

Profinite Groups

Navid Rashidian

1 Category Theory

Definition 1.1. A *category* \mathcal{C} consists of collections $\text{Obj}(\mathcal{C})$ and $\text{Mor}(\mathcal{C})$, respectively called the objects and morphisms of \mathcal{C} such that

- there are functions $s: \text{Mor}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{C})$ and $t: \text{Mor}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{C})$ assigning to each morphism a source and a target;
- for each object $X \in \text{Obj}(\mathcal{C})$ there is a distinguished morphism $\text{id}_X \in \text{Mor}(\mathcal{C})$ such that $s(\text{id}_X) = t(\text{id}_X) = X$; and
- for each pair of morphisms f, g such that $t(f) = s(g)$ there is a morphism $g \circ f$ with $s(g \circ f) = s(f)$ and $t(g \circ f) = t(g)$ called their composition;

satisfying the further conditions that $(f \circ g) \circ h = f \circ (g \circ h)$ and $\text{id}_X \circ f = f$ and $f \circ \text{id}_Y = f$ wherever these expressions make sense.

Remark 1.2. We write $f: X \rightarrow Y$ to denote a morphism $f \in \text{Mor}(\mathcal{C})$ with $s(f) = X$ and $t(f) = Y$.

Definition 1.3. Let \mathcal{C} be a category. If both $\text{Obj}(\mathcal{C})$ and $\text{Mor}(\mathcal{C})$ are sets, The category \mathcal{C} is called a *small category*. \mathcal{C} is called *locally small* if for every objects $X, Y \in \text{Obj}(\mathcal{C})$ the collection of morphisms $f \in \text{Mor}(\mathcal{C})$ with $s(f) = X$ and $t(f) = Y$ is a set. In a locally small category for objects $X, Y \in \text{Obj}(\mathcal{C})$ we define the *hom-set* of X and Y as

$$\text{Hom}_{\mathcal{C}}(X, Y) = \{f \in \text{Mor}(\mathcal{C}) : s(f) = X \text{ and } t(f) = Y\}$$

Definition 1.4. Let \mathcal{C} and \mathcal{D} be categories. A (*covariant*) *functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of maps $\text{Obj}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{D})$ and $\text{Mor}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{D})$ such that

- for every $f \in \text{Mor}(\mathcal{C})$ we have $s(F(f)) = F(s(f))$ and $t(F(f)) = F(t(f))$;
- for every $X \in \text{Obj}(\mathcal{C})$ we have $F(\text{id}_X) = \text{id}_F(X)$; and
- for every pair of morphisms $f, g \in \text{Mor}(\mathcal{C})$ with a defined composition $F(g \circ f) = F(g) \circ F(f)$.

Definition 1.5. Suppose \mathcal{C} is a category. The *opposite category* denoted by \mathcal{C}^{op} is the category with the same objects and morphisms as \mathcal{C} such that $s_{\mathcal{C}^{\text{op}}}(f) = t_{\mathcal{C}}(f)$ and $t_{\mathcal{C}^{\text{op}}}(f) = s_{\mathcal{C}}(f)$ and $g \circ_{\mathcal{C}^{\text{op}}} f = f \circ_{\mathcal{C}} g$.

Definition 1.6. A *contravariant function* $F: \mathcal{C} \rightarrow \mathcal{D}$ is a covariant functor $F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$.

Example 1.7. A *preorder* is a small category \mathcal{P} such that for every objects $X, Y \in \text{Obj}(\mathcal{P})$ we have $\#\text{Hom}_{\mathcal{P}}(X, Y) \leq 1$. If for every $X, Y \in \text{Obj}(\mathcal{P})$ there is an object $Z \in \text{Obj}(\mathcal{P})$ with morphisms $f: X \rightarrow Z$ and $g: Y \rightarrow Z$, the preorder \mathcal{P} is called a *directed set*.

