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ON TIKHONOV REGULARIZATION FOR IMAGE RECONSTRUCTION IN PARALLEL MRI

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ABSTRACT

Parallel imaging using multiple receiver coils has emerged as an effective tool to reduce imaging time in various MRI applications. When a large number of receiver channels are used to achieve large acceleration factors, the image reconstruction problem can become very ill-conditioned. This problem can be alleviated by optimizing the geometry of the coils or by mathematical regularization. Among the regularization methods, the Tikhonov scheme is most popular because of rough Gaussianity of the data noise, the easiness to incorporate prior information, as well as the existence of a closed-form solution. A central issue in implementing the Tikhonov scheme is the choice of the regularization parameter and the regularization image, which is addressed systematically in this paper. A new algorithm is also proposed for generating the regularization image and selecting the regularization parameter. Experimental results will be shown to demonstrate the performance of the algorithm.

1. INTRODUCTION

The idea of using multiple receiver coils to improve imaging speed in MRI dates back to the late 80's and early 90's [1–4]. However, practical methods for parallel imaging using multiple receiver coils emerged only recently with the development of the SMASH (Simultaneous Acquisition of Spatial Harmonics) technique [5], and the SENSE (SENSitivity Encoding) technique [6]. It has been demonstrated that the parallel imaging can also reduce various artifacts [7, 8]. Since then, many variations of SMASH and SENSE have been proposed, including PILS [9], SPACE-RIP [10], K-SENSE [11], spiral SENSE [12], which incorporate different coil sensitivity assumptions and k-space sampling patterns. A well-known problem is SENSE imaging is the amplification of data noise due to the ill-conditioned nature of the inverse problem. This problem can be alleviated by optimizing the geometry of the coils [13, 14], or by mathematical regularization [15–17]. Among the regularized methods, the Tikhonov scheme [18, 19] is most popular because the data noise is roughly Gaussian, it is easy to incorporate prior information, and there exists a closed-form solution. A central issue in implementing the Tikhonov scheme is the choice of the regularization parameter and the regularization image. Some encouraging preliminary work has been done in addressing this issue [18, 19]. This paper addresses this issue systematically so as to enhance the effectiveness of the regularization method.

The rest of the paper is organized as follows. First, the image reconstruction problem is formulated and discussed. Then, the Tikhonov regularization scheme is described, with an emphasis on the selection of the regularization parameter and image. Finally, a set of exemplary reconstruction results is shown to demonstrate the effect of different choices of the algorithm parameters.

2. DIRECT IMAGE RECONSTRUCTION

The imaging equation of parallel imaging using an array of L receiver coils with sensitivity $s_\ell(\vec{r})$ can be expressed as

$$D_\ell(\vec{k}_m) = \int \rho(\vec{r}) s_\ell(\vec{r}) e^{-i2\pi \vec{k}_m \vec{r}} d\vec{r}, \tag{1}$$

where $\rho(\vec{r})$ is the desired image function and $D_\ell(\vec{k}_m)$ is the data measured at k-space location $\vec{k}_m$ by the lth coil. The term *parallel imaging* comes from the fact that $D_\ell(\vec{k}_m)$ are acquired simultaneously for $1 \le \ell \le L$, and *sensitivity encoding* refers to the spatial encoding effect of $s_\ell(\vec{r})$ in Eq. (1). For simplicity, we consider only Cartesian sampling, and in this case the separability of the Fourier transform can be invoked to reduce the reconstruction problem to a 1D problem (e.g., along the phase encoding direction only). As a result, we can rewrite the imaging equation as

$$D_\ell(n\Delta\hat{k}) = \int_{-B/2}^{B/2} \rho(x) s_\ell(x) e^{-i2\pi n \Delta\hat{k} x} dx, \tag{2}$$

where $\rho(x)$ is assumed to be supported on $|x| < B/2$, and $\Delta\hat{k}$ is often chosen to be $R\Delta k = R/B$, with R being the data acquisition acceleration factor. Clearly, the Nyquist sampling criterion is satisfied when $R = 1$; otherwise, the k-space signal $D_\ell(n\Delta k)$ is under-sampled by a factor of R. When there is no data truncation, the image at location

