Hilbert's Program

Proof: A thematic history #4

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Class Paradoxes

1. Burali-Forti Paradox (Cantor in 1985 and Burali-Forti in 1897)
2. Cantor's Paradox (1899)
3. Russell's Paradox (Zermelo in 1899 or 1900 and Russell in 1901)

Objections to Infinity

> *A careful reader will find that the literature of mathematics is glutted with inanities and absurdities which have had their source in the infinite.*

Objections to Infinity (cont'd)

Also old objections which we supposed long abandoned still reappear in different forms. For example, the following recently appeared: Although it may be possible to introduce a concept without risk, i.e., without getting contradictions, and even though one can prove that its introduction causes no contradictions to arise, still the introduction of the concept is not thereby justified. Is not this exactly the same objection which was once brought against complex-imaginary numbers when it was said: "True, their use doesn't lead to contradictions. Nevertheless their introduction is unwarranted, for imaginary magnitudes do not exist"? If, apart from proving consistency, the question of the justification of a measure is to have any meaning, it can consist only in ascertaining whether the measure is accompanied by commensurate success. Such success is in fact essential, for in mathematics as elsewhere success is the supreme court to whose decisions everyone submits. (1925)

Current situation in mathematics

> *[T]hanks to the Herculean collaboration of Frege, Dedekind, and Cantor, the infinite was made king and enjoyed a reign of great triumph. In daring flight, the infinite had reached a dizzy pinnacle of success… But reaction was not lacking… [T]he so-called paradoxes of set theory, though at first scattered, became progressively more acute and more serious.*

Current situation in mathematics (cont'd)

Admittedly, the present state of affairs where we run up against theparadoxes is intolerable. Just think, the definitions and deductive methods which everyone learns, teaches, and uses in mathematics, the paragon of truth and certitude, lead to absurdities! If mathematical thinking isdefective, where are we to find truth and certitude?

Hilbert’s idea

> *There is, however, a completely satisfactory way of avoiding theparadoxes without betraying our science. The desires and attitudes which help us find this way and show us what direction to take are these:*
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> 1. *Wherever there is any hope of salvage, we will carefully investigate fruitful definitions and deductive methods. We will nurse them,strengthen them, and make them useful. No one shall drive us out of the paradise which Cantor has created for us.*
> 2. *We must establish throughout mathematics the same certitude for our deductions as exists in ordinary elementary number theory, which no one doubts and where contradictions and paradoxes arise only through our own carelessness.*

Finitary and infinitary statements

1. Statements are finitary which can be directly verified, e.g. 2+3=5.
2. Existential statements are finitary if their quantifier is bounded, e.g. there is a prime p among $p+1, \dots, p!+1$.
3. "In analyzing an existential statement whose content cannot be expressed by a finite disjunction, we encounter the infinite. Similarly, by negating a general statement, i.e. one which referes to arbitrary numberical symbols, we obtain a transfinite statement. For example, the statement that if $\mathfrak{a}$ is a numerical symbol, then $\mathfrak{a}+1=1+\mathfrak{a}$ is universally true, is from our finitary perspective *incapable of negation*."

Ideal statements

[I]f we remain within the domain of finitary statements, as indeed we must, we have as a rule very complicated logical laws… In short, the logical laws which Aristotle taught and which men have used ever since they began to think do not hold. We could, of course, develop logical laws which do hold for the domain of finitary statements. But it would do us no good to develop such a logic, for we do not want to give up the use of the simple laws of Aristotelian logic… Let us remember that we are mathematicians and that as mathematicians we have often been in precarious situations from which we have been rescued by the ingenious method of ideal elements… [S]imilarly, to preservev the simple formal rules of ordinary Aristotelian logic, we must supplement the finitary statements with ideal statements.

Never again!

What we have twice experienced, once with the paradoxes of the infinitesimal calculus and once with the paradoxes of set theory, will not be experienced a third time, nor ever again.

Gödel’s Theorem

Theorem XI: Let κ be an arbitrary recursive consistent class of FORMULAS. Then the SENTENCE which asserts that κ is consistent in not κ-PROVABLE; in particular, the consistency of P is unprovable in P, assuming that P is consistent (in the contrary case, of course, every statement is provable). (Gödel, 1931)

Thank you for your attention!