

# CURVE COUNTING IN GENUS ONE: ELLIPTIC SINGULARITIES & RELATIVE GEOMETRY

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ABSTRACT. We construct and study the reduced, relative, genus one Gromov–Witten theory of very ample pairs. These invariants form the principal part contribution to relative Gromov–Witten theory in genus one and are relative versions of Zinger’s reduced Gromov–Witten invariants. The relative and absolute theories are related by degeneration of the tangency conditions, and the resulting formulas generalize a well-known recursive calculation scheme put forward by Gathmann in genus 0. ~~The geometric input is a desingularization of the principal component of the moduli space of genus one logarithmic stable maps to a very ample pair, using the geometry of elliptic singularities. Our study passes through a number of general techniques for calculating integrals on logarithmic blowups of moduli spaces of stable maps, which may be of independent interest.~~

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## INTRODUCTION

The Kontsevich moduli space of stable maps from genus 0 curves to projective space exhibits **remarkable** geometry – it is a smooth orbifold with a “self-similar” normal crossings boundary. These facts have combined to produce a **remarkable** understanding of Gromov–Witten theory in genus 0.

While the higher genus situation is **substantially** more difficult, in the last several years, **substantial** theory has been developed towards a conceptual understanding of the genus one case. This began over a decade ago with the pioneering work of Li, Vakil, and Zinger [Zin09a, VZ08, LZ07, LZ09, Zin08, Zin09b], and was recently reinvigorated involving ideas from logarithmic geometry and singularity theory [BCM18, CM18, HL10, RSW17a, RSW17b, Smy11a, Vis12]. The result is a *reduced* Gromov–Witten theory, which

removes certain degenerate contributions from the ordinary theory. While the Atiyah–Bott localization for the reduced invariants was used to great effect by Zinger in his proof of the BCOV conjectures, two major components of the standard theory – relative invariants and the degeneration formula – have not been developed in this setting. Our results in this paper provide these as new structural tools in reduced genus one Gromov–Witten theory<sup>1</sup>.

**0.1. Results.** Our first contribution is a definition and construction of relative reduced Gromov–Witten theory of projective space relative to a hyperplane. The geometric content is the following, see Theorem ?? for a precise statement.

Let  $\mathcal{M}_{1,\alpha}^\circ(\mathbb{P}^N|H)$  be the moduli space of positive degree maps from smooth curves of genus 1 to  $\mathbb{P}^N$  with dimensionally transverse contact order  $\alpha \in \mathbb{Z}_{\geq 0}^n$  along  $H$ .

**Theorem A.** There exists a logarithmically smooth and proper Deligne–Mumford stack  $\mathcal{VZ}_{1,\alpha}^\circ(\mathbb{P}^N|H)$  having expected dimension, and containing  $\mathcal{M}_{1,\alpha}^\circ(\mathbb{P}^N|H)$  as a dense open subset. A point in this moduli space parameterizes maps

$$C \rightarrow \mathbb{P}^N[s] \rightarrow \mathbb{P}^N,$$

to an  $s$ -fold expansion of  $\mathbb{P}^N$  along  $H$  such that the map to the expansion and composite map  $C \rightarrow \mathbb{P}^N$  each factor nontrivially through an elliptic singularity.

The moduli space above is constructed by identifying two closed conditions that must be satisfied by the space of maps to the pair  $(\mathbb{P}^N, H)$ . Each condition requires the stable map to factor through an elliptic singularity. The behaviour mixes the geometry in [RSW17a] and [RSW17b]. Together with an analysis of the deformation theory of such maps, we are led to Theorem A.

Our second result explains how to do practical calculations on this space of stable maps.

**Theorem B.** Given a smooth pair  $(X, Y)$  with  $Y$  very ample, there is an explicit recursive algorithm to calculate

- (1) the restricted reduced genus one Gromov–Witten invariants of  $Y$ ;
- (2) the reduced genus one relative Gromov–Witten invariants of  $(X, Y)$ ;
- (3) the restricted reduced genus one rubber invariants of  $P = \mathbb{P}_Y(\mathcal{N}_{Y|X} \oplus \mathcal{O}_Y)$

from the full genus zero and reduced genus one theory of the ambient variety  $X$ .

We pursue a strategy laid out by Vakil and Gathmann, building on work of Caporaso and Harris [CH98, Gat02, Vak00]. The first step is to express the locus of maps where the contact order degenerates in terms of Chern classes of natural vector bundles. The second step is to describe that locus in terms of fiber products of moduli spaces with smaller invariants. Novelties are introduced in both steps. We achieve the first step using techniques that are entirely within tropical and logarithmic geometry, drastically simplifying Gathmann’s calculation. For the second step, we contend with the interaction of the relative splitting formula and the geometry of elliptic singularities, which is entirely new (Section REFERENCE). We express the factorization property in terms of tautological classes and combinatorial manipulations, leading to Theorem B.

<sup>1</sup>Further motivation to consider the genus one case specifically comes simply from the fact that for Calabi–Yau geometries in dimension larger than 3, Gromov–Witten theory vanishes in genera greater than 1.

**0.2. Broader contributions.** Beyond genus one, we pass through a number of seemingly useful general techniques. First, we explain how to describe degenerate moduli spaces as fiber products. This is substantially more delicate than the standard situation because of the need to work with elliptic singularities and aligned logarithmic structures. This interaction of the degeneration formula with the geometry of curve singularities is a new phenomenon. The results of [RSW17a] suggest the existence of a reduced higher genus Gromov–Witten theory formed by replacing contracted elliptic components with singularities<sup>2</sup>. The discussion here is likely to carry over to this setting.

Second, a key step in our recursion is a description of the locus of maps with higher than prescribed tangency, which was identified by Gathmann. We realize the locus as the vanishing locus of a section of a line bundle that comes from tropical geometry — from a piecewise linear function on the fan. The systematic understanding of logarithmic line bundles arising in this fashion is likely to play an important **role** in logarithmic enumerative geometry.

We rephrase Gathmann’s idea of increasing the tangency requirement within the framework of logarithmic stable maps (where maximal tangency is always assumed) by introducing *fictitious markings*, and identifying the space of maps with higher contact order with a boundary stratum inside that of lower contact order and more (fictitious) markings.

Our analysis leads naturally to **the** study of a “main component” double ramification cycle for target manifolds in genus one. The virtual geometry of this was studied in recent work of Janda, Pandharipande, Pixton, and Zvonkine [JPPZ18]. We do not know how to calculate a **closed formula** for this main component contribution, but we repurpose E. Katz’s topological recursion relations [Kat07] to complete the algorithm. We note that even in genus one, an understanding of the main component contributions of the double ramification cycle for target manifolds would be interesting.

Finally, we note that our results require us to encounter several variants of the space of relative/logarithmic stable maps, with both fixed and expanded target. The conceptual features of one space are often computational bugs, and vice versa, and we believe that the techniques developed in the genus one case here will be used repeatedly in logarithmic Gromov–Witten theory calculations. In particular, we frequently use Kato’s perspective on logarithmic blowups as subfunctors that was applied in [RSW17a, RSW17b].

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<sup>2</sup>We have been informed by Jonathan Wise that his Ph.D. candidate Sebastian Bozlee has constructed these higher genus reduced Gromov–Witten invariants. A manuscript is in preparation.

## 1. CONSTRUCTIONS AND LOGARITHMIC SMOOTHNESS

**1.1. The absolute case.** Let  $\mathfrak{M}_{1,n}$  be the logarithmic algebraic stack of  $n$ -pointed prestable curves of genus 1. Let  $\sqsubset$  be a  $n$ -pointed genus 1 tropical curve; **we formally replace the minimal subcurve of genus 1 by a single vertex, called the *core*.** Let  $C$  be a logarithmic smooth curve over  $S$ , let  $\sqsubset$  denote its tropicalization, and let  $\lambda$  be the section of  $\overline{M}_{C/S}$  giving the distance from the **core**.

We begin by recalling the logarithmic moduli spaces of genus 1 curves constructed in [RSW17a, Sections 2 & 4]. Given a family of tropical curves over a base  $\sigma$ , a central alignment is a piecewise-linearly varying choice of radius  $\delta_s \in \overline{M}_S$  for  $s \in \sigma$ , together with a consistent ordering of the vertices that lie inside the circle of radius  $\delta_s$  around the core. Precisely, let  $T$  be a geometric point with logarithmic structure.

**Definition 1.1.** A **centrally aligned curve** is a pair  $(C/T, \delta)$ , where  $\delta \in \overline{M}_T$ , such that:

- (1) the section  $\delta$  is comparable to  $\lambda(v)$  for all vertices  $v$  of  $\sqsubset$ ,
- (2) for any pair of vertices  $v$  and  $w$  at distance less than  $\delta$ , the sections  $\lambda(v)$  and  $\lambda(w)$  are comparable.

In essence, this is a family of curves where every vertex of the tropical curve  $\sqsubset$  is either inside or outside the circle of radius  $\delta$  around the core genus 1 subcurve.

The moduli stack  $\mathfrak{M}_{1,n}^{\text{cen}}$  is a logarithmic algebraic stack in the smooth topology, **and  $\mathfrak{M}_{1,n}^{\text{cen}} \rightarrow \mathfrak{M}_{1,n}$  is a logarithmic modification.** The main construction of [RSW17a] canonically associates to any **centrally** aligned family of curves  $\mathcal{C}_S$  a partial destabilization  $\widetilde{\mathcal{C}}_S$  and a contraction

$$\widetilde{\mathcal{C}}_S \rightarrow \overline{\mathcal{C}}_S,$$

where  $\overline{\mathcal{C}}_S$  may contain a Gorenstein elliptic singularity [Smy11a]. The number of branches is equal to the number of excident edges at the circle of radius  $\delta$ . This data uniquely determines the singularity.

The space of stable maps  $\overline{\mathcal{M}}_{1,n}^{\text{cen}}(\mathbb{P}^N, d)$  from the universal nodal curve  $\mathcal{C}$  over the stack  $\mathfrak{M}_{1,n}^{\text{cen}}$  is a proper and algebraic, with projective coarse moduli space. **A compatibility condition is required, namely that the circle of radius  $\delta$  passes through at least one component of positive degree.** A stable map in this space is said to satisfy the **factorization condition** if the map  $\mathcal{C} \rightarrow \mathbb{P}^N$  factors through the contraction  $\widetilde{\mathcal{C}} \rightarrow \overline{\mathcal{C}}$ .

**Theorem 1.2** ([RSW17a, Theorem B]). The substack  $\mathcal{VZ}_{1,n}(\mathbb{P}^N, d)$  of  $\overline{\mathcal{M}}_{1,n}^{\text{cen}}(\mathbb{P}^N, d)$  parameterizing maps to  $\mathbb{P}^N$  that satisfy the factorization property is smooth and proper of the expected dimension.

The remainder of this section lifts this condition to the category of relative maps to the pair  $(\mathbb{P}^N, H)$ . The core of the result is identifying the necessary factorization conditions that remove the obstructions.

**1.2. Relative geometry: compactification.** Fix a hyperplane  $H \subset \mathbb{P}^N$ . Let  $\alpha$  be a partition of the degree  $d > 0$ . Consider the moduli space  $\mathcal{M}_{1,\alpha}^{\circ}(\mathbb{P}^N|H)$  of maps from smooth elliptic curves  $C \rightarrow \mathbb{P}^N$  that meet  $H$  at finitely many marked points with vanishing orders given by the partition  $\alpha$ . This is a smooth, but non-proper Deligne–Mumford stack.

We will first compactify the space described above, and then desingularize it. For the compactification, we begin with Abramovich–Chen–Gross–Siebert’s space of logarithmic stable maps, though we will typically work with various subcategories and variants [AC14, Che14b, GS13, Kim10]. Let  $\mathbb{P}^N$  be endowed with the **divisorial logarithmic structure induced by  $H$** .

The moduli space  $\overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H)$  is a fibred category over logarithmic schemes, whose fiber over  $(S, M_S)$  is the groupoid of logarithmic curves of genus 1 over  $(S, M_S)$  equipped with a map to  $\mathbb{P}^N$  of degree  $d$  and contact order  $\alpha$ . It is a fundamental fact in the subject that this category is representable by a proper algebraic stack with logarithmic structure, parameterizing minimal objects [Che14b].

There is a representable finite logarithmic morphism to the Kontsevich space, forgetting the logarithmic structure on the target:

$$\overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H) \rightarrow \overline{\mathcal{M}}_{1,n}(\mathbb{P}^N, d).$$

The space of centrally aligned maps is a logarithmic modification

$$\overline{\mathcal{M}}_{1,n}^{\text{cen}}(\mathbb{P}^N, d) \rightarrow \overline{\mathcal{M}}_{1,n}(\mathbb{P}^N, d)$$

The fiber product leads to a fourth moduli space of **centrally aligned logarithmic maps** to  $(\mathbb{P}^n, H)$ , which we denote  $\widehat{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N|H)$ .

**1.3. Relative geometry: expansions.** Our next task is to pick out a non-singular principal component in  $\widehat{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N|H)$ . The principal component of this space, consisting of the closure of the space of maps from nonsingular curves, maps into the principal component of  $\overline{\mathcal{M}}_{1,n}^{\text{cen}}(\mathbb{P}^N, d)$ . Indeed, smoothable logarithmic maps are, in particular, smoothable as ordinary maps. An additional condition is required to isolate the principal component of the space of logarithmic maps.

In order make the obstruction theory more geometric, we expand the target. To elucidate the connection with the static target, consider a logarithmic stable map  $[C \rightarrow (\mathbb{P}|H)]$  over  $\text{Spec}(\mathbb{N} \rightarrow \mathbb{C})$ . At the level of tropicalizations, there is a map of fans

$$\square \rightarrow \mathbb{R}_{\geq 0}.$$

Choose a subdivision of  $\mathbb{R}_{\geq 0}$  whose vertices consist of the images of vertices of  $\square$ . Pull this subdivision back to  $\square$  by marking all preimages of the vertices of  $\mathbb{R}_{\geq 0}$ . Denote the resulting map  $\widetilde{\square} \rightarrow \widetilde{\mathbb{R}}_{\geq 0}$ .

These subdivisions induce logarithmic modifications

$$\widetilde{C} \rightarrow \mathbb{P}^N[s],$$

see [AW18]. Here the latter is the  $s$ -times iterated deformation to the normal cone of  $H$  in  $\mathbb{P}^N$ . The components **of the target** are in bijection with the number of vertices in  $\widetilde{\mathbb{R}}_{\geq 0}$ . The curve is modified by adding rational components corresponding to the newly introduced vertices.

The result is a logarithmic stable map to an expanded target  $\mathbb{P}^N[s]$ , together with a contraction to the main target component  $\mathbb{P}^N[s] \rightarrow \mathbb{P}^N$ . **Two maps are considered equivalent if they differ by  $\mathbb{G}_m$ -scaling the target at higher level.**

Globally, Kim constructs a moduli space of logarithmic stable maps to expanded degenerations, which on logarithmic points, reduces to the above construction [Kim10]. Indeed,

by the above description, Kim's space is identified with a subcategory of the Abramovich–Chen–Gross–Siebert space, and its dual minimal monoids are cones in a subdivision of the dual minimal cones of the unexpanded space. This process is outlined in complete generality in [Ran19, § 2].

**1.4. Relative geometry: factorization.** Let  $[C \rightarrow \mathbb{P}^N[s] \rightarrow \mathbb{P}^N]$  be a logarithmic map from a centrally aligned curve to an expansion. Recall that  $\mathbb{P}^N[s]$  consists of a union of  $\mathbb{P}^N$  with  $s$  copies of the projective bundle  $\mathbb{P}(\mathcal{O}_H \oplus \mathcal{O}_H(1))$ . We refer to these latter components as **the higher levels**. Thus, we will say that a subcurve  $D \subset C$  maps to **higher level** if the collapsed map

$$C \rightarrow \mathbb{P}^N$$

maps  $D$  into  $H \subset \mathbb{P}^N$ .

I replace  $D_1, D_2$  by the more suggestive  $D_F, D_B$ .

Let  $D_F \subset C$  be the maximal genus 1 subcurve that is mapped to higher level contracted by the map  $C \rightarrow \mathbb{P}^N[s]$  in the fiber direction. Let  $\delta_F$  be the associated radius from the circuit to the nearest non-contracted component. Let  $D_B$  be the maximal genus 1 subcurve that is contracted by the collapsed map to  $\mathbb{P}^N$ , and let  $\delta_B$  be the associated radius. This coincides with the radius of the underlying map to  $\mathbb{P}^N$ . Of course,  $\delta_F \leq \delta_B$ , so  $(C, \delta_F)$  is centrally aligned too.

The datum  $(\delta_F, \delta_B)$  determines a destabilization  $\tilde{C}$  of  $C$  together with successive contractions  $\tilde{C} \rightarrow \bar{C}_F \rightarrow \bar{C}_B$ . The contractions are constructed directly from [RSW17a, Section 3].

The following condition identifies the locus of smoothable maps.

