

# 21-355: Real Analysis 1

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Last Updated: December 2, 2013

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# 1 The Number Systems

## 1.1 The Natural Numbers

**Theorem** (existence of  $\mathbb{N}$ ): There exists a set  $\mathbb{N}$  satisfying the following properties, known as the Peano Axioms:

**PA1**  $0 \in \mathbb{N}$

**PA2** There exists a function  $S : \mathbb{N} \rightarrow \mathbb{N}$  called the successor function. In particular,  $S(n) \in \mathbb{N}$ .

**PA3**  $\forall n \in \mathbb{N}. S(n) \neq 0$

**PA4**  $S(n) = S(m) \implies n = m$  ( $S$  is injective, one-to-one)

**PA5** [Axiom of Induction] Let  $P(n)$  be a property associated to each  $n \in \mathbb{N}$ . If  $P(0)$  is true, and  $P(n) \implies P(S(n))$ , then  $P(n)$  is true  $\forall n \in \mathbb{N}$ .

**Definition:** **PA1**  $\implies 0 \in \mathbb{N}$ . **PA2**  $\implies S(0) \in \mathbb{N}$ .

Define  $1 = S(0), 2 = S(1), 3 = S(2)$ , etc.

**PA2** guarantees that  $\{0, 1, 2, \dots\} \subseteq \mathbb{N}$ .

**PA3** prevents “wraparound”: no successor can map to a “negative” number.

**PA4** prevents “stagnation”: the cycle does not terminate.

**Theorem:**  $\mathbb{N} = \{0, 1, 2, \dots\}$

*Proof:* We know that  $\{0, 1, 2, \dots\} \subseteq \mathbb{N}$ , so it suffices to prove that  $\mathbb{N} \subseteq \{0, 1, 2, \dots\}$ .

Let  $P(n)$  denote the proposition that  $n \in \{0, 1, 2, \dots\}$ . Clearly  $P(0)$  is true.

Suppose  $P(n)$  is true; then  $n \in \{0, 1, 2, \dots\} \implies S(n) \in \{0, 1, 2, \dots\}$  by construction.

Hence,  $P(S(n))$  is true. By induction, **PA5** guarantees that  $P(n)$  is true  $\forall n \in \mathbb{N}$ .

It follows that  $\mathbb{N} \subseteq \{0, 1, 2, \dots\}$ .

**Definition:** For any  $m \in \mathbb{N}$ , we define  $0 + m = m$ .

Then if  $n + m$  is defined for  $n \in \mathbb{N}$ , we set  $S(n) + m = S(n + m)$ .

**Proposition** (Properties of Addition):

1.  $\forall n \in \mathbb{N}. n + 0 = n$  (0 is the additive identity)
2.  $\forall m, n \in \mathbb{N}. n + S(m) = S(n + m)$
3.  $\forall m, n \in \mathbb{N}. m + n = n + m$  (commutativity)
4.  $\forall k, m, n \in \mathbb{N}. k + (m + n) = (k + m) + n$  (associativity)
5.  $\forall k, m, n \in \mathbb{N}. n + k = n + m \implies k = m$  (cancellation)

*Proof:*

1. Let  $P(n)$  be  $n + 0 = n$ .  
 $P(0)$  is true because  $0 + 0 = 0$  by definition.  
 Note  $P(n) \implies S(n) + 0 = S(n + 0) = S(n)$ , so  $P(S(n))$  is true. By induction, (1) is true.

2. Fix  $m \in \mathbb{N}$ . Let  $P(n)$  denote  $n + S(m) = S(n + m)$ .  
 $P(0)$  is true because  $0 + S(m) = S(m) = S(0 + m)$ .  
 $P(n) \implies S(n) + S(m) = S(n + S(m)) = S(S(n + m)) = S(S(n) + m)$ , so  $P(S(n))$  is true. By induction, since  $m \in \mathbb{N}$  was arbitrary, (2) is true.
3. Let  $m$  be fixed and  $P(n)$  denote  $n + m = m + n$ .  
 $P(0)$  is true since  $0 + m = m$  by definition, and  $m + 0 = m$  by 1, so  $0 + m = m = m + 0$ .  
 Suppose  $P(n)$ ; then  $S(n) + m = S(n + m) = S(m + n) = m + S(n)$ , so  $P(S(n))$  is true. By induction and arbitrary choice of  $m$ , (3) is true.
4. Fix  $k, m \in \mathbb{N}$  and let  $P(n)$  denote  $k + (m + n) = (k + m) + n$ .  
 $P(0)$  is true as  $k + (m + 0) = k + m = (k + m) + 0$ .  
 Suppose  $P(n)$ ; then  $k + (m + S(n)) = k + S(m + n) = S(k + (m + n)) = S(k + m) + n = (k + m) + S(n)$  by (2). By induction and arbitrary choice, (4) is true.
5. Fix  $m, n \in \mathbb{N}$  and let  $P(k)$  denote proposition 5.  
 $P(0)$  is true because  $n + 0 = n = n + m \implies m = 0 \implies k = m$ .  
 Suppose  $P(k)$ ; also, suppose  $m + S(k) = n + S(k)$ . Then  $S(m + k) = m + S(k) = n + S(k) = S(n + k) \implies m + k = n + k \implies m = n$  (by 4). By the axiom of induction, (5) is true.

### 1.1.1 Positivity

**Definition:** We say that  $n \in \mathbb{N}$  is *positive* if  $n \neq 0$ .

**Proposition** (Properties of Positivity):

1.  $\forall n, m \in \mathbb{N}$ , if  $m$  is positive, then  $m + n$  is positive.
2.  $\forall n, m \in \mathbb{N}$ , if  $m + n = 0$ , then  $m = n = 0$ .
3.  $\forall n \in \mathbb{N}$ , if  $n$  is positive, then there exists a unique  $m \in \mathbb{N}$  such that  $n = S(m)$ .

### 1.1.2 Order

**Definition:** For all  $m, n \in \mathbb{N}$ ,  $m \leq n$  or  $n \geq m$  iff  $n = m + p$  for some  $p \in \mathbb{N}$ .

$m < n$  or  $n > m$  iff  $m \leq n \wedge m \neq n$ . The relation  $\leq$  provides what is called an *order* on  $\mathbb{N}$ .

**Proposition** (Properties of Order):

Let  $j, k, m, n \in \mathbb{N}$ . Then:

1.  $n \geq n$  (reflexivity)
2.  $m \leq n \wedge k \leq m \implies k \leq n$  (transitivity)
3.  $m \geq n \wedge m \leq n \implies m = n$  (anti-symmetry)
4.  $j \leq k \wedge m \leq n \implies j + m \leq k + n$  (order preservation)
5.  $m < n \iff S(m) \leq n$
6.  $m < n \iff n = m + p$  for some positive  $p \in \mathbb{N}$ .
7.  $n \geq m \iff S(n) > m$
8.  $n = 0 \oplus 0 < n$

**Theorem** (Trichotomy of Order): Let  $m, n \in \mathbb{N}$ . Then exactly one of the following is true:

$$m < n \quad \oplus \quad m = n \quad \oplus \quad m > n$$

*Proof:* Show that no two can be true simultaneously (by definition of  $<$  and  $>$ ), and then at least one must be true (by induction on  $n$ ).

### 1.1.3 Multiplication

**Definition:** Fix  $m \in \mathbb{N}$ . Define  $0 \cdot m = 0$ . Now, if  $n \cdot m$  is defined for some  $n \in \mathbb{N}$ , we define  $S(n) \cdot m = n \cdot m + m$ .

**Proposition** (Properties of Multiplication):

Fix  $k, m, n \in \mathbb{N}$ . Then:

1.  $m \cdot n = n \cdot m$  (commutativity)
2.  $m, n$  are positive  $\implies mn$  is positive
3.  $m \cdot n = 0 \iff m = 0 \vee n = 0$  (no zero divisors)
4.  $k \cdot (m \cdot n) = (k \cdot m) \cdot n$  (associativity)
5.  $k \cdot m = k \cdot n \wedge k$  is positive  $\implies m = n$  (cancellation)
6.  $k \cdot (m + n) = (m + n) \cdot k = k \cdot m + k \cdot n$  (distributivity)
7.  $m < n \wedge k \leq l \wedge k, l$  are positive  $\implies m \cdot k < n \cdot l$

## 1.2 The Integers

Consider the following relation on the set  $\mathbb{N} \times \mathbb{N}$ :

$$(m, n) \simeq (m', n') \iff m + n' = m' + n$$

**Lemma:**  $\simeq$  is an equivalence relation.

*Proof:*

Reflexivity:  $m + n = m + n \implies (m, n) \simeq (m, n)$

Symmetry:  $(m, n) \simeq (m', n') \implies m + n' = m' + n \implies m' + n = m + n' \implies (m', n') \simeq (m, n)$

Transitivity: Suppose  $(m, n) \simeq (m', n') \wedge (m', n') \simeq (m'', n'')$ . Then:

$$\begin{aligned} m + n' &= m' + n \wedge m' + n'' = m'' + n' \\ \implies m + n'' &= m'' + n \\ \implies (m, n) &\simeq (m'', n'') \end{aligned}$$

**Definition:** Write the *equivalence class* of  $(m, n)$  as  $[(m, n)] = \{(p, q) \mid (p, q) \simeq (m, n)\}$ . Define the *integers*  $\mathbb{Z} = \{[(m, n)]\}$ .

**Lemma:** Suppose  $(m, n) \simeq (m', n'), (p, q) \simeq (p', q')$ . Then:

1.  $(m + p, n + q) \simeq (m' + p', n' + q')$

$$2. (mp + nq, mq + np) \simeq (m'p' + n'q', m'q' + n'p')$$

*Proof:* Consider equalities (a) :  $m + n' = m' + n$  and (b) :  $p + q' = p' + q$  (by definition of  $\simeq$ ).

Using linear combinations of (a) and (b), we derive the two rules of the lemma:

1. (a) + (b)
2. (a)(p' + q') + (b)(m + n)

**Definition:** Let  $[(m, n)], [(p, q)] \in \mathbb{Z}$ . Then:

1.  $[(m, n)] + [(p, q)] = [(m + p, n + q)]$  (addition of integers)
2.  $[(m, n)] \cdot [(p, q)] = [(mp + nq, mq + np)]$  (multiplication of integers)

By the lemma, these are well-defined operations.

Note that for all  $m, n \in \mathbb{N}$ :

$$\begin{aligned} [(m, 0)] &= [(n, 0)] \iff m + 0 = n + 0 \iff m = n \\ [(m, 0)] + [(n, 0)] &= [(m + n, 0)] \\ [(m, 0)] \cdot [(n, 0)] &= [(mn, 0)] \end{aligned}$$

As such, the set  $\{[(n, 0)] \mid n \in \mathbb{N}\} \subseteq \mathbb{Z}$  behaves exactly like a copy of  $\mathbb{N}$ .

**Definition:** For  $n \in \mathbb{N}$  we set  $n \in \mathbb{Z}$  to be  $n := [(n, 0)]$ .

For  $x = [(m, n)] \in \mathbb{Z}$  we define  $-x = [(n, m)]$ .

### 1.2.1 Properties of Integers

(We can see that every integer  $x \in \mathbb{Z}$  can be represented as  $x := m - n$  where  $x = [(m, n)]$ .)

**Theorem:** Every  $x \in \mathbb{Z}$  satisfies exactly one of the following:

1.  $x = n$  for some  $n \in \mathbb{N} \setminus \{0\}$
2.  $x = 0$
3.  $x = -n$  for some  $n \in \mathbb{N} \setminus \{0\}$

*Proof:* Write  $x = [(p, q)]$  for some  $p, q \in \mathbb{N}$ . By trichotomy of order on  $\mathbb{N}$  we know that  $p < q$  or  $p = q$  or  $p > q$ . Each of these correlates to one of the three properties.

**Corollary:**  $\mathbb{Z} = \{0, 1, 2, \dots\} \cup \{-1, -2, -3, \dots\}$

### 1.2.2 Algebraic Properties

**Proposition:** Let  $x, y, z \in \mathbb{Z}$ . Then the following hold:

1.  $x + y = y + x$
2.  $x + (y + z) = (x + y) + z$
3.  $x + 0 = 0 + x = x$
4.  $x + (-x) = (-x) + x = 0$
5.  $xy = yx$



6.  $(xy)z = x(yz)$
7.  $x \cdot 1 = 1 \cdot x = x$
8.  $x(y + z) = xy + xz$

**Definition:** Define  $x - y = x + (-y)$ . The usual properties hold.

**Definition:** For  $x, y \in \mathbb{Z}$ , we say  $x \leq y$  or  $y \geq x$  if  $y - x = n$  for some  $n \in \mathbb{N}$ . We say  $x < y$  if  $x \leq y \wedge x \neq y$ .

### 1.3 The Rationals and Ordered Fields

Let a relation on  $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$  be given by  $(m, n) \simeq (m', n') \iff mn' = m'n$ .

**Lemma:**  $\simeq$  is an equivalence relation. Proof follows from properties of  $\mathbb{Z}$ .

**Definition:**  $\mathbb{Q} = \{[(m, n)]\}$

1.  $[(m, n)] + [(p, q)] = [(mq + np, nq)]$  (addition)
2.  $[(m, n)] \cdot [(p, q)] = [(mp, nq)]$  (multiplication)
3.  $-[(m, n)] = [(-m, n)]$  (negation)
4. If  $m \neq 0$  we set  $[(m, n)]^{-1} = [(n, m)]$

Remark: the heuristic here is that  $\frac{m}{n} = [(m, n)]$ .

**Definition:** If  $m \in \mathbb{Z}$ , we write  $m = [(m, 1)] \in \mathbb{Q}$ ; and thus  $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$ .

1. For  $x, y \in \mathbb{Q}$ , we define  $x - y = x + (-y) \in \mathbb{Q}$
2. For  $x, y \in \mathbb{Q}, y \neq 0$  we define  $\frac{x}{y} = x(y)^{-1}$ . This is well defined because  $y = 0 \iff y = [(0, n)]$ .

**Proposition:**  $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$ .

We define and propose the trichotomy of order on  $\mathbb{Q}$ , as per the integers.

#### 1.3.1 Fields and Orders

**Definition:** A field is a set  $\mathbb{F}$  endowed with two binary operations,  $+, \cdot$ , satisfying the following axioms:

- (A1, M1)  $\forall x, y \in \mathbb{F}. x + y \in \mathbb{F}, xy \in \mathbb{F}$  (closure)
- (A2, M2)  $\forall x, y \in \mathbb{F}. x + y = y + x, xy = yx$  (commutativity)
- (A3, M3)  $\forall x, y, z \in \mathbb{F}. x + (y + z) = (x + y) + z, x(yz) = (xy)z$  (associativity)
- (A4, M4)  $\exists (0, 1) \in \mathbb{F}. \forall x \in \mathbb{F}. 0 + x = x + 0 = x, 1 \cdot x = x \cdot 1 = x$  (identity)
- (A5, M5)  $\forall x \in \mathbb{F}. \exists (-x). x + (-x) = 0; \exists x^{-1} \in \mathbb{F}. xx^{-1} = x^{-1}x = 1$  (inverse)
- (D1)  $\forall x, y, z \in \mathbb{F}. x(y + z) = xy + xz$  (distributivity)

*Remark:* Field must have at least 2 elements  $(0, 1)$  by (A/M4). To prove field, must prove 5 properties of addition and multiplication (closure, commutativity, associativity, identity, inverse) as well as distributivity.

**Definition:** Let  $E$  be a set; an *order* on  $E$  is a relation  $<$  satisfying the following:

1.  $\forall x, y \in E$  exactly one of the following is true:  $x < y$  or  $x = y$  or  $y < x$  (trichotomy)
2.  $\forall x, y, z \in E, x < y \wedge y < z \implies x < z$  (transitivity)

**Definition:** Let  $\mathbb{F}$  be a field. Then we define  $x - y = x + (-y)$  and  $\frac{x}{y} = xy^{-1}$  (for  $y \neq 0$ ).

**Theorem:**  $\mathbb{Q}$  is an ordered field with order  $<$ .

*Proof:* Follows from definitions and properties of  $\mathbb{Z}$ .

## 1.4 Problems with $\mathbb{Q}$

**Theorem:** There does not exist a  $q \in \mathbb{Q}$  such that  $q^2 = 2$ .

*Proof:* Suppose not; i.e. there does exist such a  $q \in \mathbb{Q}$ .

Consider the set  $S(q) = \{n \in \mathbb{N}^+ \mid q = \frac{m}{n} \text{ for some } m \in \mathbb{Z}\}$ . Clearly  $|S(q)| > 0$ . Then the well-ordering principle implies that  $\exists! n \in S(q)$ .  $n = \min S(q)$ .

Since  $n \in S(q)$ , we know that  $q = \frac{m}{n}$  for some  $m \in \mathbb{Z}$ . Then  $q^2 = (\frac{m}{n})^2 = \frac{m^2}{n^2} \implies m^2 = 2n^2 \implies m^2$  is even. We claim that  $m$  is also even (proof is exercise to reader).

Then  $\exists l \in \mathbb{Z}$ .  $m = 2l$ . Then  $4l^2 = (2l)^2 = m^2 = 2n^2 \implies n^2 = 2l^2 \implies n^2$  is even  $\implies n$  is even  $\implies n = 2p$  for some  $p \in \mathbb{N}^+$ .

Hence  $q = \frac{m}{n} = \frac{2l}{2p} = \frac{l}{p} \implies p \in S(q)$ . But clearly  $p < n$ , which contradicts the fact that  $n$  is the minimal element. By contradiction, the theorem must be true.

### 1.4.1 Bounds (Infimum and Supremum)

Informally,  $\mathbb{Q}$  has “holes”:

**Definition:** Let  $E$  be an ordered set with order  $<$ .

1. We say  $A \subseteq E$  is bounded above iff  $\exists x \in E. \forall a \in A. a \leq x$ . We say  $x$  is an upper bound of  $A$ .
2. We say  $A \subseteq E$  is bounded below iff  $\exists x \in E. \forall a \in A. x \leq a$ . We say  $x$  is a lower bound of  $A$ .
3. We say  $A \subseteq E$  is bounded iff it's bounded above and below.
4. We say  $x$  is a minimum of  $A$  iff  $x \in A$  and  $x$  is a lower bound of  $A$ .
5. We say  $x$  is a maximum of  $A$  iff  $x \in A$  and  $x$  is an upper bound of  $A$ .

*Remark:* If a min or max exists, then it is unique.

**Definition:** Let  $E$  be an ordered set and  $A \subseteq E$ .

1. We say  $x \in E$  is the least upper bound (*supremum*) of  $A$ , written  $x = \sup A$ , iff  $x$  is an upper bound of  $A$  and  $y \in E$  is an upper bound of  $A \implies x \leq y$ .
2. We say  $x \in E$  is the greatest lower bound (*infimum*) of  $A$ , written  $x = \inf A$ , iff  $x$  is a lower bound of  $A$  and  $y \in E$  is a lower bound of  $A \implies y \leq x$ .

*Remark:* If  $x = \min(A)$ , then  $x = \inf(A)$ . If  $x = \max(A)$ , then  $x = \sup(A)$ . But the converse is false; some sets have a supremum but no maximum, others a infimum but no minimum.

**Definition:** Let  $\mathbb{F}$  be an ordered field. We say that  $\mathbb{F}$  has the *least upper bound property* iff every  $\emptyset \neq A \subseteq \mathbb{F}$  that is bounded above has a least upper bound.

**Theorem:**  $\mathbb{Q}$  does not satisfy the least upper bound property.

*Proof:* Consider the set  $A = \{x \in \mathbb{Q} \mid x > 0, x^2 \leq 2\}$ .

Note that  $0 < 1 = 1^2 \leq 2 \implies 1 \in A$ , so  $A$  is non-empty. Also,  $2 \leq 4 = 2^2$  implies  $(x \in A \implies 0 < x^2 < 2 < 2^2) \implies x < 2$ . Then 2 is an upper bound of  $A$ .

Assume for sake of contradiction that  $\mathbb{Q}$  has the least upper bound property. Then  $A$  has a supremum. Let  $x = \sup A \in \mathbb{Q}$  and write  $x = \frac{p}{q}$  for  $p, q \in \mathbb{Z}$ .

By trichotomy,  $x^2 < 2$  or  $x^2 = 2$  or  $x^2 > 2$ . We know  $x^2 \neq 2$ .

**Case 1:** Suppose  $x^2 < 2$ . Then for any  $n \in \mathbb{N}^+$  we have  $(\frac{p}{q} + \frac{1}{n})^2 = \frac{p^2}{q^2} + \frac{2p}{qn} + \frac{1}{n^2} \leq \frac{p^2}{q^2} + \frac{1}{n}(\frac{2p+q}{q})$ . From algebra, we derive  $(\frac{p}{q} + \frac{1}{n})^2 < 2$  for some  $n \in \mathbb{N}^+$ .

Clearly  $x > 0$  since otherwise  $x \leq 0 < 1 \in A$ . Hence  $0 < x = \frac{p}{q} < \frac{p}{q} + \frac{1}{n} \in A$ . But then  $x$  is not an upper bound  $\implies$  contradiction.

**Case 2:** Suppose  $x^2 > 2$ . Considering  $(\frac{p}{q} - \frac{1}{n})^2 > 2$  and using the same logic as before, we can choose  $n$  large enough such that  $\frac{p}{q} - \frac{1}{n}$  is an upper bound of  $A$ . But  $\frac{p}{q} - \frac{1}{n} < \frac{p}{q} = x$ , which contradicts the fact that  $x = \sup A$ .

As all cases are false, we contradict trichotomy, and hence  $\mathbb{Q}$  cannot have the least upper bound property.

## 1.5 The Real Numbers

We now construct an ordered field satisfying the least upper bound property using  $\mathbb{Q}$ .

**Definition:** We say  $\mathbb{Q}$  is Archimedean iff  $\forall(x \in \mathbb{Q}). x > 0 \implies \exists(n \in \mathbb{N}). x < n$ .

**Lemma:** If  $\mathbb{Q}$  is Archimedean, then  $\forall(p < q \in \mathbb{Q}). \exists(r \in \mathbb{Q}). p < r < q$ .  
(Proofs in HW 2.)

### 1.5.1 Defining the Real Numbers: Dedekind Cuts

**Definition:** We say that  $\mathcal{C} \in \mathcal{P}(\mathbb{Q})$  is a *cut* (Dedekind cut) iff the following hold:

- (C1)  $\emptyset \neq \mathcal{C}, \mathcal{C} \neq \mathbb{Q}$
- (C2) If  $p \in \mathcal{C}$  and  $q \in \mathbb{Q}$  with  $q < p$ , then  $q \in \mathcal{C}$ .
- (C3) If  $p \in \mathcal{C}$ ,  $\exists(r \in \mathbb{Q}). p < r \wedge r \in \mathcal{C}$ .

**Lemma:** Suppose  $\mathcal{C}$  is a cut. Then:

1.  $p \in \mathcal{C}, q \notin \mathcal{C} \implies p < q$
2.  $r \notin \mathcal{C}, r < s \implies s \notin \mathcal{C}$
3.  $\mathcal{C}$  is bounded above

**Lemma:** Let  $q \in \mathbb{Q}$ . Then  $\{p \in \mathbb{Q} \mid p < q\}$  is a cut.

*Proof:* Call the set  $\mathcal{C}$ . We prove the 3 properties of a cut:

- (C1)  $q - 1 \in \mathcal{C} \implies \mathcal{C} \neq \emptyset$ ;  $q + 1 \notin \mathcal{C} \implies \mathcal{C} \neq \mathbb{Q}$ .
- (C2) If  $p \in \mathcal{C}$  and  $r \in \mathbb{Q}$  such that  $r < p$ , then  $r < p < q \implies r < q \implies r \in \mathcal{C}$ .
- (C3) Let  $p \in \mathcal{C}$  where  $p < q$ . Since  $\mathbb{Q}$  is Archimedean,  $\exists(r \in \mathbb{Q}). p < r < q \implies r \in \mathcal{C}$ .

**Definition:** Given  $q \in \mathbb{Q}$  we write  $\mathcal{C}_q = \{p \in \mathbb{Q} \mid p < q\}$ . By the above lemma,  $\mathcal{C}_q$  is a cut.

**Definition:** We write  $\mathbb{R} = \{\mathcal{C} \in \mathcal{P}(\mathbb{Q}) \mid \mathcal{C} \text{ is a cut}\} \neq \emptyset$ .

