Lie Algebra - Assignment 7

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All notations and equations are derived from reference [1].

Identities

$$[\mathbf{a}]_\times \mathbf{b} = -[\mathbf{b}]_\times \mathbf{a}$$
$$[\mathbf{A}\mathbf{b}]_\times = \mathbf{A}[\mathbf{b}]_\times \mathbf{A}^\top$$

Question 1: Derivatives

Left and Right derivatives on Lie Groups. For the following function

$$f : SO(3) \to \mathbb{R}^3; \quad f(\mathbf{R}, \mathbf{p}) = \mathbf{R}\mathbf{p}$$

calculate the left and right derivatives of f with respect to $\mathbf{R}$, by applying the definitions of the left and right derivatives:

$$\frac{{}^R D f(\mathbf{R}, \mathbf{p})}{D\mathbf{R}}$$
$$\frac{{}^L D f(\mathbf{R}, \mathbf{p})}{D\mathbf{R}}$$

Answer:
Left derivative of f with respect to R is defined as:

$$\frac{{}^L D f(\mathbf{R}, \mathbf{p})}{D\mathbf{R}} = \frac{{}^L D \mathbf{R}\mathbf{p}}{D\mathbf{R}} = \lim_{\boldsymbol{\theta} \to 0} \frac{(\boldsymbol{\theta} \oplus \boldsymbol{R})\mathbf{p} \ominus \mathbf{R}\mathbf{p}}{\boldsymbol{\theta}} = \lim_{\boldsymbol{\theta} \to 0} \frac{\operatorname{Exp}(\boldsymbol{\theta})\mathbf{R}\mathbf{p} - \mathbf{R}\mathbf{p}}{\boldsymbol{\theta}}$$

Using the distributive property of matrix multiplication:

$$\lim_{\boldsymbol{\theta} \to 0} \frac{(\mathbf{I} + [\boldsymbol{\theta}]_\times)\,\mathbf{R}\mathbf{p} - \mathbf{R}\mathbf{p}}{\boldsymbol{\theta}} = \lim_{\boldsymbol{\theta} \to 0} \frac{[\boldsymbol{\theta}]_\times \mathbf{R}\mathbf{p}}{\boldsymbol{\theta}} = \lim_{\boldsymbol{\theta} \to 0} \frac{-\boldsymbol{\theta}[\mathbf{R}\mathbf{p}]_\times}{\boldsymbol{\theta}} = -[\mathbf{R}\mathbf{p}]_\times = -\mathbf{R}[\mathbf{p}]_\times \mathbf{R}^\top \in \mathbb{R}^{3\times 3}$$

Right derivative of f with respect to R is defined as:

$$\frac{{}^R D f(\mathbf{R}, \mathbf{p})}{D\mathbf{R}} = \frac{{}^R D \mathbf{R}\mathbf{p}}{D\mathbf{R}} = \lim_{\boldsymbol{\theta} \to 0} \frac{(\mathbf{R} \oplus \boldsymbol{\theta})\mathbf{p} \ominus \mathbf{R}\mathbf{p}}{\boldsymbol{\theta}} = \lim_{\boldsymbol{\theta} \to 0} \frac{\mathbf{R}\operatorname{Exp}(\boldsymbol{\theta})\mathbf{p} - \mathbf{R}\mathbf{p}}{\boldsymbol{\theta}}$$

Expanding the first term using the distributive property of matrix multiplication, we get:

$$\lim_{\boldsymbol{\theta} \to 0} \frac{\mathbf{R}\,(\mathbf{I} + [\boldsymbol{\theta}]_\times)\,\mathbf{p} - \mathbf{R}\mathbf{p}}{\boldsymbol{\theta}} = \lim_{\boldsymbol{\theta} \to 0} \frac{\mathbf{R}[\boldsymbol{\theta}]_\times \mathbf{p}}{\boldsymbol{\theta}} = \lim_{\boldsymbol{\theta} \to 0} \frac{-\mathbf{R}[\mathbf{p}]_\times \boldsymbol{\theta}}{\boldsymbol{\theta}} = -\mathbf{R}[\mathbf{p}]_\times \in \mathbb{R}^{3\times 3}$$

Question 2: Jacobian
Given a Lie group $\mathcal{M}$ with a composition operation $\circ$, and elements $\mathcal{X}, \mathcal{Y} \in \mathcal{M}$, calculate the derivative of $\mathcal{X} \circ \mathcal{Y}$ with respect to $\mathcal{Y}$, i.e.

$$\frac{{}^{\mathcal{Y}}D\mathcal{X} \circ \mathcal{Y}}{D\mathcal{Y}}$$

Answer:
Using Jacobian law for composition, we have:

$$\mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\circ\mathcal{Y}} \triangleq \frac{{}^{\mathcal{Y}}D\mathcal{X} \circ \mathcal{Y}}{D\mathcal{Y}} \in \mathbb{R}^{m\times m}$$

