CIS 5800 Machine Perception Homework-1
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1.1

Using the perspective projection equations:

$$x = f \times \frac{X}{Z} \qquad y = f \times \frac{Y}{Z}$$

Where:

- $(X, Y, Z) = (300, 600, 1200)$ are the camera coordinates
- $(x, y) = (55, 110)$ are the image coordinates

Solving for focal length f:
From the x-coordinate equation:

$$f = x \times \frac{Z}{X} = 55 \times \frac{1200}{300} = 55 \times 4 = 220$$

From the y-coordinate equation:

$$f = y \times \frac{Z}{Y} = 110 \times \frac{1200}{600} = 110 \times 2 = 220$$

Both equations give the same result, confirming that $f = 220$ pixels.

1.2

Yes, you can determine whether the tree will hit the projection center (camera) when it falls, even without knowing the tree's height.

- Tree bottom projects to $B = (0, y_1)$
- Tree top projects to $T = (0, y_2)$
- Focal length $f = 1$
- Both points have x-coordinate $= 0$, so the tree lies on the optical axis plane x=0.
- Y_{bottom} and Y_{top} are the actual heights of the tree's bottom and top in camera coordinates.
- Z_{tree} is the distance from the camera to the tree along the Z-axis.
- Tree height: $h = Y_{\text{top}} - Y_{\text{bottom}}$
- Distance from tree base to camera: $d = \sqrt{Y_{\text{bottom}}^2 + Z_{\text{tree}}^2}$
- The tree will hit the camera if its height h is greater than or equal to the distance d.

From perspective projection (with $f = 1$ and $x = 0$ for both points):

$$\begin{aligned} \text{Bottom:} \quad y_1 &= \frac{Y_{\text{bottom}}}{Z_{\text{tree}}} \implies Y_{\text{bottom}} = y_1 Z_{\text{tree}} \\ \text{Top:} \quad y_2 &= \frac{Y_{\text{top}}}{Z_{\text{tree}}} \implies Y_{\text{top}} = y_2 Z_{\text{tree}} \end{aligned}$$

Tree height:

$$h = Y_{\text{top}} - Y_{\text{bottom}} = (y_2 - y_1) Z_{\text{tree}}$$

Distance from tree base to camera:

$$d = \sqrt{Y_{\text{bottom}}^2 + Z_{\text{tree}}^2} = Z_{\text{tree}} \sqrt{y_1^2 + 1}$$

Collision condition: The tree hits the camera if its height $\geq$ distance to camera:

$$h \geq d \implies (y_2 - y_1)Z_{\text{tree}} \geq Z_{\text{tree}}\sqrt{y_1^2 + 1}$$

$$y_2 - y_1 \geq \sqrt{y_1^2 + 1}$$

Hence!
The tree will hit the camera if and only if:

$$y_2 - y_1 \geq \sqrt{y_1^2 + 1}$$

2.1

- If $Z \neq 0$: $[X, Y, Z]$ represents an affine point.

$$(x, y) = \left(\frac{X}{Z}, \frac{Y}{Z}\right)$$

- If $Z = 0$: $[X, Y, 0]$ represents a point at infinity in the direction (X, Y).
  - All nonzero scalar multiples are the same projective point.
  - $[0, 0, 0]$ is not a valid homogeneous coordinate.
- Points at infinity correspond to where parallel lines meet (vanishing points in computer vision).

2.2

For lines through pairs of points in $\mathbb{P}^2$, we use the cross product formula: If $P_1 = [x_1, y_1, z_1]$ and $P_2 = [x_2, y_2, z_2]$, then the line $L = P_1 \times P_2 = [y_1z_2 - z_1y_2,\ z_1x_2 - x_1z_2,\ x_1y_2 - y_1x_2]$.