**Definition 1.3.** The map  $[C \rightarrow \mathbb{P}^N[s] \rightarrow \mathbb{P}^N]$  **factorizes completely** if

- the map  $C \rightarrow \mathbb{P}^N[s]$  factors through  $\bar{C}_F$  such that at least one branch of  $\bar{C}_F$  has positive degree in the fiber direction;
- the collapsed map to  $\mathbb{P}^N$  factorizes through  $\bar{C}_B$  such that at least one branch of  $\bar{C}_B$  has positive degree.

In particular, if  $[C \rightarrow \mathbb{P}^N[s] \rightarrow \mathbb{P}^N]$  is a family of centrally aligned maps over  $S$  that factorizes completely, there is a forgetful moduli map  $S \rightarrow \mathcal{VZ}_{1,n}(\mathbb{P}^N, d)$ , to the principal component of the space of absolute maps.

Let  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$  be the stack of maps from centrally aligned curves to expansions of  $\mathbb{P}^N$  that factorize completely.

**Theorem 1.4.** The stack  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$  is proper and logarithmically smooth over  $\text{Spec } \mathbb{C}$ .

*Proof.* We prove the results via the forgetful morphism

$$\nu : \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H) \rightarrow \mathcal{VZ}_{1,n}(\mathbb{P}^N, d),$$

which remembers only the stabilization of the collapsed map. The map is certainly centrally aligned, and we have already argued above that it satisfies the factorization property for  $\mathbb{P}^N$  after composition. The moduli space  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$  is proper. Indeed, the verification of the valuative criterion follows exactly as in Vakil's Lemma [Vak00, Lemma 5.9]. A direct proof that the factorization property is a closed condition may be found in [RSW17a, Theorem 4.3].

We come to logarithmic smoothness. If the elliptic curve does not map into higher level, then unobstructedness proceeds exactly as in [RSW17a, Section 4.5].

Assume now that the elliptic curve maps into higher level. Note that the relative logarithmic tangent bundle of the expansion, over the base  $\mathbb{P}^N$  is a trivial line bundle of rank 1, as the fibers are toric. Consider an  $S$ -family of maps. After replacing the source  $C$  by a destabilization, we have maps

$$\begin{array}{ccccc} C & \longrightarrow & \overline{C}_F & \longrightarrow & \overline{C}_B \\ & & \downarrow & & \downarrow \\ & & \mathbb{P}^N[s] & \longrightarrow & \mathbb{P}^N \end{array}$$

factorizing completely. Let  $f_B$  and  $f$  be the maps from  $C$  to the base and total space respectively. Examining the morphism  $\nu$ , we see that there is a map

$$\mathrm{Def}(C, f_B) \rightarrow \mathrm{Obs}(f/(C, f_B)) = H^1(\overline{C}_F, \mathcal{O}_{\overline{C}_F}),$$

where the former is the space of deformations of the curve and map to the base, as a factorized centrally aligned map. The latter is the space of obstructions to lifting a map to the base into the total space. The cokernel of this map are the absolute obstructions, which we will show vanishes, proving logarithmic smoothness.

We note the manner in which the group  $H^1(\overline{C}_F, \mathcal{O}_{\overline{C}_F})$  functions as the obstruction group for the lifting. It suffices to work near the minimal genus 1 subcurve  $D_F$  of  $C$ , since the rest of the curve is rational. This lifting is given by a rational function with prescribed orders of poles given by the slopes of the tropicalization map. That is, if  $\alpha$  be the piecewise linear function giving this tropical map, then the lifting is described by a section of the associated bundle  $\mathcal{O}_C(-\alpha)$ .

Given a strict square-zero extension  $S'$  of  $S$ , the piecewise linear function  $\alpha$  extends uniquely by strictness to any deformation of the curve. The resulting deformation of  $\mathcal{O}_C(-\alpha)$  produces an infinitesimal deformation of the trivial bundle in the Picard group, whose class in  $H^1(\overline{C}_F, \mathcal{O}_{\overline{C}_F})$ , is the obstruction to deformation. We will show that it is possible to choose a deformation of  $(C, f_B)$  that cancels out the obstruction to lifting to  $f$ , and thus the map to the obstruction space above is surjective.

The line bundle  $\mathcal{O}_C(-\alpha)$  is equivalent to the divisor  $\sum a_i x_i$  where  $x_i$  are the points connecting  $D_F$  to the rest of the curve, and  $a_i$  are the slopes of  $\alpha$  along the edges corresponding to the  $x_i$ . Infinitesimally moving the point  $x_i$  is a deformation of the curve that is unobstructed by the map  $f_B$ , since the map is constant on the interior of the **circle of radius  $\delta_B$** . By deforming the  $x_i$ , at least one of which has nonzero  $a_i$ , we produce a one-dimensional space of obstructions. The absolute obstructions therefore vanish, ~~since they are the cokernel of a surjective map~~, so the result follows.  $\square$

The above theorem guarantees that the singularities of the space of totally factorized maps to expansions is logarithmically smooth. In fact, one can say more in this case.

**Corollary 1.5.** The logarithmically smooth stack  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$  has at worst orbifold singularities, i.e. admits a non-representable cover by a smooth Deligne–Mumford stack.

*Proof.* Since  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$  is logarithmically smooth, it will suffice to show that the cones of its tropicalization are simplicial. To see this, consider a logarithmic stable map to

I don't understand the point of this Cor. Can we make it into a short remark?



an expansion  $C \rightarrow \mathbb{P}^N[s]$  without a central alignment. The tropical moduli cone obtained as the dual of the minimal base monoid can be identified with  $\mathbb{R}_{\geq 0}^s$ , see for instance [Che14a, Section 2.2]. The alignment procedure is an iterated barycentric subdivision at the level of tropical moduli spaces, as explained in [RSW17a, Section 4.6], and such subdivisions preserve the property of being simplicial. We conclude from this that the blowup  $\widetilde{\mathcal{VZ}}_{1,\alpha}^{\text{exp}}(\mathbb{P}^N|H)$  obtained by centrally aligning Kim's spaces has simplicial cones. Finally, the morphism  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H) \rightarrow \widetilde{\mathcal{VZ}}_{1,\alpha}^{\text{exp}}(\mathbb{P}^N|H)$  is strict, so the cones of  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$  are simplicial, as claimed.  $\square$

**1.5. Rubber variants.** In our implementation of the relative theory, we will need the “rubber” variant of the space of maps  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$ , where the entire curve is mapped into higher levels.

Specifically, let  $\mathbb{P}$  denote the projective bundle  $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O})$  on  $\mathbb{P}^{N-1}$ . Equip this space with the logarithmic structure coming from the 0 and  $\infty$  sections of the bundle. Consider the moduli space  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}/\mathbb{G}_m)$  of logarithmic maps

$$C \rightarrow \mathbb{P}[s] \rightarrow \mathbb{P}^{N-1},$$

which factorize completely, and where automorphisms are considered over the space of maps to  $\mathbb{P}^{N-1}$ . Specifically, two maps that differ by a  $\mathbb{G}_m$  translate in the fiber direction are considered equivalent. Total factorization, similar to the previous case, means that the map to the bundle and the map to the base  $\mathbb{P}^{N-1}$  both factorize through possibly different singularities.

**Theorem 1.6.** The stack  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}/\mathbb{G}_m)$  is logarithmically smooth.

The main point in the proof of this statement is that on the space of non-degenerate maps with smooth domains, the map from  $\mathcal{VZ}_{1,\alpha}^\circ(\mathbb{P}) \rightarrow \mathcal{VZ}_{1,\alpha}^\circ(\mathbb{P}/\mathbb{G}_m)$  is a  $\mathbb{G}_m$  torsor. Correspondingly, the map on compact moduli spaces is a nodal curve fibration. To prove this, it is convenient to work with the logarithmic multiplicative group and its torsors. Recall that the *logarithmic multiplicative group*  $\mathbb{G}_{\log}$  is the functor on logarithmic schemes whose value on a logarithmic scheme  $S$  is the group of global sections  $H^0(S, M_S^{\text{gp}})$ ; **it is a proper group functor, and it contains  $\mathbb{G}_m$  as a subfunctor**. This functor is representable only after a logarithmically étale cover. Background may be found in [MW17, RW19].

The bundle  $\mathbb{P} \rightarrow \mathbb{P}^{N-1}$  gives rise to a  $\mathbb{G}_m$ -torsor by deleting the zero and infinity sections. Replacing these fibers by their  $\mathbb{G}_{\log}$  compactifications, we obtain a non-representable functor on logarithmic schemes  $\mathbb{P}_{\log}$  and a logarithmically étale modification

$$\mathbb{P} \rightarrow \mathbb{P}_{\log}.$$

*Proof.* Consider the stack over logarithmic schemes  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}_{\log})$  of stable logarithmic maps that factorize completely. Here, stability for the map coincides with stability for the projection to  $\mathbb{P}^{N-1}$ . The arguments in the previous section show that maps to  $\mathbb{P}$  that factorize completely are unobstructed. Since this space is a logarithmically étale cover of  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}_{\log})$ , logarithmic smoothness of the latter follows.

The logarithmic multiplicative group  $\mathbb{G}_{\log}$  acts on  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}_{\log})$  by translation. Tautologically,  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}_{\log})$  is a  $\mathbb{G}_{\log}$ -torsor over the moduli problem of maps up to this  $\mathbb{G}_{\log}$  translation. It is representable after a logarithmically étale modification by a family of



nodal curves. It follows that the space  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}_{\log}/\mathbb{G}_{\log})$  of maps up to translation that factorize completely is logarithmically smooth.

To conclude the main result, we compare the stacks  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}_{\log}/\mathbb{G}_{\log})$  and  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}/\mathbb{G}_m)$ . Since a map to  $\mathbb{P}$  gives rise immediately to a map to  $\mathbb{P}_{\log}$ , there is a morphism

$$\mathcal{VZ}_{1,\alpha}(\mathbb{P}/\mathbb{G}_m) \rightarrow \mathcal{VZ}_{1,\alpha}(\mathbb{P}_{\log}/\mathbb{G}_{\log}).$$

An application of the infinitesimal lifting criterion shows that this map is logarithmically étale, so the result follows.  $\square$

**1.6. Virtual construction for very ample pairs.** Now let  $(X, Y)$  be a smooth pair with  $Y$  very ample. The definition given in §1.4 applies in this setting, producing a moduli space  $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$  together with a morphism

$$\mathcal{VZ}_{1,\alpha}(X|Y, \beta) \rightarrow \overline{\mathcal{M}}_{1,\alpha}^{\log}(X|Y, \beta)$$

obtained by performing a logarithmic modification and then passing to a closed substack. This moduli space will not in general be logarithmically smooth, but we may equip it with a virtual class as follows. The complete linear system  $|\mathcal{O}_X(Y)|$  defines an embedding  $X \hookrightarrow \mathbb{P}^N$  with  $Y = X \cap H$  for  $H$  some hyperplane.

**Lemma 1.7.** The following square is cartesian (in the category of ordinary stacks):

$$\begin{array}{ccc} \mathcal{VZ}_{1,\alpha}(X|Y, \beta) & \longrightarrow & \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d) \\ \downarrow & \square & \downarrow \\ \mathcal{VZ}_{1,n}(X, \beta) & \xrightarrow{i} & \mathcal{VZ}_{1,n}(\mathbb{P}^N, d). \end{array}$$

*Proof.* It is clear from the modular description that this square is cartesian in the category of fs logarithmic stacks. But the morphism  $i$  is strict, which implies that the square is also cartesian in the category of coherent logarithmic stacks. Since fibre products in the latter category are compatible with the functor forgetting logarithmic structures, the claim follows.  $\square$

Since  $\mathcal{VZ}_{1,n}(\mathbb{P}^N, d)$  is smooth and  $\mathcal{VZ}_{1,n}(X, \beta)$  carries a natural virtual class [RSW17a, Theorem 4.4.1] there is a diagonal pull-back morphism [BN17, Appendix C], which we use to define the virtual class on the space of maps to  $(X, Y)$ :

$$[\mathcal{VZ}_{1,\alpha}(X|Y, \beta)]^{\text{virt}} := i_{\Delta}^! [\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)].$$

Since  $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$  comes equipped with evaluation maps and cotangent line classes, we immediately arrive at a definition of reduced Gromov–Witten invariants for the pair  $(X, Y)$ . This is the same manner in which Gathmann defines his virtual classes [Gat02].

## 2. STRATIFICATION AND TROPICALIZATION

The logarithmic smoothness established in Theorem 1.4 implies that  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$  is an orbifold toroidal embedding. Consequently, the irreducible components of its boundary and their intersections form a stratification of the space. This stratification will be important in the sequel.

The tropicalizations of logarithmic maps that factorize completely satisfy a natural combinatorial condition known as well-spacedness. The version we use here is essentially identical to [RSW17b].

**Definition 2.1.** Let  $\sqsubset$  be a tropical curve of genus 1 and let  $\sqsubset_0$  be its minimal subcurve of genus 1. A tropical map  $F : \sqsubset \rightarrow \mathbb{R}_{\geq 0}$  is said to be **well-spaced** if one of the following two conditions are satisfied:

- (1) no open neighborhood of  $\sqsubset_0$  is contracted to a point in  $\mathbb{R}_{>0}$ , or
- (2) if an open neighborhood of  $\sqsubset_0$  is contracted and  $t_1, \dots, t_k$  are the ~~vertex-edge direction~~ flags whose vertex is mapped to  $F(\sqsubset_0)$ , but along which  $F$  has nonzero slope; then, the minimum of the distances from  $\sqsubset_0$  to  $t_i$  occurs for at least two indices  $i$ .

**Proposition 2.2.** Let  $[C \rightarrow \mathbb{P}^N]$  be a logarithmic stable map from a centrally aligned curve to an expansion, that factorizes completely. Then the tropicalization  $\sqsubset \rightarrow \mathbb{R}_{\geq 0}$  is well-spaced.

*Proof.* The required result is essentially contained in [RSW17b, Section 4], so we explain how to deduce the requisite result from the one in loc. cit. We may focus on a single component of the expansion  $\mathbb{P}^N[s]$  that contains the image of a contracted genus 1 subcurve, as this is the only relevant case. We may thus replace the target with the projective bundle  $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O})$  over  $\mathbb{P}^{N-1}$  equipped with the divisorial logarithmic structure from the 0 and  $\infty$  sections. Let  $p$  be the point to which the genus 1 subcurve is contracted. Passing to an open neighborhood of  $p$ , the map to the bundle is given by a rational function  $f$  on an open curve  $C^\circ$ , determined by the subgraph formed by  $\sqsubset_0$  and the flags  $t_i$  described in the definition above. To describe the tropical map to  $\mathbb{R}_{>0}$ , we observe that  $\sqsubset_0$  is contracted to a fixed point  $q \in \mathbb{R}_{>0}$ . The flags at the ~~vertex-bases~~ at  $t_i$  correspond to nodes or markings of  $C$ . The pole orders of  $f$  at these distinguished points determine the slopes of the tropical map. We are now exactly in the situation considered in [RSW17b, Second Paragraph of § 4.6], and Lemma 4.6.1 of loc. cit. guarantees the well-spacedness as required.  $\square$

**2.1. The cone complex.** To understand the stratification via combinatorial data, we will build the stratification from known objects.

**Step 1.** Let  $\Sigma^{\log}$  be the tropical moduli space of genus 1 tropical stable maps to  $\mathbb{R}_{\geq 0}$ . This is naturally identified with the tropicalization (in the logarithmic sense) of the Abramovich–Chen–Gross–Siebert space of logarithmic stable maps to the pair  $(\mathbb{P}^N, H)$ .

**Step 2.** Given such a tropical stable map, we may subdivide  $\mathbb{R}_{\geq 0}$  such that the image of every vertex of  $\sqsubset$  is a vertex of  $\mathbb{R}_{\geq 0}$ . Call this subdivision  $\widetilde{\mathbb{R}}_{\geq 0}$ . The preimages of vertices of the subdivision form a subdivision of  $\sqsubset$ . After this procedure, the images of vertices of  $\sqsubset$  which lie in  $\mathbb{R}_{>0}$  are totally ordered, in a manner extending the partial order obtained from the map to  $\mathbb{R}_{\geq 0}$ . The combinatorial types of such **image-ordered** maps produce the cones of a subdivision of  $\Sigma^{\log}$  which we denote  $\Sigma^{\text{Kim}}$ .

**Step 3.** Given a tropical map  $F : \sqsubset \rightarrow \mathbb{R}_{\geq 0}$  parameterized by  $\Sigma^{\text{Kim}}$ , there is a largest radius

doesn't picking  $\delta$  itself imply a subdivision?

$\delta$  (possibly equal to 0) such that every vertex strictly contained in the circle of radius  $\delta$  around the core has degree-marking  $d = 0$ . Let  $\Sigma^{\text{cen}}$  be the subdivision obtained by requiring that the vertices contained within the circle of radius  $\delta$  around the core of  $\square$  are totally ordered. This involves introducing cones along which certain  $\mathbb{Z}$ -linear combinations of edge lengths are identified.

