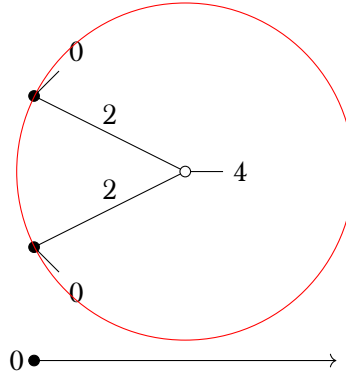
Definition 4.3. A degree-aligned genus one tropical map ϕ to $\mathbb{R}_{\geq 0}$ is ... with a circle of radius δ around the core satisfying:

- (1) the circle of radius δ passes through at least one vertex of positive degree, and no vertex of positive degree is contained in its strict interior;
- (2) every circle of radius $< \delta$ around the core passes through (either none or) at least two vertices;⁶
- (3) every edge heading out of the circle of radius δ goes directly to a vertex in $\phi^{-1}(0)$.

Remark 4.4. The core being contracted in the fiber of the tropical map is not a phenomenon that we should worry about in codimension one. Indeed, assume that the core is contracted in the fiber along a ray. Then all the edges departing from the core have expansion factor 0; call the corresponding coordinates $U = \{u_i\}_{i \in I}$. Call the remaining coordinates $E = \{e_j\}_{j \in J}$. Note that tropical continuity involves only E . Alignments on the other hand assume the form $\lambda(v) = \lambda(v')$, where $\lambda(v) = \sum_{i \in I(v)} u_i \sum_{j \in J(v)} e_j$. Pick the shortest elements of U ; then these can be shortened to zero without affecting the rest (by hypothesis, alignments can only identify them among themselves). This shows that we could not have started with a ray.

Example 4.5. We look at the following example in some detail. The ambient space is $\mathcal{VZ}_{1,(4,0,0)}(\mathbb{P}^N | H, 4)$, of dimension $4N + 3$. The underlying moduli space

⁶The subdivisions we perform before picking a ray correspond to aligning the log structure, i.e. imposing $\lambda(v) = \lambda(v')$ for some v, v' .



is $X = \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H, 2) \times_H \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H, 2) \times \mathcal{VZ}_{1,(-2,-2,4)}$ of dimension $5N+1$. Consider the fiber product:

$$\begin{array}{ccc} F & \longrightarrow & \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H, 2) \times \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H, 2) \\ \downarrow & & \downarrow \\ H & \longrightarrow & H \times H \end{array}$$

At the level of ghost sheaves, $\overline{\mathcal{M}}_F = \mathbb{N} \oplus_{\mathbb{N}^2} \overline{\mathcal{M}}_1^{\text{enl}} \oplus \overline{\mathcal{M}}_2^{\text{enl}}$, where the map $\mathbb{N}^2 \rightarrow \mathbb{N}$ is the sum, and the map $\mathbb{N}^2 \rightarrow \overline{\mathcal{M}}_1^{\text{enl}} \oplus \overline{\mathcal{M}}_2^{\text{enl}}$ generically is multiplication by 2, so $\overline{\mathcal{M}}_F = \mathbb{N}^2/(2e = 2f)$ generically. Saturation gives a finite cover $G \rightarrow F$ with $\overline{\mathcal{M}}_G = \mathbb{N}_{e=f}$ generically. Lifting this to actual log structures, what we are doing (again generically) is taking a square root of the isomorphism $T_{R_1, q_1}^{\otimes 2} \simeq T_{R_2, q_2}^{\otimes 2}$, which is obtained passing through $N_{H/\mathbb{P}^N, f(Z)}$ via $d f|_{R_i, q_i}$. This breaks when $f|_{R_i}$ is not tangent to H of order exactly 2 at q_i , for either i ; but by the maximality assumption this happens precisely along Gathmann's comb loci Δ_i . So in fact, rather than with $T_{R_i, q_i}^{\otimes 2}$, we should be working with $T_{R_i, q_i}^{\otimes 2}(-\Delta_i)$: but this is exactly $\text{ev}_i^*(-H)$ by Gathmann's genus zero formula, and the isomorphism $\text{ev}_1^*(H) = \text{ev}_2^*(H)$ holds on all of F .

On the other hand, generically on $\mathcal{VZ}_{1,(-2,-2,4)}$ we have $T_{q_1}Z \simeq T_{q_2}Z$ by exploiting the group structure on the elliptic curve. This breaks when either (but not both) is on a rational tail. Yet we have $T_{q_1}Z(\Delta_{1 \in P}) \simeq T_{q_2}Z(\Delta_{2 \in P})$ by Vakil-Zinger's construction of a universal ψ -class (i.e. by comparing both with $\pi_*\omega(\Delta)$; notice that our further blow-up has the only effect of twisting *all* the relevant line bundles by $\Delta_{1,2 \notin P}$).

