RELATIVE STABLE MAPS IN GENUS ONE VIA CENTRAL ALIGNMENTS

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ABSTRACT. For a smooth projective variety X and a smooth very ample hypersurface $Y \subseteq X$, we define moduli spaces of relative stable maps to (X, Y) in genus one, as closed substacks of the moduli space of maps from centrally aligned curves, constructed in [RSW17a]. We construct virtual classes for these moduli spaces, which we use to define *reduced relative Gromov–Witten invariants* in genus one.

[GOALS: We prove a recursion formula which allows us to completely determine these invariants in terms of the reduced Gromov–Witten invariants, as defined in [REF]. We also prove a relative version of the Li–Zinger formula, relating our invariants to the usual relative Gromov–Witten invariants. Also say something about quasimaps.]

CONTENTS

1. INTRODUCTION

Statement of the problem. Contrary to the genus zero case, the moduli space of genus one maps to projective space - with or without markings - is far from smooth; indeed it has various boundary components of different dimensions, representing maps that contract a genus one curve and have all the degree supported on a number of rational tails. The many incarnations of relative moduli spaces also suffer of the same undesirable feature.

Since the work of Vakil–Zinger and Ranganathan–Santos-Parker–Wise, it has been clear that it is possible to identify a desingularisation of the main component by adding the extra data of a contraction of the source curve

$\nu\colon C \to \bar{C}$ - where the latter is allowed to acquire a Smyth singularity - and requiring the stable map $f\colon C \to \mathbb{P}^N$ to factor through ν.

Choice of relative space and desingularisation. We focus on the space of logarithmic stable maps to $(\mathbb{P}^N|H)$, following ACGS. We perform a log modification of this space as detailed below. For a log curve $C \to S = \mathrm{Spec}(k = \bar{k})$, modify the dual graph of C by replacing the minimal genus one subcurve (in case it is a circle of $\mathbb{P}^1$) by a single vertex of genus one, called the *core* and denoted by $\circ$, and define a piecewise linear function with values in $\overline{\mathcal{M}}_S$ on such a graph by setting

$$\lambda(v) = \sum_{q \in [\circ, v]} \rho_q,$$

where the ρ_q are the smoothing parameters of the nodes q separating v from the core. Such a function is related to the log canonical bundle of $C \to S$. When the map contracts a subcurve of genus one, we endow it with the extra data of a radius $\delta \in \overline{\mathcal{M}}_S$ subject to the following compatibility condition:

(*) *the circle of radius δ around $\circ$ passes through ≥ 1 vertex of positive f-degree.* Furthermore, we require all the values of λ to be comparable with δ, and among themselves whenever they are $\leq \delta$. This is called a *centrally aligned* log structure and carries enough information to define a contraction $\nu\colon C \to \bar{C}$, possibly after a semistabilisation of (C, f) - in fact even more. The space thus obtained, $\widetilde{\mathcal{VZ}}_{1,\alpha}(\mathbb{P}^N|H, d)$, is a log modification of $\overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d)$.

The main component $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ is then identified by a double factorisation condition:

(1) If f contracts a genus one subcurve, then f is required to factor through the Smyth singularity $\nu\colon C \to \bar{C}$ determined by the contraction radius δ as above.

(2) If furthermore the core is contracted by the associated tropical map ϕ, let δ_2 be the minimal distance from $\circ$ to a vertex supporting a flag that escapes $\phi^{-1}(\phi(\circ))$; we require f to factor through $\nu_2\colon C \to \bar{C}_2$.

The main result is that

Theorem 1.1. $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ is (log) smooth.

Gathmann-type recursion. There is a forgetful morphism

$$\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d) \to \mathcal{VZ}_{1,n}(\mathbb{P}^N, d),$$

hitting Gathmann's relative space. We may therefore pullback Gathmann's line bundle and section, cutting out the locus where the k-th marking is tangent to H to order $\alpha_k + 1$. Because α was maximal ($\sum \alpha = d$) by assumption, this means that the curve has to break, and x_k has to lie on an internal component - one which is entirely mapped into H. We identify the zero locus of Gathmann's section explicitly. Here is an interesting remark: the combinatorics of such boundary loci is governed by tropical geometry, and it is otherwise very hard to districate the interaction between the relative condition and the exceptional loci of the Vakil–Zinger blow-up.