**Step 4.** Let  $\Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$  be the subcomplex of  $\Sigma^{\text{cen}}$  obtained by passing to the subcomplex of  $\Sigma^{\text{cen}}$  that consists of well-spaced tropical maps.

**Proposition 2.3.** The cone complex  $\Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$  is the fan of the toroidal embedding  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$ . In particular, the codimension  $k$  strata of  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$  are in inclusion reversing bijection with the dimension  $k$  cones in  $\Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$ .

*Proof.* The construction above has been given to mimic the construction of the space  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$ . Specifically, the fact that the cone complex  $\Sigma^{\text{cen}}$  is the cone complex attached to the logarithmic stack  $\widehat{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N|H)$  of centrally aligned maps to expansions, follows immediately from its description as a subcategory of the fibered category (over logarithmic schemes) of  $\mathcal{M}_{1,\alpha}^{\log}(\mathbb{P}^N|H)$ . To complete the result, we note that  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$  has a strict map to  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H)$ , so its cone complex is a subcomplex of  $\Sigma^{\text{cen}}$ . The fact that it must be contained in the subcomplex of well-spaced curves is immediate from Proposition 2.2.  $\square$

**2.2. Indexing the strata.** The dimension- $k$  cones in  $\Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$  can be enumerated as follows. First, the cones in  $\Sigma^{\log}$  are indexed by **combinatorial types** of tropical maps to  $\mathbb{R}_{\geq 0}$ . Here a combinatorial type encodes all of the data of a tropical map, except for the edge lengths and precise vertex positions. To be more precise, a combinatorial type  $\Delta$  consists of:

- (1) a finite graph;
- (2) genus, degree and marking assignments on the vertices;
- (3) the data of which stratum of  $\mathbb{R}_{\geq 0}$  each vertex is mapped to;
- (4) integral slopes along the edges (both finite and infinite).

The corresponding cone  $\sigma \in \Sigma^{\log}$  is then given by the resulting moduli space of tropical maps, given by choices of edge lengths and vertex positions which produce a continuous tropical map. Given  $\sigma \in \Sigma^{\log}$  we then produce all the cones in  $\Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$  mapping to  $\sigma$  by performing Steps 2–4 outlined above. This amounts to taking a particular subdivision of  $\sigma$ . Note that in this process, new cones are created which map into larger-dimensional cones of  $\Sigma^{\log}$ . The process for enumerating the codimension  $k$  strata of  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$  is therefore:

- (1) fix a combinatorial type  $\Delta$ ;
- (2) perform the subdivision of the resulting cone  $\sigma \in \Sigma^{\log}$ ;
- (3) identify the dimension  $k$  cones of that subdivision.

In §4 we will perform this analysis in the case  $k = 1$ .

In this part we prove recursion relations on the moduli space of reduced stable maps (Theorem 3.1). These are inspired by the constructions of Vakil and Gathmann, but go beyond these in several key respects, giving a full treatment of descendant invariants for very ample pairs. We apply the recursions to produce a quantum Lefschetz algorithm in

genus one, reconstructing the absolute theory of  $Y$  and the relative theory of  $(X, Y)$  from the absolute theory of  $X$ . Along the way we prove a splitting axiom for the reduced relative theory; the form of this splitting axiom is nontrivial, insofar as it involves the introduction of additional tautological classes.

In §§3–4 we prove the recursion formula for  $(\mathbb{P}^N, H)$ . In §5 we indicate how to obtain from this the recursion formula for an arbitrary very ample pair  $(X, Y)$ . Finally in §6 we describe the quantum Lefschetz algorithm.

### 3. GATHMANN’S LINE BUNDLE VIA TROPICAL GEOMETRY

Consider a reduced relative space  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ . For each marking  $x_k$  we consider the locus  $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d) \subseteq \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$  where  $x_k$  belongs to an internal component of the collapsed map. In this section, we use the logarithmic structure on  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$  to construct a line bundle  $\mathcal{L}_k$  together with a section  $s_k$  which vanishes precisely along  $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$ . We use the correspondence with tropical geometry to identify  $c_1(\mathcal{L}_k)$  in terms of tautological classes on  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ , and to compute the vanishing order of  $s_k$  along the components of  $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$ . Combined with the relative splitting axioms established in the next section, we obtain a recursion relation inside  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ .

The pair  $(\mathcal{L}_k, s_k)$  is most naturally constructed on the moduli space  $\overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H, d)$  of non-expanded logarithmic maps; the corresponding pair on  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$  will be obtained via pull-back. Consider therefore the tropicalisation  $\Sigma^{\log}$  of  $\overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H, d)$ , identified as usual with the moduli space of tropical maps to  $\mathbb{R}_{\geq 0}$ . We have a universal family

$$\begin{array}{ccc} \square & \xrightarrow{f} & \mathbb{R}_{\geq 0} \\ \nwarrow \scriptstyle x_k & \downarrow \scriptstyle \pi & \\ & \Sigma^{\log} & \end{array}$$

where  $x_k$  is the section which for every point  $\lambda \in \Sigma^{\log}$  picks out the vertex of  $\square_\lambda$  containing the leg  $x_k$ . The composition  $f \circ x_k: \Sigma^{\log} \rightarrow \mathbb{R}_{\geq 0}$  defines a piecewise-linear function on  $\Sigma^{\log}$  whose preimage over the open cone  $\mathbb{R}_{>0}$  consists of those tropical maps where  $x_k$  belongs to an internal component. This produces a section of the ghost sheaf on  $\overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H, d)$ , which in the usual way induces a line bundle and section  $(\mathcal{L}_k, s_k)$  on the moduli space, and the tropical description above shows that the zero locus of  $s_k$  is (set-theoretically) the locus where  $x_k$  belongs to an internal component.

We now calculate  $c_1(\mathcal{L}_k)$ . Choose a family of logarithmic stable maps over  $S$  and let  $\mu \in \Gamma(S, \overline{\mathcal{M}}_S)$  be the global section of the ghost sheaf constructed in the previous paragraph. This pulls back along  $\pi$  to give a global section  $\pi^b(\mu) \in \Gamma(C, \overline{\mathcal{M}}_C)$ . Interpreted as a piecewise-linear function on the tropicalisation  $\square$  with values in  $\overline{\mathcal{M}}_S$  [CCUW17, Remark 7.3], this assigns  $\mu$  to every vertex and has slope zero along every edge. By construction, the line bundle associated to this section is  $\pi^*\mathcal{L}_k$ . Consider on the other hand the generator  $1 \in \mathbb{N} = \Gamma(\mathbb{P}^N, \overline{\mathcal{M}}_{\mathbb{P}^N})$  with associated line bundle  $\mathcal{O}(H)$ . The section  $f^b(1) \in \Gamma(C, \overline{\mathcal{M}}_C)$  has associated line bundle  $f^*\mathcal{O}(H)$ . If we let  $v$  denote the vertex containing  $x_k$ , then by construction  $f^b(1)$  assigns  $\mu$  to  $v$  and has slope  $\alpha_k$  along the leg  $x_k$ . Thus if we consider the difference  $f^b(1) - \pi^b(\mu)$  then this assigns 0 to  $v$  and still has slope  $\alpha_k$  along  $x_k$ . Thus

by [RSW17a, Proposition 2.4.1] the corresponding line bundle restricted to  $C_v$  is given by

$$\mathcal{O}_{C_v} \left( \alpha_k x_k + \sum_e \mu_e x_e \right)$$

where the sum is over the edges  $e$  adjacent to  $v$  and distinct from  $x_k$ . Thus we see that:

$$(f^* \mathcal{O}(H) \otimes \pi^* \mathcal{L}_k^{-1})|_{C_v} = \mathcal{O}_{C_v} \left( \alpha_k x_k + \sum_e \mu_e x_e \right).$$

Since  $x_k$  factors through  $C_v$  we may pull back along  $x_k$  to obtain

$$\mathcal{L}_k = x_k^* \pi^* \mathcal{L}_k = x_k^* \mathcal{O}_{C_k}(-\alpha_k x_k) \otimes x_k^* f^* \mathcal{O}(H) = x_k^* \mathcal{O}_{C_k}(-\alpha_k x_k) \otimes \text{ev}_k^* \mathcal{O}(H)$$

and taking Chern classes gives:

$$c_1(\mathcal{L}_k) = \alpha_k \psi_k + \text{ev}_k^* H.$$

This gives the construction of  $(\mathcal{L}_k, s_k)$  on  $\overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H, d)$ ; the construction on  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$  is given by pull-back. The relevant piecewise-linear function is the composition:

$$\Sigma_{1,\alpha}(\mathbb{P}^N|H, d) \rightarrow \Sigma^{\log} \rightarrow \mathbb{R}_{\geq 0}.$$

Notice in particular that  $\psi_k$  should be interpreted as a **collapsed psi class** on  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ , i.e. a psi class coming from the stabilised curve of the collapsed map. In this paper, all psi classes are collapsed unless stated otherwise. We immediately arrive at:

**Theorem 3.1.** We have the following relation in the Chow ring of  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ :

$$(1) \quad (\alpha_k \psi_k + \text{ev}_k^* H) \cap [\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)] = \sum_{\mathcal{D}} \lambda_{\mathcal{D}}[\mathcal{D}].$$

The sum is over irreducible components  $\mathcal{D}$  of the divisor  $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d) \subseteq \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ , and  $\lambda_{\mathcal{D}}$  is the vanishing order of  $s_k$  along this component.

**Remark 3.2.** This construction gives the natural logarithmic analogue of Gathmann's line bundle and section [Gat02, Construction 2.1]. A benefit of the logarithmic approach is to make the computation of vanishing orders entirely combinatorial (see §4.2.4 below), circumventing the difficult technical calculation given by Gathmann.

#### 4. RECURSIVE DESCRIPTION OF BOUNDARY STRATA: THE RELATIVE SPLITTING AXIOM

In this section we provide an explicit description of the terms appearing on the right-hand side of (1) above. For each such term, we provide a combinatorial method to calculate the vanishing order  $\lambda_{\mathcal{D}}$ , and a recursive description of the stratum  $\mathcal{D}$  in terms of fibre products of moduli spaces with smaller numerical data. The form which this recursive description takes is non-obvious, in the sense that additional tautological classes appear, which must be calculated. This constitutes the most technically difficult part of the paper.

**4.1. Identifying the irreducible components of  $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$ .** Here we explain the strategy for identifying the irreducible components of  $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$  via tropical geometry.

**Lemma 4.1.** Every irreducible component of  $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$  is a codimension 1 logarithmic stratum.

*Proof.* This is not hard to see: the locus where the logarithmic structure is trivial coincides precisely with the locus where the source curve is smooth and not mapped inside  $H$ , and since  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$  is log smooth this locus is open and dense [Niz06]. By definition  $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$  is contained in the complement of this locus, which (by the very definition of logarithmic strata) can be written as a union of logarithmic strata of positive codimension. Since  $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$  has codimension 1, it must therefore be equal to a union of logarithmic divisors.  $\square$

In §2.2 we discussed a procedure for enumerating the logarithmic strata of  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ , using the cones of the tropicalisation  $\Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$ . To recap: every divisorial logarithmic stratum is obtained via a three-step process:

- (1) choose a combinatorial type  $\Delta$  of a tropical map;
- (2) subdivide the corresponding tropical moduli space  $\sigma$ ;
- (3) choose a ray in this subdivision.

Notice that this process contains some redundancies: upon choosing a ray in the tropical moduli space, some of the edge lengths or vertex positions may get set to 0. This induces a generisation of the initial combinatorial type  $\Delta$ , given by contracting the corresponding edges of the dual graph and moving the corresponding vertices from  $\mathbb{R}_{>0}$  to 0. By the combinatorial type of a stratum in  $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$  we will always mean this generisation. This is independent of the choice of  $\Delta$ , and in fact we can (and will) always choose  $\Delta$  to coincide with the generisation.

The logarithmic divisors contained in  $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$  are precisely those whose combinatorial types map the vertex of the dual graph containing  $x_k$  into the interior  $\mathbb{R}_{>0} \subseteq \mathbb{R}_{\geq 0}$ . Thus via the above procedure, we are able to enumerate the irreducible components of  $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$  in a combinatorial manner, and the combinatorial type  $\Delta$  allows us to describe the general element of such a component. What we are still lacking is a recursive description of the components, which we need in order to compute integrals over them. This is the subject of the remainder of this section.

**4.2. Recursive description of the divisors: types  $A, B$  and  $C^+$ .** Choose an irreducible component  $\mathcal{D} \subseteq \mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$ . By Lemma 4.1 this is a logarithmic divisor, and hence may be written as:

$$\mathcal{D} = \widetilde{\mathcal{D}} \cap \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$$

for a unique logarithmic stratum  $\widetilde{\mathcal{D}} \subseteq \widetilde{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N|H, d)$ . Since

$$\widetilde{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N|H, d) \rightarrow \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d)$$

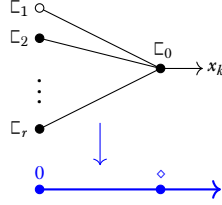
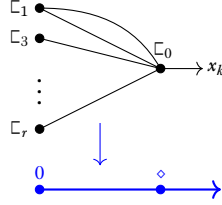
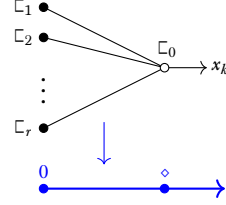
is a logarithmic modification,  $\widetilde{\mathcal{D}}$  is either exceptional or the proper transform of a logarithmic divisor. These cases correspond, respectively, to when the circuit is assigned zero

degree or nonzero degree by the combinatorial type. To begin with we will focus on the latter case. Suppose therefore that  $\tilde{\mathcal{D}}$  is the proper transform of a logarithmic divisor:

$$\mathcal{E} \subseteq \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d).$$

The birational map  $\tilde{\mathcal{D}} \rightarrow \mathcal{E}$  induces a morphism  $\mathcal{D} \rightarrow \mathcal{E}$ . In the remainder of this subsection, we will show how to interpret this morphism as a “desingularisation of the main component”. Since  $\mathcal{E}$  admits a recursive description in terms of relative and rubber moduli spaces, this will allow us to compute integrals over  $\mathcal{D}$ .

**Lemma 4.2.** Let  $\mathcal{D} \subseteq \mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$  be an irreducible component which contributes nontrivially to the recursion, and let  $\Delta$  be the corresponding combinatorial type. Suppose that  $\Delta$  assigns positive degree to the circuit. Then  $\Delta$  takes one of the following forms:

FIGURE 1.  $A$ FIGURE 2.  $B$ FIGURE 3.  $C^+$ 

The terminology is due to Vakil, and the pictures in this case are very similar to [Vak00] (we will see later that when the circuit is assigned zero degree the picture becomes *very* different). Note that in these pictures we have deliberately omitted the marked points (except for  $x_k$ ), the degree of each vertex, and the expansion factors of the edges. The omitted combinatorial data can be distributed arbitrarily, as long as:

- (1) the vertices  $\sqsubset_1, \dots, \sqsubset_r$  have positive degree;
- (2) the ellipse<sup>3</sup> has positive degree;
- (3) every vertex is stable;
- (4) the balancing condition is satisfied.

*Proof.* Let  $\sigma \in \Sigma$  be the cone corresponding to the stratum  $\mathcal{E} \subseteq \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d)$ . By the discussion above  $\mathcal{D}$  corresponds to a ray in the subdivision of  $\sigma$  given by imposing the central alignment condition. But since the ellipse is assigned positive degree, both radii are equal to 0 and the subdivision is trivial. We conclude that  $\sigma = \mathbb{R}_{\geq 0}$  (recall we are assuming that the initial and generised combinatorial types coincide). Since there must be at least one vertex mapped to higher level, we conclude that the tropical target  $\tilde{\mathbb{R}}_{\geq 0}$  is obtained from  $\mathbb{R}_{\geq 0}$  by subdividing at a single point  $\diamond \in \mathbb{R}_{>0}$ .

In order to have  $\sigma = \mathbb{R}_{\geq 0}$ , the dual graph  $\Gamma$  must be a bipartite graph with vertices over 0 and  $\diamond$ . The cases  $A, B$  and  $C^+$  enumerated above cover situations where there is a single vertex mapped to  $\diamond$ . We claim that if there is more than one such vertex, then the contribution to the recursion vanishes. To see this, we consider the stratum

$$\mathcal{E}^{\log} \subseteq \overline{\mathcal{M}}_{1,\alpha}^{\log}(\mathbb{P}^N|H, d)$$

<sup>3</sup>(Navid) Didn't we decide this was our new terminology for the circuit?



to which  $\mathcal{E}$  maps under the collapsing morphism, and examine the composition  $\mathcal{D} \rightarrow \mathcal{E} \rightarrow \mathcal{E}^{\log}$ . If we let  $\mathcal{F}^{\log}$  denote the intersection of  $\mathcal{E}^{\log}$  with the main component of the moduli space, then we obtain a factorisation  $\mathcal{D} \rightarrow \mathcal{F}^{\log} \hookrightarrow \mathcal{E}^{\log}$ . Since the moduli space is generically unobstructed along  $\mathcal{F}^{\log}$ , the codimension of  $\mathcal{F}^{\log}$  is given by the dimension of the associated cone in the tropicalisation  $\Sigma^{\log}$ . If there is more than one vertex mapped to higher level, then this cone has dimension  $\geq 2$ . Therefore the map  $\mathcal{D} \rightarrow \mathcal{F}^{\log}$  has positive-dimensional fibres, and since all insertions are pulled back from the latter space the contribution vanishes by the projection formula.  $\square$

**Remark 4.3.** The difference in dimensions between  $\mathcal{D}$  and  $\mathcal{F}^{\log}$  (or equivalently, the difference in virtual dimensions between  $\mathcal{E}$  and  $\mathcal{E}^{\log}$ ) may be interpreted as the difference in dimensions between moduli spaces of *disconnected* rubber and their images under the collapsing morphisms (see ?? below).

