RELATIVE STABLE MAPS IN GENUS ONE VIA CENTRAL ALIGNMENTS

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Abstract. For a smooth projective variety X and a smooth very ample hypersurface $Y \subseteq X$, we define moduli spaces of relative stable maps to (X, Y) in genus one, as closed substacks of the moduli space of maps from centrally aligned curves, as constructed in [RSW17]. We construct virtual classes for these moduli spaces, which we use to define *reduced relative Gromov–Witten invariants* in genus one.

[GOALS: We prove a recursion formula which allows us to completely determine these invariants in terms of the reduced Gromov–Witten invariants, as defined in [REF]. We also prove a relative version of the Li–Zinger formula, relating our invariants to the usual relative Gromov–Witten invariants. Also say something about quasimaps.]

Contents

1. The space of relative centrally aligned maps

Recall [RSW17] that the moduli space of maps from centrally aligned curves is obtained by first considering the Cartesian diagram

$$\begin{tikzcd} \widetilde{\mathcal{VZ}}^{\mathrm{ctr}}_{1,n}(X,\beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_{1,n}(X,\beta) \arrow[d] \\ \mathfrak{M}^{\mathrm{ctr}}_{1,n} \arrow[r] & \mathfrak{M}^{\dagger}_{1,n} \end{tikzcd}$$

so that objects of $\widetilde{\mathcal{VZ}}^{\mathrm{ctr}}_{1,n}(X,\beta)$ consist of

(1) a centrally aligned curve $(C, \mathcal{M}_C, \delta)$;

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(2) a stable map $f \colon C \to X$;

subject to the condition that the subcurve $C_0 \subseteq C$, consisting of those components C_v for which $\lambda(v) < \delta$, coincides with the maximal connected genus one subcurve contracted by f. They then define

$$\mathcal{VZ}^{\mathrm{ctr}}_{1,n}(X,\beta) \subseteq \widetilde{\mathcal{VZ}}^{\mathrm{ctr}}_{1,n}(X,\beta)$$

to be the closed substack consisting of maps satisfying the *factorisation condition*, namely that the map $f \colon C \to X$ factors through the associated contraction to a Smyth curve, i.e. there exists a map $\bar{f}$ making the following square commute:

$$\begin{tikzcd} \widetilde{C} \arrow[r] \arrow[d] & \overline{C} \arrow[d, dotted, "\bar{f}"'] \\ C \arrow[r, "f"] & X \end{tikzcd}$$

One should think of the factorisation condition as identifying the main component of the moduli space.

2. Definition of centrally aligned relative space

Fix a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ of tangency conditions, and suppose for the moment that $\Sigma\alpha = d$. We consider the so-called *nice locus* of relative centrally aligned maps. This is the locally closed substack of $\mathcal{VZ}^{\mathrm{ctr}}_{1,n}(\mathbb{P}^N, d)$ consisting of centrally aligned maps $(C, x_1, \ldots, x_n, \mathcal{M}_C, \delta, f)$ such that:

(1) the source curve C is smooth;
(2) C is not mapped entirely into H by f;
(3) the points of C which are mapped into H (necessarily a finite number by the previous condition) are all marked, with the map having tangency α_i to H at the marking x_i.

These conditions mean that $\delta = 0$ and that the log structure $\mathcal{M}_C$ (which is a minimal centrally aligned log structure) is equal to the minimal structure for a log smooth curve, which in this case (since there are no nodes) is just the divisorial log structure with respect to the markings $x_1, \ldots, x_n$.

This is a locally closed substack of expected codimension $\Sigma\alpha = d$, and is denoted:

$$\mathcal{VZ}^{\mathrm{ctr},\circ}_{1,\alpha}(\mathbb{P}^N | H, d)$$

It has a well-defined stack structure since it is a moduli space. Note that it is irreducible, since we can find a parametrisation for it over $\mathfrak{M}_{1,n}$.

We now define the *centrally alined relative space* to be the closure of the nice locus:

$$\mathcal{VZ}^{\mathrm{ctr}}_{1,\alpha}(\mathbb{P}^N | H, d) := \overline{\mathcal{VZ}^{\mathrm{ctr},\circ}_{1,\alpha}(\mathbb{P}^N | H, d)} \subseteq \mathcal{VZ}^{\mathrm{ctr}}_{1,n}(\mathbb{P}^N, d)$$

This inherits the natural structure of a stack from the stack structure of the nice locus. It is proper since it is a closed substack of a proper stack, and it is irreducible because it is the closure of an irreducible stack. Hence it admits a fundamental class, which we can use to define *reduced relative Gromov–Witten invariants*:

$$\langle \tau_{k_1}(\gamma_1), \ldots, \tau_{k_n}(\gamma_n) \rangle_{0,\alpha,d}^{\mathbb{P}^N|H,\mathrm{red}} := \int_{[\mathcal{VZ}_{1,\alpha}^{\mathrm{ctr}}(\mathbb{P}^N|H,d)]} \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i)\psi_i^{k_i}$$

3. Characterisation of the closed points of the relative space

It will be important for our purposes to have an understanding of the boundary of the relative space, i.e. what we add in when we take the closure of the nice locus. In this section we will describe this boundary on the level of closed points.

Proposition 3.1. A centrally aligned map over a closed point $\operatorname{Spec} \Bbbk$ is in the relative space if and only if the following two conditions are satisfied:

1. *Gathmann's relative condition:* for each connected component Z of $f^{-1}(Y)$, either:
   - (a) Z is a single point of C, in which case if $Z = x_i$ is a marking then we require that the multiplicity of f at x_i along Y is at least α_i (if Z is not a marking then there is no condition);
   - (b) Z is a subcurve $Z \subseteq C$, in which case we require that

     $$f^*[Y]|_Z - \sum_{x_i \in Z} \alpha_i x_i$$

     is an effective class in $\mathrm{A}_0(Z)$.
2. *Compatibility condition:* this applies when the minimal subcurve of genus one (the *circuit*) is contracted to a point in Y. In this case we require that there is a local morphism of monoids:

   $$Q^{\mathrm{GS}} \to \mathbb{N}$$

   such that the dashed arrow can be filled in to make the following square commutative:

   $$\begin{tikzcd} Q^{\log} \arrow[r] \arrow[d] & Q^{\mathrm{GS}} \arrow[d] \\ Q^{\mathrm{ctr}} \arrow[r, dashed] & \mathbb{N} \end{tikzcd}$$

   It is obvious from the construction that the map $Q^{\mathrm{ctr}} \to \mathbb{N}$ is unique if it exists

Remark 3.2. The monoid Q^{GS} depends on a choice of *type*, in this case a choice of the u_q (the u_p are already determined by the choice of α). Here we are not allowed to choose a random type, but have to choose one which is compatible with the underlying stable map, in the sense that the modified tropical balancing condition is satisfied. The Gathmann-style condition (1) is a necessary condition for the existence of a compatible type (on the level of $\operatorname{Pic} C$).

Remember that we get one balancing condition for each generator of $\Gamma(C, f^*\overline{\mathcal{M}}_H)$, so as well as there being a balancing condition for each internal component there is also one wherever the curve intersects the divisor in an isolate point (where for instance we get that $u_q = m_q$ at nodes connecting an external to an internal component). Also we require this condition to take place in $\mathrm{Pic}(C)$, but we are entitled at any point to restrict to components of C or take degrees to get (in general weaker) conditions.

Remark 3.3. Gathmann's condition obviously extends to the case $\Sigma\alpha \leq d$. [1] Here we try to make the condition for a curve component of $f^{-1}(Y)$ into an easily verifiable criterion: if Z is a smooth elliptic curve, recall that every line bundle of positive degree on it is effective, hence the relative condition can be rephrased as the following numerical criterion:

$$f_*[Z]\cdot[Y] + \sum_{j=1}^{r} m^{(j)} \geq \sum_{x_i\in Z} \alpha_i$$

together with the additional requirement that, when the above is an equality, there is an isomorphism of line bundles:

$$(f|_Z)^*\mathcal{O}_X(Y)\otimes\mathcal{O}_Z\left(\sum_{j=1}^{r} m^{(j)}y_j\right) = \mathcal{O}_Z\left(\sum_{x_i\in Z}\alpha_i x_i\right) \tag{1}$$

If Z is reducible, the correct extension is not to impose the same condition componentwise; rather, we should ask the numerical condition for the total degree, and, if Z is obtained by gluing a smooth elliptic curve E with a forest of rational trees T_i at roots q_i^E, then the line bundle equality (2) should be required in $\mathrm{Pic}(E)$ after counting each q_i^E towards the left hand side with multiplicity:

$$f_*[T_i]\cdot[Y] + \sum_{\text{external components attached to } T_i} m^{(j)} \geq \sum_{x_j\in T_i} \alpha_j$$

Proof of Proposition 3.1. First suppose that the conditions (1) and (2) of the Proposition are satisfied; so we have Gathmann's condition on the level of line bundles and a factorisation:

$$\begin{tikzcd} Q^{\log} \arrow[r] \arrow[d] & Q^{\mathrm{GS}} \arrow[d] \\ Q^{\mathrm{ctr}} \arrow[r] & \mathbb{N} \end{tikzcd}$$

We will construct a smoothing of the centrally aligned map, i.e. a 1-parameter family whose central fibre is the map we started with and whose general fibre is in the nice locus. The construction proceeds in the following order:

(1) construct the smoothing $\mathcal{C}$ of the source curve C, as a scheme;

[1] What is the correct generalisation of the compatibility condition?

(2) construct the extension $(\mathcal{M}_{\mathcal{C}}, \delta)$ of the central alignment on C;
(3) construct the extension $\mathcal{L}$ of the line bundle $L = f^*\mathcal{O}_{\mathbb{P}^N}(H)$ from C to $\mathcal{C}$;
(4) construct the extension of the section $u_0 \in \mathrm{H}^0(C, L)$;
(5) construct the extensions of the sections $u_1, \ldots, u_N$.

