Parallel Beam Reconstruction

SS 2022

Y. Huang, F. Wagner, J. Maier, A. Preuhs, C. Syben, A. Maier
Pattern Recognition Lab (CS 5)
Friedrich-Alexander University Erlangen-Nuremberg

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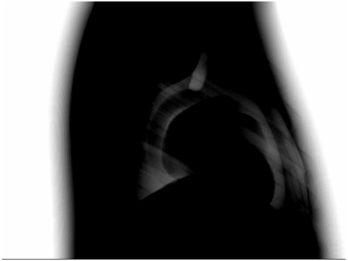
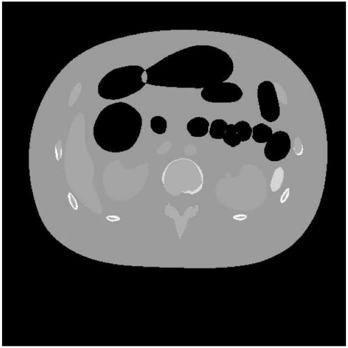
Topics

Basic Principles of Tomography

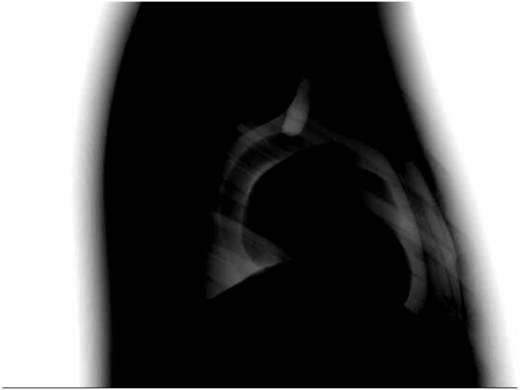
- $\pi O \mu O \sigma$ = tomos = slice





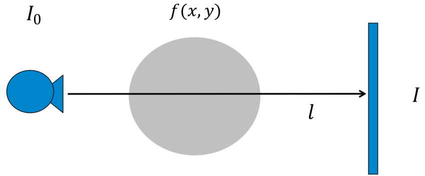
Basic Principles of Tomography (2)

- Idea: Observe object of interest from multiple sides



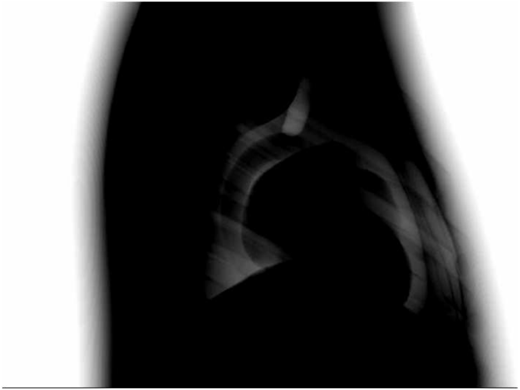
Topics

Projection – Physical Observations



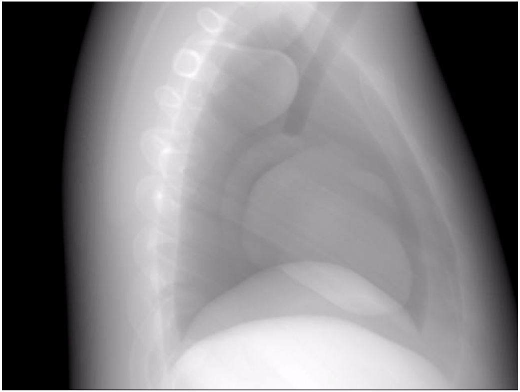
- X-ray Attenuation: $I = I_0 e^{-(\int f(x,y) \mathrm{d}l)}$
  - I_0: initial X-ray beam intensity
  - $f(x,y)$: absorption coefficient of material at position (x,y). (x,y) lies on beam line l

Projection – Physical Observations (2)



