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Fuzzy inventory without backorder for fuzzy order quantity and fuzzy total demand quantity

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Abstract

In this paper, we consider the inventory problem without backorder such that both order and the total demand quantities are triangular fuzzy numbers $\tilde{Q}=(q_1,q_0,q_2)$, and $\tilde{R}=(r_1,r_0,r_2)$, respectively, where $q_1=q_0-\Delta_1$, $q_2=q_0+\Delta_2$, $r_1=r_0-\Delta_3$, $r_2=r_0+\Delta_4$ such that $0<\Delta_1<q_0$, $0<\Delta_2$, $0<\Delta_3<r_0$, $0<\Delta_4$, and r_0 is a known positive number. Under conditions $0\leqslant q_1<q_0<q_2<r_1<r_0<r_2$ we find the membership function $\mu_{G(\tilde{Q},\tilde{R})}(z)$ of the total fuzzy cost function $G(\tilde{Q},\tilde{R})$ and their centroid, then obtain order quantity q^{**} in the fuzzy sense and the estimate of the total demand quantity.

Scope and purpose

This paper deals with the inventory problem without backorder with total cost function $F(q)=cTq/2+ar/q$, $q>0$. In the classical inventory (without backorder) model, both the total demand over the planning time period $[0,T]$ and the period from ordering to arriving are fixed. In the real situation, the total demand r and order quantity q probably will be different from the values used in the total cost function. Also, r influences the values of T. In view of this circumstances, we consider the inventory problem in which both order and total demand quantities are triangular fuzzy numbers $\tilde{Q}=(q_1,q_0,q_2)$, and $\tilde{R}=(r_1,r_0,r_2)$, respectively, where $q_1=q_0-\Delta_1$, $q_2=q_0+\Delta_2$, $r_1=r_0-\Delta_3$, $r_2=r_0+\Delta_4$ such that $0<\Delta_1<q_0$, $0<\Delta_2$, $0<\Delta_3<r_0$, $0<\Delta_4$; where r_0 is a known number and q_0 is unknown. Letting $G(\tilde{Q},\tilde{R})=cT\tilde{Q}/2+a\tilde{R}/\tilde{Q}$, we use the extension principle to find the membership function $\mu_{G(\tilde{Q},\tilde{R})}$ of the fuzzy total cost function $G(\tilde{Q},\tilde{R})$ and their centroid (see Proposition 3). Therefore, given the value of q_1, q_0, q_2, r_1 and r_2, we can find an estimate of the total cost in the fuzzy sense. Finally, we make a comparison between the crisp sense and fuzzy sense by some numerical result.

Keywords: Inventory; Membership functions; Extension principle; Fuzzy inventory without backorder; Fuzzy economic order quantity; Fuzzy demand quantity

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1. Introduction

When we discuss the classical inventory without backorder model, we get the total cost function $F(q) = cTq/2 + ar/q$, $q > 0$, where c is the cost of storing one unit for one day, a is the cost of placing an order, q is the order quantity for one cycle, T is the length of the planning time period, measure in days, and r is the total demand over the planning time period $[0, T]$. Inventory in fuzzy sense have been discussed recently in papers such as [1–6]. In [3], inventory without backorder model is discussed. They fuzzify the order quantity q to a fuzzy number. In [1,5], they use two approaches to discuss fuzzy sets concepts in the backorder model. In the first, the order quantity is fuzzified and the shortage quantity is a real number [5]. In the second, shortage quantity is fuzzified and the order quantity is a real variable [1]. In [4], they consider fuzzy demand quantity and fuzzy product quantity in which the inventory system is the factory warehouse. In [2], they consider the backorder fuzzy inventory model. They replaced the extension principle by the function principle to solve this problem. They fuzzify some variables in the numerator, but, the order quantity q, both in the numerator and denominator, is a crisp variable. In this paper, we consider the order quantity q and the total demand quantity r as fuzzy numbers in the inventory model without the backorder. Because the total cost function contains the demand quantity q in both the numerator and denominator, this problem is more complex than [2].

In the crisp case, the economic order quantity q_* is solved by $\min_{q>0} F(q) = F(q_*)$, under the condition that both the total demand over the planning time period $[0, T]$ and the period from ordering to arrival are fixed. However, in the real situation, both of them will probably be different from the values used in the total cost function. Also, the total demand r over the planning time period $[0, T]$ could change due to some uncertain influence in the market. Thus, consider the total demand r as a fuzzy number near r_0 is more reasonable than a fixed value r_0. Furthermore, given a, c, T as constants, they effect q and r. It follows that q and r are subject to some kind of uncertainty. We will consider this problem under the condition that a, c, T are fixed values. Hence, since q and r influenced $F(q)$, we express them as the vague variables instead of crisp variables to treat this problem. Therefore, corresponding to crisp order quantity $q_0 > 0$ and crisp total demand r_0, we denote $\tilde{Q} = (q_1, q_0, q_2)$ and $\tilde{R} = (r_1, r_0, r_2)$ where $q_1 = q_0 - \Delta_1$, $q_2 = q_0 + \Delta_2$ ($\tilde{Q}$ is just around the crisp order quantity q_0) such that $0 < \Delta_1 < q_0$, $0 < \Delta_2$. Similarly, $r_1 = r_0 - \Delta_3$, $r_2 = r_0 + \Delta_4$ such that $0 < \Delta_3 < r_0$, $0 < \Delta_4$. Based on the fuzzy behavior, the value of Δ_i, $i = 1, 2, 3, 4$ may be determined by the decision maker. In other words, we consider the order quantity and the total demand quantity as fuzzy numbers $\tilde{Q}$ and $\tilde{R}$ with menbership function

$$\mu_{\tilde{Q}}(q) = \begin{cases} \dfrac{q - q_1}{q_0 - q_1} & q_1 \leqslant q \leqslant q_0, \\ \dfrac{q_2 - q}{q_2 - q_0} & q_0 \leqslant q \leqslant q_2, \\ 0 & \text{elsewhere}, \end{cases} \qquad \mu_{\tilde{R}}(r) = \begin{cases} \dfrac{r - r_1}{r_0 - r_1} & r_1 \leqslant r \leqslant r_0, \\ \dfrac{r_2 - r}{r_2 - r_0} & r_0 \leqslant r \leqslant r_2, \\ 0 & \text{elsewhere}, \end{cases}$$

where $0 < q_1 < q_0 < q_2 < r_1 < r_0 < r_2$, and we suppose r_0 is a known constant. We obtain the fuzzy total cost $G(\tilde{Q}, \tilde{R}) = cT\tilde{Q}/2 + a\tilde{R}/\tilde{Q}$. In Section 2, we use the extension principle to find the membership function $\mu_{G(\tilde{Q},\tilde{R})}$ of the fuzzy total cost $G(\tilde{Q}, \tilde{R})$. In Section 3, we find the centroid of

$\mu_{G(\tilde{Q},\tilde{R})}$. This centroid is an estimate of the total cost. In Section 4, under conditions $0 \leqslant q_1 < q_0 < q_2$, $0 \leqslant r_1 < r_0 < r_2$ we use a computer program to find the total cost in fuzzy sense $M_{ijk}(q_1, q_0, q_2, r_1, r_2)$ for fuzzy order quantity (q_1, q_0, q_2) and the fuzzy total demand quantity (r_1, r_0, r_2). In Section 5, if the centroid q^{**} of fuzzy order quantity is near the optimal crisp order quantity q^* and the centroid r^{**} of fuzzy total demand quantity is near the crisp total demand quantity r_0, then the estimate total cost approaches the crisp total cost.

2. Membership function of the fuzzy total cost function

The graph in Fig. 1 shows an inventory model without the backorder. Where q denotes the order quantity for one period, t_q denotes the length of one cycle and T denotes the length of the planning time period, both are measured in days.

For convenience, we denote

1. c: the cost of storing one unit for one day.
2. a: the cost of placing an order.
3. r: the total demand over the planning time period $[0, T]$.

Then $q/t_q = r/T$ and the total storing cost is given by

$$ct_q\frac{q}{2} = \frac{cTq^2}{2r}.$$

Furthermore, r/q is the number of periods of the time interval $[0, T]$ (r/q may be not an integer, for convenience we assume it is the number of periods). Thus the total cost of the planning time period $[0, T]$ is given by

$$F(q) = \left[ct_q\frac{q}{2} + a\right]\frac{r}{q} = \frac{cTq}{2} + \frac{ar}{q}, \quad q > 0.$$

By the classical approach for inventory model without backorder, we obtain the crisp economic order quantity $q_* = \sqrt{2acrT}/cT$ and $F(q_*) = \sqrt{2acrT}$ is the minimum total cost. If $r = r_0$ (r_0 is a known positive number), then $q_* = \sqrt{2acr_0T}/cT$, $F(q_*) = \sqrt{2acr_0T}$.

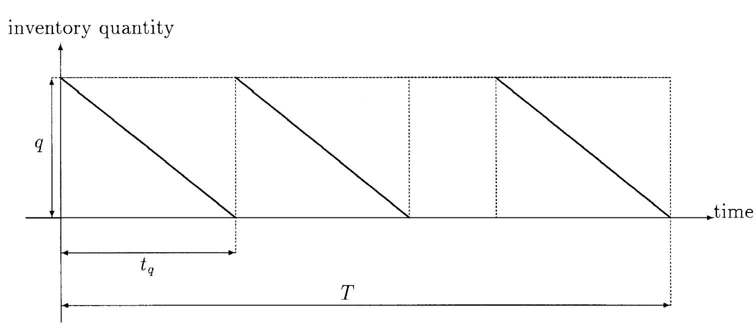

Fig. 1. Inventory without backorder.

Since the order quantity q may slightly change for some uncontrollable situations (as mentioned in the introduction) therefore, corresponding to the crisp order quantity $q_0(>0)$ which is an unknown number, we consider the fuzzy order quantity $\tilde{Q}=(q_0-\Delta_1, q_0, q_0+\Delta_2)$ which may change around q_0, where $0<\Delta_1<q_0$, $0<\Delta_2$. Suppose the membership function of $\tilde{Q}$ is given by

$$\mu_{\tilde{Q}}(q)=\begin{cases}\dfrac{q-q_1}{q_0-q_1} & q_1 \leqslant q \leqslant q_0,\\[2ex] \dfrac{q_2-q}{q_2-q_0} & q_0 \leqslant q \leqslant q_2,\\[2ex] 0 & \text{elsewhere},\end{cases} \tag{1}$$

where $q_1=q_0-\Delta_1$, $q_2=q_0+\Delta_2$, $0<\Delta_1<q_0$, $0<\Delta_2$, and Δ_1, Δ_2 are determined by the decision maker based on the uncertainty of the problem. Then

$$M_Q(q_1,q_0,q_2)=\tfrac{1}{3}(q_1+q_0+q_2)=q_0+\frac{\Delta_2-\Delta_1}{3}$$

is the centroid of $\mu_{\tilde{Q}}(q)$. Also, the total demand r is inherently extremely difficult to estimate without any error. Therefore, we consider the total demand quantity r is a fuzzy numbers $\tilde{R}$ near r_0 with membership function $\tilde{R}$ given by

$$\mu_{\tilde{R}}(r)=\begin{cases}\dfrac{r-r_1}{r_0-r_1} & r_1 \leqslant r \leqslant r_0,\\[2ex] \dfrac{r_2-r}{r_2-r_0} & r_0 \leqslant r \leqslant r_2,\\[2ex] 0 & \text{elsewhere},\end{cases} \tag{2}$$

where $0<q_1<q_0<q_2 \leqslant r_1<r_0<r_2$ (r_0 is a known constant). Similarly,

$$M_R(r_1,r_0,r_2)=\tfrac{1}{3}(r_1+r_0+r_2)$$

is the centroid of $\mu_{\tilde{R}}(r)$.