Corollary 1.2. $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$ is smooth over its Artin fan, in particular codimension one logarithmic strata can be read off from the latter.

The Artin fan is a tropical gadget. Its local structure is given by subdividing the ACGS minimal monoid according to the alignment. We are only interested in picking its rays. The upshot is that the combinatorics is slightly more involved than in the genus zero case: the alignment may force some teeth of the comb to break.

Theorem 1.3 (Gathmann-type formula, maximal tangency, $(\mathbb{P}^N|H)$ case).

$$(\alpha_k \psi_k + \mathrm{ev}_k^* H)[\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)] = [D_{1,\alpha;k}(\mathbb{P}^N|H,d)],$$

the latter being a sum of broken comb loci indexed by rays of the tropical fan.

Importantly, the broken comb loci admit a very explicit description in terms of tautological integrals on the underlying boundary of Gathmann's relative space.

Theorem 1.4. Up to a finite cover of the underlying boundary stratum - which is a combinatorially-determined fiber product of moduli spaces of genus zero and one, absolute and relative maps with lower numerical invariants - every component of $D_{\alpha,k}(\mathbb{P}^N|H,d)$ can be described as the transverse intersection of two loci in a projective bundle, where:

- the latter parametrises the possible line bundle isomorphisms imposed by the alignment of the log structure;
- the first locus is a subbundle representing the residual isomorphisms after fixing the ones virtually imposed by tropical continuity;
- the second locus is determined by the factorisation conditions.

The upshot is that we may then push the formula down to the Gathmann's space, so as to obtain multiplicities and tautological classes.

Corollary 1.5.

$$(\alpha_k \psi_k + \mathrm{ev}_k^* H)[\mathcal{VZ}_{1,\alpha}^G(\mathbb{P}^N|H,d)] = [D_{1,\alpha;k}^G(\mathbb{P}^N|H,d)],$$

the latter being expressible as a weighted sum of tautological classes on Gathmann's comb loci.

Once we have this formula, the following extensions are classical:

- A similar formula for raising the tangency holds in the non-maximal tangency case. It can be proven by adding auxiliary markings of contact order 1; forgetting them is then a $(d - \sum \alpha)! : 1$ cover because the nice locus is dense inside Gathmann's relative spaces.
- The formula holds more generally for any smooth projective target X relative to a generic hyperplane section $Y = X \cap H \subseteq \mathbb{P}^N$. This follows via virtual pullback.

Finally the recursive structure of the boundary allows us to prove the following

Theorem 1.6 (In-principle quantum Lefschetz). *The restricted reduced genus one invariants of Y can be inductively deducted from the full descendant genus zero and one (reduced) Gromov-Witten theory of X.*

The proof is more delicate than its genus zero analogue because invariants with the same numerical data appear intertwined in the last steps of the recursion.

2. A DESINGULARISATION OF THE LOG SPACE

The ultimate goal of the paper is to apply Gathmann's techniques to the Vakil-Zinger desingularisation $\mathcal{VZ}_{1,n}(\mathbb{P}^N, d)$ and to obtain a quantum Lefschetz result for reduced invariants under some positivity assumption. The key step is to study the unobstructed case $(\mathbb{P}^N|H)$. We approach the problem by lifting it to the ACGS space of log stable maps. This allows us to exploit the tools developed in [RSW17a, RSW17b]. We are in an intermediate situation between those two papers, and indeed we get an intermediate answer.

2.1. **The ACGS minimal monoid and central alignments**.

Proposition 2.1. *The map $\overline{\mathcal{M}}^{\mathrm{cen}}_{1,\alpha}(\mathbb{P}^N|H, d) \to \overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|H, d)$ is a log modification. In particular $\overline{\mathcal{M}}^{\mathrm{cen}}_{1,\alpha}(\mathbb{P}^N|H, d)$ is a log algebraic stack.*

possibly start pointing out that the log structure is already partially aligned by the map to $(\mathbb{P}^N|H)$

2.2. **Factorisation conditions**.

Proposition 2.2. *Factoring through the Smyth curve is a closed condition. In particular $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d) \subseteq_{\mathrm{cl}} \overline{\mathcal{M}}^{\mathrm{cen}}_{1,\alpha}(\mathbb{P}^N|H, d)$ is a log algebraic stack.*

2.3. **Log smoothness**.