Now, the fiber of the Vakil-Zinger blow-up over X can be described as follows. Generically it looks like

$$\mathbb{P}(T_{q_1}R_1 \otimes T_{q_1}Z \oplus T_{q_2}R_2 \otimes T_{q_2}Z)$$

but this has to be modified along the boundary:

- this has to do with the fact that the normal bundle of the strict transform is the pullback of the normal bundle twisted by the intersection

with the exceptional divisor (so it relates with previous steps of the blow-up);

- it is not globally a \mathbb{P}^1 -bundle (so it relates with further stages of the blow-up; it also has to do with a choice of compactification for the moduli space of attachments);
- it has the effect of replacing $T_{q_i}Z$ with Vakil-Zinger's universal \mathbb{T} , so that this can be factored out of the projective bundle, and in fact we are left with a projective bundle $\mathbb{P} = \mathbb{P}(T_{q_1}R_1 \oplus T_{q_2}R_2)$ over F , and its open part $\text{Iso}(T_{q_1}R_1 \oplus T_{q_2}R_2)$ represents the attachment data for a contraction to a tacnode $R_1 \sqcup_q R_2 \rightarrow \bar{C}$.

On \mathbb{P} there is a natural vector bundle map

$$s: \mathcal{O}_{\mathbb{P}}(-1) \hookrightarrow p^*(T_{q_1}R_1 \oplus T_{q_2}R_2) \xrightarrow{+df} \text{ev}_q^* T\mathbb{P}^N$$

that vanishes along the locus where f descends to \bar{C} . In general, it is not a transversal section:

- we should replace $T\mathbb{P}^N$ by TH as long as all the $m^{(i)}$ are ≥ 2 ;
- Vakil and Zinger construct a blow-up of \mathbb{P} along the vanishing loci of s of low codimension, and twist s by the exceptional divisors, so that it becomes a transverse section \tilde{s} .

On the other hand, the finite cover $G \rightarrow F$ factors through \mathbb{P} , because the two vertices are already aligned on G . We claim that the boundary locus of $\mathcal{VZ}_{1,(4,0,0)}(\mathbb{P}^N|H,4)$ corresponding to the combinatorial type of the tropical map above is the transverse intersection

$$(G \cap V(\tilde{s}) \subseteq \mathbb{P}) \times \mathcal{VZ}_{1,(-2,-2,4)}.$$

This has the expected dimension (codimension $N - 1$ with respect to X). To compute its class, we can pull \mathbb{P} back to G , and then notice that $G \hookrightarrow \mathbb{P}_G$ is the inclusion of a (trivial) subbundle.

Lemma 4.6. The class of $\mathbb{P}(\mathcal{F}) \subseteq \mathbb{P}(\mathcal{E})$ is $c_{\text{top}}(\mathcal{O}_{\mathcal{E}}(1) \otimes p^*(\mathcal{E}/\mathcal{F}))$.

See [EH16, Prop. 9.13]. It is a good time to remember that \mathcal{E} was in fact $(\bigoplus_{i=1}^r TR_{i,q_i}) \otimes \mathbb{T}$. By writing c for $c_1(\mathcal{O}_{\mathcal{E}}(1))$, ψ_i for $c_1(T^*R_{i,q_i})$, ψ_Z for Vakil-Zinger's universal psi class, and H for $\text{ev}_q^* H$, we need to compute

$$\begin{aligned} p_* \left((c - \psi_1 - \psi_2 - 2\psi_Z) [(1 + c + H)^N (1 + c)^{-1}]_{N-1} \right) = \\ p_* \left((c - \psi_1 - \psi_2 - 2\psi_Z) \left(\sum_{k=0}^{N-1} \binom{N}{1+k} c^k H^{N-1-k} \right) \right) = \\ \sum_{k=0}^{N-1} \binom{N}{k} H^{N-1-k} (s_k(\mathcal{E}) - s_{k-1}(\mathcal{E})(\psi_1 + \psi_2 + \psi_Z)) \end{aligned}$$

5. RECURSION FORMULA IN GENERAL

Now let $(X|Y)$ be a smooth pair with Y very ample. The complete linear system $|\mathcal{O}_X(Y)|$ defines an embedding $X \hookrightarrow \mathbb{P}^N$ with $Y = X \cap H$ for H some hyperplane.

Lemma 5.1. The following square is cartesian (in the category of ordinary stacks):

$$\begin{array}{ccc} \mathcal{VZ}_{1,\alpha}(X|Y, \beta) & \longrightarrow & \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d) \\ \downarrow & \square & \downarrow \\ \mathcal{VZ}_{1,n}(X, \beta) & \xrightarrow{i} & \mathcal{VZ}_{1,n}(\mathbb{P}^N, d). \end{array}$$

Since $\mathcal{VZ}_{1,n}(\mathbb{P}^N, d)$ is smooth and $\mathcal{VZ}_{1,n}(X, \beta)$ carries a natural virtual class, there is a diagonal pull-back morphism which we use to define the virtual class on the space of maps to $(X|Y)$:

$$[\mathcal{VZ}_{1,\alpha}(X|Y, \beta)]^{\text{virt}} := i_{\Delta}^! [\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)].$$

The recursion formula in $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ immediately pulls back along i to give a recursion formula in $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$.⁷

6. QUANTUM LEFSCHETZ IN GENUS ONE

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Luca Battistella

Department of Mathematics, Imperial College London

⁷(Navid) Is it clear how to compute integrals over the pulled back classes?

`l.battistella14@imperial.ac.uk`

Navid Nabijou

Department of Mathematics, Imperial College London

`navid.nabijou09@imperial.ac.uk`

Dhruv Ranganathan

Department of Mathematics, Massachusetts Institute of Technology

`dhruvr@mit.edu`