We start with (1), i.e. the construction of the source curve $\mathcal{C}$ as a family over $\operatorname{Spec} \Bbbk[\![t]\!]$. Let $E \subseteq C$ be the circuit, i.e. the minimal genus one subcurve. This is either a smooth elliptic curve or a wheel of rational curves. Look at the corresponding moduli point $[E] \in \mathfrak{M}_{1,n}$. Since this is a smooth stack, we may choose an étale chart which locally looks like the tangent space of the stack, which in this case is the deformation space of the elliptic curve. This contains a factor $\mathbb{A}^p$ where p is the number of nodes of C; the ith co-ordinate axis corresponds to a smoothing of the ith node, keeping the others fixed, while the generic point of the corresponding orthogonal hyperplane corresponds to a deformation which smooths all the nodes except the ith one.

We need to choose a map $\operatorname{Spec} \Bbbk[\![t]\!] \to \mathfrak{M}_{1,n}$ which sends the closed point to $[E]$ and the generic point to the locus of smooth curves. By the discussion of the previous paragraph, we can do this by choosing a map $\operatorname{Spec} \Bbbk[\![t]\!] \to \mathbb{A}^p$ which sends the closed point to the origin and the generic point to the complmenet $\mathbb{G}_{\mathrm{m}}^p \subseteq \mathbb{A}^p$ of the co-ordinate axes. We can certainly choose such a map; furthermore, if for each node q we choose a positive integer r_q, then we can choose the map so that it has tangency order r_q to the hyperplane of $\mathbb{A}^p$ corresponding to q. In terms of the resulting curve $\mathcal{C}$ over $\operatorname{Spec} \Bbbk[\![t]\!]$, this means that if $C_1, C_2 \subseteq \mathcal{C}$ are the two branches of C around q, then these are $\mathbb{Q}$-Cartier divisors in $\mathcal{C}$, with:

$$C_1 \cdot C_2 = 1/r_q$$

The weights r_q are chosen by looking at the map $Q^{\log} \to \mathbb{N}$; this map is local since the map $Q^{\log} \to Q^{\mathrm{GS}}$ is local[2] and the map $Q^{\mathrm{GS}} \to \mathbb{N}$ is local by assumption. Thus each generator $e_q \in Q^{\log}$ maps to a positive integer r_q, and this is the weight we choose. We have thus constructed the smoothing $\mathcal{C}$ of C.

We now move on to step (2), which involves extending the central alignment from C to $\mathcal{C}$. This is a relative trivial step: since the general fibre of $\mathcal{C}$ is smooth, there is no choice of ordering of the markings or of contraction radius, and its minimal centrally aligned monoid is trivial. We can thus take the log structures on the central fibre as charts for the log structures on the deformation. To be more precise, in the base we have a chart:

$$\begin{aligned} Q^{\mathrm{ctr}} &\to \Bbbk[\![t]\!] \\ e_q &\mapsto t^{r_q} \end{aligned}$$

Note that this is well-defined by the fact that we have the factorisation $Q^{\mathrm{ctr}} \to \mathbb{N}$. This defines a log structure $\mathcal{M}_S^{\mathrm{ctr}}$ on S. On the other hand the morphism

[2] Is this always true? Isn't this only true if there exists a log lifting? But maybe it's always true for Deligne–Faltings pairs.

$S \to \mathfrak{M}_{1,n}$ defines another log structure $\mathcal{M}_S^{\log}$ on S corresponding to the minimal structure of a log smooth curve on $\mathcal{C}$. There is a map $\mathcal{M}_S^{\log} \to \mathcal{M}_S^{\mathrm{ctr}}$ induced by the map of charts $Q^{\log} \to Q^{\mathrm{ctr}}$, and this gives a diagram:

$$\begin{tikzcd} & (C, \mathcal{M}_C^{\log}) \arrow[d] \\ (S, \mathcal{M}_S^{\mathrm{ctr}}) \arrow[r] & (S, \mathcal{M}_S^{\log}) \end{tikzcd}$$

We define $(\mathcal{C}, \mathcal{M}_{\mathcal{C}}^{\mathrm{ctr}})$ to be the fibre product. This is clearly a log smooth centrally aligned curve over $(S, \mathcal{M}_S^{\mathrm{ctr}})$, with radius δ given by the radius on the central fibre (note that $\overline{\mathcal{M}_S}^{\mathrm{ctr}}$ is supported on the closed point). This completes step (2).

We now begin step (3), that of extending the line bundle $L = f^*\mathcal{O}_{\mathbb{P}^N}(H)$ on C to a line bundle $\mathcal{L}$ on $\mathcal{C}$. In case it hasn't been mentioned already, we assume that $H = \{z_0 = 0\}$ is a co-ordinate hyperplane.

Let us write C in terms of its irreducible components as $C = C_1 \cup \ldots \cup C_m$ with $C_1, \ldots, C_r$ internal and the rest external. We define the line bundle $\mathcal{L}$ as

$$\mathcal{L} = \mathcal{O}_{\mathcal{C}}\left(\sum_{i=1}^{r} \gamma_i C_i + \sum_{i=1}^{n} \alpha_i \tilde{x}_i\right)$$

where each γ_i is a positive integer obtained as the image under $Q^{\mathrm{GS}} \to \mathbb{N}$ of the generator corresponding to C_i, and the $\tilde{x}_i$ are extensions of the Cartier divisors x_i to all of $\mathcal{C}$.

This comes with a natural section (or rather, a $\mathbb{C}^*$-equivalence class of sections) $\tilde{u}_0$ given by the divisor in brackets above, and this can be chosen in a unique way so that it restricts to u_0 on the central fibre of $\mathcal{C}$. This completes step (4).

Now we must verify that $\mathcal{L}|_C = L$. For each external component C_j, we have

$$\mathcal{L}|_{C_j} = \mathcal{O}_{C_j}\left(\sum_{x_i \in C_j} \alpha_i x_i + \sum_{q_i \in C_j} (\gamma_i / r_i) q_j\right)$$

where the latter sum is over nodes of C_j which connect C_j to an internal component C_i. We thus need

$$m_i = \gamma_i / r_i$$

in order to conclude that $\mathcal{L}|_{C_j} = L|_{C_j}$. But this is true, because in Q^{GS} the generator corresponding to the internal component C_i is identified with u_{q_i} times the generator corresponding to q_i, and $u_{q_i} = m_i$ by the fact that the type we chose to define Q^{GS} must satisfy the tropical balancing condition. Pushing forward this relation from Q^{GS} to $\mathbb{N}$ gives the equation above. This argument also applies when we have an external subcurve $Z \subseteq C$ which is a wheel of rational components.

Now let us consider an internal component C_j. We have

$$\begin{aligned}\mathcal{L}|_{C_j} &= \mathcal{O}_{C_j}\left(\sum_{x_i\in C_j}\alpha_i x_i + \sum_{\tilde{q}_i\in C_j}(\gamma_i/r_i)\tilde{q}_i\right)\otimes\mathcal{O}_{\mathcal{C}}\left(-\gamma_j\sum_{i\neq j}C_i\right)\\ &= \mathcal{O}_{C_j}\left(\sum_{x_i\in C_j}\alpha_i x_i + \sum_{q_i\in C_j}\left(\frac{\gamma_i-\gamma_j}{r_i}\right)q_i - \sum_{\tilde{q}_i\in C_j}m_i\tilde{q}_i\right)\end{aligned}$$

where the middle sum is over adjacent internal components and the final sum is over adjacent external components. The fact that $\gamma_j/r_i = m_i$ follows as in the previous case.

We must argue that $\mathcal{L}|_{C_j} = L|_{C_j}$. Suppose for the moment that C_j is genus zero, so that we only need to check the degrees. We must show:

$$\deg L|_{C_j} = \sum_{x_i\in C_j}\alpha_i - \sum_{q_i\in C_j}\left(\frac{\gamma_j-\gamma_i}{r_i}\right) - \sum_{\tilde{q}_i\in C_j}u_{\tilde{q}_i}$$

But by the definition of Q^{GS} we have:

$$(\gamma_j-\gamma_i)/r_i = u_{q_i}$$

From which the above formula holds because of the tropical balancing condition we imposed when choosing the type.

Similarly if C_j is smooth of genus one we must show:

$$L|_{C_j} = \mathcal{O}_{C_j}\left(\sum_{x_i\in C_j}\alpha_i x_i - \sum_{q_i\in C_j}u_{q_i}q_i - \sum_{\tilde{q}_i\in C_j}u_{\tilde{q}_i}\tilde{q}_i\right)$$

But this is just the balancing condition on the level of $\mathrm{Pic}(C_j)$.

What about if C_j is a wheel of rational components? Then it isn't irreducible, so let's write it as $Z = C_1\cup\ldots\cup C_k$. So there are other internal components $C_{k+1},\ldots,C_r$ and external components $C_{r+1},\ldots,C_m$. We assume that $C_1,\ldots,C_k$ are ordered so that C_i intersects C_{i-1}, C_{i+1} and none others. Notice that this means that:

$$C_i^2 = -1/r_{i-1} - 1/r_{i+1} - \sum_{q_j\in C_i}1/r_j$$

for all $i \in \{1, \ldots, k\}$, where the sum is over all nodes q_j which connect Z to the rest of the curve. Thus we obtain:

$$\begin{aligned}
\mathcal{L}|_Z &= \mathcal{O}_Z\left(\sum_{x_i \in Z} \alpha_i x_i\right) \otimes \mathcal{O}_{\mathcal{C}}\left(\sum_{i=1}^{r} \gamma_i C_i\right)\Big|_Z \\
&= \mathcal{O}_Z\left(\sum_{x_i \in Z} \alpha_i x_i + \sum_{\substack{q_i \in Z \\ q_i \text{ joins } Z \\ \text{to int. cpt.}}} (\gamma_i / r_i) q_i\right) \otimes \mathcal{O}_{\mathcal{C}}\left(\sum_{i=1}^{k} \gamma_i C_i\right)\Big|_Z \\
&= \mathcal{O}_Z\left(\sum_{x_i \in Z} \alpha_i x_i + \sum_{\substack{q_i \in Z \\ q_i \text{ joins } Z \\ \text{to int. cpt.}}} (\gamma_i / r_i) q_i + \sum_{i=1}^{k} \gamma_i \left(\frac{1}{r_{i,i-1}} q_{i-1} + \frac{1}{r_{i,i+1}} q_{i+1} + (C_i^2) s_i\right)\right)
\end{aligned}$$

where q_{i-1} is the node connecting C_{i-1} and C_i (and similarly for q_{i+1}) and s_i is some point of C_i (doesn't really matter which, since C_i is rational).

