State Estimation - Assignment 1

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August 18, 2023

1 Question1

x is a random variable of length K:

$$x = N(0,1)$$

a) What type of random variable is the following random variable?

$$y = x^T \times x$$

Sum of k squared random variables gives us a Chi-square distribution:

$y = x_1^2 + x_2^2 + \cdots + x_k^2 = \sum_{i=1}^{k} x_i^2 \Rightarrow$ sum of k random variables

b) Calculate the mean and variance of y.

$E\left[x_i^2\right] = \int_{-\infty}^{\infty} x_i^2 f\left(x_i\right) = 1 \rightarrow$ Expected value of each x_i with $N(0,1)$

$E[y] = \sum_{i=1}^{k} 1 = k = y_m \longrightarrow$ Mean

$E\left[\left(y - y_m\right)^2\right] = E\left[y^2 - 2yy_m + y_m^2\right] = E\left[y^2\right] - 2E\left[yy_m\right] + y_m^2$

$= E\left[y^2\right] - 2k\underbrace{E[y]}_{k} + k^2 = E\left[y^2\right] - k^2 = \int y^2 f(y) - k^2 \rightarrow$ *(I)*

$E\left[y^2\right] = E\left[\left(x_1^2 + \cdots + x_k^2\right)^2\right] = E\left[\sum_{i=1}^{k} x_i^4\right] + E\left[\sum_{i=1}^{k(k-1)} x_i^2 x_{k-i}^2\right]$

$= \sum_{i=1}^{k} E[x_i^4] + \sum_{i=1}^{k(k-1)} E[x_i^2 x_{k-i}^2] = 3k + k^2 - k = 2k + k^2 \rightarrow$ *(II)*

where Expectations are calculated as:

Using independence assumption $\rightarrow E[x_i^2 x_{k-i}^2] = E[x_i^2]E[x_{k-i}^2] = 1 \times 1 = 1$

Using scalar form of 2.42 in Barfoot $\rightarrow E[x_i^4] = 3 \times \sigma^2 = 3 \times 1 = 3$

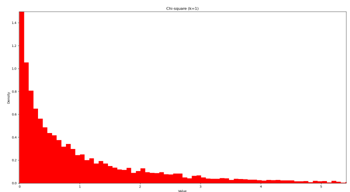
Using Equations I & II:

$$E\left[(y - y_m)^2\right] = 2k + k^2 - k^2 = 2k \longrightarrow \ \ Variance$$

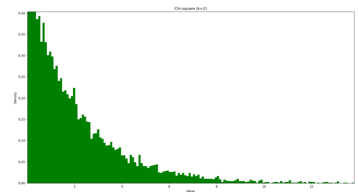
c) Using Python, plot the PDF of y for K=1, 2, 3, 10, 100.

I have plotted using two methods:

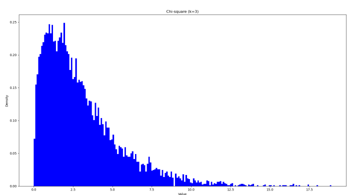
a) Sum of standard normal distributions with bar charts as in Figure 1.
b) Scipy library which plots the distribution using the corresponding equations for Chi-square distribution as in Figure 2.



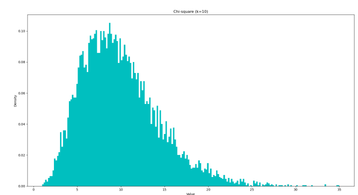
(a) k=1



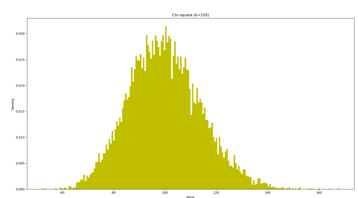
(b) k=2



(c) k=3



(d) When k=10



(e) k=100

Figure 1: Bar chart of Chi-squared distribution for different DOFs

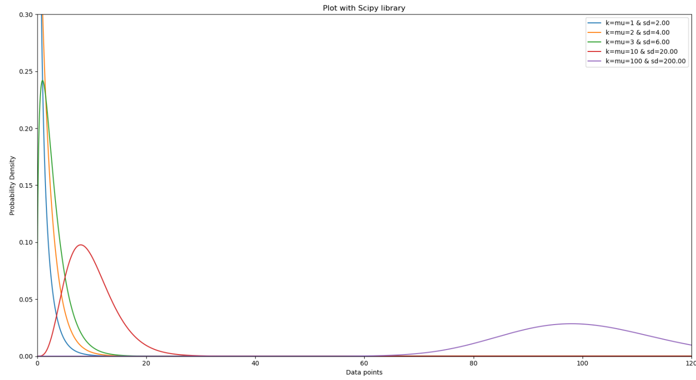


Figure 2: Comparing plots of Chi-square distribution for different values of k

2 Question 2

x is a random variable of length N:

$$x = N(\mu, \Sigma)$$

a) Assume x is transformed linearly, i.e., $y = Ax$, where A is an $N \times N$ matrix. Calculate the mean and covariance of y. Show the derivations.

$$\mu_y = E[Ax] = E\begin{bmatrix} \sum_{j=1}^{N} a_{ij} x_j \\ \sum_{i,j=1}^{N} a_{ij} x_j \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{N} a_{ij} E\left[x_j\right] \\ \vdots \\ \sum_{j=1} a_{ij} E\left[x_j\right] \end{bmatrix} = AE[x] = A\mu_x$$

$$\sum_{yy} = E\left[\left(y - \mu_y\right)\left(y - \mu_y\right)^{\top}\right] = E\left[\left(Ax - A\mu_x\right)\left(x^{\top} A^{\top} - \mu_x A^{\top}\right)\right]$$

$$= AE\left[\left(x - \mu_x\right)\left(x - \mu_x\right)^{\top}\right] A^{\top} = A\Sigma A^{\top}$$

b) Repeat a), when $y = A_1 x + A_2 x$.

$$E\left[A_1 x + A_2 x\right] = A_1 E[x] + A_2 E[x] = A_1 \mu_x + A_2 \mu_x$$

$$E\left[\left(A_1 + A_2\right) x - \left(A_1 + A_2\right) \mu_x\right)\left(\left(A_1 + A_2\right) x - \left(A_1 + A_2\right) \mu_x\right)^{\top}\right]$$

$$= \left(A_1 + A_2\right) E\left[\left(x - \mu_x\right)\left(x - \mu_x\right)^{T}\right]\left(A_1 + A_2\right)^{T} = \left(A_1 + A_2\right) \sum_{yy} \left(A_1 + A_2\right)^{T}$$

c) If x is transformed by a nonlinear differentiable function, i.e. $y = f(x)$, compute the covariance matrix of y. Show the derivation.

Using Taylor expansion, the linear approximation is:

$$f(x) \approx \mu_y + G(x - \mu_x),$$

where G is defined as:

$$G = \left.\frac{\partial f(x)}{\partial x}\right|_{x=\mu_x}$$

The mean is calculated as follows:

$$\mu_y = f(\mu_x)$$

Hence, covariance can be obtained as:

$$E\left[(y-\mu_y)(y-\mu_y))^\top\right]$$

$$= E\left[(\mu_y + G\,(x-\mu_x) - \mu_y)(\mu_y + G(x-\mu_x) - \mu_y)^\top\right]$$

$$= E\left[G\,(x-\mu_x)\,(G\,(x-\mu_x))^T\right]$$

$$= E\left[G\,(x-\mu_x)\,(x-\mu_x)^\top\,G^\top\right]$$

$$= G\Sigma G^\top$$

d) Apply c) when:

$$x = \begin{bmatrix} \rho \\ \theta \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sigma^2_{\rho\rho} & \sigma^2_{\rho\theta} \\ \sigma^2_{\rho\theta} & \sigma^2_{\theta\theta} \end{bmatrix}$$

$$y = \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \end{bmatrix}$$

Compute the covariance of y analytically. This models how range-bearing measurements in the polar coordinate frame are converted to a Cartesian coordinate frame.

$$E[J(x)x] = \begin{bmatrix} E(\rho\cos\theta - \rho\theta\sin\theta) \\ \\ E(\rho\sin\theta + \rho\theta\cos\theta) \end{bmatrix}$$

Assuming Local linearity:

$y = Ax \Rightarrow yy^T = (Ax)(Ax)^T = Axx^TA \Rightarrow E\left[yy^T\right] = AE\left[xx^T\right]A \Rightarrow \Sigma_y = A\Sigma_x A^\top \overset{A=J}{\Longrightarrow}$ $\Sigma_y = J\Sigma_x J^T$

$$\Sigma_y = \begin{bmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{bmatrix} \begin{bmatrix} \sigma^2_{\rho\rho} & \sigma^2_{\rho\theta} \\ \sigma^2_{\rho\rho} & \sigma^2_{\theta\theta} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\rho\sin\theta & \rho\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^2_{\rho\rho}\cos\theta - \sigma^2_{\rho\rho}\rho\sin\theta & \sigma^2_{\rho\theta}\cos\theta - \sigma^2_{\theta\theta}\rho\sin\theta \\ \sigma^2_{\rho\rho}\sin\theta + \sigma^2_{\rho\theta}\rho\cos\theta & \sigma^2_{\rho}\sin\theta + \sigma^2_{\theta\theta}\rho\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\rho\sin\theta & \rho\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^2_{\rho\rho}\cos^2\theta - \sigma^2_{\rho\theta}\sin 2\theta + \sigma^2_{\theta\theta}\rho^2\sin^2\theta & \sigma^2_{\rho\rho}\sin\theta\cos\theta + \sigma^2_{\rho\rho}\rho\cos 2\theta - \sigma^2_{\theta\theta}\rho^2\sin\theta\cos\theta \\ \\ \sigma^2_{\rho p}\sin\theta\cos\theta + \sigma^2_{\rho\theta}\rho\cos 2\theta - \sigma^2_{\theta}\rho^2\sin\theta\cos\theta & \sigma^2_{\rho\rho}\sin\theta + \sigma^2_{\rho\theta}\rho\sin 2\theta + \sigma^2_{\theta\theta}\rho^2\cos^2\theta \end{bmatrix}$$

e) Simulate d) using the Monte Carlo simulation, i.e. assume

$$x = \begin{bmatrix} 1m \\ 0.5^\circ \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.005 \end{bmatrix}$$

Sample 1000 points from this distribution and plot the transformed results on x-y coordinates. Plot the uncertainty ellipse, calculated from part d). Overlay the ellipse on the point samples.

The process begins by generating a multivariate normal distribution comprising 1000 samples. The resulting scatter plot in Figure 3 offers a visual representation of the distribution's pattern. Employing linear algebra techniques, including eigenvalues and eigenvectors, unveils the distribution's shape and orientation. Subsequently, an uncertainty ellipse is constructed based on these mathematical properties, effectively encapsulating the data's uncertainty. The ellipse is then integrated into the scatter plot, facilitating a direct comparison between the distribution and its associated uncertainty. By enhancing the visualization with continuous color mapping to represent the distribution's properties, the plot becomes a comprehensive tool for conveying insights. Axes labels, grid lines, and a legend are incorporated for clarity.
The procedure employed for constructing the uncertainty ellipse involved leveraging a 2 degrees of freedom (DOF) chi-square approach. This method is encapsulated within the equation $(\rho/\sigma_\rho)^2 + (\theta/\sigma_\theta)^2 = s$, where ρ and θ represent variables, and σ_ρ and σ_θ are their corresponding standard deviations. By consulting a dedicated reference table, the parameter s was determined. It's noteworthy that a confidence level of 95% was deliberately adopted for the data analysis, underpinning the reliability of the derived uncertainty ellipse.

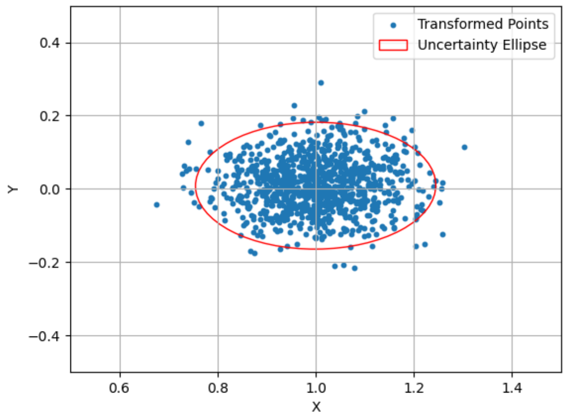


Figure 3: Uncertainty ellipse

2.1 Extension

f) Repeat part e), for the following values:

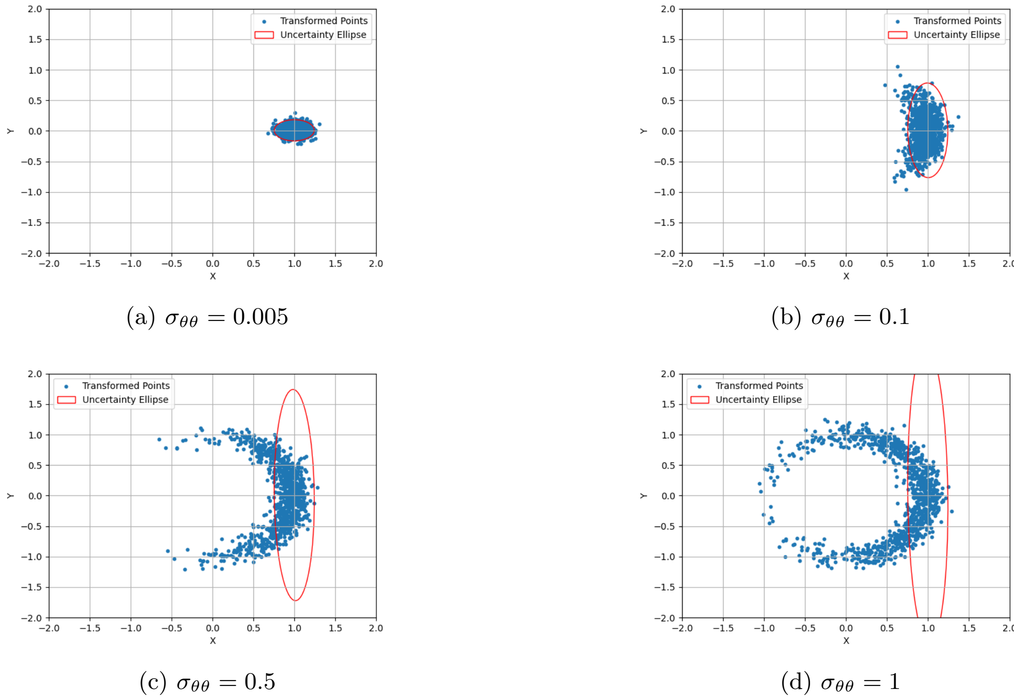
$$\mathbf{x} = \begin{bmatrix} 1\text{ m} \\ 0.5^\circ \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.005 \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} 1\text{ m} \\ 0.5^\circ \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} 1\text{ m} \\ 0.5^\circ \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} 1\text{ m} \\ 0.5^\circ \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix}.$$

For different values of $\sigma_{\theta\theta}$, the plots look like Figure 4. When $\sigma_{\theta\theta}$ is small, it indicates that the angular component of the state vector (the angle θ) is known with higher precision. A smaller $\sigma_{\theta\theta}$ means that the angular uncertainty is smaller, which results in a narrower uncertainty ellipse in the Cartesian coordinate system. The narrower ellipse implies that the uncertainty in both x and y directions is reduced. The ellipse will be elongated in the direction determined by the transformation matrix J, which is influenced by θ.
When $\sigma_{\theta\theta}$ is large, it indicates that the angular component of the state vector is less precisely known. A larger $\sigma_{\theta\theta}$ results in a wider uncertainty ellipse in the Cartesian coordinate system. The wider ellipse implies that the uncertainty in both x and y directions is larger. Again, the orientation of the ellipse is influenced by the transformation matrix J.
In summary, smaller values of $\sigma_{\theta\theta}$ lead to a narrower and more aligned uncertainty ellipse, while larger values result in a wider and less aligned uncertainty ellipse. The relationship between the angular uncertainty and the resulting uncertainty ellipse in Cartesian coordinates is influenced by the transformation matrix and the geometry of the transformation itself.



(a) $\sigma_{\theta\theta} = 0.005$

(b) $\sigma_{\theta\theta} = 0.1$

(c) $\sigma_{\theta\theta} = 0.5$

(d) $\sigma_{\theta\theta} = 1$

Figure 4: Uncertainty ellipse for different values

References

[1] Thrun, Sebastian. "Probabilistic robotics." Communications of the ACM 45, no. 3 (2002): 52-57.

[2] Barfoot, Timothy D. State estimation for robotics. Cambridge University Press, 2017.