# Profinite Groups

Navid Rashidian

## 1 Category Theory

**Definition 1.1.** A *category* $\mathcal{C}$ consists of collections $\mathrm{Obj}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$, respectively called the objects and morphisms of $\mathcal{C}$ such that

- there are functions $s\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C})$ and $t\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C})$ assigning to each morphism a source and a target;
- for each object $X \in \mathrm{Obj}(\mathcal{C})$ there is a distinguished morphism $\mathrm{id}_X \mathrm{Obj}(\mathcal{C})$ such that $s(\mathrm{id}_X) = t(\mathrm{id}_X) = X$; and
- for each pair of morphisms $f, g$ such that $t(f) = s(f)$ there is a morphism $g \circ f$ with $s(g \circ f) = s(f)$ and $t(g \circ f) = t(g)$ called their composition;

satisfying the further conditions that $(f \circ g) \circ h = f \circ (g \circ h)$ and $\mathrm{id}_X \circ f = f$ and $f \circ \mathrm{id}_X = f$ wherever these expressions make sense.

**Definition 1.2.** Let $\mathcal{C}$ be a category. If both $\mathrm{Obj}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ are sets, The category $\mathcal{C}$ is called a *small category.* $\mathcal{C}$ is called *locally small* if for every objects $X, Y \in \mathrm{Obj}(\mathcal{C})$ the collection of morphisms $f \in \mathrm{Mor}(\mathcal{C})$ with $s(f) = X$ and $t(f) = Y$ is a set. In a locally small category for objects $X, Y \in \mathrm{Obj}(\mathcal{C})$ we define the *hom-set of* $X$ *and* $Y$ as

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) = \{f \in \mathrm{Mor}(\mathcal{C}) : s(f) = X \text{ and } t(f) = Y\}$$

**Definition 1.3.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *(covariant) functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of maps $\mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$ and $\mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$ such that

- for every $f \in \mathrm{Mor}(\mathcal{C})$ we have $s(F(f)) = F(s(f))$ and $t(F(f)) = F(t(f))$;

- for every $X \in \mathrm{Obj}(\mathcal{C})$ we have $F(\mathrm{id}_X) = \mathrm{id}_F(X)$; and
- for every pair of morphisms $f, g \in \mathrm{Mor}(\mathcal{C})$ with a defined composition $F(g \circ f) = F(g) \circ F(f)$.

**Definition 1.4.** Suppose $\mathcal{C}$ is a category. The *opposite category* denoted by $\mathcal{C}^{\mathrm{op}}$ is the category with the same objects and morphisms as $\mathcal{C}$ such that $s_{\mathcal{C}^{\mathrm{op}}}(f) = t_{\mathcal{C}}(f)$ and $t_{\mathcal{C}^{\mathrm{op}}}(f) = s_{\mathcal{C}}(f)$ and $g \circ_{\mathcal{C}^{\mathrm{op}}} f = f \circ_{\mathcal{C}} g$.

**Definition 1.5.** A *contravariant function* $F \colon \mathcal{C} \to \mathcal{D}$ is a covariant functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

**Example 1.6.** A *preorder* is a small categoy $\mathcal{P}$ such that for every objects $X, Y \in \mathrm{Obj}(\mathcal{P})$ we have $\#\mathrm{Hom}_{\mathcal{P}}(X, Y) \leq 1$.

**Example 1.7.** A *partially ordered set (poset)* is a small categoy $\mathcal{P}$ such that for every objects $X, Y \in \mathrm{Obj}(\mathcal{P})$ we have $\#\big(\mathrm{Hom}_{\mathcal{P}}(X, Y) \cup \mathrm{Hom}_{\mathcal{P}}(Y, X)\big) \leq 1$. For posets $\mathcal{P}_1$ and $\mathcal{P}_2$ an order-preserving (-reversing) function from $\mathcal{P}_1$ to $\mathcal{P}_2$ is exactly a covariant (contravariant) functor $F \colon \mathcal{P}_1 \to \mathcal{P}_2$.

**Example 1.8.** Sets and functions constitute a category denoted by **Set**. Groups and group homomorphisms constitute a category denoted by **Grp**. Topological spaces and continuous functions constitute a category denoted by **Top**.