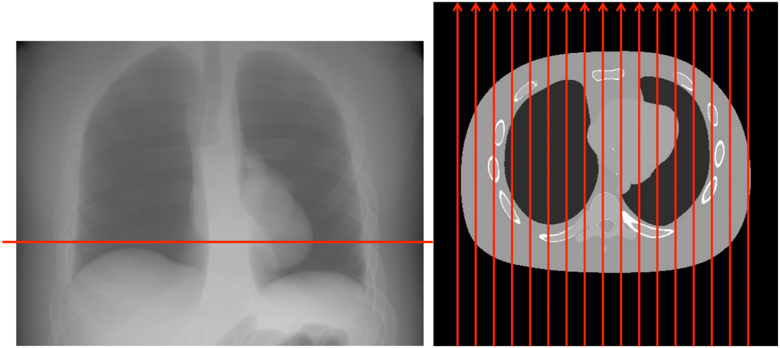
Observed Signal $I = I_0 e^{-p}$

Projection – Physical Observations (4)



Line Integral Data $p = \log(I/I_0)$

Projection Formation



Projection – Mathematical Formulation

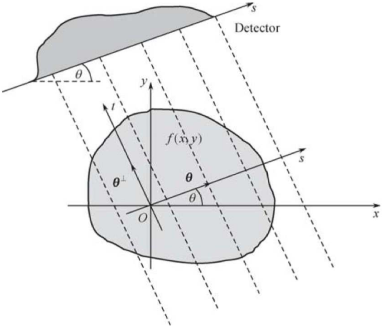
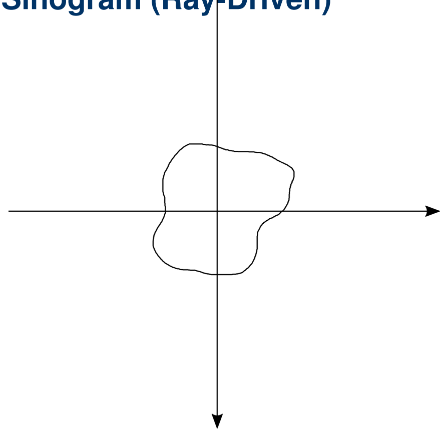


Image: Zeng, 2009

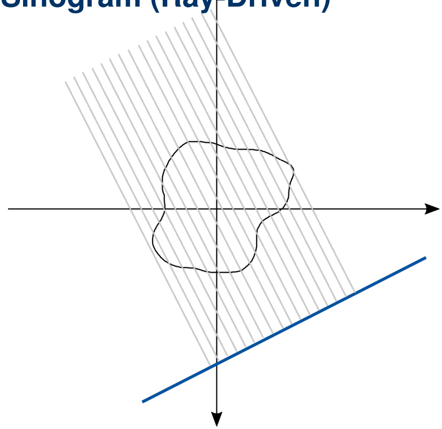
$$p(s, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos\theta + y \sin\theta - s) \mathrm{d}x \mathrm{d}y$$

Topics

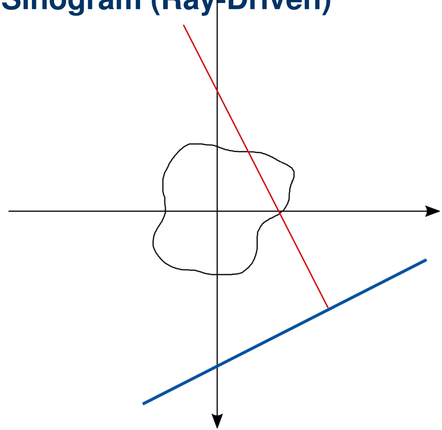
Parallel Beam Sinogram (Ray-Driven)



Parallel Beam Sinogram (Ray-Driven)

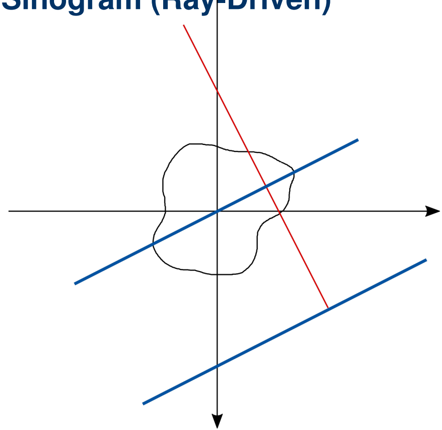


Parallel Beam Sinogram (Ray-Driven)



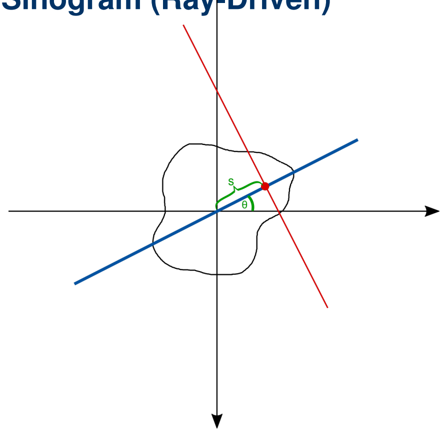
Rotation angle and detector index determines one ray.