Using the description of C_i^2 given above, we see that the terms $q_{i-1}/r_{i,i-1}$ and $q_{i+1}/r_{i,i+1}$ cancel, and we are left with

$$\begin{aligned}
\mathcal{L}|_Z &= \mathcal{O}_Z\left(\sum_{x_i \in Z} \alpha_i x_i + \sum_{\substack{q_i \in Z \\ q_i \text{ joins } Z \\ \text{to int. cpt.}}} (\gamma_i / r_i) q_i + \sum_{\substack{q_i \in Z \\ q_i \text{ joins } Z \\ \text{to rest of curve}}} (\gamma_i / r_i) q_i\right) \\
&= \mathcal{O}_Z\left(\sum_{x_i \in Z} \alpha_i x_i + \sum_{\substack{q_i \in Z \\ q_i \text{ joins } Z \\ \text{to int. cpt.}}} \left(\frac{\gamma_i - \gamma_j}{r_i}\right) q_i - \sum_{\substack{q_i \in Z \\ q_i \text{ joins } Z \\ \text{to ext. cpt}}} (\gamma_i / r_i) q_i\right)
\end{aligned}$$

and this is precisely the balancing condition on the level of $\mathrm{Pic}(Z)$. This proves that $\mathcal{L}|_C = L$, except for one case which we haven't yet considered, namely when there is a wheel of rational curves which is partially internal and partially external. In this case essentially the same argument as given above applies.

We have thus shown that we can break C up into pieces (mostly into its irreducible components, unless there is a rational wheel in which case we treat this as a single piece) such that the corresponding intersection graph has genus zero and such that $\mathcal{L}$ restricts to L on each of the pieces. The fact that $\mathcal{L}|_C = L$ then immediately follows (since the intersection graph has genus zero, so the Picard group of C is equal to the direct sum of the Picard groups of the pieces). This completes the proof of step (3).

As mentioned above, step (4) is immediate by the construction of $\mathcal{L}$. It remains to carry out step (5), i.e. to extend the sections $u_1, \ldots, u_N$. By standard results in deformation theory, obstructions to extending sections of a deformation of L are contained in $\mathrm{H}^1(C, L)$. If L is non-trivial on the circuit, then this space vanishes and so the sections extend immediately.

Thus the only case we need to check is when L is trivial on the circuit, which means that the circuit E (again, either a smooth elliptic curve or a wheel of rational components) is contracted by the map f. In this case we want to extend the sections by referring to the corresponding Smyth curve. From the centrally aligned curve $\mathcal{C}$ we obtain a diagram:

$$\begin{tikzcd} & \tilde{\mathcal{C}} \arrow[dl] \arrow[dr] & \\ \mathcal{C} & & \overline{\mathcal{C}} \end{tikzcd}$$

[3]

We have thus constructed the smoothing, which shows that conditions (1) and (2) above are sufficient to characterise points in the closure of the nice locus.

We will now show that these conditions are necessary. Suppose therefore that there is a family $\mathcal{C} \to \operatorname{Spec} \Bbbk[\![t]\!]$ and consider the central fibre $\mathcal{C}_0 = C$. Condition (1) for C follows immediately from continuity of intersection products.

To show condition (2), let $u_0 = f^*(z_0) \in \mathrm{H}^0(\mathcal{C}, f^*\mathcal{O}_{\mathbb{P}^N}(H))$. For each component C_i of the central fibre C, let γ_i be the vanishing order of u_0 along the Weil divisor C_i. For each internal component of C this defines a non-zero map $\gamma_i \colon \mathbb{N} \to \mathbb{N}$. On the other hand, for each node q consider the corresponding generator $e_q \in Q^{\mathrm{ctr}}$. A chart for the log structure on the base is given by a map

$$Q^{\mathrm{ctr}} \to \Bbbk[\![t]\!]$$

which we can compose with the valuation on $\Bbbk[\![t]\!]$ to get a map:

$$Q^{\mathrm{ctr}} \to \mathbb{N}$$

Let r_q be the image of e_q along this map. This is always non-zero because every node is smoothed in the family $\mathcal{C}$; thus we have a non-zero map $r_q \colon \mathbb{N} \to \mathbb{N}$. These, together with the γ_i defined above, give a local morphism

$$\prod_{q \in C} \mathbb{N}_q \times \prod_{C_i} \mathbb{N}_{\eta_i} \to \mathbb{N}$$

where the second product is over the internal components of C. We claim that this descends to the quotient Q^{GS} (recall again that this quotient depends on a choice of type). We have to check that for every node q we have

$$u_q r_q = \gamma_i - \gamma_j$$

[3] Finish this

where C_i and C_j are the adjacent components (this makes for all nodes, since $\gamma_i = 0$ for any external component). Remember that the u_q are not fixed a priori; thus we can think of the above formula as determining the u_q. What we then need to check is that the u_q thus defined are all integers, and that they satisfy the tropical balancing condition on the level of $\mathrm{Pic}(C)$.

Consider the dual graph of C, treating the circuit E as a single node of this graph, so that the graph has genus zero. Start from a leaf C_i of this graph, so that there is only one node q of the corresponding component. Let C_j be the unique component of C which is adjacent to C_i (via q). Then we have:

$$\deg f|_{C_i} = \deg \mathcal{L}|_{C_i} = (\gamma_i - \gamma_j)/r_q + \sum_{x_k \in C_i} \alpha_k$$

Rearranging, we see that $(\gamma_i - \gamma_i)/r_q$ is an integer. Substiting in u_q for this value, we obtain precisely the balancing condition, which we can view as happening on the level of $\mathrm{Pic}(C_j)$ (if we refrain from taking the degree first).

We now proceed by induction. The process is as follows: having determined q, we pass to the next component C_j. If this has only one other node r besides q, we apply the same argument as before to show that u_r is an integer and satisfies the balancing condition. If it contains more than one other node, we pick one of them and follow the resulting path through the dual graph until we arrive at a leaf with a single undetermined node (which must exist, since otherwise the node we started with would have been determined). We then start the process again from there.

This procedure proves that all the u_q are integers and that the balancing condition is satisfied on every component, except for those nodes and those components belonging to the circuit E. If the circuit is smooth there is nothing additional to prove, except to note that the Gathmann-like condition (1) ensures that the balancing condition will be satisfied on the level of $\mathrm{Pic}(E)$ (rather than just on the level of degree), so that it is satisfied for $\mathrm{Pic}(C)$.

It remains to consider the case when E is a wheel of rational components. Again the Gathmann-like condition (1) ensures we have balancing on the level of $\mathrm{Pic}(E)$, once we show that all the u_q for the nodes internal to E are integers. Labelling the components of E as $C_1, \ldots, C_k$ with C_i intersecting C_{i-1} and C_{i+1}, we obtain relations:

$$u_{q_i} r_{q_i} = \gamma_{i+1} - \gamma_{i-1}$$

Thus $u_{q_i} \in r_{q_i}^{-1} \cdot \mathbb{Z}$. But if we restrict $\mathcal{L}$ to each component and look at the resulting equation, we see that in fact $u_{q_i} \in (1/r_{q_j})\mathbb{Z}$ for all j. Thus $u_{q_i} \in (1/g)\mathbb{Z}$ where $g = \gcd(r_{q_1}, \ldots, r_{q_k})$. [4]

□

[4] Maybe we can fix this using saturation: note that we only need a map from the saturation of the quotient to $\mathbb{N}$, not from the quotient itself.

4. Recursion formula for $\mathbb{P}^1|\infty, \Sigma\alpha = d$

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5. Recursion formula

5.1. The case $\sum \alpha = d$.

5.2. **Definition of the boundary components.** According to Vakil's article there should be three sorts of boundary components.

(a)

$$\mathcal{Y}^a = \overline{\mathcal{M}}_{0,|\alpha^{(0)}|+r}(H, d_0)\times_{H^r}\left(\mathcal{VZ}^{\mathrm{ctr}}_{1,\alpha^{(1)}\cup\{m^{(1)}\}}(\mathbb{P}^N|H, d_1)\times\prod_{i=2}^{r}\overline{\mathcal{M}}_{0,\alpha^{(i)}\cup\{m^{(i)}\}}(\mathbb{P}^N|H, d_i)\right)$$

(b)

$$\mathcal{Y}^b = \overline{\mathcal{M}}_{0,|\alpha^{(0)}|+r}(H, d_0)\times_{H^r}\left(\overline{\mathcal{M}}_{0,\alpha^{(1)}\cup\{m^{(1)},m^{(2)}\}}(\mathbb{P}^N|H, d_1)\times\prod_{i=3}^{r}\overline{\mathcal{M}}_{0,\alpha^{(i)}\cup\{m^{(i)}\}}(\mathbb{P}^N|H, d_i)\right)$$

(c)

$$\mathcal{Y}^c \subseteq \mathcal{VZ}^{\mathrm{ctr}}_{1,|\alpha^{(0)}|+r}(H, d_0)\times_{H^r}\prod_{i=1}^{r}\overline{\mathcal{M}}_{0,\alpha^{(i)}\cup\{m^{(i)}\}}(\mathbb{P}^N|H, d_i)$$

$\mathcal{Y}^c$ is cut within the latter by Gathmann's condition on line bundles:

$$f^*_{|E}\mathcal{O}_H(1) \cong \mathcal{O}_E\left(\sum_{x_j\in E}\alpha_j x_j - \sum_{i=1}^{r} m^{(i)} y_i\right)$$

Notice that such condition depends on the choice of $m^{(i)}$ but is really just a condition on the first factor of the product, so $\mathcal{Y}^c$ itself can be expressed as a product.