We also know that:

$$\mathcal{X} = Exp(\tau)$$

$$\tau = Log(\mathcal{X})$$

Using above equations the the Jacobian is calculated as follows:

$$\mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\circ\mathcal{Y}} = \frac{{}^{\mathcal{Y}}D\mathcal{X} \circ \mathcal{Y}}{D\mathcal{Y}} = \lim_{\boldsymbol{\tau}\to 0} \frac{\mathcal{X} \circ (\mathcal{Y} \oplus \boldsymbol{\tau}) \ominus \mathcal{X} \circ \mathcal{Y}}{\boldsymbol{\tau}}$$

$$\lim_{\boldsymbol{\tau}\to 0} \frac{\log\left[(\mathcal{X}\mathcal{Y})^{-1}(\mathcal{Y}\,\mathrm{Exp}(\boldsymbol{\tau})\mathcal{X})\right]}{\boldsymbol{\tau}} = \lim_{\boldsymbol{\tau}\to 0} \frac{\log\left[\mathcal{X}^{-1}\,\mathrm{Exp}(\boldsymbol{\tau})\mathcal{X}\right]}{\boldsymbol{\tau}}$$

$$= \lim_{\boldsymbol{\tau}\to 0} \frac{\log\left[\mathcal{X}^{-1}\mathcal{X}\mathcal{X}\right]}{\boldsymbol{\tau}} = \lim_{\boldsymbol{\tau}\to 0} \frac{\log[\mathcal{X}]}{\boldsymbol{\tau}} = \lim_{\boldsymbol{\tau}\to 0} \frac{\boldsymbol{\tau}}{\boldsymbol{\tau}} = \mathbf{I}$$

Question 3: Adjoint Matrix Properties.
Given the Lie group of $M = SE(3)$ with the composition operation $\circ$, and elements $\mathcal{X}, \mathcal{Y} \in M$, show that

$$\mathbf{Ad}_{\mathcal{X}}\mathbf{Ad}_{\mathcal{Y}} = \mathbf{Ad}_{\mathcal{X}\circ\mathcal{Y}}$$

$$\mathbf{Ad}_{\mathcal{X}^{-1}} = \mathbf{Ad}_{\mathcal{X}}{}^{-1}$$

Answer:

Part a
Using adjoint formula for composition we have:

$$\mathbf{Ad}_{\mathcal{X}\circ\mathcal{Y}} = \left(\mathbf{M}_{\mathcal{X}\circ\mathcal{Y}}\boldsymbol{\tau}^{\wedge}\mathbf{M}_{\mathcal{X}\circ\mathcal{Y}}^{-1}\right)^{\vee}$$

$$\mathbf{M}_{\mathcal{X}\circ\mathcal{Y}} = \mathbf{M}_{\mathcal{X}}\mathbf{M}_{\mathcal{Y}} = \begin{bmatrix} \mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}} & \mathbf{t}_{\mathcal{X}} + \mathbf{t}_{\mathcal{Y}}\mathbf{R}_{\mathcal{X}} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$\mathbf{M}_{\mathcal{X}\circ\mathcal{Y}}^{-1} = \mathbf{M}_{\mathcal{X}}^{-1}\mathbf{M}_{\mathcal{Y}}{}^{-1} = \begin{bmatrix} \mathbf{R}_{\mathcal{X}}{}^{\top} & -\mathbf{R}_{\mathcal{X}}{}^{\top}\mathbf{t}_{\mathcal{X}} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{R}_{\mathcal{Y}}{}^{\top} & -\mathbf{R}_{\mathcal{Y}}{}^{\top}\mathbf{t}_{\mathcal{Y}} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R}_{\mathcal{X}}{}^{\top}\mathbf{R}_{\mathcal{Y}}{}^{\top} & -\mathbf{R}_{\mathcal{X}}{}^{\top}\mathbf{R}_{\mathcal{Y}}{}^{\top}\mathbf{t}_{\mathcal{Y}} - \mathbf{R}_{\mathcal{X}}{}^{\top}\mathbf{t}_{\mathcal{X}} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$\mathbf{Ad}_{\mathcal{X}\circ\mathcal{Y}} = \left(\begin{bmatrix} \mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}} & \mathbf{t}_{\mathcal{X}} + \mathbf{t}_{\mathcal{Y}}\mathbf{R}_{\mathcal{X}} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} [\boldsymbol{\theta}]_{\times} & \boldsymbol{\rho} \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{R}_{\mathcal{X}}{}^{\top}\mathbf{R}_{\mathcal{Y}}{}^{\top} & -\mathbf{R}_{\mathcal{X}}{}^{\top}\mathbf{R}_{\mathcal{Y}}{}^{\top}\mathbf{t}_{\mathcal{Y}} - \mathbf{R}_{\mathcal{X}}{}^{\top}\mathbf{t}_{\mathcal{X}} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \right)^{\vee}$$

