MA323 - Monte Carlo Simulation

Lab - 3

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1. **Implement the acceptance rejection method to generate samples from a distribution with PDF**

$$f(x) = 20x(1-x)^3 \quad \text{for} \quad 0 < x < 1.$$

Please use the smallest value of c such that f(x) ≤ cg(x) for your choice of g.

(a) What is the average of number of iterations needed to generate a random number and why?

(b) Generate 10000 random numbers from the distribution. Compute the sample mean and compare it with expectation of the PDF f.

(c) What is the approximate value of P(0.25 ≤ X ≤ 0.75) based on the generated sample in the part (b)? What is the exact value of the probability? Compare them.

(d) Keep a count of number of iterations needed to generate each of the random numbers in part (b). Compute the average of all these values and compare it with the value obtained in part (a).

(e) Draw the histogram of the sample obtained in part (b). Also, draw the PDF f on the same plot. Compare them.

(f) Repeat parts (a)–(d) above with two values of c higher than the smallest value that you have chosen. What are your observations ?

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For c: 2.109375

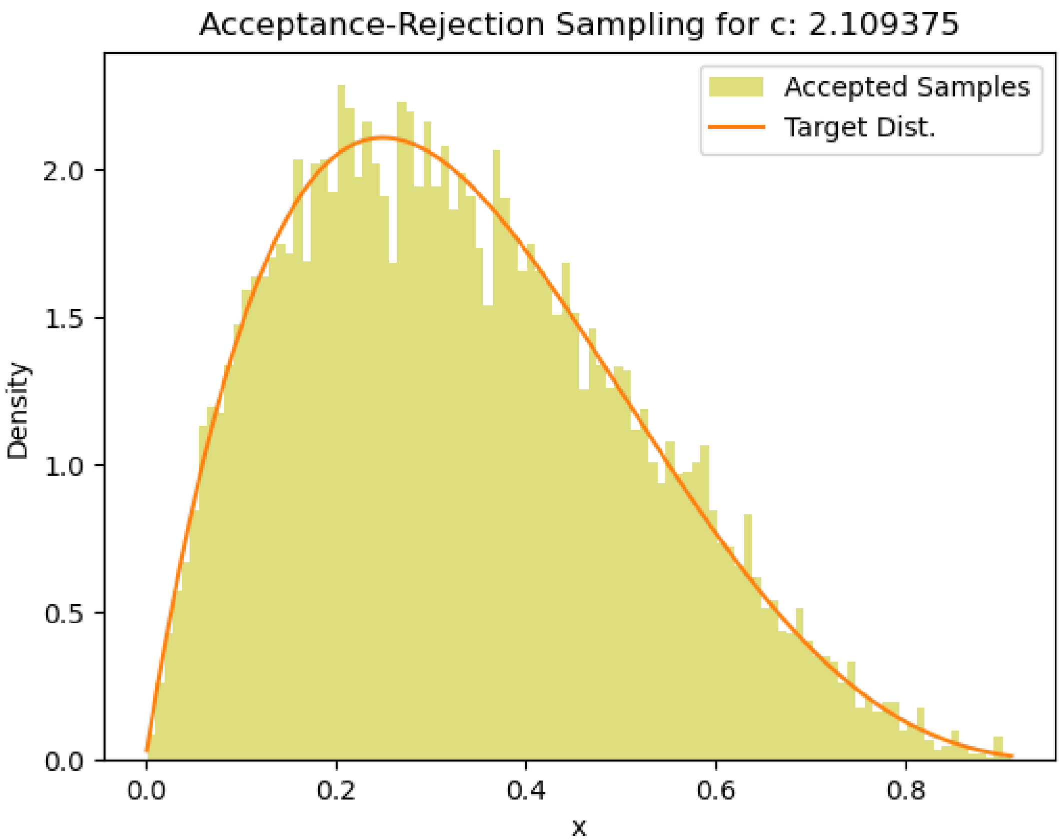
(a)
Average number of iterations needed to generate a random number: 2.109375

(b)
Empirical sample mean of the distribution: 0.33636274063079824
Expected sample mean of the distribution: 0.3333333333333333