Parallel Beam Sinogram (Ray-Driven)



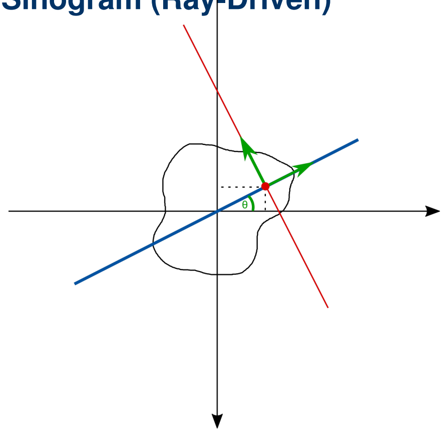
The real detector is equivalent to a detector passing origin.

Parallel Beam Sinogram (Ray-Driven)



Rotation angle θ and detector index s determines one ray.

Parallel Beam Sinogram (Ray-Driven)

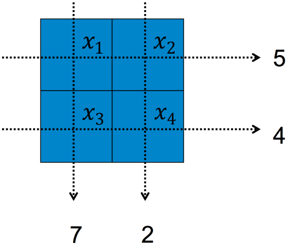


The ray orientation is orthogonal to the detector orientation, determined by θ.
Sampling along the ray for each distance Δt and sum them up as integral.

Topics

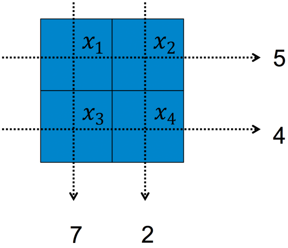
Reconstruction – Simple Example

- Solve the puzzle



Reconstruction – Simple Example

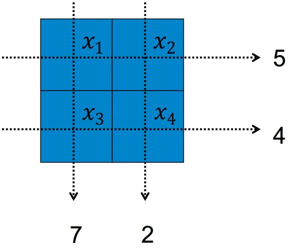
- Solve the puzzle



$$x_1 + x_3 = 7$$
$$x_2 + x_4 = 2$$
$$x_1 + x_2 = 5$$
$$x_3 + x_4 = 4$$

Reconstruction – Simple Example

- Solve the puzzle



$$x_1 + x_3 = 7$$
$$x_2 + x_4 = 2$$
$$x_1 + x_2 = 5$$
$$x_3 + x_4 = 4$$

$$x_1 = 3$$
$$x_2 = 2$$
$$x_3 = 4$$
$$x_4 = 0$$

Reconstruction – Simple Example (2)

- Projection can be formulated in matrix notation

$$\boldsymbol{P} = \boldsymbol{A}\boldsymbol{X}$$

$$\boldsymbol{P} = \begin{pmatrix} 7 \\ 2 \\ 5 \\ 4 \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad \boldsymbol{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Reconstruction – Simple Example (2)

- Solve with matrix inverse?

$$\boldsymbol{A}^{-1}\boldsymbol{P} = \boldsymbol{X}$$

- Common problem size:

$$\boldsymbol{A} \in \mathbb{R}^{512^3 \times 512^2 \times 512}$$

$$512^6 \cdot 4 \text{ Byte} = 2^{9 \cdot 6} \cdot 2^2 \text{ B} = 2^6 \cdot 2^{50} \text{ B}$$