Express the line bundle condition in terms of tautological classes?

Remark 5.1. Notice that gluing does not present the subtleties to which log geometry has made us used, since we endow the curves with the minimal centrally aligned structure, possibly after prescribing that the smoothing parameters for the newly created nodes are *bigger* than the contraction radius δ.

Remark 5.2. A dimensional computation shows that $\mathcal{Y}^b$ is never relevant to the purpose of the recursion formula. On the other hand the classes of $\mathcal{Y}^a$ and $\mathcal{Y}^c$ should be weighted with the coefficient $\frac{\prod m^{(i)}}{r!}$. Let $D_{1,\alpha,k}(\mathbb{P}^N|H)$ be the union of all the $\mathcal{Y}^a$ and $\mathcal{Y}^c$ as above, for any choice of number of external components r, distribution of the markings A, and of the degree D, provided the marked point x_k lies on the internal component $C^{(0)}$. Let each component be endowed with its fundamental class weighted as above and let $[D_{1,\alpha,k}(\mathbb{P}^N|H)]^{\mathrm{virt}}$ be the sum of all these classes.

Theorem 5.3. Assume $\sum \alpha = d$. Then

$$(\alpha_k \psi_k + \mathrm{ev}_k^* H)[\mathcal{VZ}^{\mathrm{ctr}}_{1,\alpha}(\mathbb{P}^N|H,d)] = [D_{1,\alpha,k}(\mathbb{P}^N|H)]^{\mathrm{virt}}.$$

Proof. We perform a number of reductions.

- **Reduction to the case of $\mathbb{P}^1|\{\infty\}$**: by choosing a general $N-2$-plane $A \subseteq H$ and projecting away from it. Must show that the rational map $\rho_A \colon \mathcal{VZ}^{\mathrm{ctr}}_{1,n}(\mathbb{P}^N|H,d) \dashrightarrow \mathcal{VZ}^{\mathrm{ctr}}_{1,n}(\mathbb{P}^1|\{\infty\},d)$ is defined and smooth at a general point of $D_{1,\alpha,k}(\mathbb{P}^N|H)$.
- **Reduction to an étale neighbourhood of $C^{(0)}$**: the plan is to understand the global perfect obstruction theory of $\mathcal{VZ}^{\mathrm{ctr}}_{1,n}(\mathbb{P}^1,d)$ and see that the tangent to the latter is a sum of contributions from the *special loci*, namely contracted components, ramification points, markings and nodes.

Remark 5.4. When we are studying the infinitesimal deformation theory of centrally aligned curves, notice that being centrally aligned must be checked on geometric points, and minimality is open, hence any first-order deformation as a log smooth curve will automatically be minimal centrally aligned. The tangent space is then $H^1(C, T_C^{\log})$; notice that this is independent from the log structure on the base.

Example 5.5. There are non-trivial deformations that are trivial on the underlying marked prestable curve. If $(C, \mathcal{M}_C) \to (\mathrm{Spec}(k), Q^{\mathrm{cen}})$ is a centrally aligned curve, then in order to define a deformation over $\mathrm{Spec}(k[\epsilon])$ we need to extend the log structure on the base by choosing a map $Q^{\mathrm{cen}} \to k[\epsilon]$. Notice that if we want the underlying curve to be a trivial deformation we better send all the smoothing parameters to 0 in $k[\epsilon]$; on the other hand, if Q^{cen} contains an element such as $e_i - e_j$ (making R_i further from the circuit than R_j), then we are free to map it to an invertible element in $k[\epsilon]$, which will kill it in the associated log structure; this will make R_i and R_j be *at the same distance* from the circuit in the deformation. Importantly, $Q^{\log} \to Q^{\mathrm{cen}}$ is an epimorphism only in the category of integral monoid (of which the multiplicative monoid of a ring is never an object, because it has a sink).

Invertible elements of $k[\epsilon]$ remain invertible when setting $\epsilon = 0$, which seems to kill $e_i - e_j$ on the central fiber too.

∎

Appendix A. Old

A.1. **Definition of the relative space.** We now give a definition of the relative space inside the moduli space of centrally aligned stable maps.

Definition A.1. Let X be a smooth projective variety with a smooth divisor Y. Fix a vector of tangency conditions $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\Sigma_{i=1}^n \alpha_i = Y \cdot \beta$. Then the *centrally aligned relative space*

$$\mathcal{VZ}^{\mathrm{ctr}}_{1,\alpha}(X|Y, \beta)$$

is defined to be the closed substack of $\mathcal{VZ}^{\mathrm{ctr}}_{1,n}(X, \beta)$ satisfying the following conditions:

(1) *Gathmann's relative condition:* for each connected component Z of $f^{-1}(Y)$, either:
   (a) Z is a single point of C, in which case if $Z = x_i$ is a marking then we require that the multiplicity of f at x_i along Y is at least α_i (if Z is not a marking then there is no condition);
   (b) Z is a subcurve $Z \subseteq C$, in which case we require that
   $$f^*[Y]|_Z - \sum_{x_i \in Z} \alpha_i x_i$$
   is an effective class in $\mathrm{A}_0(Z)$.

(2) *Compatibility condition:* this applies when the minimal subcurve of genus one (the *circuit* or *core*) is contracted to a point in Y. In this case we require that the following diagram of solid arrows may be completed to a commutative diagram:

$$\begin{tikzcd} Q^{\mathrm{prestable}}_{\min} \arrow[r] \arrow[d] & Q^{\mathrm{log.map}}_{\min} \\ Q^{\mathrm{cen.align}}_{\min} \arrow[ur, dashed] & \end{tikzcd}$$

Remark A.2 (Explaining Gathmann). Gathmann's condition is promptly extended to the case that $\Sigma_{i=1}^n \alpha_i \leq Y \cdot \beta$. Question: what is the correct generalisation of the compatibility condition? Here we try to make the condition for a curve component of $f^{-1}(Y)$ into an easily verifiable criterion: if Z is a smooth elliptic curve, recall that every line bundle of positive degree on it is effective, hence the relative condition can be rephrased as the following numerical criterion:

$$f_*[Z] \cdot [Y] + \sum_{j=1}^r m^{(j)} \geq \sum_{x_i \in Z} \alpha_i$$

together with the additional requirement that, when the above is an equality, there is an isomorphism of line bundles:

$$(f|_Z)^* \mathcal{O}_X(Y) \otimes \mathcal{O}_Z\left(\sum_{j=1}^r m^{(j)} y_j\right) = \mathcal{O}_Z\left(\sum_{x_i \in Z} \alpha_i x_i\right) \tag{2}$$

If Z is reducible, the correct extension is not to impose the same condition componentwise; rather, we should ask the numerical condition for the total degree, and, if Z is obtained by gluing a smooth elliptic curve E with a forest of rational trees T_i at roots q_i^E, then the line bundle equality (2) should be required in $\mathrm{Pic}(E)$ after counting each q_i^E towards the left hand side with multiplicity

$$f_*[T_i]\cdot[Y]+\sum_{\text{external components attached to } T_i} m^{(j)}-\sum_{x_j\in T_i}\alpha_j.$$

Remark A.3 (Explaining compatibility). First we need to define the monoids and maps involved. We start with $Q_{\min}^{\text{prestable}}\to Q_{\min}^{\text{log.map}}$; when looking at a geometric point $\mathrm{Spec}(k=\bar{k})$ it is well-known that

$$Q_{\min}^{\text{prestable}}=\prod_{q \text{ node}}\mathbb{N}_q.$$

Let us assume that all the nodes are internal, i.e. map to Y, for all the action happens there, and also that there is only one internal connected component Z, which is a curve of arithmetic genus one. Then we define $Q_{\min}^{\text{log.map}}$ as the saturation of the quotient of

$$\prod_{q \text{ node}}\mathbb{N}_q\times\prod_{S \text{ internal irr. compo.}}\mathbb{N}_S \tag{3}$$

by the relation generated in the groupification of the latter by

$$\langle \mathrm{Rel_q}=(\mathrm{u_q},\ldots,1,\ldots,-1,\ldots),\ \text{q internal node}\rangle$$

where 1 is in position S_q^+ (adjacent to q and further from the circuit), -1 in position S_q^- (adjacent to q and closer to the circuit), and u_q in position q is defined as to include the tropical balancing condition (I don't know whether this is standard):

$$u_q=f_*[T_q]\cdot[Y]+\sum_{R_i \text{external compo. attached to } T_q} m^{(i)}-\sum_{x_j\in T_q}\alpha_j,$$

where T_q is the tree of internal components that q separates from the circuit. Notice that the nodes between Z and an external component R_i have no 1, so the corresponding relations look like

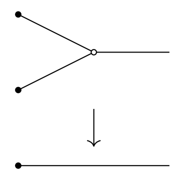
$$(u_q,\ldots,-1,\ldots).$$

The map $Q_{\min}^{\text{prestable}}\to Q_{\min}^{\text{log.map}}$ is induced by the inclusion of the node factors in (3). If we call e_q the image in $Q_{\min}^{\text{log.map}}$ of the q-th basis vector, then we can verify that the following equations hold: chosen any internal component S_i and any two external component R_i^1, R_i^2 such that their path to the circuit goes through S_i, then

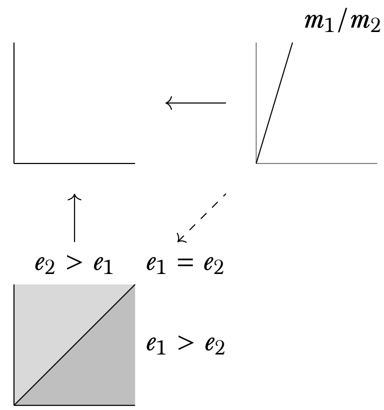
$$\sum_{q\in[R_i^1,S_i]}u_qe_q=\sum_{q\in[R_i^2,S_i]}u_qe_q. \tag{4}$$

Question: is this monoid factorisation condition a closed one?

