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Applications

Fuzzy Inventory with Backorder for Fuzzy Order Quantity

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ABSTRACT

This paper investigates a group of computing schemas for economic order quantity as fuzzy values, and the corresponding optimal stock quantity of the inventory with backorder. We express the fuzzy order quantity as the normal triangular fuzzy number (q_1, q_0, q_2), and then we solve the aforementioned optimization problem under the constraints $0 < s \leqslant q_1 < q_0 < q_2$, where s denotes the optimizing stock quantity. We find that, after defuzzification, the total cost is slightly higher than in the crisp model; however, it permits better use of the economic fuzzy quantities arising with changes in orders, deliveries, and sales.

1. INTRODUCTION

In the classical inventory with backorder model, the cost function is $F(q,s) = (aTs^2/2q) + (bT(q-s)^2/2q) + (cR/q)$, where a is the holding cost per unit quantity per time, b is the shortage cost per unit quantity per time, c is the order cost per cycle, s is the maximal stock quantity, T is the plan for the whole period, R is the total demand quantity of T, and q is the order quantity for each time interval. Chen [2] used fuzzy set concepts

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in the model that replaced the costs a, b, c and R [these parameters are not variables in $F(q,s)$] by fuzzy numbers, and solved the fuzzy order quantity problem with numerical operation based on the function principle. When we consider the crisp economic order quantity q_*; we should regard not only a, b, c, R, and T to be fixed, but also the period from the ordering goods to the arriving goods to be fixed. But there probably may be some changes for the period from the ordering goods to the arriving goods in real situation, e.g., the goods will arrive almost three months later (in the fuzzy sense), but a, b, c in $F(q,s)$ are described as before per time or per cycle, so that they will not influence directly, in the same manner as T, R in $F(q,s)$; therefore, the ordering quantity (q) and the optimal stock quantity (s) will influence directly. In order to minimize the total cost, we should not only find the crisp economic quantity; it is better to consider it under the condition that both the ordering quantity (q) and the optimal stock quantity (s) are in the fuzzy sense, but it is too complex to abort it. In this paper, we consider it under the condition that the ordering quantity q is in the fuzzy sense and the maximal stock quantity (s) is in the crisp variable; also, we shall consider it under the condition that the ordering quantity q is in the crisp variable and the stock quantity (s) is in the fuzzy sense in the other paper. So, we replace the order quantity q by the fuzzy number, and the maximal stock quantity s by the positive variable to solve the economic order quantity in the fuzzy sense; then the real problem will be suitable for any given fixed positive numbers a, b, c, R, and T. Hence, in this paper, we not only fuzzify the order quantity q, but also discuss how to solve the economic order quantity in the fuzzy sense, and the optimal stock quantity.

In Section 2, we discuss how to compute the economic order quantity (denoted by q^{**}) in the fuzzy sense, and the optimal stock quantity (s) of the inventory with the backorder model. We express the fuzzy order quantity $\tilde{Q}$ as the normal triangular fuzzy number (q_1, q_0, q_2).

From the fuzzy order quantity $\tilde{Q}$, we may induce the fuzzy total cost $F(\tilde{Q}, s)$ and generate an economic order quantity by defuzzification. Under the condition $0 < s \leqslant q_1 < q_0 < q_2$, we may find the membership function of $F(\tilde{Q}, s)$ and its centroid. We use the centroid of $F(\tilde{Q}, s)$ as the estimation of the total cost. For convenience, we consider the following four cases: 1) $s < P(s) < q_1 < q_0 < q_2$, 2) $s \leqslant q_1 < q_0 < q_2 < P(s)$, 3) $s \leqslant q_1 < P(s) < q_0 < q_2$, and 4) $s \leqslant q_1 < q_0 < P(s) < q_2$ (where $P(s) = \sqrt{((a+b)Ts^2 + 2cR/bT)}$). In Section 3, we apply the Nelder–Mead method [1, 4] to find the optimal point (q_1^*, q_0^*, q_2^*) and the optimal stock quantity s^* such that the centroid of the membership function of the fuzzy total cost of $F(\tilde{Q}, s)$ is minimal. We use the classical centroid $\frac{1}{3}(q_1^* + q_0^* + q_2^*)$ of the normal triangular fuzzy number (q_1^*, q_0^*, q_2^*) as the economic order quantity in the fuzzy sense. Also, we give some numerical examples. From these examples, we can see that the cost of economic order in the fuzzy sense is slightly higher than the crisp economic order. In Section 5, we study the sensitivity of the crisp economic order and the economic order in the fuzzy sense.

2. MEMBERSHIP FUNCTIONS OF FUZZY COST FOR FUZZY ORDER QUANTITY

Figure 1 illustrates the role of all of the parameters where

T: the plan for the whole period (month, quarter, or year)
q: order quantity per cycle
a: holding cost per unit quantity per time
b: shortage cost per unit quantity per time
c: order cost per cycle
R: total demand quantity of whole plan period
s: maximal stock quantity
t_1: duration within a cycle during which inventory is held
t_2: duration within a cycle during which a shortage exists
t_q: length of the inventory cycle, $t_q = t_1 + t_2$.

We have $t_q = (Tq/R)$, $t_1 = (st_q/q)$, $t_2 = ((q-s)t_q/q)$, the average stock during t_1 being $s/2$, and the average shortage quantity during t_2 being $((q-s)/2)$. The number of orders during the period T is approximately R/q. Then the total cost $F(q,s)$ during the plan period becomes

$$F(q,s) = \left(\frac{at_1s}{2} + \frac{bt_2(q-s)}{2} + c\right)\cdot\left(\frac{R}{q}\right) = \frac{aTs^2}{2q} + \frac{bT(q-s)^2}{2q} + \frac{cR}{q}.$$

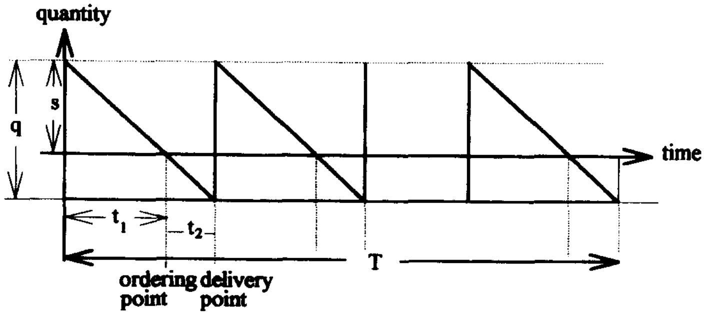


Fig. 1. Inventory with backorder model.

Let q_* and s_* be the economic order quantity and the economic maximal stock quantity, respectively. We have

$$q_* = \sqrt{(2cR/aT)}\,\sqrt{(a+b/b)}\,, \quad s_* = \sqrt{(2cR/aT)}\,\sqrt{(b/a+b)}\,;$$

the minimal total cost $F(q_*, s_*) = \sqrt{2acRT}\,\sqrt{(b/a+b)}$.

If we simulatiously consider the fuzzy order quantity $\tilde{Q}$ and the fuzzy maximal stock quantity $\tilde{S}$, then it will be very complex and inconvenient. Therefore, we only consider the fuzzy order quantity $\tilde{Q}$ and any positive variable s. If the maximal stock quantity is s, then per cycle, the order quantity $q \geqslant s$ should hold for any given positive a, b, c, T, and R.

Suppose the membership function of fuzzy order quantity $\tilde{Q}$ is as follows:

$$\mu_{\tilde{Q}}(q) = \begin{cases} \dfrac{q-q_1}{q_0-q_1}, & \text{if } q_1 \leqslant q \leqslant q_0 \\ \dfrac{q_2-q}{q_2-q_0}, & \text{if } q_0 \leqslant q \leqslant q_2 \\ 0, & \text{otherwise} \end{cases}$$

where s, q_1, q_0, q_2 are real variables, and satisfy the condition $0 < s \leqslant q_1 < q_0 < q_2$.

Since $q \geqslant s$, therefore, under the condition $(0<)s \leqslant q_1 < q_0 < q_2$, we can find the economic order quantity with defuzzification; we get the centroid of $\mu_{\tilde{Q}}(q)$ as the following formula:

$$M_Q(q_1, q_0, q_2) = \frac{\int_{-\infty}^{\infty} x\mu_{\tilde{Q}}(x)\,dx}{\int_{-\infty}^{\infty} \mu_{\tilde{Q}}(x)\,dx} = \frac{q_1+q_0+q_2}{3}.$$

For any given positive a, b, c, R, and T, and for any positive variable s, we may let the total cost $F(q,s)$ be $G_s(q)$, i.e., $G_s(q) = F(q,s)$.

If we let $G_s(q) = y(>0)$, then we have

$$bTq^2 - 2(bTs+y)q + (a+b)Ts^2 + 2cR = 0. \tag{2.1}$$

We may let two roots of (2.1) be

$$x_1(s) = \left(bTs + y - \sqrt{D(s)}\,/bT\right), \quad x_2(s) = \left(bTs + y + \sqrt{D(s)}\,/bT\right),$$

and let $z_*(s) = -bTs + \sqrt{bT[(a+b)Ts^2 + 2cR]}$, where the discriminant of (2.1) (i.e., the equation of q) $D(s) = (bTs + y)^2 - bT[(a+b)Ts^2 + 2cR] \geqslant 0$ if $y \geqslant z_*(s)$. If $y \geqslant z_*(s)$, then $G_s^{-1}(y) = \{x_1(s), x_2(s)\}$, and if $0 < y < z_*(s)$, then $G_s^{-1}(y) = 0$. By the extension principle, we obtain the membership function of $G_s(\tilde{Q})$ as follows:

$$\mu_{G_s(\tilde{Q})}(y) = \begin{cases} \sup\limits_{q \in G_s^{-1}(z)} \mu_{\tilde{Q}}(q), & \text{if } G_s^{-1}(y) \neq \varnothing \\ 0, & \text{if } G_s^{-1}(y) = \varnothing. \end{cases}$$

Hence, we have

$$\mu_{G_s(\tilde{Q})}(y) = \begin{cases} \max\{\mu_{\tilde{Q}}(x_1(s)), \mu_{\tilde{Q}}(x_2(s))\}, & \text{if } y \geqslant z_*(s) \\ 0, & \text{if } 0 < y < z_*(s). \end{cases}$$

In order to solve $\mu_{G_s(\tilde{Q})}(y)$, we show all possible situations of $x_1(s)$ and $x_2(s)$ in the following Table 1. Obviously, we have $x_1(s) \leqslant x_2(s)$.

From Table 1, in order to solve $\mu_{G_s(\tilde{Q})}(y)$ and the range of y, we need the following four inequalities.

Let

$$P(s) = \sqrt{((a+b)Ts^2 + 2cR/bT)} \quad (>s>0)$$

TABLE 1

Situations of $x_1(s), x_2(s)$

Case		q_1		q_0		q_2		$\mu_{G_s(\tilde{Q})}(y)$, if $y \geqslant z_*(s)$
(1)	$x_1(s), x_2(s)$							0
(2)		$x_1(s)$	$x_2(s)$					$(x_2(s) - q_1/q_0 - q_1)$
(3)		$x_1(s)$			$x_2(s)$			$(q_2 - x_2(s)/q_2 - q_0)$
(4)		$x_1(s)$					$x_2(s)$	0
(5)				$x_1(s), x_2(s)$				$(x_2(s) - q_1/q_0 - q_1)$
(6)			$x_1(s)$		$x_2(s)$			$\max[(x_1(s) - q_1/q_0 - q_1), (q_2 - x_2(s)/q_2 - q_0)]$
(7)			$x_1(s)$				$x_2(s)$	$(x_1(s) - q_1/q_0 - q_1)$
(8)						$x_1(s), x_2(s)$		$(q_2 - x_1(s)/q_2 - q_0)$
(9)					$x_1(s)$		$x_2(s)$	$(q_2 - x_1(s)/q_2 - q_0)$
(10)							$x_1(s), x_2(s)$	0

and

$$Q(s) = \sqrt{bT\left[(a+b)Ts^2 + 2cR\right]} \quad (>0);$$

then we have $Q(s) = bTP(s)$, and $z_*(s) = -bTs + Q(s) = bT(P(s) - s)$.