For the total cost function $F(q)$, consider q, r are variables and denote

$$G(q,r)=F(q)=\frac{cTq}{2}+\frac{ar}{q}, \quad 0<q<r,$$

where a, c and T are given positive numbers. Let $G(q,r)=z$, then

$$cTq^2-2zq+2ar=0$$

and hence

$$r=\frac{2zq-cTq^2}{2a}.$$

Therefore, for every $q \in [q_1, q_2]$, $r \geqslant 0$ if $z \geqslant cTq/2$. Hence, $z \geqslant \max_{q_1 \leqslant q \leqslant q_2} cTq/2 = cTq_2/2$ and

$$G^{-1}(z) = \left\{ (q, r) \mid \forall z \geqslant \frac{cTq_2}{2},\ z = \frac{cTq^2 + 2ar}{2q}, \quad 0 < q < r \right\}.$$

By the extension principle, the membership function of the fuzzy cost function is given by

$$\begin{aligned}\mu_{G(\tilde{Q},\tilde{R})}(z) &= \sup_{(q,r) \in G^{-1}(z)} [\mu_{\tilde{Q}}(q) \wedge \mu_{\tilde{R}}(r)] \\ &= \bigvee_{q_1 < q < q_2} \left[\mu_{\tilde{Q}}(q) \wedge \mu_{\tilde{R}}\left(\frac{2zq - cTq^2}{2a} \right) \right].\end{aligned} \tag{3}$$

In order to find $\mu_{G(\tilde{Q},\tilde{R})}(z)$, if $z \geqslant \sqrt{2acTr_j}$, $j = 1, 0, 2$ and $z \geqslant cTq_2/2$, let

$$u_j = \frac{\sqrt{z^2 - 2acTr_j} + z}{cT}, \quad j = 1, 0, 2.$$

Then $u_0 \leqslant q \leqslant u_1$ if $r_1 \leqslant (2zq - cTq^2)/2a \leqslant r_0$. Thus, by (2) we have

$$\mu_{\tilde{R}}\left(\frac{2zq - cTq^2}{2a} \right) = \begin{cases} f_2(q) & u_2 \leqslant q \leqslant u_0, \\ f_1(q) & u_0 \leqslant q \leqslant u_1, \\ 0 & \text{elsewhere,} \end{cases} \tag{4}$$

where

$$f_1(q) = \frac{-cTq^2 + 2zq - 2ar_1}{2a(r_0 - r_1)}, \qquad f_2(q) = \frac{cTq^2 - 2zq + 2ar_2}{2a(r_2 - r_0)}. \tag{5}$$

Since

$$\max_{j=1,0,2} \sqrt{2acTr_j} = \sqrt{2acTr_2},$$

so $z \geqslant \sqrt{2acTr_j}$ for all $j = 1, 0, 2$ if $z \geqslant \sqrt{2acTr_2}$, and hence z must satisfy $z \geqslant cTq_2/2$ and $z \geqslant \sqrt{2acTr_2}$. Therefore, we denote

$$a_1(q_2, r_2) = \max\left[\frac{cTq_2}{2}, \sqrt{2acTr_2} \right] (\geqslant F(q_*) = \sqrt{2acr_0T} = G(q_*, r_0)). \tag{6}$$

For convenience, under the conditions $u_0 \leqslant q_0$ and $u_1 \geqslant q_1$ the graph for Eqs. (1) and (4) are shown in Fig. 2. Also, Fig. 3 is the graph for Eqs. (1) and (4) under the conditions $u_0 \geqslant q_0$ and $u_2 \leqslant q_2$.

For $z \geqslant a_1(q_2, r_2)$ we consider $\mu_{G(\tilde{Q},\tilde{R})}(z)$ in Eq. (3) under conditions (A) and (B) as follows:

(A) $u_0 \leqslant q_0$ and $u_1 \geqslant q_1$: From Fig. 2, by the first equation in (1) and the second equation in (4), we have

$$\frac{q - q_1}{q_0 - q_1} = \frac{-cTq^2 + 2zq - 2ar_1}{2a(r_0 - r_1)}, \quad u_0 \leqslant q \leqslant u_1. \tag{7}$$

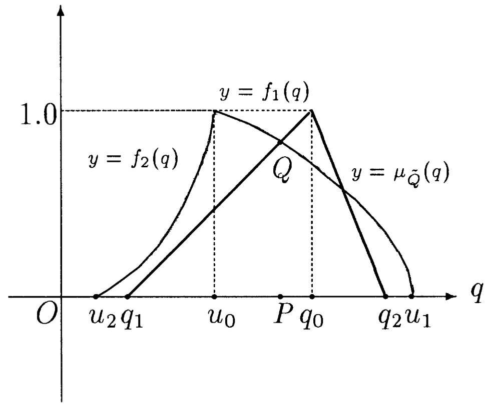

Fig. 2. Graph under (1).

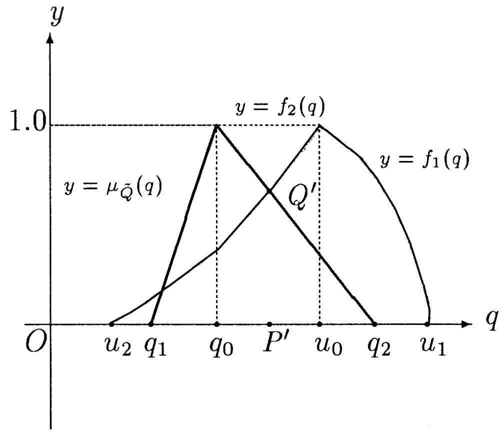

Fig. 3. Graph under (2).

Eq. (7) is equivalent to

$$cT(q_0 - q_1)q^2 - 2[(q_0 - q_1)z - a(r_0 - r_1)]q + 2a(q_0 r_1 - q_1 r_0) = 0 \tag{8}$$

The discriminant for (8) is given by

$$D_3 = [(q_0 - q_1)z - a(r_0 - r_1)]^2 - 2acT(q_0 - q_1)(q_0 r_1 - q_1 r_0). \tag{9}$$

Let $D_{30} = q_0 r_1 - q_1 r_0$ and $a_2(q_1, q_0, q_2, r_1, r_2) = \max[z_*, a_1(q_2, r_2)]$, where

$$z_* = \frac{\sqrt{2acT(q_0 - q_1)D_{30}} + a(r_0 - r_1)}{q_0 - q_1}.$$

From conditions (a) and (b) in Appendix A, if ($D_{30} \leqslant 0$ and $z \geqslant a_1(q_2, r_2)$) or ($D_{30} > 0$ and $z \geqslant a_2(q_1, q_0, q_2, r_1, r_2)$) we obtain PQ as follows:

$$PQ = \frac{(q_0 - q_1)z - a(r_0 - r_1) - cTq_1(q_0 - q_1) + \sqrt{D_3}}{cT(q_0 - q_1)^2} \equiv \mu_1(z) \quad \text{(say)} \tag{10}$$

(B) $u_0 \geqslant q_0$ and $u_2 \leqslant q_2$: From Fig. 3, by the second equation in (1) and the first equation in (4), we have

$$\frac{q_2 - q}{q_2 - q_0} = \frac{cTq^2 - 2zq + 2ar_2}{2a(r_2 - r_0)}, \quad u_2 \leqslant q \leqslant u_0. \tag{11}$$

Eq. (11) is equivalent to

$$cT(q_2 - q_0)q^2 - 2[(q_2 - q_0)z - a(r_2 - r_0)]q + 2a(q_2 r_0 - q_0 r_2) = 0. \tag{12}$$

The discriminant for (12) is given by

$$D_4 = [(q_2 - q_0)z - a(r_2 - r_0)]^2 - 2acT(q_2 - q_0)(q_2 r_0 - q_0 r_2). \tag{13}$$

Let $D_{40} = q_2 r_0 - q_0 r_2$ and $a_3(q_0, q_2, r_2) = \max[z_{**}, a_1(q_2, r_2)]$, where

$$z_{**} = \frac{\sqrt{2acT(q_2 - q_0)D_{40}} + a(r_2 - r_0)}{q_2 - q_0}.$$

Similar to (1), from conditions (c) and (d) in Appendix A, if $D_{40} \leqslant 0$ and $z \geqslant a_1(q_2, r_2)$ or $D_{40} > 0$ and $z \geqslant a_3(q_0, q_2, r_2)$, we obtain $P'Q'$ as follows:

$$P'Q' = \frac{-(q_2 - q_0)z + a(r_2 - r_0) + cTq_2(q_2 - q_0) - \sqrt{D_4}}{cT(q_2 - q_0)^2} \equiv \mu_2(z) \quad \text{(say)} \tag{14}$$

Thus, by Appendix A, we have the range of z as follows:

$$D_{30} \leqslant 0 \text{ or } D_{40} \leqslant 0\text{: } z \geqslant a_1(q_2, r_2) \text{ (from (a) and (c))}, \tag{15}$$

$$D_{30} > 0\text{: } z \geqslant a_2(q_1, q_0, q_2, r_1, r_2) \text{ (from (b))}, \tag{16}$$

$$D_{40} > 0\text{: } z \geqslant a_3(q_0, q_2, r_2) \text{ (from (d))}. \tag{17}$$

Furthermore,

$$Q_j \geqslant cTq_j \Leftrightarrow \sqrt{2acTr_j} \geqslant cTq_j, \quad \text{where } Q_j = \frac{cTq_j^2 + 2ar_j}{2q_j} \; (>0), \; j = 1, 0, 2. \tag{18}$$

Therefore, we can find the range of z for $PQ = \mu_1(z)$ and $P'Q' = \mu_2(z)$ in Section 2.1.

2.1. The range of z for $PQ = \mu_1(z)$ and $P'Q' = \mu_2(z)$

(i) $u_0 \leqslant q_0$ and $u_1 \geqslant q_1$: (from (A))
The range of z for $PQ = \mu_1(z)$ in (10) are given by the following:

- If $u_0 \leqslant q_0$, then $\sqrt{z^2 - 2acTr_0} \leqslant cTq_0 - z$ and hence

$$z \leqslant cTq_0 \quad \text{and} \quad z \leqslant Q_0. \tag{19}$$

◦ If $u_1 \geqslant q_1$, then $\sqrt{z^2 - 2acTr_1} \geqslant cTq_1 - z$ and hence

$$\begin{array}{lcl} z < cTq_1, & \text{or} & z \geqslant cTq_1, \\ z \geqslant Q_1 & & z \geqslant \sqrt{2acTr_1}. \end{array} \tag{20}$$

(ii) $u_0 \geqslant q_0$ and $u_2 \leqslant q_2$: (from (B))
Similar to (i) we find the range of z for $P'Q' = \mu_2(z)$ in (14) as follows:

◦ If $u_2 \leqslant q_2$, then

$$z \leqslant cTq_2 \quad \text{and} \quad z \leqslant Q_2. \tag{21}$$

◦ If $u_0 \geqslant q_0$, then we have

$$\begin{array}{lcl} z < cTq_0, & \text{or} & z \geqslant cTq_0, \\ z \geqslant Q_0 & & z \geqslant \sqrt{2acTr_0}. \end{array} \tag{22}$$

Under conditions $u_0 \leqslant q_0$, $u_1 \geqslant q_1$ and Eq. (15), (16), (18)–(20), since $a_1(q_2, r_2) \geqslant \sqrt{2acr_jT}$, $a_2(q_1, q_0, q_2, r_1, r_2) \geqslant \sqrt{2acr_jT}$ and $a_3(q_0, q_2, r_2) \geqslant \sqrt{2acr_jT}$, $\forall j = 1, 0, 2$; the range of z for $PQ = \mu_1(z)$ in (10) can be divided into four cases (1′)–(4′) as follows:

(1′) If $Q_0 \leqslant cTq_0$ and $Q_1 \leqslant cTq_1$, then

◦ by (18) we have $\sqrt{2acTr_1} \leqslant cTq_1$;
◦ by (19) and $Q_0 \leqslant cTq_0$ we have $z \leqslant Q_0$;
◦ by (20) and $Q_1 \leqslant cTq_1$ we have $z \geqslant Q_1$.