**Lemma:** The following hold:

1.  $\forall \mathcal{A}, \mathcal{B} \in \mathbb{R}$ , exactly one of the following holds:  $\mathcal{A} \subset \mathcal{B}$ ,  $\mathcal{A} = \mathcal{B}$ ,  $\mathcal{B} \subset \mathcal{A}$ .
2.  $\forall \mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}$ ,  $\mathcal{A} \subset \mathcal{B} \wedge \mathcal{B} \subseteq \mathcal{C} \implies \mathcal{A} \subset \mathcal{C}$ .

**Definition:** If  $\mathcal{A}, \mathcal{B} \in \mathbb{R}$  we say that  $\mathcal{A} < \mathcal{B} \iff \mathcal{A} \subset \mathcal{B}$ , and  $\mathcal{A} \leq \mathcal{B} \iff \mathcal{A} \subseteq \mathcal{B}$ . This defines an order on  $\mathbb{R}$  by the above lemma.

### 1.5.2 Defining the Real Numbers: The Least Upper Bound Property

**Lemma:** Suppose  $\emptyset \neq E \subseteq \mathbb{R}$  is bounded above. Then  $\mathcal{B} := \bigcup_{\mathcal{A} \in E} \mathcal{A} \in \mathbb{R}$ .

**Theorem:**  $\mathbb{R}$  satisfies the least upper bound property.

*Proof:* Let  $\emptyset \neq E \subseteq \mathbb{R}$  be bounded above and set  $\mathcal{B} = \bigcup_{\mathcal{A} \in E} \mathcal{A} \in \mathbb{R}$ . We claim  $\mathcal{B} = \sup E$ .

First, we show that  $\mathcal{B}$  is an upper bound of  $E$ . Let  $\mathcal{A} \in E$ . Then  $\mathcal{A} \subseteq \mathcal{B} \implies \mathcal{A} \leq \mathcal{B}$  (by definition). This is true for all  $\mathcal{A} \in E$ , so  $\mathcal{B}$  is an upper bound.

We claim that for  $\mathcal{C} \in \mathbb{R}$ ,  $\mathcal{C} < \mathcal{B} \implies \mathcal{C}$  is not an upper bound of  $E$ . If  $\mathcal{C} < \mathcal{B}$ , then  $\mathcal{C} \subset \mathcal{B}$ . This implies  $\exists b \in \mathcal{B}$ .  $b \notin \mathcal{C} \implies \exists (\mathcal{A} \in E)$ .  $b \in \mathcal{A} \wedge b \notin \mathcal{C}$ . Then  $\mathcal{A} > \mathcal{C}$  since otherwise  $\mathcal{A} \subseteq \mathcal{C} \implies b \in \mathcal{C}$ ,  $b \notin \mathcal{C}$ . Hence  $\mathcal{C} < \mathcal{A}$  and  $\mathcal{C}$  is not an upper bound of  $E$ .

By the contrapositive: if  $\mathcal{C}$  is an upper bound,  $\mathcal{C} \geq \mathcal{B}$ . Thus,  $\mathcal{B}$  is the least upper bound, and the theorem holds.

### 1.5.3 Defining the Real Numbers: Addition

**Definition:** Given  $\mathcal{A}, \mathcal{B} \in \mathbb{R}$ , set  $\mathcal{A} + \mathcal{B} = \{a + b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ .

**Lemma:** If  $\mathcal{A}, \mathcal{B} \in \mathbb{R}$ , then  $\mathcal{A} + \mathcal{B} \in \mathbb{R}$ .

**Theorem:** Define  $-\mathcal{A} = \{q \in \mathbb{Q} \mid \exists (p > q). -p \notin \mathcal{A}\}$ . Then  $\mathbb{R}, +, 0_{\mathbb{R}} = \mathcal{C}_0 = \{p \in \mathbb{Q} \mid p < 0\}$  satisfy the field axioms.

*Proof:*

- (A1)  $\mathcal{A} + \mathcal{B} \in \mathbb{R}$  by previous lemma.
- (A2)  $\mathcal{A} + \mathcal{B} = \{a + b\} = \{b + a\} = \mathcal{B} + \mathcal{A}$ .
- (A3)  $\mathcal{A} + (\mathcal{B} + \mathcal{C}) = \{a + (b + c)\} = \{(a + b) + c\} = (\mathcal{A} + \mathcal{B}) + \mathcal{C}$ .
- (A4) Show  $\forall \mathcal{A} \in \mathbb{R}$ .  $0_{\mathbb{R}} + \mathcal{A} = \mathcal{A}$ .
- (A5) Show that  $-\mathcal{A} \in \mathbb{R}$ , then  $\mathcal{A} + (-\mathcal{A}) = 0_{\mathbb{R}}$  using Archimedean property.

**Theorem (Ordered Field):** Let  $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}$ . If  $\mathcal{A} < \mathcal{B}$  then  $\mathcal{A} + \mathcal{C} < \mathcal{B} + \mathcal{C}$ .

*Proof:* It's trivial to see that  $\mathcal{A} \subseteq \mathcal{B} \implies \mathcal{A} + \mathcal{C} \subseteq \mathcal{B} + \mathcal{C} \implies \mathcal{A} + \mathcal{C} \leq \mathcal{B} + \mathcal{C}$ .

If  $\mathcal{A} + \mathcal{C} = \mathcal{B} + \mathcal{C}$ , we can add  $-\mathcal{C}$  to both sides and use the last theorem to see that  $\mathcal{A} = \mathcal{B}$ , a contradiction. Hence,  $\mathcal{A} + \mathcal{C} < \mathcal{B} + \mathcal{C}$ .

### 1.5.4 Defining the Real Numbers: Multiplication

**Lemma:** Let  $\mathcal{A}, \mathcal{B} \in \mathbb{R}$ ,  $\mathcal{A}, \mathcal{B} > 0_{\mathbb{R}}$ . Then  $\mathcal{C} = \{q \in \mathbb{Q} \mid q \leq 0\} \cup \{a \cdot b \mid a \in \mathcal{A}, b \in \mathcal{B}, a, b > 0\} \in \mathbb{R}$ .

*Proof:*

- (C1)  $0 \in \mathcal{C} \implies \mathcal{C} \neq \emptyset$ .  $\mathcal{A}, \mathcal{B}$  are bounded above by, say  $M_1, M_2$ , so  $M_1 \cdot M_2 + 1 \notin \mathcal{C}$  and  $\mathcal{C} \neq \mathbb{Q}$ .
- (C2) Let  $p \in \mathcal{C}$  and  $q < p$ . If  $q \leq 0$  then  $q \in \mathcal{C}$  by definition. If  $q > 0$  then  $0 < q < p$ , but then  $0 < p \implies p = a \cdot b$  for  $a \in \mathcal{A}, b \in \mathcal{B}, a, b > 0$ . Then  $0 < q < a \cdot b \implies \frac{q}{a} < b \implies 0 < \frac{q}{a} \in \mathcal{B}$ . Then  $q = a(\frac{q}{a}) \in \mathcal{C}$ .
- (C3) Let  $p \in \mathcal{C}$ . If  $p \leq 0$  then any  $a \cdot b$  with  $a \in \mathcal{A}, b \in \mathcal{B}, a, b > 0$  satisfies  $p < a \cdot b \in \mathcal{C}$ , so  $r = a \cdot b$  is the desired element of  $\mathcal{C}$ . However, if  $p > 0$ , then  $p = a \cdot b$  for  $a \in \mathcal{A}, b \in \mathcal{B}, a, b > 0$ . Choose  $s \in \mathcal{A}$  such that  $a < s, t \in \mathcal{B}$  such that  $t > b$ . Then  $p = a \cdot b < s \cdot t \in \mathcal{C}$ , so  $r = s \cdot t$  proves the claim.

**Definition of Multiplication:** Let  $\mathcal{A}, \mathcal{B} \in \mathbb{R}$ .

1. If  $\mathcal{A} > 0, \mathcal{B} > 0$  we set  $\mathcal{A} \cdot \mathcal{B} = \{q \in \mathbb{Q} \mid q \leq 0\} \cup \{a \cdot b \mid a \in \mathcal{A}, b \in \mathcal{B}, a, b > 0\} \in \mathbb{R}$ .
2. If  $\mathcal{A} = 0$  or  $\mathcal{B} = 0$ , we set  $\mathcal{A} \cdot \mathcal{B} = 0_{\mathbb{R}}$ .
3. If  $\mathcal{A} > 0$  and  $\mathcal{B} < 0$ , let  $\mathcal{A} \cdot \mathcal{B} = -(\mathcal{A} \cdot (-\mathcal{B}))$ .
4. If  $\mathcal{A} < 0$  and  $\mathcal{B} > 0$ , let  $\mathcal{A} \cdot \mathcal{B} = -((- \mathcal{A}) \cdot \mathcal{B})$ .
5. If  $\mathcal{A} < 0$  and  $\mathcal{B} < 0$ , let  $\mathcal{A} \cdot \mathcal{B} = (-\mathcal{A}) \cdot (-\mathcal{B})$ .

**Theorem:**  $\mathbb{R}, \cdot$  satisfies (M1-M5) with  $1_{\mathbb{R}} = \mathcal{C}_1$ , and

$\mathcal{A} > 0 \implies \mathcal{A}^{-1} = \{q \in \mathbb{Q} \mid q \leq 0\} \cup \{q \in \mathbb{Q} \mid q > 0, \exists p > q. p^{-1} \notin \mathcal{A}\} \in \mathbb{R}$ ;

$\mathcal{A} < 0 \implies \mathcal{A}^{-1} = -(-\mathcal{A})^{-1}$ .

*Proof:* HW3 (similar to addition).

**Theorem:** If  $\mathcal{A}, \mathcal{B} > 0$ , then  $\mathcal{A} \cdot \mathcal{B} > 0$ .

*Proof:* By definition  $\mathcal{C}_0 \subseteq \mathcal{A} \cdot \mathcal{B} \implies 0 \leq \mathcal{A} \cdot \mathcal{B}$ . Equality is impossible since  $\mathcal{A}, \mathcal{B} > 0$ .

### 1.5.5 Defining the Real Numbers: Distributivity

**Theorem:** Let  $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}$ . Then  $\mathcal{A} \cdot (\mathcal{B} + \mathcal{C}) = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C}$ .

*Proof:* We prove the case where all are positive. The other cases are in HW.

Let  $p \in \mathcal{A}(\mathcal{B} + \mathcal{C})$ . If  $p \leq 0$  then  $p \in \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C}$  is trivial (both products contain the interval less than 0).

If  $p > 0$ ,  $p = a(b + c)$  for  $a \in \mathcal{A}, b \in \mathcal{B}, c \in \mathcal{C}$  for  $a > 0, b + c > 0$ .

Regardless of sign of  $b$  or  $c$ ,  $a \cdot b \in \mathcal{A} \cdot \mathcal{B}, a \cdot c \in \mathcal{A} \cdot \mathcal{C}$ . Hence  $p = a(b + c) = a \cdot b + a \cdot c \in \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C}$ . So  $\mathcal{A}(\mathcal{B} + \mathcal{C}) \subseteq \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C}$ .

Finally, we show the converse is true; let  $p \in \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C} \implies p = r + s$  for  $r \in \mathcal{A} \cdot \mathcal{B}, s \in \mathcal{A} \cdot \mathcal{C}$ . Case on positivity of  $p, r, s$  to show  $p \in \mathcal{A}(\mathcal{B} + \mathcal{C})$ .

### 1.5.6 Defining the Real Numbers: Archimedean

**Theorem:** For  $p, q \in \mathbb{Q}$ , the following are true:

1.  $\mathcal{C}_{p+q} = \mathcal{C}_p + \mathcal{C}_q$
2.  $\mathcal{C}_{-p} = -\mathcal{C}_p$
3.  $\mathcal{C}_{pq} = \mathcal{C}_p \mathcal{C}_q$
4. If  $p \neq 0$  then  $\mathcal{C}_{p^{-1}} = (\mathcal{C}_p)^{-1}$
5.  $p < q \in \mathbb{Q} \iff \mathcal{C}_p < \mathcal{C}_q \in \mathbb{R}$

*Proof:* HW.

**Definition:** For  $q \in \mathbb{Q}$  we say  $\mathcal{C}_q \in \mathbb{R}$ . Then  $\mathbb{Q} \subseteq \mathbb{R}$ .

**Theorem:** There exists an ordered field satisfying the least upper bound property;  $\mathbb{R}$  is unique (for any ordered field  $\mathbb{F}$  satisfying these properties,  $\mathbb{F} = \mathbb{R}$  up to isomorphism; and  $\mathbb{R}$  is Archimedean.

*Proof:* The basic assertion is Steps (0)-(4). Step (5) proves 1, Step (6) proves 3.

## 1.6 Properties of $\mathbb{R}$

Notation: think of  $\mathbb{R}$  as numbers, not cut notation.

**Proposition:**  $\mathbb{R}$  satisfies the following:

**Theorem:** For  $p, q \in \mathbb{Q}$ , the following are true:

1.  $\mathbb{R}$  is Archimedean:  $\forall x \in \mathbb{R}, x > 0. \exists n \in \mathbb{N}. x < n$
2.  $\mathbb{N} \subset \mathbb{R}$  is not bounded above
3.  $\inf\{\frac{1}{n} \mid n \in \mathbb{N}, n \geq 1\} = 0$
4.  $\forall x \in \mathbb{R}$  the set  $B(x) = \{m \in \mathbb{Z} \mid x < m\}$  has a minimum in  $\mathbb{Z}$ .
5.  $\forall x, y \in \mathbb{R}, x < y. \exists q \in \mathbb{Q}. x < q < y$

*Remarks:*

1. (5) is interpreted as “the density of  $\mathbb{Q} \subseteq \mathbb{R}$ ”. Any element  $x \in \mathbb{R}$  can be approximated to arbitrary accuracy by elements of  $\mathbb{Q}$ .
2. (4) allows us to define the integer part of any  $x \in \mathbb{R}$ . We can set  $\lfloor x \rfloor = \min B(x) - 1 \in \mathbb{Z}$ . Then  $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ .

Next we show that  $\mathbb{R}$  does not have the “holes” we saw in  $\mathbb{Q}$ .

**Theorem:** Let  $x \in \mathbb{R}$  satisfy  $x > 0$  and  $n \in \mathbb{N}, n \geq 1$ . Then  $\exists! y \in \mathbb{R}. y > 0 \wedge y^n = x$ .

*Proof:* The case  $n = 1$  is trivial so assume  $n \geq 2$ .

Set  $E = \{z \in \mathbb{R} \mid z > 0 \wedge z^n < x\}$ . We want to show  $E \neq \emptyset$  and is bounded above. Set  $t = \frac{x}{1+x}$ ; then  $0 < t < 1$  and  $t < x$ . Hence  $0 < t^n < t < x$ , and so  $t \in E$  and  $E \neq \emptyset$ .

Set  $s = 1 + x$ . Then  $1 < s \wedge x < s \implies x < s < s^n$ ; so if  $z \in E$  then  $z^n < x < s^n \implies z < s$ . Then  $s$  is an upper bound of  $E$ .

By least upper bound property,  $\exists y \in \mathbb{R}. y = \sup E$ . Since  $t \in E$ ,  $0 < t < y$ , so  $y > 0$ . We claim that  $y^n < x$  and  $y^n > x$  are both impossible (proof is exercise), so  $y^n = x$ .

**Definition:** Let  $n \geq 1$ ; for  $x \in \mathbb{R}, x > 0$ , we write  $x^{\frac{1}{n}} = y$  where  $y^n = x$ . We set  $0^{\frac{1}{n}} = 0$ .

### 1.6.1 Absolute Value

For  $x \in \mathbb{R}$ , we define the function  $|\cdot| : \mathbb{R} \rightarrow \{r \in \mathbb{R} \mid r \geq 0\}$ :

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

**Proposition** (Properties of  $|\cdot|$ ):

1.  $\forall x \in \mathbb{R}. |x| \geq 0$  and  $|x| = 0 \iff x = 0$
2.  $\forall x, y \in \mathbb{R}. |x| < y \iff -y < x < y$
3.  $\forall x, y \in \mathbb{R}. |xy| = |x||y|$
4.  $\forall x, y \in \mathbb{R}. |x + y| \leq |x| + |y|$  (Triangle Inequality)
5.  $\forall x, y \in \mathbb{R}. ||x| - |y|| \leq |x - y|$

## 2 Sequences

Let  $E$  be a set. Then we may define a sequence  $\{a_n\}_{n=l}^{\infty} \subseteq E$  as the set of values  $a_n \equiv a(n)$  for some  $l \in \mathbb{Z}$  and some function  $a : \{n \in \mathbb{Z} \mid n \geq l\} \rightarrow E$ .

### 2.1 Convergence and Bounds

**Definition:** We say a sequence  $\{a_n\}_{n=l}^{\infty} \subseteq \mathbb{R}$  converges to  $a \in \mathbb{R}$ , i.e.  $a_n \rightarrow a$  as  $n \rightarrow \infty$  or  $\lim_{n \rightarrow \infty} a_n = a$ , if for every  $0 < \epsilon \in \mathbb{R}$ , there exists  $N \in \{m \in \mathbb{Z} \mid m \geq l\}$  such that  $n \geq N \implies |a_n - a| < \epsilon$ .

**Definition:** We say a sequence  $\{a_n\}_{n=l}^{\infty} \subseteq \mathbb{R}$  is bounded iff.  $\exists M \in \mathbb{R}, M > 0. |a_n| < M (\forall n \geq l)$ .

**Lemma:** If a sequence converges, then it is bounded.

**Definition:** Given  $\{a_n\}, \{b_n\} \subseteq \mathbb{R}$  we define  $\{a_n + b_n\} \subseteq \mathbb{R}$  to be the sequence whose elements are  $a_n + b_n$ . We similarly define  $\{ca_n\}$  for a fixed  $c \in \mathbb{R}$ ,  $\{a_nb_n\}$ , and  $\{a_n/b_n\}$  where  $b_n \neq 0, n \geq l$ .

**Theorem** (algebra of convergence): Let  $\{a_n\}, \{b_n\} \subseteq \mathbb{R}, c \in \mathbb{R}$ , and assume that  $a_n \rightarrow a, b_n \rightarrow b$  as  $n \rightarrow \infty$ . Then the following hold:

1.  $a_n + b_n \rightarrow a + b$  as  $n \rightarrow \infty$
2.  $ca_n \rightarrow ca$  as  $n \rightarrow \infty$
3.  $a_nb_n \rightarrow ab$  as  $n \rightarrow \infty$
4. If  $b_n \neq 0$  and  $b \neq 0$ , then  $a_n/b_n \rightarrow a/b$  as  $n \rightarrow \infty$ .

*Proof:* (1), (2) are in next week's HW.

(3): Note that  $|a_nb_n - ab| = |a_nb_n - ab_n + ab_n - ab| \leq |a_nb_n - ab_n| + |ab_n - ab| = |b_n||a_n - a| + |a||b_n - b|$ . Since  $b_n \rightarrow b$  we know that  $\exists M > 0. |b_n| < M (\forall n \geq l)$ .

Let  $\epsilon > 0$ . Since  $a_n \rightarrow a$  and  $b_n \rightarrow b$  we may choose  $N_1$  such that  $n \geq N_1 \implies |a_n - a| < \frac{\epsilon}{2M}$ ; and  $N_2$  where  $n \geq N_2 \implies |b_n - b| < \frac{\epsilon}{2(1+|a|)}$ .

Then set  $N = \max(N_1, N_2)$ . So if  $n \geq N$  we know that  $|a_n b_n - ab| \leq |b_n| |a_n - a| + |a| |b_n - b| < M |a_n - a| + |a| |b_n - b| < M \cdot \frac{\epsilon}{2M} + |a| \cdot \frac{\epsilon}{2(1+|a|)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ .

Since  $\epsilon$  was arbitrary, we deduce that  $a_n b_n \rightarrow ab$ .

(4): We know  $|\frac{a_n}{b_n} - \frac{a}{b}| = |\frac{a_n b - ab_n}{b_n b}| = |\frac{a_n b - ab + ab - ab_n}{b_n b}| \leq \frac{|a_n b - ab|}{|b_n| |b|} + \frac{|ab - ab_n|}{|b| |b_n|} = \frac{|a_n - a|}{|b_n|} + \frac{|a|}{|b| |b_n|} |b_n - b|$ .

Let  $\epsilon > 0$ . Since  $b_n \rightarrow b \neq 0$  we know that  $\exists N_1$  such that  $n \geq N_1 \implies |b_n - b| < \frac{|b|}{2}$ . Then  $n \geq N \implies 0 < |b| = |b - b_n + b_n| \leq |b - b_n| + |b_n| < \frac{|b|}{2} + |b_n| \implies 0 < \frac{|b|}{2} \leq |b_n| \implies 0 < \frac{1}{|b_n|} < \frac{2}{|b|}$ .

Similarly,  $a_n \rightarrow a \implies \exists N_2$ . ( $n \geq N_2 \implies |a_n - a| < \frac{\epsilon}{4} |b|$ ; and

$b_n \rightarrow b \implies \exists N_3$ . ( $n \geq N_3 \implies |b_n - b| < \frac{\epsilon |b|^2}{4(1+|a|)}$ ).

Set  $N = \max(N_1, N_2, N_3)$ . Then  $n \geq N \implies |\frac{a_n}{b_n} - \frac{a}{b}| \leq \frac{|a_n - a|}{|b_n|} + \frac{|a|}{|b_n| |b|} |b_n - b| < \frac{2}{|b| |a_n - a|} + \frac{2|a|}{|b|^2} |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} \frac{|a|}{1+|a|} < \epsilon$ .

Since  $\epsilon > 0$  was arbitrary, we deduce  $\frac{a_n}{b_n} \rightarrow \frac{a}{b}$  as  $n \rightarrow \infty$ .

**Lemma:** Let  $\{a_n\}_{n=l}^{\infty}$  converge to  $a \in \mathbb{R}$ . Then  $\forall \epsilon > 0$ .  $\exists N$ .  $m, n \geq N \implies |a_n - a_m| < \epsilon$ .

**Definition:** We say  $\{a_n\}_{n=l}^{\infty} \subseteq \mathbb{R}$  is *Cauchy* iff  $\forall \epsilon > 0$ .  $\exists N$ .  $m, n \geq N \implies |a_n - a_m| < \epsilon$ .

**Lemma:** If  $\{a_n\}$  is Cauchy, then it's bounded.

*Proof:* Let  $\epsilon = 1$ . Then  $\exists N$ .  $m, n \geq N \implies |a_m - a_n| < 1$ . Then  $n \geq N \implies |a_n - a_N| < 1 \implies |a_n| < |a_n - a_N| + |a_N| < 1 + |a_N|$ . Set  $M = \max(1 + |a_N|, k)$ , where  $k = \max\{|a_l|, \dots, |a_{N-1}|\}$ . Then  $|a_n| < M (\forall n \geq l)$ , and  $\{a_n\}$  is bounded.

**Theorem:** Let  $\{a_n\} \subseteq \mathbb{R}$ . Then  $\{a_n\}$  converges  $\iff \{a_n\}$  is Cauchy.

*Proof:*  $\implies$  is covered by 2nd-previous lemma. We show the converse:

Suppose  $\{a_n\}$  is Cauchy. Then  $|a_n| < M (\forall n \geq l)$  by the last lemma.

Set  $E = \{x \in \mathbb{R} \mid \exists N. n \geq N \implies x < a_n\}$ . Note that  $-M < a_n (\forall n \geq l)$ , and so  $-M \in E$  and  $E \neq \emptyset$ .

Also,  $x \in E \implies \exists N_x$ .  $n \geq N_x \implies x < a_n < M$ , and so  $M$  is an upper bound of  $E$ .

By the least upper bound property of  $\mathbb{R}$ ,  $\exists a = \sup E \in \mathbb{R}$ . We claim that  $a_n \rightarrow a$  as  $n \rightarrow \infty$ .

Let  $\epsilon > 0$ . Then since  $\{a_n\}$  is Cauchy,  $\exists N$ .  $m, n \geq N \implies |a_n - a_m| < \frac{\epsilon}{2}$ . In particular,  $|a_n - a_N| < \frac{\epsilon}{2}$  when  $n \geq N$ . Then  $n \geq N \implies a_N - \frac{\epsilon}{2} < a_n \implies a_N - \frac{\epsilon}{2} \in E \implies a_N - \frac{\epsilon}{2} \leq a$ .