Let $a_j(s) = (bTq_j^2 - 2bTsq_j + (a+b)Ts^2 + 2cR/2q_j)$ (>0); then we have $a_j(s) = (bTq_j^2 - 2bTsq_j + bT(P(s))^2/2q_j)$.

Since $q_j > 0$, $\forall j = 1,0,2$, so that $a_j(s) - z_*(s) = (bT/2q_j)[q_j - P(s)]^2 > 0$, $\forall j = 1,0,2$. Therefore, we have the following relations under the condition $y \geqslant z_*(s)$ for any $j = 1,0,2$:

$$a_j(s) \geqslant z_*(s) \tag{2.2}$$

$$q_j \geqslant P(s), \text{ then } z_*(s) \leqslant a_j(s) \leqslant bT(q_j - s) \tag{2.3}$$

$$\text{If } s \leqslant q_j \leqslant P(s), \text{ then } bT(q_j - s) \leqslant z_*(s) \leqslant a_j(s). \tag{2.4}$$

If $x_1(s) \leqslant q_j$ for $j = 1,0,2$, then $bTs + y - bTq_j \leqslant \sqrt{D(s)}$; since $a_j(s) > z_*(s)$, $\forall j = 1,0,2$, then we have the following two simultaneous inequalities:

$$\begin{cases} y \geqslant bT(q_j - s) \\ y \geqslant a_j(s) \end{cases} \quad \text{or} \quad \begin{cases} y < bT(q_j - s) \\ y \geqslant z_*(s). \end{cases} \tag{2.5}$$

Therefore, applying (2.2)–(2.4), we may solve (2.5) as the following:

$$\text{If } q_j \geqslant P(s), \text{ then } y \geqslant z_*(s).$$

$$\text{If } s \leqslant q_j \leqslant P(s), \text{ then } y \geqslant a_j(s).$$

Hence, we obtain the following (2).

(1) If $q_j \leqslant x_1(s)$ for $j = 1,0,2$, from $-bTq_j + bTs + y \geqslant \sqrt{D(s)}$ and $y \geqslant z_*(s)$, then we have

$$\begin{cases} y \geqslant bT(q_j - s) \\ y \leqslant a_j(s) \end{cases} \text{ and } y \geqslant z_*(s), \text{ i.e., if } q_j > P(s), \text{ then there is no solution.} \tag{2.6}$$

$$\text{If } s \leqslant q_j \leqslant P(s), \text{ then } z_*(s) \leqslant y \leqslant a_j(s). \tag{2.7}$$

(2) If $x_1(s) \leqslant q_j$, for $j=1,0,2$, then from (2.2)–(2.4) and (2.5), we have

$$\text{if } q_j \geqslant P(s), \text{ then } y \geqslant z_*(s) \tag{2.8}$$

$$\text{if } s \leqslant q_j \leqslant P(s), \text{ then } y \geqslant a_j(s). \tag{2.9}$$

(3) If $q_j \leqslant x_2(s)$, for $j=1,0,2$, i.e., $\sqrt{D(s)} \geqslant bTq_j-(bTs+y)$, then we have

$$\text{if } q_j \geqslant P(s), \text{ then } y \geqslant a_j(s) \tag{2.10}$$

$$\text{if } s \leqslant q_j \leqslant P(s), \text{ then } y \geqslant z_*(s). \tag{2.11}$$

(4) If $x_2(s) \leqslant q_j$, for $j=1,0,2$, then we obtain

$$\text{if } q_j \geqslant P(s), \text{ then } z_*(s) \leqslant y \leqslant a_j(s). \tag{2.12}$$

If $s \leqslant q_j < P(s)$, then $y \leqslant bT(q_j-s)$. From (2.4), $bT(q_j-s) \leqslant z_*(s)$. Since $\mu_{G_s(\tilde{Q})}(y)=0$ when $y \leqslant z_*(s)$, therefore, we do not consider this case, i.e.,

$$\text{when } s \leqslant q_j < P(s), \text{ then we do not consider this case.} \tag{2.13}$$

$a_i(s)-a_j(s)=(bT(q_i-q_j)/2q_iq_j)[q_iq_j-P(s)^2]$. Since $q_1<q_0<q_2$, then we have

$$\left[\begin{array}{l} \begin{cases} i=0,2 \\ j=1 \end{cases} \text{or} \begin{cases} i=2 \\ j=0 \end{cases} \text{then } a_i(s) \geqslant a_j(s) \Leftrightarrow q_iq_j \geqslant P(s)^2 \\ \begin{cases} i=1 \\ j=0,2 \end{cases} \text{or} \begin{cases} i=0 \\ j=2 \end{cases} \text{then } a_i(s) \geqslant a_j(s) \Leftrightarrow q_iq_j \leqslant P(s)^2 \end{array}\right]. \tag{2.14}$$

If $y = a_j(s) = (bT/2)q_j - bTs + (bTP(s)^2/2q_j)$, then $D(s) = ((bT/2))^2 [q_j - (P(s)^2/q_j)]^2$, and if $q_j^2 > P(s)^2$, then

$$\sqrt{D(s)} = \frac{bT}{2}\left[q_j - \frac{P(s)^2}{q_j}\right] \text{ and}$$

$$x_1(s) = \frac{bTs + y - \sqrt{D(s)}}{bT} = \frac{P(s)^2}{q_j}, \; x_2(s) = q_j. \tag{2.15}$$

If $q_j^2 < P(s)^2$, then

$$\sqrt{D(s)} = \frac{bT}{2}\left[\frac{P(s)^2}{q_j} - q_j\right] \text{ and } x_1(s) = q_j, \; x_2(s) = \frac{P(s)^2}{q_j}. \tag{2.16}$$

Since $s < P(s)$, for convenience in solving, we divide $s \leqslant q_1 < q_0 < q_2$ into four cases to solve $\mu_{G_s(\tilde{Q})}(y)$ and its centroid: 1) $(s <)P(s) < q_1 < q_0 < q_2$, 2) $s \leqslant q_1 < q_0 < q_2 < P(s)$, 3) $s \leqslant q_1 < P(s) < q_0 < q_2$, and 4) $s \leqslant q_1 < q_0 < P(s) < q_2$.

Using Table 1 and (2.2)–(2.16), we can solve $\mu_{G_s(\tilde{Q})}(y)$ and its centroid for any fixed positive real number s in the following subsection.

2.1. $\mu_{G_s(\tilde{Q})}(y)$ AND ITS CENTROID UNDER THE CONDITION $(s <)P(s) < q_1 < q_0 < q_2$

Condition

$$(s <)P(s) < q_1 < q_0 < q_2. \tag{2.1.1}$$

Since $q_0 q_1 > P(s)^2$ and $q_2 q_0 > P(s)^2$, from (2.2), (2.14), we have

$$z_*(s) \leqslant a_1(s) < a_0(s) < a_2(s). \tag{2.1.2}$$

1. The membership degrees $\mu_{G_s(\tilde{Q})}(y)$ of y for which the satisfying cases (1), (4), and (10) in Table 1 are 0, i.e., $\mu_{G_s(\tilde{Q})}(y) = 0$.

2. Case (2) in Table 1: $x_1(s) \leqslant q_1 \leqslant x_2(s) \leqslant q_0$. For $x_1(s) \leqslant q_1$, since $q_1 > P(s)$, then from (2.8), we have $y \geqslant z_*(s)$. For $q_1 \leqslant x_2(s)$, since $q_1 > P(s)$, then from (2.10), we have $y \geqslant a_1(s)$. For $x_2(s) \leqslant q_0$, since $q_0 > P(s)$, then

from (2.12), we have $z_*(s) \leqslant y \leqslant a_0(s)$. From (2.1.2), we obtain the solution of case (2) as

$$a_1(s) \leqslant y \leqslant a_0(s),\ \mu_{G_s(\tilde{Q})}(y) = \frac{x_2(s) - q_1}{q_0 - q_1}. \tag{2.1.3}$$

3. Case (3) in Table 1: $x_1(s) \leqslant q_1$ and $q_0 \leqslant x_2(s) \leqslant q_2$. Similarly to case (2), we have $y \geqslant z_*(s)$, $y \geqslant a_0(s)$ and $z_*(s) \leqslant y \leqslant a_2(s)$. From (2.1.2), we have

$$a_0(s) \leqslant y \leqslant a_2(s),\ \mu_{G_s(\tilde{Q})}(y) = \frac{q_2 - x_2(s)}{q_2 - q_0}. \tag{2.1.4}$$

4. Cases (5)–(9) in Table 1.

There is one inequality $q_j \leqslant x_1(s), j = 1, 0$, in these cases, $q_j > P(s), j = 1, 0$, so that from (2.6), there is no solution. Therefore, we need not ·consider these cases.

From (2.1.3), and (2.1.4), we have the following Theorem 2.1.

THEOREM 2.1. *Under the condition* $(s <)P(s) < q_1 < q_0 < q_2$, *the membership function* $\mu_{G_s(\tilde{Q})}(y)$ *of* $G_s(\tilde{Q})$ *is as follows*:

$$\mu_{G_s(\tilde{Q})}(y) = \begin{cases} \dfrac{x_2(s) - q_1}{q_0 - q_1}, & \textit{if } a_1(s) \leqslant y \leqslant a_0(s) \\ \dfrac{q_2 - x_2(s)}{q_2 - q_0}, & \textit{if } a_0(s) \leqslant y \leqslant a_2(s) \\ 0, & \textit{otherwise}. \end{cases} \tag{2.1.5}$$

For convenience of expression, we allow some functions as below:

$$\begin{aligned} U_1(b_1, b_2 | s) &= \frac{bT}{2} \int_{b_1}^{b_2} (x - q_1)\left(1 - \frac{P(s)^2}{x^2}\right) dx \\ &= \frac{bT}{2}\left[\frac{1}{2}(b_2^2 - b_1^2) - P(s)^2 \ln \frac{b_2}{b_1}\right. \\ &\qquad \left. - q_1(b_2 - b_1) - q_1 P(s)^2\left(\frac{1}{b_2} - \frac{1}{b_1}\right)\right] \end{aligned} \tag{2.1.6}$$

$$U_2(b_1,b_2|s)=\frac{bT}{2}\int_{b_1}^{b_2}(q_2-x)\left(1-\frac{P(s)^2}{x^2}\right)dx$$

$$=-\frac{bT}{2}\left[\frac{1}{2}(b_2^2-b_1^2)-P(s)^2\ln\frac{b_2}{b_1}-q_2(b_2-b_1)\right.$$

$$\left.-q_2P(s)^2\left(\frac{1}{b_2}-\frac{1}{b_1}\right)\right] \quad (2.1.7)$$

$$U_{11}(b_1,b_2|s)=\frac{bT}{2}\int_{b_1}^{b_2}\left(\frac{bT}{2}x-bTs+\frac{bTP(s)^2}{2x}\right)(x-q_1)\left(1-\frac{P(s)^2}{x^2}\right)dx$$

$$=-bTsU_1(b_1,b_2|s)$$

$$+\frac{b^2T^2}{4}\left\{\frac{1}{3}(b_2^3-b_1^3)+P(s)^4\left(\frac{1}{b_2}-\frac{1}{b_1}\right)\right.$$

$$\left.-\frac{q_1}{2}(b_2^2-b_1^2)-\frac{q_1P(s)^4}{2}\left(\frac{1}{b_2^2}-\frac{1}{b_1^2}\right)\right\} \quad (2.1.8)$$

$$U_{21}(b_1,b_2|s)=\frac{bT}{2}\int_{b_1}^{b_2}\left(\frac{bT}{2}x-bTs+\frac{bTP(s)^2}{2x}\right)(q_2-x)\left(1-\frac{P(s)^2}{x^2}\right)dx$$

$$=-bTsU_2(b_1,b_2|s)$$

$$-\frac{b^2T^2}{4}\left\{\frac{1}{3}(b_2^3-b_1^3)+P(s)^4\left(\frac{1}{b_2}-\frac{1}{b_1}\right)\right.$$

$$\left.-\frac{q_2}{2}(b_2^2-b_1^2)-\frac{q_2P(s)^4}{2}\left(\frac{1}{b_2^2}-\frac{1}{b_1^2}\right)\right\}. \quad (2.1.9)$$

