Problem Set 5
Linear Optimization

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Problem 8.1

Plot and solution in the Jupyter Notebook.

Problem 8.2

Plots and solutions in the Jupyter Notebook.

Problem 8.3

Let x and y denote the production of GI Bard Soldiers Joey dolls respectively. As per the problem, the revenues are given by $12x + 10y$.

Raw material cost is given by $5x + 3y$ and the overhead costs are given by $3x + 4y$. Therefore, the net profit is given by $4x + 3y$.

The finishing labour requirement is $15x + 10y$ minutes and molding labour requirement is $2x + 2y$ minutes.

The optimization problem is thus given by the following equations:

$$\begin{aligned} \max_{x,y}\ & 4x + 3y \\ \text{subject to: } & 15x + 10y \leq 1800 \\ & 2x + 2y \leq 300 \\ & y \leq 200 \end{aligned}$$

Problem 8.4

The network flow optimization problem is as follows-

$$\min_{x_{i,j}} \left(5x_{AD} + 2x_{AB} + 2x_{BD} + 7x_{BE} + 9x_{BF} + 5x_{BC} + 2x_{CF} + 4x_{DE} + 3x_{EF}\right)$$

subject to:

$$\begin{aligned}
&x_{AD} + x_{AB} = 10 \\
&x_{BC} + x_{BD} + x_{BE} + x_{BF} - x_{AB} = 1 \\
&x_{CF} - x_{BC} = -2 \\
&x_{DE} - x_{AD} - x_{BD} = -3 \\
&x_{EF} - x_{BE} - x_{BD} = 4 \\
&x_{CF} + x_{BF} + x_{EF} = 10 \\
&0 \leq x_{i,j} \leq 6 \text{ where } i, j \text{ are nodes}
\end{aligned}$$

Problem 8.5

1. The initial dictionary after adding in 3 slack variables x_3, x_4, x_5 is:

$$\zeta_1 = 3x_1 + x_2$$

$$\begin{aligned}
x_3 &= 15 - x_1 - 3x_2 \\
x_4 &= 18 - 2x_1 - 3x_2 \\
x_5 &= 4 - x_1 + x_2
\end{aligned}$$

Since the coefficient of x_1 is positive we can choose it as the entering variable and x_5 as the leaving variable as it sets the lowest bound on x_1. The new dictionary becomes:

$$\zeta_2 = 12 + 4x_2 - 3x_5$$

$$\begin{aligned}
x_1 &= 4 + x_2 - x_5 \\
x_3 &= 11 - 4x_2 + x_5 \\
x_4 &= 10 - 5x_2 + 2x_5
\end{aligned}$$

Now x_2 is the only variable with positive coefficient, therefore choosing it as the entering variable and x_4 as the leaving variable as it sets the lowest bound on x_2. The new dictionary becomes:

$$\zeta_3 = 20 - \tfrac{4}{5}x_4 - \tfrac{7}{5}x_5$$

$$\begin{aligned}
x_1 &= 6 - \tfrac{1}{5}x_4 - \tfrac{3}{5}x_5 \\
x_2 &= 2 - \tfrac{1}{5}x_4 + \tfrac{2}{5}x_5 \\
x_3 &= 3 + \tfrac{4}{5}x_4 - \tfrac{13}{5}x_5
\end{aligned}$$

Since both x_4, x_5 now appear with negative signs in the objective function, this is the optimum. The values are: $x_1 = 6, x_2 = 2$ and the value of the objective function is 20. This matches the answer in the Jupyter Notebook.

2. The initial dictionary after adding in 3 slack variables x_3, x_4, x_5 is:

$$\begin{aligned}
\zeta_1 &= 4x_1 + 6x_2 \\
\hline
x_3 &= 11 + x_1 - x_2 \\
x_4 &= 27 - x_1 - x_2 \\
x_5 &= 90 - 2x_1 - 5x_2
\end{aligned}$$

Since the coefficient of x_1 is positive we can choose it as the entering variable and x_4 as the leaving variable as it sets the lowest bound on x_1. The new dictionary becomes:

$$\begin{aligned}
\zeta_2 &= 108 + 2x_2 - 4x_4 \\
\hline
x_1 &= 27 - x_2 - x_4 \\
x_3 &= 38 - 2x_2 - x_4 \\
x_5 &= 36 - 3x_2 + 2x_4
\end{aligned}$$

Now x_2 is the only variable with positive coefficient, therefore choosing it as the entering variable and x_5 as the leaving variable as it sets the lowest bound on x_2. The new dictionary becomes:

$$\begin{aligned}
\zeta_3 &= 132 - \tfrac{8}{3}x_4 - \tfrac{2}{3}x_5 \\
\hline
x_1 &= 15 - \tfrac{5}{3}x_4 + \tfrac{1}{3}x_5 \\
x_2 &= 12 + \tfrac{2}{3}x_4 - \tfrac{1}{3}x_5 \\
x_3 &= 14 - 3x_4 + \tfrac{2}{3}x_5
\end{aligned}$$

All the variables now appear in the objective function with a negative sign. Hence the present choice is optimal. This occurs at $x = 15, y = 12$ and the value of the objective function is 132.

