Profinite Groups

Navid Rashidian

1 Category Theory

Definition 1.1. A *category* $\mathcal{C}$ consists of collections $\mathrm{Obj}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$, respectively called the objects and morphisms of $\mathcal{C}$ such that

- there are functions $s\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C})$ and $t\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C})$ assigning to each morphism a source and a target;
- for each object $X \in \mathrm{Obj}(\mathcal{C})$ there is a distinguished morphism $\mathrm{id}_X \in \mathrm{Mor}(\mathcal{C})$ such that $s(\mathrm{id}_X) = t(\mathrm{id}_X) = X$; and
- for each pair of morphisms f, g such that $t(f) = s(f)$ there is a morphism $g \circ f$ with $s(g \circ f) = s(f)$ and $t(g \circ f) = t(g)$ called their composition;

satisfying the further conditions that $(f \circ g) \circ h = f \circ (g \circ h)$ and $\mathrm{id}_X \circ f = f$ and $f \circ \mathrm{id}_X = f$ wherever these expressions make sense.

Remark 1.2. We write $f\colon X \to Y$ to denote a morphism $f \in \mathrm{Mor}(\mathcal{C})$ with $s(f) = X$ and $t(f) = Y$.

Definition 1.3. Let $\mathcal{C}$ be a category. If both $\mathrm{Obj}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ are sets, The category $\mathcal{C}$ is called a *small category*. $\mathcal{C}$ is called *locally small* if for every objects $X, Y \in \mathrm{Obj}(\mathcal{C})$ the collection of morphisms $f \in \mathrm{Mor}(\mathcal{C})$ with $s(f) = X$ and $t(f) = Y$ is a set. In a locally small category for objects $X, Y \in \mathrm{Obj}(\mathcal{C})$ we define the *hom-set of X and Y* as

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) = \{f \in \mathrm{Mor}(\mathcal{C}) : s(f) = X \text{ and } t(f) = Y\}$$

Definition 1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *(covariant) functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of maps $\mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$ and $\mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$ such that

- for every $f \in \mathrm{Mor}(\mathcal{C})$ we have $s(F(f)) = F(s(f))$ and $t(F(f)) = F(t(f))$;
- for every $X \in \mathrm{Obj}(\mathcal{C})$ we have $F(\mathrm{id}_X) = \mathrm{id}_F(X)$; and
- for every pair of morphisms $f, g \in \mathrm{Mor}(\mathcal{C})$ with a defined composition $F(g \circ f) = F(g) \circ F(f)$.

Definition 1.5. Suppose $\mathcal{C}$ is a category. The *opposite category* denoted by $\mathcal{C}^{\mathrm{op}}$ is the category with the same objects and morphisms as $\mathcal{C}$ such that $s_{\mathcal{C}^{\mathrm{op}}}(f) = t_{\mathcal{C}}(f)$ and $t_{\mathcal{C}^{\mathrm{op}}}(f) = s_{\mathcal{C}}(f)$ and $g \circ_{\mathcal{C}^{\mathrm{op}}} f = f \circ_{\mathcal{C}} g$.

Definition 1.6. A *contravariant function* $F \colon \mathcal{C} \to \mathcal{D}$ is a covariant functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Example 1.7. A *preorder* is a small categoy $\mathcal{P}$ such that for every objects $X, Y \in \mathrm{Obj}(\mathcal{P})$ we have $\#\mathrm{Hom}_{\mathcal{P}}(X, Y) \leq 1$. If for every $X, Y \in \mathrm{Obj}(\mathcal{P})$ there is an object $Z \in \mathrm{Obj}(\mathcal{P})$ with morphisms $f \colon X \to Z$ and $g \colon Y \to Z$, the preorder $\mathcal{P}$ is called a *directed set*.

Example 1.8. A *partially ordered set (poset)* is a small categoy $\mathcal{P}$ such that for every pair of objects $X, Y \in \mathrm{Obj}(\mathcal{P})$ we have

$$\#\big(\mathrm{Hom}_{\mathcal{P}}(X, Y) \cup \mathrm{Hom}_{\mathcal{P}}(Y, X)\big) \leq 1$$

. For posets $\mathcal{P}_1$ and $\mathcal{P}_2$ an order-preserving (-reversing) function from $\mathcal{P}_1$ to $\mathcal{P}_2$ is exactly a covariant (contravariant) functor $F \colon \mathcal{P}_1 \to \mathcal{P}_2$.

Example 1.9. Sets and functions constitute a category denoted by **Set**. Groups and group homomorphisms constitute a category denoted by **Grp**. Topological spaces and continuous functions constitute a category denoted by **Top**. All of these categories are locally small but none is small.

Example 1.10. The category **2** is the category consisting of two objects and only the required identity morphisms:

$$\bullet \qquad\qquad \bullet$$

Example 1.11. The following diagram describes a category consisting of three objects and two morphisms (in addition to the required identity morphisms):

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Call this category $\mathcal{J}$. A functor $F\colon \mathcal{J} \to \mathcal{C}$ describes a diagram of the following shape in the category $\mathcal{C}$:

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

This justifies us calling a functor from a small category a diagram.

Definition 1.12. Let $F\colon \mathcal{J} \to \mathcal{C}$ be a diagram. A *cone* over F is a an object $X \in \mathrm{Obj}(\mathcal{C})$ and a collection of morphisms $\alpha_J\colon X \to F(J)$ indexed by objects $J \in \mathrm{Obj}(\mathcal{J})$ such that for every morphism $f\colon J \to J'$ we have $F(f) \circ \alpha_J = \alpha_{J'}$.

Example 1.13. A diagram $F\colon \mathbf{2} \to \mathcal{C}$ is just a pair of discrete objects $X, Y \in \mathrm{Obj}(\mathcal{C})$. A cone over F is just an object $A \in \mathrm{Obj}(\mathcal{C})$ and a pair of morphisms $f\colon A \to X$ and $g\colon A \to Y$.

Example 1.14. Recall the category $\mathcal{J}$ from example 1.11. A cone over $F : \mathcal{J} \to \mathcal{C}$ is fully described by an object $A \in \mathrm{Obj}(\mathcal{C})$ and morphisms $\alpha\colon A \to X$ and $\beta\colon A \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

(Note that we don't need to explicitly describe the morphism $A \to Z$ included in the cone.)

Definition 1.15. Let $F\colon \mathcal{J} \to \mathcal{C}$ be a diagram. A *limit of* F is a cone $\langle L, \alpha_J \rangle$ over F such that for every cone $\langle N, \beta_J \rangle$ over F there is a unique morphism $u\colon N \to L$ such that for every $J \in \mathrm{Obj}(\mathcal{J})$ we have $\beta_J = \alpha_J \circ u$.

Example 1.16. Consider a diagram $F\colon \mathsf{2} \to \mathsf{Set}$ consisting of sets X and Y. It is easy to verify that the cartesian product $X \times Y$ is a limit of F. The same is true if we replace $\mathsf{2}$ with another discrete category with the requisite cardinality serving as an index set.

Example 1.17. Let $\mathcal{I}$ be a discrete category serving us an index set. A functor $F\colon \mathcal{I} \to \mathsf{Top}$ is a family $\langle X_I \rangle$ of toplogical spaces indexed by $\mathcal{I}$. It is easy to verify that the cartesian product $\prod X_I$ equipped with the product topology is a limit of F but $\prod X_I$ equipped with the box topology is in general not. (See Wikipedia: Box Topology.)