In order to solve the centroid of $\mu_{G_s(\tilde{Q})}(y)$, we consider the changing variable $x=(bTs+y+\sqrt{D(s)}\,/bT)$ $(=x_2(s))$; then we have

$$bTx^2-2(bTs+y)x+(a+b)Ts^2+2cR=0, \text{ i.e.,}$$

$$y=\frac{bT}{2}x-bTs+\frac{bTP(s)^2}{2x}, \text{ and } dy=\frac{bT}{2}\left(1-\frac{P(s)^2}{x^2}\right)dx.$$

Since $q_j^2 > P(s)^2$, hence, from (2.15), if $y = a_j(s)$, then $x = q_j$ for $j = 1, 0, 2$; therefore, we have

$$\int_{-\infty}^{\infty} \mu_{G_s(\tilde{Q})}(y)\, dy$$

$$= \frac{bT}{2(q_0 - q_1)} \int_{q_1}^{q_0} (x - q_1)\left(1 - \frac{P(s)^2}{x^2}\right) dx$$

$$+ \frac{bT}{2(q_2 - q_0)} \int_{q_0}^{q_2} (q_2 - x)\left(1 - \frac{P(s)^2}{x^2}\right) dx$$

$$= \frac{1}{q_0 - q_1} U_1(q_1, q_0 | s) + \frac{1}{q_2 - q_0} U_2(q_0, q_2 | s) \equiv P_1 \text{ (say)} \quad (2.1.10)$$

$$\int_{-\infty}^{\infty} y \mu_{G_s(\tilde{Q})}(y)\, dy$$

$$= \frac{1}{q_0 - q_1} U_{11}(q_1, q_0 | s) + \frac{1}{q_2 - q_0} U_{21}(q_0, q_2 | s) \equiv R_1 \text{ (say)}. \quad (2.1.11)$$

From (2.1.10) and (2.1.11), we obtain the centroid of $\mu_{G_s(\tilde{Q})}(y)$ for (2.1.5) as the following Theorem 2.2.

THEOREM 2.2. *The centroid* $M_1(q_1, q_0, q_2, s)$ *of* $\mu_{G_s(\tilde{Q})}(y)$ *for* (2.1.5) *under the condition* $(s <) P(s) < q_1 < q_0 < q_2$ *is*

$$M_1(q_1, q_0, q_2, s) = \frac{R_1}{P_1}. \quad (2.1.12)$$

We regard this value as the estimation of the total cost under this condition.

2.2. $\mu_{G_s(\tilde{Q})}(y)$ *AND ITS CENTROID UNDER THE CONDITION* $s \leqslant q_1 < q_0 < q_2 < P(s)$

The condition

$$s \leqslant q_1 < q_0 < q_2 < P(s). \quad (2.2.1)$$

Since $q_1 q_0 < P(s)^2$ and $q_0 q_2 < P(s)^2$, from (2.2), (2.14), we have

$$z_*(s) \leqslant a_2(s) < a_0(s) < a_1(s).$$

In the same way as in Section 2.1, we obtain $\mu_{G_s(\tilde{Q})}(y)$ as the following Theorem 2.3.

THEOREM 2.3. *Under the condition* $s \leqslant q_1 < q_0 < q_2 < P(s)$, *the membership function* $\mu_{G_s(\tilde{Q})}(y)$ *of* $G_{(s)}(\tilde{Q})$ *is as follows*:

$$\mu_{G_s(\tilde{Q})}(y) = \begin{cases} \dfrac{q_2 - x_1(s)}{q_2 - q_0}, & \text{if } a_2(s) \leqslant y \leqslant a_0(s) \\ \dfrac{x_1(s) - q_1}{q_0 - q_1}, & \text{if } a_0(s) \leqslant y \leqslant a_1(s) \\ 0, & \text{otherwise}. \end{cases} \tag{2.2.2}$$

In the same way as in Section 2.1, integration by changing the variable $x = (bTs + y - \sqrt{D(s)}/bT)$ $(= x_1(s))$, by (2.16), we have that if $y = a_j(s)$, then $x = q_j$ for $j = 1, 0, 2$, and the centroid of $\mu_{G_s(\tilde{Q})}(y)$ for (2.2.2) as follows.

THEOREM 2.4. *The centroid* $M_2(q_1, q_0, q_2, s)$ *of* $\mu_{G_s(\tilde{Q})}(y)$ *for* (2.2.2) *under the condition* $s \leqslant q_1 < q_0 < q_2 < P(s)$ *is*

$$M_2(q_1, q_0, q_2, s) = \frac{-R_1}{-P_1} = M_1(q_1, q_0, q_2, s). \tag{2.2.3}$$

We regard this value as the estimation of total cost under this condition.

2.3. $\mu_{G_s(\tilde{Q})}(y)$ AND ITS CENTROID UNDER THE CONDITION $s \leqslant q_1 < P(s) < q_0 < q_2$

The condition

$$s \leqslant q_1 < P(s) < q_0 < q_2. \tag{2.3.1}$$

From (2.3.1), (2.2)–(2.14), and Table 1, we have the following. Since $q_2 q_0 > P(s)^2$, and from (2.2), (2.14), we have

$$z_*(s) \leqslant a_0(s) \leqslant a_2(s). \tag{2.3.2}$$

From (2.3.2) and (2.14), the permutations of $a_1(s), a_0(s), a_2(s)$ are only the following three cases:

$$a_1(s) < a_0(s) < a_2(s), \qquad \text{if } q_0 q_1 > P(s)^2 \tag{2.3.3}$$

$$a_0(s) < a_1(s) < a_2(s), \qquad \text{if } q_2 q_1 > P(s)^2 \text{ and } q_1 q_0 < P(s)^2 \tag{2.3.4}$$

$$a_0(s) < a_2(s) < a_1(s), \qquad \text{if } q_1 q_2 < P(s)^2. \tag{2.3.5}$$

With the same method as Sections 2.1 and 2.2, we obtain $\mu_{G_s(\tilde{Q})}(y)$ as Theorem 2.5 (see Appendix A).

By Appendix A, we have the following notations:

$$D_1(s) = A(s)^2 - 4(q_2 - q_0)(q_0 - q_1)B(s)$$

where

$$A(s) = -(q_2 - q_1)^2 bT(s - q_0) + bTs(q_2 + q_1 - 2q_0)^2$$

$$B(s) = b^2T^2\left[(q_2 - q_1)^2(s - q_0)^2 + \left(P(s)^2 - s^2\right)(q_2 + q_1 - 2q_0)^2\right].$$

If $D_1(s) > 0$, then we let $t_1(s) = (A(s) - \sqrt{D_1(s)}/4(q_2 - q_0)(q_0 - q_1))$, $t_2(s) = (A(s) + \sqrt{D_1(s)}/4(q_2 - q_0)(q_0 - q_1))$; if $D_1(s) = 0$, then we let $t_3(s) = (A(s)/4(q_2 - q_0)(q_0 - q_1))$.

THEOREM 2.5. *Under the condition $s \leqslant q_1 < P(s) < q_0 < q_2$, the membership function $\mu_{G_s(\tilde{Q})}(y)$ is as follows*:

1. *If $q_0 q_1 > P(s)^2$, then*

$$\mu_{G_s(\tilde{Q})}(y) = \begin{cases} \dfrac{x_2(s) - q_1}{q_0 - q_1}, & \text{if } z_*(s) \leqslant y \leqslant a_0(s) \\ \dfrac{q_2 - x_2(s)}{q_2 - q_0}, & \text{if } a_0(s) \leqslant y \leqslant a_2(s) \\ 0, & \text{otherwise}. \end{cases} \tag{2.3.6}$$

2. *If* $q_2 q_1 > P(s)^2$ *and* $q_1 q_0 < P(s)^2$ *and* (2-1) *if* $D_1(s) < 0$ *or* ($D_1(s) = 0$ *and* $a_0(s) < t_3(s) < a_1(s)$), *then*

$$\mu_{G_s(\tilde{Q})}(y) = \begin{cases} \dfrac{x_2(s) - q_1}{q_0 - q_1}, & \text{if } z_*(s) \leqslant y \leqslant a_0(s) \\ \dfrac{q_2 - x_2(s)}{q_2 - q_0}, & \text{if } a_0(s) \leqslant y \leqslant a_2(s) \\ 0, & \text{otherwise.} \end{cases} \tag{2.3.7}$$

(2-2) *if* $D_1(s) > 0$ *and* $a_0(s) < t_1(s) < t_2(s) < a_1(s)$, *then*

$$\mu_{G_s(\tilde{Q})}(y) = \begin{cases} \dfrac{x_2(s) - q_1}{q_0 - q_1}, & \text{if } z_*(s) \leqslant y \leqslant a_0(s) \\ \dfrac{q_2 - x_2(s)}{q_2 - q_0}, & \text{if } a_0(s) \leqslant y \leqslant t_1(s) \\ \dfrac{x_1(s) - q_1}{q_0 - q_1}, & \text{if } t_1(s) \leqslant y \leqslant t_2(s) \\ \dfrac{q_2 - x_2(s)}{q_2 - q_0}, & \text{if } t_2(s) \leqslant y \leqslant a_2(s) \\ 0, & \text{otherwise.} \end{cases} \tag{2.3.8}$$

3. *If* $q_1 q_2 < P(s)^2$, $D_1(s) = 0$, *and* $a_0(s) < t_3(s) < a_2(s)$, *then*

$$\mu_{G_s(\tilde{Q})}(y) = \begin{cases} \dfrac{x_2(s) - q_1}{q_0 - q_1}, & \text{if } z_*(s) \leqslant y \leqslant a_0(s) \\ \dfrac{q_2 - x_2(s)}{q_2 - q_0}, & \text{if } a_0(s) \leqslant y \leqslant t_3(s) \\ \dfrac{x_1(s) - q_1}{q_0 - q_1}, & \text{if } t_3(s) \leqslant y \leqslant a_1(s) \\ 0, & \text{otherwise.} \end{cases} \tag{2.3.9}$$

Let

$$I_A = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}, \qquad E_1 = \left\{(q_1, q_0, q_2, s) | q_0 q_1 > P(s)^2\right\},$$

$$E_2 = \left\{(q_1, q_0, q_2, s) | q_2 q_1 > P(s)^2 \text{ and } q_1 q_0 < P(s)^2\right\},$$

$$E_3 = \left\{(q_1, q_0, q_2, s) | q_1 q_2 < P(s)^2\right\},$$

$E_{21}=\{(q_1,q_0,q_2,s)|D_1(s)<0 \text{ or } (D_1(s)=0,\ a_0(s)<t_3(s)<a_1(s))\}$,

$E_{22}=\{(q_1,q_0,q_2,s)|D_1(s)>0,\ a_0(s)<t_1(s)<t_2(s)<a_1(s)\}$,

$E_{33}=\{(q_1,q_0,q_2,s)|D_1(s)=0,\ a_0(s)<t_3(s)<a_2(s)\}$.

[C] Defuzzify $\mu_{G_s(\tilde{Q})}(y)$ in Theorem 2.5:

Since $q_2<P(s)$ and $P(s)<q_j$ for $j=1,0$, by (2.15), (2.16), we obtain the following:

$$\text{if } y=a_1(s), \qquad \text{then } x_1(s)=q_1,\ x_2(s)=\frac{P(s)^2}{q_1},$$

$$\text{if } y=a_j(s), \qquad \text{for } j=0,2, \text{ then } x_1(s)=\frac{P(s)^2}{q_j}q_1, x_2(s)=q_j,$$

$$\text{if } y=z_*(s), \qquad \text{then } x_1(s)=x_2(s)=\frac{Q(s)}{bT}.$$

We consider changing the variable $x=x_1(s)$ or $x=x_2(s)$ in the following integrations. Then $y=(bT/2)x-bTs+(bTP(s)^2/2x)$.

We defuzzify $\mu_{G_s(\tilde{Q})}(y)$ in (2.3.6)–(2.3.9) in Theorem 2.5 by the centroid method as follows.