$B/2-\hat{B} < x < B/2$ obtained from the ℓth channel is given by

$$d_\ell(x) = \sum_{m=0}^{R-1} \rho(x - m\hat{B}) s_\ell(x - m\hat{B}), \tag{3}$$

where $\ell = 1, 2, \cdots, L$, and $\hat{B} = B/R$. Equation (3) can be rewritten in matrix form as

$$\mathbf{S}\vec{\rho} = \vec{d}, \tag{4}$$

where

$$\mathbf{S} = \begin{bmatrix} s_1(x) & s_1(x-\hat{B}) & \cdots & s_1(x-(R-1)\hat{B}) \\ s_2(x) & s_2(x-\hat{B}) & \cdots & s_2(x-(R-1)\hat{B}) \\ \vdots & \vdots & & \vdots \\ s_L(x) & s_L(x-\hat{B}) & \cdots & s_L(x-(R-1)\hat{B}) \end{bmatrix},$$

$$\vec{\rho} = \begin{bmatrix} \rho(x) \\ \rho(x-\hat{B}) \\ \vdots \\ \rho(x-(R-1)\hat{B}) \end{bmatrix}, \quad \text{and} \quad \vec{d} = \begin{bmatrix} d_1(x) \\ d_2(x) \\ \vdots \\ d_L(x) \end{bmatrix}.$$

Taking into account data uncertainties, Eq. (4) becomes

$$\mathbf{S}(\vec{\rho} + \Delta\vec{\rho}) = \vec{d} + \Delta\vec{d}. \tag{5}$$

There are two primary methods to solve for $\vec{\rho}$: the least-squares (LS) and the minimum-variance (MV) methods (assuming $\Delta\vec{d}$ is a Gaussian random vector and its covariance matrix $\mathbf{\Psi}$ is known). The LS solution is given by [20]

$$\vec{\rho}_{\text{LS}} = (\mathbf{S}^H\mathbf{S})^{-1}\mathbf{S}^H\vec{d}, \tag{6}$$

and the MV solution is [20]

$$\vec{\rho}_{\text{MV}} = (\mathbf{S}^H\mathbf{\Psi}^{-1}\mathbf{S})^{-1}\mathbf{S}^H\mathbf{\Psi}^{-1}\vec{d}. \tag{7}$$

Some known results about $\vec{\rho}_{\text{LS}}$ and $\vec{\rho}_{\text{MV}}$ are summarized below.

Remark 1: $\vec{\rho}_{\text{LS}} = \vec{\rho}_{\text{MV}}$ *if* $\mathbf{S}$ *is a square matrix of full rank and* $\mathbf{\Psi}$ *is non-singular, or if* $\mathbf{\Psi}$ *is an identity matrix.*

Remark 2: *The variance of the reconstruction error* ($\Delta\rho$) *at location* x *is given by*

$$\sigma_{\text{LS}}(x) = \sqrt{[(\mathbf{S}^H\mathbf{S})^{-1}\mathbf{S}^H\mathbf{\Psi}\mathbf{S}(\mathbf{S}^H\mathbf{S})^{-1}]_x} \tag{8}$$

for the LS *method, and by*

$$\sigma_{\text{MV}}(x) = \sqrt{[(\mathbf{S}^H\mathbf{\Psi}^{-1}\mathbf{S})^{-1}]_x} \tag{9}$$

for the MV *method.*

Remark 3: σ_{LS} *and* σ_{MV} *increases as* $\mathbf{S}$ *becomes more ill-conditioned.*

3. REGULARIZED RECONSTRUCTION

The regularized solution to Eq. (5) in the Tikhonov framework is the minimizer of the following functional:

$$\vec{\rho}_{\text{reg}} = \arg\min_{\rho} \left\{ \|\mathbf{S}\vec{\rho} - \vec{d}\|^2 + \lambda^2 \|\mathbf{A}(\vec{\rho} - \vec{\rho}_r)\|^2 \right\}, \tag{10}$$

where the regularization parameter λ is chosen to balance the data fitting error and the penalty (or regularization) term formed from the difference between the expected solution and a prior image $\vec{\rho}_r$ known as the regularization image. A closed-form solution for $\vec{\rho}_{\text{reg}}$ exists and is given by

$$\vec{\rho}_{\text{reg}} = \vec{\rho}_r + (\mathbf{S}^H\mathbf{S} + \lambda^2\mathbf{A}^H\mathbf{A})^{-1}\mathbf{S}^H(\vec{d} - \mathbf{S}\vec{\rho}_r). \tag{11}$$

We next discuss how to select λ and $\vec{\rho}_r$. The $\mathbf{A}$ is assumed to be an identity matrix in this paper.

