RELATIVE STABLE MAPS IN GENUS ONE VIA RADIAL ALIGNMENTS

LUCA BATTISTELLA AND NAVID NABIJOU

1. Relative space equals closure of the nice locus

Since the log structures only come into play when the source curve is reducible, it follows that the nice locus in the radially aligned setting is the same as the nice locus in the ordinary setting. In particular, it is irreducible.

We now want to show that the relative space in the radially aligned setting is equal to the closure of the nice locus; irreducibility follows immediately. One direction is clear: [WHY?]

It thus remains to show that, given a relative radially aligned map, we can smooth it to one in the nice locus. This is done by considering different cases locally, then gluing.

Case 1: non-contracted genus one internal component. Assume that the curve takes the form

$$C = C_0 \cup C_1 \cup \ldots \cup C_k$$

where all the C_i are smooth, C_0 has genus one, all the other C_i have genus zero, and for $i \in \{1, \ldots, k\}$, C_i intersects C_0 at a single node (denoted q_i) and does not intersect any other components.

Suppose furthermore that C_0 is a non-contracted *internal component*, meaning that it is mapped into H via f, and that $C_1, \ldots, C_k$ are *external components*, meaning that they are not mapped into H via f. The picture is:

[FIGURE]

Suppose that this is a relative stable map. This means that [BLAH]. We claim that it can be smoothed to a relative stable map in the nice locus. The construction depends on choosing an appropriate smoothing of the curve C, so that the map also smooths.

We start with $W = C_0 \times \mathbb{A}^1_t$ (where t denotes a fixed co-ordinate on the affine line). This is a smooth surface, fibred over $\mathbb{A}^1_t$, with fibre equal to the elliptic curve C_0. Consider the points $q_1, \ldots, q_k$ on C_0. We will perform a series of weighted blow-ups at the points $(q_i, 0) \in W$, in order to obtain a surface whose general fibre is smooth (in fact, isomorphic to C_0) and whose central fibre is isomorphic to C.

Fix $i \in \{1, \ldots, k\}$ and let m_i be the multiplicity of f with H at $q_i \in C_i$. We define:

$$l = \operatorname{lcm}(m_1, \ldots, m_k) \qquad r_i = l/m_i$$

We now blow-up the surface W at the points $(q_i, 0)$ with weight r_i in the horizontal direction and weight 1 in the vertical direction: if x_i is a local co-ordinate for the fibre around q_i, this means that we blow-up in the ideal (x_i, t^{r_i}).

The result is a fibred surface $W' \to \mathbb{A}^1_t$ with general fibre equal to C_0 and central fibre $W'_0 \cong C$. The total space of W' is no longer smooth (its singular points are [BLAH]), but this is not a problem since the projection to $\mathbb{A}^1_t$ is still flat. The central fibre is a linearly trivial Cartier divisor:

$$W'_0 = C_0 + C_1 + \ldots + C_k = 0 \in \operatorname{Pic} W'$$

For $i \in \{1, \ldots, k\}$ we have that $r_i C_i$ is Cartier, although the same is not necessarily true of C_i. Furthermore, since

$$lC_0 = -\sum_{i=1}^{k} lC_i = -\sum_{i=1}^{k} m_i (r_i C_i)$$

in $A_1(W')$, it follows that lC_0 is Cartier. Finally, a local computation shows that

$$r_i C_i \cdot C_0 = 1$$

for $i \in \{1, \ldots, k\}$. Now, let $x_1, \ldots, x_n$ denote the marked points of C. These are smooth points of the central fibre W'_0, and hence can be extended to Cartier divisors $\tilde{x}_1, \ldots, \tilde{x}_n$ on W'. Consider the line bundle:

$$\tilde{L} = \mathcal{O}_{W'}(lC_0 + \Sigma_{j=1}^{n} \alpha_j \tilde{x}_j)$$

on W'. We claim that this gives a smoothing of the line bundle $L = f^*\mathcal{O}(1)$ on C, i.e. that $\tilde{L}|_{W'_0} = L$. We show this by first restricting $\tilde{L}$ to each of the components C_i of $W'_0 \cong C$. For $i \in \{1, \ldots, k\}$, we have