Remark A.4. Here is a tidy example: start with an elliptic curve E contracted to Y, with two external components R_1, R_2 attached to it and a single marking $x \in E$. The tropical picture would be:



The dual monoids in the compatibility condition give us the following nice picture:



So, for example, the factorisation exists for the subcone $e_2 > e_1$ only when $m_1 > m_2$, which is compatible with the fact that $\mathbb{N}^2 = Q^{\text{prestable}}_{\text{min}} \to Q^{\text{log.map}}_{\text{min}} = \mathbb{N}$ is given by $e_1 \mapsto m_2, e_2 \mapsto m_1$.

A.2. **The relative space is the closure of the nice locus for** $(\mathbb{P}^N, H)$. In general we do not know very much about the space we have just defined. The aim of this section is to show that, in the case where $X = \mathbb{P}^N$ and $Y = H$ is a hyperplane, the space is proper and irreducible of the expected dimension. As such it has a fundamental class, which we can use to define reduced relative Gromov–Witten invariants in genus one.

Remark A.5. In the case of a general pair (X, Y) we will see that the moduli space is still proper, but is not in general irreducible or even equidimensional. Nevertheless, we can equip it with a virtual class by "pulling back" from the case of $(\mathbb{P}^N, H)$; for details, see §[REF].

The strategy is as follows. We define (Definition A.6) an open subspace

$$\mathcal{VZ}^{\text{ctr}}_{1,\alpha}(\mathbb{P}^N|H, d)^\circ \subseteq \mathcal{VZ}^{\text{ctr}}_{1,\alpha}(\mathbb{P}^N|H, d)$$

called the *nice locus*, on which the source curve and the map take a particularly simple form. Because of this simplicity, it is easy to show that the nice locus is irreducible (Lemma A.7). We then prove that $\mathcal{VZ}^{\text{ctr}}_{1,\alpha}(\mathbb{P}^N|H, d)$ is equal to the closure of the nice locus inside $\mathcal{VZ}^{\mathrm{ctr}}_{1,n}(\mathbb{P}^N, d)$. Thus it is proper, since it is a closed subspace of a proper space, and it is irreducible, since it is the closure of an irreducible space.

Definition A.6. The *nice locus* is defined as the open substack

$$\mathcal{VZ}^{\mathrm{ctr}}_{1,\alpha}(\mathbb{P}^N|H, d)^\circ \subseteq \mathcal{VZ}^{\mathrm{ctr}}_{1,\alpha}(\mathbb{P}^N|H, d)$$

of centrally aligned maps satisfying the following two conditions:

(1) the source curve C is irreducible;
(2) f does not map C inside H.

Lemma A.7. The nice locus $\mathcal{VZ}^{\mathrm{ctr}}_{1,\alpha}(\mathbb{P}^N|H, d)^\circ$ is irreducible.

Proof. By definition the contraction radius δ is compatible with the map, i.e. the subcurve $C_0 \subseteq C$ where $\lambda < \delta$ is equal to the maximal connected genus one subcurve contracted by f. Hence when the source curve is irreducible we must have $\delta = 0$, and we see that the central alignment on C is uniquely and trivially determined. Thus to specify a point in the nice locus we only need to specify the source curve C (as a scheme) and the map f. A parametrisation can be given from the vector bundle:

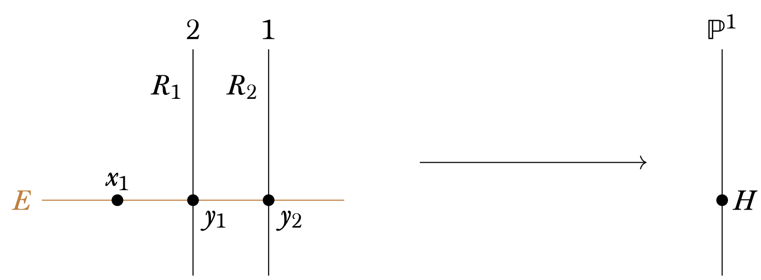
$$\mathrm{Vb}\left(\pi_*\mathcal{O}_{\mathcal{E}}(\sum_{j=n+1}^{n+\delta}\sigma_j) \oplus \pi_*\mathcal{O}_{\mathcal{E}}(\sum_{j=1}^{n+\delta}\sigma_j)^{\oplus r}\right) \quad \text{on} \quad \mathcal{M}_{1,n+\delta}$$

where $\pi\colon \mathcal{E} \to \mathcal{M}$ is the universal curve and $\delta = d - \sum \alpha_i$. □

A.3. **Justifying the novel condition by dimensional reasons.** Here is a dimensional computation in an explicit example which shows that being in the closure of the nice locus must impose some condition on the log structure.

Example A.8. Consider $\overline{\mathcal{M}}_{1,(3)}(\mathbb{P}^1|H, 3)$. This moduli space has virtual dimension $6 + 1 - 3 = 4$. Here is a parametrisation of the nice locus: choose an object $(E, p) \in \mathcal{M}_{1,1}$, and let s_0 be the natural section $s_0\colon \mathcal{O}_E \hookrightarrow \mathcal{O}_E(3p)$ and s_1 any other section of $\mathcal{O}_E(3p)$ not vanishing at p (notice that $\mathrm{h}^0(E, \mathcal{O}_E(3p)) = 3$). Then $(E, p, [\lambda s_0, s_1])$ gives a well-defined element of the nice locus for $\lambda \neq 0$.

Consider now the following weighted graph for a map in the boundary



where the brown line represents a contracted genus 1 curve. Now (E, x_1, y_1, y_2) is a point of $\mathcal{M}_{1,3}$ subject to the divisorial condition $3x_1 - 2y_1 - y_2 = 0 \in \mathrm{A}_0(E)$; furthermore we have to choose the second branch point of the $2:1$ map from

R_1 to $\mathbb{P}^1$. This already makes up for a 3-dimensional moduli space of degenerate relative maps corresponding to such a graph. The minimal log structure for this curve has a chart from $\mathbb{N}^2$, with generators e_1 and e_2 corresponding to the smoothing parameters of the two nodes. If we allowed e_1 and e_2 to be identified in the characteristic sheaf, then we would get an extra $\mathbb{G}_{\mathrm{m}}$ of choices for the log structure, so in total a 4-dimensional moduli space. Thus if we don't impose the novel condition then we get a whole other component of the relative space, which of course for dimensional reasons cannot be contained in the closure of the nice locus. □

Remark A.9. For the reader who is unfamiliar with centrally aligned log curves, it might be convenient to recall that they provide a modular interpretation of the iterated blow-up procedure of Vakil and Zinger. In particular it is natural that not all of the exceptional divisors be contained in the closure of the nice locus for every α; this geometric information is contained in the compatibility condition.

A.4. **Proof sketch**. The compatibility condition should imply that the map admits a log-enhancement; if we can prove that there is a smooth, connected and proper moduli space above, then we automatically get that the Gathmann-like relative space is closed and irreducible, and it obviously contains the nice locus; hence, if we can prove that the latter is dense, we win.

It thus remains to show that, given a relative centrally aligned map, we can smooth it to one in the nice locus. The only interesting case should be that the core is contracted to a point of Y. Here is how we prove it:

(1) we consider the core (say it's a smooth elliptic curve) marked with all the special points (markings and nodes); consider $E \times \Delta$ and blow it up in all the points of the central fiber that originally corresponded to nodes; we carefully and iteratively blow up smooth points of the central fiber in order to reconstruct the curve C that we started with. In fact we may perform weighted instead of ordinary blow-ups, picking the ideal (z, t^r) where z is a vertical coordinate; this has the effect of introducing an A_{r-1} surface singularity at the corresponding node of the central fiber. We do so by choosing a homomorphism $Q^{\mathrm{log.map}}_{\mathrm{min}} \to \mathbb{N}$ and composing it with $Q^{\mathrm{prestable}}_{\mathrm{min}} \to Q^{\mathrm{log.map}}_{\mathrm{min}}$ to determine the relevant r for every node.

(2) Let $E, S_1, \ldots, S_k$ be the irreducible components of $f^{-1}(Y)$; we define a line bundle on the smoothing by

$$\mathcal{L} = \mathcal{O}_{\mathcal{C}}\left(\sum \gamma_i S_i\right)\left(\sum \alpha_i x_i\right).$$

Using intersection theory on the surface (recall that the local multiplicity of intersection of the two branches of the node underlying an A_{r-1} singularity is $\frac{1}{r}$) we check that the compatibility condition is precisely what we need in order to be able to choose γ_i adequately so that the following equations hold:

$$\begin{aligned} 0 &= \deg(\mathcal{L}_{|S_i}) \\ m^{(i)} &= \deg(\mathcal{L}_{|R_i}) \\ \mathcal{O}_E &\cong \mathcal{L}_{|E} \end{aligned}$$

the last from Gathmann's condition.

(3) It is not clear that sections of $\mathcal{L}$ are unobstructed. In order to extend them we pass to $\overline{\mathcal{C}}$.

A.5. **Basic facts about centrally aligned curves.**

Proposition A.10 ([RSW17, Proposition 4.6.2.2]). The morphism $\mathfrak{M}_{1,n}^{\mathrm{ctr}} \to \mathfrak{M}_{1,n}^{\dagger}$ is a log-modification.

Explanation: this is a local statement so I can probably reduce to an atomic neighbourhood S of a point p. S and the curve over it are endowed with the minimal log structure; let $P = \overline{\mathcal{M}}_p$ determine a chart for this log structure. Observe that the subcurve $\sqsubset_0$ of the tropicalisation $\sqsubset$ of C_p determines a subset MinPos of the set of vertices, namely those adjacent to $\sqsubset_0$. Perform the following log-blowups: consider the set of primitive values of the function $\lambda\colon \sqsubset \to P$, and blow up the ideal that they generate; now locally the set of values of λ is principal with generator p: blow up the ideal generated by $\{\lambda(v) - p\} \setminus \{-p\}$. Keep going until $\lambda(v_i)$ is reached for some $v_i \in$ MinPos; at this point stop and declare the contraction radius $\delta := \lambda(v_i)$. Finish by adjoining $\lambda(v) - \delta$ for all the vertices untouched to this stage. This shows that the choice of δ is not an extra degree of freedom.

do I sound like a physicist?