Problem 8.6

After adding 3 slack variables x_1, x_2, x_3 and simplifying the constraints after taking out the common factors, the initial dictionary is:

$$\zeta_1 = 4x + 3y$$

$$\begin{aligned} x_1 &= 360 - 3x - 2y \\ x_2 &= 150 - x - y \\ x_3 &= 200 - y \end{aligned}$$

Since the coefficient of x is positive we can choose it as the entering variable and x_1 as the leaving variable as it sets the lowest bound on x_1. The new dictionary becomes:

$$\zeta_2 = 480 + \frac{1}{3}y - \frac{4}{3}x_1$$

$$\begin{aligned} x &= 120 - \frac{2}{3}y - \frac{1}{3}x_1 \\ x_2 &= 30 - \frac{1}{3}y + \frac{1}{3}x_1 \\ x_3 &= 200 - y \end{aligned}$$

Now y is the only variable with positive coefficient, therefore choosing it as the entering variable and x_2 as the leaving variable as it sets the lowest bound on x_2.

$$\zeta_3 = 510 - 10x_2 - \frac{5}{3}x_1$$

$$\begin{aligned} x &= 60 + 20x_2 + \frac{1}{3}x_1 \\ y &= 90 - 30x_2 - x_1 \\ x_3 &= 110 + 30x_2 + x_1 \end{aligned}$$

As all the terms in the objective function appear with a negative sign, we are at the optimum. The value of the objective function i.e profit is \$510 and $x = 60, y = 90$

Problem 8.7

The Jupyter notebook provides the plot for all three problems below.

1. The origin is not part of the feasible set. This can be seen from the Jupyter Notebook where the feasible set has been plotted.Therefore the problem needs to be set up an auxiliary problem first by subtracting x_0 from all the constraints. The dictionary for the auxiliary problem is:

$$\zeta_1 = -x_0$$

$$\begin{aligned} x_3 &= -8 + 4x_1 + 2x_2 + x_0 \\ x_4 &= 6 + 2x_1 - 3x_2 + x_0 \\ x_5 &= 3 - x_1 + x_0 \end{aligned}$$

We pivot x_0 and x_1. The new dictionary becomes:

$$\zeta_2 = -x_0$$

$$\begin{aligned} x_1 &= 2 - \tfrac{1}{2}x_2 + \tfrac{1}{4}x_3 - \tfrac{1}{4}x_0 \\ x_4 &= 10 - 4x_2 + \tfrac{1}{2}x_3 + \tfrac{1}{2}x_0 \\ x_5 &= 1 + \tfrac{5}{2}x_2 - \tfrac{1}{4}x_3 + \tfrac{5}{4}x_0 \end{aligned}$$

We can see that this dictionary is optimal as all points are feasible and the objective function is 0 Thus $x_1 = 2, x_2 = 0$, is a feasible point for the original problem. We can remove x_0 from the main problem and replace x_1 in terms of the non-basic variables. The new dictionary becomes:

$$\zeta_3 = 2 + \tfrac{3}{2}x_2 + \tfrac{1}{4}x_3$$

$$\begin{aligned} x_1 &= 2 - \tfrac{1}{2}x_2 + \tfrac{1}{4}x_3 \\ x_4 &= 10 - 4x_2 + \tfrac{1}{2}x_3 \\ x_5 &= 1 + \tfrac{1}{2}x_2 - \tfrac{1}{4}x_3 \end{aligned}$$

Since the coefficient for x_3 is positive and x_5 provides the lowest bound for it, we pivot around x_5. The new dictionary becomes:

$$\zeta_4 = 3 + 2x_2 - x_5$$

$$\begin{aligned} x_1 &= 3 - x_5 \\ x_4 &= 12 - 3x_2 - 2x_5 \\ x_3 &= 4 + 2x_2 - 4x_5 \end{aligned}$$

Now x_2 is the only variable with positive coefficient, x_4 provides a positive lower bound on x_2, therefore taking a pivot around it we obtain the dictionary:

$$\frac{\zeta_5 = 11 - \frac{2}{3}x_4 - \frac{7}{3}x_5}{\begin{aligned} x_1 &= 3 - x_5 \\ x_2 &= 4 - \tfrac{1}{3}x_4 - \tfrac{2}{3}x_5 \\ x_3 &= 12 - \tfrac{2}{3}x_4 - \tfrac{16}{3}x_5 \end{aligned}}$$

This dictionary is optimal as all the terms appear with a negative sign. The optimal values are $x_1 = 3, x_2 = 4$. This can also be confirmed from the plot in the Jupyter Notebook.