(a) Points: [−2, 5, 3] and [1, 3, 4]

$$\begin{aligned} \text{First component:} &\quad 5 \times 4 - 3 \times 3 = 20 - 9 = 11 \\ \text{Second component:} &\quad 3 \times 1 - (-2) \times 4 = 3 + 8 = 11 \\ \text{Third component:} &\quad (-2) \times 3 - 5 \times 1 = -6 - 5 = -11 \end{aligned}$$

Line equation: $11x + 11y - 11z = 0$
$x + y - z = 0$

(b) Points: $[a,\ 0,\ b]$ and $[0,\ c,\ b]$

$$\begin{aligned} \text{First component:} \quad & 0 \times b - b \times c = -bc \\ \text{Second component:} \quad & b \times 0 - a \times b = -ab \\ \text{Third component:} \quad & a \times c - 0 \times 0 = ac \end{aligned}$$

Line equation: $-bcx - aby + acz = 0$
Or: $bcx + aby - acz = 0$

(c) Points: $[a,\ 0,\ 0]$ and $[0,\ 0,\ a]$

$$\begin{aligned} \text{First component:} \quad & 0 \times a - 0 \times 0 = 0 \\ \text{Second component:} \quad & 0 \times 0 - a \times a = -a^2 \\ \text{Third component:} \quad & a \times 0 - 0 \times 0 = 0 \end{aligned}$$

Line equation: $-a^2 y = 0$
Or: $y = 0$

2.3

(a) $3x - y + 2w = 0$ and $x + 5y - w = 0$

Line vectors,

$$\mathbf{l}_1 = [3, -1, 2]^T$$
$$\mathbf{l}_2 = [1, 5, -1]^T$$

Next, we compute their cross product to find the intersection point $\mathbf{p}$:

$$\begin{aligned} \mathbf{p} = \mathbf{l}_1 \times \mathbf{l}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 1 & 5 & -1 \end{vmatrix} \\ &= \mathbf{i}((-1)(-1) - (2)(5)) - \mathbf{j}((3)(-1) - (2)(1)) + \mathbf{k}((3)(5) - (-1)(1)) \\ &= \mathbf{i}(1 - 10) - \mathbf{j}(-3 - 2) + \mathbf{k}(15 + 1) \\ &= [-9, 5, 16]^T \end{aligned}$$

The point of intersection is $[\mathbf{-9}, \mathbf{5}, \mathbf{16}]$.

(b) $2x - 6w = 0$ and $5x - 2y = 0$

$2x + 0y - 6w = 0 \implies \mathbf{l}_1 = [2, 0, -6]^T,$ which can be simplified to $[1, 0, -3]^T$.
$5x - 2y + 0w = 0 \implies \mathbf{l}_2 = [5, -2, 0]^T$.

Now, we compute the cross product using the simplified vector for $\mathbf{l}_1$:

$$\begin{aligned}
\mathbf{p} = \mathbf{l}_1 \times \mathbf{l}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -3 \\ 5 & -2 & 0 \end{vmatrix} \\
&= \mathbf{i}((0)(0) - (-3)(-2)) - \mathbf{j}((1)(0) - (-3)(5)) + \mathbf{k}((1)(-2) - (0)(5)) \\
&= \mathbf{i}(0 - 6) - \mathbf{j}(0 + 15) + \mathbf{k}(-2 - 0) \\
&= [-6, -15, -2]^T
\end{aligned}$$

We can simplify this point by multiplying by -1. The point of intersection is $[\mathbf{6}, \mathbf{15}, \mathbf{2}]$.