If  $x \in E$ , then  $\exists E_x$ . ( $n \geq N_x \implies x < a_n < a_N + \frac{\epsilon}{2}$ ). Hence  $a_N + \frac{\epsilon}{2}$  is an upper bound of  $E \implies a \leq a_N + \frac{\epsilon}{2}$ . Then  $|a - a_N| < \frac{\epsilon}{2}$ .

But if  $n \geq N$ , then  $|a_n - a| \leq |a_n - a_N| + |a_N - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ . Hence  $a_n \rightarrow a$ .

### 2.1.1 Squeeze Lemma

**Lemma:** Let  $\{a_n\}_{n=l}^{\infty}, \{b_n\}, \{c_n\} \subseteq \mathbb{R}$  and suppose that  $a_n \rightarrow a, c_n \rightarrow a$  as  $n \rightarrow \infty$ . If  $\exists k \geq l$  such that  $a_n \leq b_n \leq c_n (\forall n \geq k)$ , then  $b_n \rightarrow a$  as  $n \rightarrow \infty$ .

Examples:



1. Suppose  $a_n \rightarrow 0$  and  $\{b_n\}$  is bounded, i.e.  $|b_n| \leq M (\forall n \geq l)$ . Then  $|a_n b_n| = |a_n| |b_n| \leq |a_n| M$ . But  $c_n \rightarrow 0 \iff |c_n| \rightarrow 0$ . Then  $0 \leq |a_n b_n| \leq |a_n| M$ , both sides of which go to 0; and by the squeeze lemma,  $|a_n b_n| \rightarrow 0 \implies a_n b_n \rightarrow 0$ .
2. Fix  $k \in \mathbb{N}$  with  $k \geq 1$ . Set  $a_n = \frac{1}{n^k}, n \geq 1$ . Then  $0 \leq \frac{1}{n^k} \leq \frac{1}{n}$ , and by squeeze lemma  $\frac{1}{n^k} \rightarrow 0$ .
3. Fix  $k \in \mathbb{N}$  with  $k \geq 2$ . Let  $a_n = \frac{1}{k^n}, n \geq 0$ . We know  $\forall n \in \mathbb{N}. n \leq k^n$  (proof by induction). Then  $0 \leq \frac{1}{k^n} \leq \frac{1}{n}$ , and by squeeze  $\frac{1}{k^n} \rightarrow 0$ .

## 2.2 Monotonicity and limsup, liminf

**Definition:** Let  $\{a_n\}_{n=l}^\infty \subseteq \mathbb{R}$ . We say  $\{a_n\}$  is:

1. increasing iff.  $a_n < a_{n+1} (\forall n \geq l)$ ,
2. non-decreasing iff.  $a_n \leq a_{n+1} (\forall n \geq l)$ ,
3. decreasing iff.  $a_{n+1} < a_n (\forall n \geq l)$ ,
4. non-increasing iff.  $a_{n+1} \leq a_n (\forall n \geq l)$ .

We say  $\{a_n\}$  is *monotone* iff. it is either non-increasing or non-decreasing.

*Remark:* increasing  $\implies$  non-decreasing, decreasing  $\implies$  non-increasing.

**Theorem:** Suppose that  $\{a_n\}_{n=l}^\infty \subseteq \mathbb{R}$  is monotone. Then  $\{a_n\}$  is bounded iff  $\{a_n\}$  is convergent.

*Proof:*  $\Leftarrow$  is done in a previous lemma.

$\implies$  : We'll prove when the sequence is non-decreasing (other case handled by similar argument).

Set  $E = \{a_n \mid n \geq l\} \subseteq \mathbb{R}$ . Clearly  $E \neq \emptyset$ . Also, since  $\{a_n\}$  is bounded,  $E$  is as well (in particular above). By least upper bound property of  $\mathbb{R}$ ,  $\exists a = \sup(E) \in \mathbb{R}$ . We claim that  $a = \lim_{n \rightarrow \infty} a_n$ .

Let  $\epsilon > 0$ . Since  $a = \sup(E)$  we know that  $a - \epsilon$  is not an upper bound of  $E$ ; hence  $\exists (N \geq l). a - \epsilon < a_N$ . Also, since the sequence is non-decreasing,  $a_n \leq a_{n+1} (\forall n \geq l)$ , and so  $n \geq N \implies a_N \leq a_n$ . Then  $n \geq N \implies a - \epsilon < a_N \leq a_n \leq a$  because  $a$  is an upper bound of  $E$ .

So  $n \geq N \implies -\epsilon < a_n - a \leq 0 \implies |a_n - a| < \epsilon$ . Since  $\epsilon > 0$  was arbitrary, we deduce that  $a_n \rightarrow a$  as  $n \rightarrow \infty$ .

**Lemma:** Suppose that  $\{a_n\}$  is bounded. Set  $S_m = \sup\{a_n \mid n \geq m\}$  and  $I_m = \inf\{a_n \mid n \geq m\}$ . Then  $S_m, I_m \in \mathbb{R}$  are well-defined  $\forall m \geq l$ ;  $\{S_m\}$  is non-increasing; and  $\{I_m\}$  is non-decreasing. Both sequences are bounded.

**Definition:** Suppose  $\{a_n\} \subseteq \mathbb{R}$  is bounded. We set  $\limsup_{n \rightarrow \infty} a_n = \lim_{m \rightarrow \infty} S_m \in \mathbb{R}$  and  $\liminf_{n \rightarrow \infty} a_n = \lim_{m \rightarrow \infty} I_m \in \mathbb{R}$ . Both limits exist by the lemma and previous theorem. We know that  $\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$  from HW.

## 2.3 Subsequences

**Definition:** Let  $\phi : \{n \in \mathbb{Z} \mid n \geq l\} \rightarrow \{n \in \mathbb{Z} \mid n \geq l\}$  be order preserving (increasing), i.e.  $m < n$  then  $\phi(m) < \phi(n)$ . Let  $\{a_n\}_{l=k}^\infty \subseteq \mathbb{R}$  be a sequence. We say  $\{a_{\phi(k)}\}_{k=l}^\infty$  is a *subsequence* of  $\{a_n\}$ .

*Remarks:*

1.  $\phi(k) = k$  is order preserving, so every sequence is a subsequence of itself.
2. Not every  $a_n$  has to be in the subsequence  $\{a_{\phi(k)}\}$ .  
For example, if  $l = 0$  then  $\phi(k) = 2k$  is order preserving. In this case  $a_n, n$  odd does not appear in the subsequence  $\{a_{\phi(k)}\}$ .
3. We will often write  $n_k = \phi(k)$  to simplify notation, so  $\{a_{n_k}\}$  denotes a subsequence.
4. From HW1, we know  $k \leq \phi(k)$  ( $\forall k \geq l$ ).

**Proposition:** Suppose  $\{a_n\}$  satisfies  $a_n \rightarrow a \in \mathbb{R}$  as  $n \rightarrow \infty$ . Then any subsequence of  $\{a_n\}$  also converges to  $a$ .

*Proof:*

Let  $\{a_{\phi(k)}\}$  be a subsequence of  $\{a_n\}$ . Let  $\epsilon > 0$ . Since  $a_n \rightarrow a$  as  $n \rightarrow \infty$ , we know  $\exists N \geq l. n \geq N \implies |a_n - a| < \epsilon$ . We claim  $\exists K \geq l. k \geq K \implies \phi(k) \geq N$ .

If not, then  $\phi(k) < N (\forall k \geq l)$ ; but  $k \leq \phi(k) < N (\forall k \geq l)$  is a contradiction. Then the claim is true, and  $k \geq K \implies \phi(k) \geq N \implies |a_{\phi(k)} - a| < \epsilon$ . Since  $\epsilon > 0$  was arbitrary, we deduce  $\{a_{\phi(k)}\} \rightarrow a$  as  $k \rightarrow \infty$ .

*Remark:* Converse fails. Example:  $a_n = (-1)^n$ ;  $a_{2n} = +1 \rightarrow +1$ , but  $a_{2n+1} = -1 \rightarrow -1$ .

### 2.3.1 Limsup Theorem

**Theorem:** Let  $\{a_n\} \subseteq \mathbb{R}$  be bounded. The following hold:

1. Every subsequence of  $\{a_n\}$  is bounded.
2. If  $\{a_{n_k}\}$  is a subsequence, then  $\limsup_{k \rightarrow \infty} a_{n_k} \leq \limsup_{n \rightarrow \infty} a_n$ .
3. If  $\{a_{n_k}\}$  is a subsequence, then  $\liminf_{n \rightarrow \infty} a_n \leq \liminf_{k \rightarrow \infty} a_{n_k}$ .
4. There exists a subsequence  $\{a_{n_k}\}$  such that  $\lim_{k \rightarrow \infty} a_{n_k} = \limsup_{n \rightarrow \infty} a_n$ .
5. There exists a subsequence  $\{a_{n_k}\}$  such that  $\lim_{k \rightarrow \infty} a_{n_k} = \liminf_{n \rightarrow \infty} a_n$  ( $\neq$  (4)).

*Proof:*

1. Trivial.
2. Since  $k \leq \phi(k)$ ,  $\{a_{\phi(n)} \mid n \geq k\} \subseteq \{a_n \mid n \geq k\}$  for every order-preserving  $\phi$ . Hence  $S_k = \sup\{a_{\phi(n)} \mid n \geq k\} \subseteq \sup\{a_n \mid n \geq k\} = T_k$ . But:  
 $\limsup_{n \rightarrow \infty} a_{\phi(n)} = \limsup_{k \rightarrow \infty} \{a_{\phi(n)} \mid n \geq k\} \leq \limsup_{k \rightarrow \infty} \{a_n \mid n \geq k\} = \limsup_{n \rightarrow \infty} a_n$ .
3. Similar to (2); exercise to reader.
4. Too lazy to L<sup>A</sup>T<sub>E</sub>X; exercise to reader.
5. Exercise to reader.

**Theorem:** Suppose  $\{a_n\} \subseteq \mathbb{R}$ ; the following are equivalent:

1.  $a_n \rightarrow a$  as  $n \rightarrow \infty$
2.  $\{a_n\}$  is bounded, and every convergent subsequence converges to  $a$ .
3.  $\{a_n\}$  is bounded, and  $\limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n$ .

*Proof:* (1)  $\implies$  (2) proven already.

(2)  $\implies$  (3)

Limsup theorem (4,5)  $\implies \exists \{a_{\phi(k)}\}, \{a_{\gamma(k)}\}$  subsequences such that  $a_{\phi(k)} \rightarrow \limsup_{n \rightarrow \infty} a_n, a_{\gamma(k)} \rightarrow \liminf_{n \rightarrow \infty} a_n$  as  $k \rightarrow \infty$ . By (2) the limits must agree.

(3)  $\implies$  (1)

Limsup theorem (1-3)  $\implies \forall \{a_{\phi(k)}\}. \liminf_{n \rightarrow \infty} a_n \leq \liminf_{k \rightarrow \infty} a_{\phi(k)} \leq \limsup_{k \rightarrow \infty} a_{\phi(k)} \leq \limsup_{n \rightarrow \infty} a_n$ . As the first and last are equal, by transitivity it follows all subsequences satisfy  $\liminf_{k \rightarrow \infty} a_{\phi(k)} = \limsup_{k \rightarrow \infty} a_{\phi(k)}$ . As  $a_n$  is a subsequence of itself, it therefore converges to some  $a$  as  $n \rightarrow \infty$ .

**Theorem** (Bolzano-Weierstrass): If  $\{a_n\} \subseteq \mathbb{R}$  is bounded then there exists a convergent subsequence. Proof from (4) or (5) of Limsup Theorem.

## 2.4 Special Sequences

**Definition:** Given  $a_n \in \mathbb{R}$  for  $0 \leq k \leq n, n \in \mathbb{N}$  we define  $\sum_{k=0}^n a_k = a_0 + a_1 + \cdots + a_n$ .

**Lemma** (Binomial Theorem): Let  $x, y \in \mathbb{R}$  and  $n \in \mathbb{N}$ . Then  $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$ , where  $\binom{n}{k} := \frac{n!}{k!(n-k)!} \in \mathbb{N}$ .

**Theorem:** In the following assuming that  $n \geq 1$ :

1. Let  $x \in \mathbb{R}, x > 0$ . Then  $a_n = \frac{1}{n^x} \rightarrow 0$  as  $n \rightarrow \infty$ .
2. Let  $x \in \mathbb{R}, x > 0$ . Then  $a_n = x^{1/n} \rightarrow 1$  as  $n \rightarrow \infty$ .
3. Let  $a_n = n^{1/n}$ ; then  $a_n \rightarrow 1$  as  $n \rightarrow \infty$ .
4. Let  $a, x \in \mathbb{R}, x > 0$ . Then  $\frac{n^a}{(1+x)^a} \rightarrow 0$  as  $n \rightarrow \infty$ .
5. Let  $x \in \mathbb{R}, |x| < 1$ . Then  $a_n = x^n \rightarrow 0$  as  $n \rightarrow \infty$ .

## 3 Series

**Definition:** Let  $\{a_n\}_{n=l}^{\infty} \subseteq \mathbb{R}$ ; for  $p < q$  we write  $\sum_{n=p}^q a_n = (a_p + \cdots + a_q)$ .

1. We define, for each  $n \geq l$ ,  $S_n = \sum_{k=l}^n a_k \in \mathbb{R}$  to be the  $n^{\text{th}}$  partial sum of  $\{a_n\}_{n=l}^{\infty}$ .
2. If  $\exists s \in \mathbb{R}. S_n \rightarrow s$  as  $n \rightarrow \infty$ , then  $\sum_{n=l}^{\infty} a_n = s$ . We say the "infinite series"  $\sum_{n=l}^{\infty} a_n$  converges.
3. If the series does not converge, it diverges.

### Examples

1. Let  $a_n = x^n$  for  $n \geq 0, x \in \mathbb{R}$ . Then  $S_n = \sum_{k=0}^n x^k$ . Notice that  $(1-x)S_n = \sum_{k=0}^n x^k - \sum_{k=0}^n x^{k+1} = \sum_{k=0}^n x^k - \sum_{k=1}^{n+1} x^k = 1 - x^{n+1}$ .  
So  $S_n = \sum_{k=0}^n x^k = \left(\frac{1-x^{n+1}}{1-x}\right)$ . If  $|x| < 1$  then  $S_n \rightarrow \frac{1}{1-x}$  by special seq (5).
2. Suppose  $\{b_n\}_{n=0}^{\infty} \subseteq \mathbb{R}$  where  $b_n \rightarrow b$  as  $n \rightarrow \infty$ . Set  $a_n = b_{n+1} - b_n$  for  $n \geq 0$ . Then the series  $\sum_{n=0}^{\infty} a_n$  converges and in fact  $\sum_{n=0}^{\infty} a_n = b - b_0$ .

### 3.1 Convergence Results

We develop tools that will let us deduce the convergence of a series without knowing its value.

**Theorem:** Suppose  $\sum_{n=l}^{\infty} a_n$  converges. Then  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

*Proof:* Notice that  $a_n = S_n - S_{n-1}$  and so  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (S_n - S_{n-1}) = S - S = 0$ .

**Corollary:**  $\sum_{n=0}^{\infty} (-1)^n$  and  $\sum_{n=0}^{\infty} n$  diverge, as neither sequences converge to 0.

**Corollary:** The series  $\sum_{n=0}^{\infty} x^n$  converges  $\iff |x| < 1$ .

*Proof:*  $|x| \geq 1 \implies |x^n| = |x|^n \geq 1 (\forall n \in \mathbb{N})$ . The converse was proved last time.

Next, we provide a characterization of convergence in terms of the size of the “tails” of the series.

**Theorem:**  $\sum_{n=l}^{\infty} a_n$  converges  $\iff \forall \epsilon > 0. \exists N \geq l. m \geq k \geq N \implies |\sum_{n=k}^m a_n| < \epsilon$ .

*Proof:*  $\sum_{n=l}^{\infty} a_n$  converges  $\iff S_k = \sum_{n=l}^k a_n$  converges  $\iff \{S_k\}$  is Cauchy.

This is useful in practice because we can guarantee a series converges without knowing its value.

**Theorem:**

1. If  $\forall n \geq k. |a_n| \leq b_n$  for some  $k \geq l$ , and  $\sum_{n=l}^{\infty} b_n$  converges, then  $\sum_{n=l}^{\infty} a_n$  converges.
2. If  $\forall n \geq k. 0 \leq a_n \leq b_n$  for some  $k \geq l$ , and  $\sum_{n=l}^{\infty} a_n$  diverges, then  $\sum_{n=l}^{\infty} b_n$  diverges.

*Proof:* (1) Let  $\epsilon > 0$  and prove with previous theorem and induction on triangle inequality. (2) follows from contrapositive.

**Examples:**

1.  $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$  converges because  $|\frac{(-1)^n}{2^n}| = \frac{1}{2^n}$  and  $\sum_{n=0}^{\infty} \frac{1}{2^n}$  converges ( $\frac{1}{2} < 1$ ).
2. Suppose  $\sum_{n=0}^{\infty} a_n$  converges and  $a_n \geq 0 \forall n \geq 0$ . Let  $\{b_n\} \subseteq \mathbb{R}$  be bounded, i.e.  $|b_n| \leq M \forall n$ . Then  $|a_n b_n| = |a_n| |b_n| \leq M a_n$ . Then  $M S_n = M \sum_{k=0}^n a_k = \sum_{k=0}^n M a_k$ , so by the theorem,  $\sum_{n=0}^{\infty} a_n b_n$  converges.
3.  $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \cdot \frac{n!}{n^n} \cdot \frac{3n^2}{4n^2+2}$  converges because the product is bounded.

**Theorem:** Suppose  $\forall n \geq l. a_n \geq 0$ . Then  $\sum_{n=l}^{\infty} a_n$  converges  $\iff \{S_n\}_{n=l}^{\infty}$  is bounded.

*Proof:* Since  $a_n \geq 0$ , the sequence  $S_n = \sum_{k=l}^n a_k$  is non-decreasing:  $S_{n+1} = a_{n+1} + S_n \geq S_n$ . Since  $S_n$  is monotone and converges, it is bounded.

#### 3.1.1 Cauchy Criterion Theorem

**Theorem:** Suppose that  $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$  satisfies  $\forall n \geq l. a_n \geq 0$  and  $\forall n \geq 1. a_{n+1} \leq a_n$ . Then  $\sum_{n=1}^{\infty} a_n$  converges  $\iff \sum_{n=0}^{\infty} 2^n a_{2^n}$  converges.

*Proof:*

Let  $S_n = \sum_{k=1}^n a_k$  and  $T_n = \sum_{n=0}^m 2^n a_{2^n}$ . Notice that if  $m \leq 2^k$  then  $S_m = a_1 + a_2 + \dots + a_{2^k} \leq a_1 + (a_2 + a_3) + \dots + (a_{2^k} + \dots + a_{2^{k+1}-1}) \leq a_1 + 2a_2 + \dots + 2^k a_{2^k} = T_k$ .

On the other hand, if  $m \geq 2^k$ ,  $S_m \geq a_1 + \cdots + a_{2^k} = a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}-1} + \cdots + a_{2^k}) \geq \frac{1}{2}a_1 + a_2 + \cdots + 2^{k-1}a_{2^k} = \frac{1}{2}T_k$ .

Now, if  $\sum_{n=0}^{\infty} 2^n a_{2^n}$  converges, then  $T_n \rightarrow T$  as  $n \rightarrow \infty$  and so  $S_m \leq \lim_{n \rightarrow \infty} T_m = T$ , which means  $\{S_m\}$  is bounded and  $\sum_{n=1}^{\infty} a_n$  converges.

Similarly, if  $\sum_{n=1}^{\infty} a_n$  converges, then  $T_k \leq 2 \lim_{n \rightarrow \infty} S_n \implies \{T_k\}$  is bounded  $\implies \sum_{n=0}^{\infty} 2^n a_{2^n}$  converges.

**Theorem:** Let  $p \in \mathbb{R}$ . Then  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges  $\iff p > 1$ .

*Proof:*

If  $p \leq 0$  the result is trivial since  $\frac{1}{n^p} \geq 1$  (the sequence converges to 0). Assume that  $p > 0$ . Then  $\frac{1}{(n+1)^p} \leq \frac{1}{n^p}$ , so we can apply the Cauchy criterion:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges } \iff \sum_{n=0}^{\infty} \frac{2^n}{(2^n)^p} \text{ converges.}$$

But  $\sum_{n=0}^{\infty} \frac{2^n}{(2^n)^p} = \sum_{n=0}^{\infty} \frac{1}{(2^{p-1})^n}$ , and this series converges  $\iff \frac{1}{2^{p-1}} < 1 \iff p > 1$ .

Notice  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent, but  $\sum_{n=1}^{\infty} \frac{1}{n^{1+r}}$  converges  $\forall r > 0$ . To try to find intermediate series, we need the logarithm.

### 3.1.2 Logarithm

**Definition:** From Supplemental Reading 3, for every  $1 < b \in \mathbb{R}$ , we define a function  $\log_b : \{x \in \mathbb{R} \mid x > 0\} \rightarrow \mathbb{R}$  such that

1.  $b^{\log_b x} = x$  ( $\forall x > 0$ )
2.  $\log_b(1) = 0$ ,  $\log_b b = 1$
3.  $0 < x < y \iff \log_b x < \log_b y$
4.  $\log_b(x^z) = z \log_b(x)$  ( $\forall x > 0, \forall z \in \mathbb{R}$ )
5.  $\log_b$  is a bijection
6.  $\lim_{n \rightarrow \infty} \frac{\log_b n}{n^r} = 0$  ( $\forall r \in \mathbb{R}, r > 0$ )

Then from (6), for large  $n$  and  $p > 0$  we know:

$$n \leq n(\log_b n)^p \leq n \cdot n^p = n^{1+p} \implies \frac{1}{n^{1+p}} \leq \frac{1}{n(\log_b n)^p} \leq \frac{1}{n}.$$

So  $\frac{1}{n(\log_b n)^p}$  is such an “intermediate series.”

**Theorem:** Let  $b > 1$ .  $\sum_{n=2}^{\infty} \frac{1}{n(\log_b n)^p}$  converges  $\iff p > 1$ . ( $n \geq 2 \implies \log_b n > 0$ )

*Proof:*

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1}{n(\log_b n)^p} \text{ converges } &\iff \sum_{n=1}^{\infty} \frac{2^n}{2^n(\log_b 2^n)^p} \text{ converges by Cauchy criterion, but} \\ \sum_{n=1}^{\infty} \frac{1}{(\log_b 2)^p n^p} &= \frac{1}{(\log_b 2)^p} \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges } \iff p > 1. \end{aligned}$$

In particular,  $\sum_{n=2}^{\infty} \frac{1}{n \log_b n}$  is divergent.

### 3.2 The number $e$

**Lemma:**  $\sum_{n=0}^{\infty} \frac{1}{n!}$  converges.

*Proof:* If  $n \geq 2$  then:

$$\begin{aligned} S_n &= \sum_{k=0}^n \frac{1}{k!} = 1 + 1 + \frac{1}{2 \cdot 1} + \cdots + \frac{1}{n(n-1) \cdots 2 \cdot 1} \\ &\leq 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 2} + \cdots + \frac{1}{2^{n-1}} \\ &\leq 1 + \sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + 2 = 3 \end{aligned}$$

Since  $S_n$  is increasing and bounded, we know that  $\sum_{n=0}^{\infty} \frac{1}{n!}$  converges.

**Definition:** We set  $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ . Note that  $e > 1$ .

**Theorem:**  $e = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$ .

*Proof:* Let  $S_n = \sum_{k=0}^n \frac{1}{k!}$ ,  $T_n = (1 + \frac{1}{n})^n$ . Then by the Binomial Theorem:

$$\begin{aligned} T_n &= (1 + \frac{1}{n})^n = \sum_{k=0}^n \frac{n!}{k!(n-k)!} \frac{1}{n^k} \\ &= 1 + 1 + \frac{1}{2!} \frac{n(n-1)}{n^2} + \cdots + \frac{1}{n!} \frac{n(n-1) \cdots 1}{n^n} \\ &= 1 + 1 + \frac{1}{2!} (1 - \frac{1}{n}) + \frac{1}{3!} (1 - \frac{1}{n})(1 - \frac{2}{n}) + \cdots + \frac{1}{n!} (1 - \frac{1}{n}) \cdots (1 - \frac{n-1}{n}) \\ &\leq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} = S_n \end{aligned}$$

Hence,  $\limsup_{n \rightarrow \infty} T_n \leq \limsup_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} S_n = e$ .