Theorem 2.3. *$\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ is log smooth.*

Proof. We reduce to the situation dealt with in [RSW17b] by adding generic extra hyperplanes $H_1, \ldots, H_N$.

First, note that, for divisors $D_1 \subseteq D_2$ in X, there is a morphism of log schemes $(X, \mathcal{M}_{D_2}) \to (X, \mathcal{M}_{D_1})$, or equivalently a morphism of log structures $\mathcal{M}_{D_1} \to \mathcal{M}_{D_2}$ over id_X, because functions invertible off D_1 are in particular invertible off D_2 as well, and divisorial log structures are subsheaves of the structure sheaf $\mathcal{M}_D \subseteq \mathcal{O}_X$.

Now fix a point $[(C, f)] \in \mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$. Choose hyperplanes $H_1, \ldots, H_N$ in such a way that they intersect the image of f transversally, namely $f^{-1}(H_1 \cup \ldots \cup H_N)$ is a reduced collection of points $\{q^i_j\}_{\substack{i=1,\ldots,N \\ j=1,\ldots,d}}$ in the smooth locus of C. This condition will then hold in an open neighbourhood of $[(C, f)]$. Mark C at these points, end endow it with the pullback along f of the divisorial log structure $(\mathbb{P}^N, \Delta)$, where $\Delta = H + \sum_{i=1}^{N} H_i$. Then

$$f : (C, \{p_k\}_{k=1,\ldots,n} \{q^i_j\}_{\substack{i=1,\ldots,N \\ j=1,\ldots,d}}) \to (\mathbb{P}^N, \Delta)$$

is a lift of $[(C, f)]$ to $\overline{\mathcal{M}}_{1,\alpha}(\mathbb{P}^N|\Delta, d)$ (under the forgetful morphism discussed in the previous paragraph).

Looking at the associated tropical map ϕ, observe that:

- new flags have been attached only to vertices of positive degree, and these already have a flag escaping $\phi^{-1}\phi(v)$, because the sum of the incoming slopes is not zero (by modified balancing);
- the image of the new tropical map $\tilde{\phi}$ is entirely contained in the ray of the tropicalisation of $(\mathbb{P}^N, \Delta)$ corresponding to H, with new flags going off to infinity in all the new ray directions from every vertex of positive degree.

In particular, for every quotient N' of the lattice N, the associated tropical map $\tilde{\phi}'$ will either

(1) have image contained in the ray corresponding to H, isomorphically to the original ϕ, so the contraction radius can be seen to coincide with δ_2, or

(2) collapse the entire curve to the zero-cell of the fan, in which case we argue from the previous remarks that the contraction radius is δ.

Hence the lift of $[(C, f)]$ is centrally aligned and satisfies the factorisation property for every subtorus $H < T$, therefore it is well-spaced (see [RSW17b, Definition 3.4.2]) and it belongs to $\mathcal{VZ}_{1,\tilde{\alpha}}(\mathbb{P}^N|\Delta, d)$. Note that the deformation spaces of (C, f) and its lift are isomorphic by construction, as can be checked by the infinitesimal criterion - an infinitesimal deformation of (C, f) brings along a unique deformation of the $\{q_j^i\}$ compatible with the map to $(\mathbb{P}^N, \Delta)$. At the logarithmic level, observe that the ACGS minimal monoid is the same, because no component of C is entirely mapped into any of the newly added hyperplanes; since the global contraction radius δ is the same, the subdivisions corresponding to the alignment procedure do coincide as well. This shows that the forgetful morphism is (log) étale in a neighbourhood of the lift of $[(C, f)]$, hence we may conclude by appealing to [RSW17b, Theorem 3.5.1]. □

Corollary 2.4. $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H, d)$ is smooth over its Artin fan.

describe the latter as explicitly as possible; comment on the cones that are (possibly?) not there because of the compatibility of the alignment with the log map (this is probably awkward, useless, and superceded by saying "we align subdivide the ACGS minimal dual monoid") and because of smoothability/factorisation (this is probably related to tropical well-spacedness)