3.1. Construction of $\vec{\rho}_r$

There are basically three schemes to construct $\vec{\rho}_r$: (a) setting $\vec{\rho}_r = 0$, (b) recycling an initial SENSE reconstruction to create $\vec{\rho}_r$, and (c) collecting additional data to generate $\vec{\rho}_r$.

Scheme a corresponds to, perhaps, the simplest version of the Tikhonov regularization scheme. It was used in [15] with some success. In Scheme b, the conventional SENSE algorithm is used to obtain an initial reconstruction, which is then filtered by a median filter to suppress any residual aliasing artifacts [18]. However, if the matrix $\mathbf{S}$ is highly ill-conditioned within a large region, the filtering step may not be effective in suppressing the aliasing artifacts. Scheme c is the focus of the paper. We propose to use a generalized series (GS) model [21] to derive $\vec{\rho}_r$. Specifically, during the data acquisition stage, each coil collects several (say, 8) additional encodings at the Nyquist rate in the center of k-space. An image is then reconstructed from the k-space data in each coil using the GS model [21] whose basis functions are formed from the reference data collected at the Nyquist rate. Details of the GS model-based image reconstruction algorithm can be found in [21]. Let $\rho_\ell^{\text{GS}}(x)$ be the GS image obtained from the ℓth coil. We have

$$\rho_\ell^{\text{GS}}(x) \approx \rho(x)s_\ell(x). \tag{12}$$

A least-squares estimate of $\rho(x)$ from $\rho_\ell^{\text{GS}}(x)$ is used for $\rho_r(x)$, which is given by

$$\rho_r(x) = \frac{\sum_{\ell=1}^{L}[\rho_\ell^{\text{GS}}(x)s_\ell^*(x)]}{\sum_{\ell=1}^{L}[s_\ell(x)s_\ell^*(x)]}. \tag{13}$$

This method is particularly suitable for dynamic imaging applications, where Eq. (12) holds because the GS model can often generate high-quality images with a small number of encodings [21].

3.2. Selection of λ

According to Eqs. (10) and (11), the regularization parameter λ balances the tradeoff between the conditioning of the reconstruction and the data misfit. A straightforward way to select the regularization parameter is to set λ heuristically as a constant over the entire image. A more elaborate way is to use the L-curve method [19]. Different from these approaches where the λ is selected for each equation independently, our method chooses λ in a spatially dependent fashion, taking into account all the equations simultaneously. More specifically, we set $\lambda(x)$ to be a linear function of the local condition number of $\mathbf{S}$, i.e.,

$$\lambda(x) = \alpha\kappa(\mathbf{S}) + \beta, \tag{14}$$

for $B/2 - \hat{B} < x < B/2$. This scheme is motivated by the fact that the larger $\kappa(\mathbf{S})$, the heavier the regularization is needed for Eq. (11). To determine α and β, we rewrite Eq. (14) as

$$\lambda(x) = \frac{\kappa(\mathbf{S}) - \kappa_{\min}(\mathbf{S})}{\kappa_{\max}(\mathbf{S}) - \kappa_{\min}(\mathbf{S})}(\lambda_{\max} - \lambda_{\min}) + \lambda_{\min}. \tag{15}$$

Determination of $\lambda_{\min}$ and $\lambda_{\max}$ is essential for the proposed algorithm, which is done as follows.

(a) Choose $\lambda_{\min}$ such that the minimal requirement is satisfied to prevent $\mathbf{S}$ from being overly ill-conditioned. Note that

$$\kappa((\mathbf{S}^H\mathbf{S} + \lambda^2\mathbf{I})^{-1}\mathbf{S}^H) = \frac{\max_i \sigma_i}{\min_i\{\sigma_i + \lambda^2/\sigma_i\}}, \tag{16}$$

where σ_i is the ith singular value of $\mathbf{S}$. Because $\kappa((\mathbf{S}^H\mathbf{S} + \lambda^2\mathbf{I})^{-1}\mathbf{S}^H)$ characterizes the potential magnification of $\Delta\vec{d}$ in the final reconstruction, it is useful to bound it to a user-specified constant K (say, 10). This condition can be satisfied by setting $\lambda_{\min}$ to

$$\lambda_{\min} = \arg\min_{\lambda}\left\{\frac{\max_i \sigma_i}{\min_i(\sigma_i + \lambda^2/\sigma_i)} < K\right\}. \tag{17}$$

(b) Choose $\lambda_{\max}$ such that the averaged fitting error is bounded to a user-specified constant ϵ. Specifically, we set $\lambda_{\max}$ to

$$\lambda_{\max} = \arg\max_{\lambda}\left\{\sum_x \|\mathbf{S}\vec{\rho}_{\text{reg}} - \vec{d}\| \leq \epsilon\right\}, \tag{18}$$

where ϵ is determined by the number of equations, the number of coils and the noise variance.