A.6. **Moral justification of the novel condition.**

Example A.11. In this example we will give further moral justification for the novel condition, using as motivation the expanded degenerations approach to relative stable maps. This is not strictly necessary for what we want to do (since our spaces don't involve expanded degenerations), but it helps to explain where this condition comes from.

So, let us pretend that we have a good definition of the "centrally aligned Li space" of maps from centrally aligned curves to expanded degenerations:

$$\mathcal{VZ}_{1,\alpha}^{\mathrm{ctr,Li}}(X|D,\beta)$$

Objects of this moduli space should consist of maps $C \to X_l$ where $X_l \to X$ is an expansion (of some length l) of the pair (X, D), together with a central alignment on C which is compatible with this map. The factorisation property should say that the following diagram commutes:

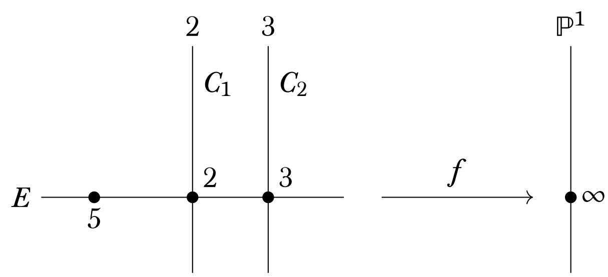
$$
\begin{tikzcd}
& \tilde{C} \arrow[dl, "\nu"'] \arrow[dr, "\tau"] & \\
C \arrow[d, "f"'] & & \overline{C} \arrow[d, dotted, "\overline{f}"] \\
X_l \arrow[dr, "\pi"] & & X_l \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}
$$

Morally speaking, the projection map

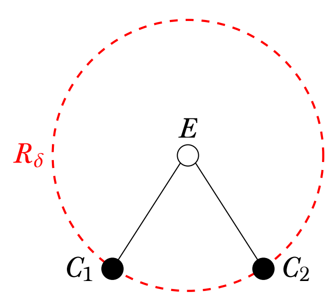
$$\pi_*\colon \mathcal{VZ}^{\mathrm{ctr,Li}}_{1,\alpha}(X|D,\beta) \to \mathcal{VZ}^{\mathrm{ctr}}_{1,n}(X,\beta)$$

should have as its image our space $\mathcal{VZ}^{\mathrm{ctr}}_{1,\alpha}(X|D,\beta)$. We will see that everything in the image of this space must satisfy the novel condition, thus providing further justification for us imposing it.

We proceed by contradiction. Consider a genus one map $C \to \mathbb{P}^1$ of the following form

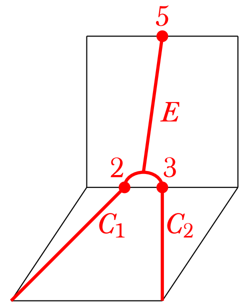


and equip the source curve C with a central alignment which identifies the lengths of the components C_1 and C_2:

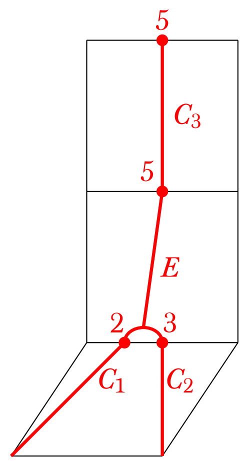


This defines an "object" of $\mathcal{VZ}^{\mathrm{ctr}}_{1,(5)}(\mathbb{P}^1|\infty,5)$, satisfying all the necessary conditions *except for the novel condition*. We will show that this cannot belong to the image of π_*.

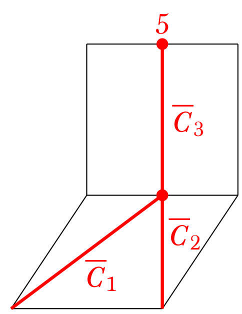
First we have to identify what the possible lifts of this object to the centrally aligned Li space are. The map $C \to X_1$ takes the following form:



(Here we use Navid's convention for drawing maps to expanded degenerations, so that the box at level 1 represents a *single fibre* of the expanded degeneration, i.e. E is mapped into a fibre and has zero horizontal degree.) Now, $\widetilde{C}$ is obtained from C by bubbling a rational component C_3 at the single marking, and so the map $\widetilde{C} \to X_2$ is given by:



The map $\widetilde{C} \to \overline{C}$ contracts the elliptic component E, and so the map $\overline{C} \to X_1$ *should* take the form:

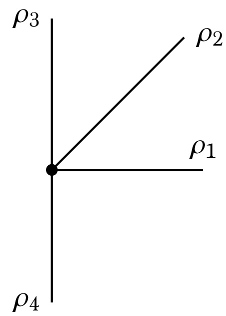


We will now see that, in this example, the map drawn above cannot actually exist, and that this occurs precisely because we have violated the novel condition.

We start by choosing a one-parameter smoothing $\mathcal{C} \to \mathbb{A}^1$ of the centrally aligned curve C. This induces one-parameter smoothings $\widetilde{\mathcal{C}}$ of $\widetilde{C}$ and $\overline{\mathcal{C}}$ of $\overline{C}$.

Since the factorisation condition must also hold in families, we must have a map $\overline{\mathcal{C}} \to \mathfrak{X}$, whose general fibre maps to the space $X_0 = X$ (with no expansion) and whose general fibre maps to $X_1 = X \sqcup_D Y$.

We thus have a map $\overline{\mathcal{C}} \to \mathfrak{X}$, where $\mathfrak{X}$ is the total space of the expanded degeneration. This is obtained as the toric blow-up of $\mathbb{P}^1 \times \mathbb{A}^1$ at the point $(\infty, 0)$. Thus it is also a smooth toric variety, with fan given by:



There is a map $\mathfrak{X} \to \mathbb{A}^1$ whose general fibre is isomorphic to X and whose central fibre is isomorphic to X_1. We see from the fan above that $\mathrm{Pic}(\mathfrak{X})$ is generated by $\mathcal{O}_{\mathfrak{X}}(D_{\rho_i})$ for $i = 1, \ldots, 4$, subject to the relations:

$$\mathcal{O}_{\mathfrak{X}}(D_{\rho_1}) \otimes \mathcal{O}_{\mathfrak{X}}(D_{\rho_2}) \cong \mathcal{O}_{\mathfrak{X}}$$
$$\mathcal{O}_{\mathfrak{X}}(D_{\rho_2}) \otimes \mathcal{O}_{\mathfrak{X}}(D_{\rho_3}) \cong \mathcal{O}_{\mathfrak{X}}(D_{\rho_4})$$

Choosing as $\{\mathcal{O}_{\mathfrak{X}}(D_{\rho_1}), \mathcal{O}_{\mathfrak{X}}(D_{\rho_3})\}$ our minimal set of generators, we see that a map $\overline{\mathcal{C}} \to \mathfrak{X}$ is given by the data of line bundes L_1 and L_3 on $\overline{\mathcal{C}}$ together with sections:

$$s_1 \in \mathrm{H}^0(\overline{\mathcal{C}}, L_1)$$
$$s_2 \in \mathrm{H}^0(\overline{\mathcal{C}}, L_1^{-1})$$
$$s_3 \in \mathrm{H}^0(\overline{\mathcal{C}}, L_3)$$
$$s_4 \in \mathrm{H}^0(\overline{\mathcal{C}}, L_1^{-1} \otimes L_3)$$

The divisors D_{ρ_1} and D_{ρ_2} are, respectively, the level-0 and level-1 pieces of the central fibre of $\mathfrak{X}$. Therefore we must have

$$L_1 \cong \mathcal{O}_{\overline{\mathcal{C}}}(a_1\overline{C}_1 + a_2\overline{C}_2)$$
$$L_1^{-1} \cong \mathcal{O}_{\overline{\mathcal{C}}}(a_3\overline{C}_3)$$

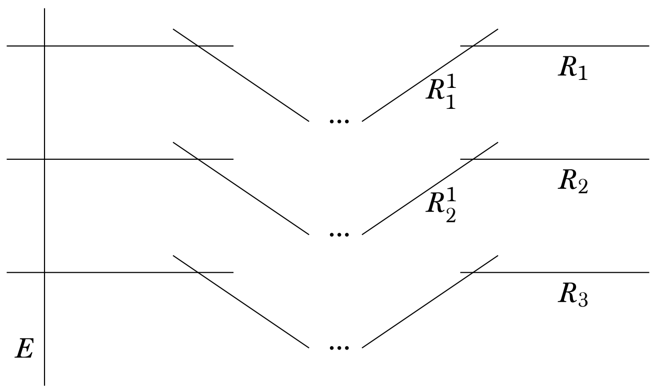
for some positive integers a_1, a_2 and a_3. Here we are using the fact that each $\overline{C}_i$ is a $\mathbb{Q}$-Cartier divisor on $\overline{\mathcal{C}}$, so that (after possibly multipltying by some positive integer) the corresponding line bundle makes sense. Furthermore, the intersection multiplicites of $\overline{C}_1$ and $\overline{C}_2$ with the level-1 piece mean that we must have:

$$\deg(L_1^{-1}|_{\overline{C}_1}) = 2$$
$$\deg(L_1^{-1}|_{\overline{C}_2}) = 3$$

We thus obtain:

$$a_3(\overline{C}_1 \cdot \overline{C}_3) = 2$$
$$a_3(\overline{C}_2 \cdot \overline{C}_3) = 3$$

However, we claim that in fact $\overline{C}_1 \cdot \overline{C}_3 = \overline{C}_2 \cdot \overline{C}_3$, so that the above equations cannot hold. To calculate these intersection numbers, we pass to the semistable model of $\overline{\mathcal{C}}$. This is obtained by performing further blow-ups to $\widetilde{\mathcal{C}}$, resulting in a semistable model $\widetilde{\mathcal{C}}^{\mathrm{ss}}$of the following form:



Here the component labeled E is smooth elliptic and all the other components are smooth rational. The projection $\pi : \widetilde{\mathcal{C}}^{\mathrm{ss}} \to \overline{\mathcal{C}}$ sends each R_i to $\overline{C}_i$ and contracts all the other components. The semistable model has the advantage that the total space is regular, so the intersections are precisely those that you "see" in the picture. In particular, any two adjacent components have intersection 1, and the self-intersection of any component is equal to minus the number of adjacent components.