$$\begin{aligned}\tilde{L}|_{C_i} &= \mathcal{O}_{C_i}\left((lC_0 \cdot C_i) q_i + \sum_{x_j \in C_i} \alpha_j x_j \right) = \mathcal{O}_{C_i}\left((l/r_i) q_i + \sum_{x_j \in C_i} \alpha_j x_j \right) \\ &= \mathcal{O}_{C_i}\left(m_i q_i + \sum_{x_j \in C_i} \alpha_j x_j \right) = L|_{C_i}\end{aligned}$$

while for $i = 0$ we have:

$$\tilde{L}|_{C_0} = \mathcal{O}_{C_0}\left(-\sum_{i=1}^{k} (lC_i \cdot C_0) q_i + \sum_{x_j \in C_0} \alpha_j x_j \right) = \mathcal{O}_{C_0}\left(-\sum_{i=1}^{k} m_i q_i + \sum_{x_j \in C_0} \alpha_j x_j \right) = L|_{C_0}$$

Finally the fact that $\tilde{L}|_{W'_0} = L$ follows from the fact that the dual intersection graph of C has genus zero.

Now, $\tilde{L}$ comes with a unique section whose restriction to $W'_0 \cong C$ is s_0. After we extend the sections $s_1, \ldots, s_N$, it is clear that the resulting stable map is in the nice locus (i.e. that it is not mapped into H).

In order to extend the sections $s_1, \ldots, s_N$, we simply check that they are unobstructed. The space containing the obstructions to extending the sections is:

$$\mathrm{H}^1(C, L)$$

By taking the normalisation exact sequence for C, tensoring with L and passing to cohomology, we obtain an exact sequence:

Is this true even when C is reducible

$$0 \to \mathrm{H}^0(C, L) \to \bigoplus_{i=0}^{k} \mathrm{H}^0(C_i, L) \xrightarrow{\theta} \bigoplus_{i=1}^{k} L_{q_i} \to$$
$$\to \mathrm{H}^1(C, L) \to \bigoplus_{i=0}^{k} \mathrm{H}^1(C_i, L) \to 0$$

Now, each of $C_1, \ldots, C_k$ is isomorphic to $\mathbb{P}^1$ and $L|_{C_i}$ has non-negative degree; hence the map θ is surjective. Thus the map

$$\mathrm{H}^1(C, L) \to \bigoplus_{i=0}^{k} \mathrm{H}^1(C_i, L)$$

is an isomorphism. But $\mathrm{H}^1(C_i, L) = 0$ for $i \in \{1, \ldots, k\}$ since $C_i \cong \mathbb{P}^1$ and $L|_{C_i}$ has non-negative degree; also we have by Serre duality

$$\mathrm{H}^1(C_0, L) \cong \mathrm{H}^0(C_0, L^\vee \otimes \omega_{C_0}) = \mathrm{H}^0(C_0, L^\vee) = 0$$

where the penultimate equality holds because $\mathrm{g}(C_0) = 1$ and the last equality holds because $L|_{C_0}$ has *strictly* positive degree (here we are using the fact that $f|_{C_0}$ is non-constant).

To conclude, we have a family $\tilde{C} = W'$ of nodal curves and a map from this family to $\mathbb{P}^N$

$$\begin{tikzcd} \tilde{C} \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & \mathbb{P}^N \\ \mathbb{A}^1_t & \end{tikzcd}$$

such that when we restrict to $0 \in \mathbb{A}^1_t$ we recover the map $f \colon C \to \mathbb{P}^N$ and such that the general fibre is an element of the nice locus.

Luca Battistella
Department of Mathematics, Imperial College London
l.battistella14@imperial.ac.uk

Navid Nabijou
Department of Mathematics, Imperial College London
navid.nabijou09@imperial.ac.uk