4. RESULTS AND DISCUSSION

Figure 1 shows a set of exemplary reconstructions from real experimental data acquired with 4 receiver coils and $R = 4$. As can be seen from Fig. 1(a), the basic SENSE reconstruction has noticeable residual aliasing artifacts and a loss of signal-to-noise ratio. Reconstructions with different $\vec{\rho}_r$ and λ are shown in (b)-(f). In (b), $\vec{\rho}_r = 0$, and λ was a constant. In (c), (d) and (e), $\lambda(x)$ was chosen in a spatially adaptive manner as described in Section 3, but with different regularization images. Specifically, we set, respectively, $\vec{\rho}_r = 0$, $\vec{\rho}_r =$ median-filtered SENSE, and $\vec{\rho}_r =$ GS reconstruction with 8 additional encodings. In (f), λ was chosen by the L-curve method with $\vec{\rho}_r$ the same as (e). As can be seen, both λ and the quality of $\vec{\rho}_r$ have direct effects on the final reconstruction. The proposed algorithm for selecting λ and $\vec{\rho}_r$ yields superior reconstruction results than existing Tikhonov regularization schemes.

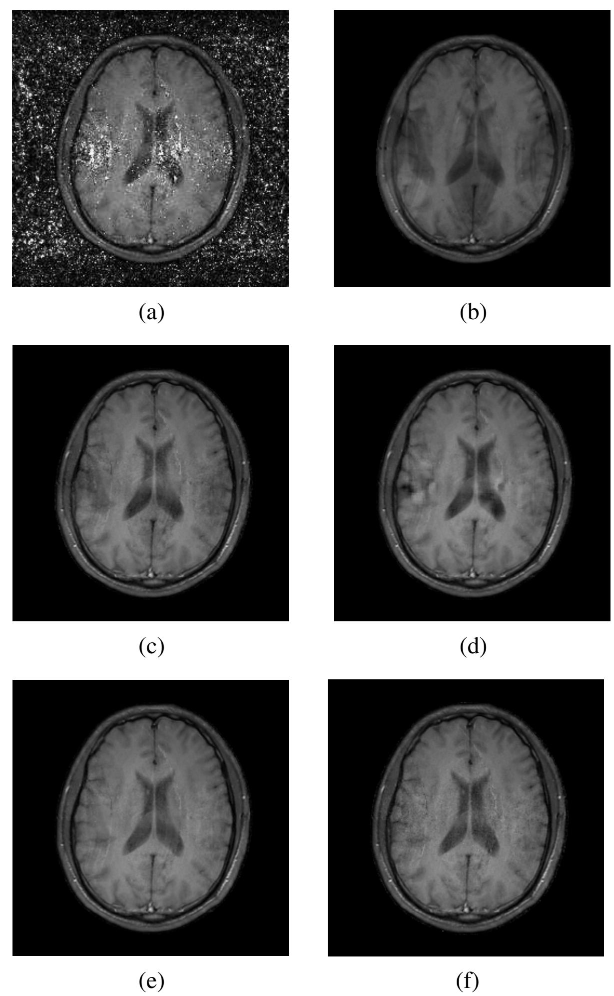

Fig. 1. SENSE reconstructions from a real data set acquired with 4 coils and $R = 4$. (a) Basic SENSE reconstruction; and regularized SENSE reconstructions with: (b) $\vec{\rho}_r = 0$ and λ being a constant, (c) $\vec{\rho}_r = 0$ and spatially adaptive λ, (d) $\vec{\rho}_r =$ median-filtered SENSE, (e) the proposed method, and (f) λ is chosen by L-curve.

5. CONCLUSION

The paper has presented an in-depth analysis of the Tikhonov scheme for regularized image reconstruction in parallel MRI. An improved algorithm has also been proposed for selection of the regularization parameter and the generation of the regularization image. The algorithm should prove useful for parallel MRI with a large phased array to achieve high acceleration factors.

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