$$= 64 \text{ PB} = 65536 \text{ TB}$$

Reconstruction – Example Projection

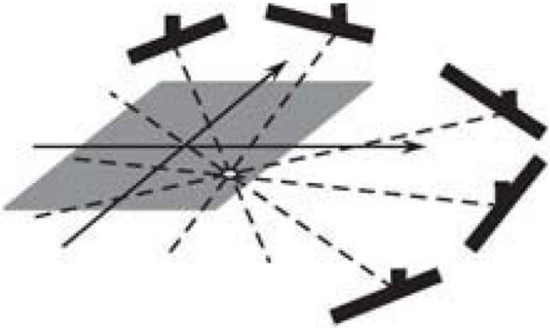


Image: Zeng, 2009

Reconstruction – Example Backprojection

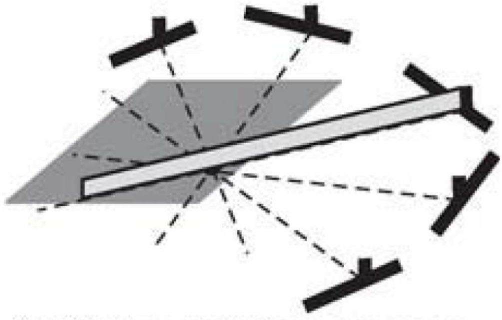


Image: Zeng, 2009

Reconstruction – Example Backprojection (2)

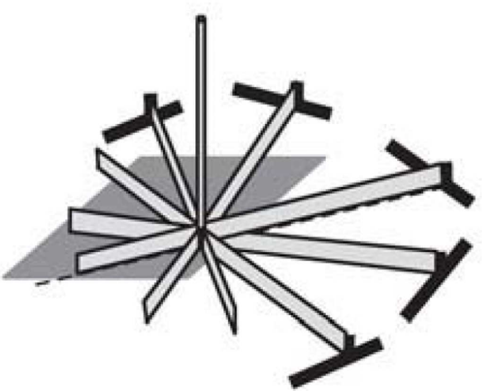


Image: Zeng, 2009

Reconstruction – Example Backprojection (3)

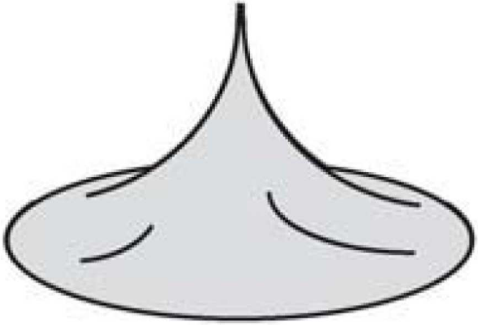


Image: Zeng, 2009

Reconstruction – Example "Negative Wings"

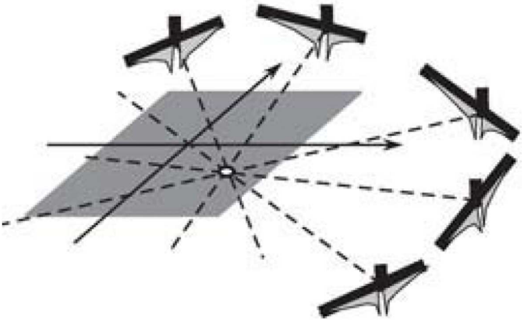


Image: Zeng, 2009

Reconstruction – Example Reconstruction

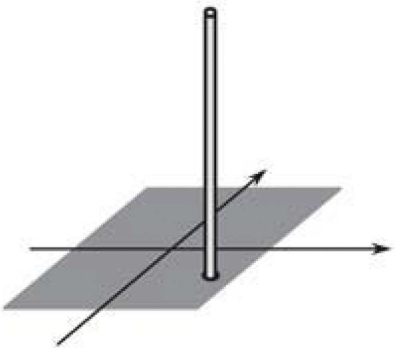


Image: Zeng, 2009

Topics

Fourier Transform

- 1D Fourier transform:

$$P(\omega) = \int_{-\infty}^{\infty} p(s) e^{-2\pi i s \omega} \mathrm{d}s$$

- 1D inverse Fourier transform:

$$p(s) = \int_{-\infty}^{\infty} P(\omega) e^{2\pi i s \omega} \mathrm{d}\omega$$

Convolution

- Convolution:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\mathrm{d}\tau$$

- Convolution theorem:

$$q(s) \quad = \quad f(s) * g(s)$$

$$Q(\omega) \quad = \quad F(\omega) \cdot G(\omega)$$

Topics

Central Slice Theorem

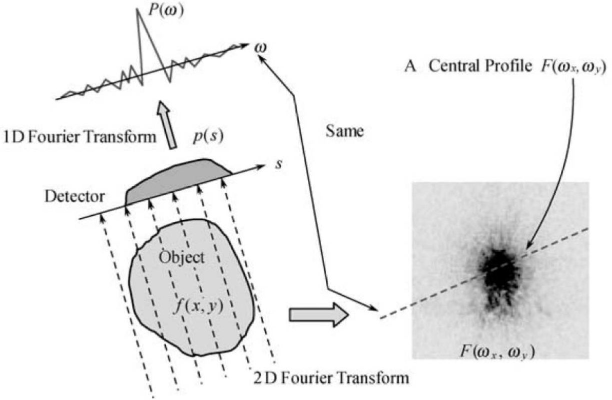


Image: Zeng, 2009

Idea for Reconstruction

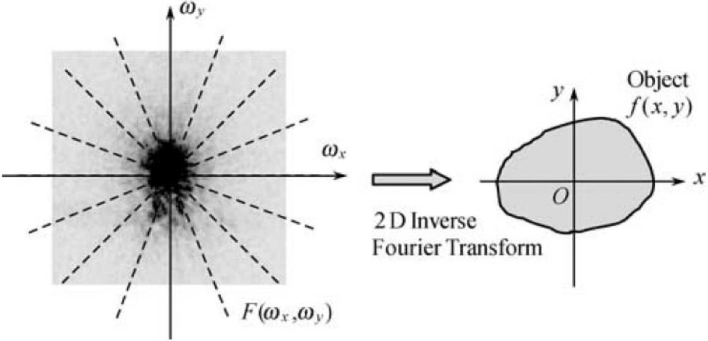


Image: Zeng, 2009

Topics

Filtered Backprojection

- Transform between polar coordinates and Cartesian coordinates, the Jacobian coefficient ω

$$\omega_x = \omega \cos\theta, \quad \omega_y = \omega \sin\theta$$

- Fourier transform in polar coordinates:

$$f(x, y) = \int_0^{2\pi} \int_0^{\infty} F_{\text{polar}}(\omega, \theta) e^{2\pi i \omega (x \cos\theta + y \sin\theta)} \omega \mathrm{d}\omega \mathrm{d}\theta$$

- Multiplication with $H(\omega) = |\omega|$ in Fourier domain

$$||$$

Convolution with $h(s)$ in image domain

Filtered Backprojection – Practical Algorithm

- Apply Filter on the detector row:

$$q(s,\theta) = h(s) * p(s,\theta)$$

- Backproject $q(s,\theta)$:

$$f(x,y) = \int_0^{\pi} q(s,\theta)\big|_{s=x\cos\theta+y\sin\theta} \mathrm{d}\theta$$

- **Backprojection means to add the detector value to each pixel $\rightarrow$ For each pixel at each rotation, you need to find the corresponding detector position and read its value.**

Discrete Spatial Form of the Ramp Filter

- Find the inverse Fourier transform of $|\omega|$
- Set cut-off frequency of the ramp filter at $\omega = \frac{1}{2}$

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Discrete Spatial Form of the Ramp Filter

- Find the inverse Fourier transform of $|\omega|$
- Set cut-off frequency of the ramp filter at $\omega = \frac{1}{2}$

$$h(s) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\omega| e^{2\pi i \omega s} \mathrm{d}\omega$$

Discrete Spatial Form of the Ramp Filter

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$$h(s) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\omega| e^{2\pi i \omega s} \mathrm{d}\omega$$

$$h(s) = \frac{1}{2} \frac{\sin \pi s}{\pi s} - \frac{1}{4} \left[\frac{\sin\left(\frac{\pi s}{2}\right)}{\frac{\pi s}{2}} \right]^2$$

Discrete Spatial Form of the Ramp Filter (2)

$$h(s) = \frac{1}{2} \quad \frac{\sin \pi s}{\pi s} - \frac{1}{4} \left[\frac{\sin\left(\frac{\pi s}{2}\right)}{\frac{\pi s}{2}} \right]^2$$

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$$h(s) = \frac{1}{2} \frac{\sin \pi s}{\pi s} - \frac{1}{4} \left[\frac{\sin\left(\frac{\pi s}{2}\right)}{\frac{\pi s}{2}} \right]^2$$

- Convert to discrete form: Let $s = n \cdot s_0$ (n integer, s_0 spacing)

$$h(n \cdot s_0) = \begin{cases} \frac{1}{4 s_0^2} & n = 0 \\ 0 & n \text{ even} \\ -\frac{1}{n^2 \pi^2 s_0^2} & n \text{ odd} \end{cases}$$

Discrete Spatial Form of the Ramp Filter (2)