[C1] The centroid of $\mu_{G_s(\tilde{Q})}(y)$ in (2.3.6):

$$\begin{aligned}
&\int_{-\infty}^{\infty}\mu_{G_s(\tilde{Q})}(y)\,dy \\
&\quad=\frac{bT}{2(q_0-q_1)}\int_{Q(s)/bT}^{q_0}(x-q_1)\left(1-\frac{P(s)^2}{x^2}\right)dx \\
&\qquad+\frac{bT}{2(q_2-q_0)}\int_{q_0}^{q_2}(q_2-x)\left(1-\frac{P(s)^2}{x^2}\right)dx \\
&\quad=\frac{1}{q_0-q_1}U_1\left(\frac{Q(s)}{bT},q_0|s\right)+\frac{1}{q_2-q_0}U_2(q_0,q_2|s)\equiv P_{31}\ (\text{say}) \qquad (2.3.10)
\end{aligned}$$

$$\begin{aligned}
&\int_{-\infty}^{\infty}y\mu_{G_s(\tilde{Q})}(y)\,dy \\
&\quad=\frac{1}{q_0-q_1}U_{11}\left(\frac{Q(s)}{bT},q_0|s\right)+\frac{1}{q_2-q_0}U_{21}(q_0,q_2|s)\equiv R_{31}\ (\text{say}).
\end{aligned}$$

(2.3.11)

LEMMA 2.1. *The centroid* $M_{31}(q_1, q_0, q_2, s)$ *of* $\mu_{G_s(\tilde{Q})}(y)$ *for* (2.3.6) *under the condition* $s \leqslant q_1 < P(s) < q_0 < q_2$, *and* $(q_1, q_0, q_2, s) \in E_1$ *is*

$$M_{31}(q_1, q_0, q_2, s) = \frac{R_{31}}{P_{31}}. \tag{2.3.12}$$

We regard this value as the estimation of total cost under this condition.

[C2] The centroid of $\mu_{G_s(\tilde{Q})}(y)$ in (2.3.7), (2.3.8) in Theorem 2.5: We have that if $y = t_j(s)$, then $x_1(s) = c_j(s)$, $x_2(s) = d_j(s)$ where

$$c_j(s) = \frac{bTs + t_j(s) - \sqrt{\left(bTs + t_j(s)\right)^2 - b^2T^2P(s)^2}}{bT}$$

$$d_j(s) = \frac{bTs + t_j(s) + \sqrt{\left(bTs + t_j(s)\right)^2 - b^2T^2P(s)^2}}{bT}, \qquad \text{for } j = 1,2,3.$$

We consider the following under the condition $s \leqslant q_1 < P(s) < q_0 < q_2$, and $(q_1, q_0, q_2, s) \in E_2$.

[C21] By (2.3.7), we have the same as (2.3.6).

LEMMA 2.2. *The centroid* $M_{321}(q_1, q_0, q_2, s)$ *of* $\mu_{G_s(\tilde{Q})}(y)$ *in* (2.3.7) *in Theorem* 2.5 *under the condition* $s \leqslant q_1 < P(s) < q_0 < q_2$, $(q_1, q_0, q_2, s) \in E_2 \cap E_{21}$ *is*

$$M_{321}(q_1, q_0, q_2, s) = \frac{R_{31}}{P_{31}}. \tag{2.3.13}$$

[C22] For (2.3.8), we have:

$$\begin{aligned}\int_{-\infty}^{\infty} \mu_{G_s(\tilde{Q})}(y)\, dy &= \frac{1}{q_0 - q_1}\left[U_1\left(\frac{Q(s)}{bT}, q_0|s\right) + U_1(c_1(s), c_2(s)|s)\right] \\ &\quad + \frac{1}{q_2 - q_0}\left[U_2(q_0, d_1(s)|s) + U_2(d_2(s), q_2|s)\right] \\ &\equiv P_{322} \text{ (say)}\end{aligned} \tag{2.3.14}$$

$$\int_{-\infty}^{\infty} y\mu_{G_s(\tilde{Q})}(y)\,dy = \frac{1}{q_0-q_1}\left[U_{11}\left(\frac{Q(s)}{bT},q_0|s\right)+U_{11}(c_1(s),c_2(s)|s)\right]$$

$$+\frac{1}{q_2-q_0}\left[U_{21}(q_0,d_1(s)|s)+U_{21}(d_2(s),q_2|s)\right]$$

$$\equiv R_{322}\ (\text{say}). \qquad (2.3.15)$$

LEMMA 2.3. *The centroid* $M_{322}(q_1,q_0,q_2,s)$ *of* $\mu_{G_s(\tilde{Q})}(y)$ *in* (2.3.8) *under the condition* $(q_1,q_0,q_2,s)\in E_2\cap E_{22}$ *is*

$$M_{322}(q_1,q_0,q_2,s)=\frac{R_{322}}{P_{322}}. \qquad (2.3.16)$$

We regard this value as the estimation of total cost under this condition.

[C3] The centroid of $\mu_{G_s(\tilde{Q})}(y)$ in (2.3.9):

$$\int_{-\infty}^{\infty} \mu_{G_s(\tilde{Q})}(y)\,dy = \frac{1}{q_0-q_1}U_1\left(\frac{Q(s)}{bT},q_0|s\right)$$

$$+\frac{1}{q_2-q_0}U_2(q_0,d_3(s)|s)+\frac{1}{q_0-q_1}U_1(c_3(s),q_1|s)$$

$$\equiv P_{33}\ (\text{say}) \qquad (2.3.17)$$

$$\int_{-\infty}^{\infty} y\mu_{G_s(\tilde{Q})}(y)\,dy = \frac{1}{q_0-q_1}U_{11}\left(\frac{Q(s)}{bT},q_0|s\right)$$

$$+\frac{1}{q_2-q_0}U_{21}(q_0,d_3(s)|s)+\frac{1}{q_0-q_1}U_{11}(c_3(s),q_1|s)$$

$$\equiv R_{33}\ (\text{say}). \qquad (2.3.18)$$

LEMMA 2.4. *The centroid* $M_{33}(q_1,q_0,q_2,s)$ *of* $\mu_{G_s(\tilde{Q})}(y)$ *for* (2.3.9) *under the condition* $(q_1,q_0,q_2,s)\in E_3\cap E_{33}$ *is*

$$M_{33}(q_1,q_0,q_2,s)=\frac{R_{33}}{P_{33}}. \qquad (2.3.19)$$

We regard this value as the estimation of total cost under this condition.

By Lemmas 2.1–2.4, we have the following Theorem 2.6.

THEOREM 2.6. *The centroid of* $\mu_{G_s(\tilde{Q})}(y)$ *under the condition* $s \leqslant q_1 < P(s) < q_0 < q_2$ *is*

$$M_3(q_1, q_0, q_2, s) = M_{31}(q_1, q_0, q_2, s) I_{E_1} + M_{321}(q_1, q_0, q_2, s) I_{E_2 \cap E_{21}} + M_{322}(q_1, q_0, q_2, s) I_{E_2 \cap E_{22}} + M_{33}(q_1, q_0, q_2, s) I_{E_3 \cap E_{33}}.$$

We regard this value as the estimation of total cost under this condition.

2.4. $\mu_{G_s(\tilde{Q})}(y)$ AND ITS CENTROID UNDER THE CONDITION $s \leqslant q_1 < q_0 < P(s) < q_2$

Condition

$$s \leqslant q_1 < q_0 < P(s) < q_2. \tag{2.4.1}$$

Since $q_1 q_0 < P(s)^2$, from (2.2)–(2.14), we have

$$z_*(s) \leqslant a_0(s) < a_1(s). \tag{2.4.2}$$

From (2.4.2) and (2.14), there are only three permutations of $a_1(s), a_0(s)$, $a_2(s)$ as follows:

$$a_0(s) < a_1(s) < a_2(s), \qquad \text{if } q_2 q_1 > P(s)^2 \tag{2.4.3}$$

$$a_0(s) < a_2(s) < a_1(s), \qquad \text{if } q_2 q_0 > P(s)^2 \text{ and } q_1 q_2 < P(s)^2 \tag{2.4.4}$$

$$\text{if } a_2(s) < a_0(s) < a_1(s), \qquad \text{if } q_0 q_2 < P(s)^2. \tag{2.4.5}$$

So, as in Section 2.3, under these conditions, $\mu_{G_s(\tilde{Q})}(y)$ and their centroids are as follows in Theorem 2.7 (see Appendix B).

THEOREM 2.7. *Under the condition* $s \leqslant q_1 < q_0 < P(s) < q_2$, *the membership function* $\mu_{G_s(\tilde{Q})}(y)$ *is as follows*:

(1) *If* $q_2 q_1 > P(s)^2$, $(D_1(s) = 0$ *and* $a_0(s) < t_3(s) < a_1(s))$, *then*

$$\mu_{G_s(\tilde{Q})}(y) = \begin{cases} \dfrac{q_2 - x_1(s)}{q_2 - q_0}, & \text{if } z_*(s) \leqslant y \leqslant a_0(s) \\ \dfrac{x_1(s) - q_1}{q_0 - q_1}, & \text{if } a_0(s) \leqslant y \leqslant t_3(s) \\ \dfrac{q_2 - x_2(s)}{q_2 - q_0}, & \text{if } t_3(s) \leqslant y \leqslant a_2(s) \\ 0, & \text{otherwise}. \end{cases} \tag{2.4.6}$$

(2) *If* $q_2 q_0 > P(s)^2$ *and* $q_1 q_2 < P(s)^2$ *and* (2-1) *if* $D_1(s) < 0$, *or* $(D_1(s) = 0$ *and* $a_0(s) < t_3(s) < a_2(s))$, *then*

$$\mu_{G_s(\tilde{Q})}(y) = \begin{cases} \dfrac{q_2 - x_1(s)}{q_2 - q_0}, & \text{if } z_*(s) \leqslant y \leqslant a_0(s) \\ \dfrac{x_1(s) - q_1}{q_0 - q_1}, & \text{if } a_0(s) \leqslant y \leqslant a_1(s) \\ 0, & \text{otherwise}. \end{cases} \tag{2.4.7}$$

(2-2) *if* $D_1(s) > 0$ *and* $a_0(s) < t_1(s) < t_2(s) < a_2(s)$, *then*

$$\mu_{G_s(\tilde{Q})}(y) = \begin{cases} \dfrac{q_2 - x_1(s)}{q_2 - q_0}, & \text{if } z_*(s) \leqslant y \leqslant a_0(s) \\ \dfrac{x_1(s) - q_1}{q_0 - q_1}, & \text{if } a_0(s) \leqslant y \leqslant t_1(s) \\ \dfrac{q_2 - x_2(s)}{q_2 - q_0}, & \text{if } t_1(s) \leqslant y \leqslant t_2(s) \\ \dfrac{x_1(s) - q_1}{q_0 - q_1}, & \text{if } t_2(s) \leqslant y \leqslant a_1(s) \\ 0, & \text{otherwise}. \end{cases} \tag{2.4.8}$$

(3) *if* $q_0 q_2 < P(s)^2$, *then*

$$\mu_{G_s(\tilde{Q})}(y) = \begin{cases} \dfrac{q_2 - x_1(s)}{q_2 - q_0}, & \text{if } z_*(s) \leqslant y \leqslant a_0(s) \\ \dfrac{x_1(s) - q_1}{q_0 - q_1}, & \text{if } a_0(s) \leqslant y \leqslant a_1(s) \\ 0, & \text{otherwise.} \end{cases} \tag{2.4.9}$$

Let

$$G_1 = \left\{(q_1, q_0, q_2, s) | q_2 q_1 > P(s)^2\right\},$$

$$G_2 = \left\{(q_1, q_0, q_2, s) | q_2 q_0 > P(s)^2, \text{ and } q_1 q_2 < P(s)^2\right\},$$

$$G_3 = \left\{(q_1, q_0, q_2, s) | q_0 q_2 < P(s)^2\right\},$$

$$G_{11} = \{(q_1, q_0, q_2, s) | D_1(s) = 0,\ a_0(s) < t_3(s) < a_1(s)\},$$

$$G_{21} = \{(q_1, q_0, q_2, s) | D_1(s) < 0 \text{ or } (D_1(s) = 0,\ a_0(s) < t_3(s) < a_2(s))\},$$

$$G_{22} = \{(q_1, q_0, q_2, s) | D_1(s) > 0,\ a_0(s) < t_1(s) < t_2(s) < a_2(s)\}.$$

[C] Defuzzification of $\mu_{G_s(\tilde{Q})}(y)$ in Theorem 2.7:

We defuzzify $\mu_{G_s(\tilde{Q})}(y)$ in (2.4.6)–(2.4.9) by the centroid method as follows. In the same way as in Sections 2.2, 2.3, if there is $x_1(s)$ in the integration, then by changing the variable $x = (bTs + y - \sqrt{D(s)}/bT)$ $(= x_1(s))$. If there is $x_2(s)$ in the integration, then by changing the variable $x = (bTs + y + \sqrt{D(s)}/bT)$ $(= x_2(s))$ we have that, for $j = 0, 1$, if $y = a_j(s)$, then by (2.16), $x_1(s) = q_j$, $x_2 = (P(s)^2/q_j)$; for $j = 2$, if $y = a_2(s)$, then by (2.15), $x_1(s) = (P(s)^2/q_2)$, $x_2(s) = q_2$. If $y = z_*(s)$, then $x_1(s) = x_2(s) = (Q(s)/bT)$. By the same method as in Section 2.3, if $y = t_j(s)$, then $x_1(s) = c_j(s)$, $x_2(s) = d_j(s)$, for $j = 1, 2, 3$.