We now investigate the three types  $A, B, C^+$  separately, giving a recursive description of the boundary divisor in each case. For the remainder of this subsection, therefore, we fix a one-dimensional cone  $\tau \in \Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$ , let  $\mathcal{D} \subseteq \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$  be the associated logarithmic divisor, and assume that  $\mathcal{D}$  is contained in  $\mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$  and is of type  $A, B$  or  $C^+$ . We let  $\mathcal{E} \subseteq \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d)$  be the logarithmic stratum into which  $\mathcal{D}$  is mapped; this is indexed by a cone  $\epsilon \in \Sigma$  corresponding to a combinatorial type  $\Delta$ , and  $\epsilon$  is one-dimensional since we are restricting to the type  $A, B, C^+$  cases.

4.2.1. *Type A.* Suppose  $\Delta$  is of type  $A$ . Then  $\mathcal{E}$  admits a finite splitting morphism onto the fibre product:

$$\mathcal{E} \rightarrow \left( \overline{\mathcal{M}}_{1,\alpha^{(1)} \cup (m_1)}(\mathbb{P}^N|H, d_1) \times \prod_{i=2}^r \overline{\mathcal{M}}_{0,\alpha^{(i)} \cup (m_i)}(\mathbb{P}^N|H, d_i) \right) \times_{H^r} \overline{\mathcal{M}}_{0,\alpha^{(0)} \cup (-m_1, \dots, -m_r)}^{\leftrightarrow}(P|H_0+H_\infty, d_0).$$

**Lemma 4.4.**  $\mathcal{D}$  admits a natural splitting morphism

$$\mathcal{D} \xrightarrow{\rho} \left( \mathcal{VZ}_{1,\alpha^{(1)} \cup (m_1)}(\mathbb{P}^N|H, d_1) \times \prod_{i=2}^r \overline{\mathcal{M}}_{0,\alpha^{(i)} \cup (m_i)}(\mathbb{P}^N|H, d_i) \right) \times_{H^r} \overline{\mathcal{M}}_{0,\alpha^{(0)} \cup (-m_1, \dots, -m_r)}^{\leftrightarrow}(P|H_0+H_\infty, d_0)$$

such that the map  $\mathcal{D} \rightarrow \mathcal{E}$  covers the map on fibre products obtained by desingularising the main component of the factor corresponding to  $\square_1$ .

*Proof.* The choices involved in gluing log maps to expansions (which accounts for the degrees of the splitting morphisms above) is not affected by the introduction of alignments, so for the purposes of this proof we may pretend as if these splitting morphisms are the identity maps.

Note that the map  $\mathcal{D} \rightarrow \mathcal{E}$  is an isomorphism away from the blown-up locus, which is contained inside the locus where the circuit is contracted. Given an element of  $\mathcal{D}$  we can split it along the nodes  $q_1, \dots, q_r$  as in []. It is then clear that  $\square_1$  is aligned. We claim that  $\square$  satisfies the factorisation property if and only if  $\square_1$  does. This immediately implies the lemma.

On  $\square$  there are associated contraction radii  $\delta_1, \delta_2$ . Let  $\delta \in \{\delta_1, \delta_2\}$ . By Lemma 4.5 below, we have  $\lambda(\square') > \delta$  for any component  $\square'$  of  $\square_0$ . This means that  $\square_0, \square_2, \square_3, \dots, \square_r$  lie outside both contraction radii, and so the aligned curve  $\square$  satisfies the factorisation condition if and only if  $\square_1$  does.  $\square$

**Lemma 4.5.**  $\lambda(\square') > \delta$  for any component  $\square'$  of  $\square_0$  (with notation as in the proof above).

*Proof.* The basic point is that  $\mathcal{D}$  is obtained as the disjoint union of the locally closed logarithmic strata contained in the closure of the stratum where all of the  $\square_i$  are irreducible. If we look at one of these boundary strata, then by definition the associated cone  $\sigma \in \Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$  contains a ray  $\tau$  corresponding to the stratum where all the  $\square_i$  are irreducible; this amounts to setting all edge lengths other than  $e_1, \dots, e_r$  to zero. If  $f_1, \dots, f_l$  are some collection of additional edge lengths in  $\sigma$  (corresponding to internal nodes in degenerations of the  $\square_i$ ) then since  $\sigma$  is adjacent to  $\tau$  we must have (by construction of the subdivision)  $f_1 + \dots + f_l < e_j$  for all  $j \in \{1, \dots, r\}$ , since  $f_1 = \dots = f_l = 0$  and  $e_j \neq 0$  on  $\tau$ . In particular, if  $\square_1$  is degenerate and if  $\delta$  denotes the minimal distance from the circuit to a non-contracted vertex of  $\square_1$  (which certainly exists since  $\square_1$  has positive degree) then  $\delta < e_1 \leq \lambda(\square')$ , as claimed.  $\square$

Lemma 4.4 provides a means to calculate integrals over the class  $\lambda_{\mathcal{D}}[\mathcal{D}]$  appearing on the right-hand side of Theorem 3.1, provided that we can calculate  $\lambda_{\mathcal{D}}$  and the degree of the splitting morphism. Simple closed formulae for these are given in §4.2.4 below.

4.2.2. *Type B.* Now suppose  $\Delta$  is of type  $B$ . Note that in this case, it is impossible for the circuit to be contracted. Hence  $\mathcal{E}$  is disjoint from the blown-up locus, and the map  $\mathcal{D} \rightarrow \mathcal{E}$  is an isomorphism. We thus obtain the following description, entirely in terms of genus zero data:

**Lemma 4.6.**  $\mathcal{D}$  admits a finite splitting morphism:

$$\mathcal{D} \xrightarrow{\rho} \left( \overline{\mathcal{M}}_{0,\alpha^{(1)} \cup (m_1, m_2)}(\mathbb{P}^N|H, d_1) \times \prod_{i=3}^r \overline{\mathcal{M}}_{0,\alpha^{(i)} \cup (m_i)}(\mathbb{P}^N|H, d_i) \right) \times_{H^r} \overline{\mathcal{M}}_{0,\alpha^{(0)} \cup (-m_1, \dots, -m_r)}^{\leftrightarrow}(P|H_0 + H_{\infty}, d_0).$$

4.2.3. *Type  $C^+$ .* Finally suppose that  $\Delta$  is of type  $C^+$ . As before we have a finite morphism:

$$\mathcal{E} \rightarrow \left( \prod_{i=1}^r \overline{\mathcal{M}}_{0,\alpha^{(i)} \cup (m_i)}(\mathbb{P}^N|H, d_i) \right) \times_{H^r} \overline{\mathcal{M}}_{1,\alpha^{(0)} \cup (-m_1, \dots, -m_r)}^{\leftrightarrow}(P|H_0 + H_{\infty}, d_0).$$

The same arguments as in §4.2.1 then apply to give:

**Lemma 4.7.**  $\mathcal{D}$  admits a finite splitting morphism

$$\mathcal{D} \xrightarrow{\rho} \left( \prod_{i=1}^r \overline{\mathcal{M}}_{0,\alpha^{(i)} \cup (m_i)}(\mathbb{P}^N|H, d_i) \right) \times_{H^r} \mathcal{VZ}_{1,\alpha^{(0)} \cup (-m_1, \dots, -m_r)}^{\leftrightarrow}(P|H_0 + H_{\infty}, d_0)$$

such that  $\mathcal{D} \rightarrow \mathcal{E}$  corresponds to the obvious map on fibre products.

Here the first factor in the fibre product is the logarithmic blow-up of the moduli space of rubber maps constructed in §??. We will compute integrals over such spaces as part of the recursion (see §6).

4.2.4. *Splitting degree and vanishing order.* In each of the subsections 4.2.1–4.2.3 above, we obtained a finite splitting morphism  $\rho$  from  $\mathcal{D}$  to a fibre product of moduli spaces with smaller numerical data. Here we describe the degree of  $\rho$  and calculate the vanishing order  $\lambda_{\mathcal{D}}$  of the section  $s_k$  constructed in §3. This gives a complete recursive description of the terms appearing on the right-hand side of Theorem 3.1 which are of type  $A, B$  or  $C^+$ .

**Lemma 4.8.** The degree of  $\rho$  is given by:

$$\frac{\prod_{i=1}^r m_i}{\text{lcm}(m_1, \dots, m_r)}.$$

*Proof.* This calculation is by now standard, and is given by examining the base order of the corresponding family of targets. See, for instance [Che14a], [ACGS17].  $\square$

**Lemma 4.9.**  $\lambda_{\mathcal{D}} = \text{lcm}(m_1, \dots, m_r)$ .

*Proof.* Let  $\tau \in \Sigma_{1,\alpha}(\mathbb{P}^N|H, d)$  be the cone corresponding to  $\mathcal{D}$  and let

$$\varphi: \Sigma_{1,\alpha}(\mathbb{P}^N|H, d) \rightarrow \Sigma^{\log} \rightarrow \mathbb{R}_{\geq 0}$$

be the piecewise-linear function constructed in §3. It follows from the tropical description that  $\lambda_{\mathcal{D}}$  is equal to the index of the map of integral cones  $\tau \rightarrow \mathbb{R}_{\geq 0}$  obtained by restricting  $\varphi$ . By tropical continuity we have  $\tau \subseteq \mathbb{R}_{\geq 0}^r$  with integral generator:

$$w = \left( \frac{\text{lcm}(m_1, \dots, m_r)}{m_1}, \dots, \frac{\text{lcm}(m_1, \dots, m_r)}{m_r} \right).$$

The map  $\tau \rightarrow \mathbb{R}_{\geq 0}$  is given by projecting onto the  $i$ th factor and then multiplying by  $m_i$  (this is independent of  $i$  by continuity of the tropical map). Thus we conclude that the index is equal to  $\text{lcm}(m_1, \dots, m_r)$  as claimed.  $\square$

In summary, then: the contribution of each term  $\lambda_{\mathcal{D}}[\mathcal{D}]$  is given by integrating over the appropriate fibre product and then multiplying the result by  $\lambda_{\mathcal{D}} \cdot \deg \rho = \prod_{i=1}^r m_i$ .

#### 4.3. Recursive description of the divisors: type $C^0$ .

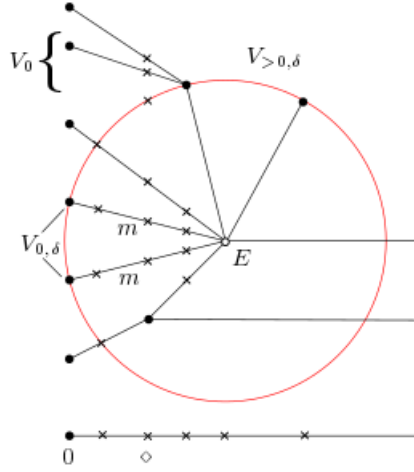
**4.3.1. Combinatorial types.** Recall that a boundary divisor  $\mathcal{D} \subseteq \mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$  is said to be of **type  $C^0$**  if the core is assigned zero degree by the combinatorial type. In this case, the combinatorial type is not as tightly constrained as previously (it needs not be a bipartite graph with a single interior vertex). Nevertheless, we have the following lemmas:

**Lemma 4.10.** Let  $\mathcal{D} \subseteq \mathcal{D}_{1,\alpha,k}(\mathbb{P}^N|H, d)$  be of type  $C^0$  and let  $\Delta$  be the corresponding combinatorial type. Then, for every vertex  $\diamond \neq 0$  of the expanded target, the fibre of the tropical map over  $\diamond$  contains exactly one stable vertex (together with a number of trivial bubbles), and at least one vertex which lies on the circle (which may or may not be stable).

*Proof.* By stability, we always need at least one stable vertex lying over each vertex of the target. If there was more than one, then (since both the insertions and the factorisation condition are pulled back from the moduli space of maps without expansions) it could be seen, as in the proof of Lemma 4.2, that the contribution would vanish.

Furthermore, for each vertex of the target there must always be a vertex in the preimage which lies on the circle, since otherwise there would be at least two independent tropical parameters, and so  $\mathcal{D}$  would not be a divisor.  $\square$

**Lemma 4.11.** If, for any nonzero vertex  $\diamond$  of the expanded target, the only stable vertex in  $\phi^{-1}(\diamond)$  is not the core, nor it lies on the circle, then the corresponding combinatorial type  $\Delta$  does not contribute to the invariants.

FIGURE 4. A ray of type  $C^0$ .

*Proof.* By the previous Lemma, there must be a vertex  $v$  lying on the circle and over  $\diamond$ ; such a vertex is strictly semistable by assumption, in particular the map  $f$  is constant along the corresponding component, therefore the fact that the factorisation condition is satisfied is unaffected by  $\mathbb{G}_m$ -scaling the coordinate corresponding to  $v$  in the moduli space of attachments. We deduce that the generic fiber of  $\mathcal{D} \rightarrow \mathcal{E}$  has positive dimension.  $\square$

**Lemma 4.12.** If the maximal contact order at the core is  $m = 1$ , the corresponding combinatorial type does not contribute to the invariants.

*Proof.* It follows once again from a dimensional argument.  $\square$

We observe that the cone  $\sigma \in \Sigma^{\text{Kim}}$  corresponding to the stratum  $\mathcal{E} \subseteq \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d)$  is isomorphic to  $\mathbb{N}^r$  where  $r$  is the number of target expansions. The discrepancy between the dimension of  $\sigma$  and the dimension of the ray  $\tau$  is to be expected, since the loci of type  $C^0$  are contained in the exceptional locus of the blow-up.

We write  $V_0$  for the collection of stable vertices of the combinatorial type which are mapped to 0 by the tropical map, and  $V_{>0}$  for the stable vertices which are mapped to  $\mathbb{R}_{>0}$ . For each vertex  $v$  there is associated discrete data  $\Gamma_v$ , which defines a moduli space of relative stable maps if  $v \in V_0$  and a moduli space of rubber maps if  $v \in V_{>0}$ . Then  $\mathcal{E}$  admits a finite splitting morphism onto the fibre product

$$\mathcal{E} \xrightarrow{\rho} \left( \prod_{v \in V_0} \overline{\mathcal{M}}_{\Gamma_v}(\mathbb{P}^N|H) \times \prod_{v \in V_{>0}} \overline{\mathcal{M}}_{\Gamma_v}^{\leftrightarrow}(P|H_0 + H_{\infty}) \right) \times_{H^{2E}} H^E$$

where  $E$  is the set of edges of the (stabilised) dual graph, and the fibre product is taken over the appropriate evaluation maps. Note that only the factor corresponding to the core vertex has non-zero genus, and that it has zero degree by assumption.

**Remark 4.13.** As in §4.2.4 above, we should calculate the degree of  $\rho$  and the vanishing order  $\lambda_{\mathcal{D}}$  of the section  $s_k$  along  $\mathcal{D}$ . The degree of  $\rho$  is given exactly as in Lemma 4.8,

where now we obtain one such factor for each bounded edge of the tropical target. The vanishing order  $\lambda_{\mathcal{D}}$  is slightly more delicate, but may be computed as the index of an explicit morphism of one-dimensional lattices, as explained in the proof of Lemma 4.9.

Typically,  $\mathcal{E}$  will have true dimension larger than that of  $\mathcal{D}$ . We will study the map  $\mathcal{D} \rightarrow \mathcal{E}$  in two steps: first, we will describe the map  $\widetilde{\mathcal{D}} \rightarrow \mathcal{E}$ , and then we will explain how to express the class of the substack  $\mathcal{D} \subseteq \widetilde{\mathcal{D}}$  in terms of tautological classes.

