RELATIVE STABLE MAPS IN GENUS ONE VIA CENTRAL ALIGNMENTS

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Abstract. For a smooth projective variety X and a smooth very ample hypersurface $Y \subseteq X$, we define moduli spaces of relative stable maps to (X, Y) in genus one, as closed substacks of the moduli space of maps from centrally aligned curves, constructed in [RSW17]. We construct virtual classes for these moduli spaces, which we use to define *reduced relative Gromov–Witten invariants* in genus one.

[GOALS: We prove a recursion formula which allows us to completely determine these invariants in terms of the reduced Gromov–Witten invariants, as defined in [REF]. We also prove a relative version of the Li–Zinger formula, relating our invariants to the usual relative Gromov–Witten invariants. Also say something about quasimaps.]

Contents

1. The space of relative centrally aligned maps

Recall [RSW17] that the moduli space of maps from centrally aligned curves is obtained by first considering the Cartesian diagram

$$\begin{tikzcd}
\widetilde{\mathcal{VZ}}^{\mathrm{ctr}}_{1,n}(X,\beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_{1,n}(X,\beta) \arrow[d] \\
\mathfrak{M}^{\mathrm{ctr}}_{1,n} \arrow[r] & \mathfrak{M}^{\dagger}_{1,n}
\end{tikzcd}$$

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and then defining

$$\mathcal{VZ}^{\mathrm{ctr}}_{1,n}(X,\beta) \subseteq \widetilde{\mathcal{VZ}}^{\mathrm{ctr}}_{1,n}(X,\beta)$$

to be the closed substack consisting of maps satisfying the *factorisation condition*, namely that the map $f\colon C \to X$ factors through the associated contraction to a Smyth curve, i.e. there exists a map $\bar{f}$ making the following square commute:

$$\begin{tikzcd} \widetilde{C} \arrow[r] \arrow[d] & \overline{C} \arrow[d, dotted, "\bar{f}"'] \\ C \arrow[r, "f"] & X \end{tikzcd}$$

1.1. **Definition of the relative space.** We now give the central definition of this article.

Definition 1.1. Fix a vector of tangency conditions $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\Sigma_i \alpha_i \leq Y \cdot \beta$. Then the *centrally aligned relative space*

$$\mathcal{VZ}^{\mathrm{ctr}}_{1,\alpha}(X|Y,\beta)$$

is defined to be the closed substack of $\mathcal{VZ}^{\mathrm{ctr}}_{1,n}(X,\beta)$ satisfying the following conditions:

(1) *Gathmann's relative condition:* for each connected component Z of $f^{-1}(Y)$, either:
   (a) Z is a single point of C, in which case if $Z = x_i$ is a marking then we require that the multiplicity of f at x_i along Y is at least α_i (if Z is not a marking then there is no condition);
   (b) Z is a subcurve $Z \subseteq C$, in which case we require that

$$f^*[Y]|_Z - \sum_{x_i \in Z} \alpha_i x_i$$

   is an effective class in $\mathrm{A}_1(Z)$. Since Z is at most genus one and any line bundle of strictly positive degree on a Gorenstein genus one curve is effective, this condition can be rephrased as the following numerical criterion

$$f_*[Z] \cdot [Y] + \sum_{j=1}^{r} m^{(j)} \geq \sum_{x_i \in Z} \alpha_i$$

   together with the additional requirement that, when the above is an equality, there is an isomorphism of line bundles:

$$(f|_Z)^* \mathcal{O}_X(Y) \otimes \mathcal{O}_Z\left(\sum_{j=1}^{r} m^{(j)} y_j\right) = \mathcal{O}_Z\left(\sum_{x_i \in Z} \alpha_i x_i\right)$$

(2) *Novel condition:* if v_1, v_2 are two vertices belonging to R_δ, then the corresponding tangencies to the divisor $m^{(j_1)}$ and $m^{(j_2)}$ must be equal. Here R_δ is the set of vertices $v \in \sqsubset$ with $\lambda(v) = \delta$.