Hence:

(1′.1) If $D_{30} \leqslant 0$ (implies $z \geqslant a_1(q_2, r_2)$) and $\max[a_1(q_2, r_2), Q_1] \leqslant Q_0$, since $z \leqslant Q_0$ and $z \geqslant Q_1$ we have

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \mu_1(z) \quad \text{for } \max[a_1(q_2, r_2), Q_1] \leqslant z \leqslant Q_0. \tag{23}$$

(1′.2) If $D_{30} > 0$ (implies $z \geqslant a_2(q_1, q_0, q_2, r_1, r_2)$) and $\max[a_2(q_1, q_0, q_2, r_1, r_2), Q_1] \leqslant Q_0$, since $z \leqslant Q_0$ and $z \geqslant Q_1$, we have

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \mu_1(z) \quad \text{for } \max[a_2(q_1, q_0, q_2, r_1, r_2), Q_1] \leqslant z \leqslant Q_0. \tag{24}$$

(2′) If $Q_0 \leqslant cTq_0$ and $Q_1 \geqslant cTq_1$, similar to (1′), we have:

(2′.1) If $D_{30} \leqslant 0$ and $a_1(q_2, r_2) \leqslant Q_0$, since $a_1(q_2, r_2) \geqslant \sqrt{2acTr_1}$, $z \leqslant Q_0$ and $z \geqslant \sqrt{2acTr_1}$, we have

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \mu_1(z) \quad \text{for } a_1(q_2, r_2) \leqslant z \leqslant Q_0. \tag{25}$$

(2′.2) If $D_{30} > 0$ (implies $z \geqslant a_2(q_1, q_0, q_2, r_1, r_2)$) and $a_2(q_1, q_0, q_2, r_1, r_2) \leqslant Q_0$, (since $a_2(q_1, q_0, q_2, r_1, r_2) \geqslant \sqrt{2acTr_1}$), $z \leqslant Q_0$ and $z \geqslant \sqrt{2acTr_1}$, we have

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \mu_1(z) \quad \text{for } a_2(q_1, q_0, q_2, r_1, r_2) \leqslant z \leqslant Q_0. \tag{26}$$

(3′) If $Q_0 \geqslant cTq_0$ and $Q_1 \leqslant cTq_1$, similar to (1′) we get:

(3′.1) If $D_{30} \leqslant 0$ and $\max[a_1(q_2, r_2), Q_1] \leqslant cTq_0$, then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \mu_1(z) \quad \text{for } \max[a_1(q_2, r_2), Q_1] \leqslant z \leqslant cTq_0. \tag{27}$$

(3′.2) If $D_{30} > 0$ and $\max[a_2(q_1, q_0, q_2, r_1, r_2), Q_1] \leqslant cTq_0$, then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \mu_1(z) \quad \text{for } \max[a_2(q_1, q_0, q_2, r_1, r_2), Q_1] \leqslant z \leqslant cTq_0. \tag{28}$$

(4′) If $Q_0 \geqslant cTq_0$ and $Q_1 \geqslant cTq_1$, then we have:
(4′.1) If $D_{30} \leqslant 0$ and $a_1(q_2, r_2) \leqslant cTq_0$, then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \mu_1(z) \quad \text{for } a_1(q_2, r_2) \leqslant z \leqslant cTq_0. \tag{29}$$

(4′.2) If $D_{30} > 0$ and $a_2(q_1, q_0, q_2, r_1, r_2) \leqslant cTq_0$, then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \mu_1(z) \quad \text{for } a_2(q_1, q_0, q_2, r_1, r_2) \leqslant z \leqslant cTq_0. \tag{30}$$

Similarly, in cases (1′)–(4′), we replace D_{30} with D_{40} (denoted by $D_{30} \to D_{40}$); furthermore, take $a_2(q_1, q_0, q_2, r_1, r_2) \to a_3(q_0, q_2, r_2)$, $Q_0 \to Q_2$, $q_0 \to q_2$, $Q_1 \to Q_0$, $q_1 \to q_0$. Thus, under conditions $u_0 \geqslant q_0$ and $u_2 \leqslant q_2$, from Eqs. (15), (17), (18), (21) and (22), the range of z for $P'Q' = \mu_2(z)$ in (14) can be divided into four cases (5′)–(8′) as follows:
(5′) If $Q_2 \leqslant cTq_2$ and $Q_0 \leqslant cTq_0$, then we have:
(5′.1) If $D_{40} \leqslant 0$ (implies $z \geqslant a_1(q_2, r_2)$) and $\max[a_1(q_2, r_2), Q_0] \leqslant Q_2$, we have

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \mu_2(z) \quad \text{for } \max[a_1(q_2, r_2), Q_0] \leqslant z \leqslant Q_2. \tag{31}$$

(5′.2) If $D_{40} > 0$ (implies $z \geqslant a_3(q_0, q_2, r_2)$) and $\max[a_3(q_0, q_2, r_2), Q_0] \leqslant Q_2$, we have

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \mu_2(z) \quad \text{for } \max[a_3(q_0, q_2, r_2), Q_0] \leqslant z \leqslant Q_2. \tag{32}$$

(6′) If $Q_2 \leqslant cTq_2$ and $Q_0 \geqslant cTq_0$, we get:
(6′.1) If $D_{40} \leqslant 0$ and $a_1(q_2, r_2) \leqslant Q_2$, then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \mu_2(z) \quad \text{for } a_1(q_2, r_2) \leqslant z \leqslant Q_2. \tag{33}$$

(6′.2) If $D_{40} > 0$ and $a_3(q_0, q_2, r_2) \leqslant Q_2$, then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \mu_2(z) \quad \text{for } a_3(q_0, q_2, r_2) \leqslant z \leqslant Q_2. \tag{34}$$

(7′) If $Q_2 \geqslant cTq_2$ and $Q_0 \leqslant cTq_0$, we get:
(7′.1) If $D_{40} \leqslant 0$ and $\max[a_1(q_2, r_2), Q_0] \leqslant cTq_2$, then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \mu_2(z) \quad \text{for } \max[a_1(q_2, r_2), Q_0] \leqslant z \leqslant cTq_2. \tag{35}$$

(7′.2) If $D_{40} > 0$ and $\max[a_3(q_0, q_2, r_2), Q_0] \leqslant cTq_2$, then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \mu_2(z) \quad \text{for } \max[a_3(q_0, q_2, r_2), Q_0] \leqslant z \leqslant cTq_2. \tag{36}$$

(8′) If $Q_2 \geqslant cTq_2$ and $Q_0 \geqslant cTq_0$, we get:
(8′.1) If $D_{40} \leqslant 0$ and $a_1(q_2, r_2) \leqslant cTq_2$, then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \mu_2(z) \quad \text{for } a_1(q_2, r_2) \leqslant z \leqslant cTq_2. \tag{37}$$

(8′.2) If $D_{40} > 0$ and $a_3(q_0, q_2, r_2) \leqslant cTq_2$, then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \mu_2(z) \quad \text{for } a_3(q_0, q_2, r_2) \leqslant z \leqslant cTq_2. \tag{38}$$

2.2. The range of z in $\mu_{G(\tilde{Q},\tilde{R})}(z)$

In order to describe the range of z in $\mu_{G(\tilde{Q},\tilde{R})}(z)$, we denote the following sets:

$$S_1 = \{(q_1, q_0, q_2, r_1, r_2) \mid Q_1 \leqslant cTq_1, Q_0 \leqslant cTq_0, Q_2 \leqslant cTq_2\} \leftrightarrow \text{case } (1'), (5'),$$

$$S_2 = \{(q_1, q_0, q_2, r_1, r_2) \mid Q_1 \leqslant cTq_1, Q_0 \leqslant cTq_0, Q_2 \geqslant cTq_2\} \leftrightarrow \text{case } (1'), (7'),$$

$$S_3 = \{(q_1, q_0, q_2, r_1, r_2) \mid Q_1 \geqslant cTq_1, Q_0 \leqslant cTq_0, Q_2 \leqslant cTq_2\} \leftrightarrow \text{case } (2'), (5'),$$

$$S_4 = \{(q_1, q_0, q_2, r_1, r_2) \mid Q_1 \geqslant cTq_1, Q_0 \leqslant cTq_0, Q_2 \geqslant cTq_2\} \leftrightarrow \text{case } (2'), (7'),$$

$$S_5 = \{(q_1, q_0, q_2, r_1, r_2) \mid Q_1 \leqslant cTq_1, Q_0 \geqslant cTq_0, Q_2 \leqslant cTq_2\} \leftrightarrow \text{case } (3'), (6'),$$

$$S_6 = \{(q_1, q_0, q_2, r_1, r_2) \mid Q_1 \leqslant cTq_1, Q_0 \geqslant cTq_0, Q_2 \geqslant cTq_2\} \leftrightarrow \text{case } (3'), (8'),$$

$$S_7 = \{(q_1, q_0, q_2, r_1, r_2) \mid Q_1 \geqslant cTq_1, Q_0 \geqslant cTq_0, Q_2 \leqslant cTq_2\} \leftrightarrow \text{case } (4'), (6'),$$

$$S_8 = \{(q_1, q_0, q_2, r_1, r_2) \mid Q_1 \geqslant cTq_1, Q_0 \geqslant cTq_0, Q_2 \geqslant cTq_2\} \leftrightarrow \text{case } (4'), (8'),$$

$$T_1 = \{(q_1, q_0, q_2, r_1, r_2) \mid D_{30} \leqslant 0 \text{ and } D_{40} \leqslant 0\},$$

$$T_2 = \{(q_1, q_0, q_2, r_1, r_2) \mid D_{30} \leqslant 0 \text{ and } D_{40} > 0\},$$

$$T_3 = \{(q_1, q_0, q_2, r_1, r_2) \mid D_{30} > 0 \text{ and } D_{40} \leqslant 0\},$$

$$T_4 = \{(q_1, q_0, q_2, r_1, r_2) \mid D_{30} > 0 \text{ and } D_{40} > 0\}.$$

Also, for $j = 1, 2, \ldots, 8$, and $i = 1, 2, 3, 4$, we denote

$$E_{ji} = \{E(j,i,l,1) \leqslant E(j,i,l,2) \quad l = 1,2 \quad \text{and} \quad E(j,i,1,2) \leqslant E(j,i,2,1)\}.$$

Combine (1′)–(4′) and (5′)–(8′), we obtain $\mu_{G(\tilde{Q},\tilde{R})}(z)$ and the corresponding range of z as follows:

(a) For $i = 1, 2, 3, 4$, if $(q_1, q_0, q_2, r_1, r_2) \in S_1 \cap E_{1i} \cap T_i$, then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \begin{cases} \mu_1(z) & \text{if } E(1,i,1,1) \leqslant z \leqslant E(1,i,1,2), \\ \mu_2(z) & \text{if } E(1,i,2,1) \leqslant z \leqslant E(1,i,2,2), \\ 0 & \text{elsewhere,} \end{cases} \tag{39}$$

where by (1′) and (5′):

- for $i = 1$, by (1′.1), $E(1,1,1,1) = \max[a_1(q_2, r_2), Q_1]$ and $E(1,1,1,2) = Q_0$; by (5′.1), $E(1,1,2,1) = \max[a_1(q_2, r_2), Q_0]$ and $E(1,1,2,2) = Q_2$;
- for $i = 2$, by (1′.1), we have $E(1,2,1,k) = E(1,1,1,k)$, $k = 1,2$; by (5′.2), $E(1,2,2,1) = \max[a_3(q_0, q_2, r_2), Q_0]$ and $E(1,2,2,2) = Q_2$;
- for $i = 3$, by (1′.2) we have $E(1,3,1,1) = \max[a_2(q_1, q_0, q_2, r_1, r_2), Q_1]$ and $E(1,3,1,2) = Q_0$; by (5′.1), $E(1,3,2,k) = E(1,1,2,k)$, $k = 1,2$;
- for $i = 4$, by (1′.2) and (5′.2), we have $E(1,4,1,k) = E(1,3,1,k)$, $k = 1,2$ and $E(1,4,2,k) = E(1,2,2,k)$, $k = 1,2$.