4.3.2. *From  $\mathcal{E}$  to  $\widetilde{\mathcal{D}}$ .* The reader should bear the following construction from toric geometry in mind: suppose we subdivide a fan, and we want to describe the relation between the varieties corresponding to a cone  $\rho$  of the subdivision, and to the original cone  $\sigma$  containing it, respectively. For any cone  $\tau$ , the fan of  $X_\tau$  is obtained by projecting the star of  $\tau$  to the quotient  $N_\tau = N_{\mathbb{R}}/\langle \tau \rangle_{\mathbb{R}}$  of the ambient vector space by the linear span of  $\tau$ .<sup>4</sup> Notice that the dimension of the vector spaces  $N_\rho$  and  $N_\sigma$  (or the corresponding toric varieties) is off by the difference  $\epsilon$  between the dimension of the cones. The image of  $\Sigma_\rho$  provides a subdivision of  $\Sigma_\sigma$ . The geometric content is that  $X_\rho$  is obtained as an appropriate compactification of a toric bundle of dimension  $\epsilon$  over  $X_\sigma$ .

We start by describing the modification of  $\mathcal{E}$  corresponding to the subdivision of the last paragraph; really, we only need to unravel the combinatorics partially, in order to describe an intermediate blowup  $\widetilde{\mathcal{E}} \rightarrow \mathcal{E}$ , such that  $\mathcal{D}$  factors through  $\widetilde{\mathcal{E}}$ , and all the structures that we need to name in what follows make sense on  $\widetilde{\mathcal{E}}$  already.

Let  $V_{0,\delta} \subseteq V_0$  be the subset of vertices lying on the circle (besides being sent to 0 by the tropical map). Notice that they all have the same expansion factor  $m$ , therefore they are at the same distance from the core, generically on  $\mathcal{E}$ .<sup>5</sup> Another way to see the generic alignment is the following: the normal bundles corresponding to these nodes can be expressed as  $\mathbb{T}_{R_v, q_v} \otimes \mathbb{T}_{E, q_v}$ ; for two different vertices, the right-hand factors are canonically identified thanks to the group structure on the elliptic curve, whereas the  $m$ -th powers of the left-hand factors are canonically identified by comparison with the target tangent line, and  $\rho$  can be seen as taking an  $m$ -th root of this preferred isomorphism. On the other hand, the alignment does not persist when the curves degenerate (see Figure 5).

We modify  $\mathcal{E}$ <sup>6</sup> along the boundary as follows:

- (1) radially aligning the core, and imposing fiberwise factorisation, thus replacing

$$\overline{\mathcal{M}}_{1,\mu}^{\leftarrow}(P|H_0 + H_\infty, d = 0) \quad \text{by} \quad \mathcal{VZ}_{1,\mu}^{\leftarrow}(P|H_0 + H_\infty, d = 0)$$

in the product above, thus introducing a reduced rubber with point target;

- (2) requiring that any degeneration of  $V_{0,\delta}$  (resp.  $\forall v \in V_{>0,\delta}$ ) remains centrally aligned with respect to the core;<sup>7</sup> as a consequence, on the whole of  $\widetilde{\mathcal{E}}$  - including the boundary -, for each  $v \in \{V_{0,\delta}\} \cup V_{>0,\delta}$ , it is always possible to tell the components of the curve that are at distance  $\delta$  from the core along the path denominated  $v$ , and, if there is more

<sup>4</sup>This can be encoded in the formalism of extended tropicalisations, see [ACP15] and references therein.

<sup>5</sup>Indeed, generically, these vertices do not produce any subdivision of the target, so they contribute to the relative dimension of  $\widetilde{\mathcal{D}} \rightarrow \mathcal{E}$  as one.

<sup>6</sup>With the logarithmic structure obtained by pulling back from Kim's space and modding out by  $\mathbb{N}^{\text{codim}(\sigma)}$ , i.e. considering the edges of  $E$  to have *infinite length*.

<sup>7</sup>Notice that this requires us to blow up some diagonal subloci of the product above, see Figure 5

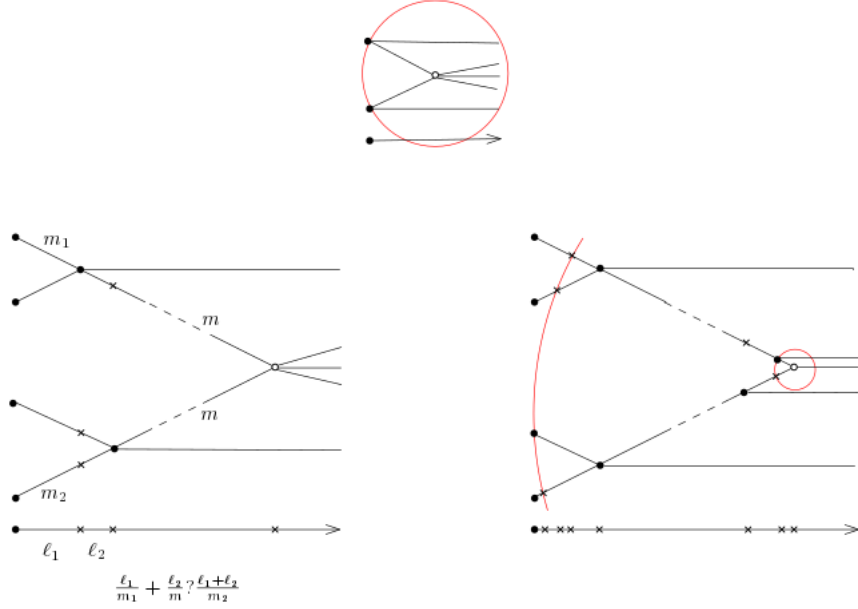


FIGURE 5. Two degenerations of the combinatorial type above, which represents a ray. On the left-hand side, the alignment holds only generically. On the right-hand side, it is not enough to align each individual factor.

than one for the same  $v$ , the lengths of these paths are identified, so as to define line bundles  $\mathcal{O}(\delta_v)$ ,  $\forall v \in \{V_{0,\delta}\} \cup V_{>0,\delta}$ .

It is clear that  $\tilde{\mathcal{D}} \rightarrow \tilde{\mathcal{E}}$  is generically a torus bundle of relative dimension  $\text{codim}(\sigma) - 1$ ; the open locus is represented by the moduli space of attachments for the elliptic singularities.

**Lemma 4.14.** Generically on  $\mathcal{E}$ ,  $\tilde{\mathcal{D}} \rightarrow \tilde{\mathcal{E}}$  is isomorphic to  $\mathcal{P} = \mathbb{P}_{\tilde{\mathcal{E}}}(\bigoplus_{v \in \{V_{0,\delta}\} \cup V_{>0,\delta}} \mathcal{O}(\delta_v))$ .

*Proof.* All of the line bundles  $\mathcal{O}(\delta_v)$  share a common factor  $\mathbb{T}_E$ , the universal tangent line to the elliptic curve (appropriately twisted by bubbling loci as in [VZ08, §2.3]); removing this, when the curve has the prescribed dual graph,  $\mathcal{O}(\delta_v)$  is isomorphic to the tangent line to the rational tail  $R_v$  at the gluing node  $q_v$  (with the caveat that all those corresponding to  $V_{0,\delta}$  are canonically identified). This is the open stratum of the moduli space of attaching data [Smy11b, §§2.2-2.3]. On the fiber boundary, the circle of radius  $\delta$  will pass through a subset of  $\{V_{0,\delta}\} \cup V_{>0,\delta}$ , while cutting the complementary subset along the edge between the core and  $v$ ; correspondingly, there is a destabilisation of the curve, and an elliptic singularity containing strictly semistable branches. Tropically, pushing some of the vertices off the circle creates new finite-length edges with coordinates  $\ell_v \in \mathbb{R}_{\geq 0}$  (see Figure 6); not all of them can be strictly positive, because the circle needs to pass through at least one vertex (of positive horizontal degree). This is patently the fan of a projective space.  $\square$

Notice that  $\mathcal{P}$  is well-defined on the whole of  $\tilde{\mathcal{E}}$ , while  $\tilde{\mathcal{D}} \rightarrow \mathcal{P}$  is in fact a modification along the boundary, as dictated by the alignment procedure. In the next section, we extrapolate what we need from this combinatorial setup in order to describe  $\mathcal{D}$  explicitly.



FIGURE 6. Moving vertices off the circle creates new tropical parameters.

4.3.3. *From  $\tilde{\mathcal{D}}$  to  $\mathcal{D}$ .* Let  $F = \bigoplus_{v \in \{V_{0,\delta}\} \cup V_{>0,\delta}} \mathcal{O}(\delta_v)$ . On  $\mathcal{P}$  we have a tautological sequence:

$$\mathcal{O}_F(-1) \hookrightarrow p^*(F) \xrightarrow{+df} \text{ev}_E^*(TH),$$

where  $p: \mathcal{P} = \mathbb{P}(F) \rightarrow \tilde{\mathcal{E}}$ , and  $+df$  should be interpreted as follows: for every  $v \in \{V_{0,\delta}\} \cup V_{>0,\delta}$ , as many copies of  $\mathcal{O}(\delta_v)$  are considered as there are components at distance  $\delta$  from the core along the path  $v$ , and they are identified according to the preferred isomorphisms; then, the differential of the universal map to  $\mathbb{P}^N$  is applied to each tangent vector (possibly including some contracted rational bubbles on the path), and the sum of these contributions is taken in  $T_{f(E)}\mathbb{P}^N$  - it actually lies in  $T_{f(E)}H$  because all the normal contributions can be assumed to be trivial by Lemma 4.12. The composite, regarded as a section  $s$  of  $G = \mathcal{O}_F(1) \otimes \text{ev}_E^*(TH)$ , vanishes precisely where  $f$  factors through the Smyth's singularity determined by the point in the fiber of  $p$ . Generically on  $\tilde{\mathcal{E}}$ , this is exactly what we are after. On the boundary, though, it may happen that a simultaneous degeneration in the base and the fiber makes this factorisation trivial, in the sense that  $\tilde{f}$  would be constant on all the branches of the elliptic singularity.

**Lemma 4.15.** For every  $V_B \subseteq V_\delta$  and  $V_F = V_\delta \setminus V_B$ ,  $s$  vanishes along the loci where the factors corresponding to  $V_B$  in  $\tilde{\mathcal{E}}$  degenerate so that  $f_v: R_v \rightarrow \mathbb{P}^N$  is constant in a neighbourhood of  $q_v$ , and the fiber coordinates  $(\lambda_{v'})_{v' \in V_F}$  along  $p: \mathcal{P} \rightarrow \tilde{\mathcal{E}}$  are set to be 0.

Call these loci  $\Xi_i$ ,  $i \in I$  (compare with [VZ08, §3.2] for an analogous phenomenon in the non-relative case); it is clear that they may be in excess with respect to the expected codimension of  $V(s)$ , that is  $N - 1$ . These loci are logarithmic substrata; therefore, there exists a logarithmic blow-up that principalizes them. We claim that  $\tilde{\mathcal{D}}$  is (possibly a refinement of) such a blow-up.

**Lemma 4.16.** The pullback of  $\Xi_i$  is principal in  $\tilde{\mathcal{D}}$ .



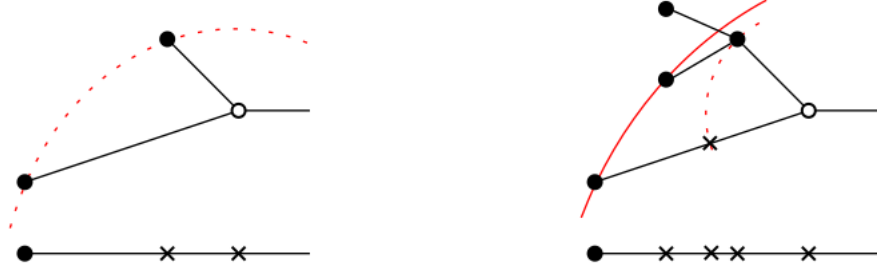


FIGURE 7. On the left, a generic point of  $\mathcal{P}$ . On the right, a point of  $\Xi$ . The dotted circle is the one we draw on  $\mathcal{P}$ . The continuous circle represents one of the possible choices of  $\delta$  enlarging the dotted one (a stratum of  $\tilde{\mathcal{D}}$ ).

*Proof.* Let us analyse the ideal of  $\Xi_i$ : it is generated by

- (1) equations on  $\tilde{\mathcal{E}}$  cutting the locus where  $R_v$  degenerates so that  $f_v$  is constant on a neighbourhood of  $q_v$ ,  $v \in V_B$ : these correspond to the tropical functions  $\delta_v - \text{dist}(q_v, \circ)$ ;
- (2) equations on the fiber of  $p$  for the linear subspace corresponding to  $v \in V_F$ : tropically, these correspond to the  $\ell_v$  discussed above.

By construction, on  $\tilde{\mathcal{D}}$  it is always possible to tell which of these tropical lengths is the shortest (the circle we have drawn on  $\Xi$  is degenerate - it passes through no vertex of positive horizontal degree -, therefore we need to decide for a strictly larger circle in order to lift to  $\tilde{\mathcal{D}}$ ), which corresponds to principalising the ideal (see Figure 7).  $\square$

Finally, on  $\tilde{\mathcal{D}}$  (or, in fact, on an intermediate blow-up  $\tilde{\mathcal{P}} \rightarrow \mathcal{P}$ ) there are exceptional divisors  $\tilde{\Xi}_i$  corresponding to  $\Xi_i$  in  $\mathcal{P}$ . Since the  $\Xi_i$  exhaust all the excess-dimensional components of the vanishing locus of  $s \in \Gamma(\mathcal{P}, G)$ , the latter induces a section  $\tilde{s} \in \Gamma(\tilde{\mathcal{D}}, G(-\sum_i \tilde{\Xi}_i))$  by pullback and twisting, whose vanishing locus is dimensionally transverse to the boundary, and whose open part coincides with the generic point of  $\mathcal{D}$ , therefore  $V(\tilde{s}) = \mathcal{D}$ .

Notice that, expressing the pushforward of the Chern classes of  $\mathcal{O}_F(1)$  and  $\mathcal{O}(\tilde{\Xi}_i)$  in terms of tautological classes on the fiber product underlying  $\mathcal{E}$ , we get:

**Corollary 4.17.** After pushforward to  $\mathcal{E}$ , the contribution of  $\mathcal{D}$  to the invariants can be computed in terms of tautological integrals on moduli spaces of maps with lower numerics.

**Lemma 4.18.** The logarithmic modification  $\psi$  restricts to a birational map:

$$\mathcal{VZ}_{1,\alpha^{(0)} \cup (-m_1, \dots, -m_r)}^{\text{punct}}(H, d) \rightarrow \overline{\mathcal{M}}_{1,\alpha^{(0)} \cup (-m_1, \dots, -m_r)}^{\text{punct}, \circ}(H, d).$$

Note that the latter space (the main component of the moduli space of punctured maps) is birational to the main component of the double ramification locus. In Lemma 4.19 below we explain how to compute integrals over this cd

*Proof.* If it is true, we might be able to prove it via deformation theory. For this it might be useful to notice that there is a morphism  $\overline{\mathcal{M}}_{g,\alpha}^{\text{punct}}(H, d) \rightarrow \overline{\mathcal{M}}_{g,\alpha}^{\text{punct}}(\mathbb{P}^N, d)$  (because  $H \subseteq \mathbb{P}^N$  is strict), which is probably a closed immersion, and the loci with smooth source curve are isomorphic under this map. It seems plausible that the main component is log smooth over  $B\mathbb{G}_m \subseteq [\mathbb{A}^1/\mathbb{G}_m]$  (or the standard logarithmic point).  $\square$

Integrals over the main component of the double ramification locus can be computed using the following lemma.

**Lemma 4.19.** The morphism which forgets a marking

$$\text{fgt}_i: \overline{\mathcal{M}}_{1,n}^{\alpha, \circ}(H, d) \rightarrow \overline{\mathcal{M}}_{1,n-1}^{\circ}(H, d)$$

is generically finite, of degree  $\alpha_i^2$  (except in one special case, described in the proof).

*Proof.* Let us consider the case  $d = 0$  first. We may assume the source curve is smooth elliptic  $E$ . The map

$$\phi: E \rightarrow \text{Pic}^0(E), \quad x \mapsto \mathcal{O}_E \left( \alpha_i x + \sum_{j=1, \dots, i, \dots, n} \alpha_j p_j \right)$$

is an isogeny of degree  $\alpha_i^2$ . The locus of  $(C, p_1, \dots, \hat{p}_i, \dots, p_n)$  such that the kernel of  $\phi$  contains one of the points  $\{p_1, \dots, \hat{p}_i, \dots, p_n\}$  is itself a double ramification locus inside  $\overline{\mathcal{M}}_{1,n}$ , hence non-generic - with one exception: namely, when  $n = 2$ ,  $x = p_1$  is always a solution, but the curve lying above such point bubbles off a  $\mathbb{P}^1$ . To see that it does not belong to the closure of the nice locus, notice that the rational function trivialising  $\mathcal{O}_C(\alpha_1 p_1 - \alpha_2 p_2)$  should descend to the cusp, thus having a ramification point at the node; yet its ramification profile is determined by Riemann-Hurwitz, and it is entirely supported on  $p_1$  and  $p_2$ .

