RELATIVE STABLE MAPS IN GENUS ONE VIA RADIAL ALIGNMENTS

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1. Relative space equals closure of the nice locus

Since the log structures only come into play when the source curve is reducible, it follows that the nice locus in the radially aligned setting is the same as the nice locus in the ordinary setting. In particular, it is irreducible.

We now want to show that the relative space in the radially aligned setting is equal to the closure of the nice locus; irreducibility follows immediately. One direction is clear: [WHY?]

It thus remains to show that, given a relative radially aligned map, we can smooth it to one in the nice locus. This is done by considering different cases locally, then gluing.

Case 1: non-contracted genus one internal component. Assume that the curve takes the form

$$C = C_0 \cup C_1 \cup \ldots \cup C_k$$

where all the C_i are smooth, C_0 has genus one, all the other C_i have genus zero, and for $i = 1, \ldots, k$, C_i intersects C_0 at a single node (denoted q_i) and does not intersect any other components.

Suppose furthermore that C_0 is a non-contracted *internal component*, meaning that it is mapped into H via f, and that $C_1, \ldots, C_k$ are *external components*, meaning that they are not mapped into H via f. The picture is:

[FIGURE]

Suppose that this is a relative stable map. This means that [BLAH]. We claim that it can be smoothed to a relative stable map in the nice locus. The construction depends on choosing an appropriate smoothing of the curve C, so that the map also smooths.

We start with $W = C_0 \times \mathbb{A}^1_t$ (where t denotes a fixed co-ordinate on the affine line). This is a smooth surface, fibred over $\mathbb{A}^1_t$, with fibre equal to the elliptic curve C_0. Consider the points $q_1, \ldots, q_k$ on C_0. We will perform a series of weighted blow-ups at the points $(q_i, 0) \in W$, in order to obtain a surface whose general fibre is smooth (in fact, isomorphic to C_0) and whose central fibre is isomorphic to C.

Fix $i = 1, \ldots, k$ and let m_i be the multiplicity of f with H at $q_i \in C_i$. We define:

$$\beta = \prod_{i=1}^{k} m_i \qquad r_i = \beta / m_i$$

We now blow-up the surface W at the points $(q_i, 0)$ with weight r_i in the horizontal direction and weight 1 in the vertical direction: if x_i is a local co-ordinate for the fibre around q_i, this means that we blow-up in the ideal (x_i, t^{r_i}).

The result is a fibred surface $W' \to \mathbb{A}^1_t$ with general fibre isomorphic to C_0 and central fibre isomorphic to C. The total space of W' is no longer smooth, but this is fine; the projection to $\mathbb{A}^1_t$ is still flat. The central fibre is a linearly trivial Cartier divisor:

$$W'_0 = C_0 + C_1 + \ldots + C_k = 0 \in \operatorname{Pic} W'$$

The C_i are not themselves Cartier, but $r_i C_i$ is Cartier for $i = 1, \ldots, k$. We have:

$$r_i C_i \cdot C_0 = 1$$

Now, let $x_1, \ldots, x_n$ denote the marked points of C. These are divisors on the central fibre W'_0, and hence can be extended to divisors $\tilde{x}_1, \ldots, \tilde{x}_n$ on W'. Consider the line bundle:

$$\tilde{L} = \mathcal{O}_{W'}(\beta C_0 + \Sigma_{i=1}^{n} \alpha_i \tilde{x}_i)$$

Note that βC_0 is indeed a Cartier divisor, since

$$\beta C_0 = -\sum_{i=1}^{k} \beta C_i = -\sum_{i=1}^{k} m_i (r_i C_i)$$

and the $r_i C_i$ are Cartier. We claim that when we restrict $\tilde{L}$ to the central fibre we recover the line bundle $L = f^* \mathcal{O}(1)$ that we started with.

For $i = 1, \ldots, k$, we have:

$$\begin{aligned}
\tilde{L}|_{C_i} &= \mathcal{O}_{C_i}\left(\beta (C_0 \cdot C_i) q_i + \Sigma_{x_j \in C_i} \alpha_j x_j\right) \\
&= \mathcal{O}_{C_i}\left((\beta / r_i) q_i + \Sigma_{x_j \in C_i} \alpha_j x_j\right) \\
&= \mathcal{O}_{C_i}\left(m_i q_i + \Sigma_{x_j \in C_i} \alpha_j x_j\right) \\
&= L|_{C_i}
\end{aligned}$$

On the other hand

$$\begin{aligned}\tilde{L}|_{C_0} &= \mathcal{O}_{C_0}\left(-\sum_{i=1}^{k}\beta(C_i\cdot C_0)q_i + \sum_{x_j\in C_0}\alpha_j x_j\right)\\ &= \mathcal{O}_{C_0}\left(-\sum_{i=1}^{k} m_i q_i + \sum_{x_j\in C_0}\alpha_j x_j\right)\\ &= L|_{C_0}\end{aligned}$$

and the fact that $\tilde{L}|_{W_0'} = L$ follows from the fact that the dual intersection graph of C has genus zero.

Now, $\tilde{L}$ comes with a unique section whose restriction to $W_0' \cong C$ is s_0. After we extend the sections $s_1, \ldots, s_N$, it is clear that the resulting stable map is in the nice locus (i.e. that it is not mapped into H).

Now, to extend the sections $s_1, \ldots, s_N$, we simply check that they are unobstructed. The space containing the obstructions is:

$$\mathrm{H}^1(C, L)$$

By taking the normalisation exact sequence for C, tensoring with L and passing to cohomology, we obtain:

Is this true even when C is reducible

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