2. The relative space is equal to the closure of the nice locus for $(\mathbb{P}^N, H)$

In general we do not know very much about the space we have just defined. The aim of this section is to show that, in the case where $X = \mathbb{P}^N$ and $Y = H$ is a hyperplane, the space is proper and irreducible of the expected dimension. As such it has a fundamental class, which we can use to define reduced relative Gromov–Witten invariants in genus one.

Remark 2.1. In the case of a general pair (X, Y) we will see that the moduli space is still proper, but is not in general irreducible or even equidimensional. Nevertheless, we can equip it with a virtual class by 'pulling back" from the case of $(\mathbb{P}^N, H)$; for details, see §[REF].

The strategy is as follows. We define (Definition 2.2)an open subspace

$$\mathcal{VZ}^{\mathrm{ctr}}_{1,\alpha}(\mathbb{P}^N|H, d)^\circ \subseteq \mathcal{VZ}^{\mathrm{ctr}}_{1,\alpha}(\mathbb{P}^N|H, d)$$

called the *nice locus*, on which the source curve and the map take a particularly simple form. Because of this simplicity, it is easy to show that the nice locus is irreducible (Lemma 2.3). We then prove that $\mathcal{VZ}^{\mathrm{ctr}}_{1,\alpha}(\mathbb{P}^N|H, d)$ is equal to the closure of the nice locus inside $\mathcal{VZ}^{\mathrm{ctr}}_{1,n}(\mathbb{P}^N, d)$. Thus it is proper, since it is a closed space of a proper space, and it is irreducible, since it is the closure of an irreducible space inside an irreducible space.

Definition 2.2. The *nice locus* is defined as the open substack

$$\mathcal{VZ}^{\mathrm{ctr}}_{1,\alpha}(\mathbb{P}^N|H, d)^\circ \subseteq \mathcal{VZ}^{\mathrm{ctr}}_{1,\alpha}(\mathbb{P}^N|H, d)$$

of centrally aligned maps satisfying the following two conditions:

(1) the source curve C is irreducible;
(2) f does not map C inside H.

Lemma 2.3. The nice locus $\mathcal{VZ}^{\mathrm{ctr}}_{1,\alpha}(\mathbb{P}^N|H, d)^\circ$ is irreducible.

Proof. By definition the contraction radius δ is compatible with the map, i.e. the subcurve $C_0 \subseteq C$ where $\lambda < \delta$ is equal to the maximal connected genus one subcurve contracted by f. Hence when the source curve is irreducible we must have $\delta = 0$, and we see that the central alignment on C is uniquely and trivially determined. Thus to specify a point in the nice locus we only need to specify the source curve C (as a scheme) and the map f. A parametrisation can be given from the vector bundle:

I'm sure there's a simpler way to say this - Navid

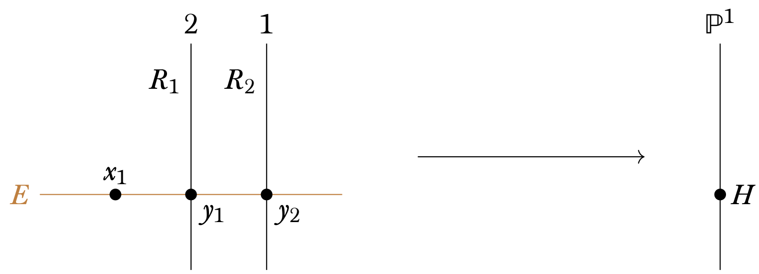
$$\mathrm{Vb}\left(\pi_*\mathcal{O}_{\mathcal{E}}(\sum_{j=n+1}^{n+\delta}\sigma_j) \oplus \pi_*\mathcal{O}_{\mathcal{E}}(\sum_{j=1}^{n+\delta}\sigma_j)^{\oplus r}\right) \quad \text{on} \quad \mathcal{M}_{1,n+\delta}$$

where $\pi\colon \mathcal{E} \to \mathcal{M}$ is the universal curve and $\delta = d - \sum \alpha_i$. □

2.1. **Justifying the novel condition**. Remember that we are trying to impose conditions which determine the closure of the nice locus inside the moduli space of centrally aligned maps. Here is one example where we show by a dimensional computation that the novel condition must be included if we hope to determine the closure of the nice locus.

