

# Profinite Groups

Navid Rashidian

## 1 Category Theory

**Definition 1.1.** A *category*  $\mathcal{C}$  consists of collections  $\text{Obj}(\mathcal{C})$  and  $\text{Mor}(\mathcal{C})$ , respectively called the objects and morphisms of  $\mathcal{C}$  such that

- there are functions  $s: \text{Mor}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{C})$  and  $t: \text{Mor}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{C})$  assigning to each morphism a source and a target;
- for each object  $X \in \text{Obj}(\mathcal{C})$  there is a distinguished morphism  $\text{id}_X \in \text{Mor}(\mathcal{C})$  such that  $s(\text{id}_X) = t(\text{id}_X) = X$ ; and
- for each pair of morphisms  $f, g$  such that  $t(f) = s(g)$  there is a morphism  $g \circ f$  with  $s(g \circ f) = s(f)$  and  $t(g \circ f) = t(g)$  called their composition;

satisfying the further conditions that  $(f \circ g) \circ h = f \circ (g \circ h)$  and  $\text{id}_X \circ f = f$  and  $f \circ \text{id}_Y = f$  wherever these expressions make sense.

*Remark 1.2.* We write  $f: X \rightarrow Y$  to denote a morphism  $f \in \text{Mor}(\mathcal{C})$  with  $s(f) = X$  and  $t(f) = Y$ .

**Definition 1.3.** Let  $\mathcal{C}$  be a category. If both  $\text{Obj}(\mathcal{C})$  and  $\text{Mor}(\mathcal{C})$  are sets, The category  $\mathcal{C}$  is called a *small category*.  $\mathcal{C}$  is called *locally small* if for every objects  $X, Y \in \text{Obj}(\mathcal{C})$  the collection of morphisms  $f \in \text{Mor}(\mathcal{C})$  with  $s(f) = X$  and  $t(f) = Y$  is a set. In a locally small category for objects  $X, Y \in \text{Obj}(\mathcal{C})$  we define the *hom-set* of  $X$  and  $Y$  as

$$\text{Hom}_{\mathcal{C}}(X, Y) = \{f \in \text{Mor}(\mathcal{C}) : s(f) = X \text{ and } t(f) = Y\}$$

**Definition 1.4.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A (*covariant*) *functor*  $F: \mathcal{C} \rightarrow \mathcal{D}$  consists of maps  $\text{Obj}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{D})$  and  $\text{Mor}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{D})$  such that

- for every  $f \in \text{Mor}(\mathcal{C})$  we have  $s(F(f)) = F(s(f))$  and  $t(F(f)) = F(t(f))$ ;
- for every  $X \in \text{Obj}(\mathcal{C})$  we have  $F(\text{id}_X) = \text{id}_F(X)$ ; and
- for every pair of morphisms  $f, g \in \text{Mor}(\mathcal{C})$  with a defined composition  $F(g \circ f) = F(g) \circ F(f)$ .

**Definition 1.5.** Suppose  $\mathcal{C}$  is a category. The *opposite category* denoted by  $\mathcal{C}^{\text{op}}$  is the category with the same objects and morphisms as  $\mathcal{C}$  such that  $s_{\mathcal{C}^{\text{op}}}(f) = t_{\mathcal{C}}(f)$  and  $t_{\mathcal{C}^{\text{op}}}(f) = s_{\mathcal{C}}(f)$  and  $g \circ_{\mathcal{C}^{\text{op}}} f = f \circ_{\mathcal{C}} g$ .

**Definition 1.6.** A *contravariant function*  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a covariant functor  $F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ .

**Example 1.7.** A *preorder* is a small category  $\mathcal{P}$  such that for every objects  $X, Y \in \text{Obj}(\mathcal{P})$  we have  $\#\text{Hom}_{\mathcal{P}}(X, Y) \leq 1$ . If for every  $X, Y \in \text{Obj}(\mathcal{P})$  there is an object  $Z \in \text{Obj}(\mathcal{P})$  with morphisms  $f: X \rightarrow Z$  and  $g: Y \rightarrow Z$ , the preorder  $\mathcal{P}$  is called a *directed set*.