For a different proof: notice that we should obtain the class of the main component from the full double ramification cycle by subtracting the boundary class  $[D_{1, \emptyset | 0, \{1, \dots, n\}}]$ . The latter pushes forward to 0 under  $\text{fgt}_i$ , unless  $n = 2$ . Therefore we may apply  $\text{fgt}_{i,*}$  to the Hain-Pixton formula:

$$DR_1(A) = \frac{1}{2} \left( \sum_{i=1}^n a_i^2 \psi_i - \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I| \geq 2}} a_I^2 [D_{1, I^c | 0, I}] - \frac{1}{12} \delta_0 \right),$$

where  $a_I = \sum_{i \in I} a_i$  and  $\delta_0 = \text{glue}_*([\overline{\mathcal{M}}_{0,n+2}])$ . From  $\psi_j = \text{fgt}_i^* \psi_j + [D_{i,j}]$  for  $i \neq j$ , and the dilaton equation, we see that  $\text{fgt}_{i,*} \psi_j = 1$  and  $\text{fgt}_{i,*} \psi_i = n - 1$ . On the other hand, the only surviving boundary classes are  $[D_{i,j}]$ , and they push down to 1. Hence the formula pushes down to  $\frac{1}{2} \left( \sum_{j \neq i} \alpha_j^2 + (n - 1) \alpha_i^2 - \sum_{j \neq i} (\alpha_j + \alpha_i)^2 \right) = -\alpha_i \left( \sum_{j \neq i} \alpha_j \right) = \alpha_i^2$ .  $\square$

**Definition 4.20.** A combinatorial type  $\Delta$  of a tropical curve consists of the following data:

- (1)  $G$  a finite graph, with a set  $V(G)$  of vertices, a set  $F(G)$  of flags and an involution  $\iota: F(G) \rightarrow F(G)$  with  $\iota^2 = \text{Id}$ . We let  $L(G)$  denote the fixed set of  $\iota$ , and think of its elements as infinite legs; we let  $E(G)$  denote the set of equivalence classes of elements not fixed by  $\iota$ , and think of them as finite edges connecting two (possibly equal) vertices of  $G$ ;
- (2) a genus assignment  $g: V(G) \rightarrow \mathbb{Z}_{\geq 0}$ ;
- (3) a degree assignment  $\deg: V(G) \rightarrow \mathbb{Z}_{\geq 0}$ ;
- (4) a weight assignment  $u: F(G) \rightarrow \mathbb{Z}$  such that for every edge  $e \in E(G)$  we have  $u(f_1) = -u(f_2)$  where  $e = [f_1] = [f_2]$ , and  $u(l) \geq 0$  for any  $l \in L(G)$ ;

<sup>8</sup> Review here the process of gluing for punctured maps. In particular the basic monoid should be modified in order to make the relevant evaluations log morphisms. Claim: evaluations are strict. Consequence: the fiber product in the category of log stacks is fine. We have then to apply saturation. This is a finite morphism of degree... (I think it could be  $\prod \frac{m^{(i)}}{lc(m^{(i)})}$ ).

**Lemma 4.21** (Virtual pushforward). The following hold.

- $\text{fgt}_*[\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H, d)] = [\overline{\mathcal{M}}_{0,\alpha}^G(\mathbb{P}^N|H, d)]$  (follows from [Gat03, AMW14]).
- $\text{fgt}_*[\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)]$  computes the reduced relative invariants by definition.
- $\text{fgt}_*[\overline{\mathcal{M}}_{0,\mu}(H, d_0)^\sim] = [\overline{\mathcal{M}}_{0,|\mu|}(H, d_0)]$  (follows from [Gat03] and... comparison of punctured with rubber invariants).
- $\text{fgt}_*[\mathcal{VZ}_{1,\mu}(H, d_0)^\sim]$  here we should need a variation on Pixton's DRC formula; hopefully it's enough to avoid the graphs that tropical well-spacedness discards.

**4.4. Recursive description of the divisors: type  $C_0$ .** Consider now a combinatorial type  $\Delta$  such that the circuit is contracted into the divisor. The corresponding stratum  $\mathcal{D}_{\mathcal{M}} \subseteq \mathcal{M}$  is contained in the blown-up locus. Our task is to identify the logarithmic divisors in the preimage of  $\mathcal{D}_{\mathcal{M}}$  under the map  $\mathcal{VZ} \rightarrow \mathcal{M}$ . As discussed, these are indexed by choices of ray in the subdivision of the tropical moduli space. Suppose we have fixed such data.

**Claim 1.** The teeth of the comb can break at most once, along the circuit.

*Proof.* If not there would be more parameters in the tropical moduli space.  $\square$

**Proposition 4.22.** Let  $\Delta$  be a combinatorial type as above. Then:

- (1) the circuit is a single vertex, and is mapped to the interior  $\mathbb{R}_{>0} \subseteq \mathbb{R}_{\geq 0}$ ;
- (2) the degree of the circuit is zero;
- (3) there is a vertex which is adjacent to the circuit and mapped to the interior  $0 \in \mathbb{R}_{\geq 0}$ ;
- (4) let  $e \in E(\square)$  be an edge which is adjacent to the circuit, and such that  $u(e)$  is maximal among the edges adjacent to the circuit. Then the other vertex  $v$  contained in  $e$  is mapped to  $0 \in \mathbb{R}_{\geq 0}$ , and  $v$  lies on the (absolute) radius;
- (5) if  $v \in V(\square)$  and  $\lambda(v) > \delta$  then  $f(v) = 0$ .

On the other hand, when there is a contracted elliptic subcurve - and it will be contracted into the hyperplane, because otherwise it wouldn't be generic, by density of the nice locus

<sup>8</sup>(Luca) to be made homogeneous with what comes earlier

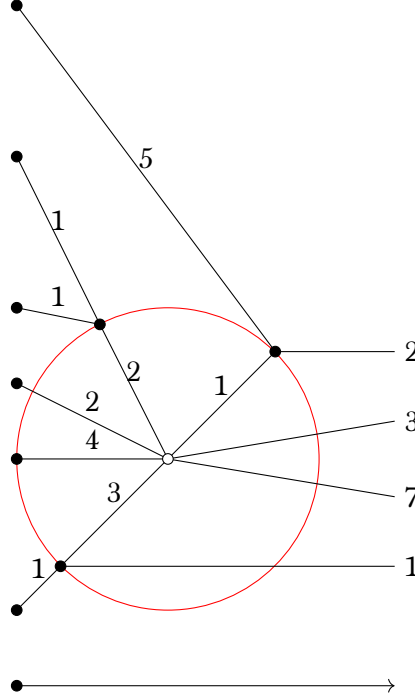


FIGURE 8. The degrees can be figured out from the balancing equation.

in  $\mathcal{VZ}_{1,\alpha'}(\mathbb{P}^N|H, d')$  - the picture becomes more complicated due to the alignment. The combs may break. We label these loci  $\mathcal{Y}_C^0$ .

In the following we describe the rays of the tropical moduli space.

**Lemma 4.23.** A one-parameter tropical map  $\phi$  to  $\mathbb{R}_{\geq 0}$  is a decorated tree (with expansion factors - *contact orders* - along edges and legs, and degrees on vertices, satisfying the balancing condition) with a circle (of radius  $\delta$ ) around the root (sometimes called the *core* and denoted by  $\circ$ ) satisfying:

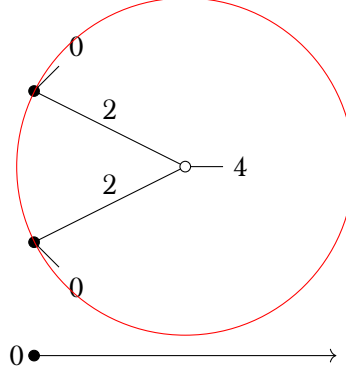
- (1) the circle of radius  $\delta$  passes through at least one vertex of  $\phi^{-1}(0)$  - which necessarily has positive degree - call  $m$  its contact order with  $H$ ;
- (2) teeth may break only when they intersect the circle of radius  $\delta$ ; in particular,  $\circ$  is the only vertex contained in its strict interior, and every edge heading out from the circle goes directly to a vertex of  $\phi^{-1}(0)$ ;
- (3) every tooth that starts with contact order  $m$  goes directly to a vertex of  $\text{circle}(\circ, \delta) \cap \phi^{-1}(0)$ , and every other tooth starts with contact order  $< m$  (possibly negative).

*Proof.* Otherwise there would be more than one parameters.  $\square$

**Remark 4.24.** The core being contracted in the fiber of the tropical map is not a phenomenon that we should worry about in codimension one. Indeed, assume that the core is contracted in the fiber along a ray. Then all the edges departing from the core have expansion factor 0; call the corresponding coordinates  $U = \{u_i\}_{i \in I}$ . Call the remaining coordinates  $E = \{e_j\}_{j \in J}$ . Note that tropical continuity involves only  $E$ . Alignments on the other hand assume the form  $\lambda(v) = \lambda(v')$ , where  $\lambda(v) = \sum_{i \in I(v)} u_i \sum_{j \in J(v)} e_j$ . Pick the

shortest elements of  $U$ ; then these can be shortened to zero without affecting the rest (by hypothesis, alignments can only identify them among themselves). This shows that we could not have started with a ray.

**Example 4.25.** We look at the following example in some detail. The ambient space is



$\mathcal{VZ}_{1,(4,0,0)}(\mathbb{P}^N|H, 4)$ , of dimension  $4N+3$ . The underlying moduli space is  $X = \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H, 2) \times_H \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H, 2) \times \mathcal{VZ}_{1,(-2,-2,4)}$  of dimension  $5N+1$ . Consider the fiber product:

$$\begin{array}{ccc} F & \longrightarrow & \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H, 2) \times \overline{\mathcal{M}}_{0,(2,0)}(\mathbb{P}^N|H, 2) \\ \downarrow & & \downarrow \\ H & \longrightarrow & H \times H \end{array}$$

At the level of ghost sheaves,  $\overline{\mathcal{M}}_F = \mathbb{N} \oplus_{\mathbb{N}^2} \overline{\mathcal{M}}_1^{\text{enl}} \oplus \overline{\mathcal{M}}_2^{\text{enl}}$ , where the map  $\mathbb{N}^2 \rightarrow \mathbb{N}$  is the sum, and the map  $\mathbb{N}^2 \rightarrow \overline{\mathcal{M}}_1^{\text{enl}} \oplus \overline{\mathcal{M}}_2^{\text{enl}}$  generically is multiplication by 2, so  $\overline{\mathcal{M}}_F = \mathbb{N}^2/(2e = 2f)$  generically. Saturation gives a finite cover  $G \rightarrow F$  with  $\overline{\mathcal{M}}_G = \mathbb{N}_{e=f}$  generically. Lifting this to actual log structures, what we are doing (again generically) is taking a square root of the isomorphism  $T_{R_1, q_1}^{\otimes 2} \simeq T_{R_2, q_2}^{\otimes 2}$ , which is obtained passing through  $N_{H/\mathbb{P}^N, f(Z)}$  via  $d f|_{R_i, q_i}$ . This breaks when  $f|_{R_i}$  is not tangent to  $H$  of order exactly 2 at  $q_i$ , for either  $i$ ; but by the maximality assumption this happens precisely along Gathmann's comb loci  $\Delta_i$ . So in fact, rather than with  $T_{R_i, q_i}^{\otimes 2}$ , we should be working with  $T_{R_i, q_i}^{\otimes 2}(-\Delta_i)$ ; but this is exactly  $\text{ev}_i^*(-H)$  by Gathmann's genus zero formula, and the isomorphism  $\text{ev}_1^*(H) = \text{ev}_2^*(H)$  holds on all of  $F$ .

On the other hand, generically on  $\mathcal{VZ}_{1,(-2,-2,4)}$  we have  $T_{q_1}Z \simeq T_{q_2}Z$  by exploiting the group structure on the elliptic curve. This breaks when either (but not both) is on a rational tail. Yet we have  $T_{q_1}Z(\Delta_{1 \in P}) \simeq T_{q_2}Z(\Delta_{2 \in P})$  by Vakil-Zinger's construction of a universal  $\psi$ -class (i.e. by comparing both with  $\pi_*\omega(\Delta)$ ; notice that our further blow-up has the only effect of twisting *all* the relevant line bundles by  $\Delta_{1,2 \notin P}$ ).

Now, the fiber of the Vakil-Zinger blow-up over  $X$  can be described as follows. Generically it looks like

$$\mathbb{P}(T_{q_1}R_1 \otimes T_{q_1}Z \oplus T_{q_2}R_2 \otimes T_{q_2}Z)$$

but this has to be modified along the boundary:

- this has to do with the fact that the normal bundle of the strict transform is the pullback of the normal bundle twisted by the intersection with the exceptional divisor (so it relates with previous steps of the blow-up);
- it is not globally a  $\mathbb{P}^1$ -bundle (so it relates with further stages of the blow-up; it also has to do with a choice of compactification for the moduli space of attachments);
- it has the effect of replacing  $T_{q_i}Z$  with Vakil-Zinger's universal  $\mathbb{T}$ , so that this can be factored out of the projective bundle, and in fact we are left with a projective bundle  $\mathbb{P} = \mathbb{P}(T_{q_1}R_1 \oplus T_{q_2}R_2)$  over  $F$ , and its open part  $\text{Iso}(T_{q_1}R_1 \oplus T_{q_2}R_2)$  represents the attachment data for a contraction to a tacnode  $R_1 \sqcup_q R_2 \rightarrow \bar{C}$ .

On  $\mathbb{P}$  there is a natural vector bundle map

$$s: \mathcal{O}_{\mathbb{P}}(-1) \hookrightarrow p^*(T_{q_1}R_1 \oplus T_{q_2}R_2) \xrightarrow{+df} \text{ev}_q^* T\mathbb{P}^N$$

that vanishes along the locus where  $f$  descends to  $\bar{C}$ . In general, it is not a transversal section:

- we should replace  $T\mathbb{P}^N$  by  $TH$  as long as all the  $m^{(i)}$  are  $\geq 2$ ;
- Vakil and Zinger construct a blow-up of  $\mathbb{P}$  along the vanishing loci of  $s$  of low codimension, and twist  $s$  by the exceptional divisors, so that it becomes a transverse section  $\tilde{s}$ .

On the other hand, the finite cover  $G \rightarrow F$  factors through  $\mathbb{P}$ , because the two vertices are already aligned on  $G$ . We claim that the boundary locus of  $\mathcal{VZ}_{1,(4,0,0)}(\mathbb{P}^N|H, 4)$  corresponding to the combinatorial type of the tropical map above is the transverse intersection

$$(G \cap V(\tilde{s}) \subseteq \mathbb{P}) \times \mathcal{VZ}_{1,(-2,-2,4)}.$$

This has the expected dimension (codimension  $N - 1$  with respect to  $X$ ). To compute its class, we can pull  $\mathbb{P}$  back to  $G$ , and then notice that  $G \hookrightarrow \mathbb{P}_G$  is the inclusion of a (trivial) subbundle.

**Lemma 4.26.** The class of  $\mathbb{P}(\mathcal{F}) \subseteq \mathbb{P}(\mathcal{E})$  is  $c_{\text{top}}(\mathcal{O}_{\mathcal{E}}(1) \otimes p^*(\mathcal{E}/\mathcal{F}))$ .

See [EH16, Prop. 9.13]. It is a good time to remember that  $\mathcal{E}$  was in fact  $(\bigoplus_{i=1}^r TR_{i,q_i}) \otimes \mathbb{T}$ . By writing  $c$  for  $c_1(\mathcal{O}_{\mathcal{E}}(1))$ ,  $\psi_i$  for  $c_1(T^*R_{i,q_i})$ ,  $\psi_Z$  for Vakil-Zinger's universal psi class, and  $H$  for  $\text{ev}_q^* H$ , we need to compute

$$\begin{aligned} p_* \left( (c - \psi_1 - \psi_2 - 2\psi_Z) [(1 + c + H)^N (1 + c)^{-1}]_{N-1} \right) &= \\ p_* \left( (c - \psi_1 - \psi_2 - 2\psi_Z) \left( \sum_{k=0}^{N-1} \binom{N}{1+k} c^k H^{N-1-k} \right) \right) &= \\ \sum_{k=0}^{N-1} \binom{N}{k} H^{N-1-k} (s_k(\mathcal{E}) - s_{k-1}(\mathcal{E})(\psi_1 + \psi_2 + \psi_Z)) & \end{aligned}$$

We now generalise this picture. Recall that the map  $C \rightarrow \bar{C}$  is given by a (generic) line in the sum of the tangent spaces to the rational tails at the nodes that join them to the contracted curve of genus one. This is equivalent to an alignment, and it is parametrised by an open subset of a projective bundle over the moduli space for the tails corresponding to vertices of the dual graph lying on the circle of radius  $\delta$ . Yet, notice that those vertices

lying in  $\phi^{-1}(0)$  are already aligned among themselves. This is why we find it convenient to distinguish among four groups of vertices:

- (1) the core;
- (2) vertices on  $\phi^{-1}(0) \cap \text{circle}(\circ, \delta)$ ;
- (3) vertices on  $\text{circle}(\circ, \delta) \setminus \phi^{-1}(0)$ ;
- (4) vertices on  $\phi^{-1}(0) \setminus \text{circle}(\circ, \delta)$ .