Example 2.4. Consider $\overline{\mathcal{M}}_{1,(3)}(\mathbb{P}^1|H,3)$. The virtual dimension is $7-3=4$. Here is a parametrisation of the nice locus: choose an element (E,p) of $\overline{\mathcal{M}}_{1,1} \setminus \partial\overline{\mathcal{M}}_{1,1}$ (which has dimension 1), and let s_0 be the natural section $s_0\colon \mathcal{O}_E \hookrightarrow \mathcal{O}_E(3p)$ and s_1 any other section of $\mathcal{O}_E(3p)$ not vanishing at p (notice that $\mathrm{h}^0(E,\mathcal{O}_E(3p)) = 3$). Then $(E,p,[\lambda s_0, s_1])$ gives a well-defined element of the nice locus for $\lambda \neq 0$.

Consider now the following weighted graph for a map in the boundary



where the brown line represents a contracted genus 1 curve. Now (E,x_1,y_1,y_2) is a point of $\overline{\mathcal{M}}_{1,3}$ subject to the divisorial condition $3x_1-2y_1-y_2=0 \in \mathrm{A}_0(E)$; furthermore we have to choose the second branch point of the $2\colon 1$ map from R_1 to $\mathbb{P}^1$. This already makes up for a 3-dimensional moduli space of degenerate relative maps corresponding to such a graph. The minimal log structure for this curve has a chart from $\mathbb{N}^2$, with generators e_1 and e_2 corresponding to the smoothing parameters of the two nodes. If we allowed e_1 and e_2 to be identified in the characteristic sheaf, then we would get an extra $\mathbb{G}_{\mathrm{m}}$ of choices for the log structure, so in total a 4-dimensional moduli space. Thus if we don't impose the novel condition then we get a whole other component of the relative space, which of course for dimensional reasons cannot be contained in the closure of the nice locus.

Example 2.5. In this example we will give further justification for the novel condition, using as motivation the expanded degeneration approach to relative stable maps.

2.2. **Proof**. We now want to show that the relative space in the centrally aligned setting is equal to the closure of the nice locus; irreducibility then follows immediately. We start with one direction.

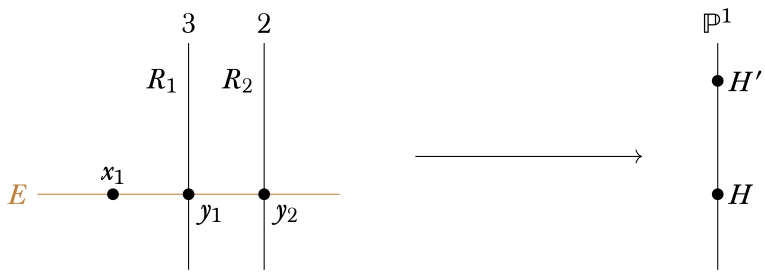
Lemma 2.6. The closure of the nice locus is contained in the relative space.

Proof. We address the relative conditions one at a time. Notice that they are all obviously satisfied on the nice locus.

The factorisation property is closed: see [RSW17, Theorem 4.3].

Gathmann's relative condition is closed: see [Vak00, Proposition 4.9].

It remains to show that the novel condition is satisfied on the closure of the nice locus. Here is no proof but rather some heuristics. Consider for example a map similar to the one above



and assume that it is in the closure of the nice locus; I want to argue that the two smoothing parameters cannot be identified. By the previous points I may assume that the factorisation property and Gathmann's relative conditions hold. I have a diagram:

$$\begin{tikzcd} \mathcal{C}^{\mathrm{ss}} \arrow[r] \arrow[dr] & \widetilde{\mathcal{C}} \arrow[r] \arrow[d] & \overline{\mathcal{C}} \arrow[d, "\bar{f}"] \\ & \mathcal{C} \arrow[r, "f"] & X \end{tikzcd}$$

where I have included the semistable model of $\mathcal{C}$ (and $\widetilde{\mathcal{C}}$), thought of as a curve marked with $f^{-1}(H')$ as well, where $H \neq H' \in \mathbb{P}^1$.