OTOH, fix  $m \in \mathbb{N}$ . Then for  $n \geq m$ :

$$\begin{aligned} T_n &\geq 1 + 1 + \frac{1}{2!} (1 - \frac{1}{n}) + \cdots + \frac{1}{m!} (1 - \frac{1}{n}) \cdots (1 - \frac{m-1}{n}) \\ \implies \liminf_{n \rightarrow \infty} T_n &\geq \liminf_{n \rightarrow \infty} \text{RHS} \geq 1 + 1 + \frac{1}{2!} \liminf_{n \rightarrow \infty} (1 - \frac{1}{n}) + \cdots + \frac{1}{m!} \liminf_{n \rightarrow \infty} (1 - \frac{1}{n}) \cdots (1 - \frac{m-1}{n}) = 1 + 1 + \cdots \end{aligned}$$

Then, letting  $m \rightarrow \infty$ ,  $e = \lim_{m \rightarrow \infty} S_m \leq \liminf_{n \rightarrow \infty} T_n$ .

Thus,  $e \leq \liminf_{n \rightarrow \infty} T_n \leq \limsup_{n \rightarrow \infty} T_n \leq e \implies \lim_{n \rightarrow \infty} T_n = e$ .

**Theorem:**  $\forall n \geq 1$ .  $0 < e - S_n < \frac{1}{n \cdot n!}$ . Also,  $e \in \mathbb{R} \setminus \mathbb{Q}$  is irrational.

*Proof:* Since  $S_n$  is increasing,  $0 < e - S_n$  is clear. The other side can be seen from algebra.

Now, suppose  $e \in \mathbb{Q}$ ; then  $e = \frac{p}{q}$  for  $p, q \in \mathbb{N}, p, q \geq 1$ .

Then  $0 < q!(e - S_q) < \frac{1}{q}$  ( $\forall q \geq 1$ ). Notice that  $q!e = q!\frac{p}{q} = (q-1)!p \in \mathbb{N}$  and  $q!(1 + \frac{1}{2!} + \cdots + \frac{1}{q!}) \in \mathbb{N}$ .

Hence  $q!(e - S_q) \in \mathbb{Z}$ ; but this yields an integer between 0 and 1, a contradiction. So  $e$  is irrational.

*Remark:* In fact,  $e$  is transcendental.

### 3.3 More Convergence Results

**Theorem (Root Test):** Suppose  $\{a_n\}_{n=l}^{\infty} \subseteq \mathbb{R}$  and  $\{|a_n|^{1/n}\}$  is bounded. Let  $0 \leq \alpha = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$ . Then the following holds:

1. If  $\alpha < 1$ , then  $\sum_{n=l}^{\infty} a_n$  converges.
2. If  $\alpha > 1$ , then  $\sum_{n=l}^{\infty} a_n$  diverges.
3. if  $\alpha = 1$ , both convergence and divergence are possible.

**Theorem (Ratio Test):** Let  $\{a_n\}_{n=l}^{\infty} \subseteq \mathbb{R}$ . Then  $\sum_{n=l}^{\infty} a_n$ :

1. converges if  $\{|\frac{a_{n+1}}{a_n}|\}_{n=l}^{\infty}$  is bounded and  $\limsup_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} < 1$ .
2. diverges if  $\exists k \geq l$ .  $|a_k| \neq 0$  and  $|a_{n+1}| \geq |a_n| (\forall n \geq k)$ .

**Lemma (Summation of Parts):** Let  $\{a_n\}_{n=0}^{\infty} \subseteq \mathbb{R}$  and define:

$$A_n = \begin{cases} \sum_{k=0}^n a_k & \text{if } n \geq 0 \\ 0 & \text{if } n = -1 \end{cases}$$

Then if  $0 \leq p < q$ :

$$\sum_{n=p}^q a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

**Theorem (Dirichlet Test):** Suppose  $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty} \subseteq \mathbb{R}$  satisfy:

1. The sequence  $A_n = \sum_{k=0}^n a_k$  is bounded.
2.  $0 \leq b_{n+1} \leq b_n (\forall n \in \mathbb{N})$
3.  $\lim_{n \rightarrow \infty} b_n = 0$

Then  $\sum_{n=0}^{\infty} a_n b_n$  converges.

**Corollary (Alternating Series):** Suppose  $0 \leq a_{n+1} \leq a_n, a_n \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $\sum_{n=l}^{\infty} (-1)^n a_n$  converges. Proof follows from Dirichlet Test.

**Corollary (Abel's Test):** Suppose  $\sum_{n=l}^{\infty} a_n$  converges,  $b_{n+1} \leq b_n (\forall n \geq l)$  and  $b_n \rightarrow b$  as  $n \rightarrow \infty$ . Then  $\sum_{n=l}^{\infty} a_n b_n$  converges.

### 3.4 Algebra of Series

**Theorem:** If  $A = \sum_{n=l}^{\infty} a_n, B = \sum_{n=l}^{\infty} b_n$ , then

$$(1) A + B = \sum_{n=l}^{\infty} (a_n + b_n) \qquad (2) cA = \sum_{n=l}^{\infty} ca_n \quad (\forall c \in \mathbb{R})$$

**Theorem:** Suppose  $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty} \in \mathbb{R}$  satisfy:

$$(1) \sum_{n=0}^{\infty} |a_n| \text{ converges} \qquad (2) \sum_{n=0}^{\infty} b_n = B \qquad (3) c_n = \sum_{k=0}^n a_k b_{n-k} \text{ for } n \geq 0$$

Then  $\sum_{n=0}^{\infty} c_n = A \cdot B$  converges.

**Definition:** The series  $\sum_{n=0}^{\infty} c_n$ , where  $c_n = \sum_{k=0}^n a_k b_{n-k}$ , is called the *Cauchy product* of the series  $\sum_{n=0}^{\infty} a_n$ ,  $\sum_{n=0}^{\infty} b_n$ .

*Remark:* If  $\sum a_n$ ,  $\sum b_n$  converge,  $\sum c_n$  does not necessarily converge if neither series has convergent absolute values.

### 3.5 Absolute Convergence and Rearrangements

**Proposition:** If  $\sum_{n=l}^{\infty} |a_n|$  converges, then  $\sum_{n=l}^{\infty} a_n$  converges. Proof is trivial.

**Definition:** Suppose  $\sum_{n=l}^{\infty} a_n$  converges. If  $\sum_{n=l}^{\infty} |a_n|$  converges, the series converges *absolutely*. If  $\sum |a_n|$  diverges, the series is *conditionally convergent*.

*Example:*  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  is conditionally convergent, while  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$  is absolutely convergent.

Let's try to manipulate the series without being careful.

$$\begin{aligned} \gamma &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \\ &= \lim_{k \rightarrow \infty} (S_k = \sum_{n=0}^k \frac{(-1)^{n+1}}{n}) = \lim_{k \rightarrow \infty} (S_{2k} = \sum_{n=0}^{2k} \frac{(-1)^{n+1}}{n}) \\ \text{but: } S_{2k} &= (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4} + \cdots + (\frac{1}{2k-1} - \frac{1}{2k})) > 0 \end{aligned}$$

Hence,  $\gamma > 0$ . But the next step is questionable:

$$\begin{aligned} 2\gamma &= \sum_{n=1}^{\infty} \frac{(2)(-1)^{n+1}}{n} \stackrel{?}{=} \sum_{k=0}^{\infty} \frac{2}{2k+1} - \sum_{k=1}^{\infty} \frac{2}{2k} \\ &\stackrel{?}{=} \sum_{k=0}^{\infty} \frac{2}{2k+1} - \sum_{k=1}^{\infty} \frac{1}{k} = \sum_{k=0}^{\infty} \frac{1}{2k+1} - \sum_{k=1}^{\infty} \frac{1}{2k} = \gamma \\ &\implies 2\gamma = \gamma \wedge \gamma > 0 \quad \text{a contradiction!} \end{aligned}$$

Problem: rearrangement is a delicate issue.

**Definition:** Let  $\gamma : \{m \in \mathbb{Z} \mid m \geq l\} \rightarrow \{m \in \mathbb{Z} \mid m \geq l\}$  be a bijection. The series  $\sum_{n=l}^{\infty} a_{\gamma(n)}$  is called a rearrangement of  $\sum_{n=l}^{\infty} a_n$ .

**Theorem:** If  $\sum_{n=l}^{\infty} a_n$  is absolutely convergent, then every rearrangement converges to  $\sum_{n=l}^{\infty} a_n$ .

*Proof:* Let  $\epsilon > 0$ .

Since  $\sum_{n=l}^{\infty} a_n$  converges absolutely,  $\exists N \geq l. k \geq m \geq N \implies \sum_{n=m}^k |a_n| < \frac{\epsilon}{2}$ .

Let  $k \rightarrow \infty : \sum_{n=m}^{\infty} |a_n| \leq \frac{\epsilon}{2} < \epsilon$ .

Now choose  $M \geq N$  such that  $\{l, l+1, \dots, N\} \subseteq \{\gamma(l), \gamma(l+1), \dots, \gamma(M)\}$ . Then  $m \geq M \implies |\sum_{n=l}^m a_n - \sum_{n=l}^m a_{\gamma(n)}| \leq \sum_{n=N}^{\infty} |a_n| < \epsilon$ .

Hence  $\lim_{m \rightarrow \infty} (\sum_{n=l}^m a_n - \sum_{n=l}^m a_{\gamma(n)}) = 0$  and from this we deduce  $\lim_{m \rightarrow \infty} \sum_{n=l}^m a_{\gamma(n)} = \lim_{m \rightarrow \infty} \sum_{n=l}^m a_n = \sum_{n=l}^{\infty} a_n$ .



When a series is only conditionally convergent, the situation is vastly worse.

**Theorem:** Suppose  $\sum_{n=0}^{\infty} a_n$  is conditionally convergent. Let  $c \in \mathbb{R}$ .

There exists a rearrangement (bijection)  $\gamma : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\sum_{n=0}^{\infty} a_{\gamma(n)} = c$ .

**Lemma:** Suppose  $\sum_{n=0}^{\infty} a_n$  is conditionally convergent and set:

$$b_n = \begin{cases} a_n & \text{if } a_n > 0 \\ 0 & \text{if } a_n \leq 0 \end{cases} \quad c_n = \begin{cases} -a_n & \text{if } a_n < 0 \\ 0 & \text{if } a_n \geq 0 \end{cases}$$

Then  $\sum_{n=0}^{\infty} b_n$  and  $\sum_{n=0}^{\infty} c_n$  both diverge.

*Proof:* Suppose not; one of the series is convergent. If  $\sum b_n$  converges, then  $c_n = b_n - a_n \implies \sum c_n = \sum b_n - \sum a_n$ ; but  $|a_n| = b_n + c_n$  and so  $\sum |a_n| = \sum b_n + \sum c_n$  is convergent, a contradiction. A similar argument holds if  $\sum c_n$  converges.

**Rearrangement Theorem Proof:**

Let  $\{a_n^+\}_{n=0}^{\infty}$  denote the subsequence of  $\{b_n \mid b_n > 0 \text{ or } b_n = 0 \wedge a_n = 0\}$ . Let  $\{a_n^-\}_{n=0}^{\infty}$  denote the subsequence of  $\{c_n \mid c_n > 0\}$  (from last lemma). Note:

1.  $a_n^+ \rightarrow 0, a_n^- \rightarrow 0$  since  $a_n \rightarrow 0 \implies b_n \rightarrow 0, c_n \rightarrow 0$ .
2.  $\sum a_n^+$  and  $\sum a_n^-$  both diverge because they differ by 0 from  $\sum b_n, \sum c_n$  respectively.

Set  $m_0 = n_0 = -1$ . Since  $\sum a_n^+$  diverges we may use the well-ordering principle:  $\exists m_1 = \min\{k \in \mathbb{N} \mid \sum_{n=0}^k a_n^+ > c\}$ . Similarly,  $\exists n_1 = \min\{k \in \mathbb{N} \mid \sum_{n=0}^{m_1} a_n^+ - \sum_{n=0}^k a_n^- < c\}$ .

Next, if  $m_p$  and  $n_p$  are known, we set:

$$m_{p+1} = \min \left\{ k \in \mathbb{N} \mid \sum_{l=0}^{p-1} \sum_{j=1+m_l}^{m_l} a_j^+ - \sum_{l=0}^{p-1} \sum_{j=1+n_l}^{n_l} a_j^- + \sum_{j=1+m_p}^k a_j^+ > c \right\}$$

$$n_{p+1} = \min \left\{ k \in \mathbb{N} \mid \sum_{l=0}^{p-1} \sum_{j=1+m_l}^{m_l} a_j^+ - \sum_{l=0}^{p-1} \sum_{j=1+n_l}^{n_l} a_j^- + \sum_{j=1+m_p}^{m_{p+1}} a_j^+ - \sum_{j=1+n_p}^k a_j^- < c \right\}$$

Consider the series  $(a_1^+ + \cdots + a_{m_1}^+) - (a_1^- + \cdots + a_{n_1}^-) + (a_{1+m_1}^+ + \cdots + a_{m_2}^+) - (a_{1+n_1}^- + \cdots + a_{n_2}^-) + \cdots$ . This is clearly a rearrangement of  $\sum_{n=0}^{\infty} a_n$ .

Write  $A_p = \sum_{l=1+m_p}^{m_{p+1}} a_l^+, A_p^- = \sum_{l=1+n_p}^{n_{p+1}} a_l^-$ , and let  $S_j$  denote the  $j^{\text{th}}$  partial sum of the rearrangement.

By construction,  $\limsup_{j \rightarrow \infty} S_j = \limsup_{p \rightarrow \infty} (\sum_{l=0}^{p+1} A_l^+ - \sum_{l=0}^p A_l^-)$  and  $\liminf_{j \rightarrow \infty} S_j = \liminf_{p \rightarrow \infty} (\sum_{l=0}^p A_l^+ + \sum_{l=0}^p A_l^-)$ .

Also,  $c < \sum_{l=0}^{p+1} A_l^+ - \sum_{l=0}^p A_l^- < c + a_{m_{p+1}}^+$  and  $c - a_{n_{p+1}}^- < \sum_{l=0}^{p+1} A_l^+ - \sum_{l=0}^{p+1} A_l^- < c$ .

Thus, by the squeeze lemma,  $\lim_{p \rightarrow \infty} (\sum_{l=0}^{p+1} A_l^+ - \sum_{l=0}^p A_l^-) = \lim_{p \rightarrow \infty} (\sum_{l=0}^p A_l^+ - \sum_{l=0}^p A_l^-) = c$ , and so  $\lim_{j \rightarrow \infty} S_j = c \implies \sum_{n=0}^{\infty} a_{\gamma(n)} = c$ .

*Remark:* One can also rearrange such that  $\sum a_{\gamma(n)} = \pm\infty$ .

## 4 Topology of $\mathbb{R}$

Our goal in Section 4 is to develop some tools for understanding the “topology” of  $\mathbb{R}$ , which is a sort of generalized qualitative geometry.

### 4.1 Open and Closed Sets

#### 4.1.1 Open Sets

**Definition:**

1. For  $a, b \in \mathbb{R}$  with  $a \leq b$ , we define:

$$\begin{aligned} (a, b) &= \{x \in \mathbb{R} \mid a < x < b\} & [a, b) &= \{x \in \mathbb{R} \mid a \leq x < b\} \\ (a, b] &= \{x \in \mathbb{R} \mid a < x \leq b\} & [a, b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\} \end{aligned}$$

2. For  $x \in \mathbb{R}$  and  $\epsilon > 0$ , we set  $B(x, \epsilon) = (x - \epsilon, x + \epsilon)$  and  $B[x, \epsilon] = [x - \epsilon, x + \epsilon]$ . We call the set  $B(x, \epsilon)$  a *neighborhood* of  $x$  or a “ball of radius  $\epsilon$  centered at  $x$ ”.
3. A set  $E \subseteq \mathbb{R}$  is *open* if  $\forall x \in E. \exists \epsilon > 0. B(x, \epsilon) \subseteq E$ .  
In other words, every point in  $E$  has a neighborhood contained in  $E$ .

*Examples:*

1.  $\emptyset$  is vacuously open.
2.  $\mathbb{R}$  is open because  $\forall x \in \mathbb{R}. B(x, 1) \subseteq \mathbb{R}$ .
3. If  $a < b$  then  $(a, b)$  is open.  
*Proof:* Fix  $x \in (a, b)$  and let  $\epsilon = \min\{x - a, b - x\} > 0$ . Then  $a \leq x - \epsilon < x < x + \epsilon \leq b$  by construction, and  $B(x, \epsilon) \subseteq (a, b)$ .
4. If  $a < b$  then  $[a, b)$  is not open.  
*Proof:* For  $x = a$  we know that  $\forall \epsilon > 0. a - \epsilon \notin [a, b)$  and hence  $B(a, \epsilon) \not\subseteq [a, b)$ .
5.  $[a, b]$  is not open, nor is  $(a, b]$  by previous argument.
6.  $E = \{a\}$  is not open.
7.  $E = \{\frac{1}{n} \mid n \in \mathbb{N}, n \geq 1\}$  is not open:  $\forall \epsilon > 0. B(1, \epsilon) \not\subseteq E$ .

**Lemma:** If  $E_\alpha \subseteq \mathbb{R}$  is open  $\forall \alpha \in A$  (some index set), then  $\bigcup_{\alpha \in A} E_\alpha$  is open.

*Proof:* Let  $x \in \bigcup_{\alpha \in A} E_\alpha$ . Then  $x \in E_{\alpha_0}$  for some  $\alpha_0 \in A$ . Since  $E_{\alpha_0}$  is open,  $\exists \epsilon > 0. B(x, \epsilon) \subseteq E_{\alpha_0} \subseteq \bigcup_{\alpha \in A} E_\alpha$ .

**Lemma:** If  $E_i \subseteq \mathbb{R}$  is open for  $i \in [n], n \in \mathbb{N}$ , then  $\bigcap_{i=1}^n E_i$  is open.

*Remark:* Infinite intersections of open sets need not be open. Let  $E_n = (\frac{-1}{n}, \frac{1}{n}), n \geq 1$ . Then  $\bigcap_{n=1}^\infty E_n = \{0\}$  which is closed.

#### 4.1.2 Closed Sets

**Definition:** We say  $E \subseteq \mathbb{R}$  is *closed* iff  $E^c = \mathbb{R} \setminus E$  is open.

**Lemma:**  $E$  is open  $\iff E^c$  is closed (by definition).

*Examples:*

1.  $\emptyset$  is closed because  $\emptyset^c = \mathbb{R}$  is open.
2.  $\mathbb{R}$  is closed because  $\mathbb{R}^c = \emptyset$  is open.
3.  $[a, b]$  is closed because  $[a, b]^c = (-\infty, a) \cup (b, \infty)$  is the union of open sets, and thus open.
4.  $[a, b)$  and  $(a, b]$  are not closed because  $[a, b)^c = (-\infty, a) \cup [b, \infty)$  and  $B(b, \epsilon) \not\subseteq [a, b)^c$  ( $\forall \epsilon > 0$ ).
5.  $\{a\}$  is closed since  $\{a\}^c = (-\infty, a) \cup (a, \infty)$ , both open sets.
6. Suppose  $E \subseteq \mathbb{R}$  is finite. Write  $E = \{a_i \mid i \in [n]\}$  where  $a_1 < a_2 < \dots < a_n$ . Then  $E^c = (-\infty, a_1) \cup (a_1, a_2) \cup \dots \cup (a_{n-1}, a_n) \cup (a_n, \infty)$ , all of which are open.
7.  $E = \{\frac{1}{n} \mid n \in \mathbb{N}, n \geq 1\}$  is not closed.  $E^c = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n}) \cup (1, \infty)$  is not open because  $B(0, \epsilon) \cap E = \{\frac{1}{n} \mid \frac{1}{\epsilon} < n\} \neq \emptyset \implies B(0, \epsilon) \not\subseteq E^c$ .
8.  $E = \{0\} \cup \{\frac{1}{n} \mid n \geq 1\}$  is closed, as  $E^c = (-\infty, 0) \cup \bigcup_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n}) \cup (1, \infty)$  is open.

**Lemma:**

1. If  $E_\alpha \subseteq \mathbb{R}$  is closed  $\forall \alpha \in A$ , then  $\bigcap_{\alpha \in A} E_\alpha$  is closed.
2. If  $E_i \subseteq \mathbb{R}$  is closed  $\forall i \in [n]$  then  $\bigcup_{i=1}^n E_i$  is closed.

*Proof:* The complement is the union of  $E_\alpha^c$  (open by claim), which is open by previous lemma.

*Remark:* Example (7) shows that infinite unions of closed sets need not be closed.

### 4.1.3 Limit Points

**Definition:** Let  $E \subseteq \mathbb{R}$ .

1. A point  $x \in \mathbb{R}$  is a *limit point* of  $E$  iff.  $\forall \epsilon > 0$ .  $(B(x, \epsilon) \cap E) \setminus \{x\} \neq \emptyset$ .
2. A point  $x \in E$  is called *isolated* if it is not a limit point.

*Example:*  $E = \{\frac{1}{n} \mid n \geq 1\}$ . 0 is a limit point, but  $\frac{1}{n} \in E$  is isolated, since  $B(\frac{1}{n}, \frac{1}{n(n+1)}) \cap E = \{\frac{1}{n}\}$ .

**Theorem:** Let  $E \subseteq \mathbb{R}$ .  $E$  is closed  $\iff$  every limit point of  $E$  is contained in  $E$ .

*Proof:*

$\implies$  :

Assume  $E$  is closed and  $x \in \mathbb{R}$  is a limit point of  $E$ . If  $x \in E^c$  then, since  $E^c$  is open,  $\exists \epsilon > 0$ .  $B(x, \epsilon) \subseteq E^c \implies B(x, \epsilon) \cap E = \emptyset$ . But this contradicts the fact that  $x$  is a limit point of  $E$ ; thus  $x \in E$ .

$\impliedby$  :

Suppose  $E$  is not closed; then  $E^c$  is not open and so  $\forall \epsilon > 0$ .  $\exists x \in E^c$ .  $B(x, \epsilon) \cap E \neq \emptyset$ . Since  $x \in E^c$ ,  $(B(x, \epsilon) \cap E) \setminus \{x\} = B(x, \epsilon) \cap E \neq \emptyset$  and hence  $x$  is a limit point of  $E$ . Then  $x \in E \cap E^c$ , a contradiction; and  $E$  is closed.

**Definition:** Let  $\{x_n\}_{n=l}^{\infty} \subseteq S$  for some set  $S$ . We say  $\{x_n\}$  is *eventually constant* if  $\exists N \geq l$ .  $x_n = x_N$  ( $\forall n \geq N$ ).

**Proposition:** Let  $E \subseteq \mathbb{R}$ . Then  $x$  is a limit point of  $E \iff \exists \{x_n\}_{n=1}^{\infty} \subseteq E$  such that the sequence is not eventually constant and  $x_n \rightarrow x$  as  $n \rightarrow \infty$ .

*Proof:*

$\implies$  :

Suppose  $x$  is a limit point of  $E$ , i.e.  $\forall \epsilon > 0$ .  $(B(x, \epsilon) \cap E) \setminus \{x\} \neq \emptyset$ . Set  $r_1 = 1$  and choose  $x_1 \in E$  such that  $x_1 \in (B(x, r) \cap E) \setminus \{x\}$ .

Set  $r_n = \min(\frac{1}{n}, |x - x_{n-1}|)$  and choose  $x_n \in (B(x_1, r_n) \cap E) \setminus \{x\}$ .

Then  $\forall n \geq 1$ .  $\{x_n\}_{n=1}^\infty \subseteq E$  and  $|x - x_{n-1}| < |x - x_n|$  and  $|x - x_n| < \frac{1}{n}$ . It follows  $\{x_n\}$  is not eventually constant, and  $x_n \rightarrow x$  as  $n \rightarrow \infty$ .