3. Description of the boundary

Note that f has contact order with H exactly equal to α_k at the marking p_k, or else the irreducible component of C on which p_k lies is mapped entirely inside H. By pulling back along f the equation defining H, and taking its α_k-th derivative at p_k (which makes sense because all the lower order derivatives do vanish by assumption) we single out the latter locus. By staring at the exact

sequence of jet bundles

$$0 \to p_k^* \Omega_C^{\otimes \alpha_k} \otimes \mathrm{ev}_k^* \mathcal{O}_{\mathbb{P}^N}(H) \to p_k^* \mathcal{J}^{\alpha_k}(f^* \mathcal{O}_{\mathbb{P}^N}(H)) \to p_k^* \mathcal{J}^{\alpha_k-1}(f^* \mathcal{O}_{\mathbb{P}^N}(H)) \to 0$$

we realise that there is on $\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)$ a line bundle (the leftmost term in the exact sequence above) with a natural section cutting the locus where the curve breaks, and the piece containing p_k is mapped inside H.

Lemma 3.1.

$$(\alpha_k \psi_k + \mathrm{ev}_k^* H)[\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)] = [D_{1,\alpha;k}(\mathbb{P}^N|H,d)],$$

In what follows we shall give an explicit (recursive) description of the right-hand side, in terms of tautological classes on spaces of maps with lower numerical invariants.

Remark 3.2. The divisors we are after are a union of logarithmic strata, because the locus where the log structure is trivial coincides precisely with the nice locus.

By Corollary 2.4, the logarithmic strata of codimension one correspond to rays of the Artin fan.

3.1. **Combinatorial description.** As long as the minimal subcurve of genus one has positive degree, the picture is classical: the dual graph of the curve is bipartite - vertices mapped inside H are distinguished from the others. In fact there is only one such vertex, because, if there were more, the relative position of their image in $\mathbb{R}_{\geq 0}$ under the tropical map would be unconstrained, therefore spanning a locus larger than a ray in the tropical moduli space. We follow in R. Vakil's footsteps in setting the following notation:

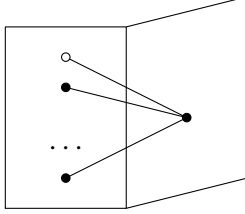say something about rigid tropical curves as in ACGS and KLR
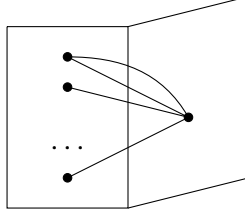

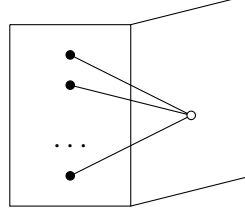FIGURE 1. $\mathcal{Y}_a$ FIGURE 2. $\mathcal{Y}_b$ FIGURE 3. $\mathcal{Y}_c^+$

Notice that continuity of the tropical map determines the mutual relationship among the edge lengths.

On the other hand, when there is a contracted elliptic subcurve - and it will be contracted into the hyperplane, because otherwise it wouldn't be generic, by density of the nice locus in $\mathcal{VZ}_{1,\alpha'}(\mathbb{P}^N|H,d')$ - the picture becomes more complicated due to the alignment. The combs may break (multiple times) and they may point in opposite directions. We label these loci $\mathcal{Y}_c^0$.

Definition 3.3. A degree-aligned genus one tropical map ϕ to $\mathbb{R}_{\geq 0}$ is ... with a circle of radius δ around the core satisfying:
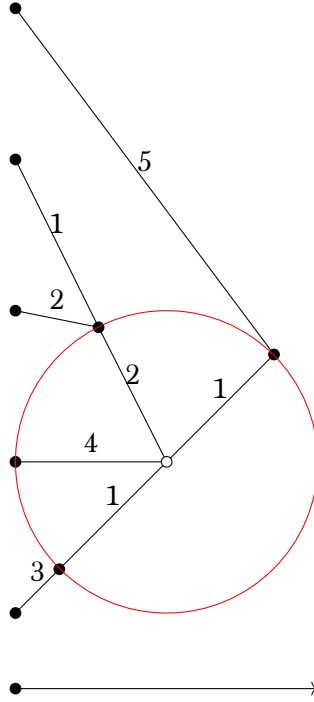
(1) the circle of radius δ passes through at least one vertex of positive degree, and no vertex of positive degree is contained in its strict interior;

(2) every circle of radius $< \delta$ around the core passes through (either none or) at least two vertices;[1]

(3) every edge heading out of the circle of radius δ goes directly to a vertex in $\phi^{-1}(0)$.