We shall first argue that gluing of log maps can be performed separately for the exterior and interior of the circle (the analogous classical picture is that, since the core is contracted, this moduli space is the product of a genus one curve, and a fiber product of genus zero maps under evaluation morphisms).

**Lemma 4.27.**  $\mathcal{VZ}_{1, \alpha^{(0)} \cup (-m_1, \dots, -m_r)}^{\text{punct}}(H, 0) \simeq \mathcal{VZ}_{1, \alpha^{(0)} \cup (-m_1, \dots, -m_r)}^{\text{punct}}(\text{Spec}(k \oplus \mathbb{N})) \times H$ .<sup>9</sup>

Let us now deal with vertices of type 2.

**Lemma 4.28.** Consider the following fiber product in the category of fs log stacks:

$$\begin{array}{ccc} F & \longrightarrow & \prod_{i=1}^r \overline{\mathcal{M}}_{0, \alpha^{(i)} \cup \{m\}}(\mathbb{P}^N | H, d_i) \\ \downarrow & & \downarrow \text{ev}_{q_i} \\ H & \longrightarrow & H^r \end{array}$$

On  $F$  there is a canonical isomorphism  $\mathbb{L}_{q_i} \cong \mathbb{L}_{q_j}$ <sup>10</sup>, the latter being the cotangent line bundles at the gluing markings on two different components  $i$  and  $j$ . Furthermore,  $\underline{F}$  is a finite cover of degree  $m^{r-1}$  of the fiber product of the underlying stacks.

*Proof.* Recall that each  $\overline{\mathcal{M}}_{0, \alpha^{(i)} \cup \{m\}}(\mathbb{P}^N | H, d_i)$  is endowed with the log structure induced by pulling back along  $q_i$  the divisorial log structure of the universal curve at the image of  $q_i$  itself. Thus  $\text{ev}_{q_i}$  is made into a log morphism to  $H$  with its induced DF(1) log structure. We claim that the subtext of such morphism is Gathmann's formula; namely, the log morphism to  $H$  corresponds to an isomorphism between  $\text{ev}_{q_i}^* \mathcal{O}_H(-H)$  on one side, and  $\mathcal{I}_{D_i} \otimes \mathbb{L}_{q_i}^m$  on the other, where  $\mathcal{I}_{D_i}$  is the ideal sheaf of the union of the comb loci  $D_i$  in  $\overline{\mathcal{M}}_{0, \alpha^{(i)} \cup \{m\}}(\mathbb{P}^N | H, d_i)$ .

On the fiber product there is a canonical isomorphism between  $\text{ev}_{q_i}^* H$  and  $\text{ev}_{q_j}^* H$ .

Say something about integrality.

Saturation is a local operation, as much as computing the degree, hence we can concentrate on the dense open locus where all the curves we are gluing are smooth, and they are not mapped entirely into  $H$ . There the minimal log structure on  $\overline{\mathcal{M}}_{0, \alpha^{(i)} \cup \{m\}}(\mathbb{P}^N | H, d_i)$  is trivial, therefore the isomorphism between  $\text{ev}_{q_i}^* H$  and  $\text{ev}_{q_j}^* H$  translates into an isomorphism  $\mathbb{L}_{q_i}^m \cong \mathbb{L}_{q_j}^m$ . The saturation  $F$  is obtained by taking an  $m$ -th root of this isomorphism.  $\square$

Let us denote by  $\mathbb{L}_F$  the universal cotangent line at  $q$ .

<sup>9</sup>(Luca) check log structure, could be fibered over std log point

<sup>10</sup>(Luca) check how it degenerates



The projective bundle we are seeking has base

$$\mathcal{X} = \left( F \times \prod_{i=1}^s \overline{\mathcal{M}}_{0, \tilde{\alpha}^{(i)} \cup \{m_i\}}^{\text{punct}}(H, \tilde{d}_i) \right) \times_{H^{s+1}} H$$

(with  $m_i < m$ ) and it is

$$\mathbb{P} = \text{Proj}_{\mathcal{X}} \left( \mathbb{T}_F \oplus \bigoplus_{i=1}^s \mathbb{T}_{\tilde{q}_i} \right).$$

We are interested in the vanishing locus of the section

$$s: \mathcal{O}_{\mathbb{P}}(-1) \hookrightarrow p^* \left( \mathbb{T}_F \oplus \bigoplus_{i=1}^s \mathbb{T}_{\tilde{q}_i} \right) \xrightarrow{+df_q} f^*(T \mathbb{P}^N)_q,$$

because it represents the geometric condition that  $f: C \rightarrow \mathbb{P}^N$  factors through the normalisation map  $C \rightarrow \bar{C}$  prescribed by the given point of  $\mathbb{P}$ . As is,  $s$  is not transverse to the zero section. First of all, unless  $m = 1$ ,  $T \mathbb{P}^N$  can be replaced by  $TH$  in the definition of  $s$  above, because the projection of all  $df_i(T_{q_i} R_i)$  to  $N_{H/\mathbb{P}^N}$  is zero. The case  $m = 1$  has to be dealt with separately and it turns out that, once  $s$  is made transverse, the dimension of its zero locus is smaller than the expected dimension, hence the corresponding combinatorial types are in fact irrelevant.

The procedure to make  $s$  transverse is the same as described in [VZ08, §3], namely we need to blow up inside  $\mathbb{P}$  the projective subbundle  $\mathbb{P}(\mathcal{E})$  of  $\mathbb{P}|_{\mathcal{X}_\sigma}$ , where  $\mathcal{X}_\sigma$  is the closed substack of  $\mathcal{X}$  where some of the tails degenerate so that the corresponding gluing marking lies on a component contracted by  $f$ , and  $\mathcal{E}$  is the sum of the tangent line bundles at such subset of the gluing markings. Because  $\overline{\mathcal{M}}_{0, \tilde{\alpha}^{(i)} \cup \{m_i\}}^{\text{punct}}(H, \tilde{d}_i)$  is isomorphic to  $\overline{\mathcal{M}}_{0, \tilde{n}_i}(H, d_i)$ , the construction of Vakil and Zinger in the section “A blowup of a moduli space of genus-zero maps” of their paper carries through unchanged. The result of the blow-up is to replace  $T_{q_i} R_i$  with  $T_{q_i} R_i \otimes \bigoplus_{j=1}^k T_{\tilde{q}_j} R_i \otimes T_{\tilde{q}_j} S_{ij}$ . We interpret this in terms of alignments.<sup>11</sup>

We claim that  $\tilde{s}$  obtained from  $s$  by twisting by the exceptional divisors of the blowup is a transverse section. We also claim that comb loci of type  $\mathcal{Y}_c^0$  can be described as a fibered product of

- moduli of genus one punctured maps to the standard log point, radially aligned and satisfying factorisation;
- $V(\tilde{s})$ ;
- moduli of genus zero maps relative to  $(\mathbb{P}^N|H)$ , corresponding to vertices of type 4 above.

Finally, we claim that integrals of psi and evaluation classes over these loci can be translated into tautological integrals, i.e. descendant Gromov–Witten invariants, whose numerics is governed by the combinatorial type of the tropical map.

<sup>11</sup>(Luca) this has to be written properly

5. RECURSION FOR GENERAL  $(X, Y)$ 

## 6. QUANTUM LEFSCHETZ ALGORITHM

Consider a smooth pair  $(X, Y)$  with  $Y$  very ample, and let  $P$  be the projective bundle  $P = \mathbb{P}_Y(N_{Y|X} \oplus \mathcal{O}_Y)$ . We assume that the genus zero and reduced genus one Gromov–Witten theories of  $X$  are known. From this starting data, we will apply our recursion formula to compute:

- (1) the genus one **reduced restricted absolute Gromov–Witten theory** of  $Y$ ;
- (2) the genus one **reduced relative Gromov–Witten theory** of the pair  $(X, Y)$ ;
- (3) the genus one **reduced rubber theory** of  $P$ .

**6.1. Reduced absolute, relative and rubber invariants.** To be precise: by a reduced invariant of  $Y$  we mean an integral over  $\mathcal{VZ}_{1,n}(Y, \beta)$  of products of pullbacks of evaluation and psi classes along morphisms which forget a subset  $S$  of the marked points (taking  $S = \emptyset$  gives the ordinary evaluation and psi classes). Here the evaluation maps are viewed as mapping into  $X$  (hence the adjective “restricted”). Reduced relative invariants of  $(X, Y)$  are defined in the same way, except now the forgetful morphism maps into a space of absolute maps:

$$\text{fgt}_S: \mathcal{VZ}_{1,\alpha}(X|Y, \beta) \rightarrow \mathcal{VZ}_{1,m-\#S}(X, \beta).$$

In particular, all the psi classes which we consider are *collapsed psi classes*, meaning that they are relative cotangent line classes for the corresponding collapsed stable map. Note that, unlike in the absolute case, in the relative case it may well be the case that the entire insertion is pulled back along a single forgetful map  $\text{fgt}_S$ . The reduced rubber invariants of  $P$  are defined similarly (again using collapsed psi classes).

The systems of invariants defined above are equivalent to the classical systems of invariants (which do not use forgetful morphisms) by well-known topological recursion relations.

**6.2. Fictitious and true markings.** The recursion procedure is rather delicate. Roughly speaking, we will induct on the degree (meaning  $Y \cdot \beta$ ), number of marked points and total tangency. To get the correct notion of number of markings and total tangency in the relative setting, we introduce the concept of **fictitious markings**. Consider a moduli space  $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$  of reduced relative stable maps and a corresponding integrand  $\gamma$ . We let  $F \subseteq \{1, \dots, m\}$  be the maximal subset of marked points such that:

- (1)  $\alpha_i = 1$  for all  $i \in F$ ;
- (2) the entire integrand  $\gamma$  is pulled back along  $\text{fgt}_F$ .

This subset is uniquely determined, and its elements are referred to as **fictitious markings**. Markings which are not fictitious are referred to as **true**. When inducting on relative invariants we will always be interested in the number of true markings (as opposed to the total number of markings) and the true tangency

$$\sum_{i \notin F} \alpha_i \leq d = Y \cdot \beta$$

as opposed to the total tangency, which is always  $d$ . This formalises the idea that relative invariants with non-maximal tangency  $t < d$  can be obtained by adding  $d - t$  fictitious markings of tangency 1; see [Gat02, Lemma 1.15(i)].

**6.3. Structure of the recursion.** Given the genus-zero Gromov–Witten theory of  $X$ , the arguments of [Gat02] give an effective algorithm to reconstruct the genus-zero theories of  $Y$  and  $(X, Y)$ ; moreover, the genus-zero rubber theory of  $P$  is identical to the genus-zero theory of  $Y$  [Gat03]. Thus we may assume that all genus-zero data is known. We assume in addition that we know the genus one reduced theory of  $X$ . The structure of the recursion is then as follows:

```

for  $d \geq 0$  :
  for  $n \geq 0$  :
    for  $t \geq 0$  :
      | Step 1: Compute forgetful relative invariants of  $(X, Y)$  (degree  $d$ ,  $n + 1$ 
      | true markings, true tangency  $t$ ); see below.
    Step 2: Compute absolute invariants of  $Y$  (degree  $d$ ,  $n$  markings).
    for  $t \geq 0$  :
      | Step 3: Compute relative invariants of  $(X, Y)$  (degree  $d$ ,  $n$  true
      | markings, true tangency  $t$ ).
    for  $n \geq 0$  :
      for  $m \geq 0$  :
        | Step 4: Compute rubber invariants of  $P$  (degree  $d$ ,  $n$  relative markings,
        |  $m$  non-relative markings).

```

Although the loops involving  $d$ ,  $n$  and  $m$  have infinite length, in order to compute any single invariant it is only necessary to iterate the preceding loops for a finite amount of time. A **forgetful relative invariant** of  $(X, Y)$  is by definition a reduced relative invariant with a marked point  $x_0$  such that all of the insertions are pulled back along  $\text{fgt}_{x_0}$ . In our recursion, we first deal with this special class of relative invariants (with  $n + 1$  true markings), before computing the absolute invariants (with  $n$  markings) and then returning to compute all of the relative invariants (with  $n$  true markings). This need to treat separately a particular subclass of the relative invariants is an inescapable feature of the genus one recursion.

The base terms of the recursion all have  $d = 0$  and as such are easy to compute: the relative invariants of  $(X, Y)$  are nothing but absolute invariants of  $X$ , the absolute invariants of  $Y$  are given by obstruction bundle integrals over Deligne–Mumford spaces, and the rubber invariants of  $P$  are also given by integrals over Deligne–Mumford spaces, using the formula for the double ramification cycle [Hai13, JPPZ17] in terms of tautological classes. We will now explain how to perform each of the four inductive steps outlined above.

**Notation 6.1.** Given tuples  $\mathbf{a} = (a_1, \dots, a_n)$  and  $\mathbf{b} = (b_1, \dots, b_n)$ , we say that  $\mathbf{a} < \mathbf{b}$  if there exists an  $i \in \{1, \dots, n\}$  such that  $a_j = b_j$  for  $j < i$  and  $a_i < b_i$ .

**Step 1.** Suppose we are given a forgetful relative invariant to compute. That is, we have a relative space  $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$  of degree  $d$ , with  $n + 1$  true markings and true tangency  $t$ , and a marking  $x_0$  such that the insertion  $\gamma$  is pulled back along  $\text{fgt}_{x_0}$ . We assume inductively that every absolute, relative and rubber invariant with  $(d', n') < (d, n)$  is known, and also

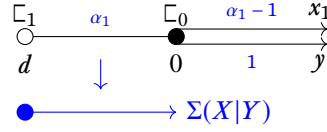
that every forgetful relative invariant with  $(d', n', t') < (d, n + 1, t)$  is known. Choose a true marking  $x_1$  with  $\alpha_1 \geq 1$  and consider the space:

$$\mathcal{VZ}_{1,(\alpha-e_1)\cup(1)}(X|Y, \beta).$$

Denote the newly-introduced marking by  $y$  and consider the integrand  $\tilde{\gamma}$  obtained from  $\gamma$  by introducing  $\text{fgt}_y^*$  everywhere. Applying our recursion formula to  $x_1$ , we obtain:

$$(2) \quad ((\alpha_1 - 1)\psi_1 + \text{ev}_1^* Y) \tilde{\gamma} \cap [\mathcal{VZ}_{1,(\alpha-e_1)\cup(1)}(X|Y, \beta)] = \tilde{\gamma} \cap [\mathcal{D}(1)].$$

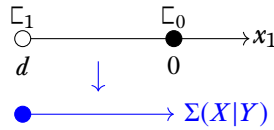
Let us first examine the left-hand side. The class  $\psi_1$  differs from  $\text{fgt}_y^* \psi_1$  by the locus where  $x_1$  and  $y$  are contained on a contracted rational bubble. This locus consists of reduced relative stable maps of the form



with all other marked points contained on  $C_1$ . This is isomorphic to  $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$  and when we restrict  $\tilde{\gamma}$  to this locus we obtain precisely the class  $\gamma$  which we started with. Thus the left-hand side of (2) may be written as

$$(\alpha_1 - 1)I + \text{fgt}_y^* ((\alpha_1 - 1)\psi_1 + \text{ev}_1^* Y) \tilde{\gamma} \cap [\mathcal{VZ}_{1,(\alpha-e_1)\cup(1)}(X|Y, \beta)]$$

where  $I$  is the invariant we are trying to compute. The second term is a forgetful relative invariant with the same degree and number of true markings (since  $y$  is fictitious), and strictly smaller true tangency; hence it is recursively known. We now examine the right-hand side of (2). Recall that all of the genus zero data has already been computed, so we only need to focus on the genus one pieces. First consider the type  $A$  loci. The genus one piece has strictly smaller degree (and hence is recursively known) except in the following situation (with some stable distribution of the remaining markings):



In this situation,  $C_1$  contains at most  $n + 1$  true markings. If it has  $n - 1$  or fewer, then it is known recursively. If it has exactly  $n$  this means that  $C_0$  contains exactly one true marking (besides  $x_1$ , which may be true or fictitious). We claim that in this situation we must have  $y \in C_1$ , since otherwise we would have a moduli space for  $C_0$  given by  $\overline{\mathcal{M}}_{0,k}$  with  $k \geq 4$ , and applying the projection formula with  $\text{fgt}_y$  we would conclude that the contribution is zero. Thus we have  $y \in C_1$ , and so the genus one contribution is a forgetful relative invariant with  $n$  true markings, and hence is recursively known.

Finally, if  $C_1$  contains exactly  $n + 1$  true markings, then the only additional markings on  $C_0$  are fictitious, and by the same argument as in the previous paragraph there can only be one. Thus for each fictitious marked point we obtain a locus isomorphic to  $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$  and  $\tilde{\gamma}$  restricts to  $\gamma$  here (from the point of view of computing invariants, the fictitious marked points are indistinguishable, meaning that the contributions are all the same).