(b) For $i = 1, 2, 3, 4$, if $(q_1, q_0, q_2, r_1, r_2) \in S_2 \cap E_{2i} \cap T_i$, then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \begin{cases} \mu_1(z) & \text{if } E(2,i,1,1) \leqslant z \leqslant E(2,i,1,2), \\ \mu_2(z) & \text{if } E(2,i,2,1) \leqslant z \leqslant E(2,i,2,2), \\ 0 & \text{elsewhere,} \end{cases} \tag{40}$$

where:

- by (1′), $E(2,i,1,k) = E(1,i,1,k)$, $i = 1,2,3,4$ for $k = 1,2$;
- by (7′), $E(2,i,2,1) = E(1,i,2,1)$, $E(2,i,2,2) = cTq_2$ for $i = 1,2,3,4$.

(c) For $i = 1, 2, 3, 4$, if $(q_1, q_0, q_2, r_1, r_2) \in S_3 \cap E_{3i} \cap T_i$, then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \begin{cases} \mu_1(z) & \text{if } E(3,i,1,1) \leqslant z \leqslant E(3,i,1,2), \\ \mu_2(z) & \text{if } E(3,i,2,1) \leqslant z \leqslant E(3,i,2,2), \\ 0 & \text{elsewhere,} \end{cases} \tag{41}$$

where:

- by (2′.1), $E(3,1,1,1) = a_1(q_2, r_2)$, $E(3,1,1,2) = Q_0$, $E(3,2,1,k) = E(3,1,1,k)$ for $k = 1,2$;
- by (2′.2), $E(3,3,1,1) = a_2(q_1, q_0, q_2, r_1, r_2)$, $E(3,3,1,2) = Q_0$, $E(3,4,1,k) = E(3,3,1,k)$ for $k = 1,2$;
- by (5′), $E(3,i,2,k) = E(1,i,2,k)$ for $i = 1,2,3,4$ and $k = 1,2$.

(d) For $i = 1, 2, 3, 4$, if $(q_1, q_0, q_2, r_1, r_2) \in S_4 \cap E_{4i} \cap T_i$, then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \begin{cases} \mu_1(z) & \text{if } E(4,i,1,1) \leqslant z \leqslant E(4,i,1,2), \\ \mu_2(z) & \text{if } E(4,i,2,1) \leqslant z \leqslant E(4,i,2,2), \\ 0 & \text{elsewhere,} \end{cases} \tag{42}$$

where:

- by (2′), $E(4,i,1,k) = E(3,i,1,k)$ for $i = 1,2,3,4$ and $k = 1,2$;
- by (7′), $E(4,i,2,k) = E(2,i,2,k)$ for $i = 1,2,3,4$ and $k = 1,2$.

(e) For $i = 1, 2, 3, 4$, if $(q_1, q_0, q_2, r_1, r_2) \in S_5 \cap E_{5i} \cap T_i$, then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \begin{cases} \mu_1(z) & \text{if } E(5,i,1,1) \leqslant z \leqslant E(5,i,1,2), \\ \mu_2(z) & \text{if } E(5,i,2,1) \leqslant z \leqslant E(5,i,2,2), \\ 0 & \text{elsewhere,} \end{cases} \tag{43}$$

where similar to (a), we have:

$$E(5,1,1,1) = \max[a_1(q_2, r_2), Q_1],\ E(5,1,1,2) = cTq_0,$$
$$E(5,1,2,1) = a_1(q_2, r_2),\ E(5,1,2,2) = Q_2.$$
$$E(5,2,1,k) = E(5,1,1,k),\ k = 1,2,\ E(5,2,2,1) = a_3(q_0, q_2, r_2),\ E(5,2,2,2) = Q_2,$$
$$E(5,3,1,1) = \max[a_2(q_1, q_0, q_2, r_1, r_2), Q_1],$$
$$E(5,3,1,2) = cTq_0,\ E(5,3,2,k) = E(5,1,2,k),\ k = 1,2,$$
$$E(5,4,1,k) = E(5,3,1,k),\ k = 1,2,\ E(5,4,2,k) = E(5,2,2,k),\ k = 1,2.$$

(f) For $i = 1,2,3,4$, if $(q_1, q_0, q_2, r_1, r_2) \in S_6 \cap E_{6i} \cap T_i$, then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \begin{cases} \mu_1(z) & \text{if } E(6,i,1,1) \leqslant z \leqslant E(6,i,1,2), \\ \mu_2(z) & \text{if } E(6,i,2,1) \leqslant z \leqslant E(6,i,2,2), \\ 0 & \text{elsewhere,} \end{cases} \tag{44}$$

where:

- by (3′), $E(6,i,1,k) = E(5,i,1,k)$ for $i = 1,2,3,4$ and $k = 1,2$;
- by (8′), $E(6,i,2,1) = E(5,i,2,1)$, $E(6,i,2,2) = cTq_2$ for $i = 1,2,3,4$.

(g) For $i = 1,2,3,4$, if $(q_1, q_0, q_2, r_1, r_2) \in S_7 \cap E_{7i} \cap T_i$, then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \begin{cases} \mu_1(z) & \text{if } E(7,i,1,1) \leqslant z \leqslant E(7,i,1,2), \\ \mu_2(z) & \text{if } E(7,i,2,1) \leqslant z \leqslant E(7,i,2,2), \\ 0 & \text{elsewhere,} \end{cases} \tag{45}$$

where:

- by (4′.1), $E(7,1,1,1) = E(3,1,1,1)$, $E(7,1,1,2) = cTq_0$, $E(7,2,1,k) = E(7,1,1,k)$ for $k = 1,2$;
- by (4′.2), $E(7,3,1,1) = E(3,3,1,1)$, $E(7,3,1,2) = cTq_0$, $E(7,4,1,k) = E(7,3,1,k)$ for $k = 1,2$;
- by (6′), $E(7,i,2,k) = E(5,i,2,k)$ for $i = 1,2,3,4$ and $k = 1,2$.

(h) For $i = 1,2,3,4$, if $(q_1, q_0, q_2, r_1, r_2) \in S_8 \cap E_{8i} \cap T_i$, then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \begin{cases} \mu_1(z) & \text{if } E(8,i,1,1) \leqslant z \leqslant E(8,i,1,2), \\ \mu_2(z) & \text{if } E(8,i,2,1) \leqslant z \leqslant E(8,i,2,2), \\ 0 & \text{elsewhere,} \end{cases} \tag{46}$$

where:

- by (4′), $E(8,i,1,k) = E(7,i,1,k)$ for $i = 1,2,3,4$ and $k = 1,2$;
- by (8′), $E(8,i,2,k) = E(6,i,2,k)$ for $i = 1,2,3,4$ and $k = 1,2$.

Next, we will find the range of z to preserve $0 \leqslant \mu_i(z) \leqslant 1$, $i = 1,2$ in Sections 2.3 and 2.4.

2.3. The range of z for $0 \leqslant \mu_1(z) \leqslant 1$

From Eq. (10), we have

$$\mu_1(z) = \frac{(q_0 - q_1)z - a(r_0 - r_1) - cTq_1(q_0 - q_1) + \sqrt{D_3}}{cT(q_0 - q_1)^2}.$$

Thus, $0 \leqslant \mu_1(z) \leqslant 1$ is equivalent to

$$0 \leqslant (q_0 - q_1)z - a(r_0 - r_1) - cTq_1(q_0 - q_1) + \sqrt{D_3} \leqslant cT(q_0 - q_1)^2. \tag{47}$$

From (a) and (b) in Appendix A, we note that

$$\text{If } D_{30} \leqslant 0 \quad \text{and} \quad z \geqslant a_1(q_2, r_2), \text{ then } D_3 \geqslant 0. \tag{48}$$

$$\text{If } D_{30} > 0, z \geqslant a_1(q_2, r_2) \quad \text{and} \quad z \geqslant z_*, \text{ then } D_3 \geqslant 0. \tag{49}$$

Therefore, we will discuss inequality (47) as follows: (B1) When

$$-(q_0-q_1)z+a(r_0-r_1)+cTq_1(q_0-q_1)\leqslant\sqrt{D_3}, \tag{50}$$

under (48) and (49), denote

$$A_{11}=\max\left[a_1(q_2,r_2),\frac{2aq_1(r_0-r_1)+cTq_1^2(q_0-q_1)+2aD_{30}}{2q_1(q_0-q_1)}\right],$$

$$A_{12}=\max\left[a_2(q_1,q_0,q_2,r_1,r_2),\frac{2aq_1(r_0-r_1)+cTq_1^2(q_0-q_1)+2aD_{30}}{2q_1(q_0-q_1)}\right],$$

$$B_{11}=\max\left[a_1(q_2,r_2),\frac{a(r_0-r_1)+cTq_1(q_0-q_1)}{q_0-q_1}\right],$$

$$B_{12}=\max\left[a_2(q_1,q_0,q_2,r_1,r_2),\frac{a(r_0-r_1)+cTq_1(q_0-q_1)}{q_0-q_1}\right],$$

$$H_{11}=\left\{(q_1,q_0,q_2,r_1,r_2)\,|\left(A_{11}\leqslant\frac{a(r_0-r_1)+cTq_1(q_0-q_1)}{q_0-q_1}\right)\wedge(D_{30}\leqslant 0)\right.$$

or

$$\left.\left(A_{12}\leqslant\frac{a(r_0-r_1)+cTq_1(q_0-q_1)}{q_0-q_1}\right)\wedge(D_{30}>0)\right\},$$

$$H_{12}=\left\{(q_1,q_0,q_2,r_1,r_2)\,|\left(B_{11}<\frac{2aq_1(r_0-r_1)+cTq_1^2(q_0-q_1)+2aD_{30}}{2q_1(q_0-q_1)}\right)\wedge(D_{30}\leqslant 0),\right.$$

or

$$\left.\left(B_{12}<\frac{2aq_1(r_0-r_1)+cTq_1^2(q_0-q_1)+2aD_{30}}{2q_1(q_0-q_1)}\right)\wedge(D_{30}>0)\right\}.$$

By (S1) and (S2) in Appendix B, since $a_2(q_1,q_0,q_2,r_1,r_2)=\max[z_*,a_1(q_2,r_2)]$, we have

∘ If $(q_1,q_0,q_2,r_1,r_2,)\in H_{11}$, then

$$A_{11}\leqslant z\leqslant\frac{a(r_0-r_1)+cTq_1(q_0-q_1)}{q_0-q_1}\quad\text{when } D_{30}\leqslant 0, \tag{51}$$

$$A_{12}\leqslant z\leqslant\frac{a(r_0-r_1)+cTq_1(q_0-q_1)}{q_0-q_1}\quad\text{when } D_{30}>0. \tag{52}$$

∘ If $(q_1,q_0,q_2,r_1,r_2,)\in H_{12}$, then

$$B_{11}\leqslant z\quad\text{when } D_{30}\leqslant 0, \tag{53}$$

$$B_{12}\leqslant z\quad\text{when } D_{30}>0. \tag{54}$$

(B2) When

$$\sqrt{D_3}\leqslant\ -(q_0-q_1)z+a(r_0-r_1)+cTq_1(q_0-q_1)+cT(q_0-q_1)^2, \tag{55}$$

let

$$C_1^* = \min\left[\frac{a(r_0 - r_1) + cTq_0(q_0 - q_1)}{(q_0 - q_1)}, \frac{2aq_0(r_0 - r_1) + cTq_0^2(q_0 - q_1) + 2aD_{30}}{2q_0(q_0 - q_1)}\right].$$