(c) $7x + y - w = 0$ and $w = 0$

$$\begin{aligned}
\mathbf{l}_1 &= [7, 1, -1]^T \\
\mathbf{l}_2 &= [0, 0, 1]^T
\end{aligned}$$

The line $w = 0$ is the **line at infinity**. The intersection of any finite line with the line at infinity gives the "direction" of that line, represented as a point at infinity. Let's compute the cross product:

$$\begin{aligned}
\mathbf{p} = \mathbf{l}_1 \times \mathbf{l}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} \\
&= \mathbf{i}((1)(1) - (-1)(0)) - \mathbf{j}((7)(1) - (-1)(0)) + \mathbf{k}((7)(0) - (1)(0)) \\
&= \mathbf{i}(1 - 0) - \mathbf{j}(7 - 0) + \mathbf{k}(0 - 0) \\
&= [1, -7, 0]^T
\end{aligned}$$

The point of intersection is $[\mathbf{1}, -\mathbf{7}, \mathbf{0}]$, which is a point at infinity as expected.

2.4

Given lines:

$$\ell_1 : w = 0 \implies [0,\ 0,\ 1]$$

$$\ell_2 : x + 2y + w = 0 \implies [1,\ 2,\ 1]$$

Intersection point of given two lines:

$$\ell_1 \cap \ell_2 = [0,\ 0,\ 1] \times [1,\ 2,\ 1] = [-2,\ 1,\ 0]$$

This is a point at infinity: $[-2,\ 1,\ 0]$.

Now, checking if this point lies on each ℓ_3:

(a) $\ell_3 : x + 2y + 6w = 0 \implies [1,\ 2,\ 6]$

Dot product: $[-2,\ 1,\ 0] \cdot [1,\ 2,\ 6] = -2 + 2 + 0 = 0$
Yes, $[-2,\ 1,\ 0]$ lies on ℓ_3.

(b) $\ell_3 : -3x - 6y + 6w = 0 \implies [-3,\ -6,\ 6]$

Dot product: $[-2,\ 1,\ 0] \cdot [-3,\ -6,\ 6] = 6 - 6 + 0 = 0$
Yes, $[-2,\ 1,\ 0]$ lies on ℓ_3.

(c) $\ell_3 : 7x + y - w = 0 \implies [7,\ 1,\ -1]$

Dot product: $[-2,\ 1,\ 0] \cdot [7,\ 1,\ -1] = -14 + 1 + 0 = -13 \neq 0$
No, $[-2,\ 1,\ 0]$ does not lie on ℓ_3.

2.5

Let T be a 3×3 projective transformation matrix. The action of T on the standard basis vectors gives its columns:

$$T \times [1,\ 0,\ 0]^T = \text{first column of } T = [1,\ -1,\ 1]^T$$

$$T \times [0,\ 1,\ 0]^T = \text{second column of } T = [1,\ -2,\ 2]^T$$

$$T \times [0,\ 0,\ 1]^T = \text{third column of } T = [-1,\ 2,\ -1]^T$$

Thus, the projective transformation T is:

$$T = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -2 & 2 \\ 1 & 2 & -1 \end{pmatrix}$$

More generally, the parametric family of projective transformations is:

$$T(\lambda_1, \lambda_2, \lambda_3) = \begin{pmatrix} \lambda_1 & \lambda_2 & -\lambda_3 \\ -\lambda_1 & -2\lambda_2 & 2\lambda_3 \\ \lambda_1 & 2\lambda_2 & -\lambda_3 \end{pmatrix}$$

where λ_1, λ_2, λ_3 are any nonzero real numbers such that $\det(T) \neq 0$.

Valid solutions:

- $\lambda_1 = 1,\ \lambda_2 = 1,\ \lambda_3 = 1 \qquad$ (shown above)
- $\lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = 1$
- $\lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = 3$

2.6

1. **Horizontal lines remain horizontal**

   In homogeneous coordinates, all horizontal lines (e.g., $y = c$) intersect at the same point at infinity, which has coordinates $\mathbf{p}_h = [1, 0, 0]^T$. If horizontal lines are mapped to horizontal lines, this point must be mapped to itself (or a scaled version of itself).

   Let $H = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}$.