Remark 3.4. The core being contracted in the fiber of the tropical map is not a phenomenon that we should worry about in codimension one. Indeed, assume that the core is contracted in the fiber along a ray. Then all the edges departing from the core have expansion factor 0; call the corresponding coordinates $U = \{u_i\}_{i \in I}$. Call the remaining coordinates $E = \{e_j\}_{j \in J}$. Note that tropical continuity involves only E. Alignments on the other hand assume the form $\lambda(v) = \lambda(v')$, where $\lambda(v) = \sum_{i \in I(v)} u_i \sum_{j \in J(v)} e_j$. Pick the shortest elements of U; then these can be shortened to zero without affecting the rest (by hypothesis, alignments can only identify them among themselves). This shows that we could not have started with a ray.

3.2. **Explicit realisation.** The description of the first three types is relatively easy:

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[1]The subdivisions we perform before picking a ray correspond to aligning the log structure, i.e. imposing $\lambda(v) = \lambda(v')$ for some v, v'.

(a)

$$\mathcal{Y}_a = \overline{\mathcal{M}}_{0,\{-m^{(1)},\ldots,-m^{(r)}\}\cup\alpha^{(0)}}(H,d_0)^{\widetilde{}}\times_{H^r}\left(\mathcal{VZ}_{1,\alpha^{(1)}\cup\{m^{(1)}\}}(\mathbb{P}^N|H,d_1)\times\prod_{i=2}^{r}\overline{\mathcal{M}}_{0,\alpha^{(i)}\cup\{m^{(i)}\}}(\mathbb{P}^N|H,d_i)\right)$$

(b)

$$\mathcal{Y}_b = \overline{\mathcal{M}}_{0,\{-m^{(1)},\ldots,-m^{(r)}\}\cup\alpha^{(0)}}(H,d_0)^{\widetilde{}}\times_{H^r}\left(\overline{\mathcal{M}}_{0,\alpha^{(1)}\cup\{m^{(1)},m^{(2)}\}}(\mathbb{P}^N|H,d_1)\times\prod_{i=3}^{r}\overline{\mathcal{M}}_{0,\alpha^{(i)}\cup\{m^{(i)}\}}(\mathbb{P}^N|H,d_i)\right)$$

(c)

$$\mathcal{Y}_c^+ = \mathcal{VZ}_{1,\{-m^{(1)},\ldots,-m^{(r)}\}\cup\alpha^{(0)}}(H,d_0)^{\widetilde{}}\times_{H^r}\prod_{i=1}^{r}\overline{\mathcal{M}}_{0,\alpha^{(i)}\cup\{m^{(i)}\}}(\mathbb{P}^N|H,d_i)$$

where $\overline{\mathcal{M}}_{0,\mu}(H,d)^{\widetilde{}}$ denotes the space of genus zero punctured maps, while $\mathcal{VZ}_{0,\mu}(H,d)$ denotes the desingularisation of the main component of the space of punctured maps discussed in Section ???. The numerical data run over all splitting A of the tangency conditions (resp. B of the degree) in $r+1$ parts, such that:

(1) $d_0 + \sum_{i=1}^{r} m^{(i)} = \sum \alpha^{(0)}$;
(2) $d_i = m^{(i)} + \sum \alpha^{(i)}$;
(3) $d_0 > 0$ in case c.

Review here the process of gluing for punctured maps. In particular the basic monoid should be modified in order to make the relevant evaluations log morphisms. Claim: evaluations are strict. Consequence: the fiber product in the category of log stacks is fine. We have then to apply saturation. This is a finite morphism of degree... (I think it could be $\frac{\prod m^{(i)}}{lcm(m^{(i)})}$).

Lemma 3.5 (Virtual pushforward). The following hold.

- $\mathrm{fgt}_*[\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d)] = [\overline{\mathcal{M}}^{\mathrm{G}}_{0,\alpha}(\mathbb{P}^N|H,d)]$ (follows from [Gat03, AMW14]).
- $\mathrm{fgt}_*[\mathcal{VZ}_{1,\alpha}(\mathbb{P}^N|H,d)]$ computes the reduced relative invariants by definition.
- $\mathrm{fgt}_*[\overline{\mathcal{M}}_{0,\mu}(H,d_0)^{\widetilde{}}] = [\overline{\mathcal{M}}_{0,|\mu|}(H,d_0)]$ (follows from [Gat03] and... comparison of punctured with rubber invariants).
- $\mathrm{fgt}_*[\mathcal{VZ}_{1,\mu}(H,d_0)^{\widetilde{}}]$ here we should need a variation on Pixton's DRC formula; hopefully it's enough to avoid the graphs that tropical well-spacedness discards.