2. The origin is not part of the feasible set as the third constraint appears with a negative sign. The auxiliary problem is:

$$\frac{\zeta_1 = -x_0}{\begin{aligned} x_3 &= 15 - 5x_1 - 3x_2 + x_0 \\ x_4 &= 15 - 3x_1 - 5x_2 + x_0 \\ x_5 &= -12 - 4x_1 + 3x_2 + x_0 \end{aligned}}$$

We pivot x_0, x_5 and get the new dictionary as:

$$\frac{\zeta_2 = 12 + 4x_1 - 3x_2 + x_5}{\begin{aligned} x_0 &= 12 + 4x_1 - 3x_2 + x_5 \\ x_3 &= 27 - x_1 - 6x_2 + x_5 \\ x_4 &= 27 + x_1 - 8x_2 + x_5 \end{aligned}}$$

Again, we pivot x_2, x_4 and obtain:

$$\frac{\zeta_3 = -\frac{15}{8} - \frac{29}{8}x_1 - \frac{3}{8}x_4 - \frac{5}{8}x_5}{\begin{aligned} x_3 &= \tfrac{27}{4} - \tfrac{7}{4}x_1 + \tfrac{3}{4}x_4 + \tfrac{1}{4}x_5 \\ x_2 &= \tfrac{27}{8} + \tfrac{1}{8}x_1 - \tfrac{1}{8}x_4 + \tfrac{1}{8}x_5 \\ x_0 &= \tfrac{15}{8} + \tfrac{29}{8}x_1 + \tfrac{3}{8}x_4 + \tfrac{5}{8}x_5 \end{aligned}}$$

This dictionary is optimal since all the coefficients in the objective function are negative. However, at this optimum, $x_0 \neq 0$. Hence the problem is infeasible. It is clear from the Jupyter notebook as well where it can be seen that the feasible set is null.

3. After adding in the slack variables, the initial dictionary is :

$$\zeta_1 = -3x_1 + x_2$$

$$\begin{aligned} x_3 &= 4 - x_2 \\ x_4 &= 6 + 2x_1 - 3x_2 \end{aligned}$$

   We pivot, x_2, x_4 and obtain the following dictionary:

$$\zeta_2 = 2 - \frac{7}{3}x_1 - \frac{1}{3}x_4$$

$$\begin{aligned} x_2 &= 2 + \frac{2}{3}x_1 - \frac{1}{3}x_4 \\ x_3 &= 2 - \frac{2}{3}x_1 + \frac{1}{3}x_3 \end{aligned}$$

   This dictionary is optimal as all the coefficients in the objective function have negative signs. The optimal solution is $x_1 = 0, x_2 = 2$ and the value of the objective function is 2.

Problem 8.8

Problem 8.7(iii) provides a hint for this kind of case. The plot for it in the Jupyter notebook shows how such a region can exist and there still can be a unique feasible solution. Extending the 8.7(iii) example in 3-D.

$$\begin{aligned} \max_{x,y,z} -3x + y + 3z \\ \text{subject to: } y \leq 4 \\ -2x + 3y + 4z \leq 10 \end{aligned}$$

I think this should do.

Problem 8.9

As per proposition 8.3.1, if the coefficient of the variable in objective function is positive and the coefficients in the constraints are non-negative then the optimization problem will be unbounded.Consider the problem below

$$\begin{aligned} \max_{x,y,z} ax + 5y + 3z \\ \text{subject to: } mx - y - z \leq 1 \\ 3y + 4z \leq 1 \end{aligned}$$

If a, m are positive then the problem would be unbounded.

Problem 8.10

As seen in the problem 8.7(ii), if the constraints are such that their solution set is null then the optimization problem will be infeasible.

$$\begin{aligned}
\max_{x,y,z} 2x + 9y + 3z \\
\text{subject to: } x + y + z \leq 1 \\
3x + 2y + 5z \leq 4 \\
x - y - z \leq -10000
\end{aligned}$$

The last constraint ensures that the intersection between the three constraints is null.Hence, the optimization is infeasible.