**Example 1.8.** A *partially ordered set (poset)* is a small category  $\mathcal{P}$  such that for every pair of objects  $X, Y \in \text{Obj}(\mathcal{P})$  we have

$$\#(\text{Hom}_{\mathcal{P}}(X, Y) \cup \text{Hom}_{\mathcal{P}}(Y, X)) \leq 1$$

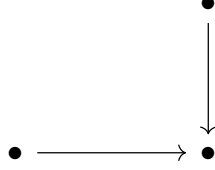
. For posets  $\mathcal{P}_1$  and  $\mathcal{P}_2$  an order-preserving (-reversing) function from  $\mathcal{P}_1$  to  $\mathcal{P}_2$  is exactly a covariant (contravariant) functor  $F: \mathcal{P}_1 \rightarrow \mathcal{P}_2$ .

**Example 1.9.** Sets and functions constitute a category denoted by **Set**. Groups and group homomorphisms constitute a category denoted by **Grp**. Topological spaces and continuous functions constitute a category denoted by **Top**. All of these categories are locally small but none is small.

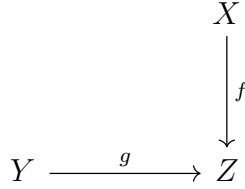
**Example 1.10.** The category **2** is the category consisting of two objects and only the required identity morphisms:

$$\bullet \qquad \bullet$$

**Example 1.11.** The following diagram describes a category consisting of three objects and two morphisms (in addition to the required identity morphisms):



Call this category  $\mathcal{J}$ . A functor  $F: \mathcal{J} \rightarrow \mathcal{C}$  describes a diagram of the following shape in the category  $\mathcal{C}$ :

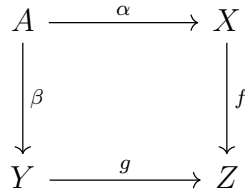


This justifies us calling a functor from a small category a diagram.

**Definition 1.12.** Let  $F: \mathcal{J} \rightarrow \mathcal{C}$  be a diagram. A *cone* over  $F$  is a an object  $X \in \text{Obj}(\mathcal{C})$  and a collection of morphisms  $\alpha_J: X \rightarrow F(J)$  indexed by objects  $J \in \text{Obj}(\mathcal{J})$  such that for every morphism  $f: J \rightarrow J'$  we have  $F(f) \circ \alpha_J = \alpha_{J'}$ .

**Example 1.13.** A diagram  $F: \mathbf{2} \rightarrow \mathcal{C}$  is just a pair of discrete objects  $X, Y \in \text{Obj}(\mathcal{C})$ . A cone over  $F$  is just an object  $A \in \text{Obj}(\mathcal{C})$  and a pair of morphisms  $f: A \rightarrow X$  and  $g: A \rightarrow Y$ .

**Example 1.14.** Recall the category  $\mathcal{J}$  from example 1.11. A cone over  $F: \mathcal{J} \rightarrow \mathcal{C}$  is fully described by an object  $A \in \text{Obj}(\mathcal{C})$  and morphisms  $\alpha: A \rightarrow X$  and  $\beta: A \rightarrow Y$  such that the following diagram commutes:



(Note that we don't need to explicitly describe the morphism  $A \rightarrow Z$  included in the cone.)

**Definition 1.15.** Let  $F: \mathcal{J} \rightarrow \mathcal{C}$  be a diagram. A *limit of  $F$*  is a cone  $\langle L, \alpha_J \rangle$  over  $F$  such that for every cone  $\langle N, \beta_J \rangle$  over  $F$  there is a unique morphism  $u: N \rightarrow L$  such that for every  $J \in \text{Obj}(\mathcal{J})$  we have  $\beta_J = \alpha_J \circ u$ .

**Example 1.16.** Consider a diagram  $F: \mathbf{2} \rightarrow \mathbf{Set}$  consisting of sets  $X$  and  $Y$ . It is easy to verify that the cartesian product  $X \times Y$  is a limit of  $F$ . The same is true if we replace  $\mathbf{2}$  with another discrete category with the requisite cardinality serving as an index set.

**Example 1.17.** Let  $\mathcal{I}$  be a discrete category serving us an index set. A functor  $F: \mathcal{I} \rightarrow \mathbf{Top}$  is a family  $\langle X_I \rangle$  of topological spaces indexed by  $\mathcal{I}$ . It is easy to verify that the cartesian product  $\prod X_I$  equipped with the product topology is a limit of  $F$  but  $\prod X_I$  equipped with the box topology is in general not. (See Wikipedia: Box Topology.)