[C41] The centroid of $\mu_{G_s(\tilde{Q})}(y)$ in (2.4.6): we have

$$\int_{-\infty}^{\infty} \mu_{G_s(\tilde{Q})}(y)\, dy = \frac{1}{q_2 - q_0}\left[U_2\left(\frac{Q(s)}{bT}, q_0 | s\right) + U_2(d_3(s), q_2 | s)\right] + \frac{1}{q_0 - q_1} U_1(q_0, c_3(s) | s)$$

$$\equiv P_{41} \text{ (say)} \tag{2.4.10}$$

$$\int_{-\infty}^{\infty} y\mu_{G_s(\tilde{Q})}(y)\,dy = \frac{1}{q_0-q_1}U_{11}(q_0,c_3(s)|s)$$

$$+\frac{1}{q_2-q_0}\left[U_{21}\left(\frac{Q(s)}{bT},q_0|s\right)+U_{21}(d_3(s),q_2|s)\right]$$

$$\equiv R_{41}\ (\text{say}). \tag{2.4.11}$$

LEMMA 2.6. *The centroid* $M_{41}(q_1,q_0,q_2,s)$ *of* $\mu_{G_s(\tilde{Q})}(y)$ *in* (2.4.6) *under the condition* $s \leqslant q_1 < q_0 < P(s) < q_2, (q_1,q_0,q_2,s) \in G_1 \cap G_{11}$ *is*

$$M_{41}(q_1,q_0,q_2,s) = \frac{R_{41}}{P_{41}}. \tag{2.4.12}$$

We regard this value as the estimation of total cost under this condition.

[C42] The centroid of $\mu_{G_s(\tilde{Q})}(y)$ in (2.4.7)–(2.4.8):

From (2.4.7)–(2.4.8), we have

[C421] If $(q_1,q_0,q_2,s) \in G_2 \cap G_{21}$, then from (2.4.7) we obtain

$$\int_{-\infty}^{\infty} \mu_{G_s(\tilde{Q})}(y)\,dy = \frac{1}{q_2-q_0}U_2\left(\frac{Q(s)}{bT},q_0|s\right)+\frac{1}{q_0-q_1}U_1(q_0,q_1|s)$$

$$\equiv P_{421}\ (\text{say}) \tag{2.4.13}$$

$$\int_{-\infty}^{\infty} y\mu_{G_s(\tilde{Q})}(y)\,dy = \frac{1}{q_0-q_1}U_{11}(q_0,q_1|s)+\frac{1}{q_2-q_0}U_{21}\left(\frac{Q(s)}{bT},q_0|s\right)$$

$$\equiv R_{421}\ (\text{say}). \tag{2.4.14}$$

LEMMA 2.7. *The centroid* $M_{421}(q_1,q_0,q_2,s)$ *of* $\mu_{G_s(\tilde{Q})}(y)$ *in* (2.4.7) *under the condition* $s \leqslant q_1 < q_0 < P(s) < q_2, (q_1,q_0,q_2,s) \in G_2 \cap G_{21}$ *is*

$$M_{421}(q_1,q_0,q_2,s) = \frac{R_{421}}{P_{421}}. \tag{2.4.15}$$

We regard this value as the estimation of total cost under this condition.

[C422] If $(q_1,q_0,q_2,s) \in G_2 \cap G_{22}$, then from (2.4.8), we obtain

$$\int_{-\infty}^{\infty} \mu_{G_s(\tilde{Q})}(y)\,dy = \frac{1}{q_0-q_1}\left[U_1(q_0,c_1(s)|s)+U_1(c_2(s),q_1|s)\right]$$

$$+\frac{1}{q_2-q_0}\left[U_2\left(\frac{Q(s)}{bT},q_0|s\right)+U_2(d_1(s),d_2(s)|s)\right]$$

$$\equiv P_{422}\ (\text{say}) \tag{2.4.16}$$

$$\int_{-\infty}^{\infty} y\mu_{G_s(\tilde{Q})}(y)\,dy = \frac{1}{q_0-q_1}\left[U_{11}(q_0, c_1(s)|s) + U_{11}(c_2(s), q_1|s)\right]$$

$$+\frac{1}{q_2-q_0}\left[U_{21}\left(\frac{Q(s)}{bT}, q_0|s\right) + U_{21}(d_1(s), d_2(s)|s)\right]$$

$$\equiv R_{422}\text{ (say)}. \tag{2.4.17}$$

LEMMA 2.8. *The centroid $M_{422}(q_1, q_0, q_2, s)$ of $\mu_{G_s(\tilde{Q})}(y)$ in (2.4.8) under the condition $s \leqslant q_1 < q_0 < P(s) < q_2, (q_1, q_0, q_2, s) \in G_2 \cap G_{22}$ is*

$$M_{422}(q_1, q_0, q_2, s) = \frac{R_{422}}{P_{422}}. \tag{2.4.18}$$

We regard this value as the estimation of total cost under this condition.

[C43] The centroid of $\mu_{G_s(\tilde{Q})}(y)$ in (2.4.9): we obtain the same as (2.4.7).

LEMMA 2.9. *The centroid $M_{43}(q_1, q_0, q_2, s)$ of $\mu_{G_s(\tilde{Q})}(y)$ in (2.4.9) under the condition $s \leqslant q_1 < q_0 < P(s) < q_2, (q_1, q_0, q_2, s) \in G_3$ is*

$$M_{43}(q_1, q_0, q_2, s) = \frac{R_{421}}{P_{421}}. \tag{2.4.19}$$

We regard this value as the estimation of total cost under this condition.

From Lemmas 2.6–2.9, we have the following Theorem 2.8.

THEOREM 2.8. *The centroid of $\mu_{G_s(\tilde{Q})}(y)$ under the condition $s \leqslant q_1 < q_0 < P(s) < q_2$ is*

$$M_4(q_1, q_0, q_2, s) = M_{41}(q_1, q_0, q_2, s) I_{G_1 \cap G_{11}} + M_{421}(q_1, q_0, q_2, s) I_{G_2 \cap G_{21}}$$

$$+ M_{422}(q_1, q_0, q_2, s) I_{G_2 \cap G_{22}} + M_{43}(q_1, q_0, q_2, s) I_{G_3}. \tag{2.4.20}$$

We regard this value as the estimation of total cost under this condition.

3. OPTIMAL ORDER QUANTITY

Let

$$Q_1 = \{(q_1, q_0, q_2, s) | s \leqslant P(s) < q_1 < q_0 < q_2\},$$

$$Q_2 = \{(q_1, q_0, q_2, s) || s \leqslant q_1 < q_0 < q_2 < P(s)\},$$

$$Q_3 = \{(q_1, q_0, q_2, s) || s \leqslant q_1 < P(s) < q_0 < q_2\}$$

$$Q_4 = \{(q_1, q_0, q_2, s) || s \leqslant q_1 < q_0 < P(s) < q_2\}.$$

Then from Theorems 2.2, 2.4, 2.6, and 2.8, we have the following theorem.

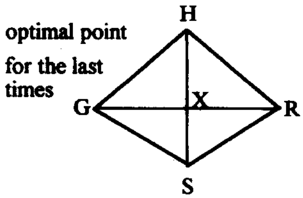
THEOREM 3. *The total cost estimation under the condition* $0 < s \leqslant q_1 < q_0 < q_2$ *is*

$$M(q_1, q_0, q_2, s) = \sum_{j=1}^{4} M_j(q_1, q_0, q_2, s) I_{Q_j}. \tag{3.1}$$

We regard this value as the estimation of total cost under the condition $0 < s \leqslant q_1 < q_0 < q_2$.

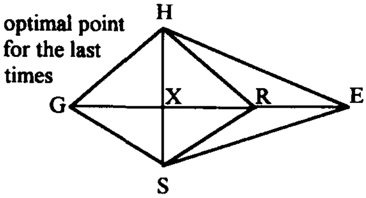
In order to minimize $M(q_1, q_0, q_2, s)$, we apply the Nelder–Mead method [4]. But in our paper, the q_1, q_0, q_2, and s should satisfy $0 < s \leqslant q_1 < q_0 < q_2$; therefore, we apply the Nelder–Mead simplex algorithm [1], the two transformations (3.2) and (3.3) shown in Figures 2 and 3 instead of the two transformations of Algorithm 6.5 of Nelder–Mead method [4].

In this paper, we denote q_1 for $R(1)$, $X(1)$, $G(1)$, and $E(1)$, q_0 for $R(2)$, $X(2)$, $G(2)$, and $E(2)$, q_2 for $R(3)$, $X(3)$, $G(3)$, and $E(3)$, s for $R(4)$, $X(4)$,



$$\begin{aligned} R &= X + d\,(X - G) \\ &= (1 + d)\,X - d\,G \\ &\text{where } 0 < d \leq 1 \end{aligned} \tag{3.2}$$

Fig. 2. Contraction step.



$$\begin{aligned} E &= X + e(R - X) \\ &= (1 - e)X + eR \end{aligned} \qquad (3.3)$$
where $e > 1$

Fig. 3. Expansion step.

$G(4)$, and $E(4)$ instead of the symbols in Algorithm 6.5 of the Nelder–Mead method [4]. Given $X(1)<X(2)<X(3)<X(4)$ and $G(1)<G(2)<G(3)<G(4)$. We let

$$H(k+1,k)=X(k)-X(k+1)-G(k)+G(k+1);$$

$$I(H(k+1,k))=\begin{cases}1, & \text{if } H(k+1,k)>0\\ 0, & \text{if } H(K+1,k)\leqslant 0\end{cases} \qquad \text{for } k=1,2,3, \text{ and}$$

$$H_*=\min\left[\frac{X(2)-X(1)}{H(2,1)}I(H(2,1)),\frac{X(3)-X(2)}{H(3,2)}I(H(3,2)),\right.$$
$$\left.\frac{X(4)-X(3)}{H(4,3)}I(H(4,3)),1\right].$$

If we take d in (3.2) satisfying

$$0<d<H_* \qquad (3.4)$$

then it is easy to show that $R(1)<R(2)<R(3)<R(4)$.

We let $L(k+1,k)=X(k+1)-X(k)+R(k)-R(k+1)$, for $k=1,2,3$, and

$$L_*=\begin{cases}\min\limits_{1\leqslant k\leqslant 3}\dfrac{X(k+1)-X(k)}{L(k+1,k)}, \\ \qquad \text{if } L(k+1,k)>0, \quad \text{for } k=1,2,3\\ \min\left[\dfrac{X(k+1)-X(k)}{L(k+1,k)},\dfrac{X(j+1)-X(j)}{L(j+1,j)}\right], \\ \qquad \text{if } k\neq j,\ k,j=1,2,3,\ L(k+1,k)>0,\ L(j+1,j)>0\\ \dfrac{X(k+1)-X(k)}{L(k+1,k)}, \\ \qquad \text{if } L(k+1,k)>0,\ k=1, \text{ or } 2, \text{ or } 3\\ \infty, \quad \text{if } L(K+1,k)<0, \quad \text{for } k=1,2,3.\end{cases}$$

If we take e in (3.3) satisfying

$$1 < e < L_* \tag{3.5}$$

then it is easy to show that $E(1) < E(2) < E(3) < E(4)$.