   The mapping is $H\mathbf{p}_h \propto \mathbf{p}_h$:

   $$H \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} h_{11} \\ h_{21} \\ h_{31} \end{pmatrix} \propto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

   This implies $h_{21} = 0$ and $h_{31} = 0$.

2. **Vertical lines map to lines through** $(0, 3)$

   All vertical lines ($x = c$) intersect at the point at infinity $\mathbf{p}_v = [0,\ 1,\ 0]^T$. The transformation maps this point to a new location. Since all the resulting lines pass through the point $(0, 3)$, the image of $\mathbf{p}_v$ must be the point $(0, 3)$. The homogeneous coordinates for $(0, 3)$ are $[0,\ 3,\ 1]^T$.

   The mapping is $H\mathbf{p}_v \propto [0,\ 3,\ 1]^T$:

   $$H \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} h_{12} \\ h_{22} \\ h_{32} \end{pmatrix} \propto \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

   This implies $h_{12} = 0$ and the ratio $h_{22} : h_{32} = 3 : 1$. So, $h_{22} = 3k$, $h_{32} = k$ for some scalar k.

3. **The point** $(0, 0)$ **is a fixed point**

   The point $(0, 0)$ is $[0,\ 0,\ 1]^T$ in homogeneous coordinates. A fixed point is mapped to itself.

   The mapping is $H[0,\ 0,\ 1]^T \propto [0,\ 0,\ 1]^T$:

   $$H \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} h_{13} \\ h_{23} \\ h_{33} \end{pmatrix} \propto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   This implies $h_{13} = 0$, $h_{23} = 0$. Let $h_{33} = m$.

   So far, our matrix looks like:

   $$H = \begin{pmatrix} h_{11} & 0 & 0 \\ 0 & 3k & 0 \\ 0 & k & m \end{pmatrix}$$

4. **The point** $(1, 1)$ **is a fixed point**

   The point $(1, 1)$ is $[1,\ 1,\ 1]^T$ in homogeneous coordinates. This point is also fixed.

   The mapping is $H[1,\ 1,\ 1]^T \propto [1,\ 1,\ 1]^T$:

   $$H \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} h_{11} \\ 3k \\ k + m \end{pmatrix} \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For the output vector to be proportional to $[1,\ 1,\ 1]^T$, its components must be equal:

$$\begin{aligned} h_{11} &= 3k \\ 3k &= k + m \implies m = 2k \end{aligned}$$

Substituting back, the matrix is:

$$H = \begin{pmatrix} 3k & 0 & 0 \\ 0 & 3k & 0 \\ 0 & k & 2k \end{pmatrix} = k \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

Since a homography is defined up to scale, we can set $k = 1$:

$$\boxed{H = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}}$$

2.7

Let H be a general 3×3 homography matrix and p a generic point on the line at infinity:

$$H = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}, \qquad p = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

Transforming p by H:

$$p' = Hp = \begin{pmatrix} h_{11}x + h_{12}y \\ h_{21}x + h_{22}y \\ h_{31}x + h_{32}y \end{pmatrix}$$

For p' to also lie on the line at infinity, its third component must be zero for all x, y:

$$h_{31}x + h_{32}y = 0 \quad \forall\, x, y$$

This is only possible if $h_{31} = 0$ and $h_{32} = 0$.

Thus, the matrix for an affine transformation has the form:

$$H_{\text{affine}} = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ 0 & 0 & h_{33} \end{pmatrix}$$

This structure ensures that finite points remain finite and points at infinity remain at infinity.