Now, $\overline{C}_i \cdot \overline{C}_j = (\pi^*\overline{C}_i) \cdot (\pi^*\overline{C}_j)$, so we want to show that:

$$(\pi^*\overline{C}_1) \cdot (\pi^*\overline{C}_3) = (\pi^*\overline{C}_2) \cdot (\pi^*\overline{C}_3)$$

For each i, $\pi^*\overline{C}_i$ is equal to R_i plus a non-negative linear combination of the components in $\widetilde{\mathcal{C}}^{\mathrm{ss}}$ contracted by π. By the projection formula, any component contracted by π will multiply to zero with a class pulled back along π. Thus

$$(\pi^*\overline{C}_1) \cdot (\pi^*\overline{C}_3) = R_1 \cdot (\pi^*\overline{C}_3) = a_3(R_1^1)$$

the coefficient of R_1^1 in $\pi^*\overline{C}_3$. Similarly

$$(\pi^*\overline{C}_2) \cdot (\pi^*\overline{C}_3) = a_3(R_2^1)$$

the coefficient of R_2^1 in $\pi^*\overline{C}_3$. So we need to show that:

$$a_3(R_1^1) = a_3(R_1^2)$$

Now, we have

$$0 = R_1^1 \cdot (\pi^*\overline{C}_3) = a_3(R_1^2) - 2a_3(R_1^1)$$

because $(R_1^1)^2 = -2$. So $a_3(R_1^1) = a_3(R_1^2)/2$. Similarly:

$$0 = R_1^2 \cdot (\pi^* \overline{C}_3) = a_3(R_1^1) - 2a_3(R_1^2) + a_3(R_1^3) = -3a_3(R_1^1) + a_3(R_1^3)$$

So $a_3(R_1^1) = a_3(R_1^3)/3$. Continuing in this way, we find that

$$a_3(R_1^1) = a_3(E)/l_1$$

where l_1 is the length of the rational tail connecting E to R_1. Similarly $a_3(R_1^2) = a_3(E)/l_2$. But by Smyth's balancing condition for the semistable model [Smy11, Proposition 2.11], we must have $l_1 = l_2$. This concludes the proof that $\overline{C}_1 \cdot \overline{C}_3 = \overline{C}_2 \cdot \overline{C}_3$, which shows the necessity of the novel condition in this example.

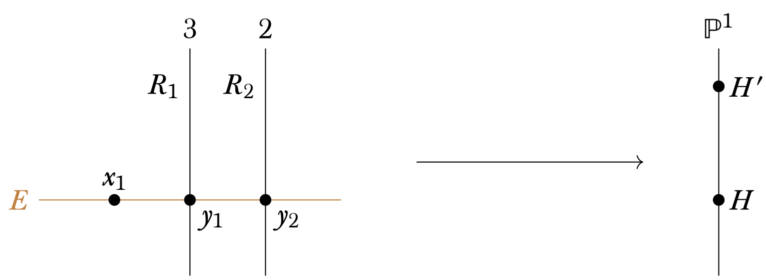
Notice that here we used the fact that the rational tails $\overline{C}_1$ and $\overline{C}_2$ are adjacent to the same irreducible component (namely E) of the core. □

A.7. **Proof that the relative space equals the closure of the nice locus.** We now want to show that the relative space in the centrally aligned setting is equal to the closure of the nice locus; irreducibility then follows immediately. We start with one direction:

Lemma A.12. The closure of the nice locus is contained in the relative space.

Proof. We address the relative conditions one at a time. Notice that they are all obviously satisfied on the nice locus. Also, Gathmann's relative condition is obviously closed; see [Vak00, Proposition 4.9].

It remains to show that the novel condition is satisfied on the closure of the nice locus. Here is no proof but rather some heuristics. Consider for example a map similar to the one above



and assume that it is in the closure of the nice locus; I want to argue that the two smoothing parameters cannot be identified. By the previous points I may assume that the factorisation property and Gathmann's relative conditions hold. I have a diagram:

$$\begin{tikzcd} \mathcal{C}^{\mathrm{ss}} \arrow[r] \arrow[dr] & \widetilde{\mathcal{C}} \arrow[r] \arrow[d] & \overline{\mathcal{C}} \arrow[d, "\bar{f}"] \\ & \mathcal{C} \arrow[r, "f"] & X \end{tikzcd}$$

where I have included the semistable model of $\mathcal{C}$ (and $\widetilde{\mathcal{C}}$), thought of as a curve marked with $f^{-1}(H')$ as well, where $H \neq H' \in \mathbb{P}^1$.

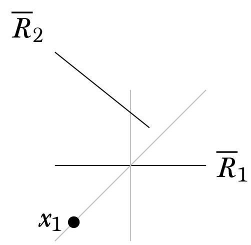
Notice that $\mathcal{C}$ is a normal surface with at worst singular points at the nodes of $C = \mathcal{C}_0$ (the central fibre $C = E + R_1 + R_2$ is Cartier and a variety is smooth at any smooth point of a Cartier divisor) and the singularities are of type A_{n_i}, $i = 1, 2$ (from the deformation theory of nodal curves).

I assume maximal multiplicity $\sum \alpha = d$, i.e. I am looking at the moduli space $\overline{\mathcal{M}}_{1,(5)}(\mathbb{P}^1|H, 5)$. Hence the line bundle and s_0 are determined as $f^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_{\mathcal{C}}(5x_1) \otimes \mathcal{O}_{\mathcal{C}}(\beta E)$ for some (positive rational) β and s_0 is the natural inclusion of $\mathcal{O}_{\mathcal{C}}$ (up to $\mathbb{G}_{\mathrm{m}}$).

The map is totally ramified at y_i by Gathmann's condition. We conclude:

$$\frac{\beta}{n_i + 1} = (\beta E) \cdot R_i = m^{(i)}$$

Hence, having fixed the multiplicities, the two possible singularities of $\mathcal{C}$ determine each other. In our example we may pick $n_1 = 1$, $n_2 = 2$, $\beta = 6$. But knowing the singularity determines the semistable model: in our case y_1 is replaced by a (-2)-curve and y_2 by a chain of two (-2)-curves. Now we know from the work of Smyth [Smy11, Proposition 2.12] that the exceptional locus of $\mathcal{C}^{\mathrm{ss}} \to \overline{\mathcal{C}}$ is balanced, therefore we may deduce that $\bar{f}$ is constant on the branch of the genus 1 singularity to which R_2 is joined, so $\overline{C}$ looks like this:



where a gray line is contracted by the map. On the other hand if $\delta = \lambda(v_1) = \lambda(v_2)$ then the prescription of [RSW17, Proposition 3.6.1] implies that $\tilde{C} \to \overline{C}$ looks like:

which is a contradiction. □

A.8. **Novel Condition 2.0**. This is done with $(\mathbb{P}^1, H)$ in mind; in general it will need to be modified to account for internal components of positive degree.

Novel condition 2.0: if v_1, v_2 are two vertices belonging to R_δ, then the corresponding tangencies $m^{(j_1)}$ and $m^{(j_2)}$ to H must be equal *if the corresponding* *rational tails are attached to the same irreducible component of* $\sqsubset_0$. Here R_δ is the set of vertices $v \in \sqsubset$ with $\lambda(v) = \delta$ and $\sqsubset_0$ is the set of vertices with $\lambda(v) < \delta$.

Sketch of "sufficient" direction: Denote by $S_0 = E, S_1, \dots, S_k$ the irreducible components of Z and assume E is smooth elliptic (or the core). Every S_i for $i \geq 0$ comes with a bunch of special points: divide them into $B^-(i) =$ the singleton representing the only node separating S_i from S_0 (this is not defined for $i = 0$); $B^+(i) =$ the set of remaining nodes; and $A(i) =$ the set of markings on S_i. Start with the trivial family $E \times \mathbb{A}^1$, where E is marked with the points of $B(0) = B^+(0)$ and $A(0)$. Now perform a weighted blow-up at the points $\{q \in B(0)\} \times \{0 \in \mathbb{A}^1\}$ of weight r_q; call each resulting rational tail S_i correspondingly to the original picture. On S_i we may mark points $B^+(i)$ and $A(i)$ in such a way that the resulting point of $\overline{\mathcal{M}}_{0,1+|B^+(i)|+|A(i)|}$ corresponds to the one we started with. Now blow up this family in all the $B^+(i) \times \{0\}$ with a weight. Keep going until you have recreated C. Call R_j the first external components, namely those corresponding to the vertices with $\lambda(v) \geq \delta$, $\nexists v'$ with $\lambda(v) > \lambda(v') \geq \delta$.

We define the line bundle $\mathcal{L} = \mathcal{O}_{\mathcal{C}}(\sum \beta_i S_i)(\sum \alpha_i x_i)$; let's look at the equations that it gives us, starting from the outside.

(1) $m^{(j)} = \mathcal{L}_{|R_j} = \frac{\beta^-(j)}{r_{q^-(j)}}$;
(2) on every $S_i, i \geq 1$ we can prove inductively that:
$$0 = \mathcal{L}_{|S_i} = \frac{\beta^-(i) - \beta_i}{r_{q^-(i)}} + \sum_{x_h \in A(i)} \alpha_i + \sum_{y_h \in B^+(i)} M^{(h)},$$
where I have denoted by $M^{(h)}$ the sum of all the contributions (both α_l and $m^{(l)}$) coming from the rational tree attached to the corresponding point of $B^+(i)$;
(3) it turns out that $\mathcal{O}_{S_0} \simeq \mathcal{L}_{|S_0}$ is precisely Gathmann's condition (in the numeric form if S_0 is a circle of rational curves).