$\impliedby$  :

Let  $\epsilon > 0$ .  $\exists N \geq 1$ .  $n \geq N \implies |x - x_n| < \epsilon$ . Then  $\{x_n \mid n \geq N\} \subseteq B(x, \epsilon) \cap E$ . If  $\{x_n \mid n \geq N\} = \{x\}$  then  $\{x_n\}$  is eventually constant, a contradiction. Hence  $\emptyset \neq \{x_n \mid n \geq N\} \setminus \{x\} \subseteq (B(x, \epsilon) \cap E) \setminus \{x\} \implies (B(x, \epsilon) \cap E) \setminus \{x\} \neq \emptyset$ , and hence  $x$  is a limit point.

**Corollary:** Let  $E \subseteq \mathbb{R}$ . The following are equivalent (proof follows from last theorem):

1.  $E$  is closed.
2. If  $x \in \mathbb{R}$  is a limit point of  $E$ ,  $x \in E$ .
3. If  $\{x_n\}_{n=l}^\infty \subseteq E$  is such that  $x_n \rightarrow x$  as  $n \rightarrow \infty$ , then  $x \in E$ .

**Corollary:** Let  $E \subseteq \mathbb{R}$  and  $E \neq \emptyset$ . Suppose  $E$  is closed.

1. If  $E$  is bounded above, then  $\sup E \in E$ , i.e.  $\sup E = \max E$ .
2. If  $E$  is bounded below, then  $\inf E \in E$ , i.e.  $\inf E = \min E$ .

#### 4.1.4 Closure, Interior, and Boundary Sets

**Definition:** Let  $E \subseteq \mathbb{R}$ .

1. Let  $\mathcal{O}(E) = \{V \subseteq \mathbb{R} \mid V \subseteq E \text{ and } V \text{ is open}\} \subseteq \mathcal{P}(\mathbb{R})$   
 $\mathcal{C}(E) = \{C \subseteq \mathbb{R} \mid E \subseteq C \text{ and } C \text{ is closed}\} \subseteq \mathcal{P}(\mathbb{R})$ .  
 Note that  $\emptyset \in \mathcal{O}(E)$  and  $\mathbb{R} \in \mathcal{C}(E)$ .
2. We define  $E^0 = \bigcup_{V \in \mathcal{O}(E)} V$ , and call this set the *interior* of  $E$ .  
 We define  $\bar{E} = \bigcap_{C \in \mathcal{C}(E)} C$ , and call this set the *closure* of  $E$ .
3. We define  $\partial E = E \setminus E^0$  to be the *boundary* of  $E$ .

**Theorem:** Let  $E \subseteq \mathbb{R}$ . The following hold:

1.  $E^0 \subseteq E \subseteq \bar{E}$
2.  $E^0$  is open and  $\bar{E}, \partial E$  are closed.
3. For every  $x \in E$ ,  $x \in E^0 \oplus x \in \partial E$ .
4.  $\partial E = \{x \in \mathbb{R} \mid \forall \epsilon > 0. B(x, \epsilon) \cap E \neq \emptyset \text{ and } B(x, \epsilon) \cap E^c \neq \emptyset\}$ .
5.  $E$  is open  $\iff E = E^0$ ,  $E$  is closed  $\iff E = \bar{E}$ .

*Proof:*

1. Trivial.
2.  $E^0$  is an arbitrary union of open sets and thus open;  $\bar{E}$  is an arbitrary intersection of closed sets, so it's closed.  $\partial E = \bar{E} \setminus E^0 = \bar{E} \cap (\mathbb{R} \setminus E^0)$  is the intersection of two closed sets, so it's closed.

3. Trivial.
4. Suppose  $x \in \partial E$ . Show the two properties of the set are satisfied via contradiction. Next, assume  $x$  in the set, and show that  $x \in \partial E$ .
5. Trivial.

*Corollary:* Let  $E \subseteq \mathbb{R}$ . Then  $E$  is closed  $\iff \partial E \subseteq E$ .

*Proof:*  $E$  is closed  $\implies E = \bar{E} \implies \partial E \subseteq \bar{E} \subseteq E$ . On the other hand, if  $\partial E \subseteq E$  then  $E \subseteq \bar{E} = E^0 \cup \partial E \subseteq E$ , so  $E = \bar{E}$ .

**Theorem (Bolzano-Weierstass, Part 2):** Let  $E \subseteq \mathbb{R}$  be infinite and bounded. Then  $E$  has a limit point.

*Proof:* Since  $E$  is infinite we may construct a non-eventually-constant sequence  $\{x_n\}_{n=0}^\infty \subseteq E$ . We do so by choosing  $x_0 \in E$  arbitrarily, and  $x_n \in E \setminus \{x_0, \dots, x_{n-1}\}$  for any  $n \in \mathbb{N}^+$ . Since  $E$  is bounded, the sequence is too, so B-W implies there exists a convergent subsequence  $\{x_{n_k}\}_{k=0}^\infty \subseteq E$ . This subsequence is not eventually constant by construction, so its limit is a limit point.

## 4.2 Compact Sets

**Definition:**

1. Let  $A$  be some index set and assume  $\forall \alpha \in A. V_\alpha \subseteq \mathbb{R}$ . We write  $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$  for the collection of all of these subsets.
2. If  $E \subseteq \mathbb{R}$  and  $E \subseteq \bigcup_{\alpha \in A} V_\alpha$ , then we say  $\mathcal{V}$  is a *cover* of  $E$ .
3. If  $V_\alpha \subseteq \mathbb{R}$  is open  $\forall \alpha \in A$  and  $\mathcal{V}$  is a cover of  $E$ , we say  $\mathcal{V}$  is an *open cover*.
4. Let  $\mathcal{V}$  be a cover of  $E$ . We say  $\mathcal{W} = \{V_\alpha\}_{\alpha \in A'}$  is a *subcover* of  $E$  if  $A' \subseteq A$  and  $\mathcal{W}$  is a cover of  $E$ .
5. Let  $\mathcal{V}$  be a cover of  $E$ . If  $A$  is finite, then  $\mathcal{W} = \{V_\alpha\}_{\alpha \in A'}$  is a *finite subcover* of  $E$ , if  $\mathcal{W}$  is a subcover of  $E$ .

*Examples:*

1. Every  $E \subseteq \mathbb{R}$  admits a cover:  $E = \bigcup_{x \in E} \{x\}$ .
2. Every  $E \subseteq \mathbb{R}$  admits an open cover:  $E \subseteq \bigcup_{x \in E} B(x, \epsilon)$  for  $\epsilon > 0$ .
3. If  $E$  is finite and  $\mathcal{V}$  is an open cover, we claim there is a finite open subcover. Indeed, write  $E = \{a_i \mid 1 \leq i \leq n\}$  and choose  $V_{\alpha_i}$  such that  $a_i \in V_{\alpha_i}$ . Then  $E \subseteq \bigcup_{i=1}^n V_{\alpha_i}$  and  $\{V_{\alpha_i}\}_{i=1}^n \subseteq \{V_\alpha\}_{\alpha \in A}$ . Hence every open cover of a finite set admits a finite open subcover.
4.  $E = \{\frac{1}{n} \mid n \geq 1\}$ .  $\mathcal{V} = \{B(\frac{1}{n}, \frac{1}{n(n+1)})\}_{n=1}^\infty$  is an open cover of  $E$ . Note that  $B(\frac{1}{n}, \frac{1}{n(n+1)}) \cap E = \{\frac{1}{n}\}$ , so there does not exist a finite subcover.
5.  $E = \{0\} \cup \{\frac{1}{n} \mid n \geq 1\}$ . Suppose  $\mathcal{V}$  is an open cover of  $E$ . Since  $0 \in E$ ,  $\exists \alpha_0 \in A. 0 \in V_{\alpha_0}$ . Since  $V_{\alpha_0}$  is open,  $\exists \epsilon > 0. B(0, \epsilon) \subseteq V_{\alpha_0}$ . Then  $B(0, \epsilon) \cap E = \{\frac{1}{n} \mid n \geq N\}$  where  $N = \min\{n \in \mathbb{N} \mid n \geq \frac{1}{\epsilon}\}$ . Hence  $E \setminus B(0, \epsilon) = \{\frac{1}{n} \mid 1 \leq n \leq N\}$ . There exist  $V_{\alpha_n}$  for  $n \in [N]$  such that  $\frac{1}{n} \in V_{\alpha_n}$ . Then  $E \subseteq \bigcup_{n=0}^N V_{\alpha_n}$  and  $E$  has a finite subcover.
6. Let  $a < b$  and  $E = (a, b)$ . Then  $\mathcal{V} = \{(a + \frac{1}{n+1}, b - \frac{1}{n+1})\}_{n \in \mathbb{N}}$  is an open cover of  $E$ . Since these intervals are nested, there cannot be a finite subcover.

**Definition:** Let  $E \subseteq \mathbb{R}$ . We say that  $E$  is *compact* if every open cover of  $E$  admits a finite subcover.

*Examples:*

1.  $\emptyset$  is trivially compact.
2.  $\mathbb{R}$  is not compact because  $\mathcal{V} = \{B(0, n)\}_{n \in \mathbb{N}}$  is an open cover that clearly does not admit a finite subcover of  $\mathbb{R}$ .
3. Any finite set  $E \subseteq \mathbb{R}$  is compact.
4.  $(a, b)$  for  $a < b$  is not compact.
5.  $\{\frac{1}{n} \mid n \geq 1\}$  is not compact.
6.  $\{0\} \cup \{\frac{1}{n} \mid n \geq 1\}$  is compact.

Notice in each of our examples of compact sets that the set is closed and bounded.

#### 4.2.1 Heine-Borel Theorem

**Theorem:** Let  $K \subseteq \mathbb{R}$ . Then  $K$  is compact  $\iff K$  is closed and bounded.

*Proof:*

$\implies$  Suppose  $K$  is compact.

Notice that  $\bigcup_{n=1}^{\infty} B(0, n) = \mathbb{R}$  (since  $\mathbb{R}$  is Archimedean) and so  $K \subseteq \mathbb{R} = \bigcup_{n=1}^{\infty} B(0, n)$ . Then  $\{B(0, n)\}_{n=1}^{\infty}$  is an open cover of  $K$ . Since  $K$  is compact,  $\exists$  a finite subcover:  $K \subseteq \bigcup_{i=1}^m B(0, n_i)$  for some  $m \in \mathbb{N}$ .

Set  $r = \max_{i \in [m]} n_i$ . Then  $K \subseteq \bigcup_{i=1}^m B(0, n_i) \subseteq B(0, r) \implies K$  is bounded.

Now we show  $K$  is closed. Let  $x \in K^C$ . For each  $y \in K$  we set  $r_y = \frac{1}{2}|x - y| > 0$ . Then  $B(y, r_y) \cap B(x, r_y) = \emptyset$  ( $\forall y \in K$ ). Also,  $\{B(y, r_y)\}_{y \in K}$  is an open cover.

$K$  compact  $\implies \exists$  a finite subcover:  $K \subseteq \bigcup_{i=1}^n B(y_i, r_{y_i})$ . Set  $r = \min_{i \in [n]} r_i > 0$  and notice that  $B(y_i, r_{y_i}) \cap B(x, r) = \emptyset$ . Hence  $\bigcup_{i=1}^n B(y_i, r_{y_i}) \cap B(x, r) = \emptyset \implies K \cap B(x, r) = \emptyset \implies B(x, r) \subseteq K^C$ . This means that  $K^C$  is open and so  $K$  is closed.

$\Leftarrow$  (**Heine-Borel**) Suppose  $K$  is closed and bounded. If  $K = \emptyset$  we're done, so suppose  $K \neq \emptyset$ .

Notice that  $K$  bounded  $\implies \inf K, \sup K \in \mathbb{R}$ , and  $K$  closed  $\implies \inf K, \sup K \in K$ . In particular,  $\sup K = \max K, \inf K = \min K$ . Let  $\mathcal{V}$  be an open cover of  $K$ .

Let  $E = \{x \in K \mid \mathcal{V} \text{ admits a finite subcover of } K \cap [\inf K, x]\} \subseteq K$ . Notice that  $K \cap [\inf K, \inf K] = \{\inf K\}$  is a finite set and hence compact; thus  $\mathcal{V}$  admits a finite subcover of  $K \cap [\inf K, \inf K]$ . Hence  $\inf K \in E$ , and so  $E \neq \emptyset$ . Clearly  $E$  is bounded above by  $\sup K$ . By LUB property,  $\exists \sup E \in \mathbb{R}$  and  $\sup E \leq \sup K$ .

We want to show  $\sup E = \sup K = \max E$ . Notice that  $\forall n \geq 1, \exists x_n \in E \subseteq K$  such that  $\sup E - \frac{1}{n} < x_n \leq \sup E$ . Then  $x_n \rightarrow \sup E$  as  $n \rightarrow \infty$ , and so  $\sup E \in K$  (since  $K$  is closed).

Write  $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$ . Since  $\sup E \in K, \exists \alpha_0 \in A$  such that  $\sup E \in V_{\alpha_0}$ . But  $V_{\alpha_0}$  is open so  $\exists \epsilon > 0, B(\sup E, \epsilon) \subseteq V_{\alpha_0}$ . By definition,  $\exists x \in E, \sup E - \epsilon < x \leq \sup E$ . Hence  $\mathcal{V}$  admits a finite subcover of  $K \cap [\inf K, x]$ , i.e.  $K \cap [\inf K, x] \subseteq \bigcup_{i=1}^n V_{\alpha_i}$ . Then  $K \cap [\inf K, \sup E] \subseteq \bigcup_{i=1}^n V_{\alpha_i} \implies \sup E \in E \implies \sup E = \max E$ .

Assume for sake of contradiction that  $\max E < \max K$ . Let  $K' = K \setminus \bigcup_{i=0}^n V_{\alpha_i}$ .  $K'$  is closed since it's the intersection of closed sets.  $K' \neq \emptyset$  since otherwise  $K \subseteq \bigcup_{i=0}^n V_{\alpha_i} \implies \max E = \max K$ .

Let  $y = \inf K' = \min K'$  (since  $K'$  is closed) and note that  $y > \max E$ . Then  $K \cap [\inf K, y] = K \cap [\inf K, \min K'] \subseteq \bigcup_{i=0}^n V_{\alpha_i} \cup \{y\}$ . But since  $y \in K' \subseteq K$ ,  $\exists V_{\alpha_{n+1}} \in \mathcal{V}$  such that  $y \in V_{\alpha_{n+1}}$ . Hence  $K \cap [\inf K, y] \subseteq \bigcup_{i=0}^{n+1} V_{\alpha_i} \implies y \in E \implies \max E < y \leq \max E$ , a contradiction. We then deduce that  $\max E = \max K \implies K = K \cap [\min K, \max K]$  is covered by a finite subcover of  $\mathcal{V}$ ; thus,  $K$  is compact.

#### Corollary:

1. If  $K \subseteq \mathbb{R}$  is compact and  $E \subseteq \mathbb{R}$  is closed, then  $E \cap K$  is compact.
2. If  $K \subseteq \mathbb{R}$  is compact and  $E \subseteq K$  is closed, then  $E$  is compact.
3. If  $K_i \subseteq \mathbb{R}$  is compact for  $i \in [n]$ , then  $\bigcup_{i=1}^n K_i$  is compact.
4. If  $K_\alpha \subseteq \mathbb{R}$  is compact  $\forall \alpha \in A$ , then  $\bigcap_{\alpha \in A} K_\alpha$  is compact.

### 4.3 Connected Sets

**Definition:** We say two sets  $A, B \subseteq \mathbb{R}$  are *separated* if  $A \cap \bar{B} = \bar{A} \cap B = \emptyset$ .

A set  $E \subseteq \mathbb{R}$  is *disconnected* if  $E = A \cup B$  such that  $A \neq \emptyset, B \neq \emptyset$  and  $A, B$  are separated. If a set  $E \subseteq \mathbb{R}$  is not disconnected, we say it's *connected*.

*Examples:*

1.  $(0, 1)$  and  $[1, 2)$  are not separated, though they are disjoint, since  $\overline{(0, 1)} \cap [1, 2) = [0, 1] \cap [1, 2) = \{1\} \neq \emptyset$ .
2.  $(a, b)$  and  $(b, c)$  for  $a < b < c$  are separated, since  $\overline{(a, b)} \cap (b, c) = \emptyset = (a, b) \cap \overline{(b, c)}$ . Then  $(a, c) \setminus \{b\}$  is disconnected, since  $(a, c) \setminus \{b\} = (a, b) \cup (b, c)$ .
3. Similarly,  $\forall a \in \mathbb{R}$ .  $(-\infty, a)$  and  $(a, \infty)$  are separated. Then  $\mathbb{R} \setminus \{a\} = (-\infty, a) \cup (a, \infty)$  is disconnected.

**Theorem:** Let  $E \subseteq \mathbb{R}$ . Then  $E$  is connected  $\iff (x, y \in E \text{ and } x < z < y \implies z \in E)$ .

*Proof:*

$\neg 2 \implies \neg 1$ :

If (2) is false then  $\exists x, y \in E$  and  $z \in (x, y)$  such that  $z \notin E$ . Then  $E = L_z \cup R_z$  for  $L_z = E \cap (-\infty, z)$  and  $R_z = E \cap (z, \infty)$ . Since  $x \in L_z, y \in R_z$ , and  $L_z \subseteq (-\infty, z)$  and  $R_z \subseteq (z, \infty)$ , it follows that  $L_z$  and  $R_z$  are separated. Hence  $E$  is disconnected.

$\neg 1 \implies \neg 2$

Suppose  $E$  is disconnected. Write  $E = A \cup B$  with  $A, B \neq \emptyset$  and  $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ . Let  $x \in A$  and  $y \in B$ . Without loss of generality, we assume  $x < y$ .

Let  $z = \sup(A \cap [x, y])$ . Clearly  $z \in \bar{A}$  and so  $z \notin B \implies z \neq y \implies x \leq z \leq y$ . If  $z \notin A$  then  $z \neq x \implies x < z < y$  and  $z \notin A \cup B = E$ . Otherwise, if  $z \in A$ , then  $z \notin \bar{B}$ .  $\bar{B}$  is closed, so  $\bar{B}^C$  is open; and hence we can find  $w$  such that  $z < w < y$ ,  $w \notin B$ , and  $w \notin A$ . Then  $x < w < y$  and  $w \notin A \cup B = E$ . In all cases, then,  $\neg 2$  is true.

*Corollary:*  $\mathbb{R}, (-\infty, a), (-\infty, a], (a, \infty), [a, \infty), (a, b), (a, b], [a, b), \text{ and } [a, b]$  are all connected.

## 5 Continuity

### 5.1 Limits of Functions

**Definition:** Let  $E \subseteq \mathbb{R}$ ,  $f : E \rightarrow \mathbb{R}$ , and  $p \in \mathbb{R}$  be a limit point. Let  $q \in \mathbb{R}$ .

We say  $\lim_{x \rightarrow p} f(x) = q$  or  $f(x) \rightarrow q$  as  $x \rightarrow p$  iff  $\forall \epsilon > 0. \exists \delta > 0. x \in E \wedge 0 < |x - p| < \delta \implies |f(x) - q| < \epsilon$ .

*Examples:*

1.  $E = [0, 1], f(x) = x$ . Let  $p = \frac{1}{2}$ .  $\lim_{x \rightarrow \frac{1}{2}} f(x) = \frac{1}{2}$ .  
*Proof:* Let  $\epsilon > 0$ ; choose  $\delta = \epsilon > 0$ . Then  $x \in [0, 1]$  and  $0 < |x - \frac{1}{2}| < \delta \implies |f(x) - \frac{1}{2}| < \epsilon$ .
2.  $E = [0, 1], f(x) = x$  (for  $x \neq \frac{1}{2}$ ),  $f(x) = 37$  (for  $x = \frac{1}{2}$ ).  
 By the proof of (1), the claim still holds.
3.  $f(x) = x^n$  on  $E = (0, 1)$  for  $2 \leq n \in \mathbb{N}$ . 0 is a limit point of  $E$ ; we claim  $\lim_{x \rightarrow 0} x^n = 0$ .  
*Proof:* Let  $\epsilon > 0$ ; choose  $\delta = \epsilon^{1/n} > 0$ . Then  $x \in (0, 1)$  and  $0 < |x - 0| < \delta \implies 0 < x < \delta \implies 0 < x^n < \delta^n = \epsilon \implies |f(x) - 0| = x^n < \epsilon$ .
4.  $\lim_{x \rightarrow p} x = p$  whenever  $p$  is a limit point of  $E$ .
5. If  $\forall x \in E. f(x) = 1$  then  $\lim_{x \rightarrow p} f(x) = 1$  whenever  $p$  is a limit point of  $E$ .
6. Let  $E = \mathbb{R}$  and  $f(x) = \cos(x)$ . From HW6,  $|\cos(x) - 1| \leq x^2 e^{x^2}$ . We claim  $\lim_{x \rightarrow 0} \cos(x) = 1$ .  
*Proof:* Let  $\epsilon > 0$ . Choose  $\delta = \min(1, \sqrt{\epsilon/e}) > 0$ . Then for  $x \in \mathbb{R}$ ,  $0 < |x - 0| < \delta \implies |x| < \min(1, \sqrt{\epsilon/e}) \implies |\cos(x) - 1| \leq x^2 e^{x^2}$  (since  $|x|^2 < \delta \leq 1 \implies e^{|x|^2} \leq e^1$ )  $\implies |\cos(x) - 1| < \delta^2 e \leq (\sqrt{\epsilon/e})^2 e = \epsilon$ .
7.  $E = \{\frac{1}{n} \mid n \geq 1\}, p = 0$ . Let  $f(x) = \frac{1}{x}$  for  $x \in E$ . We claim  $\lim_{x \rightarrow 0} f(x)$  does not exist.  
*Proof:* Suppose not. Then for  $\epsilon = 1$ .  $\exists \delta > 0. x \in E, 0 < |x - 0| < \delta \implies |f(x) - q| < 1$ . But  $x \in E, |x| < \delta \implies x = \frac{1}{n}, \frac{1}{\delta} < n$ , and  $|f(x) - q| = |\frac{1}{1/n} - q| = |n - q| < 1$ , which is a contradiction.

**Definition:** Let  $f : E \rightarrow \mathbb{R}$  for some  $E \subseteq \mathbb{R}$ . If  $A \subseteq E$  we define  $f(A) = \{f(x) \mid x \in A\} \subseteq \mathbb{R}$  as the *image* of  $A$  under  $f$ . If  $B \subseteq \mathbb{R}$  we define  $f^{-1}(B) = \{x \in E \mid f(x) \in B\}$  as the *pre-image* of  $B$  under  $f$ .

**Lemma:** Suppose  $f : E \rightarrow \mathbb{R}$ . Then  $A \subseteq B \subseteq E \implies f(A) \subseteq f(B)$ , and  $A \subseteq B \subseteq \mathbb{R} \implies f^{-1}(A) \subseteq f^{-1}(B) \subseteq E$ .

#### 5.1.1 Divergence Criteria

**Theorem (Divergence Criteria):** Let  $E \subseteq \mathbb{R}$ ,  $f : E \rightarrow \mathbb{R}$ ,  $p$  be a limit point of  $E$ ,  $q \in \mathbb{R}$ . The following are equivalent:

1.  $\lim_{x \rightarrow p} f(x) = q$
2. For every open set  $V \subseteq \mathbb{R}$  such that  $q \in V$ ,  $\exists$  an open set  $U \subseteq \mathbb{R}$  with  $p \in U$  such that  $f(U \cap E \setminus \{p\}) \subseteq V$ . (*Topological characterization*)



3. If  $\{x_n\}_{n=l}^{\infty} \subseteq E$  satisfies  $x_n \neq p$  ( $\forall n \geq l$ ) and  $x_n \rightarrow p$  as  $n \rightarrow \infty$ , the sequence  $\{f(x_n)\}_{n=l}^{\infty} \subseteq \mathbb{R}$  converges and  $f(x_n) \rightarrow q$  as  $n \rightarrow \infty$ . (*Sequential characterization*)

*Proof:*

(1)  $\implies$  (2) :

Assume (1) and let  $V \subseteq \mathbb{R}$  be open with  $q \in V$ . Since  $V$  is open,  $\exists \epsilon > 0$ .  $B(q, \epsilon) \subseteq V$ . Since  $\lim_{x \rightarrow p} f(x) = q$ ,  $\exists \delta > 0$ .  $x \in E \wedge 0 < |x - p| < \delta \implies |f(x) - q| < \delta$ . Let  $U = B(p, \delta)$  (an open set). Then  $x \in U \cap E \setminus \{p\} \implies x \in E \wedge |x - p| < \delta \implies |f(x) - q| < \epsilon \implies f(x) \in B(q, \epsilon) \subseteq V$ . So  $f(U \cap E \setminus \{p\}) \subseteq V$  as desired.