3.1

Given points $E(-1,-1)$ and $F(0,2)$:

Slope:

$$m = \frac{2-(-1)}{0-(-1)} = \frac{3}{1} = 3$$

Line equation:

$$y = 3x + 2 \implies 3x - y + 2 = 0$$

Coefficients: $[a,b,c] = [3,-1,2]$

Given points $G(2,2)$ and $H(3,-1)$:

Slope:

$$m = \frac{-1-2}{3-2} = \frac{-3}{1} = -3$$

Line equation:

$$y = -3x + 8 \implies 3x + y - 8 = 0$$

Coefficients: $[a,b,c] = [3,1,-8]$

3.2

From 3.1 we have:

EF: $3x - y + 2 = 0$

GH: $3x + y - 8 = 0$

Intersection of $EF \cap GH$:

Solving:

$$\begin{cases} 3x - y + 2 = 0 \\ 3x + y - 8 = 0 \end{cases}$$

Add the equations:

$$(3x - y + 2) + (3x + y - 8) = 0 \implies 6x - 6 = 0 \implies x = 1$$

Substitute $x = 1$ into $3x - y + 2 = 0$:

$$3(1) - y + 2 = 0 \implies 5 - y = 0 \implies y = 5$$

So the intersection is (1, 5).

Coefficients of Lines EG and FH:
$E(-1,-1),\ G(2,2)$: Slope $m = \frac{2-(-1)}{2-(-1)} = 1$
Equation: $y = x$ or $x - y = 0$ (coefficients [1, −1, 0])
$F(0,2),\ H(3,-1)$: Slope $m = \frac{-1-2}{3-0} = -1$
Equation: $y = -x + 2$ or $x + y - 2 = 0$ (coefficients [1, 1, −2])

Intersection $EG \cap FH$:
Solving:

$$\begin{cases} x - y = 0 \\ x + y - 2 = 0 \end{cases}$$

From $x - y = 0$, $x = y$. Substitute into $x + y - 2 = 0$:

$$x + x - 2 = 0 \implies 2x = 2 \implies x = 1,\ y = 1$$

So the intersection is (1, 1).

3.3

Euclidean coordinates of the points in homogeneous form:

$$a = [0,\ 0,\ 1]^T \to e = [-1,\ -1,\ 1]^T$$
$$b = [0,\ 2,\ 1]^T \to f = [0,\ 2,\ 1]^T$$
$$c = [2,\ 2,\ 1]^T \to g = [2,\ 2,\ 1]^T$$
$$d = [2,\ 0,\ 1]^T \to h = [3,\ -1,\ 1]^T$$

The transformation for each point is $x' \sim Mx$, which means $x' = k \cdot Mx$ for some scalar k.

Linear Dependency in $\mathbb{P}^2$

Since we are in $\mathbb{P}^2$, any four points are linearly dependent. Let's express point d as a combination of a, b, and c:

$$d = c_1 a + c_2 b + c_3 c$$

$$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

Solving this system gives $c_1 = 1$, $c_2 = -1$, $c_3 = 1$. So, $d = a - b + c$.

Solving for Scaling Factors

This relationship must be preserved for the destination points, up to individual scaling factors k_i:

$$k_4 h = k_1 e - k_2 f + k_3 g$$

$$k_4 \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = k_1 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} - k_2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + k_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

Since the overall transformation has a scale ambiguity, we can fix one scalar, for instance, $k_4 = 1$. This gives a system of three linear equations:

$$\begin{aligned} 3 &= -k_1 + 2k_3 \\ -1 &= -k_1 - 2k_2 + 2k_3 \\ 1 &= k_1 - k_2 + k_3 \end{aligned}$$

Solving this system yields:

$$k_1 = 1, \quad k_2 = 2, \quad k_3 = 2$$

Constructing the Matrix M

Now we have three definitive mappings:

$$\begin{aligned} Ma = k_1 e &= [-1, -1, 1]^T \\ Mb = k_2 f &= [0, 4, 2]^T \\ Mc = k_3 g &= [4, 4, 2]^T \end{aligned}$$

We can write this as a single matrix equation: $M[a\ b\ c] = [k_1 e\ k_2 f\ k_3 g]$. Let

$$P_{\text{src}} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \qquad P_{\text{dst}} = \begin{pmatrix} -1 & 0 & 4 \\ -1 & 4 & 4 \\ 1 & 2 & 2 \end{pmatrix}$$

Then $M = P_{\text{dst}} \cdot (P_{\text{src}})^{-1}$.