(c)
Empirical value of P(0.25 ≤ X ≤ 0.75): 0.5977526641857913
Exact value of P(0.25 ≤ X ≤ 0.75): 0.617188

(d)
Average number of iterations needed to generate a random number from part(a):
2.109374985410495
Expected number of iterations by averaging iterations per sample: 2.1171
```

(e)

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For c: 5

(a)
Average number of iterations needed to generate a random number: 5

(b)
Empirical sample mean of the distribution: 0.3333260307264442
Expected sample mean of the distribution: 0.3333333333333333

(c)
Empirical value of P(0.25 ≤ X ≤ 0.75): 0.5990406215399493
Exact value of P(0.25 ≤ X ≤ 0.75): 0.617188

(d)
Average number of iterations needed to generate a random number from part(a):
2.1093749984370516
Expected number of iterations by averaging iterations per sample: 4.9558



For c: 10

(a)
Average number of iterations needed to generate a random number: 10

(b)
Empirical sample mean of the distribution: 0.33382695586956895
Expected sample mean of the distribution: 0.3333333333333333
```

```
(c)
Empirical value of P(0.25 ≤ X ≤ 0.75): 0.6043540815850008
Exact value of P(0.25 ≤ X ≤ 0.75): 0.617188

(d)
Average number of iterations needed to generate a random number from part(a):
2.1093749999157603
Expected number of iterations by averaging iterations per sample: 9.8473
```

Observations and reasoning:

For minimum value of c:

The function chosen to bound f(x), i.e. g(x) is uniform distribution U(0, 1), PDF. The value of c chosen to bound is the maximum value of f(x) in (0, 1). Therefore c = 2.109 is chosen as the minimum value empirically.

(a) : The probability of acceptance is given by the ratio of area under f(x) to the area under cg(x) (the ratio of target distribution to proposal distribution).

$$p = \frac{\int_0^1 f(x)dx}{\int_0^1 cg(x)dx}$$

In this case, p = 64 / 135. The expected number of iterations (E) needed to get an expected sample can be calculated as:

$$E = \frac{1}{p}$$

In this case, E = 135 / 64 = 2.109.

With the same reasoning given for c = 2.109, the expected number of iterations to get an accepted sample is 5 and 10 for c = 5 and c = 10 respectively. This is higher than c = 2.109, suggesting that g(x) wasn't a very good upper bound for f(x).

(b) : The empirical mean and expected mean are very close for all c = 5, c = 10 and c = 2.109.

(c) : The empirical value and the exact value for `P(0.25 ≤ X ≤ 0.75)` :are very close for all c = 5, c = 10 and c = 2.109.

(d) : The empirical number of iterations and expected iterations are very close for both c = 5 and c = 10 and higher than that of c = 2.109. Choosing a value of c higher than the minimum value provides a poor envelope for the target distribution.

(e) : The histogram and actual PDF f(x) have almost the same distribution.

2. **Implement acceptance rejection method to generate random number from the PDF**

$$f(x) \propto x^{\alpha-1}e^{-x} \quad \text{for} \quad 0 < x < 1.$$

Generate 10000 random numbers form the above PDF. Please mention the rejection constant and dominating PDF in the report.

Target distribution is given by f(x) / 0.842701 for alpha = 0.5 where f(x) is given by,

$$f(x) = \frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}$$

f(x) is divided by 0.842701 so that target distribution is PDF and it's CDF tends to 1 as x -> 1.

This is gamma distribution function with beta (scale) = 1 with alpha (shape) = 0.5 chosen for 10,000 samples.

Dominating PDF is given by

$$g(x) = \begin{cases} \frac{x^{\alpha-1}}{A} & \text{if } 0 < x < 1 \\ \frac{e^{-x}}{A} & \text{if } x \geq 1, \end{cases}$$

such that

$$f(x) \leq cg(x)$$

where,

rejection constant (c = 1.3359) is given by $c = \frac{A}{\Gamma(\alpha)}$.

A is given by $A = \frac{1}{\alpha} + \frac{1}{e}$