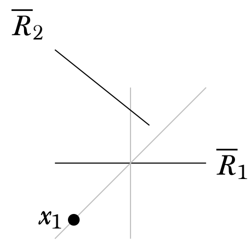
Notice that $\mathcal{C}$ is a normal surface with at worst singular points at the nodes of $C = \mathcal{C}_0$ (the central fibre $C = E + R_1 + R_2$ is Cartier and a variety is smooth at any smooth point of a Cartier divisor) and the singularities are of type A_{n_i}, $i = 1, 2$ (from the deformation theory of nodal curves).

I assume maximal multiplicity $\sum \alpha = d$, i.e. I am looking at the moduli space $\overline{\mathcal{M}}_{1,(5)}(\mathbb{P}^1|H, 5)$. Hence the line bundle and s_0 are determined as $f^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_{\mathcal{C}}(5x_1) \otimes \mathcal{O}_{\mathcal{C}}(\beta E)$ for some (positive rational) β and s_0 is the natural inclusion of $\mathcal{O}_{\mathcal{C}}$ (up to $\mathbb{G}_{\mathrm{m}}$).

The map is totally ramified at y_i by Gathmann's condition. We conclude:

$$\frac{\beta}{n_i + 1} = (\beta E) \cdot R_i = m^{(i)}$$

Hence, having fixed the multiplicities, the two possible singularities of $\mathcal{C}$ determine each other. In our example we may pick $n_1 = 1$, $n_2 = 2$, $\beta = 6$. But knowing the singularity determines the semistable model: in our case y_1 is replaced by a (-2)-curve and y_2 by a chain of two (-2)-curves. Now we know from the work of Smyth [Smy11, Proposition 2.12] that the exceptional locus of $\mathcal{C}^{\mathrm{ss}} \to \overline{\mathcal{C}}$ is balanced, therefore we may deduce that $\bar{f}$ is constant on the branch of the genus 1 singularity to which R_2 is joined, so $\overline{C}$ looks like this:



where a gray line is contracted by the map. On the other hand if $\delta = \lambda(v_1) = \lambda(v_2)$ then the prescription of [RSW17, Proposition 3.6.1] implies that $\tilde{C} \to \overline{C}$ looks like:

which is a contradiction. □

It thus remains to show that, given a relative radially aligned map, we can smooth it to one in the nice locus. This is done by considering different cases locally, then gluing.

What do we exactly mean by this? How does gluing work in the centrally aligned setting?

Case 1: non-contracted genus one internal component. Assume that the curve takes the form

$$C = C_0 \cup C_1 \cup \ldots \cup C_k$$

where all the C_i are smooth, C_0 has genus one, all the other C_i have genus zero, and for $i \in \{1, \ldots, k\}$, C_i intersects C_0 at a single node (denoted q_i) and does not intersect any other components.

Suppose furthermore that C_0 is a non-contracted *internal component*, meaning that it is mapped into H via f, and that $C_1, \ldots, C_k$ are *external components*, meaning that they are not mapped into H via f. The picture is:

[FIGURE]

Suppose that this is a relative stable map. This means that [BLAH]. We claim that it can be smoothed to a relative stable map in the nice locus. The construction depends on choosing an appropriate smoothing of the curve C, so that the map also smooths.

We start with $W = C_0 \times \mathbb{A}^1_t$ (where t denotes a fixed co-ordinate on the affine line). This is a smooth surface, fibred over $\mathbb{A}^1_t$, with fibre equal to the elliptic curve C_0. Consider the points $q_1, \ldots, q_k$ on C_0. We will perform a series of weighted blow-ups at the points $(q_i, 0) \in W$, in order to obtain a surface whose general fibre is smooth (in fact, isomorphic to C_0) and whose central fibre is isomorphic to C.

Fix $i \in \{1, \ldots, k\}$ and let m_i be the multiplicity of f with H at $q_i \in C_i$. We define:

$$l = \operatorname{lcm}(m_1, \ldots, m_k) \qquad r_i = l/m_i$$

We now blow-up the surface W at the points $(q_i, 0)$ with weight r_i in the horizontal direction and weight 1 in the vertical direction: if x_i is a local co-ordinate for the fibre around q_i, this means that we blow-up in the ideal (x_i, t^{r_i}).