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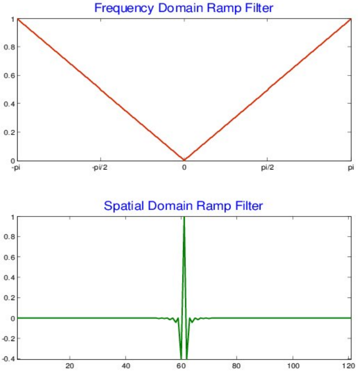
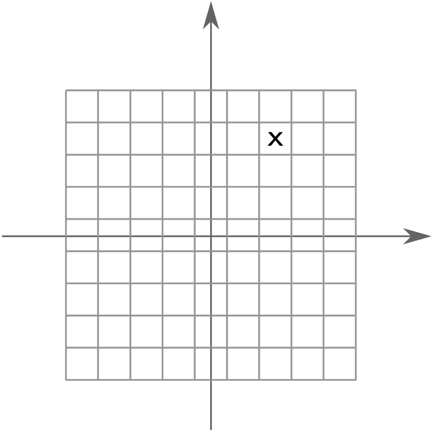
- Also known as the “Ramachandran-Lakshminarayanan” convolver or “Ram-Lak” convolver

Discrete Spatial Form of the Ramp Filter (3)

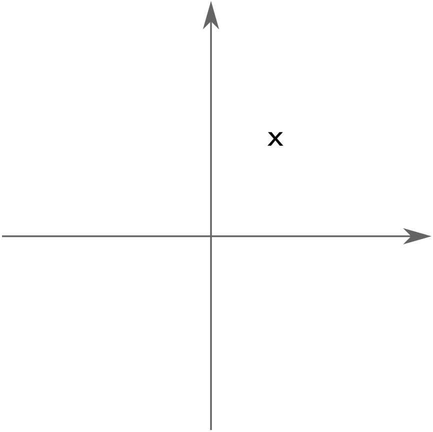


Topics

Parallel Beam Backprojection (Pixel-Driven)

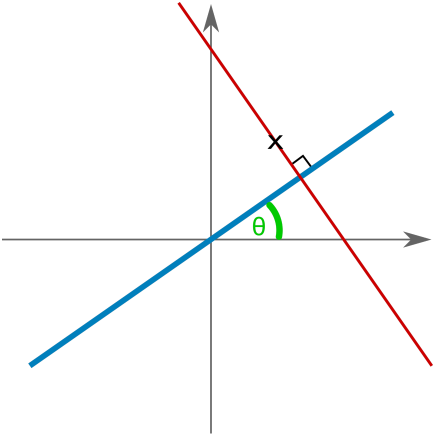


Parallel Beam Backprojection (Pixel-Driven)



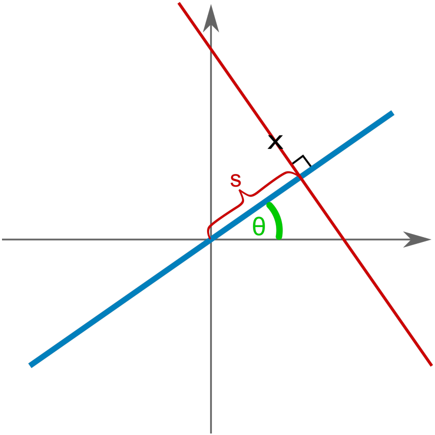
Get each pixel position x, y.

Parallel Beam Backprojection (Pixel-Driven)



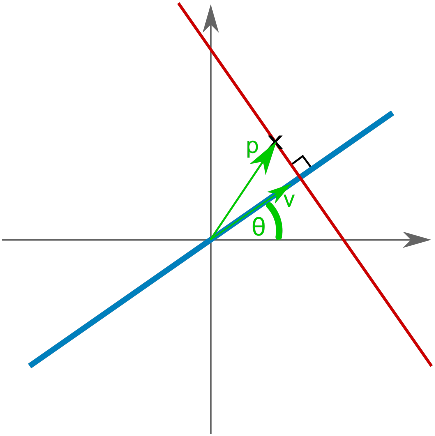
According to rotation angle θ, determine detector orientation.

Parallel Beam Backprojection (Pixel-Driven)



Want to calculate the position *s*, where the ray hits the detector.

Parallel Beam Backprojection (Pixel-Driven)



s can be calculated simply as the vector *p* projected to the direction vector *v*.