Now notice that $\lambda(v_j) = \lambda(v'_j)$ may happen only at the first external components; in this case $r_{q^-(j)} = r_{q^-(j')}$, and from the first equation we see that $m^{(j)} = m^{(j')}$ if they are attached to the same component ($\beta^-(j) = \beta^-(j')$). The second equation can be made to hold for every S_i by appropriately choosing $\beta^-(i), r_{q^-(i)}$. The last equation is automatically satisfied.

is this true? if not, probably there are contributions $M^{(h)}$ and $M^{(h')}$ to equate

Extending sections is the usual business of exploiting the fatorisation through the elliptic m-fold.

Sketch of "necessary" direction: pick a smoothing $\mathcal{C}$ of (C, f). Suppose that $\lambda(v) = \lambda(v')$ and the corresponding R, R' are attached to the same S_i, i.e. they share the same the tree separating them from C_0. Notice that R and R' map to two branches of the Smyth singularity. Looking at the semistable model of $\mathcal{C}$, it follows that (the strict transforms of) R and R' still share the same tree there. Then by writing $(f^{\mathrm{ss}})^* \mathcal{O}_{\mathbb{P}}(1) = \mathcal{O}_{\mathcal{C}^{\mathrm{ss}}}(\sum \beta_i S_i)(\sum \alpha_i x_i)$ and intersecting with R, R', we see that they must have the same contact order to H.

A.9. **Details of the proof.**

Case 1: non-contracted genus one internal component. Assume that the curve takes the form
$$C = C_0 \cup C_1 \cup \ldots \cup C_k$$
where all the C_i are smooth, C_0 has genus one, all the other C_i have genus zero, and for $i \in \{1, \ldots, k\}$, C_i intersects C_0 at a single node (denoted q_i) and does not intersect any other components.

Suppose furthermore that C_0 is a non-contracted *internal component*, meaning that it is mapped into H via f, and that $C_1, \ldots, C_k$ are *external components*, meaning that they are not mapped into H via f. The picture is:

[FIGURE]

Suppose that this is a relative stable map. This means that [BLAH]. We claim that it can be smoothed to a relative stable map in the nice locus. The construction depends on choosing an appropriate smoothing of the curve C, so that the map also smooths.

We start with $W = C_0 \times \mathbb{A}^1_t$ (where t denotes a fixed co-ordinate on the affine line). This is a smooth surface, fibred over $\mathbb{A}^1_t$, with fibre equal to the elliptic curve C_0. Consider the points $q_1, \ldots, q_k$ on C_0. We will perform a series of weighted blow-ups at the points $(q_i, 0) \in W$, in order to obtain a surface whose general fibre is smooth (in fact, isomorphic to C_0) and whose central fibre is isomorphic to C.

Fix $i \in \{1, \ldots, k\}$ and let m_i be the multiplicity of f with H at $q_i \in C_i$. We define:
$$l = \operatorname{lcm}(m_1, \ldots, m_k) \qquad r_i = l/m_i$$
We now blow-up the surface W at the points $(q_i, 0)$ with weight r_i in the horizontal direction and weight 1 in the vertical direction: if x_i is a local co-ordinate for the fibre around q_i, this means that we blow-up in the ideal (x_i, t^{r_i}).

The result is a fibred surface $W' \to \mathbb{A}^1_t$ with general fibre equal to C_0 and central fibre $W'_0 \cong C$. The total space of W' is no longer smooth (its singular points are [BLAH]), but this is not a problem since the projection to $\mathbb{A}^1_t$ is still flat. The central fibre is a linearly trivial Cartier divisor:
$$W'_0 = C_0 + C_1 + \ldots + C_k = 0 \in \operatorname{Pic} W'$$
For $i \in \{1, \ldots, k\}$ we have that $r_i C_i$ is Cartier, although the same is not necessarily true of C_i. Furthermore, since
$$lC_0 = -\sum_{i=1}^{k} lC_i = -\sum_{i=1}^{k} m_i(r_i C_i)$$
in $A_1(W')$, it follows that lC_0 is Cartier. Finally, a local computation shows that
$$r_i C_i \cdot C_0 = 1$$
for $i \in \{1, \ldots, k\}$. Now, let $x_1, \ldots, x_n$ denote the marked points of C. These are smooth points of the central fibre W'_0, and hence can be extended to Cartier

divisors $\tilde{x}_1, \ldots, \tilde{x}_n$ on W'. Consider the line bundle:

$$\tilde{L} = \mathcal{O}_{W'}(lC_0 + \Sigma_{j=1}^n \alpha_j \tilde{x}_j)$$

on W'. We claim that this gives a smoothing of the line bundle $L = f^*\mathcal{O}(1)$ on C, i.e. that $\tilde{L}|_{W_0'} = L$. We show this by first restricting $\tilde{L}$ to each of the components C_i of $W_0' \cong C$. For $i \in \{1, \ldots, k\}$, we have

$$\begin{aligned} \tilde{L}|_{C_i} &= \mathcal{O}_{C_i}\left((lC_0 \cdot C_i)q_i + \sum_{x_j \in C_i} \alpha_j x_j\right) = \mathcal{O}_{C_i}\left((l/r_i)q_i + \sum_{x_j \in C_i} \alpha_j x_j\right) \\ &= \mathcal{O}_{C_i}\left(m_i q_i + \sum_{x_j \in C_i} \alpha_j x_j\right) = L|_{C_i} \end{aligned}$$

while for $i = 0$ we have:

$$\tilde{L}|_{C_0} = \mathcal{O}_{C_0}\left(-\sum_{i=1}^k (lC_i \cdot C_0)q_i + \sum_{x_j \in C_0} \alpha_j x_j\right) = \mathcal{O}_{C_0}\left(-\sum_{i=1}^k m_i q_i + \sum_{x_j \in C_0} \alpha_j x_j\right) = L|_{C_0}$$

Finally the fact that $\tilde{L}|_{W_0'} = L$ follows from the fact that the dual intersection graph of C has genus zero.

Now, $\tilde{L}$ comes with a unique section whose restriction to $W_0' \cong C$ is s_0. After we extend the sections $s_1, \ldots, s_N$, it is clear that the resulting stable map is in the nice locus (i.e. that it is not mapped into H).

In order to extend the sections $s_1, \ldots, s_N$, we simply check that they are unobstructed. The space containing the obstructions to extending the sections is [Wan12, Theorem 3.1]:

$$\mathrm{H}^1(C, L)$$

By taking the normalisation exact sequence for C, tensoring with L and passing to cohomology, we obtain an exact sequence:

$$\begin{aligned} 0 \to \mathrm{H}^0(C, L) \to \bigoplus_{i=0}^k \mathrm{H}^0(C_i, L) \xrightarrow{\theta} \bigoplus_{i=1}^k L_{q_i} \to \\ \to \mathrm{H}^1(C, L) \to \bigoplus_{i=0}^k \mathrm{H}^1(C_i, L) \to 0 \end{aligned}$$

Now, each of $C_1, \ldots, C_k$ is isomorphic to $\mathbb{P}^1$ and $L|_{C_i}$ has non-negative degree; hence the map θ is surjective. Thus the map

$$\mathrm{H}^1(C, L) \to \bigoplus_{i=0}^k \mathrm{H}^1(C_i, L)$$

is an isomorphism. But $\mathrm{H}^1(C_i, L) = 0$ for $i \in \{1, \ldots, k\}$ since $C_i \cong \mathbb{P}^1$ and $L|_{C_i}$ has non-negative degree; also we have by Serre duality

$$\mathrm{H}^1(C_0, L) \cong \mathrm{H}^0(C_0, L^\vee \otimes \omega_{C_0}) = \mathrm{H}^0(C_0, L^\vee) = 0$$

where the penultimate equality holds because $\mathrm{g}(C_0) = 1$ and the last equality holds because $L|_{C_0}$ has *strictly* positive degree (here we are using the fact that $f|_{C_0}$ is non-constant).

To conclude, we have a family $\tilde{C} = W'$ of nodal curves and a map from this family to $\mathbb{P}^N$

$$\begin{tikzcd} \tilde{C} \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & \mathbb{P}^N \\ \mathbb{A}^1_t & \end{tikzcd}$$

such that when we restrict to $0 \in \mathbb{A}^1_t$ we recover the map $f\colon C \to \mathbb{P}^N$ and such that the general fibre is an element of the nice locus.

Address the case that C_0 is not irreducible, and that $C_i,\ i \neq 0$ are chains of $\mathbb{P}^1$'s.

This should follow from gluing, whatever that means.

Case 2: contracted genus one internal component. This is similar to the previous case, but slightly more delicate. The smoothing W', the line bundle $\tilde{L}$ and the section s_0 are constructed as before. Now, however, $\mathrm{H}^1(C, L) \neq 0$, so we need to work harder to show that we can extend the other sections.

Before extending the sections let us first extend the centrally aligned log structure. On the base $\mathbb{A}^1_t$ the chart sends $e_i \in \mathbb{N}^r$ to t^{r_i}. By the novel condition, if $e_i = e_j$ in the minimal centrally aligned structure that we start with on C, then $r_i = r_j$, so the chart that we have just defined factors indeed through the sharpening of the submonoid of $\mathbb{Z}^r$ generated by $\mathbb{N}^r$ and the differences $\lambda(v) - \lambda(w)$. We declare the contraction radius δ to be e_i for those i's corresponding to branches of the genus 1 singularity on which $\bar{f}$ is non-constant.

actually this sounds like bs and I probably have δ already?

The central alignment provides us with a diagram:

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r] \arrow[d] & \overline{\mathcal{C}} \\ \mathcal{C} & \end{tikzcd}$$

We can now extend the sections $\bar{s}_1, \ldots, \bar{s}_N$ onto $\overline{\mathcal{C}}$ because $\mathrm{H}^1(\overline{\mathcal{C}}, \bar{L}) = 0$, so we get a map $\bar{F}\colon \overline{\mathcal{C}} \to \mathbb{P}^N$ deforming $\bar{f}$. Finally we claim that $F\colon \mathcal{C} \to \mathbb{P}^N$ is the stable model of $\bar{F}$.

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