Problem 8.11

Section 8.3.2 provides the condition for the origin $\mathbf{0}$ vector to be infeasible. The intuition is that $Ax \preceq b$ has to be true for $x = 0$ and if vector b has any one component has a negative value then the origin becomes infeasible. Problem 8.7(i) is an example of such a case in 2-dimensions.

The dual problem for 8.18 can be an example for a 3-D case. I try to give another solution below:-

$$\begin{aligned}
\max_{x,y,z} 2x + 9y + 3z \\
\text{subject to: } x + y + z \leq 1 \\
3x + 2y + 5z \leq -1 \\
x - 2y - 3z \leq -4
\end{aligned}$$

The last two constraints ensure that some components of b vector are negative. Hence the origin will be infeasible.

As I have already solved 8.18 dual problem. See it as an example for auxiliary problem method.

Problem 8.12

The initial dictionary after adding in the slack variables is:

$$\begin{aligned}
&\zeta_1 = 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
\hline
&x_5 = -0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\
&x_6 = -0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\
&x_7 = 1 - x_1
\end{aligned}$$

Using Bland's rule, we choose lowest coefficient basic and non-basic variables for the pivot x_1, x_5. The new dictionary is:

$$\zeta_2 = -27x_2 + x_3 - 44x_4 - 20x_5$$

$$\begin{aligned} x_1 &= 3x_2 + x_3 - 2x_4 - 2x_5 \\ x_6 &= 4x_2 + 2x_3 - 8x_4 + x_5 \\ x_7 &= 1 - 3x_2 - x_3 + 2x_4 + 2x_5 \end{aligned}$$

We now pivot x_3, x_7 and obtain:

$$\zeta_3 = 1 - 30x_2 - 42x_4 - 18x_5 - x_7$$

$$\begin{aligned} x_1 &= 1 - x_7 \\ x_6 &= 2 - 2x_2 - 4x_4 + 5x_5 - 2x_7 \\ x_3 &= 1 - 3x_2 + 2x_4 + 2x_5 - x_7 \end{aligned}$$

This dictionary is optimal as all the coefficients appear with negative sign in the objective function. The optimal points are $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0$ and the value of the objective function is 1.

Problem 8.15

Using the definitions of primal and dual problems where $\boldsymbol{x}, \boldsymbol{y}$ are feasible points of the primal and the dual respectively, we have

$$\boldsymbol{A^T y} \succeq \boldsymbol{c}$$

and

$$\boldsymbol{Ax} \preceq \boldsymbol{b}$$

Using these two results we can do the following basic linear algebra operations:

$$\begin{aligned} &\boldsymbol{A^T y} \succeq \boldsymbol{c} \\ &\Rightarrow \boldsymbol{x^T A^T y} \geq \boldsymbol{x^T c} \\ &\Rightarrow (\boldsymbol{Ax})^{\boldsymbol{T}} \boldsymbol{y} \geq (\boldsymbol{x^T c}) \\ &\Rightarrow \boldsymbol{b^T y} \geq (\boldsymbol{Ax})^{\boldsymbol{T}} \boldsymbol{y} \geq \boldsymbol{x^T c} \\ &\Rightarrow \boldsymbol{b^T y} \geq \boldsymbol{x^T c} \\ &= \boldsymbol{c^T x} \\ &\Rightarrow \boldsymbol{b^T y} \geq \boldsymbol{c^T x} \end{aligned}$$

Problem 8.17

Consider the primal problem

$$\max_x \mathbf{c^T x}$$
$$\text{subject to } A\mathbf{x} \preceq \mathbf{b}$$
$$\mathbf{x} \succeq 0$$

The dual of the primal problem is

$$\min_y \mathbf{b^T y}$$
$$\text{subject to } A^T\mathbf{y} \succeq \mathbf{c}$$
$$\mathbf{y} \succeq 0$$

The dual can be re-written as

$$\max_y -\mathbf{b^T y}$$
$$\text{subject to } -A^T\mathbf{y} \preceq \textbf{-c}$$
$$\mathbf{y} \succeq 0$$

The dual of the dual problem can be written as follows

$$\min_x -\mathbf{c^T x}$$
$$\text{subject to } -(A^T)^T\mathbf{x} \succeq \textbf{-b}$$
$$\mathbf{x} \succeq 0$$

Which can be simplified to the primal problem:

$$\max_x \mathbf{c^T x}$$
$$\text{subject to } A\mathbf{x} \preceq \mathbf{b}$$
$$\mathbf{x} \succeq 0$$

Which is nothing but the primal problem. Hence the dual of the dual is the primal problem.