We modify $R(k) = 2 \cdot M(k) - V(Hi, k)$ in the subroutine Newpoints of Algorithm 6.5 [4] to be $R(k) = (1+d) \cdot M(k) - d \cdot V(Hi, k)$ where d satisfies (3.4). Also, we modify $E(k) = 2 \cdot R(k) - M(k)$ to be $E(k) = e \cdot R(k) + (1-e) \cdot M(k)$ where e satisfies (3.5). Use the modified Algorithm 6.5 [4]; we can find q_1^*, q_0^*, q_2^*, and s^* such that $M(q_1^*, q_0^*, q_2^*, s^*)$ is the local minimal value.

When we find q_1^*, q_0^*, q_2^*, and s^* such that $M(q_1^*, q_0^*, q_2^*, s^*)$ is the local minimal value, then $q^{**} = \frac{1}{3}(q_1^* + q_0^* + q_2^*)$ is the economic order quantity in the fuzzy sense and s^* is the optimal stock quantity and $M(q_1^*, q_0^*, q_2^*, s^*)$ is the minimal total cost in the fuzzy sense.

4. NUMERICAL EXAMPLE IMPLEMENTATION

In this section, we apply Theorem 3 for some numerical examples to find the economic order quantity q^{**} in the fuzzy sense, and the optimal stock quantity s^* such that the total cost $M(q_1^*, q_0^*, q_2^*, s^*)$ is the minimum.

Let q_1, q_0, q_2, and s be any initial points, q_1^*, q_0^*, q_2^*, and s^* the coordinates of local minimum, $q^{**} = (q_1^* + q_0^* + q_2^*/3)$ the centroid (the optimal order) for the triangular fuzzy number (q_1^*, q_0^*, q_2^*),

$$q_* = \sqrt{(2cR/aT)}\,\sqrt{(a+b/b)}$$

the crisp economic order quantity,

$$s_* = \sqrt{(2cR/aT)}\,\sqrt{(b/a+b)}\,, \qquad F(q_*, s_*) = \sqrt{2acRT}\,\sqrt{(b/a+b)}$$

the total cost for order quantity (q_*) and stock quantity (s_*). We also let $r_q = (q^{**} - q_*/q_*) \times 100\%$, $r_c = (M(q_1^*, q_0^*, q_2^*, s^*) - F(q_*, s_*)/F(q_*, s_*)) \times 100\%$ be the relative error of order quantity in the fuzzy sense, and the relative error of fuzzy total cost, respectively. From the definition of $\mu_{G,(\tilde{Q})}(q^{**})$, we know that it is the membership degree of q^{**}.

EXAMPLE 4.1. Since there are four variables in $M(q_1, q_0, q_2, s)$, by the algorithm discussed in Section 3, when we run the program to solve the optimal solution for $M(q_1, q_0, q_2, s)$, we should assign a set of five initial points for (q_1, q_0, q_2, s) which satisfies $0 < s \leqslant q_1 < q_0 < q_2$.

Given $a = 10$, $b = 20$, $c = 200$, $R = 2000$, $T = 12$, we get $q_* = 100$, $s_* = 66.666667$, and $F(q_*, s_*) = 8000$. We give five sets (4.1.1)–(4.1.5) of initial points values of (q_1, q_0, q_2, s), and we obtain the computing results for each case as follows:

(4.1.1): $(q_1, q_0, q_2$ are near q_*, and s is $s_*)$

q_1	q_0	q_2	s			
101	102	103	66.666667	we have		
101	103	104	66.666667	$q_1^* = 100.003364$	$q^{**} = 101.003689$	$\mu_{G_s(\tilde{Q})}(q^{**}) = 0.9994$
101	102	104	66.666667	$q_0^* = 101.003085$	$M(q_1^*, q_0^*, q_2^*, s^*) = 8001.724438$	
100	101	102	66.666667	$q_2^* = 102.004617$	$r_q = 1.004\%$	$r_c = 0.022\%$
101	102	104	66.666667	$s^* = 66.666667$		

(4.1.2): $(q_1, q_0, q_2$ are near q_*, and s is far from $s_*)$

q_1	q_0	q_2	s			
98	99	100	55	we have		
97	98	99	50	$q_1^* = 100.999926$	$q^{**} = 101.999926$	$\mu_{G_s(\tilde{Q})}(q^{**}) = 1.0$
96	98	99	50	$q_0^* = 101.999926$	$M(q_1^*, q_0^*, q_2^*, s^*) = 8114.999891$	
101	102	103	60	$q_2^* = 102.999926$	$r_q = 2.0\%$	$r_c = 1.437\%$
96	97	98	50	$s^* = 59.99987$		

(4.1.3): $(q_1, q_0, q_2$ are near q_*, and s is near $s_*)$

q_1	q_0	q_2	s			
101	102	103	67	we have		
101	103	104	67	$q_1^* = 101.00000$	$q^{**} = 102.01801$	$\mu_{G_s(\tilde{Q})}(q^{**}) = 0.98314$
101	102	104	67	$q_0^* = 102.00110$	$M(q_1^*, q_0^*, q_2^*, s^*) = 8003.881534$	
101	103	105	67	$q_2^* = 103.00430$	$r_q = 2.002\%$	$r_c = 0.049\%$
101	102	105	67	$s^* = 66.666667$		

(4.1.4): $(q_1, q_0, q_2$ are near q_*, and s is far from $s_*)$

q_1	q_0	q_2	s			
98	99	100	90	we have		
97	98	99	85	$q_1^* = 97.445160$	$q^{**} = 98.630152$	$\mu_{G_s(\tilde{Q})}(q^{**}) = 0.9276$
96	98	99	85	$q_0^* = 98.722647$	$M(q_1^*, q_0^*, q_2^*, s^*) = 8697.957806$	
101	102	103	88	$q_2^* = 99.722647$	$r_q = -1.37\%$	$r_c = 8.724\%$
96	97	98	85	$s^* = 85.282626$		

(4.1.5): (q_1, q_0, q_2 are far from q_*, and $s = s_*$)

q_1	q_0	q_2	s			
132	134	135	66.666667	we have		
130	132	136	66.666667	$q_1^* = 130.000003$	$q^{**} = 132.666668$	$\mu_{G_s(\tilde{Q})}(q^{**}) = 0.8333$
140	143	144	66.666667	$q_0^* = 132.000003$	$M(q_1^*, q_0^*, q_2^*, s^*) = 8967.621679$	
142	144	146	66.666667	$q_2^* = 135.999999$	$r_q = 32.667\%$	$r_c = 12.095\%$
135	137	139	66.666667	$s^* = 66.666667$.		

Since we cannot apply the analysis method to solve the critical point $(q_1^*, q_0^*, q_2^*, s^*)$ such that $M(q_1^*, q_0^*, q_2^*, s^*)$ in Theorem 3 is the minimum, therefore, we apply the above-discussed numerical analytic method to find the approximation critical point $(q_1^*, q_0^*, q_2^*, s^*)$. Also, from the above discussion, since there are four variables q_1, q_0, q_2, and s, we should assign five initial points values to run the computer program.

From the above (4.1.1)–(4.1.5), $r_c = 0.022\%$ in (4.1.1) is lesser than in the other cases; we know that the minimal total cost in the fuzzy sense is $M(q_1^*, q_0^*, q_2^*, s^*) = 8001.724438$ in (4.1.1). Therefore, we use $q^{**} = 101.003689$ as our economic order quantity in the fuzzy sense. In this case, the total cost in the fuzzy sense is more than the crisp optimal total cost $F(q_*, s_*)0.022\%$, and q^{**} more than $q_*1.004\%$. Since we have the good local minimal result, hence we need not consider the absolute minimal value problem.

If the initial points q_1, q_0, q_2 are in small neighbors of q_*, and s is a small neighbor of s_*, then the solved q^{**} and s^* are such that the total cost is minimal in the fuzzy environment; otherwise, if q_1, q_0, q_2 are far from q_* and s is far from s_*, then the resulting total cost is higher.

For each set of given a, b, c, R, T, we should run some sets of initial points in order that we can obtain an ideal solution.

5. CONCLUSION

The crisp economic order quantity q_* and the optimal stock quantity s_* are determined under the conditions that the demand for each cycle is certainty and the period from ordering to delivery for each cycle is certainty. But there probably may be a small change in the demand or the period from the ordering to delivery for each cycle in the real situation. But we consider this problem under the condition that a, b, c, R, and T are fixed values (as described in the Introduction). Hence, q, s in $F(q, s)$ are vague variables. But if we fuzzify both q and s, then it will not only be very complex, but also not practical. Therefore, we fuzzify the crisp q to a fuzzy number $\tilde{Q}$ in the fuzzy sense and regard s to be a crisp variable, and we solve the optimal solution. We use the normal triangular fuzzy number (q_1, q_0, q_2) to represent $\tilde{Q}$. The domain of the membership function $\mu_{\tilde{Q}}(q)$ of $\tilde{Q}$ is $\{q \mid 0 < q_1 \leqslant q \leqslant q_0$, or $q_0 \leqslant q \leqslant q_2, 0 < s \leqslant q_1 < q_0 < q_2\}$ containing q_* and s_*. We assign the initial values points (q_1, q_0, q_2, s) satisfying $0 < s \leqslant q_1 < q_0 < q_2$ to solve the optimal points $(q_1^*, q_0^*, q_2^*, s^*)$ such that the estimated total cost $M(q_1^*, q_0^*, q_2^*, s^*)$ (in the fuzzy sense) is the minimum; then it is probable that q_1^*, q_0^*, q_2^* may be q_*, and s^* may be s_*. We guess in the fuzzy sense that the economic order quantity in the fuzzy sense q^{**} should approximate q_*, and s^* should approximate s_*. Hence, when we run the computer program, we assign the five initial points (q_1, q_0, q_2) to be near the q_*, and s to be near s_* is the best. From Example 4.1, we know that the best solution is $q_1^* = 100.003364$, $q_0^* = 101.003085$, $q_2^* = 102.004617$ in the case where (4.1.1) is very near $q_* = 100$. Also, since $q_0^* - q_1^* = 0.999721$, $q_2^* - q_0^* = 1.001532$, the basis side of the triangle is not so wide. The economic order quantity $q^{**} = 101.003689$, $s^* = 66.666667$ and the estimated cost $M(q_1^*, q_0^*, q_2^*, s^*) = 8001.724438$ (in the fuzzy sense) is near the crisp $q_* = 100$, $s_* = 66.666667$, and $F(q_*, s^*) = 8000$, respectively. Therefore, we have that whenever we solved the problem either in the fuzzy sense or in crisp, the results are very approximate. It indicates the reliability of the crisp case in the fuzzy sense.

APPENDIX A: PROVE $\mu_{G_s(\tilde{Q})}(y)$ IN THEOREM 2.5

1. The membership degrees $\mu_{G_s(\tilde{Q})}(y)$ of y satisfying cases (1), (4), and (10) in Table 1 are 0, i.e., $\mu_{G_s(\tilde{Q})}(y) = 0$.