(2)  $\implies$  (3):

Assume (2) and let  $\{x_n\}_{n=l}^{\infty} \subseteq E$  satisfy  $x_n \neq p, x_n \rightarrow p$ . Let  $\epsilon > 0$  and set  $V = B(q, \epsilon)$  (open). From (2),  $\exists$  open  $U$  such that  $f(U \cap E \setminus \{p\}) \subseteq V$  and  $p \in U$ . Since  $U$  is open,  $\exists \delta > 0$ .  $B(p, \delta) \subseteq U$ . Since  $x_n \rightarrow p$  as  $n \rightarrow \infty$ ,  $\exists N \geq l$ .  $n \geq N \implies |x_n - p| < \delta \implies x_n \in U \cap E \setminus \{p\} \implies f(x_n) \in V = B(q, \epsilon)$ . Hence  $n \geq N \implies |f(x_n) - q| < \epsilon$ , and  $f(x) \rightarrow q$  as  $n \rightarrow \infty$ .

$\neg(1) \implies \neg(3)$ :

Suppose (1) is false; then  $\exists \epsilon > 0$ .  $\forall \delta > 0$ .  $\exists x \in E$  with  $0 < |x - p| < \delta$  such that  $|f(x) - q| \geq \epsilon$ . For  $n \in \mathbb{N}, n \geq 1$ , set  $\delta = \frac{1}{n}$  to find  $x_n \in E$  such that  $0 < |x_n - p| < \frac{1}{n}$  and  $|f(x_n) - q| \geq \epsilon$ . Clearly,  $\{x_n\}_{n=1}^{\infty} \subseteq E$  satisfies  $x_n \neq p, x_n \rightarrow p$ . But  $f(x_n)$  does not converge to  $q$ . Hence (3) fails.

**Corollary:** If  $E \subseteq \mathbb{R}, f : E \rightarrow \mathbb{R}, p$  is a limit point of  $E$ , and  $\lim_{x \rightarrow p} f(x) = q$ , then  $q$  is unique.

*Proof:* Limits of sequences are unique, so this follows from (3) in Divergent Criteria theorem.

**Corollary (Algebra of limits):** Let  $E \subseteq \mathbb{R}, f, g : E \rightarrow \mathbb{R}, p$  be a limit point of  $E$ . Assume  $\lim_{x \rightarrow p} f(x) = q_1, \lim_{x \rightarrow p} g(x) = q_2$ . The following hold:

1. If  $\alpha, \beta \in \mathbb{R}$  then  $\lim_{x \rightarrow p} (\alpha f(x) + \beta g(x)) = \alpha q_1 + \beta q_2$
2.  $\lim_{x \rightarrow p} f(x)g(x) = q_1 q_2$
3. If  $q_2 = \lim_{x \rightarrow p} g(x) \neq 0$ , then  $\frac{f}{g} : E \setminus g^{-1}(\{0\}) \rightarrow \mathbb{R}$  is well-defined,  $p$  is a limit point of  $E \setminus g^{-1}(\{0\})$ , and  $\lim_{x \rightarrow p} \frac{f(x)}{g(x)} = \frac{q_1}{q_2}$

*Proof:* All follow from the algebra of sequential limits and (3) in the Theorem.

As an application of this, we get a large class of limit examples.

**Corollary:** Let  $P : E \rightarrow \mathbb{R}$  be a polynomial, i.e.  $P(x) = a_0 + a_1x + \cdots + a_nx^n$  for some  $n \in \mathbb{N}, a_i \in \mathbb{R}$  for  $i \in [n]$ . If  $p$  is a limit point of  $E$ , then  $\lim_{x \rightarrow p} P(x) = P(p)$ .

*Proof:* We know  $\lim_{x \rightarrow p} 1 = 1, \lim_{x \rightarrow p} x = p$ . Algebra of limits (2) and simple induction show  $\lim_{x \rightarrow p} x^k = p^k$  ( $\forall k \in \mathbb{N}^+$ ). Then algebra of limits (1) and another induction argument prove  $\lim_{x \rightarrow p} P(x) = \lim_{x \rightarrow p} (a_0 + a_1x + \cdots + a_nx^n) = \lim_{x \rightarrow p} (a_0 + a_1p + \cdots + a_np^n) = P(p)$ .

## 5.2 Continuous Functions

**Definition:** Let  $E \subseteq \mathbb{R}, f : E \rightarrow \mathbb{R}, p \in E$ . We say  $f$  is *continuous* at  $p$  iff:

$$\forall \epsilon > 0. \exists \delta > 0. x \in E \wedge |x - p| < \delta \implies |f(x) - f(p)| < \epsilon$$

If  $f : E \rightarrow \mathbb{R}$  is continuous at each  $p \in E$  we say  $f$  is continuous on  $E$ .

*Remarks:*

1. In order to be continuous at  $p \in E$ ,  $f$  must be defined at  $p$ . Contrast this to  $\lim_{x \rightarrow p} f(x)$ , in which case  $p$  need only be a limit point of  $E$ .
2. Informally one can think of continuous functions as those approximated well “near  $p$ ” by  $f(p)$ , i.e.  $f(x) \approx f(p)$  when  $x \approx p$ .
3. In the definition, the value of  $\delta$  may depend on the point  $p$ . If a function is continuous on  $E$  then for a given  $\epsilon > 0$  the  $\delta = \delta(p)$  may vary greatly as  $p$  varies.
4. If  $p \in E$  is isolated (not a limit point of  $E$ ), then  $f$  is vacuously continuous at  $p$ :  $x \in E, |x - p| < \delta$  for  $\delta$  small enough  $\implies x = p$ .

*Example:*

We saw last time that  $\lim_{x \rightarrow p} P(x) = P(p)$  for all polynomials  $P : \mathbb{R} \rightarrow \mathbb{R}$ . Hence  $\forall \epsilon > 0. \exists \delta > 0. x \in \mathbb{R}, 0 < |x - p| < \delta \implies |P(x) - P(p)| < \epsilon$ . Hence  $P$  is continuous at  $p$ .

**Theorem:** Let  $E \subseteq \mathbb{R}, f : E \rightarrow \mathbb{R}, p \in E$  be a limit point of  $E$ . Then:

$$f \text{ is continuous at } p \iff \lim_{x \rightarrow p} f(x) = f(p)$$

**Corollary (Algebra of Continuity):** Let  $E \subseteq \mathbb{R}, f, g : E \rightarrow \mathbb{R}$ , and  $p \in E$ . Assume that  $f, g$  are continuous at  $p$ . Then the following hold:

1. If  $\alpha, \beta \in \mathbb{R}$  then  $\alpha f + \beta g$  is continuous at  $p$ .
2.  $fg$  is continuous at  $p$ .
3. If  $g(p) \neq 0$  then  $\frac{f}{g} : E \setminus g^{-1}(\{0\}) \rightarrow \mathbb{R}$  is well-defined and continuous at  $p$ .

*Proof:* If  $p$  is isolated, the claim is vacuously true. Assume  $p$  is not isolated, i.e.  $p$  is a limit point of  $E$ . Then the last theorem and algebra of limits gives the result.

**Corollary:** Let  $E \subseteq \mathbb{R}, f, g : E \rightarrow \mathbb{R}$ . If  $f, g$  are continuous on  $E$ , then:

1. If  $\alpha, \beta \in \mathbb{R}$  then  $\alpha f + \beta g$  is continuous on  $E$ .
2.  $fg$  is continuous on  $E$ .
3. If  $g(x) \neq 0$  ( $\forall x \in E$ ), then  $\frac{f}{g}$  is continuous on  $E$ .

**Theorem:** Let  $E, F \subseteq \mathbb{R}, f : E \rightarrow \mathbb{R}, g : F \rightarrow \mathbb{R}$ . Assume  $f(E) \subseteq F$ ,  $f$  is continuous at  $p \in E$ , and  $g$  is continuous at  $f(p) \in F$ . Then  $g \circ f : E \rightarrow \mathbb{R}$  (where  $(g \circ f)(x) = g(f(x))$ ) is continuous at  $p$ . Moreover, if  $f$  is continuous on  $E$  and  $g$  is continuous on  $F$ , then  $g \circ f$  is continuous on  $E$ .

*Proof:* Let  $\epsilon > 0$ .

Since  $g$  is continuous at  $f(p)$ ,  $\exists \eta > 0. y \in F$  and  $|y - f(p)| < \eta \implies |g(y) - g(f(p))| < \epsilon$ .

Since  $f$  is continuous at  $p$ ,  $\exists \delta > 0. x \in E, |x - p| < \delta \implies |f(x) - f(p)| < \eta$ .

Since  $f(E) \subseteq F$  we know that  $x \in E, |x - p| < \delta \implies f(x) \in F, |f(x) - f(p)| < \eta \implies |g(f(x)) - g(f(p))| < \epsilon$ . Hence,  $g \circ f$  is continuous by definition.

*Examples:*

1.  $\exp, \cos, \sin : \mathbb{R} \rightarrow \mathbb{R}$  are continuous on  $\mathbb{R}$  (proof in HW). Also,  $\log : (0, \infty) \rightarrow \mathbb{R}$  is continuous on  $(0, \infty)$ .
2. Let  $\alpha \in \mathbb{R}$  and set  $f : (0, \infty) \rightarrow \mathbb{R}$  via  $f(x) = x^\alpha$ . Notice that  $f(x) = \exp(\alpha \log x)$ . Since  $\log$  and  $\exp$  are continuous,  $f(x) = x^\alpha$  is continuous.

**Definition:** Let  $E \subseteq \mathbb{R}$  and  $A \subseteq E$ . We say  $A$  is *relatively open* in  $E$  iff  $A = U \cap E$  for some open set  $U \subseteq \mathbb{R}$ . Similarly, we say  $A$  is *relatively closed* in  $E$  iff  $A = C \cap E$  for some closed  $C \subseteq \mathbb{R}$ .

**Proposition:** Let  $A \subseteq E \subseteq \mathbb{R}$ . The following hold:

1.  $A$  is relatively open in  $E \iff \forall x \in A. \exists \epsilon > 0. B(x, \epsilon) \cap A \subseteq E$ .
2.  $A$  is relatively closed in  $E \iff A = B^C \cap E$  for some relatively open  $B \subseteq E$ .

**Theorem (Continuity Criteria):** Let  $E \subseteq \mathbb{R}, f : E \rightarrow \mathbb{R}$ . The following are equivalent:

1.  $f$  is continuous on  $E$ .
2. If  $p \in E$  is a limit point of  $E$ , then  $\lim_{x \rightarrow p} f(x) = f(p)$ .
3. If  $p \in E$  is a limit point of  $E$  and  $\{x_n\}_{n=1}^\infty \subseteq E$  satisfies  $x_n \rightarrow p$  as  $n \rightarrow \infty$ , then  $f(x_n) \rightarrow f(p)$  as  $n \rightarrow \infty$ .
4. If  $V \subseteq \mathbb{R}$  is open, then  $f^{-1}(V) \subseteq E$  is relatively open in  $E$ .
5. If  $C \subseteq \mathbb{R}$  is closed, then  $f^{-1}(C) \subseteq E$  is relatively closed in  $E$ .

*Proof:*

- (1)  $\iff$  (2)  $\iff$  (3) follows from the sequential criterion of limits, previous theorem.  
 (4)  $\iff$  (5) follows since  $f^{-1}(V^C) = (f^{-1}(V))^C \cap E$ .

(1)  $\implies$  (4):

Let  $V \subseteq \mathbb{R}$  be open and choose  $p \in f^{-1}(V)$ . Since  $V$  is open,  $\exists \epsilon > 0. B(f(p), \epsilon) \subseteq V$ . It suffices to show, via previous proposition, that  $\exists \delta > 0. B(p, \delta) \cap E \subseteq f^{-1}(V)$ . Since  $f$  is continuous on  $E$ ,  $\exists \delta > 0. x \in E, |x - p| < \delta \implies |f(x) - f(p)| < \epsilon$ . That is,  $x \in B(p, \delta) \cap E \implies |f(x) - f(p)| < \epsilon \implies f(x) \in B(f(p), \epsilon) \subseteq V$ . Hence  $B(p, \delta) \cap E \subseteq f^{-1}(V)$ .

(4)  $\implies$  (1):

Let  $p \in E, \epsilon > 0$ , and  $V = B(f(p), \epsilon)$ . Then  $f^{-1}(B(f(p), \epsilon)) \subseteq E$  is relatively open in  $E \implies$  (by previous proposition)  $\exists \delta > 0. B(p, \delta) \cap E \subseteq f^{-1}(B(f(p), \epsilon))$ . Then  $x \in E$  and  $|x - p| < \delta \implies f(x) \in B(f(p), \epsilon) \implies |f(x) - f(p)| < \epsilon$ . Since  $\epsilon, p$  were arbitrary, we deduce  $f$  is continuous on  $E$ .

### 5.3 Compactness and Continuity

**Theorem:** Suppose  $K \subseteq \mathbb{R}$  is compact and  $f : K \rightarrow \mathbb{R}$  is continuous on  $K$ . Then  $f(K)$  is compact.

*Proof:*

Note that for  $E \subseteq \mathbb{R}$ ,  $f(f^{-1}(E)) \subseteq E$  and  $E \subseteq f^{-1}(f(E))$ . Let  $\{V_\alpha\}_{\alpha \in A}$  be an open cover of  $f(K)$ . Since  $f$  is continuous and  $V_\alpha$  is open,  $f^{-1}(V_\alpha)$  is relatively open in  $K \implies f^{-1}(V_\alpha) = U_\alpha \cap K$  for some open  $U_\alpha \subseteq \mathbb{R}$ .

Since  $\{V_\alpha\}_{\alpha \in A}$  cover  $f(K)$ , we see that  $\{f^{-1}(V_\alpha)\}_{\alpha \in A}$  is a cover of  $K$ . Then  $\{U_\alpha\}_{\alpha \in A}$  is an open cover of  $K$ . Since  $K$  is compact, there exists a finite subcover:  $K \subseteq \bigcup_{i=1}^n U_{\alpha_i}$ . Then  $K \subseteq \bigcup_{i=1}^n U_{\alpha_i} \cap K = \bigcup_{i=1}^n f^{-1}(V_{\alpha_i}) \implies f(K) \subseteq \bigcup_{i=1}^n f(f^{-1}(V_{\alpha_i})) \subseteq \bigcup_{i=1}^n V_{\alpha_i}$ . As we have extracted a finite open subcover of  $f(K)$ ,  $f(K)$  is compact.

**Extreme Value Theorem:** Let  $K \subseteq \mathbb{R}$  be compact and  $f : K \rightarrow \mathbb{R}$  be continuous. Then  $\exists x_0, x_1 \in K$  such that  $f(x_0) \leq f(x) \leq f(x_1)$  ( $\forall x \in K$ ). That is,  $f(x_0) = \min_{x \in K} f(x) = \min f(K)$  and  $f(x_1) = \max_{x \in K} f(x) = \max f(K)$ .

*Proof:* From last theorem, we know  $f(K)$  is compact, so it's closed and bounded. From a previous theorem, closed and bounded sets contain their infimum and supremum (and thus min, max).

**Definition:** Let  $E \subseteq \mathbb{R}$  and  $f : E \rightarrow \mathbb{R}$ . We say  $f$  is *uniformly continuous* on  $E$  iff:

$$\forall \epsilon > 0. \exists \delta > 0. x, y \in E \wedge |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

*Remarks:*

1.  $f$  is uniformly continuous on  $E \implies f$  is continuous on  $E$ .
2. The key difference is that for uniform continuity,  $\delta > 0$  works for all points in  $E$ .

*Examples:*

1. Let  $E = (0, 1)$  and  $f(x) = \frac{1}{x}$ . It's trivial that  $f$  is continuous on  $E$ , but it is not uniformly continuous.

*Proof:* Suppose it is; then for  $\epsilon = \frac{1}{2}, \exists \delta > 0. x, y \in (0, 1) \wedge |x - y| < \delta \implies |f(x) - f(y)| < \frac{1}{2}$ . Choose  $N \in \mathbb{N}$  such that  $\frac{1}{\sqrt{\delta}} < N$ . Then  $x = \frac{1}{n}, y = \frac{1}{n+1}$  satisfy  $|x - y| = \frac{1}{n(n+1)} \leq \frac{1}{n^2} < \delta$  if  $n \geq N$ . Then  $\frac{1}{2} > |f(x) - f(y)| = |n - (n+1)| = 1$ , a contradiction.

**Definition:** A function  $f : E \rightarrow \mathbb{R}$  is *Lipschitz* if  $\forall x, y \in E. \exists k > 0. |f(x) - f(y)| \leq k|x - y|$ .

**Claim:** If  $f$  is Lipschitz, it is uniformly continuous. Proof: let  $\delta = \frac{\epsilon}{k}$ .

**Theorem:** Let  $K \subseteq \mathbb{R}$  be compact and  $f : K \rightarrow \mathbb{R}$  be continuous. Then  $f$  is uniformly continuous on  $K$ .

## 5.4 Continuity and Connectedness

**Theorem:** Let  $E \subseteq \mathbb{R}$  be connected and  $f : E \rightarrow \mathbb{R}$  be continuous on  $E$ . If  $X \subseteq E$  is connected, then  $f(X)$  is connected.

**Intermediate Value Theorem:** Let  $a < b \in \mathbb{R}$ . Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. If  $f(a) < c < f(b)$  or  $f(b) > c > f(a)$  for some  $c \in \mathbb{R}$ , then  $\exists x \in (a, b). f(x) = c$ .

## 5.5 Discontinuities

**Lemma:** If  $p$  is a limit point of  $E \subseteq \mathbb{R}$  then  $p$  is a limit point of  $E_p^+ = E \cap (p, \infty)$  or  $E_p^- = E \cap (-\infty, p)$ .

**Definition:** Let  $E \subseteq \mathbb{R}, f : E \rightarrow \mathbb{R}, p$  be a limit point of  $E, q \in \mathbb{R}$ .

1. If  $p$  is a limit point of  $E_p^-$ , we say  $\lim_{x \rightarrow p^-} f(x) = q \iff \forall \epsilon > 0. \exists \delta > 0. x \in E_p^-, 0 < p - \delta \implies |f(x) - q| < \epsilon$ .
2. If  $p$  is a limit point of  $E_p^+$ , then  $\lim_{x \rightarrow p^+} f(x) = q \iff \forall \epsilon > 0. \exists \delta > 0. x \in E_p^+, 0 < x - p < \delta \implies |f(x) - q| < \epsilon$ .

**Proposition:** If  $p$  is not a limit point of  $E_p^+$  then  $\lim_{x \rightarrow p} f(x) = \lim_{x \rightarrow p^-} f(x)$ . If  $p$  is not a limit point of  $E_p^-$  then  $\lim_{x \rightarrow p} f(x) = \lim_{x \rightarrow p^+} f(x)$ .

**Proposition:** If  $p$  is both a limit point of either  $E_p^+$  or  $E_p^-$ , then

$$\lim_{x \rightarrow p} f(x) = q \iff \lim_{x \rightarrow p^-} f(x) = \lim_{x \rightarrow p^+} f(x) = q$$

**Definition:** Suppose  $E \subseteq \mathbb{R}$ ,  $f : E \rightarrow \mathbb{R}$ ,  $p \in E$  is a limit point of  $E$ . Suppose further that  $p$  is not a point of continuity of  $f$ .

1. We say  $f$  has a *simple discontinuity* of  $p$  if
  - $p$  is not a limit point of  $E_p^+$  and  $\lim_{x \rightarrow p^-} f(x)$  exists,
  - $p$  is not a limit point of  $E_p^-$  and  $\lim_{x \rightarrow p^+} f(x)$  exists, or
  - $p$  is a limit point of  $E_p^+$  and  $E_p^-$  and  $\lim_{x \rightarrow p^+} f(x)$ ,  $\lim_{x \rightarrow p^-} f(x)$  both exist.
2. Otherwise, we say  $f$  has an *essential discontinuity* of  $p$ .

## 5.6 Monotone Functions

**Definition:** Let  $E \subseteq \mathbb{R}$  and  $f : E \rightarrow \mathbb{R}$ . We say:

$f$  is *non-decreasing* (increasing) if  $x, y \in E$  and  $x < y \implies f(x) \leq f(y)$  ( $f(x) < f(y)$ ), and  $f$  is *non-increasing* (decreasing) if  $x, y \in E$  and  $x < y \implies f(y) \leq f(x)$  ( $f(y) < f(x)$ ).

If  $f$  is non-increasing or non-decreasing,  $f$  is *monotone*.

**Theorem:** Suppose  $f : (a, b) \rightarrow \mathbb{R}$  is monotone, and let  $p \in (a, b)$ . Then  $\lim_{x \rightarrow p^-} f(x)$  and  $\lim_{x \rightarrow p^+} f(x)$  both exist. Moreover, if  $f$  is non-decreasing, then

$$\lim_{x \rightarrow p^-} f(x) = \sup f((a, p)) \leq f(p) \leq \inf f((p, b)) = \lim_{x \rightarrow p^+} f(x)$$

**Corollary:** If  $f : (a, b) \rightarrow \mathbb{R}$  is monotone, then  $f$  has no essential discontinuities.

*Example:*  $f(x) = \lfloor x \rfloor$  is non-decreasing and  $f$  has countably many simple discontinuities.

**Theorem:** If  $f : (a, b) \rightarrow \mathbb{R}$  is monotone, then  $f$  has at most countably many simple discontinuities.

## 6 Differentiation

### 6.1 The Derivative

**Definition:** Assume  $f : [a, b] \rightarrow \mathbb{R}$  for  $a < b \in \mathbb{R}$ . For all  $x \in [a, b]$ , the function  $\phi : (a, b) \setminus \{x\} \rightarrow \mathbb{R}$  via  $\phi(t) = \frac{f(t) - f(x)}{t - x}$  is well-defined, and  $x$  is a limit point of  $(a, b) \setminus \{x\}$ . If  $\lim_{t \rightarrow x} \phi(t)$  exists we write  $f'(x) = \lim_{t \rightarrow x} \phi(t)$  and say that  $f$  is *differentiable* at  $x$ .

We define  $f' : \{x \in [a, b] \mid x \text{ is differentiable at } x\} \rightarrow \mathbb{R}$  to be the *derivative* of  $f$ . If  $f$  is differentiable  $\forall x \in E \subseteq [a, b]$ , we say  $f$  is differentiable on  $E$ .

**Definition (General):** Let  $E \subseteq \mathbb{R}$ ,  $f : E \rightarrow \mathbb{R}$ , and  $x \in E$  be a limit point of  $E$ . Define  $\phi : E \setminus \{x\} \rightarrow \mathbb{R}$  by  $\phi(t) = \frac{f(t) - f(x)}{t - x}$ . If  $\lim_{t \rightarrow x} \phi(t)$  exists we say  $f$  is differentiable at  $x$ , and write  $f'(x) = \lim_{t \rightarrow x} \phi(t)$ .

**Proposition (locality of derivative):** Suppose  $f : E \rightarrow \mathbb{R}$ ,  $g : F \rightarrow \mathbb{R}$ ,  $x \in E \cap F$  is a limit point of  $E \cap F$ , and that  $f$  and  $g$  are differentiable at  $x$ . If  $f = g$  on  $E \cap F$  then  $f'(x) = g'(x)$ . This shows that  $f'(x)$  only depends on the value of  $f$  “near  $x$ ”.

**Proposition (Newtonian approximation):** Let  $f : E \rightarrow \mathbb{R}$ ,  $x \in E$  be a limit point of  $E$ , and  $L \in \mathbb{R}$ . Then the following are equivalent:

1.  $f$  is differentiable at  $x$  and  $f'(x) = L$
2.  $\forall \epsilon > 0. \exists \delta > 0. t \in E \wedge |x - t| < \delta \implies |f(t) - (f(x) + L(t - x))| < \epsilon|t - x|$

Proof follows from definition of  $\lim_{t \rightarrow x} \phi(t)$ . Newton's approximation says differentiable functions are those that can be "well-approximated" by affine functions  $\alpha + \beta x$ . Continuous functions are those well-approximated by constants, while differentiable functions are well-approximated by the "next" simplest function.