The inverse of P_{src} is

$$(P_{\text{src}})^{-1} = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Multiplying the matrices gives:

$$M = P_{\text{dst}} \cdot (P_{\text{src}})^{-1} = \begin{pmatrix} -1 & 0 & 4 \\ -1 & 4 & 4 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 4 & 1 & -2 \\ 0 & 5 & -2 \\ 0 & 1 & 2 \end{pmatrix}$$

or, up to scale,

$$M \sim \begin{pmatrix} 4 & 1 & -2 \\ 0 & 5 & -2 \\ 0 & 1 & 2 \end{pmatrix}$$

3.4

The transformation matrix is

$$M = \begin{pmatrix} 4 & 1 & -2 \\ 0 & 5 & -2 \\ 0 & 1 & 2 \end{pmatrix}$$

The coordinates of point D are $(2, 0)$, which in homogeneous form is

$$d = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

The coordinates of point H are $(3, -1)$, which in homogeneous form is

$$h = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

We multiply the matrix M by the vector d:

$$Md = \begin{pmatrix} 4 & 1 & -2 \\ 0 & 5 & -2 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \times 2 + 1 \times 0 + (-2) \times 1 \\ 0 \times 2 + 5 \times 0 + (-2) \times 1 \\ 0 \times 2 + 1 \times 0 + 2 \times 1 \end{pmatrix} = \begin{pmatrix} 8 - 2 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix}$$

Check for proportionality:

$$\begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix} = 2 \times \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

So Md is a nonzero scalar multiple of the target vector h.

Hence! The transformation M correctly maps d to h up to scale.

4.1

Vanishing Points and Vanishing Line

Horizontal lines converge at the vanishing point $V_x = (-b, 0)$ and vertical lines at $V_y = (0, h)$. The line connecting these points is the vanishing line (horizon) for the facade.

In homogeneous coordinates:

$$v_x = [-b,\ 0,\ 1]^T, \qquad v_y = [0,\ h,\ 1]^T$$

The vanishing line is given by the cross product:

$$l_v = v_x \times v_y = [-h,\ b,\ -bh]^T$$

(up to scale for simplicity, $[h,\ -b,\ bh]^T$).

Transformation Constraints

We seek a homography H that maps the vanishing line l_v to the line at infinity $l_\infty = [0,\ 0,\ 1]^T$.

Thus, the last row of H must be proportional to $(h,\ -b,\ bh)$.

Fixed Points

The origin $(0, 0)$ is fixed:

$$H \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \propto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This requires $h_{13} = h_{23} = 0$.

The point $(1, 1)$ is also fixed:

$$H \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} h_{11} + h_{12} \\ h_{21} + h_{22} \\ h - b + bh \end{pmatrix} \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

So $h_{11} + h_{12} = h_{21} + h_{22} = h - b + bh$.

Vanishing Points Mapping

V_x maps to $[1,\ 0,\ 0]^T$:

$$H \begin{pmatrix} -b \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -bh_{11} \\ -bh_{21} \\ 0 \end{pmatrix} \propto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\implies h_{21} = 0$

V_y maps to $[0,\ 1,\ 0]^T$:

$$H \begin{pmatrix} 0 \\ h \\ 1 \end{pmatrix} = \begin{pmatrix} hh_{12} \\ hh_{22} \\ 0 \end{pmatrix} \propto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\implies h_{12} = 0$

Final Matrix

With $h_{12} = h_{21} = h_{13} = h_{23} = 0$, and $h_{11} = h_{22} = h - b + bh$, the homography is:

$$H = \begin{pmatrix} h - b + bh & 0 & 0 \\ 0 & h - b + bh & 0 \\ h & -b & bh \end{pmatrix}$$

(up to scale for simplicity, $[h,\ -b,\ bh]^T$).