2. Case (2) in Table 1: $x_1(s) \leqslant q_1 \leqslant x_2(s) \leqslant q_0$. For $x_1(s) \leqslant q_1, s \leqslant q_1 < P(s)$, then by (2.9), we have $y \geqslant a_1(s)$. For $q_1 \leqslant x_2(s), s \leqslant q_1 < P(s)$, then by (2.11), we have $y \geqslant z_*(s)$. For $x_2(s) \leqslant q_0, q_0 > P(s)$, then by (2.12), we have $z_*(s) \leqslant y \leqslant a_0(s)$. From (2.3.3), we obtain that the solution of case (2) is

$$\text{if } q_0 q_1 > P(s)^2, \text{ then } a_1(s) \leqslant y \leqslant a_0(s),\ \mu_{G_s(\tilde{Q})}(y) = \frac{x_2(s) - q_1}{q_0 - q_1}. \quad \text{(A.1)}$$

3. Case (3) in Table 1: $x_1(s) \leqslant q_1$ and $q_0 \leqslant X_2(s) \leqslant q_2$. Similarly to case (2), from (2.9), (2.10), (2.12), we have $y \geqslant a_1(s)$, $y \geqslant a_0(s)$, and $z_*(s) \leqslant y \leqslant$

$a_2(s)$. From (2.3.4), (2.3.3), we obtain that the solutions of case (3) are

$$\text{if } q_2 q_1 > P(s)^2 \text{ and } q_1 q_0 < P(s)^2, \text{ then } a_1(s) \leqslant y \leqslant a_2(s) \quad \text{(A.2)}$$

$$\text{if } q_0 q_1 > P(s)^2, \text{ then } a_0(s) \leqslant y \leqslant a_2(s) \quad \text{(A.3)}$$

$$\mu_{G_s(\tilde{Q})}(y) = \frac{q_2 - x_2(s)}{q_2 - q_0}. \quad \text{(A.4)}$$

4. Case (5) in Table 1: $q_1 \leqslant x_1(s)$ and $x_2(s) \leqslant q_0$. From (2.7), (2.12), we have $z_*(s) \leqslant y \leqslant a_1(s)$, and $z_*(s) \leqslant y \leqslant a_0(s)$. From (2.3.3), (2.3.4), (2.3.5), we obtain that the solutions of case (5) are

$$\text{if } q_0 q_1 > P(s)^2, \text{ then } z_*(s) \leqslant y \leqslant a_1(s) \quad \text{(A.5)}$$

$$\text{if } q_2 q_1 > P(s)^2 \text{ and } q_1 q_0 < P(s)^2, \text{ then } z_*(s) \leqslant y \leqslant a_0(s) \quad \text{(A.6)}$$

$$\text{if } q_1 q_2 < P(s)^2, \text{ then } z_*(s) \leqslant y \leqslant a_0(s) \quad \text{(A.7)}$$

$$\mu_{G_s(\tilde{Q})}(y) = \frac{x_2(s) - q_1}{q_0 - q_1}.$$

5. Case (6) in Table 1: $q_1 \leqslant x_1(s) \leqslant q_0 \leqslant x_2(s) \leqslant q_2$.
From (2.7), (2.8), (2.10), (2.12), we have $z_*(s) \leqslant y \leqslant a_1(s)$, $y \geqslant z_*(s)$, $y \geqslant a_0(s)$, and $z_*(s) \leqslant y \leqslant a_2(s)$. From (2.3.4), (2.3.5), we obtain that the solutions of case (6) are

$$\text{if } q_2 q_1 > P(s)^2 \text{ and } q_1 q_0 < P(s)^2, \text{ then } a_0(s) \leqslant y \leqslant a_1(s) \quad \text{(A.8)}$$

$$\text{if } q_1 q_2 < P(s)^2, \text{ then } a_0(s) \leqslant y \leqslant a_2(s) \quad \text{(A.9)}$$

$$\mu_{G_s(\tilde{Q})}(y) = \max\left\{\frac{x_1(s) - q_1}{q_0 - q_1}, \frac{q_2 - x_2(s)}{q_2 - q_0}\right\}. \quad \text{(A.10)}$$

6. Case (7) in Table 1: $q_1 \leqslant x_1(s) \leqslant q_0$ and $q_2 \leqslant x_2(s)$. From (2.7), (2.8), (2.10), we have $z_*(s) \leqslant y \leqslant a_1(s)$, $y \geqslant z_*(s)$, and $y \geqslant a_2(s)$. From (2.3.5), we obtain that the solution of case (7) is

$$\text{if } q_1 q_2 < P(s)^2, \text{ then } a_2(s) \leqslant y \leqslant a_1(s),\ \mu_{G_s(\tilde{Q})}(y) = \frac{x_1(s) - q_1}{q_0 - q_1}. \quad \text{(A.11)}$$

7. Case (8), (9) in Table 1: There is one inequality $q_0 \leqslant x_1(s)$ in these cases, and $q_0 > P(s)$; from (2.6), these cases could not occur.

From (A.1)–(A.11), we obtain $\mu_{G_s(\tilde{Q})}(y)$ as follows:

[A] Under the condition $s \leqslant q_1 < P(s) < q_0 < q_2$:

[A31] If $q_0 q_1 > P(s)^2$, then

$$\mu_{G_s(\tilde{Q})}(y) = \begin{cases} \dfrac{x_2(s) - q_1}{q_0 - q_1}, & \text{if } z_*(s) \leqslant y \leqslant a_0(s) \text{ (A.1), (A.5)} \\ \dfrac{q_2 - x_2(s)}{q_2 - q_0}, & \text{if } a_0(s) \leqslant y \leqslant a_2(s) \text{ (A.3)} \\ 0, & \text{otherwise.} \end{cases} \tag{A.12}$$

[A32] If $q_2 q_1 > P(s)^2$ and $q_1 q_0 < P(s)^2$, then

$$\mu_{G_s(\tilde{Q})}(y) = \begin{cases} \dfrac{x_2(s) - q_1}{q_0 - q_1}, & \text{if } z_*(s) \leqslant y \leqslant a_0(s) \text{ (A.6)} \\ \max\left\{\dfrac{x_1(s) - q_1}{q_0 - q_1}, \dfrac{q_2 - x_2(s)}{q_2 - q_1}\right\}, & \text{if } a_0(s) \leqslant y \leqslant a_1(s) \text{ (A.8)} \\ \dfrac{q_2 - x_2(s)}{q_2 - q_0}, & \text{if } a_1(s) \leqslant y \leqslant a_2(s) \text{ (A.2)} \\ 0, & \text{otherwise.} \end{cases} \tag{A.13}$$

[A33] If $q_1 q_2 < P(s)^2$, then

$$\mu_{G_2(\tilde{Q})}(y) = \begin{cases} \dfrac{x_2(s) - q_1}{q_0 - q_1}, & \text{if } z_*(s) \leqslant y \leqslant a_0(s) \text{ (A.7)} \\ \max\left\{\dfrac{x_1(s) - q_1}{q_0 - q_1}, \dfrac{q_2 - x_2(s)}{q_2 - q_0}\right\}, & \text{if } a_0(s) \leqslant y \leqslant a_2(s) \text{ (A.9)} \\ \dfrac{x_1(s) - q_1}{q_0 - q_1}, & \text{if } a_2(s) \leqslant y \leqslant a_1(s) \text{ (A.11)} \\ 0, & \text{otherwise.} \end{cases} \tag{A.14}$$

(A.12) is (2.3.6) in Theorem 2.5.

[B] We consider $\max\{(x_1(s)-q_1/q_0-q_1),(q_2-x_2(s)/q_2-q_0)\}$ in (A.13), (A.14) as follows: [B1] Simplify max in (A.13) for $a_0(s)\leqslant y\leqslant a_1(s)$: Let

$$f_s(y)=\frac{x_1(s)-q_1}{q_0-q_1}=\frac{1}{bT(q_0-q_1)}\left[bTs+y-\sqrt{D(s)}-q_1bT\right],$$

$$a_0(s)\leqslant y\leqslant a_1(s).$$

$$g_s(y)=\frac{q_2-x_2(s)}{q_2-q_0}=\frac{1}{bT(q_2-q_0)}\left[bTq_2-bTs-y-\sqrt{D(s)}\right],$$

$$a_0(s)\leqslant y\leqslant a_1(s).$$

$$D(s)=(bTs+y)^2-b^2T^2P(s)^2>0, \qquad \text{for every } a_0(s)\leqslant y\leqslant a_1(s)$$

then

$$f_s'(y)=\frac{1}{bT(q_0-q_1)}\left(1-\frac{bTs+y}{\sqrt{D(s)}}\right)<0, \qquad \text{for every } a_0(s)<y<a_1(s)$$

$$f_s''(y)=\frac{1}{bT(q_0-q_1)}\left(\frac{b^2T^2P(s)^2}{D(s)^{3/2}}\right)>0$$

$$g_s'(y)=\frac{1}{bT(q_2-q_0)}\left(-1-\frac{bTs+y}{\sqrt{D(s)}}\right)<0,$$

$$g_s''(y)=\frac{1}{bT(q_2-q_0)}\left(\frac{b^2T^2P(s)^2}{D(s)^{3/2}}\right)>0.$$

So, $u=f_s(y)$, $u=g_s(y)$ are decreasing, concave upward, and continuous on $a_0(s)\leqslant y\leqslant a_1(s)$.

For $j=0,2$, $q_j^2>P(s)^2$, if $y=a_j(s)$, then by (2.15), we have $x_1(s)=(P(s)^2/q_j)$, $x_2(s)=q_j$; for $j=1$, $q_1^2>P(s)^2$, if $y=a_1(s)$, then by (2.16), we have $x_1(s)=q_1$, $x_2(s)=(P(s)^2/q_1)$.

If $q_2 q_1 > P(s)^2$, $q_1 q_0 < P(s)^2$, and $s \leqslant q_1 < P(s) < q_0 < q_2$, then we have:

$$f_s(a_0(s)) = \frac{P(s)^2 - q_0 q_1}{q_0(q_0 - q_1)} > 0, \quad g_s(a_0(s)) = \frac{q_2 - q_0}{q_2 - q_0} = 1,$$

$$f_s(a_0(s)) < g_s(a_0(s)),$$

$$f_s(a_1(s)) = 0, \qquad g_s(a_1(s)) = \frac{q_1 q_2 - P(s)^2}{q_1(q_2 - q_0)} > 0,$$

$$f_s(a_1(s)) < g_s(a_1(s)).$$

Simplifying $(x_1(s) - q_1/q_0 - q_1) = (q_2 - x_2(s)/q_2 - q_0)$, we obtain the following equation:

$$4(q_2 - q_0)(q_0 - q_1)y^2 - 2A(s)y + B(s) = 0 \tag{A.15}$$

where

$$A(s) = -(q_2 - q_1)^2 bT(s - q_0) + bTs(q_2 + q_1 - 2q_0)^2$$

$$B(s) = b^2T^2\left((q_2 - q_1)^2(s - q_0)^2 + \left(P(s)^2 - s^2\right)(q_2 + q_1 - 2q_0)^2\right).$$

$$\text{Discriminant } D_1(s) = A(s)^2 - 4(q_2 - q_0)(q_0 - q_1)B(s). \tag{A.16}$$

If $D_1(s) > 0$, then Eq. (A.15) has two roots:

$$t_1(s) = \frac{A(s) - \sqrt{D_1(s)}}{4(q_2 - q_0)(q_0 - q_1)}, \quad t_2(s) = \frac{A(s) + \sqrt{D_1(s)}}{4(q_2 - q_0)(q_0 - q_1)};$$

else, if $D_1(s) = 0$, then Eq. (A.15) has one root, $t_3(s) = (A(s)/4(q_2 - q_0)(q_0 - q_1))$.

From the above discussion, we have that all possible graphs of $u = f_s(y)$, and $u = g_s(y)$ are as shown in Figure 4(a)–(c).

Therefore, we have the three cases for $a_0(s) \leqslant y \leqslant a_1(s)$ in (A.13) as follows:

[B11] If $D_1(s) < 0$ or ($D_1(s) = 0$, and $a_0(s) < t_3(s) < a_1(s)$), then $f_s(y) \leqslant g_s(y)$. From Figure 4(a), (c) and (A.13), we have

$$\max\left[\frac{x_1(s) - q_1}{q_0 - q_1}, \frac{q_2 - x_2(s)}{q_2 - q_0}\right] = \frac{q_2 - x_2(s)}{q_2 - q_0}, \qquad a_0(s) \leqslant y \leqslant a_1(s). \tag{A.17}$$

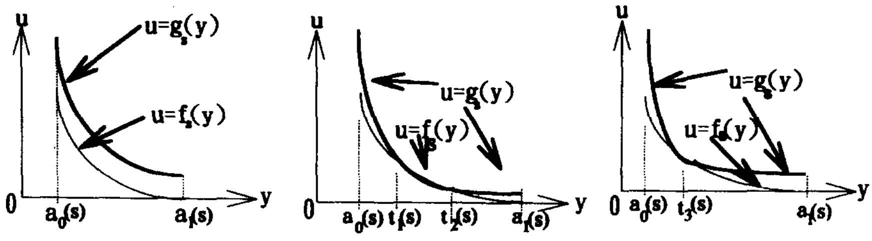


Fig. 4. (a) ($D_1(s) < 0$), no intersection. (b) ($D_1(s) > 0$), two intersections. (c) ($D_1(s) = 0$), one intersection.