The result is a fibred surface $W' \to \mathbb{A}^1_t$ with general fibre equal to C_0 and central fibre $W'_0 \cong C$. The total space of W' is no longer smooth (its singular points are [BLAH]), but this is not a problem since the projection to $\mathbb{A}^1_t$ is still flat. The central fibre is a linearly trivial Cartier divisor:

$$W'_0 = C_0 + C_1 + \ldots + C_k = 0 \in \operatorname{Pic} W'$$

For $i \in \{1, \ldots, k\}$ we have that $r_i C_i$ is Cartier, although the same is not necessarily true of C_i. Furthermore, since

$$lC_0 = -\sum_{i=1}^{k} lC_i = -\sum_{i=1}^{k} m_i(r_i C_i)$$

in $A_1(W')$, it follows that lC_0 is Cartier. Finally, a local computation shows that

$$r_i C_i \cdot C_0 = 1$$

for $i \in \{1, \ldots, k\}$. Now, let $x_1, \ldots, x_n$ denote the marked points of C. These are smooth points of the central fibre W'_0, and hence can be extended to Cartier divisors $\tilde{x}_1, \ldots, \tilde{x}_n$ on W'. Consider the line bundle:

$$\tilde{L} = \mathcal{O}_{W'}(lC_0 + \Sigma_{j=1}^{n} \alpha_j \tilde{x}_j)$$

on W'. We claim that this gives a smoothing of the line bundle $L = f^*\mathcal{O}(1)$ on C, i.e. that $\tilde{L}|_{W'_0} = L$. We show this by first restricting $\tilde{L}$ to each of the components C_i of $W'_0 \cong C$. For $i \in \{1, \ldots, k\}$, we have

$$\begin{aligned}\tilde{L}|_{C_i} &= \mathcal{O}_{C_i}\left((lC_0 \cdot C_i)q_i + \sum_{x_j \in C_i} \alpha_j x_j\right) = \mathcal{O}_{C_i}\left((l/r_i)q_i + \sum_{x_j \in C_i} \alpha_j x_j\right) \\ &= \mathcal{O}_{C_i}\left(m_i q_i + \sum_{x_j \in C_i} \alpha_j x_j\right) = L|_{C_i}\end{aligned}$$

while for $i = 0$ we have:

$$\tilde{L}|_{C_0} = \mathcal{O}_{C_0}\left(-\sum_{i=1}^{k}(lC_i \cdot C_0)q_i + \sum_{x_j \in C_0} \alpha_j x_j\right) = \mathcal{O}_{C_0}\left(-\sum_{i=1}^{k} m_i q_i + \sum_{x_j \in C_0} \alpha_j x_j\right) = L|_{C_0}$$

Finally the fact that $\tilde{L}|_{W'_0} = L$ follows from the fact that the dual intersection graph of C has genus zero.

Now, $\tilde{L}$ comes with a unique section whose restriction to $W'_0 \cong C$ is s_0. After we extend the sections $s_1, \ldots, s_N$, it is clear that the resulting stable map is in the nice locus (i.e. that it is not mapped into H).

In order to extend the sections $s_1, \ldots, s_N$, we simply check that they are unobstructed. The space containing the obstructions to extending the sections is [Wan12, Theorem 3.1]:

$$\mathrm{H}^1(C, L)$$

By taking the normalisation exact sequence for C, tensoring with L and passing to cohomology, we obtain an exact sequence:

$$0 \to \mathrm{H}^0(C, L) \to \bigoplus_{i=0}^{k} \mathrm{H}^0(C_i, L) \xrightarrow{\theta} \bigoplus_{i=1}^{k} L_{q_i} \to$$
$$\to \mathrm{H}^1(C, L) \to \bigoplus_{i=0}^{k} \mathrm{H}^1(C_i, L) \to 0$$

Now, each of $C_1, \ldots, C_k$ is isomorphic to $\mathbb{P}^1$ and $L|_{C_i}$ has non-negative degree; hence the map θ is surjective. Thus the map

$$\mathrm{H}^1(C, L) \to \bigoplus_{i=0}^{k} \mathrm{H}^1(C_i, L)$$

is an isomorphism. But $\mathrm{H}^1(C_i, L) = 0$ for $i \in \{1, \ldots, k\}$ since $C_i \cong \mathbb{P}^1$ and $L|_{C_i}$ has non-negative degree; also we have by Serre duality

$$\mathrm{H}^1(C_0, L) \cong \mathrm{H}^0(C_0, L^\vee \otimes \omega_{C_0}) = \mathrm{H}^0(C_0, L^\vee) = 0$$

where the penultimate equality holds because $\mathrm{g}(C_0) = 1$ and the last equality holds because $L|_{C_0}$ has *strictly* positive degree (here we are using the fact that $f|_{C_0}$ is non-constant).