Then we have the following results:

$$\text{If } D_{30} \leqslant 0 \quad \text{and} \quad a_1(q_2, r_2) \leqslant C_1^*, \text{ then } a_1(q_2, r_2) \leqslant z \leqslant C_1^*. \tag{56}$$

$$\text{If } D_{30} > 0 \quad \text{and} \quad a_2(q_1, q_0, q_2, r_1, r_2,) \leqslant C_1^*, \text{ then } a_2(q_1, q_0, q_2, r_1, r_2,) \leqslant z \leqslant C_1^*. \tag{57}$$

Furthermore, denote

$$C_1 = \begin{cases} \min\left[\dfrac{a(r_0 - r_1) + cTq_1(q_0 - q_1)}{(q_0 - q_1)}, C_1^*\right] & \text{if } (q_1, q_0, q_2, r_1, r_2,) \in H_{11}, \\ C_1^* & \text{if } (q_1, q_0, q_2, r_1, r_2,) \in H_{12}. \end{cases}$$

By (51)–(54) and (56)–(57), we have the range of z in which $0 \leqslant \mu_1(z) \leqslant 1$ as follows:

1. If $(q_1, q_0, q_2, r_1, r_2,) \in H_{11}$, then

$$A_{11} \leqslant z \leqslant C_1 \quad \text{when } D_{30} \leqslant 0 \text{ and } a_1(q_2, r_2) \leqslant C_1, \tag{58}$$

$$A_{12} \leqslant z \leqslant C_1 \quad \text{when } D_{30} > 0 \text{ and } a_2(q_1, q_0, q_2, r_1, r_2,) \leqslant C_1. \tag{59}$$

2. If $(q_1, q_0, q_2, r_1, r_2,) \in H_{12}$, then

$$B_{11} \leqslant z \leqslant C_1 \quad \text{when } D_{30} \leqslant 0 \text{ and } a_1(q_2, r_2) \leqslant C_1, \tag{60}$$

$$B_{12} \leqslant z \leqslant C_1 \quad \text{when } D_{30} > 0 \text{ and } a_2(q_1, q_0, q_2, r_1, r_2,) \leqslant C_1. \tag{61}$$

2.4. *The range of z for* $0 \leqslant \mu_2(z) \leqslant 1$

From (14) we have

$$\mu_2(z) = \frac{-(q_2 - q_0)z + a(r_2 - r_0) + cTq_2(q_2 - q_0) - \sqrt{D_4}}{cT(q_2 - q_0)^2}.$$

Thus, $0 \leqslant \mu_2(z) \leqslant 1$ is equivalent to

$$0 \leqslant -(q_2 - q_0)z + a(r_2 - r_0) + cTq_2(q_2 - q_0) - \sqrt{D_4} \leqslant cT(q_2 - q_0)^2.$$

Similar to Section 2.3, let

$$A_{21} = \max\left[a_1(q_2, r_2), \frac{2aq_0(r_2 - r_0) + cTq_0^2(q_2 - q_0) + 2aD_{40}}{2q_0(q_2 - q_0)}\right],$$

$$A_{22} = \max\left[a_3(q_0, q_2, r_2), \frac{2aq_0(r_2 - r_0) + cTq_0^2(q_2 - q_0) + 2aD_{40}}{2q_0(q_2 - q_0)}\right],$$

$$B_{21}=\max\left[a_1(q_2,r_2),\frac{a(r_2-r_0)+cTq_0(q_2-q_0)}{q_2-q_0}\right],$$

$$B_{22}=\max\left[a_3(q_0,q_2,r_2),\frac{a(r_2-r_0)+cTq_0(q_2-q_0)}{q_2-q_0}\right],$$

$$C_2^*=\min\left[\frac{a(r_2-r_0)+cTq_2(q_2-q_0)}{(q_2-q_0)},\frac{2aq_2(r_2-r_0)+cTq_2^2(q_2-q_0)+2aD_{40}}{2q_2(q_2-q_0)}\right],$$

$$H_{21}=\left\{(q_1,q_0,q_2,r_1,r_2)\left|\left(A_{21}\leqslant\frac{a(r_2-r_0)+cTq_0(q_2-q_0)}{q_2-q_0}\right)\wedge(D_{40}\leqslant 0),\right.\right.$$

or

$$\left.\left(A_{22}\leqslant\frac{a(r_2-r_0)+cTq_0(q_2-q_0)}{q_2-q_0}\right)\wedge(D_{40}>0)\right\},$$

$$H_{22}=\left\{(q_1,q_0,q_2,r_1,r_2)\,|\left(B_{21}<\frac{2aq_0(r_2-r_0)+cTq_0^2(q_2-q_0)+2aD_{40}}{2q_0(q_2-q_0)}\right)\wedge(D_{40}\leqslant 0),\right.$$

or

$$\left.\left(B_{22}<\frac{2aq_0(r_2-r_0)+cTq_0^2(q_2-q_0)+2aD_{40}}{2q_0(q_2-q_0)}\right)\wedge(D_{40}>0)\right\},$$

$$C_2=\begin{cases}\min\left[\dfrac{a(r_2-r_0)+cTq_0(q_2-q_0)}{(q_2-q_0)},C_2^*\right] & \text{if } (q_1,q_0,q_2,r_1,r_2,)\in H_{21},\\ C_2^* & \text{if } (q_1,q_0,q_2,r_1,r_2,)\in H_{22}.\end{cases}$$

Therefore, we have the range of z in which $0\leqslant\mu_2(z)\leqslant 1$ as follows:

1. If $(q_1,q_0,q_2,r_1,r_2,)\in H_{21}$, then

$$A_{21}\leqslant z\leqslant C_2\quad\text{when } D_{40}\leqslant 0 \text{ and } a_1(q_2,r_2)\leqslant C_2, \tag{62}$$

$$A_{22}\leqslant z\leqslant C_2\quad\text{when } D_{40}>0 \text{ and } a_3(q_0,q_2,r_2,)\leqslant C_2. \tag{63}$$

2. If $(q_1,q_0,q_2,r_1,r_2,)\in H_{22}$, then

$$B_{21}\leqslant z\leqslant C_2\quad\text{when } D_{40}\leqslant 0 \text{ and } a_1(q_2,r_2)\leqslant C_2, \tag{64}$$

$$B_{22}\leqslant z\leqslant C_2\quad\text{when } D_{40}>0 \text{ and } a_3(q_0,q_2,r_2,)\leqslant C_2. \tag{65}$$

2.5. More about the range of z in $\mu_{G(\tilde{Q},\tilde{R})}(z)$

Corresponding to T_i, $i=1,2,3,4$, and Eqs. (58)–(65), we let

$$V_1=\{(q_1,\,q_0,q_2,r_1,r_2)\,|\,a_1(q_2,r_2)\leqslant C_1,a_1(q_2,r_2)\leqslant C_2\},$$

$$V_2=\{(q_1,q_0,q_2,r_1,r_2)\,|\,a_1(q_2,r_2)\leqslant C_1,a_3(q_0,q_2,r_2)\leqslant C_2\},$$

$$V_3 = \{(q_1, q_0, q_2, r_1, r_2) \mid a_2(q_1, q_0, q_2, r_1, r_2) \leqslant C_1, a_1(q_2, r_2) \leqslant C_2\},$$

$$V_4 = \{(q_1, q_0, q_2, r_1, r_2) \mid a_2(q_1, q_0, q_2, r_1, r_2) \leqslant C_1, a_3(q_0, q_2, r_2) \leqslant C_2\}.$$

Combining the range of z for $0 \leqslant \mu_1(z) \leqslant 1$ (in (58)–(61)) and $0 \leqslant \mu_2(z) \leqslant 1$ (in (62)–(65)), and based on the sign of D_{30} and D_{40}, we rewrite the range of z as follows:

1. In case $(D_{30} \leqslant 0) \wedge (a_1(q_2, r_2) \leqslant C_1)$ and $(D_{40} \leqslant 0) \wedge (a_1(q_2, r_2) \leqslant C_2)$, for every $j = 1, 2, \ldots, 8$, by (58), (60), (62) and (64), we denote

$$E^*_{j11}(1,1) = E^*_{j12}(1,1) = \max(A_{11}, E(j,1,1,1)),$$

$$E^*_{j11}(1,2) = E^*_{j12}(1,2) = \min(C_1, E(j,1,1,2)),$$

$$E^*_{j13}(1,1) = E^*_{j14}(1,1) = \max(B_{11}, E(j,1,1,1)),$$

$$E^*_{j13}(1,2) = E^*_{j14}(1,2) = \min(C_1, E(j,1,1,2)),$$

$$E^*_{j11}(2,1) = E^*_{j13}(2,1) = \max(A_{21}, E(j,1,2,1)),$$

$$E^*_{j11}(2,2) = E^*_{j13}(2,2) = \min(C_2, E(j,1,2,2)),$$

$$E^*_{j12}(2,1) = E^*_{j14}(2,1) = \max(B_{21}, E(j,1,2,1)),$$

$$E^*_{j12}(2,2) = E^*_{j14}(2,2) = \min(C_2, E(j,1,2,2)).$$

- By (58) and (62), $(q_1, q_0, q_2, r_1, r_2) \in H_{11} \cap H_{21} \cap V_1$. Let

$$K_{j11} = \{(q_1, q_0, q_2, r_1, r_2) \mid E^*_{j11}(1,1) \leqslant E^*_{j11}(1,2) \leqslant E^*_{j11}(2,1) \leqslant E^*_{j11}(2,2)\}.$$

- By (58) and (64), $(q_1, q_0, q_2, r_1, r_2) \in H_{11} \cap H_{22} \cap V_1$. Let

$$K_{j12} = \{(q_1, q_0, q_2, r_1, r_2) \mid E^*_{j12}(1,1) \leqslant E^*_{j12}(1,2) \leqslant E^*_{j12}(2,1) \leqslant E^*_{j12}(2,2)\}.$$

- By (60) and (62), $(q_1, q_0, q_2, r_1, r_2) \in H_{12} \cap H_{21} \cap V_1$. Let

$$K_{j13} = \{(q_1, q_0, q_2, r_1, r_2) \mid E^*_{j13}(1,1) \leqslant E^*_{j13}(1,2) \leqslant E^*_{j13}(2,1) \leqslant E^*_{j13}(2,2)\}.$$

- By (60) and (64), $(q_1, q_0, q_2, r_1, r_2) \in H_{12} \cap H_{22} \cap V_1$. Let

$$K_{j14} = \{(q_1, q_0, q_2, r_1, r_2) \mid E^*_{j14}(1,1) \leqslant E^*_{j14}(1,2) \leqslant E^*_{j14}(2,1) \leqslant E^*_{j14}(2,2)\}.$$

Similarly:

2. In case $(D_{30} \leqslant 0) \wedge (a_1(q_2, r_2) \leqslant C_1)$ and $(D_{40} > 0) \wedge (a_3(q_0, q_2, r_2) \leqslant C_2)$, for every $j = 1, 2, \ldots, 8$, $k = 1, 2, 3, 4$, by (58), (60), (63) and (65) we let

$$K_{j2k} = \{(q_1, q_0, q_2, r_1, r_2) \mid E^*_{j2k}(1,1) \leqslant E^*_{j2k}(1,2) \leqslant E^*_{j2k}(2,1) \leqslant E^*_{j2k}(2,2)\},$$

where

$$E^*_{j21}(1,1) = E^*_{j22}(1,1) = \max(A_{11}, E(j,2,1,1)),$$

$$E^*_{j21}(1,2) = E^*_{j22}(1,2) = \min(C_1, E(j,2,1,2)),$$

$$E^*_{j23}(1,1) = E^*_{j24}(1,1) = \max(B_{11}, E(j,2,1,1)),$$

$$E^*_{j23}(1,2) = E^*_{j24}(1,2) = \min(C_1, E(j,2,1,2)),$$

$$E^*_{j21}(2,1) = E^*_{j23}(2,1) = \max(A_{22}, E(j,2,2,1)),$$

$$E^*_{j21}(2,2) = E^*_{j23}(2,2) = \min(C_2, E(j,2,2,2)),$$

$$E^*_{j22}(2,1) = E^*_{j24}(2,1) = \max(B_{22}, E(j,2,2,1)),$$

$$E^*_{j22}(2,2) = E^*_{j24}(2,2) = \min(C_2, E(j,2,2,2)).$$

3. In case $(D_{30} > 0) \wedge (a_2(q_1, q_0, q_2, r_1, r_2,) \leqslant C_1)$ and $(D_{40} \leqslant 0) \wedge (a_1(q_2, r_2) \leqslant C_2)$, for every $j = 1, 2, \ldots, 8$, $k = 1, 2, 3, 4$, by (59), (61), (62) and (64) we let

$$K_{j3k} = \{(q_1, q_0, q_2, r_1, r_2) \mid E^*_{j3k}(1,1) \leqslant E^*_{j3k}(1,2) \leqslant E^*_{j3k}(2,1) \leqslant E^*_{j3k}(2,2)\},$$

where

$$E^*_{j31}(1,1) = E^*_{j32}(1,1) = \max(A_{12}, E(j,3,1,1)),$$

$$E^*_{j31}(1,2) = E^*_{j32}(1,2) = \min(C_1, E(j,3,1,2)),$$

$$E^*_{j33}(1,1) = E^*_{j34}(1,1) = \max(B_{12}, E(j,3,1,1)),$$

$$E^*_{j33}(1,2) = E^*_{j34}(1,2) = \min(C_1, E(j,3,1,2)),$$

$$E^*_{j31}(2,1) = E^*_{j33}(2,1) = \max(A_{21}, E(j,3,2,1)),$$

$$E^*_{j31}(2,2) = E^*_{j33}(2,2) = \min(C_2, E(j,3,2,2)),$$

$$E^*_{j32}(2,1) = E^*_{j34}(2,1) = \max(B_{21}, E(j,3,2,1)),$$

$$E^*_{j32}(2,2) = E^*_{j34}(2,2) = \min(C_2, E(j,3,2,2)).$$

4. In case $(D_{30} > 0) \wedge (a_2(q_1, q_0, q_2, r_1, r_2,) \leqslant C_1)$ and $(D_{40} > 0) \wedge (a_3(q_0, q_2, r_2,) \leqslant C_2)$, for every $j = 1, 2, \ldots, 8$, $k = 1, 2, 3, 4$ by (59), (61), (63) and (65) we let

$$K_{j4k} = \{(q_1, q_0, q_2, r_1, r_2) \mid E^*_{j4k}(1,1) \leqslant E^*_{j4k}(1,2) \leqslant E^*_{j4k}(2,1) \leqslant E^*_{j4k}(2,2)\},$$

where

$$E^*_{j41}(1,1) = E^*_{j42}(1,1) = \max(A_{12}, E(j,4,1,1)),$$

$$E^*_{j41}(1,2) = E^*_{j42}(1,2) = \min(C_1, E(j,4,1,2)),$$

$$E^*_{j43}(1,1) = E^*_{j44}(1,1) = \max(B_{12}, E(j,4,1,1)),$$

$$E^*_{j43}(1,2) = E^*_{j44}(1,2) = \min(C_1, E(j,4,1,2)),$$

$$E^*_{j41}(2,1) = E^*_{j43}(2,1) = \max(A_{22}, E(j,4,2,1)),$$

$$E^*_{j41}(2,2) = E^*_{j43}(2,2) = \min(C_2, E(j,4,2,2)),$$

$$E^*_{j42}(2,1) = E^*_{j44}(2,1) = \max(B_{22}, E(j,4,2,1)),$$

$$E^*_{j42}(2,2) = E^*_{j44}(2,2) = \min(C_2, E(j,4,2,2)).$$

Then, for a summary of the above discussion, we have $\mu_{G(\tilde{Q},\tilde{R})}(z)$ in Proposition 1.

Proposition 1. *For every* $j = 1,2,\ldots,8$, $i = 1,2,3,4$ *and*

$$(q_1, q_0, q_2, r_1, r_2) \in S_j \cap E_{ji} \cap T_i,$$

the membership function of $G(\tilde{Q},\tilde{R})$ *is described as follows*:
(1′) *If* $(q_1, q_0, q_2, r_1, r_2) \in H_{11} \cap H_{21} \cap V_i \cap K_{ji1}$, *then*

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \begin{cases} \mu_1(z) & \text{if } E^*_{ji1}(1,1) \leqslant z \leqslant E^*_{ji1}(1,2), \\ \mu_2(z) & \text{if } E^*_{ji1}(2,1) \leqslant z \leqslant E^*_{ji1}(2,2), \\ 0 & \text{elsewhere.} \end{cases}$$

(2′) *If* $(q_1, q_0, q_2, r_1, r_2) \in H_{11} \cap H_{22} \cap V_i \cap K_{ji2}$, *then*

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \begin{cases} \mu_1(z) & \text{if } E^*_{ji2}(1,1) \leqslant z \leqslant E^*_{ji2}(1,2), \\ \mu_2(z) & \text{if } E^*_{ji2}(2,1) \leqslant z \leqslant E^*_{ji2}(2,2), \\ 0 & \text{elsewhere.} \end{cases}$$

(3′) *If* $(q_1, q_0, q_2, r_1, r_2) \in H_{12} \cap H_{21} \cap V_i \cap K_{ji3}$, *then*

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \begin{cases} \mu_1(z) & \text{if } E^*_{ji3}(1,1) \leqslant z \leqslant E^*_{ji3}(1,2), \\ \mu_2(z) & \text{if } E^*_{ji3}(2,1) \leqslant z \leqslant E^*_{ji3}(2,2), \\ 0 & \text{elsewhere.} \end{cases}$$

(4′) *If* $(q_1, q_0, q_2, r_1, r_2) \in H_{12} \cap H_{22} \cap V_i \cap K_{ji4}$, *then*

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \begin{cases} \mu_1(z) & \text{if } E^*_{ji4}(1,1) \leqslant z \leqslant E^*_{ji4}(1,2), \\ \mu_2(z) & \text{if } E^*_{ji4}(2,1) \leqslant z \leqslant E^*_{ji4}(2,2), \\ 0 & \text{elsewhere.} \end{cases}$$

Denote $H_1 = H_{11} \cap H_{21}$, $H_2 = H_{11} \cap H_{22}$, $H_3 = H_{12} \cap H_{21}$ and $H_4 = H_{12} \cap H_{22}$ then Proposition 1 can be rewritten as follows:

Proposition 2. *For every* $j = 1,2,\ldots,8$, $k, i = 1,2,3,4$, *if*

$$(q_1, q_0, q_2, r_1, r_2) \in S_j \cap E_{ji} \cap T_i \cap H_k \cap V_i \cap K_{jik},$$

then

$$\mu_{G(\tilde{Q},\tilde{R})}(z) = \begin{cases} \mu_1(z) & \text{if } E^*_{jik}(1,1) \leqslant z \leqslant E^*_{jik}(1,2), \\ \mu_2(z) & \text{if } E^*_{jik}(2,1) \leqslant z \leqslant E^*_{jik}(2,2), \\ 0 & \text{elsewhere.} \end{cases} \tag{66}$$

3. The centroid of $\mu_{G(\tilde{Q},\tilde{R})}(z)$

From Eqs. (C.1)–(C.10) in Appendix C, we can find the centroid for $\mu_{G(\tilde{Q},\tilde{R})}(z)$ in Proposition 2 as follows:

For every $j = 1, 2, \ldots, 8,\ i, k = 1, 2, 3, 4$, if

$$(q_1, q_0, q_2, r_1, r_2) \in S_j \cap E_{ji} \cap T_i \cap H_k \cap V_i \cap K_{jik},$$

let

$$\begin{aligned} P(j,i,k) &= \int_{-\infty}^{\infty} \mu_{G(\tilde{Q},\tilde{R})}(z)\,\mathrm{d}z \\ &= \frac{1}{cT(q_0 - q_1)^2} \int_{E^*_{jik}(1,1)}^{E^*_{jik}(1,2)} H_3(z)\,\mathrm{d}z + \frac{1}{cT(q_2 - q_0)^2} \int_{E^*_{jik}(2,1)}^{E^*_{jik}(2,2)} H_4(z)\,\mathrm{d}z, \end{aligned} \tag{67}$$

$$\begin{aligned} Q(j,i,k) &= \int_{-\infty}^{\infty} z\mu_{G(\tilde{Q},\tilde{R})}(z)\,\mathrm{d}z \\ &= \frac{1}{cT(q_0 - q_1)^2} \int_{E^*_{jik}(1,1)}^{E^*_{jik}(1,2)} zH_3(z)\,\mathrm{d}z + \frac{1}{cT(q_2 - q_0)^2} \int_{E^*_{jik}(2,1)}^{E^*_{jik}(2,2)} zH_4(z)\,\mathrm{d}z, \end{aligned} \tag{68}$$

where

$$H_3(z) = (q_0 - q_1)z - a(r_0 - r_1) - cTq_1(q_0 - q_1) + \sqrt{D_3},$$

$$H_4(z) = -(q_2 - q_0)z + a(r_2 - r_0) + cTq_2(q_2 - q_0) - \sqrt{D_4}.$$

For convenience, we give the following notations:

1. If $i = 1, 2$, by T_i we have $D_{30} \leqslant 0$ and hence
   - By (C.5) and (C.2) we denote

$$\begin{aligned} P_{11}(j,i,k) &= \int_{E^*_{jik}(1,1)}^{E^*_{jik}(1,2)} H_3(z)\,\mathrm{d}z \\ &= R_1(E^*_{jik}(1,1), E^*_{jik}(1,2)) + W_{12}(E^*_{jik}(1,1), E^*_{jik}(1,2)). \end{aligned} \tag{69}$$

   - By (C.9) and (C.7) we denote

$$\begin{aligned} Q_{11}(j,i,k) &= \int_{E^*_{jik}(1,1)}^{E^*_{jik}(1,2)} zH_3(z)\,\mathrm{d}z \\ &= R_{12}(E^*_{jik}(1,1), E^*_{jik}(1,2)) + W_{122}(E^*_{jik}(1,1), E^*_{jik}(1,2)). \end{aligned} \tag{70}$$

2. If $i = 1, 3$, by T_i we have $D_{40} \leqslant 0$ and hence
   - By (C.6) and (C.4) we denote

$$P_{12}(j,i,k) = \int_{E^*_{jik}(2,1)}^{E^*_{jik}(2,2)} H_4(z)\,\mathrm{d}z$$

$$= R_2(E^*_{jik}(2,1), E^*_{jik}(2,2)) - W_{22}(E^*_{jik}(2,1), E^*_{jik}(2,2)). \tag{71}$$

   - By (C.10) and (C.8) we denote

$$Q_{12}(j,i,k) = \int_{E^*_{jik}(2,1)}^{E^*_{jik}(2,2)} zH_4(z)\,\mathrm{d}z$$

$$= R_{22}(E^*_{jik}(2,1), E^*_{jik}(2,2)) - W_{222}(E^*_{jik}(2,1), E^*_{jik}(2,2)). \tag{72}$$

3. If $i = 3, 4$, by T_i we have $D_{30} > 0$ and hence
   - By (C.5) and (C.1) we denote

$$P_{21}(j,i,k) = \int_{E^*_{jik}(1,1)}^{E^*_{jik}(1,2)} H_3(z)\,\mathrm{d}z$$

$$= R_1(E^*_{jik}(1,1), E^*_{jik}(1,2)) + W_{11}(E^*_{jik}(1,1), E^*_{jik}(1,2)). \tag{73}$$

   - By (C.9) and (C.7) we denote

$$Q_{21}(j,i,k) = \int_{E^*_{jik}(1,1)}^{E^*_{jik}(1,2)} zH_3(z)\,\mathrm{d}z$$

$$= R_{12}(E^*_{jik}(1,1), E^*_{jik}(1,2)) + W_{112}(E^*_{jik}(1,1), E^*_{jik}(1,2)). \tag{74}$$