Example 1.8. A *partially ordered set (poset)* is a small category \mathcal{P} such that for every pair of objects $X, Y \in \text{Obj}(\mathcal{P})$ we have

$$\#(\text{Hom}_{\mathcal{P}}(X, Y) \cup \text{Hom}_{\mathcal{P}}(Y, X)) \leq 1$$

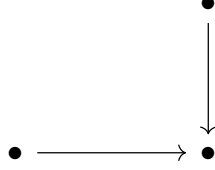
. For posets \mathcal{P}_1 and \mathcal{P}_2 an order-preserving (-reversing) function from \mathcal{P}_1 to \mathcal{P}_2 is exactly a covariant (contravariant) functor $F: \mathcal{P}_1 \rightarrow \mathcal{P}_2$.

Example 1.9. Sets and functions constitute a category denoted by **Set**. Groups and group homomorphisms constitute a category denoted by **Grp**. Topological spaces and continuous functions constitute a category denoted by **Top**. All of these categories are locally small but none is small.

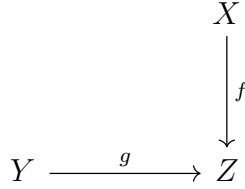
Example 1.10. The category **2** is the category consisting of two objects and only the required identity morphisms:

$$\bullet \qquad \bullet$$

Example 1.11. The following diagram describes a category consisting of three objects and two morphisms (in addition to the required identity morphisms):



Call this category \mathcal{J} . A functor $F: \mathcal{J} \rightarrow \mathcal{C}$ describes a diagram of the following shape in the category \mathcal{C} :

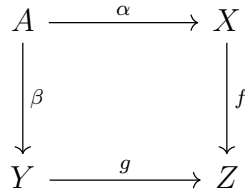


This justifies us calling a functor from a small category a diagram.

Definition 1.12. Let $F: \mathcal{J} \rightarrow \mathcal{C}$ be a diagram. A *cone* over F is a an object $X \in \text{Obj}(\mathcal{C})$ and a collection of morphisms $\alpha_J: X \rightarrow F(J)$ indexed by objects $J \in \text{Obj}(\mathcal{J})$ such that for every morphism $f: J \rightarrow J'$ we have $F(f) \circ \alpha_J = \alpha_{J'}$.

Example 1.13. A diagram $F: \mathbf{2} \rightarrow \mathcal{C}$ is just a pair of discrete objects $X, Y \in \text{Obj}(\mathcal{C})$. A cone over F is just an object $A \in \text{Obj}(\mathcal{C})$ and a pair of morphisms $f: A \rightarrow X$ and $g: A \rightarrow Y$.

Example 1.14. Recall the category \mathcal{J} from example 1.11. A cone over $F: \mathcal{J} \rightarrow \mathcal{C}$ is fully described by an object $A \in \text{Obj}(\mathcal{C})$ and morphisms $\alpha: A \rightarrow X$ and $\beta: A \rightarrow Y$ such that the following diagram commutes:



(Note that we don't need to explicitly describe the morphism $A \rightarrow Z$ included in the cone.)

Definition 1.15. Let $F: \mathcal{J} \rightarrow \mathcal{C}$ be a diagram. A *limit of F* is a cone $\langle L, \alpha_J \rangle$ over F such that for every cone $\langle N, \beta_J \rangle$ over F there is a unique morphism $u: N \rightarrow L$ such that for every $J \in \text{Obj}(\mathcal{J})$ we have $\beta_J = \alpha_J \circ u$.

Example 1.16. Consider a diagram $F: \mathbf{2} \rightarrow \mathbf{Set}$ consisting of sets X and Y . It is easy to verify that the cartesian product $X \times Y$ is a limit of F . The same is true if we replace $\mathbf{2}$ with another discrete category with the requisite cardinality serving as an index set.

Example 1.17. Let \mathcal{I} be a discrete category serving us an index set. A functor $F: \mathcal{I} \rightarrow \mathbf{Top}$ is a family $\langle X_I \rangle$ of topological spaces indexed by \mathcal{I} . It is easy to verify that the cartesian product $\prod X_I$ equipped with the product topology is a limit of F but $\prod X_I$ equipped with the box topology is in general not. (See Wikipedia: Box Topology.)

Example 1.18. This is an example.