4. THE SPACE OF RELATIVE CENTRALLY ALIGNED MAPS

Recall [RSW17a] that the moduli space of maps from centrally aligned curves is obtained by considering the Cartesian diagram

$$\widetilde{\mathcal{VZ}}^{\mathrm{ctr}}_{1,n}(X,\beta) \longrightarrow \overline{\mathcal{M}}_{1,n}(X,\beta)$$

$$\downarrow \qquad\qquad \square \qquad\qquad \downarrow$$

$$\mathfrak{M}^{\mathrm{ctr}}_{1,n} \longrightarrow \mathfrak{M}^{\dagger}_{1,n}$$

so that objects of $\widetilde{\mathcal{VZ}}^{\mathrm{ctr}}_{1,n}(X,\beta)$ consist of

(1) a centrally aligned curve (C, M_C, δ);
(2) a stable map $f \colon C \to X$;

subject to the condition that the subcurve $C_0 \subseteq C$, consisting of those components C_v for which $\lambda(v) < \delta$, coincides with the maximal connected genus one subcurve contracted by f. They then define

$$\mathcal{VZ}_{1,n}(X,\beta) \subseteq \widetilde{\mathcal{VZ}}^{\mathrm{ctr}}_{1,n}(X,\beta)$$

to be the closed substack consisting of maps satisfying the *factorisation condition*, namely that the map $f \colon C \to X$ factors through the associated contraction to a Smyth curve, i.e. there exists a map $\bar{f}$ making the following square commute:

$$\widetilde{C} \longrightarrow \overline{C}$$

$$\downarrow \qquad \bar{f} \Big\downarrow$$

$$C \xrightarrow{\ f\ } X$$

One should think of the factorisation condition as identifying the main component of the moduli space.

5. Definition of centrally aligned relative space

6. Characterisation of the closed points of the relative space

7. Recursion formula

7.1. **The case $\sum \alpha = d$.**

7.2. **Definition of the boundary components.** According to Vakil's article there should be three sorts of boundary components.

(a)

$$\mathcal{Y}^a = \overline{\mathcal{M}}_{0,|\alpha^{(0)}|+r}(H,d_0) \times_{H^r} \left(\mathcal{VZ}_{1,\alpha^{(1)}\cup\{m^{(1)}\}}(\mathbb{P}^N|H,d_1) \times \prod_{i=2}^{r} \overline{\mathcal{M}}_{0,\alpha^{(i)}\cup\{m^{(i)}\}}(\mathbb{P}^N|H,d_i) \right)$$

(b)

$$\mathcal{Y}^b = \overline{\mathcal{M}}_{0,|\alpha^{(0)}|+r}(H,d_0) \times_{H^r} \left(\overline{\mathcal{M}}_{0,\alpha^{(1)}\cup\{m^{(1)},m^{(2)}\}}(\mathbb{P}^N|H,d_1) \times \prod_{i=3}^{r} \overline{\mathcal{M}}_{0,\alpha^{(i)}\cup\{m^{(i)}\}}(\mathbb{P}^N|H,d_i) \right)$$

(c)

$$\mathcal{Y}^c \subseteq \mathcal{VZ}_{1,|\alpha^{(0)}|+r}(H,d_0) \times_{H^r} \prod_{i=1}^{r} \overline{\mathcal{M}}_{0,\alpha^{(i)}\cup\{m^{(i)}\}}(\mathbb{P}^N|H,d_i)$$

$\mathcal{Y}^c$ is cut within the latter by Gathmann's condition on line bundles:

$$f_{|E}^* \mathcal{O}_H(1) \cong \mathcal{O}_E \left(\sum_{x_j \in E} \alpha_j x_j - \sum_{i=1}^r m^{(i)} y_i \right)$$

Express the line bundle condition in terms of tautological classes?

Notice that such condition depends on the choice of $m^{(i)}$ but is really just a condition on the first factor of the product, so $\mathcal{Y}^c$ itself can be expressed as a product.

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