**Theorem:** Suppose  $f : E \rightarrow \mathbb{R}$ ,  $x \in E$  is a limit point of  $E$ , and  $f$  is differentiable at  $x$ . Then  $f$  is continuous at  $x$ .

*Proof:* By definition, if  $t \in E \setminus \{x\}$  then  $f(t) - f(x) = \phi(t)(t - x)$ . Then  $f(t) = f(x) + \phi(t)(t - x)$  and hence  $\lim_{t \rightarrow x} f(t) = f(x) + \lim_{t \rightarrow x} \phi(t)(t - x) = f(x) + f'(x)0 = f(x)$ . By the limit characterization of continuity, we deduce that  $f$  is continuous at  $x$ .

*Remark:* The converse fails. Let  $f(x) = |x|$  on  $\mathbb{R}$ . Since  $||x| - |y|| \leq |x - y|$ ,  $f$  is Lipschitz and hence uniformly continuous. However, for  $x = 0$ ,  $t > 0 \implies \phi(t) = \frac{|t| - 0}{t - 0} = 1$  and  $t < 0 \implies \phi(t) = \frac{-t - 0}{t - 0} = -1$ . Then  $\lim_{t \rightarrow 0^-} \phi(t) = -1 \neq \lim_{t \rightarrow 0^+} \phi(t) = 1$ , so  $f'(0)$  does not exist.

**Theorem (Algebra of Derivatives):** Let  $f, g : E \rightarrow \mathbb{R}$  be differentiable at  $x \in E$ . Then:

1.  $f + g : E \rightarrow \mathbb{R}$  is differentiable at  $x$  and  $(f + g)'(x) = f'(x) + g'(x)$
2.  $fg : E \rightarrow \mathbb{R}$  is differentiable at  $x$  and  $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$
3. If  $g(x) \neq 0$  then  $\frac{f}{g} : E \setminus g^{-1}(\{0\}) \rightarrow \mathbb{R}$  is differentiable at  $x$  and  $(\frac{f}{g})'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

*Examples:*

1.  $f(x) = \alpha + \beta x$  on  $\mathbb{R} \implies f'(x) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} = \beta \ (\forall x \in \mathbb{R})$ .
2.  $f(x) = x^n$  for  $n \in \mathbb{N} \implies f'(x) = nx^{n-1}$ . Proof by induction.
3. Every polynomial  $P(x) = \sum_{n=0}^N a_n x^n$  is differentiable, and  $P'(x) = \sum_{n=0}^N n a_n x^{n-1}$ .
4.  $R(x) = \frac{P(x)}{Q(x)}$  is differentiable when  $P, Q$  are polynomials at points  $p \in \mathbb{R}$  where  $Q(p) \neq 0$ .

**Theorem (Chain Rule):** Suppose  $f : E \rightarrow \mathbb{R}$  is differentiable at  $x \in E$ ,  $f(E) \subseteq F$ , and  $g : F \rightarrow \mathbb{R}$  is differentiable at  $f(x) \in F$ . Then  $g \circ f : E \rightarrow \mathbb{R}$  is differentiable at  $x$  and  $(g \circ f)'(x) = g'(f(x))f'(x)$ .

## 6.2 Mean Value Theorems

**Definition:** Let  $f : E \rightarrow \mathbb{R}$ . We say that  $f$  has a *local maximum* at  $x \in E$  if  $\exists \delta > 0. t \in E$  and  $|x - t| < \delta \implies f(t) \leq f(x)$ . We say  $f$  has a *local minimum* at  $x \in E$  if  $-f$  has a local maximum.

If  $f$  has either a local max or min at  $x \in E$ , we say  $f$  has a *local extremum* at  $x$ .

**Theorem:** Suppose  $f : E \rightarrow \mathbb{R}$  is differentiable at  $x \in E$  and  $x$  is a limit point of both  $E_x^+$  and  $E_x^-$ . If  $f$  has a local extremum at  $x$ , then  $f'(x) = 0$ .

*Proof:* It suffices to assume that  $f$  has a local max at  $x$ . Let  $\delta > 0$  such that  $t \in E$  and  $|x - t| < \delta \implies f(t) \leq f(x)$ . Then  $t \in E, 0 < x - t < \delta \implies \frac{f(t) - f(x)}{t - x} \geq 0$  and  $0 < t - x < \delta \implies \frac{f(t) - f(x)}{t - x} \leq 0$ . So  $\lim_{t \rightarrow x^-} \frac{f(t) - f(x)}{t - x} = f'(x) \geq 0$ ,  $\lim_{t \rightarrow x^+} \frac{f(t) - f(x)}{t - x} = f'(x) \leq 0$ , and thus  $f'(x) = 0$ .

*Remark:* The result is false if  $x$  is not a limit point of either  $E_x^+$  or  $E_x^-$ . Consider  $f(x) = x$  on  $E = [0, 1]$ ;  $f$  has a local min at  $x = 0$ , local max at  $x = 1$ , but  $f'(x) = 1 \forall x \in [0, 1]$ .

**Theorem (Monotonicity part 1):** Let  $f : E \rightarrow \mathbb{R}$  be differentiable at  $x \in E$ .

1. If  $f$  is non-decreasing on  $E$ , then  $f'(x) \geq 0$ .
2. If  $f$  is non-increasing on  $E$ , then  $f'(x) \leq 0$ .

**Cauchy's Mean Value Theorem:** Suppose that  $f, g : [a, b] \rightarrow \mathbb{R}$  are continuous on  $[a, b]$ , differentiable on  $(a, b)$ . Then  $\exists x \in (a, b)$ .  $(g(b) - g(a))f'(x) = (f(b) - f(a))g'(x)$ .

*Proof:*

Consider  $h : [a, b] \rightarrow \mathbb{R}$  via  $h(x) = (g(b) - g(a))f(x) - (f(b) - f(a))g(x)$ . To prove the result, it suffices to find  $x \in (a, b)$  such that  $h'(x) = 0$  (since  $h$  is cont, diff on  $[a, b]$  and  $(a, b)$ ).

Notice that  $h(a) = g(b)f(a) - g(a)f(b) = h(b)$ .

If  $h$  is constant, then  $h'(x) = 0$  trivially and we're done. Assume  $h$  is not constant; then  $\exists t \in (a, b)$ .  $h(t) > h(a)$  or  $h(t) < h(a)$ .

If  $h(t) > h(a)$ , then Extreme Value Theorem guarantees that  $\exists x \in (a, b)$ .  $h(x) = \max h([a, b])$  and Local Extremum Theorem  $\implies h'(x) = 0$ .

If  $h(t) < h(a)$  then EVT guarantees  $\exists x \in (a, b)$ .  $h(x) = \min h([a, b])$  and LET  $\implies h'(x) = 0$ .

**Corollary (Mean Value Theorem):** If  $f : [a, b] \rightarrow \mathbb{R}$  is cont and diff on  $[a, b], (a, b)$  then  $\exists x \in (a, b)$ .  $f(b) - f(a) = f'(x)(b - a)$ . *Proof:* Set  $g(x) = x$ .

**Corollary (Monotonicity part 2):** Suppose  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable on  $(a, b)$ . The following hold:

1.  $\forall x \in (a, b)$ .  $f'(x) > 0 \implies f$  is increasing.
2.  $\forall x \in (a, b)$ .  $f'(x) \geq 0 \implies f$  is non-decreasing.
3.  $\forall x \in (a, b)$ .  $f'(x) = 0 \implies f$  is constant.
4.  $\forall x \in (a, b)$ .  $f'(x) \leq 0 \implies f$  is non-increasing.
5.  $\forall x \in (a, b)$ .  $f'(x) < 0 \implies f$  is decreasing.

*Proof:* by MVT, if  $a < x_1 < x_2 < b$ , then  $f(x_2) - f(x_1) = f'(x)(x_2 - x_1)$ .

## 6.3 Darboux's Theorem

**Definition:** We say a function  $g : \mathbb{R} \rightarrow \mathbb{R}$  is periodic with period  $p > 0$  if  $g(x + p) = g(x)$  ( $\forall x \in \mathbb{R}$ ).

**Theorem (Darboux):** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is differentiable on  $[a, b]$  and  $f'(a) < \gamma < f'(b)$ . Then  $\exists x \in (a, b)$ .  $f'(x) = \gamma$ .

**Corollary:** If  $f : [a, b] \rightarrow \mathbb{R}$  is differentiable on  $[a, b]$ , then  $f'$  has no simple discontinuities.

## 6.4 L'Hôpital's Rule

**Theorem:** Suppose  $f, g : [a, b] \rightarrow \mathbb{R}$  are continuous on  $[a, b]$ , differentiable on  $(a, b)$ , and  $g'(x) \neq 0$  ( $\forall x \in (a, b)$ ). Assume that  $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L$ . If  $f(a) = g(a) = 0$ , then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L$ .

*Proof:*

We claim first that  $g(x) \neq 0$  for  $x \in (a, b]$ . Otherwise,  $g(x) = 0$  for some  $x \in (a, b] \implies 0 = \frac{g(x)-g(a)}{x-a} = g'(z)$  for some  $z \in (a, x)$ , a contradiction. So  $\frac{f}{g} : (a, b] \rightarrow \mathbb{R}$  is well-defined.

Let  $\{x_n\}_{n=l}^{\infty} \subseteq (a, b]$  satisfy  $x_n \rightarrow a$  as  $n \rightarrow \infty$ . We claim that  $\lim_{n \rightarrow \infty} \frac{f(x_n)}{g(x_n)} = L$ . Once this is established, the sequential characterization of limits yields the desired result.

To prove the claim, we apply Cauchy's Mean Value Theorem on  $[a, x_n]$ :  $\exists y_n \in (a, x_n)$  such that  $f'(y_n)g(x_n) = f'(y_n)(g(x_n) - g(a)) = g'(x_n)(f(x_n) - f(a)) = g'(x_n)f(x_n)$ . Then  $\forall n \geq l$ ,  $\frac{f(x_n)}{g(x_n)} = \frac{f'(x_n)}{g'(x_n)}$ . Since  $a < y_n < x_n$ , the squeeze lemma implies  $y_n \rightarrow a$ .

Hence  $\lim_{n \rightarrow \infty} \frac{f(x_n)}{g(x_n)} = \lim_{n \rightarrow \infty} \frac{f'(x_n)}{g'(x_n)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L$ .

**Remarks:**

1. The theorem is also true if we take limits at  $t$ .
2. If  $f, g : (a, b] \rightarrow \mathbb{R}$  and  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ , then the theorem still works.

## 6.5 Higher Derivatives and Taylor's Theorem

**Definition:** Suppose  $f : E \rightarrow \mathbb{R}$  is differentiable at  $x \in E$ , and  $x$  is a limit point of  $\{y \in E \mid f'(y) \text{ exists}\}$ . We say  $f$  is twice differentiable at  $x$  if  $f' : \{y \in E \mid f'(y) \text{ exists}\} \rightarrow \mathbb{R}$  is differentiable at  $x$ ; and  $f''(x) = f^{(2)}(x) = (f')'(x)$ . Similarly, for  $n \in \mathbb{N}$  with  $n > 2$ , we say  $f$  is  $n$ -times differentiable at  $x$  if  $x$  is a limit point of  $\{y \in E \mid f^{(n-1)}(y) \text{ exists}\}$  and  $f^{(n-1)}$  is differentiable at  $x$ , in which case  $f^{(n)}(x) = (f^{(n-1)})'(x)$ .

If  $f^{(n)}$  exists  $\forall n \in \mathbb{N}$ ,  $n \geq 1$  we say  $f$  is infinitely differentiable at  $x$ .

**Theorem (Taylor):** Suppose  $f : [a, b] \rightarrow \mathbb{R}$ . Assume  $f^{(n-1)}$  is continuous on  $[a, b]$  and  $f^{(n)}$  exists on  $(a, b)$ . Let  $x, y \in [a, b]$  with  $x \neq y$ . Then  $\exists z \in (\min\{x, y\}, \max\{x, y\})$  such that

$$f(y) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} (y-x)^k + \frac{f^{(n)}(z)}{n!} (y-x)^n$$

(called the Taylor polynomial or Taylor approximation).

*Proof:*

Suppose  $x < y$  ( $y < x$  is handled without loss of generality). Let  $P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} (t-x)^k$ , and set  $M = \frac{f(y)-P(y)}{(y-x)^n}$ . It suffices to prove that  $M = \frac{f^{(n)}(z)}{n!}$  for some  $z \in (x, y)$ .

Define  $g(t) = f(t) - P(t) - M(t-x)^n$ , and notice that  $g^{(n)}(t) = f^{(n)}(t) - n!M$ . As such, it suffices to show that  $g^{(n)}(z) = 0$  for some  $z \in (x, y)$ .

By construction,  $g^{(k)}(x) = 0$  ( $\forall k = 0, \dots, n-1$ ), and  $g(y) = 0$  (by choice of  $M$ ).

By Mean Value Theorem,  $\exists x_1 \in (x, y)$ .  $g'(x_1) = \frac{g(y)-g(x)}{y-x} = 0$ . Similarly,  $\exists x_2 \in$

$(x, x_1)$ .  $g''(x_2) = \frac{g'(x_1)-g'(x)}{x_1-x} = 0$ . Iterating, we eventually find  $x_{n-1} \in (x, y)$ .  $g^{(n-1)}(x_{n-1}) =$

0. Then  $0 = g^{(n)}(z) = \frac{g^{(n-1)}(x_{n-1})-g^{(n-1)}(x)}{x_{n-1}-x} = 0$  for some  $z \in (x, x_{n-1})$ .



## 7 Riemann-Stieltjes Integration

### 7.1 The R-S Integral

**Definition:** Let  $a, b \in \mathbb{R}$  with  $a \leq b$ . A partition of  $[a, b]$  is a finite ordered set  $P = \{x_0, \dots, x_n\}$  such that  $a = x_0 \leq x_1 \leq \dots \leq x_n = b$ . Write  $\Pi[a, b] = \{P \mid P \text{ is a partition of } [a, b]\}$ . For brevity we'll write  $\Pi = \Pi[a, b]$ .

**Universal Assumptions:** Throughout §7 we will always assume that:

1.  $f : [a, b] \rightarrow \mathbb{R}$  is bounded:  $\forall x \in [a, b]. m \leq f(x) \leq M$ , where  $m = \inf f([a, b])$ ,  $M = \sup f([a, b])$
2.  $\alpha : [a, b] \rightarrow \mathbb{R}$  (the integrator or weight function) is non-decreasing (in particular,  $\alpha$  is also bounded)

**Definition:** For each  $P \in \Pi[a, b]$  we associate to  $f$  the following quantities ( $P = \{x_0, \dots, x_n\}$ ):

1.  $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$  for  $i \in [n]$
2.  $M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$  for  $i \in [n]$
3.  $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \geq 0$  for  $i \in [n]$

We write  $U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta\alpha_i$ ,  $L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta\alpha_i$ .

$U$  is the upper Riemann-Stieltjes sum, and  $L$  is the lower R-S sum.

*Remark:* Clearly  $m(\alpha(b) - \alpha(a)) = \sum_{i=1}^n m(\alpha(x_i) - \alpha(x_{i-1})) = \sum_{i=1}^n m \Delta\alpha_i \leq \sum_{i=1}^n M_i \Delta\alpha_i \leq M \sum_{i=1}^n \Delta\alpha_i = M(\alpha(b) - \alpha(a))$ . Hence  $\forall P \in \Pi[a, b]. m(\alpha(b) - \alpha(a)) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq M(\alpha(b) - \alpha(a))$ .

**Definition of Integral:** We define

$$\int_a^b f d\alpha = \sup\{L(P, f, \alpha) \mid P \in \Pi[a, b]\}, \text{ and } \overline{\int_a^b f d\alpha} = \inf\{U(P, f, \alpha) \mid P \in \Pi[a, b]\}.$$

Both are well-defined by the remark.

If  $\int_a^b f d\alpha = \overline{\int_a^b f d\alpha}$  then we say  $f$  is R-S integrable with respect to  $\alpha$ , and write

$$\int_a^b f d\alpha = \underline{\int_a^b f d\alpha} = \overline{\int_a^b f d\alpha}.$$

We write  $\mathcal{R}([a, b]; \alpha) = \{f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is bounded, } f \text{ is R-S integrable with respect to } \alpha\}$ .

When  $\alpha(x) = x$ ,  $\int_a^b f dx$  is the Riemann integral and we write  $\mathcal{R}([a, b])$ .

*Heuristics:* The function  $\alpha$  assigns different weights to different points in  $[a, b]$ . The intuition is that  $\int_a^b f d\alpha$  is a “weighted Riemann integral”. If  $\alpha$  is continuous then we have a geometric interpretation of  $\int_a^b f d\alpha$ : consider the curve in  $\mathbb{R}^2$  parameterized by  $(x(t), y(t)) = (\alpha(t), f(t))$ ;  $\int_a^b f d\alpha = \text{area under this curve}$ .

**Lemma:** Let  $f(x) = C$  ( $\forall x \in [a, b]$ ). Then  $f \in \mathcal{R}([a, b]; \alpha)$  and  $\int_a^b f d\alpha = C(\alpha(b) - \alpha(a))$ .

*Proof:* For any  $P \in \Pi[a, b]$  we have  $m_i = M_i = C$ . Hence  $U(P, f, \alpha) = L(P, f, \alpha) = \sum_{i=1}^n C \Delta\alpha_i = C(\alpha(b) - \alpha(a))$ . So  $\underline{\int_a^b f d\alpha} = \sup\{L(P, f, \alpha)\} = C(\alpha(b) - \alpha(a)) = \inf\{U(P, f, \alpha)\} = \overline{\int_a^b f d\alpha}$ .

**Definition:** If  $P, P' \in \Pi[a, b]$  and every point in  $P$  is in  $P'$ , we say  $P'$  is a *refinement* of  $P$ .

If  $P_1, P_2 \in \Pi[a, b]$  we define the *common refinement*  $P_1 \# P_2 \in \Pi[a, b]$  by  $P_1 \# P_2 = P_1 \cup P_2$ , ordered appropriately.

**Proposition:** If  $P, P' \in \Pi[a, b]$  and  $P'$  is a refinement of  $P$ , then  $L(P, f, \alpha) \leq L(P', f, \alpha) \leq U(P', f, \alpha) \leq U(P, f, \alpha)$ .

**Theorem:**  $\int_a^b f d\alpha \subseteq \overline{\int_a^b f d\alpha}$ .

*Proof:* Let  $P_1, P_2 \in \Pi[a, b]$ . Then  $L(P_1, f, \alpha) \leq L(P_1 \# P_2, f, \alpha) \leq U(P_1 \# P_2, f, \alpha) \leq U(P_2, f, \alpha)$  by last proposition. Hence  $\int_a^b f d\alpha = \sup\{L(P_1, f, \alpha) \mid P_1 \in \Pi[a, b]\} \leq U(P_2, f, \alpha)$ . Then  $\int_a^b f d\alpha \leq \inf\{U(P_2, f, \alpha) \mid P_2 \in \Pi[a, b]\} = \overline{\int_a^b f d\alpha}$ .

## 7.2 Integrability Criteria

**Theorem (Riemann)\*:**  $f \in \mathcal{R}([a, b]; \alpha) \iff \forall \epsilon > 0. \exists P \in \Pi[a, b]. U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ .

*Proof:*

(1)  $\implies$  (2):

Let  $\epsilon > 0$ . By definition,  $\exists P_1, P_2 \in \Pi[a, b]. \int_a^b f d\alpha - \frac{\epsilon}{2} < L(P_1, f, \alpha)$ , and  $U(P_2, f, \alpha) \leq \overline{\int_a^b f d\alpha} + \frac{\epsilon}{2}$ . Let  $P = P_1 \# P_2$ . Then  $U(P_1, f, \alpha) \leq U(P_2, f, \alpha) < \overline{\int_a^b f d\alpha} + \frac{\epsilon}{2}$ . Hence  $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ .

(2)  $\implies$  (1):

For any partition  $P \in \Pi[a, b]$  we know  $\overline{\int_a^b f d\alpha} \leq U(P_1, f, \alpha)$  and  $L(P, f, \alpha) \leq \int_a^b f d\alpha$ . We also know  $\int_a^b f d\alpha \leq \overline{\int_a^b f d\alpha}$ . Then (2) implies that for  $\epsilon > 0$  we have  $P \in \Pi[a, b]. U(P_1, f, \alpha) - L(P_1, f, \alpha) < \epsilon$ . But then  $0 \leq \overline{\int_a^b f d\alpha} - \int_a^b f d\alpha < \epsilon$  ( $\forall \epsilon > 0$ ) and so  $\int_a^b f d\alpha = \overline{\int_a^b f d\alpha} \implies f \in \mathcal{R}([a, b]; \alpha)$ .

**Lemma:** Let  $P \in \Pi[a, b]$ . The following are true:

1. If  $P'$  is a refinement of  $P$ , then  $U(P', f, \alpha) - L(P', f, \alpha) \leq U(P, f, \alpha) - L(P, f, \alpha)$ .
2. If  $s_i, t_i \in [x_{i-1}, x_i]$  ( $\forall i \in [n]$ ), then  $0 \leq \sum_{i=1}^n |f(s_i) - f(t_i)| \Delta \alpha_i \leq U(P, f, \alpha) - L(P, f, \alpha)$ .
3. If  $f \in \mathcal{R}([a, b]; \alpha)$ , then  $t_i \in [x_{i-1}, x_i] \implies |\sum_{i=1}^n f(t_i) \Delta \alpha_i - \int_a^b f d\alpha| \leq U(P, f, \alpha) - L(P, f, \alpha)$ .

**Theorem:** If  $f$  is continuous on  $[a, b]$ , then  $f \in \mathcal{R}([a, b]; \alpha)$ .

*Proof:*

Note that EVT implies  $f$  is bounded. Let  $\epsilon > 0$  and choose  $k > 0$  so  $k(\alpha(b) - \alpha(a)) < \epsilon$ . Since  $f$  is cont on compact  $[a, b]$ ,  $f$  is uniformly continuous. Then  $\exists \delta > 0. x, y \in [a, b]$  and  $|x - y| < \delta \implies |f(x) - f(y)| < k$ .

Choose a partition  $P \in \Pi[a, b]$  such that  $x_i - x_{i-1} < \delta$  ( $\forall i \in [n]$ ). Then by EVT,  $\exists s_i, t_i \in [x_{i-1}, x_i]$  such that  $m_i = f(s_i), M_i = f(t_i)$ . Then  $U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i = \sum_{i=1}^n (f(t_i) - f(s_i)) \Delta \alpha_i \leq \sum_{i=1}^n k \Delta \alpha_i = k \sum_{i=1}^n \Delta \alpha_i = k(\alpha(b) - \alpha(a)) < \epsilon$ . Since  $\epsilon > 0$  was arbitrary, we deduce via Riemann's Theorem that  $f \in \mathcal{R}([a, b]; \alpha)$ .

**Theorem:** Suppose that  $f$  is monotone and  $\alpha$  is continuous on  $[a, b]$ ; then  $f \in \mathcal{R}([a, b]; \alpha)$ .

*Proof:*

Since  $\alpha$  is continuous,  $\alpha([a, b])$  is connected. Then for every  $n \in \mathbb{N}$  with  $n \geq 1$ , we know  $\alpha(a) + \frac{i}{n}(\alpha(b) - \alpha(a)) \in \alpha([a, b])$  for  $i \in [n]$ . In particular,  $\exists x_i \in [a, b]$  such that  $\alpha(x_i) = \alpha(a) + \frac{i}{n}(\alpha(b) - \alpha(a))$ . Since  $\alpha$  is non-decreasing, we may choose these  $x_i$  such that  $a = x_0 \leq x_1 \leq \dots \leq x_n = b$ . Then  $P_n = \{x_0, x_1, \dots, x_n\}$  is a partition of  $[a, b]$ .

Assume  $f$  is non-decreasing (the case of  $f$  non-increasing is similar). Let  $\epsilon > 0$ .