4.2

Horizon = line through the two vanishing points.

Vanishing points:

$$V_x = (-b,\ 0,\ 1), \qquad V_y = (0,\ h,\ 1)$$

Line through them (in homogeneous form) is

$$\ell = V_x \times V_y$$

Compute the cross product:

$$\ell = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -b & 0 & 1 \\ 0 & h & 1 \end{vmatrix} = [-h,\, b,\, -bh]$$

(up to scale, this is equivalent to $[h,\ -b,\ bh]$).

$$\ell = hx - by + bh = 0$$

5.1 Bonus: Affine Transforms (15pts)

Conic Sections and the Line at Infinity

In projective geometry, conic sections are distinguished by their intersection with the line at infinity ℓ_∞:

- **Ellipse (including circles):** Does not intersect ℓ_∞ at real points
- **Parabola:** Intersects ℓ_∞ at exactly one real point (tangent)
- **Hyperbola:** Intersects ℓ_∞ at exactly two distinct real points

Key Property of Affine Transformations

From Problem 2.7, we established that affine transformations have the form:

$$H = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ 0 & 0 & h_{33} \end{pmatrix}$$

The crucial property is that **affine transformations preserve the line at infinity**.

Proof: Points at infinity $[x, y, 0]$ transform as:

$$H \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} h_{11}x + h_{12}y \\ h_{21}x + h_{22}y \\ 0 \end{pmatrix}$$

The third coordinate remains 0, so points at infinity map to points at infinity.

Circle → Ellipse is Possible

Claim: Any circle can be transformed into any ellipse via an affine transformation.

Proof:

1. A circle has equation $x^2 + y^2 = r^2$ (centered at origin for simplicity)
2. Consider the affine transformation:
$$H = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
where $a, b > 0$ and $a \neq b$.
3. Under this transformation, a point (x, y) maps to (ax, by)
4. The circle $x^2 + y^2 = r^2$ becomes:
$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = r^2$$
$$\frac{x'^2}{a^2r^2} + \frac{y'^2}{b^2r^2} = 1$$
5. This is an ellipse with semi-axes ar and br.
6. **Key insight:** Both circle and ellipse have no real intersections with ℓ_∞, so the transformation preserves this topological property.

Ellipse → Hyperbola/Parabola is Impossible

Claim: No affine transformation can map an ellipse to a hyperbola or parabola.

Proof by Topological Invariant:
Intersection Analysis

- **Ellipse:** The general ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in homogeneous coordinates becomes:
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z^2$$
At the line at infinity ($z = 0$), we get $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$, which has no real solutions.

- **Hyperbola:** The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ becomes:

  $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z^2$$

  At $z = 0$: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \Rightarrow y = \pm\frac{b}{a}x$

  This gives two distinct real intersection points: $[a : b : 0]$ and $[a : -b : 0]$.

- **Parabola:** The parabola $y = x^2$ becomes $yz = x^2$ in homogeneous form. At $z = 0$: $0 = x^2 \Rightarrow x = 0$, giving one real intersection point $[0 : 1 : 0]$.

Invariance Under Affine Transformations

Since affine transformations preserve the line at infinity, they preserve the number of real intersections between any conic and ℓ_∞.

Intersection Count Invariant:

- Ellipse: 0 real intersections with ℓ_∞
- Parabola: 1 real intersection with ℓ_∞
- Hyperbola: 2 real intersections with ℓ_∞

Conclusion

Since the number of real intersections with the line at infinity is preserved under affine transformations:

- **Ellipse $\to$ Parabola:** Impossible (0 intersections $\not\to$ 1 intersection)
- **Ellipse $\to$ Hyperbola:** Impossible (0 intersections $\not\to$ 2 intersections)

Also euclidean classification by Discriminant,

A conic section $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ has discriminant $\Delta = B^2 - 4AC$.