$$= \left(\begin{bmatrix} \mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}}[\boldsymbol{\theta}]_{\times}\mathbf{R}_{\mathcal{X}}{}^{\top}\mathbf{R}_{\mathcal{Y}}{}^{\top} & -\mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}}[\boldsymbol{\theta}]_{\times}\mathbf{R}_{\mathcal{X}}{}^{\top}\mathbf{R}_{\mathcal{Y}}{}^{\top}\mathbf{t}_{\mathcal{Y}} - \mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}}[\boldsymbol{\theta}]_{\times}\mathbf{R}_{\mathcal{X}}{}^{\top}\mathbf{t}_{\mathcal{X}} + \boldsymbol{\rho}\mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right)^{\vee}$$

$$= \left(\begin{bmatrix} [\mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}}\boldsymbol{\theta}]_{\times} & -[\mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}}\boldsymbol{\theta}]_{\times}\mathbf{t}_{\mathcal{Y}} - \mathbf{R}_{\mathcal{Y}}[\mathbf{R}_{\mathcal{X}}\boldsymbol{\theta}]_{\times}\mathbf{t}_{\mathcal{X}} + \boldsymbol{\rho}\mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right)^{\vee}$$

$$= \left(\begin{bmatrix} [\mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}}\boldsymbol{\theta}]_{\times} & +\mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}}\boldsymbol{\theta}[\mathbf{t}_{\mathcal{Y}}]_{\times} + \mathbf{R}_{\mathcal{Y}}\mathbf{R}_{\mathcal{X}}\boldsymbol{\theta}[\mathbf{t}_{\mathcal{X}}]_{\times} + \boldsymbol{\rho}\mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right)^{\vee}$$

$$= \begin{bmatrix} [\mathbf{t}_{\mathcal{Y}}]_{\times}\mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}}\boldsymbol{\theta} + [\mathbf{t}_{\mathcal{X}}]_{\times}\mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}}\boldsymbol{\theta} + \mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}}\boldsymbol{\rho} \\ \mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}}\boldsymbol{\theta} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}} & [\mathbf{t}_{\mathcal{Y}}]_{\times}\mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}} + [\mathbf{t}_{\mathcal{X}}]_{\times}\mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}} \\ \mathbf{0} & \mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix}$$

We know that $[\mathbf{R}\boldsymbol{\theta}]_{\times} = \mathbf{R}[\boldsymbol{\theta}]_{\times}\mathbf{R}^{\top}$ and $[\mathbf{a}]_{\times}\mathbf{b} = -[\mathbf{b}]_{\times}\mathrm{a}$. So the adjoint matrix is calculated as follows:

$$\begin{bmatrix} \mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}} & [\mathbf{t}_{\mathcal{Y}}]_{\times}\mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}} + [\mathbf{t}_{\mathcal{X}}]_{\times}\mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}} \\ \mathbf{0} & \mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}} \end{bmatrix}$$

Calculating the other side of the equation yields:

$$\mathbf{Ad}_{\mathcal{X}}\mathbf{Ad}_{\mathcal{Y}} = \begin{bmatrix} \mathbf{R}_{\mathcal{X}} & [\mathbf{t}_{\mathcal{X}}]_{\times}\mathbf{R}_{\mathcal{X}} \\ 0 & \mathbf{R}_{\mathcal{X}} \end{bmatrix} \begin{bmatrix} \mathbf{R}_{\mathcal{Y}} & [\mathbf{t}_{\mathcal{Y}}]_{\times}\mathbf{R}_{\mathcal{Y}} \\ 0 & \mathbf{R}_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} \mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}} & [\mathbf{t}_{\mathcal{Y}}]_{\times}\mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}} + [\mathbf{t}_{\mathcal{X}}]_{\times}\mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}} \\ \mathbf{0} & \mathbf{R}_{\mathcal{X}}\mathbf{R}_{\mathcal{Y}} \end{bmatrix}$$

Thus, we have the same value for the left and right sides of the equation.

Part b

$$\mathbf{Ad}_{\mathcal{X}} = \left[\mathcal{X}\tau^{\wedge}\mathcal{X}^{-1}\right]^{\vee}$$

$$\mathbf{Ad}_{\mathcal{X}}^{-1} = \left[\left[\mathcal{X}\tau^{\wedge}\mathcal{X}^{-1}\right]^{\vee}\right]^{-1} = \left[\left[\mathcal{X}\tau^{\wedge}\mathcal{X}^{-1}\right]^{-1}\right]^{\vee} = \left[\mathcal{X}^{-1}\tau^{\wedge}\mathcal{X}\right]^{\vee}$$

$$\mathbf{Ad}_{\mathcal{X}^{-1}} = \left[\mathcal{X}^{-1}\tau^{\wedge}\mathcal{X}\right]^{\vee}$$

So, It can be proved that:

$$\mathbf{Ad}_{\mathcal{X}^{-1}} = \mathbf{Ad}_{\mathcal{X}}{}^{-1}$$

References

[1] Joan Sola, Jeremie Deray, and Dinesh Atchuthan. A micro lie theory for state estimation in robotics. *arXiv preprint arXiv:1812.01537*, 2018.