Since  $f$  is non-decreasing,  $f(x_i) = M_i$ ,  $f(x_{i-1}) = m_i$  for  $i \in [n]$ . Then

$$\begin{aligned} U(P_n, f, \alpha) - L(P_n, f, \alpha) &= \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i \\ &= \sum_{i=1}^n (f(x_i) - f(x_{i-1})) \Delta \alpha_i \\ &= \sum_{i=1}^n (f(x_i) - f(x_{i-1})) \frac{(\alpha(b) - \alpha(a))}{n} \\ &= \frac{\alpha(b) - \alpha(a)}{n} (f(b) - f(a)) \end{aligned}$$

Choose  $n$  such that  $n > \frac{(\alpha(b) - \alpha(a))(f(b) - f(a))}{\epsilon}$ . Then  $U(P_n, f, \alpha) - L(P_n, f, \alpha) < \epsilon$ .

*Remarks:*

1. If  $\alpha = x$ , then all monotone functions  $f$  are in  $\mathcal{R}([a, b])$ , i.e. all monotone functions are Riemann integrable.
2. Monotone functions don't have to be continuous, so  $\exists f$  with simple discontinuities in  $\mathcal{R}([a, b]; \alpha)$  when  $\alpha$  is continuous.
3. Monotone functions can have countably infinite sets of discontinuity; R-S integrals can handle infinite discontinuities.

Next, we show that R-S integration can handle more general discontinuities.

**Theorem:** Suppose that  $f$  is continuous on  $[a, b] \setminus E$ , where  $E \subseteq [a, b]$  is finite. Assume that  $\alpha$  is continuous on  $E$ ; then  $f \in \mathcal{R}([a, b]; \alpha)$ .

*Remark:* There is a much more powerful version known as Lebesgue's Theorem, which says that:

$$f \in \mathcal{R}([a, b]; \alpha) \iff (E = \{x \in [a, b] \mid f \text{ is not continuous at } x\} \implies E \text{ is } \alpha\text{-null})$$

Here, a set  $E$  is  $\alpha$ -null if  $\forall \epsilon > 0$ .  $\exists$  intervals  $(u_j, v_j) \subseteq \mathbb{R}$  for  $j \geq 1 \in \mathbb{N}$  such that  $E \subseteq \bigcup_{j=1}^{\infty} (u_j, v_j)$  and  $\sum_{j=1}^{\infty} \alpha(v_j) - \alpha(u_j) < \epsilon$ .

In particular, if  $\alpha(x) = x$ , then all countable sets are  $\alpha$ -null, and hence if  $f$  has discontinuities only on a countable set, then  $f \in \mathcal{R}([a, b])$ .

**Theorem:** Let  $I \in \mathbb{R}$ . Then the following are equivalent:

1.  $f \in \mathcal{R}([a, b]; \alpha)$  and  $\int_a^b f d\alpha = I$
2.  $\forall \epsilon > 0$ .  $\exists P \in \Pi[a, b]$ .  $(\forall i \in [n]. t_i \in [x_{i-1}, x_i] \implies |\sum_{i=1}^n f(t_i) \Delta \alpha_i - I| < \epsilon)$ .

### 7.3 Properties of $\mathcal{R}([a, b]; \alpha)$

**Theorem:** If  $f_1, f_2 \in \mathcal{R}([a, b]; \alpha)$  and  $c_1, c_2 \in \mathbb{R}$ , then  $c_1 f_1 + c_2 f_2 \in \mathcal{R}([a, b]; \alpha)$  and  $\int_a^b (c_1 f_1 + c_2 f_2) d\alpha = c_1 \int_a^b f_1 d\alpha + c_2 \int_a^b f_2 d\alpha$ .

*Remark:* This theorem gives us two important consequences:

1.  $\mathcal{R}([a, b]; \alpha)$  is a linear space;
2. The integral is a linear function, i.e.  $\int_a^b d\alpha : \mathcal{R}([a, b]; \alpha) \rightarrow \mathbb{R}$  given by  $f \mapsto \int_a^b f d\alpha$  is a linear map.

**Theorem:** Suppose  $f \in \mathcal{R}([a, b]; \alpha_i)$  for  $i \in \{1, 2\}$ . Let  $c_1, c_2 \geq 0$ . Then  $f \in \mathcal{R}([a, b]; c_1 \alpha_1 + c_2 \alpha_2)$  and  $\int_a^b f d(c_1 \alpha_1 + c_2 \alpha_2) = c_1 \int_a^b f d\alpha_1 + c_2 \int_a^b f d\alpha_2$ .

**Theorem:** Suppose  $f \in \mathcal{R}([a, b]; \alpha)$ . Let  $c \in (a, b)$ ; then  $f \in \mathcal{R}([a, c]; \alpha)$  and  $f \in \mathcal{R}([c, b]; \alpha)$ ; i.e.  $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$ .

**Theorem:** Suppose  $f \in \mathcal{R}([a, b]; \alpha)$  and  $g : [m = \inf f, M = \sup f] \rightarrow \mathbb{R}$  is continuous on  $[m, M]$ . Then  $g \circ f \in \mathcal{R}([a, b]; \alpha)$ .

**Theorem:** Suppose  $f, g \in \mathcal{R}([a, b]; \alpha)$ . Then  $fg \in \mathcal{R}([a, b]; \alpha)$ .

*Proof:* The function  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  where  $\phi(x) = x^2$  is cont. on  $\mathbb{R}$ . By the last theorem, if  $h \in \mathcal{R}([a, b]; \alpha)$  then  $h^2 \in \mathcal{R}([a, b]; \alpha)$ . Note that  $fg = \frac{1}{4}(f+g)^2 - \frac{1}{4}(f-g)^2 \in \mathcal{R}([a, b]; \alpha)$  since  $f \pm g \in \mathcal{R}([a, b]; \alpha)$ .

This result shows that  $\mathcal{R}([a, b]; \alpha)$  has even more structure than a linear space; it is an algebra, i.e. a vector space closed under product.

### 7.4 Integration and Order

**Theorem:** Suppose  $f_1, f_2 \in \mathcal{R}([a, b]; \alpha)$  and  $\forall x \in [a, b]. f_1(x) \leq f_2(x)$ . Then  $\int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha$ .

**Theorem:** Let  $f \in \mathcal{R}([a, b]; \alpha)$ . Then  $|f| \in \mathcal{R}([a, b]; \alpha)$  and  $|\int_a^b f d\alpha| \leq \int_a^b |f| d\alpha$ .

*Proof:*  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  where  $\psi(x) = |x|$  is continuous on  $\mathbb{R}$ , so the composition theorem guarantess that  $|f| \in \mathcal{R}([a, b]; \alpha)$ . Let  $c = 1$  if  $\int_a^b f d\alpha \geq 0$  and  $-1$  otherwise. Then  $|\int_a^b f d\alpha| = c \int_a^b f d\alpha = \int_a^b cf d\alpha$ , but  $cf \leq |cf| = |c||f| = |f|$  so the last theorem gives the desired result.

**Definition:** Given a function  $f : E \rightarrow \mathbb{R}$  ( $E \subseteq \mathbb{R}$  we define the *positive part* of  $f$  as  $f^+ : E \rightarrow [0, \infty)$  given by  $f^+(x) = \max\{f(x), 0\}$ . Similarly,  $f^- : E \rightarrow [0, \infty)$  via  $f^-(x) = \max\{-f(x), 0\} = (-f)^+(x)$ .

**Lemma:**  $f = f^+ - f^-$  and  $|f| = f^+ + f^-$ . Proof is trivial.

**Theorem:** Let  $f \in \mathcal{R}([a, b]; \alpha)$ . Then  $f^+, f^- \in \mathcal{R}([a, b]; \alpha)$ .

*Proof:*  $f^+ = \frac{f+|f|}{2}, f^- = \frac{f-|f|}{2}$ .

This leads to a theorem  $f, g \in \mathcal{R}([a, b]; \alpha) \implies f \wedge g, f \vee g \in \mathcal{R}([a, b]; \alpha)$ , where  $(f \wedge g)(x) = \min\{f(x), g(x)\}$  and  $(f \vee g)(x) = \max\{f(x), g(x)\}$ . (Proof on HW.)

### 7.5 Fundamental Theorem of Calculus

**Lemma:** Suppose  $f \in \mathcal{R}([a, b]; \alpha)$ . Define  $F : [a, b] \rightarrow \mathbb{R}$  via  $F(x) = \int_a^x f d\alpha$ . Let  $M = \sup\{|f(x)| \mid x \in [a, b]\}$ . Then  $|F(x) - F(y)| \leq M|\alpha(x) - \alpha(y)|$  ( $\forall x, y \in [a, b]$ ).

**Theorem:** Suppose  $f \in \mathcal{R}([a, b]; \alpha)$  and define  $F(x) = \int_a^x f d\alpha$ . The following hold:

1. If  $\alpha$  is continuous at a point  $x \in [a, b]$ , then  $F$  is continuous at  $x$ .
2. If  $\alpha$  is continuous on  $[a, b]$ , then  $F$  is uniformly continuous at  $[a, b]$ .
3. If  $\alpha$  is Lipschitz on  $[a, b]$  ( $|\alpha(x) - \alpha(y)| \leq K|x - y|$ ,  $\forall x, y \in [a, b]$ ), then  $F$  is Lipschitz.

**Theorem (FTC1):** Assume that  $f$  is continuous at  $x \in [a, b]$  and that  $\alpha$  is differentiable at  $x$ . Let  $F(s) = \int_a^s f d\alpha$ . Then  $F$  is differentiable at  $x$  and  $F'(x) = f(x)\alpha'(x)$ .

*Proof:*

For  $t \in [a, b] \setminus \{x\}$  set  $u = \min\{t, x\}$ ,  $v = \max\{t, x\}$ . Then:

$$\frac{F(t) - F(x)}{t - x} - f(x) \left( \frac{\alpha(t) - \alpha(x)}{t - x} \right) = \frac{1}{v - u} \int_u^v (f - f(x)) d\alpha$$

Hence

$$\begin{aligned} \left| \frac{F(t) - F(x)}{t - x} - f(x)\alpha'(x) \right| &\leq \frac{1}{v - u} \left| \int_u^v (f - f(x)) d\alpha \right| + \left| f(x) \left( \frac{\alpha(t) - \alpha(x)}{t - x} \right) - f(x)\alpha'(x) \right| \\ &\leq \frac{1}{v - u} \int_u^v |f - f(x)| d\alpha + |f(x)| \left| \frac{\alpha(t) - \alpha(x)}{t - x} - \alpha'(x) \right| \end{aligned}$$

Let  $\epsilon > 0$ . Since  $f$  is cont. at  $x$  and  $\alpha$  is diff. at  $x$ , we can find  $\delta > 0$  such that

$$|x - y| < \delta \implies |f(y) - f(x)| < \frac{\epsilon}{2(1 + |\alpha'(x)|)} \wedge \left| \frac{\alpha(y) - \alpha(x)}{y - x} - \alpha'(x) \right| < \min\left\{1, \frac{\epsilon}{2(1 + |f(x)|)}\right\}$$

Then, if  $|x - t| < \delta$  we can deduce:

$$\begin{aligned} \left| \frac{F(t) - F(x)}{t - x} - f(x)\alpha'(x) \right| &\leq \frac{1}{v - u} \int_u^v \frac{\epsilon}{2(1 + |\alpha'(x)|)} d\alpha + |f(x)| \frac{\epsilon}{2(1 + |f(x)|)} \\ &= \frac{\epsilon}{2(1 + |\alpha'(x)|)} \left| \frac{\alpha(t) - \alpha(x)}{t - x} \right| + \frac{\epsilon}{2} \\ &\leq \frac{\epsilon}{2(1 + |\alpha'(x)|)} (1 + |\alpha'(x)|) + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Hence,  $F$  is differentiable at  $x$  and  $F'(x) = f(x)\alpha'(x)$ .

**Theorem (FTC2):** If  $f \in \mathcal{R}([a, b])$  and  $\exists F : [a, b] \rightarrow \mathbb{R}$  that is continuous on  $[a, b]$  and differentiable on  $(a, b)$  and  $\forall x \in (a, b)$ .  $F'(x) = f(x)$ , then  $\int_a^b f dx = F(b) - F(a)$ .

*Proof:*

Let  $\epsilon > 0$ . Then  $\exists P \in \Pi[a, b]$  such that  $|\sum_{i=1}^n f(t_i)\Delta x_i - \int_a^b f dx| < \epsilon$  for any  $t_i \in [x_{i-1}, x_i]$  for  $i \in [n]$ .

The Mean Value Theorem guarantees that  $\exists t_i \in [x_{i-1}, x_i]$  such that  $F(x_i) - F(x_{i-1}) = (x_i - x_{i-1})F'(t_i) = f(t_i)\Delta x_i$ . Hence:

$$\left| F(b) - F(a) - \int_a^b f dx \right| = \left| \sum_{i=1}^n F(x_i) - F(x_{i-1}) - \int_a^b f dx \right| = \left| \sum_{i=1}^n f(t_i)\Delta x_i - \int_a^b f dx \right| < \epsilon$$

Since this is true  $\forall \epsilon > 0$ , we deduce  $\int_a^b f dx = F(b) - F(a)$ .

**Theorem (Integration by Parts):** Suppose  $f, g : [a, b] \rightarrow \mathbb{R}$  are differentiable on  $[a, b]$ , and  $f', g' \in \mathcal{R}([a, b])$ . Then  $\int_a^b f'g dx = f(b)g(b) - f(a)g(a) - \int_a^b fg' dx$ .

*Proof:* Let  $h(x) = f(x)g(x)$ . Then  $h'(x) = f(x)g'(x) + f'(x)g(x)$ , and  $h' \in \mathcal{R}([a, b])$  (since  $f, g$  diff.  $\implies f, g$  cont). Applying FTC2 to  $h$  yields the desired result.

## 7.6 Advanced Results in R-S Integration

**Theorem:** Suppose  $\alpha$  is diff. on  $[a, b]$  and  $\alpha' \in \mathcal{R}([a, b])$ . Then:

$$f \in \mathcal{R}([a, b]; \alpha) \iff fa' \in \mathcal{R}([a, b]) \text{ and } \int_a^b f dx = \int_a^b f \alpha' dx$$

*Proof:* Write  $K = \sup |f|$ . For any  $P \in \Pi[a, b]$  we know for  $t_i, s_i \in [x_{i-1}, x_i]$  for  $i \in [n]$ ,

$$\sum_{i=1}^n |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i \leq U(P, \alpha', x) - L(P, \alpha', x)$$

(since  $\alpha' \in \mathcal{R}([a, b])$ ). The Mean Value Theorem allows us to find  $s_i \in (x_{i-1}, x_i)$  such that  $\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = \alpha'(s_i)(x_i - x_{i-1}) = \alpha'(s_i) \Delta x_i$ . Then for any choice of  $t_i \in [x_{i-1}, x_i]$ , we may estimate

$$\begin{aligned} \left| \sum_{i=1}^n f(t_i) \Delta \alpha_i - \sum_{i=1}^n f(t_i) \alpha'(t_i) \Delta x_i \right| &= \left| \sum_{i=1}^n f(t_i) \alpha'(s_i) \Delta x_i - \sum_{i=1}^n f(t_i) \alpha'(t_i) \Delta x_i \right| \\ &= \left| \sum_{i=1}^n f(t_i) (\alpha'(s_i) - \alpha'(t_i)) \Delta x_i \right| \\ &\leq K \sum_{i=1}^n |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i \\ &\leq K (U(P, \alpha', x) - L(P, \alpha', x)) \end{aligned}$$

Now, (1)  $\implies$  (2):

Note  $f \in \mathcal{R}([a, b]; \alpha)$  and  $\alpha' \in \mathcal{R}([a, b]) \implies \exists P_1, P_2. U(P_1, f, \alpha) - L(P_1, f, \alpha) < \frac{\epsilon}{2}$  and  $U(P_2, \alpha', x) - L(P_2, \alpha', x) < \frac{\epsilon}{2(1+K)}$ . Then

$$\begin{aligned} \left| \sum_{i=1}^n f(t_i) \alpha'(t_i) \Delta x_i - \int_a^b f d\alpha \right| &\leq \left| \sum_{i=1}^n f(t_i) \alpha'(t_i) \Delta x_i - \sum_{i=1}^n f(t_i) \Delta \alpha_i \right| + \left| \sum_{i=1}^n f(t_i) \Delta \alpha_i - \int_a^b f d\alpha \right| \\ &\leq K (U(P, \alpha', x) - L(P, \alpha', x)) + U(P, f, \alpha) - L(P, f, \alpha) \\ &< K \left( \frac{\epsilon}{2(K+1)} \right) + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

and the last theorem in §7.2 guarantees that  $f\alpha' \in \mathcal{R}([a, b])$  and  $\int_a^b f\alpha' dx = \int_a^b f d\alpha$ .

The proof that (2)  $\implies$  (1) is similar (exercise to reader).

**Theorem (Change of Variables):** Let  $A < B$ . Suppose  $\phi : [A, B] \rightarrow [a, b]$  is an increasing bijection. Let  $f \in \mathcal{R}([a, b]; \alpha)$  and define  $g : [A, B] \rightarrow \mathbb{R}$  via  $g = f \circ \phi$  and  $\beta : [A, B] \rightarrow \mathbb{R}$  via  $\beta = \alpha \circ \phi$ . Then  $g \in \mathcal{R}([A, B]; \beta)$  and  $\int_A^B g d\beta = \int_a^b f d\alpha$ .

*Proof:*

Note that  $\phi$  induces a bijection from  $\Pi[A, B]$  to  $\Pi[a, b]$ : if  $P = \{x_0, x_1, \dots, x_n\} \in \Pi[A, B]$  then  $\phi(P) := \{\phi(x_0), \phi(x_1), \dots, \phi(x_n)\} \in \Pi[a, b]$ .

Also,  $\sup\{g(x) \mid x \in [x_{i-1}, x_i]\} = \sup\{(f \circ \phi)(x) \mid x \in [x_{i-1}, x_i]\} = \sup\{f(y) \mid y \in [\phi(x_{i-1}), \phi(x_i)]\}$ . Similarly,  $\inf\{g(x) \mid x \in [x_{i-1}, x_i]\} = \inf\{f(y) \mid y \in [\phi(x_{i-1}), \phi(x_i)]\}$ . Hence, since  $\beta = \alpha \circ \phi$ ,  $U(P, g, \beta) = U(\phi(P), f, \alpha)$  and  $L(P, g, \beta) = L(\phi(P), f, \alpha)$   $\forall P \in \Pi[A, B]$ .

Let  $\epsilon > 0$ . Since  $f \in \mathcal{R}([a, b]; \alpha)$ ,  $\exists Q \in \Pi[a, b]$ .  $U(Q, f, \alpha) - L(Q, f, \alpha) < \frac{\epsilon}{2}$ . Writing  $P = \phi^{-1}(Q)$  satisfies  $U(P, g, \beta) - L(P, g, \beta) = U(Q, f, \alpha) - L(Q, f, \alpha) < \frac{\epsilon}{2}$ . By Riemann's Theorem, we deduce that  $g \in \mathcal{R}([A, B]; \beta)$ .

Moreover:

$$\begin{aligned} \left| \int_A^B g d\beta - \int_a^b f d\alpha \right| &\leq \left| \int_A^B g d\beta - \sum_{i=1}^n g(x_i) \Delta\beta_i \right| + \left| \sum_{i=1}^n g(x_i) \Delta\beta_i - \int_a^b f d\alpha \right| \\ &= \left| \int_A^B g d\beta - \sum_{i=1}^n g(x_i) \Delta\beta_i \right| + \left| \sum_{i=1}^n f(\phi(x_i)) (\alpha(\phi(x_i)) - \alpha(\phi(x_{i-1}))) - \int_a^b f d\alpha \right| \\ &\leq U(P, g, \beta) - L(P, g, \beta) + U(\phi(P), f, \alpha) - L(\phi(P), f, \alpha) \\ &= 2(U(Q, f, \alpha) - L(Q, f, \alpha)) \leq \frac{2\epsilon}{2} = \epsilon \end{aligned}$$

Since  $\epsilon > 0$  was arbitrary, we deduce that  $\int_A^B g d\beta = \int_a^b f d\alpha$ .

We can combine this with the last theorem to deduce the “usual” Riemmanian change of variables:

**Corollary:** Assume  $\alpha(x) = x$  and  $\phi : [A, B] \rightarrow [a, b]$  is increasing, differentiable on  $[A, B]$ , and  $\phi' \in \mathcal{R}([A, B])$ . If  $f \in \mathcal{R}([a, b])$  then  $f \circ \phi \in \mathcal{R}([A, B])$  and  $\int_a^b f(x) dx = \int_A^B f(\phi(x)) \phi'(x) dx$ .

*Proof:*  $\int_A^B g d\beta = \int_A^B g \phi' dx \forall g \in \mathcal{R}([A, B]; \beta)$  with  $\beta = \alpha \circ \phi = \phi$ .

## 8 Sequences and Series of Functions

### 8.1 Convergence

**Definition:** Let  $\emptyset \neq E \subseteq \mathbb{R}$ . Suppose  $\forall n \geq l \in \mathbb{Z}$ .  $f_n : E \rightarrow \mathbb{R}$ . Then  $\{f_n\}_{n=l}^\infty$  is a *sequence of functions* on  $E$ .

1. If  $\exists f : E \rightarrow \mathbb{R}$  such that  $\forall x \in E$ .  $f_n(x) \rightarrow f(x)$  as  $n \rightarrow \infty$ , then  $f_n$  converges to  $f$  *pointwise* and write  $f_n \rightarrow f$  as  $n \rightarrow \infty$ .
2. If  $\exists f : E \rightarrow \mathbb{R}$  such that  $\forall n \geq l$ .  $\{|f_n(x) - f(x)| \mid x \in E\}$  is bounded, and  $\lim_{n \rightarrow \infty} \sup\{|f_n(x) - f(x)|\} = 0$ , then  $f_n$  converges to  $f$  *uniformly* on  $E$ , and we write  $f_n \rightrightarrows f$  as  $n \rightarrow \infty$ .

**Lemma:**

1. Let  $f_n, f : E \rightarrow \mathbb{R}$ . Then  $f_n \rightrightarrows f \iff \forall \epsilon > 0. \exists N \geq l. n \geq N \implies |f_n(x) - f(x)| < \epsilon$  for every  $x \in E$ .
2. If  $f_n \rightrightarrows f$  then  $f_n \rightarrow f$ . The converse fails.

**Examples:**

1.  $E = (0, \frac{1}{2})$ ,  $f_n(x) = x^n$ .  
 $f_n \rightarrow 0$  pointwise is trivial, and  $|f_n(x)| = |x^n| = |x|^n \leq \frac{1}{2^n}$ . So  $\sup\{|f_n(x)|\} \leq \frac{1}{2^n}$   
and so  $\lim_{n \rightarrow \infty} \sup\{|f_n - 0|\} = 0$ . Thus,  $f_n \Rightarrow 0$ .
2.  $E = \mathbb{R}$ ,  $f_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in [n, n+1] \\ 0 & \text{otherwise} \end{cases}$   
 $\sup\{|f_n(x)|\} = \frac{1}{n} \rightarrow 0$ , so  $f_n \Rightarrow 0$  as  $n \rightarrow \infty$ .
3.  $E = \mathbb{R}$ ,  $f_n(x) = \begin{cases} n & \text{if } x \in [n, n+1] \\ 0 & \text{otherwise} \end{cases}$   
Fix  $x \in \mathbb{R}$ . For sufficiently large  $N$ ,  $x < N$ , so  $x \notin [n, n+1]$  for  $n \geq N$ . Hence  
 $f_n(x) = 0$  for  $n \geq N$  and  $f_n \rightarrow 0$  pointwise on  $\mathbb{R}$ . However,  $\sup\{|f_n(x)|\} = n \not\rightarrow 0$   
and so it does not converge uniformly.

### 8.1.1 Big Questions

1. If  $\forall n$ .  $f_n : E \rightarrow \mathbb{R}$  is continuous on  $E$ , and  $f_n$  converges to  $f : E \rightarrow \mathbb{R}$ , is it true that  $f$  is continuous?
2. Same question for differentiability? If so, is  $f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$ ?
3. If  $\forall n$ .  $\{f_n\}_{n=l}^\infty \subseteq \mathcal{R}([a, b]; \alpha)$  and  $f_n$  converges to  $f$ , is  $f \in \mathcal{R}([a, b]; \alpha)$ ? If so, is  $\int_a^b f d\alpha = \lim_{n \rightarrow \infty} \int_a^b f_n d\alpha$ ?