- $\Delta < 0$: Ellipse (including circles when $A = C$ and $B = 0$)
- $\Delta = 0$: Parabola

- $\Delta > 0$: Hyperbola

Under an affine transformation $H = \begin{pmatrix} a & b & p \\ c & d & q \\ 0 & 0 & 1 \end{pmatrix}$, the discriminant transforms as:

$$\Delta' = \Delta \cdot (\det(A))^2$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the linear part.

Since $\det(A) \neq 0$ for invertible transformations, the sign of Δ is preserved:

- $\Delta < 0 \Rightarrow \Delta' < 0$ (ellipse stays ellipse)
- $\Delta = 0 \Rightarrow \Delta' = 0$ (parabola stays parabola)
- $\Delta > 0 \Rightarrow \Delta' > 0$ (hyperbola stays hyperbola)

Conclusion: Affine transformations preserve the topological type of conic sections. Therefore:

- ✓ Circle → Ellipse: **Possible** (both have $\Delta < 0$)
- × Ellipse → Hyperbola: **Impossible** ($\Delta < 0 \not\to \Delta > 0$)
- × Ellipse → Parabola: **Impossible** ($\Delta < 0 \not\to \Delta = 0$)

5.2 Bonus: Vanishing Point Computation (10pts)

Given Setup

We have a 2×2 homography matrix acting on points in $\mathbb{P}^1$ (the projective line):

$$H_{2\times 2} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

Points on the projective line are represented in homogeneous coordinates as $[a, b]^T$, where:

- When $b \neq 0$: the point corresponds to the affine coordinate $\frac{a}{b}$
- When $b = 0$: the point $[a, 0]^T$ represents the point at infinity

Vanishing Point

The **vanishing point** is where parallel lines in the world appear to meet in the image. In the context of a 1D projective transformation, the vanishing point is the image of the point at infinity from the world line.

The point at infinity in $\mathbb{P}^1$ is represented as $[1, 0]^T$.

Computing the Vanishing Point

1. Apply the homography to the point at infinity:

$$y_{\text{vanishing}} = H_{2\times 2}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}h_{11}\\h_{21}\end{pmatrix}$$

2. Convert to affine coordinates (if possible):
Case 1: If $h_{21} \neq 0$, the vanishing point in affine coordinates is:

$$\boxed{v = \frac{h_{11}}{h_{21}}}$$

Case 2: If $h_{21} = 0$ and $h_{11} \neq 0$, the vanishing point is at infinity in the image, represented as $[1, 0]^T$.

Case 3: If $h_{21} = h_{11} = 0$, this would mean the first column of H is zero, making H singular (non-invertible), which is not a valid homography.

Geometric Interpretation

The homography matrix can be written as:

$$H_{2\times 2} = \begin{pmatrix}h_{11} & h_{12}\\h_{21} & h_{22}\end{pmatrix}$$

The transformation maps:

- World point $[x, 1]^T$ (affine coordinate x) to image point $[h_{11}x+h_{12}, h_{21}x+h_{22}]^T$
- World point at infinity $[1, 0]^T$ to image point $[h_{11}, h_{21}]^T$ (the vanishing point)

General Matrix Form

If we write the homography as:

$$H_{2\times 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then the vanishing point is:

$$\boxed{\text{Vanishing point} = \begin{cases} \frac{a}{c} & \text{if } c \neq 0 \\ \text{point at infinity} & \text{if } c = 0, a \neq 0 \\ \text{undefined (singular)} & \text{if } c = a = 0 \end{cases}}$$

Final Answer

For a 2×2 homography matrix $H_{2\times 2} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$, the vanishing point is:

$$\boxed{\text{Vanishing Point} = \frac{h_{11}}{h_{21}} \quad (\text{provided } h_{21} \neq 0)}$$

This represents the affine coordinate where the point at infinity from the world line appears in the image line.