To conclude, we have a family $\tilde{C} = W'$ of nodal curves and a map from this family to $\mathbb{P}^N$

$$\begin{tikzcd} \tilde{C} \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & \mathbb{P}^N \\ \mathbb{A}^1_t & \end{tikzcd}$$

such that when we restrict to $0 \in \mathbb{A}^1_t$ we recover the map $f \colon C \to \mathbb{P}^N$ and such that the general fibre is an element of the nice locus.

Address the case that C_0 is not irreducible, and that $C_i,\ i \neq 0$ are chains of $\mathbb{P}^1$'s.

This should follow from gluing, whatever that means.

Case 2: contracted genus one internal component. This is similar to the previous case, but slightly more delicate. The smoothing W', the line bundle $\tilde{L}$ and the section s_0 are constructed as before. Now, however, $\mathrm{H}^1(C, L) \neq 0$, so we need to work harder to show that we can extend the other sections.

Before extending the sections let us first extend the centrally aligned log structure. On the base $\mathbb{A}^1_t$ the chart sends $e_i \in \mathbb{N}^r$ to t^{r_i}. By the novel condition, if $e_i = e_j$ in the minimal centrally aligned structure that we start with on C, then $r_i = r_j$, so the chart that we have just defined factors indeed through the sharpening of the submonoid of $\mathbb{Z}^r$ generated by $\mathbb{N}^r$ and the differences $\lambda(v) - \lambda(w)$. We declare the contraction radius δ to be e_i for those i's corresponding to branches of the genus 1 singularity on which $\bar{f}$ is non-constant.

actually this sounds like bs and I probably have δ already?

The central alignment provides us with a diagram:

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r] \arrow[d] & \overline{\mathcal{C}} \\ \mathcal{C} & \end{tikzcd}$$

We can now extend the sections $\bar{s}_1, \ldots, \bar{s}_N$ onto $\overline{\mathcal{C}}$ because $\mathrm{H}^1(\overline{\mathcal{C}}, \bar{L}) = 0$, so we get a map $\bar{F} \colon \overline{\mathcal{C}} \to \mathbb{P}^N$ deforming $\bar{f}$. Finally we claim that $F \colon \mathcal{C} \to \mathbb{P}^N$ is the stable model of $\bar{F}$.

Appendix A. Basic facts about centrally aligned curves

Proposition A.1 ([RSW17, Proposition 4.6.2.2])**.** The morphism $\mathfrak{M}_{1,n}^{\mathrm{ctr}} \to \mathfrak{M}_{1,n}^{\dagger}$ is a log-modification.

Explanation: this is a local statement so I can probably reduce to an atomic neighbourhood S of a point p. S and the curve over it are endowed with the minimal log structure; let $P = \overline{\mathcal{M}}_p$ determine a chart for this log structure. Observe that the subcurve $\sqsubset_0$ of the tropicalisation $\sqsubset$ of C_p determines a subset MinPos of the set of vertices, namely those adjacent to $\sqsubset_0$. Perform the following log-blowups: consider the set of primitive values of the function $\lambda \colon \sqsubset \to P$, and blow up the ideal that they generate; now locally the set of values of λ is principal with generator p: blow up the ideal generated by $\{\lambda(v) - p\} \setminus \{-p\}$. Keep going until $\lambda(v_i)$ is reached for some $v_i \in$ MinPos; at this point stop and declare the contraction radius $\delta := \lambda(v_i)$. Finish by adjoining $\lambda(v) - \delta$ for all the vertices untouched to this stage. This shows that the choice of δ is not an extra degree of freedom.

do I sound like a physicist?

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