Computational MR imaging
Laboratory 7: Parallel Imaging III: Non-Cartesian Imaging and Iterative Reconstruction

Report is due on Wednesday before the next lab session at 23:50. Please upload your report on StudOn.

Learning objectives

- deal with iterative parallel imaging reconstruction for non-Cartesian sampling patterns, using gradient descent.
- explore the conjugate gradient SENSE method.

1. Derivation of gradient descent (analytical):

In gradient descent methods, we need to calculate the gradient of our objective function. In the lecture (slide 21), we said that the gradient is:

$$\frac{\partial}{\partial x}\|Ax - b\|_2^2 = 2A^T(Ax - b)$$

Show that this is indeed true. For this exercise, assume that A, x and b are real valued (the derivation for complex numbers is more involved). To simplify the notation, you can do the derivation with the assumption that A is a 2x2 matrix without loss of generality. You should only need standard linear algebra and vector analysis for this proof. Hints:

i. Remember these expressions: $\|Ax - b\|_2^2 = (Ax - b)^T(Ax - b)$

$$x^T A^T b = (b^T A x)^T$$

ii. It will be useful during the derivation to use the following substitutions to simplify the notation: $\underbrace{2b^T A}_{c^T} x$ and $x^T \underbrace{A^T A}_{B} x$

2. Iterative image reconstruction with gradient descent

You will find the following items in the data file data_radial_brain_4ch.mat:

kdata (512,64,4): radial k-space data, 64 spokes, 512 readout points, 4 channels
c (256,256,4): receive coil sensitivity maps, 4 channel coil
k (512,64): radial trajectory
w *(512,64):* density compensation
img_senscomb (256,256): Sensitivity combined fully sampled ground truth

2.1. Plot the data and at the sampling trajectory.

2.2. Build a NUFFT operator in the same way as in lab 4 and do a simple gridding reconstruction using density compensation.

2.3. Implement a gradient descent reconstruction as described in the lecture:
   2.3.1. Build the forward and adjoint operators
   2.3.2. Choose a stepsize $t=10^{-2}$
   2.3.3. Initialize u with a zeros matrix
   2.3.4. Implement the gradient descent update equation
   2.3.5. Run for e.g. 100 iterations

2.4. Plot the fully sampled ground truth, the regridding reconstruction, the gradient descent reconstructed image and the difference image between the reconstructed image and the ground truth.

2.5. Plot the MSE to the ground truth (*img_senscomb)* and the l2 norm of the gradient over the iterations.

2.6. Play around with hyperparameters, such as a stepsize and the number of iterations, and find the optimal hyperparameters. Plot MSEs like the task 2.5 for different hyperparameter setups and discuss convergences of them.

3. CG-SENSE

3.1. Implement CG-SENSE algorithm in CG-SENSE.py. Look at the appendix 1. Build a NUFFT operator in the same way as in lab 4 and do a simple gridding reconstruction using density compensation.

3.2. Perform a conjugate gradient SENSE reconstruction using the provided code. Compare the convergence behavior of CG to that of gradient descent in exercise You should see convergence in 20-30 iterations.

3.3. Plot the fully sampled ground truth, the regridding reconstruction, the CG reconstructed image and the difference image between the CG image and the ground truth.

3.4. Repeat CG reconstruction with noise instead of the k-space data and plot the result in k-space. Perform at least 500 iterations. What did you just obtain?

Appendix 1

Conjugate gradient algorithm

$\mathbf{x}_0 = 0$	Initializing $\mathbf{x}_0$ to zeros
$\mathbf{r}_0 := \mathbf{b} - \mathbf{A}\mathbf{x}_0$	Initializing residual $\mathbf{r}_0$
$\mathbf{p}_0 := \mathbf{r}_0$	Initializing $\mathbf{p}_0$
$k := 0$	
repeat	
$\quad \alpha_k := \frac{\mathbf{r}_k^T \mathbf{r}_k}{\mathbf{p}_k^T \mathbf{A} \mathbf{p}_K}$	Step length
$\quad \mathbf{x}_{k+1} := \mathbf{x}_k + \alpha_k \mathbf{p}_k$	Update step
$\quad \mathbf{r}_{k+1} := \mathbf{r}_k - \alpha_k \mathbf{A} \mathbf{p}_k$	Update residual
$\quad$ if $\mathbf{r}_{k+1} < \epsilon$	
$\qquad$ break	
$\quad \beta_k := \frac{\mathbf{r}_{k+1}^T \mathbf{r}_{k+1}}{\mathbf{r}_k^T \mathbf{r}_k}$	
$\quad \mathbf{p}_{k_1} := \mathbf{r}_{k+1} + \beta_k \mathbf{p}_k$	CG Search direction
$\quad k := k + 1$	
return $\mathbf{x}_{k+1}$	