Thus for each fictitious marked point (of which there is at least one, namely  $y$ ) we get a contribution of  $\alpha_1 I$  to the right-hand side of (2).

The contributions of the type  $B$  loci only involve genus zero data and hence are known. The contributions of the type  $C^0$  loci are determined by genus zero data and tautological integrals on Deligne–Mumford space, hence are also known. It remains to consider type  $C^+$  loci. If the degree of the genus one piece is less than  $d$  then we have a rubber invariant of strictly smaller degree. The only other possibility is that the entire curve is mapped into the divisor. In this case we may apply the projection formula with  $\text{fgt}_y$  to identify this with an integral over  $\mathcal{VZ}_{1,m}(Y, \beta)$  for some (possibly large) number  $m$  of marked points. But by assumption there is another marked point  $x_0$  with all of the insertions pulled back along  $\text{fgt}_{x_0}$ , so a further application of the projection formula shows that this contribution vanishes. To conclude, we may rearrange (2) to obtain an expression of the form

$$\lambda I = \text{recursively known terms}$$

where  $\lambda$  is an explicit scalar which is always nonzero; we have thus determined  $I$ , which completes Step 1.

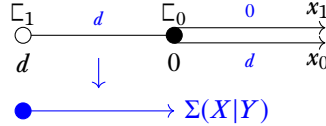
**Step 2.** Consider now the absolute space  $\mathcal{VZ}_{1,n}(Y, \beta)$  with an insertion  $\gamma$ , and suppose inductively that we have computed all forgetful relative invariants with  $(d', n') \leq (d, n+1)$ , all relative invariants and absolute invariants with  $(d', n') < (d, n)$ , and all special rubber invariants with  $d' < d$ . Consider the following moduli space with  $n+1$  markings:

$$\mathcal{VZ}_{1,(d,0,\dots,0)}(X|Y, \beta).$$

Let  $x_0$  denote the relative marking and consider the integrand  $\tilde{\gamma}$  obtained from  $\gamma$  by introducing  $\text{fgt}_{x_0}^*$  everywhere. Now recurse at  $x_1$  to obtain:

$$(3) \quad \text{ev}_1^* Y \cdot \tilde{\gamma} \cap [\mathcal{VZ}_{1,(d,0,\dots,0)}(X|Y, \beta)] = \tilde{\gamma} \cap [\mathcal{D}(1)].$$

The left-hand side is a forgetful relative invariant of degree  $d$  and  $\leq n+1$  true markings, and so has already been computed. For the right-hand side, let us begin with loci of type  $A$ . The genus one contributions from each loci have  $(d', n') < (d, n)$  except in the following case



which gives a contribution of

$$d \cdot \gamma \cap [\mathcal{VZ}_{1,(d,0,\dots,0)}(X|Y, \beta)]^{12}$$

to the right-hand side of (3). Here  $\gamma$  is the insertion we started with; the difference is that we are now considering relative maps to  $(X, Y)$  with maximal tangency at  $x_1$ , rather than absolute maps to  $Y$ . The type  $B$  and  $C^0$  loci are recursively determined as in Step 1, and similarly the type  $C^+$  loci are recursively determined except for the locus where the

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entire curve is mapped into the divisor. On this locus we may apply  $\text{fgt}_{x_0}$  and identify the contribution with

$$d^2 \cdot \gamma \cap [\mathcal{VZ}_{1,n}(Y, \beta)] = d^2 \cdot I$$

where  $I$  is the invariant we are trying to compute. Putting this all together, we obtain

$$(4) \quad I = (-\gamma/d) \cap [\mathcal{VZ}_{1,(d,0,\underbrace{\dots}_{n-1},0)}(X|Y, \beta)] + \text{recursively known terms}$$

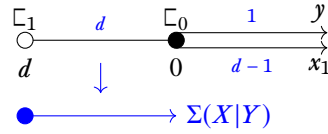
where on the right-hand side there are  $n-1$  non-relative markings  $x_2, \dots, x_n$ , and a relative marking  $x_1$ . We now apply the recursion again to the right-hand side, by considering the space

$$\mathcal{VZ}_{1,(d-1,1,0,\dots,0)}(X|Y, \beta)$$

where  $x_1$  now has tangency  $d-1$  and we have introduced a new marking  $y$  with tangency 1. We consider a new insertion, denoted  $\tilde{\gamma}$  as usual, by introducing  $\text{fgt}_y^*$  everywhere. Recursing at  $x_1$  we obtain:

$$(5) \quad ((d-1)\psi_1 + \text{ev}_1^* Y) \cdot (-\tilde{\gamma}/d) \cap [\mathcal{VZ}_{1,(d-1,1,0,\dots,0)}(X|Y, \beta)] = (-\tilde{\gamma}/d) \cap [\mathcal{D}(1)].$$

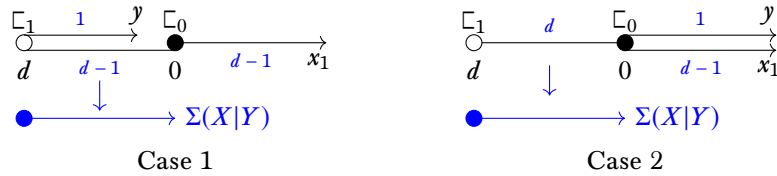
The difference between  $\psi_1$  and  $\text{fgt}_y^* \psi_1$  is given by the locus where  $x_1$  and  $y$  belong to a collapsed rational bubble:



The contribution of this locus to the left-hand side of (5) is:

$$(d-1) \cdot (-\gamma/d) \cap [\mathcal{VZ}_{1,(d,0,\underbrace{\dots}_{n-1},0)}(X|Y, \beta)].$$

What remains on the left-hand side is a forgetful relative invariant of degree  $d$  and  $\leq n+1$  true markings ( $y$  being the “forgetful” marking), hence is recursively known. On the right-hand side, the type  $A$  loci are recursively known except possibly in the following special cases (with some stable distribution of the remaining non-relative markings):



In Case 1 the contribution from  $C_1$  is a forgetful relative invariant with  $\leq n$  true markings, hence is recursively known. In Case 2, we first note that there cannot be any more markings on  $C_0$  (since otherwise we could apply  $\text{fgt}_y$  and conclude that the contribution vanishes). Thus we obtain a single locus, which contributes precisely

$$d \cdot (-\gamma/d) \cap [\mathcal{VZ}_{1,(d,0,\underbrace{\dots}_{n-1},0)}(X|Y, \beta)] = -\gamma \cap [\mathcal{VZ}_{1,(d,0,\underbrace{\dots}_{n-1},0)}(X|Y, \beta)]$$

which gives us the first term on the right-hand side of (5). As usual the type  $B$  and  $C^0$  contributions are known recursively, and the only type  $C^+$  contribution not known

recursively occurs when the entire curve is mapped into the divisor, in which case we apply  $\text{fgt}_y$  to calculate the contribution as:

$$(-\gamma/d) \cap [\mathcal{VZ}_{1,n}(Y, \beta)] = -I/d.$$

Substituting into (5) we end up with

$$I(1 - d^{-1}) = \text{recursively known terms}$$

which completes the recursion step as long as  $d \neq 1$ . But since  $\mathcal{VZ}_{1,n}(H, 1) = \emptyset$  it follows that  $\mathcal{VZ}_{1,n}(Y, \beta) = \emptyset$  if  $d = Y \cdot \beta = 1$ , so we may always assume  $d \geq 2$  in the recursion.

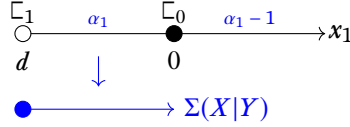
**Step 3.** Now suppose we are given a relative space  $\mathcal{VZ}_{1,\alpha}(X|Y, \beta)$  with an insertion  $\gamma$ , and suppose inductively that we have computed all forgetful relative invariants with  $(d', n') \leq (d, n+1)$ , all absolute invariants with  $(d', n') \leq (d, n)$ , all relative invariants with  $(d', n', t') < (d, n, t)$  and all rubber invariants with  $d' < d$ . Choose a true marking  $x_1$  with  $\alpha_1 \geq 1$  and consider the moduli space

$$\mathcal{VZ}_{1,(\alpha-\epsilon_1)\cup(1)}(X|Y, \beta)$$

where  $y$  is the newly-introduced marking. As usual consider the insertion  $\tilde{\gamma}$  obtained from  $\gamma$  by introducing  $\text{fgt}_y$  everywhere. Recursing at  $x_1$  we obtain:

$$((\alpha_1 - 1)\psi_1 + \text{ev}_1^* H) \tilde{\gamma} \cap [\mathcal{VZ}_{1,(\alpha-\epsilon_1)\cup(1)}(X|Y, \beta)] = \tilde{\gamma} \cap [\mathcal{D}(1)].$$

The left-hand side is a relative invariant with the same degree and number of true markings, but smaller true tangency: hence it is recursively known. On the right-hand side, the type  $A$  contributions are recursively known except for those of the following form



where  $C_0$  contains a single fictitious marking (if it had multiple fictitious markings then the contribution would vanish by projection formula) and all the other markings are on  $C_1$ . Thus for each fictitious marking we get a contribution of  $\alpha_1 I$  where  $I$  is the invariant we are trying to compute. Note that  $\alpha_1 \neq 0$ , and that this term appears at least once since  $y$  is a fictitious marking; so we get a nonzero multiple of  $I$ .

As usual the type  $B$  and  $C^0$  contributions are recursively known. The type  $C^+$  contributions are determined by lower-degree rubber invariants, except for when the whole curve is mapped into  $Y$ ; however in this case we may apply  $\text{fgt}_y$  to identify the contribution with an absolute invariant of  $Y$  with degree  $d$  and  $n$  markings, which is also recursively known. Thus we have determined the invariant  $I$ .

**Step 4.** Finally, consider a rubber space  $\mathcal{VZ}_{1,\alpha}(P|Y_0 + Y_\infty, \beta)_{\mathbb{G}_m}$  with insertion  $\gamma$ . Suppose inductively that we have computed all absolute and relative invariants with  $d' \leq d$  and all rubber invariants with  $(d', n', m') < (d, n, m)$ .

We first note the following important reduction: if there exists a relative marking  $x_k$  such that all insertions are pulled back along  $\text{fgt}_k$ , then we may apply the projection formula, together with the fact that

$$(\text{fgt}_k)_* [\mathcal{VZ}_{1,\alpha}(P|Y_0 + Y_\infty, \beta)_{\mathbb{G}_m}] = \alpha_k^2 \cdot [\mathcal{VZ}_{1,n+m-1}(Y, \beta)]$$



to identify the rubber invariant with a multiple of a reduced invariant of  $Y$ , which has the same degree and hence is known recursively.

We will deal with the general case by reducing to the one above. Consider the moduli space

$$\mathcal{VZ}_{1,\alpha \cup (0)}(P|Y_0 + Y_\infty, \beta)_{\mathbb{G}_m}$$

obtained by introducing a marked point  $y$  with no tangency. Let  $x_1$  be a positive-tangency marking (such a marking always exists since  $d \geq 2$ ) and let  $\tilde{\gamma}$  be the insertion obtained from  $\gamma$  by replacing  $\text{ev}_1$  and  $\psi_1$  by  $\text{ev}_y$  and  $\psi_y$ , and then introducing  $\text{fgt}_1^\star$  everywhere.

We will make use of a recursion formula for rubber spaces analogous to the recursion formula for relative spaces used in Steps 1–3. Following [Kat07], there is a line bundle  $L_y^{\text{bot}}$  on  $\mathcal{VZ}_{1,\alpha \cup (0)}(P|Y_0 + Y_\infty, \beta)_{\mathbb{G}_m}$ , together with a section  $s_y^{\text{bot}}$  whose vanishing locus consists of the locus  $\mathcal{D}(y)$  of rubber maps where  $y$  is not mapped into the bottom level of the expanded target. As in §??, we can give a logarithmic interpretation of this: it corresponds to the piecewise-linear function on the tropical moduli space which associates, to every rubber tropical map, the distance between  $\varphi(\mathbb{C}_y)$  and the leftmost vertex of the tropical target, where  $\mathbb{C}_y$  is the vertex of the source curve containing the flag corresponding to  $y$ . Using this tropical description, we may easily calculate the vanishing orders of  $s_y^{\text{bot}}$  along the various components of  $\mathcal{D}(y)$ , and show that  $c_1(L_y^{\text{bot}}) = \Psi_0 - \text{ev}_y^\star Y$  (see also [Kat05], where similar results in the non-reduced setting are obtained, using different methods). From this, we obtain a rubber recursion formula

$$(6) \quad (\Psi_0 - \text{ev}_y^\star Y) \tilde{\gamma} \cap [\mathcal{VZ}_{1,\alpha \cup (0)}(P|Y_0 + Y_\infty, \beta)_{\mathbb{G}_m}] = \tilde{\gamma} \cap [\mathcal{D}(y)]$$

where the fundamental class  $[\mathcal{D}(y)]$  is weighted by vanishing orders on the components. We will first show that the left-hand side is recursively known. By construction the class  $\tilde{\gamma}$  is pulled back via  $\text{fgt}_1^\star$ , and the same is true for  $\text{ev}_y^\star Y$ . It remains to examine  $\Psi_0$ . If there exists a negative-tangency marking  $x_2$ , then we have [Gat03, Construction 5.1.17]

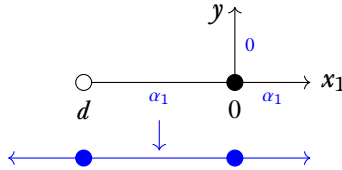
$$\Psi_0 = -\alpha_2 \hat{\psi}_2 - \text{ev}_2^\star Y$$

where  $\hat{\psi}_2$  is a *non-collapsed* psi class. (If there are no negative-tangency markings, then the construction given in [MP06, §1.5.2] shows that  $\Psi_0 = 0$ .) Now,  $\hat{\psi}_2 - \psi_2$  is given by the loci where  $x_2$  belongs to a trivial bubble. This entails a splitting of the curve into pieces, each of which contributes a rubber integral. Typically each of these pieces will have  $(g', d', n', m') < (1, d, n, m)$  and hence be recursively known. The one exception is when all of the genus and degree is concentrated on the top level of the expansion, with the bottom level containing only a single non-relative marking in addition to all the negative-tangency markings. But in this case the contribution is a rubber invariant where all of the insertions are pulled back via  $\text{fgt}_1^\star$ , and hence we may apply the projection formula to identify this with an absolute invariant of  $Y$  which is recursively known. We conclude that, up to recursively known terms, we may replace  $\hat{\psi}_2$  by  $\psi_2$  in the left-hand side of (6). If we now compare  $\psi_2$  with  $\text{fgt}_1^\star \psi_2$  we see that the difference is given by the locus where  $x_1$  and  $x_2$  belong to a collapsed rational piece. The contribution of this locus consists of rubber invariant with strictly fewer relative markings, and hence is recursively known. So up to recursively-known terms (6) becomes

$$\text{fgt}_1^\star(-\alpha_2 \psi_2 - \text{ev}_y^\star Y) \tilde{\gamma} \cap [\mathcal{VZ}_{1,\alpha \cup (0)}(P|Y_0 + Y_\infty, \beta)_{\mathbb{G}_m}] = \tilde{\gamma} \cap [\mathcal{D}(y)]$$

and now the left-hand side is also recursively known, by the projection formula. Let us now examine the right-hand side. The components of  $\mathcal{D}(y)$  are indexed by splittings of the curve, and certainly the contributions are recursively known unless there is a piece of the curve carrying all of the genus and degree, so we may restrict to examining these contributions.

Let us denote the piece of the curve carrying all of the genus and degree by  $C' \subseteq C$ . If  $C'$  is mapped to top level, then either it contains  $x_1$ , in which case we apply  $\text{fgt}_1$  to compute the contribution, or it does not contain  $x_1$ , in which case it has fewer than  $n$  relative markings and is known recursively. If on the other hand  $C'$  is not mapped to top level, then generically it is mapped to the bottom level (since generically the circuit is not contracted on this locus, so the desingularisation process does nothing). One possible contribution is given by the following locus



which contributes a nonzero multiple of the invariant  $I$  we are trying to compute. For the other possibilities, first note that, unless every component at the top level contains a single positive-tangency marking and a single node (together with possibly some tangency-zero markings), then  $C'$  has fewer than  $n$  relative markings and hence the contribution is known recursively. On the other hand, if any non-relative marking other than  $y$  is mapped to the top level,  $C'$  has  $\leq n$  relative markings and less than  $m$  non-relative markings, and hence again the contribution is known recursively. The only remaining possibilities are when  $x_1$  is replaced by another positive-tangency marking  $x_k$  in the above picture; but then the contribution can be calculated by applying the projection formula to  $\text{fgt}_1$ . We therefore conclude that the only contribution to the right-hand side of (6) which is not known recursively is a nonzero multiple of the invariant we were trying to compute. This completes the recursion step.

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