4. If $i = 2, 4$, by T_i we have $D_{40} > 0$ and hence
   - By (C.6) and (C.3) we denote

$$P_{22}(j,i,k) = \int_{E^*_{jik}(2,1)}^{E^*_{jik}(2,2)} H_4(z)\,\mathrm{d}z$$

$$= R_2(E^*_{jik}(2,1), E^*_{jik}(2,2)) - W_{21}(E^*_{jik}(2,1), E^*_{jik}(2,2)). \tag{75}$$

   - By (C.10) and (C.8) we denote

$$Q_{22}(j,i,k) = \int_{E^*_{jik}(2,1)}^{E^*_{jik}(2,2)} zH_4(z)\,\mathrm{d}z$$

$$= R_{22}(E^*_{jik}(2,1), E^*_{jik}(2,2)) - W_{212}(E^*_{jik}(2,1), E^*_{jik}(2,2)). \tag{76}$$

Therefore, by (67)–(76) we obtain the following:

$$P(j,1,k)=\frac{P_{11}(j,1,k)}{cT(q_0-q_1)^2}+\frac{P_{12}(j,1,k)}{cT(q_2-q_0)^2},\qquad Q(j,1,k)=\frac{Q_{11}(j,1,k)}{cT(q_0-q_1)^2}+\frac{Q_{12}(j,1,k)}{cT(q_2-q_0)^2},$$

$$P(j,2,k)=\frac{P_{11}(j,2,k)}{cT(q_0-q_1)^2}+\frac{P_{22}(j,2,k)}{cT(q_2-q_0)^2},\qquad Q(j,2,k)=\frac{Q_{11}(j,2,k)}{cT(q_0-q_1)^2}+\frac{Q_{22}(j,2,k)}{cT(q_2-q_0)^2},$$

$$P(j,3,k)=\frac{P_{21}(j,3,k)}{cT(q_0-q_1)^2}+\frac{P_{12}(j,3,k)}{cT(q_2-q_0)^2},\qquad Q(j,3,k)=\frac{Q_{21}(j,3,k)}{cT(q_0-q_1)^2}+\frac{Q_{12}(j,3,k)}{cT(q_2-q_0)^2},$$

$$P(j,4,k)=\frac{P_{21}(j,4,k)}{cT(q_0-q_1)^2}+\frac{P_{22}(j,4,k)}{cT(q_2-q_0)^2},\qquad Q(j,4,k)=\frac{Q_{21}(j,4,k)}{cT(q_0-q_1)^2}+\frac{Q_{22}(j,4,k)}{cT(q_2-q_0)^2}.$$

Proposition 3. *For every $j=1,2,\dots,8$ and $i,k=1,2,3,4$, if*

$$(q_1,q_0,q_2,r_1,r_2)\in S_j\cap E_{ji}\cap T_i\cap H_k\cap V_i\cap K_{jik},$$

then the centroid for $\mu_{G(\tilde{Q},\tilde{R})}(z)$ is given by

$$M_{jik}(q_1,q_0,q_2,r_1,r_2)=\frac{Q_{(j,\,i,\,k)}}{P_{(j,\,i,\,k)}} \tag{77}$$

which is an estimate of the total cost.

4. Numerical example

Let $a=3$, $c=4$, $r_0=30$ and $T=5$, then $q_*=3$ is the crisp optimal order quantity and $G(q_*,r_0)=F(q_*)=60$ is the crisp minimum total cost. In the fuzzy sense, we use a FORTRAN program to find $M^{**}=M_{jik}(q_1,q_0,q_2,r_1,r_2)$, which is an estimate of the total cost under the fuzzy order quantity (q_1,q_0,q_2) and the fuzzy total demand quantity (r_1,r_0,r_2). Also, let $q^{**}=(q_1+q_0+q_2)/3$ and $r^{**}=(r_1+r_0+r_2)/3$ be the centroid of (q_1,q_0,q_2) and (r_1,r_0,r_2), respectively. Then the relative error in the fuzzy sense for order quantity, total demand quantity and total cost are given by

$$Rel\,Q=\frac{q^{**}-q^*}{q^*},\quad Rel\,R=\frac{r^{**}-r_0}{r_0},\quad Rel\,C=\frac{M^{**}-G(q_*,r_0)}{G(q_*,r_0)},$$

respectively. For some sets of (q_1,q_0,q_2) and (r_1,r_0,r_2), we have the numerical results in Table 1 as follows:

Roughly speaking, Table 1 displays that the value of M^{**} approaches the crisp minimum cost $G(q_*,r_0)$ as both q^{**} approaches the crisp order quantity q_* and r^{**} approaches the crisp total demand quantity r_0. That is, $Rel\,C\to 0$ as $Rel\,Q\to 0$ and $Rel\,R\to 0$. Therefore, if both the relative error for order quantity and total demand quantity approaches zero, then the relative error for total cost approaches zero. Otherwise, if q^{**} are far from q_* and r^{**} is far from r_0, then the relative error for the total cost is large.

Table 1
Numerical Results for $a=3$, $c=4$, $r_0=30$, $T=5$, $q_*=3$, $G(q_*, r_0)=60$

Input data					Output data					
q_1	q_0	q_2	r_1	r_2	q^{**}	r^{**}	M^{**}	*Rel Q*	*Rel R*	*Rel C*
1.0	4.1	4.7	29.0	31.0	3.27	30.00	63.12	0.0889	0.0000	0.0520
2.2	4.1	4.7	23.0	31.0	3.67	28.00	63.12	0.2222	− 0.0667	0.0520
2.2	4.1	5.3	26.0	31.0	3.87	29.00	64.29	0.2889	− 0.0333	0.0715
2.2	4.1	5.3	29.0	31.0	3.87	30.00	64.32	0.2889	0.0000	0.0719
1.0	4.1	5.0	25.0	33.0	3.37	29.33	65.20	0.1222	− 0.0222	0.0866
2.8	4.1	5.9	29.0	31.0	4.27	30.00	65.69	0.4222	0.0000	0.0948
2.8	4.7	5.3	20.0	34.0	4.27	28.00	66.89	0.4222	− 0.0667	0.1148
2.8	4.7	5.3	29.0	34.0	4.27	31.00	66.93	0.4222	0.0333	0.1154
1.0	4.7	5.9	29.0	34.0	3.87	31.00	68.16	0.2889	0.0333	0.1360
2.8	4.7	5.9	29.0	34.0	4.47	31.00	68.21	0.4889	0.0333	0.1369
2.9	4.9	5.9	21.0	34.0	4.57	28.33	68.40	0.5222	− 0.0556	0.1400
2.8	4.7	6.5	29.0	31.0	4.67	30.00	69.74	0.5556	0.0000	0.1624
2.2	4.7	6.5	20.0	34.0	4.47	28.00	70.21	0.4889	− 0.0667	0.1702
2.8	4.7	6.5	20.0	34.0	4.67	28.00	70.22	0.5556	− 0.0667	0.1703
2.9	4.9	6.7	27.0	34.0	4.83	30.33	72.03	0.6111	0.0111	0.2005
4.9	6.9	7.9	25.0	32.0	6.57	29.00	83.36	1.1889	− 0.0333	0.3893
4.9	6.9	7.9	28.0	33.0	6.57	30.33	83.49	1.1889	0.0111	0.3915
4.9	6.9	7.9	19.0	37.0	6.57	28.67	83.98	1.1889	− 0.0444	0.3996
4.9	6.9	7.9	19.0	40.0	6.57	29.67	84.36	1.1889	− 0.0111	0.4060
4.9	6.9	7.9	28.0	41.0	6.57	33.00	84.51	1.1889	0.1000	0.4085
6.5	7.9	8.1	28.8	35.9	7.50	31.57	87.94	1.5000	0.0522	0.4657
6.5	7.9	8.1	23.4	44.0	7.50	32.47	88.64	1.5000	0.0822	0.4773
7.1	7.9	8.1	27.0	38.6	7.70	31.87	89.37	1.5667	0.0622	0.4895
7.1	7.9	8.9	28.8	35.9	7.97	31.57	93.16	1.6556	0.0522	0.5526
7.1	8.7	8.9	23.4	37.7	8.23	30.37	95.00	1.7444	0.0122	0.5833
7.1	8.7	8.9	23.4	44.0	8.23	32.47	95.66	1.7444	0.0822	0.5943
6.5	8.7	9.7	28.8	36.8	8.30	31.87	100.81	1.7667	0.0622	0.6801
6.5	9.5	9.7	23.4	39.5	8.57	30.97	102.29	1.8556	0.0322	0.7049

5. Conclusions

In the classical inventory without the backorder model, we consider the order quantity q and the total demand quantity r are fuzzy numbers. If the centroid q^{**} of fuzzy order quantity near the optimal crisp order quantity q^* and the centroid r^{**} of fuzzy total demand quantity near the crisp total demand quantity r_0, then the estimate total cost in Proposition 3 approach the crisp optimal total cost $F(q_*)$.

In brief, when the fuzzy situation is known, we can consider this inventory model in the fuzzy sense. This method may also be used in comparison with the crisp case.

Appendix A

(a) If $D_{30} \leqslant 0$ then $D_3 \geqslant 0$. Therefore for $z \geqslant a_1(q_2, r_2)$ the maximum positive root for (8) is given by

$$q = \frac{(q_0 - q_1)z - a(r_0 - r_1) + \sqrt{D_3}}{cT(q_0 - q_1)} \quad (>0). \quad (*)$$

When $D_{30} \leqslant 0$ and $z \geqslant a_1(q_2, r_2)$, substituting this q (in (*)) in the left-hand side of (7), we obtain

$$PQ = \frac{(q_0 - q_1)z - a(r_0 - r_1) - cTq_1(q_0 - q_1) + \sqrt{D_3}}{cT(q_0 - q_1)^2} \equiv \mu_1(z) \quad \text{(say).}$$

(b) If $D_{30} > 0$ then $z \geqslant a_1(q_2, r_2)$ and $z \geqslant z_*$ imply $D_3 \geqslant 0$, where

$$z_* = \frac{\sqrt{2acT(q_0 - q_1)D_{30}} + a(r_0 - r_1)}{q_0 - q_1} \quad (>0).$$

Also,

$$z \geqslant z_* \Rightarrow (q_0 - q_1)z - a(r_0 - r_1) \geqslant (q_0 - q_1)z_* - a(r_0 - r_1) = \sqrt{2acT(q_0 - q_1)D_{30}} > 0.$$

Similarly, the maximum positive root for (8) is given by (*) if $z \geqslant a_1(q_2, r_2)$ and $z \geqslant z_*$. Denote $a_2(q_1, q_0, q_2, r_1, r_2) = \max\{z_*, a_1(q_2, r_2)]$. Therefore,

$$\text{if } D_{30} > 0 \quad \text{and } z \geqslant a_2(q_1, q_0, q_2, r_1, r_2) \quad \text{then } PQ = \mu_1(z).$$

(c) If $D_{40} \leqslant 0$ then $D_4 \geqslant 0$, for $z \geqslant a_1(q_2, r_2)$. The maximum positive root for (12) is given by

$$q = \frac{(q_2 - q_0)z - a(r_2 - r_0) + \sqrt{D_4}}{cT(q_2 - q_0)} \quad (>0).$$

Substituting q in the left-hand side of (11), when $D_{40} \leqslant 0$ and $z \geqslant a_1(q_2, r_2)$, we obtain

$$P'Q' = \frac{-(q_2 - q_0)z + a(r_2 - r_0) + cTq_2(q_2 - q_0) - \sqrt{D_4}}{cT(q_2 - q_0)^2} \equiv \mu_2(z) \quad \text{(say).}$$

(d) If $D_{40} > 0$ then $z \geqslant a_1(q_2, r_2)$ and $z \geqslant z_{**}$ imply $D_4 \geqslant 0$, where

$$z_{**} = \frac{\sqrt{2acT(q_2 - q_0)D_{40}} + a(r_2 - r_0)}{q_2 - q_0} \quad (>0)$$

and

$$z \geqslant z_{**} \Rightarrow (q_2 - q_0)z - a(r_2 - r_0) \geqslant (q_2 - q_0)z_{**} - a(r_2 - r_0) = \sqrt{2acT(q_2 - q_0)D_{40}} > 0.$$