[B12] If $D_1(s) > 0$ and $a_0(s) < t_1(s) < t_2(s) < a_1(s)$, then, from Figure 4(b), we have

$$\max\left[\frac{x_1(s)-q_1}{q_0-q_1}, \frac{q_2-x_2(s)}{q_2-q_0}\right] = \begin{cases} \dfrac{q_2-x_2(s)}{q_2-q_0}, & a_0(s) \leqslant y \leqslant t_1(s) \\ \dfrac{x_1(s)-q_1}{q_0-q_1}, & t_1(s) \leqslant y \leqslant t_2(s) \\ \dfrac{q_2-x_2(s)}{q_2-q_0}, & t_2(s) \leqslant y \leqslant a_1(s) \\ 0 \end{cases} \tag{A.18}$$

(A.13), (A.17), and (A.18) are (2.3.7), (2.3.8) in Theorem 2.5, respectively. Similarly, simplify max in (A.14); since $f_s(a_0(s)) < g_s(a_0(s))$ and $g_s(a_2(s)) < f_s(a_2(s))$, therefore, we have (2.3.9) in Theorem 2.5.

APPENDIX B: PROVE $\mu_{G_s(\tilde{Q})}(y)$ IN THEOREM 2.7

Similarly to Appendix A, we have the following: Since the inequalities $q_j \geqslant P(s), j = 0, 1$, and $q_2 \leqslant P(s)$ in (1)–(4) do not satisfy (2.4.1), therefore, we do not consider them.

1. Cases (1), (4), and (10) in Table 1: the membership degrees $\mu_{G_s(\tilde{Q})}(y)$ of y which satisfy cases (1), (4), and (10) in Table 1 are 0, i.e., $\mu_{G_s(\tilde{Q})}(y) = 0$.

2. Cases (2), (5) in Table 1: For $x_2(s) \leqslant q_0$, and $s \leqslant q_0 < P(s)$, from (2.13), we do not consider these cases.

3. Case (3) in Table 1: $x_1(s) \leqslant q_1$ and $q_0 \leqslant x_2(s) \leqslant q_2$. By (2.9), (2.11), and (2.12), we have $y \geqslant a_1(s)$, $y \geqslant z_*(s)$, $z_*(s) \leqslant y \leqslant a_2(s)$. From (2.4.3), we have

$$\text{if } q_2 q_1 \geqslant P(s)^2, \text{then } a_1(s) \leqslant y \leqslant a_2(s),\ \mu_{G_s(\tilde{Q})}(y) = \frac{q_2 - x_2(s)}{q_2 - q_0}. \quad \text{(B.1)}$$

4. Case (6) in Table 1: By (2.7), (2.9), (2.11), and (2.12), we have $z_*(s) \leqslant y \leqslant a_1(s)$, $y \geqslant a_0(s)$, $y \geqslant z_*(s)$, $z_*(s) \leqslant y \leqslant a_2(s)$. From (2.4.3), (2.4.4), we have

$$\text{if } q_2 q_1 > P(s)^2, \text{then } a_0(s) \leqslant y \leqslant a_1(s) \quad \text{(B.2)}$$

$$\text{if } q_1 q_2 < P(s)^2 \text{ and } q_2 q_0 > P(s)^2, \text{then } a_0(s) \leqslant y \leqslant a_2(s) \quad \text{(B.3)}$$

$$\mu_{G_s(\tilde{Q})}(y) = \max\left\{ \frac{x_1(s) - q_1}{q_0 - q_1}, \frac{q_2 - x_2(s)}{q_2 - q_0} \right\}. \quad \text{(B.4)}$$

5. Case (7) in Table 1: By (2.7), (2.9), and (2.10), we have $z_*(s) \leqslant y \leqslant a_1(s)$, $y \geqslant a_0(s)$, $y \geqslant a_2(s)$. From (2.4.4), (2.4.5), we have

$$\text{if } q_2 q_0 \geqslant P(s)^2 \text{ and } q_1 q_2 < P(s)^2, \text{then } a_2(s) \leqslant y \leqslant a_1(s) \quad \text{(B.5)}$$

$$\text{if } q_0 q_2 < P(s)^2, \text{then } a_0(s) \leqslant y \leqslant a_1(s) \quad \text{(B.6)}$$

$$\mu_{G_2(\tilde{Q})}(y) = \frac{x_1(s) - q_1}{q_0 - q_1}. \quad \text{(B.7)}$$

6. Case (8) in Table 1: By (2.7) and (2.12), we have $z_*(s) \leqslant y \leqslant a_0(s)$, $z_*(s) \leqslant y \leqslant a_2(s)$. From (2.4.3)–(2.4.5), we have

$$\text{if } q_2 q_1 > P(s)^2, \text{then } z_*(s) \leqslant y \leqslant a_0(s) \quad \text{(B.8)}$$

$$\text{if } q_2 q_0 > P(s)^2 \text{ and } q_1 q_2 < P(s)^2, \text{then } z_*(s) \leqslant y \leqslant a_0(s) \quad \text{(B.9)}$$

$$\text{if } q_0 q_2 < P(s)^2, \text{ then } z_*(s) \leqslant y \leqslant a_2(s) \tag{B.10}$$

$$\mu_{G_s(\tilde{Q})}(y) = \frac{q_2 - x_1(s)}{q_2 - q_0}. \tag{B.11}$$

7. Case (9) in Table 1: By (2.7), (2.8), and (2.10), we have $z_*(s) \leqslant y \leqslant a_0(s)$, $y \geqslant z_*(s)$, $y \geqslant a_2(s)$. From (2.4.5), we obtain that the solution of case (9) is

$$q_0 q_2 < P(s)^2, \text{ then } a_2(s) \leqslant y \leqslant a_0(s),\ \mu_{G_s(\tilde{Q})}(y) = \frac{q_2 - x_1(s)}{q_2 - q_0}. \tag{B.12}$$

From (B.1)–(B.12), we have the following.

[A4] Under the condition $s \leqslant q_1 < q_0 < P(s) < q_2$

[A41] If $q_2 q_1 > P(s)^2$, then

$$\mu_{G_s(\tilde{Q})}(y) = \begin{cases} \dfrac{q_2 - x_1(s)}{q_2 - q_0}, & \text{if } z_*(s) \leqslant y \leqslant a_0(s) \text{ (B.8)} \\ \max\left\{ \dfrac{x_1(s) - q_1}{q_0 - q_1}, \dfrac{q_2 - x_2(s)}{q_2 - q_0} \right\}, & \text{if } a_0(s) \leqslant y \leqslant a_1(s) \text{ (B.2)} \\ \dfrac{q_2 - x_2(s)}{q_2 - q_0}, & \text{if } a_1(s) \leqslant y \leqslant a_2(s) \text{ (B.1)} \\ 0, & \text{otherwise}. \end{cases} \tag{B.13}$$

[A42] If $q_2 q_0 > P(s)^2$, and $q_1 q_2 < P(s)^2$, then

$$\mu_{G_s(\tilde{Q})}(y) = \begin{cases} \dfrac{q_2 - x_1(s)}{q_2 - q_0}, & \text{if } z_*(s) \leqslant y \leqslant a_0(s) \text{ (B.9)} \\ \max\left\{ \dfrac{x_1(s) - q_1}{q_0 - q_1}, \dfrac{q_2 - x_2(s)}{q_2 - q_0} \right\}, & \text{if } a_0(s) \leqslant y \leqslant a_2(s) \text{ (B.3)} \\ \dfrac{x_1(s) - q_1}{q_0 - q_1}, & \text{if } a_2(s) \leqslant y \leqslant a_1(s) \text{ (B.5)} \\ 0, & \text{otherwise}. \end{cases} \tag{B.14}$$

[A43] If $q_0 q_2 < P(s)^2$, then

$$\mu_{G_s(\tilde{Q})}(y)=\begin{cases}\dfrac{q_2-x_1(s)}{q_2-q_0}, & \text{if } z_*(s)\leqslant y\leqslant a_0(s)\ (\text{B.10}),(\text{B.12})\\ \dfrac{x_1(s)-q_1}{q_0-q_1}, & \text{if } a_0(s)\leqslant y\leqslant a_1(s)\ (\text{B.6})\\ 0, & \text{otherwise.}\end{cases} \tag{B.15}$$

From (B.15), we have (2.4.9) in Theorem 2.7.

[B4] Simplify $\max\{(x_1(s)-q_1/q_0-q_1),(q_2-x_2(s)/q_2-q_0)\}$ in (B.13) and (B.14) as follows:

[B41] Simplify max in (B.13) for $a_0(s)\leqslant y\leqslant a_1(s)$ (the same as [B] in Appendix A): $u=f_s(y)$, $u=g_s(y)$ are decreasing, concave upward, and continuous on $a_0(s)\leqslant y\leqslant a_1(s)$. For $j=1,0$, by (2.16), if $y=a_j$, then $x_1(s)=q_j$, $x_2(s)=(P(s)^2/q_j)$ so that $f_s(a_0(s))=1>g_s(a_0(s)), f_s(a_1(s))=0<g_s(a_1(s))$. Therefore, the $\mu_{G_s(\tilde{Q})}(y)$ in (B.13) is as (2.4.6) in Theorem 2.7.

[B42] Simplify max in (B.14) for $a_0(s)\leqslant y\leqslant a_2(s)$ (the same manner as [B] in Appendix A).

We have $u=f_s(y)$, $u=g_s(y)$ are decreasing, concave upward, and continuous on $a_0(s)\leqslant y\leqslant a_2(s)$, $f_s(a_0(s))=1>g_s(a_0(s))$, $f_s(a_2(s))>g_s(a_2(s))=0$.

There are three possible cases for $a_0(s)\leqslant y\leqslant a_2(s)$ in (B.14) as follows:

[B421] If $D_1(s)<0$ or ($D_1(s)$ and $a_0(s)\leqslant t_3(s)\leqslant a_2(s)$), $g_s(y)<f_s(y)$, then from (B.14), we have

$$\max\left[\frac{x_1(s)-q_1}{q_0-q_1},\frac{q_2-x_2(s)}{q_2-q_0}\right]=\frac{x_1(s)-q_1}{q_0-q_1},\qquad a_0(s)\leqslant y\leqslant a_2(s). \tag{B.16}$$

[B422] If $D_1(s)>0$ and $a_0(s)<t_1(s)<t_2(s)<a_2(s)$, then from (B.14), we have

$$\max\left[\frac{x_1(s)-q_1}{q_0-q_1},\frac{q_2-x_2(s)}{q_2-q_0}\right]=\begin{cases}\dfrac{x_1(s)-q_1}{q_0-q_1}, & \text{if } a_0(s)\leqslant y\leqslant t_1(s)\\ \dfrac{q_2-x_2(s)}{q_2-q_0}, & \text{if } t_1(s)\leqslant y\leqslant t_2(s)\\ \dfrac{x_1(s)-q_1}{q_0-q_1}, & \text{if } t_2(s)\leqslant y\leqslant a_2(s)\\ 0, & \text{otherwise.}\end{cases} \tag{B.17}$$

From (B.14), (B.16), (B.17), we have (2.4.7), (2.4.8) in Theorem 2.7, respectively.

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REFERENCES

1. J. L. Buchanan and P. R. Turner, *Numerical Methods and Analysis*, McGraw-Hill, New York, 1992.
2. S. H. Chen and C. C. Wang, Backorder fuzzy inventory model, in *Proc. Joint Conf. on Information Sciences Proceedings* [*JCIS'1994B*], North Carolina, Nov. 14–16, 1994, pp. 389–392.
3. A. Kaufmann and M. M. Gupta, *Introduction to Fuzzy Arithematic Theory and Applications*, Van Nostrand Reinhold, New York, 1991.
4. J. H. Mathews, *Numerical Methods for Mathematics, Science, and Engineering*, Prentice-Hall International, Inc., London, 1992.
5. H.-J. Zimmermann, *Fuzzy Set Theory and Its Application*, 2nd rev. ed., Kluwer Academic Publishers, Boston/Dordrecht/London, 1991.

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