Problem 8.18

We first solve the primal problem using the simplex method. The initial dictionary is

$$
\begin{aligned}
\zeta_1 &= x_1 + x_2 \\
\hline
x_3 &= 3 - 2x_1 - x_2 \\
x_4 &= 5 - x_1 - 3x_2 \\
x_5 &= 4 - 2x_1 - 3x_2
\end{aligned}
$$

As coefficient for x_1 is positive, and the slack variable x_3 gives the lowest bound on x_1, we choose it as pivot. This gives the new dictionary as:

$$
\begin{aligned}
\zeta_2 &= \tfrac{3}{2} + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 \\
\hline
x_1 &= \tfrac{3}{2} - \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 \\
x_4 &= \tfrac{7}{2} - \tfrac{5}{2}x_2 + \tfrac{1}{2}x_3 \\
x_5 &= 1 - 2x_2 + x_3
\end{aligned}
$$

As coefficient for x_2 is positive, and the slack variable x_4 gives the lowest bound on x_2, we choose it as pivot. This gives the new dictionary as:

$$
\begin{aligned}
\zeta_3 &= \tfrac{7}{4} - \tfrac{1}{4}x_3 - \tfrac{1}{4}x_5 \\
\hline
x_1 &= \tfrac{5}{4} - \tfrac{3}{4}x_3 + \tfrac{1}{4}x_5 \\
x_4 &= \tfrac{9}{4} - \tfrac{3}{4}x_3 + \tfrac{5}{4}x_5 \\
x_2 &= \tfrac{1}{2} + \tfrac{1}{2}x_3 - \tfrac{1}{2}x_5
\end{aligned}
$$

This dictionary is optimal as the coefficients of the variables in the objective function appear with a negative sign. The optimal values are $x_1 = \frac{5}{4}, x_2 = \frac{1}{2}$ and the objective function is $\frac{7}{4}$.

The dual problem in minima can be written as a maximization problem in the following fashion:

$$\begin{aligned} \max_{x,y,z} \; & -3x - 5y - 4z \\ \text{subject to:} \; & -2x - y - 2z \leq -1 \\ & -x - 3y - 3z \leq -1 \end{aligned}$$

Since both the constants of the constraints are negative, the origin is not a feasible set for the solution here and we need to write the auxiliary problem

$$\begin{array}{l} \zeta_2 = -w_0 \\ \hline w_1 = -1 + 2x + y + 2z + w_0 \\ w_2 = -1 + x + 3y + 3z + w_0 \end{array}$$

Pivoting between w_1 and w_0 gets us the dictionary:

$$\begin{array}{l} \zeta_3 = -1 + 2x + y + 2z - w_1 \\ \hline w_0 = 1 - 2x - y - 2z + w_1 \\ w_2 = -x + 2y + z + w_1 \end{array}$$

Pivoting between y and w_0 gets us the dictionary:

$$\begin{array}{l} \zeta_4 = -w_0 \\ \hline y = 1 - 2x - 2z + w_1 - w_0 \\ w_2 = 2 - 5x - 3z + 3w_1 - 2w_0 \end{array}$$

This shows that $y = 1$ can be taken as the starting point for the feasible solution.Pivoting between y and z gets us the dictionary:

$$\begin{array}{l} \zeta_5 = -2 + x - 3y - 2w_1 \\ \hline z = \frac{1}{2} - x - \frac{1}{2}y + \frac{1}{2}w_1 \\ w_2 = \frac{1}{2} - 2x + \frac{3}{2}y + \frac{3}{2}w_1 \end{array}$$

Pivoting between x and w_2 gets us the dictionary:

$$\begin{array}{c} \zeta_6 = -\frac{7}{4} - \frac{3}{2}y - \frac{5}{4}w_1 - \frac{1}{2}w_2 \\ \hline z = \frac{1}{4} - 2\frac{3}{2}y - \frac{1}{4}w_1 + \frac{1}{2}w_2 \\ x = \frac{1}{4} + \frac{3}{2}y + \frac{3}{4}w_1 - \frac{1}{2}w_2 \end{array}$$

The optimal will be achieved at:$(\frac{1}{4}, 0, \frac{1}{4})$ with the optimal value:$\frac{7}{4}$. The solutions to the primal and dual problems prove the strong duality theorem as the duality gap is $\mathbf{0}$.