Thus, the maximum positive root for (12) is given by

$$q = \frac{(q_2 - q_0)z - a(r_2 - r_0) + \sqrt{D_4}}{cT(q_2 - q_0)} \quad (>0).$$

Denote $a_3(q_0, q_2, r_2) = \max[z_{**}, a_1(q_2, r_2)]$. Therefore,

$$\text{if } D_{40} > 0 \quad \text{and} \quad z \geqslant a_3(q_0, q_2, r_2) \quad \text{then } P'Q' = \mu_2(z).$$

Appendix B

(S1) If $-(q_0 - q_1)z + a(r_0 - r_1) + cTq_1(q_0 - q_1) \geqslant 0$, then inequality (50) implies

$$2aq_1(r_0 - r_1) + cTq_1^2(q_0 - q_1) + 2aD_{30} \leqslant 2q_1(q_0 - q_1)z.$$

Therefore,

- If $D_{30} \leqslant 0$, then $z \geqslant a_1(q_2, r_2)$,

$$z \leqslant \frac{a(r_0 - r_1) + cTq_1(q_0 - q_1)}{(q_0 - q_1)}$$

and

$$z \geqslant \frac{2aq_1(r_0 - r_1) + cTq_1^2(q_0 - q_1) + 2aD_{30}}{2q_1(q_0 - q_1)}.$$

- If $D_{30} > 0$, then $z \geqslant a_1(q_2, r_2)$, $z \geqslant z_*$,

$$z \leqslant \frac{a(r_0 - r_1) + cTq_1(q_0 - q_1)}{(q_0 - q_1)}$$

and

$$z \geqslant \frac{2aq_1(r_0 - r_1) + cTq_1^2(q_0 - q_1) + 2aD_{30}}{2q_1(q_0 - q_1)}.$$

(S2) If $-(q_0 - q_1)z + a(r_0 - r_1) + cTq_1(q_0 - q_1) < 0$, under conditions (48) and (49), inequality (50) holds. Then we have

- If $D_{30} \leqslant 0$, then $z \geqslant a_1(q_2, r_2)$ and

$$z > \frac{a(r_0 - r_1) + cTq_1(q_0 - q_1)}{(q_0 - q_1)}.$$

- If $D_{30} > 0$, then $z \geqslant a_1(q_2, r_2)$, $z \geqslant z_*$ and

$$z > \frac{a(r_0 - r_1) + cTq_1(q_0 - q_1)}{(q_0 - q_1)}.$$

Appendix C

In order to find the centroid for the membership function $\mu_{G(\tilde{Q},\,\tilde{R})}(z)$ in Proposition 2, we consider the following integrals:

1. If $D_{30} > 0$ denote

$$W_{11}(a_1, a_2) = \int_{a_1}^{a_2} \sqrt{D_3}\,\mathrm{d}z$$

$$= \int_{a_1}^{a_2} \sqrt{[(q_0 - q_1)z - a(r_0 - r_1)]^2 - 2acT(q_0 - q_1)D_{30}}\,\mathrm{d}z,$$

where $D_{30} = (q_0 r_1 - q_1 r_0)$.

Substituting $y = (q_0 - q_1)z - a(r_0 - r_1)$ and $y = \sqrt{2acT(q_0 - q_1)D_{30}}\sec\theta$ in the above equation we get

$$W_{11}(a_1, a_2) = acTD_{30}\left[\sec\theta_2\tan\theta_2 - \sec\theta_1\tan\theta_1 - \ln\left|\frac{\sec\theta_2 + \tan\theta_2}{\sec\theta_1 + \tan\theta_1}\right|\right], \tag{C.1}$$

where

$$\sec\theta_j = \frac{a_j^*}{\sqrt{2acT(q_0 - q_1)D_{30}}} \quad \text{and} \quad a_j^* = (q_0 - q_1)a_j - a(r_0 - r_1).$$

2. If $D_{30} < 0$ we have

$$W_{12}^*(a_1, a_2) = \int_{a_1}^{a_2} \sqrt{D_3}\,\mathrm{d}z$$

$$= -acTD_{30}\left[\sec\theta_2^*\tan\theta_2^* - \sec\theta_1^*\tan\theta_1^* + \ln\left|\frac{\sec\theta_2^* + \tan\theta_2^*}{\sec\theta_1^* + \tan\theta_1^*}\right|\right],$$

where

$$\tan\theta_j^* = \frac{a_j^*}{\sqrt{-2acT(q_0 - q_1)D_{30}}}.$$

3. If $D_{30} = 0$ let

$$W_{13}^*(a_1, a_2) = \int_{a_1}^{a_2} |(q_0 - q_1)z - a(r_0 - r_1)|\,\mathrm{d}z$$

then

$$W^*_{13}(a_1,a_2)=\frac{1}{q_0-q_1}\int_{a_1^*}^{a_2^*}|y|\,\mathrm{d}y$$

$$=\begin{cases}\frac{1}{2(q_0-q_1)}[(a_2^*)^2-(a_1^*)^2] & \text{if } 0\leqslant a_1^*,\\ \frac{1}{2(q_0-q_1)}[(a_2^*)^2+(a_1^*)^2] & \text{if } a_1^*<0\leqslant a_2^*,\\ \frac{1}{2(q_0-q_1)}[(a_1^*)^2-(a_2^*)^2] & \text{if } a_1^*<a_2^*\leqslant 0.\end{cases}$$

For convenience, denote

$$W_{12}(a_1,a_2)=\begin{cases}W^*_{12}(a_1,a_2) & \text{if } D_{30}<0,\\ W^*_{13}(a_1,a_2) & \text{if } D_{30}=0.\end{cases} \tag{C.2}$$

On the other hand, consider the signs of D_{40}, we obtain the following:

4. If $D_{40}>0$, then

$$W_{21}(a_1,a_2)=\int_{a_1}^{a_2}\sqrt{D_4}\,\mathrm{d}z$$

$$=\int_{a_1}^{a_2}\sqrt{[(q_2-q_0)z-a(r_2-r_0)]^2-2acT(q_2-q_0)D_{40}}\,\mathrm{d}z$$

$$=acTD_{40}\left[\sec\varphi_2\tan\varphi_2-\sec\varphi_1\tan\varphi_1-\ln\left|\frac{\sec\varphi_2+\tan\varphi_2}{\sec\varphi_1+\tan\varphi_1}\right|\right], \tag{C.3}$$

where

$$\sec\varphi_j=\frac{c_j^*}{\sqrt{2acT(q_2-q_0)D_{40}}}\quad\text{and}\quad c_j^*=(q_2-q_0)a_j-a(r_2-r_0).$$

5. If $D_{40}<0$, then

$$W^*_{22}(a_1,a_2)=\int_{a_1}^{a_2}\sqrt{D_4}\,\mathrm{d}z$$

$$=-acTD_{40}\left[\sec\varphi_2^*\tan\varphi_2^*-\sec\varphi_1^*\tan\varphi_1^*+\ln\left|\frac{\sec\varphi_2^*+\tan\varphi_2^*}{\sec\varphi_1^*+\tan\varphi_1^*}\right|\right],$$

where

$$\tan\varphi_j^*=\frac{c_j^*}{\sqrt{-2acT(q_2-q_0)D_{40}}}.$$

6. If $D_{40}=0$, then

$$W^*_{23}(a_1,a_2)=\frac{1}{q_2-q_0}\int_{c_1^*}^{c_2^*}|y|\,\mathrm{d}y$$

$$= \begin{cases} \frac{1}{2(q_2-q_0)}[(c_2^*)^2 - (c_1^*)^2] & \text{if } 0 \leqslant c_1^*, \\ \frac{1}{2(q_2-q_0)}[(c_2^*)^2 + (c_1^*)^2] & \text{if } c_1^* < 0 \leqslant c_2^*, \\ \frac{1}{2(q_2-q_0)}[(c_1^*)^2 - (c_2^*)^2] & \text{if } c_1^* < c_2^* \leqslant 0. \end{cases}$$

For convenience, denote

$$W_{22}(a_1, a_2) = \begin{cases} W_{22}^*(a_1, a_2) & \text{if } D_{40} < 0, \\ W_{23}^*(a_1, a_2) & \text{if } D_{40} = 0. \end{cases} \tag{C.4}$$

7.

$$R_1(a_1, a_2) = \int_{a_1}^{a_2} [(q_0 - q_1)z - a(r_0 - r_1) - cTq_1(q_0 - q_1)]\, \mathrm{d}z$$

$$= (a_2 - a_1)\left[\frac{q_0 - q_1}{2}(a_2 + a_1) - a(r_0 - r_1) - cTq_1(q_0 - q_1)\right]. \tag{C.5}$$

8.

$$R_2(a_1, a_2) = \int_{a_1}^{a_2} [-(q_2 - q_0)z + a(r_2 - r_0) + cTq_2(q_2 - q_0)]\, \mathrm{d}z$$

$$= (a_2 - a_1)\left[-\frac{q_2 - q_0}{2}(a_2 + a_1) + a(r_2 - r_0) + cTq_2(q_2 - q_0)\right]. \tag{C.6}$$

9. Substituting $z = [y + a(r_0 - r_1)]/(g_0 - g_1)$ in the following integration we get

$$W_{1j2}(a_1, a_2) = \int_{a_1}^{a_2} z\sqrt{D_3}\, \mathrm{d}z \quad (D_{30} > 0 \text{ for } j = 1 \text{ and } D_{30} \leqslant 0 \text{ for } j = 2)$$

$$= \frac{1}{(q_0 - q_1)^2}\int_{a_1^*}^{a_2^*} [y + a(r_0 - r_1)]\sqrt{y^2 - 2acT(q_0 - q_1)D_{30}}\, \mathrm{d}y$$

$$= \frac{a(r_0 - r_1)}{q_0 - q_1} W_{1j}(a_1, a_2) + \frac{1}{3(q_0 - q_1)^2}$$

$$\times \{[(a_2^*)^2 - 2acT(q_0 - q_1)D_{30}]^{3/2} - [(a_1^*)^2 - 2acT(q_0 - q_1)D_{30}]^{3/2}\}. \tag{C.7}$$

Similarly,

10.

$$W_{2j2}(a_1, a_2) = \int_{a_1}^{a_2} z\sqrt{D_4}\, \mathrm{d}z \quad (D_{40} > 0 \text{ for } j = 1 \text{ and } D_{40} \leqslant 0 \text{ for } j = 2)$$

$$= \frac{a(r_2 - r_0)}{q_2 - q_0} W_{2j}(a_1, a_2) + \frac{1}{3(q_2 - q_0)^2}$$

$$\times \{[(c_2^*)^2 - 2acT(q_2 - q_0)D_{40}]^{3/2} - [(c_1^*)^2 - 2acT(q_2 - q_0)D_{40}]^{3/2}\}. \tag{C.8}$$

11.

$$R_{12}(a_1, a_2) = \int_{a_1}^{a_2} z[(q_0 - q_1)z - a(r_0 - r_1) - cTq_1(q_0 - q_1)]\,dz$$
$$= \frac{1}{3}(q_0 - q_1)(a_2^3 - a_1^3) - \frac{(a_2^2 - a_1^2)}{2}[a(r_0 - r_1) + cTq_1(q_0 - q_1)]. \tag{C.9}$$

12.

$$R_{22}(a_1, a_2) = \int_{a_1}^{a_2} z[-(q_2 - q_0)z + a(r_2 - r_0) + cTq_2(q_2 - q_0)]\,dz$$
$$= -\frac{1}{3}(q_2 - q_0)(a_2^3 - a_1^3) + \frac{(a_2^2 - a_1^2)}{2}[a(r_2 - r_0) + cTq_2(q_2 - q_0)]. \tag{C.10}$$

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