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Theory and Methodology

Inventory models with general backorder costs

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Abstract: Under the prevalent assumption of linear inventory holding and backorder costs, important results have been established in many basic inventory models. However, the assumption is often too restrictive for many applications. In this paper, we study stochastic inventory models under more general backorder costs, namely, the penalty costs for each unit of backorder consisting of fixed and proportional components. Under this backorder cost structure, we derive a necessary and sufficient condition for the quasiconvexity of the one-step loss function. This condition is satisfied by a wide spectrum of demand distributions, but it is not distribution-free. Consequently, existing results based on quasiconvex loss functions are applicable in many, but not all, models with the above cost structure.

Keywords: Inventory; Markov decision programming; Backorder costs; Likelihood ratio ordering

1. Introduction

In many inventory systems, stockouts are unavoidable due to various uncertainties. When excess demand is backordered, the penalty costs are usually, in inventory theory literature, assumed to accumulate at a constant rate over time which is proportional to the volume of the backorders on the books. Under this and other standard assumptions, many important results have been established. However, this linearity assumption of backorder costs is made primarily for mathematical convenience at the cost of modelling power. In real world inventory systems, backorder costs are typically nonlinear. One way to model this nonlinearity is by introducing a fixed component in the backorder costs. In other words, the total penalty costs for each unit of backorder are $\pi + pt$, where t is the time the backorder unit lasts, and π and p are some positive constants. Hadley and Whitin [7] have a detailed discussion of this fixed-plus-linear backorder cost structure. It is the purpose of this paper to study inventory models with this more general backorder cost structure. Throughout this paper we retain the linearity assumption of inventory holding costs. The combination of the fixed-plus-linear backorder costs and the linear holding costs will be hereafter referred to as 'the general cost structure' for convenience.

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Backorder costs are subject to various intangible factors in many practical situations. The difficulties in measuring backorder costs have led to alternative methods in modelling stochastic inventory systems. Among these methods, the most popular ones use various constraints on backorders, including the constraints on the average fraction of demand that can be satisfied from shelves (the so-called fill rate) and on the average backorders. In a cost minimization model with these service constraints, by relaxing these constraints using Lagrangian multipliers, the penalty cost terms appearing in the objective function take the fixed-plus-linear structure. More specifically, the fixed component in the backorder costs corresponds to the multiplier associated with the fill rate constraint, whereas the linear component corresponds to the average backorder constraint. In other words, both types of backorder costs are surrogates for the popular service constraints, therefore neither one should be ignored.

The structure of holding and backorder costs affects the property of the one-step loss function (hereafter referred to as 'loss function'), which arises in almost every stochastic inventory model. For example, a fundamental single-item, single-location, stochastic inventory model can be described by the following dynamic program:

$$f_n(x) = \min_{y \geq x} \{K 1(y - x > 0) + c(y - x) + G(y) + \alpha E f_{n+1}(y - D_n)\}$$

where K is the fixed ordering cost, $1(\cdot)$ is the indicator function, c is the unit purchase cost, $G(\cdot)$ is the one-period expected holding and backorder costs (loss function), α is the discount factor, D_n is the demand in period n, E is the expectation operator, and $f_n(x)$ is the minimal expected discounted total costs over the rest of the planning horizon starting from period n with x units of stock. See, e.g., Scarf [10] for a fuller description of this dynamic program.

Furthermore, the convexity or quasiconvexity of the loss function plays a central role in establishing many important results. If $G(\cdot)$ is convex, the optimality of (s, S) policies was established by Scarf [10] in finite horizon models, by Iglehart [8] in infinite horizon models, and by Beckmann [2] in continuous review models; furthermore, algorithms for computing optimal (s, S) policies were provided by Archibald and Silver [1] and Veinott and Wagner [12]. If, less restrictively, $G(\cdot)$ is quasiconvex, the optimality of (s, S) policies was established by Veinott [11] for a class of periodic review models; and efficient algorithms were provided by Federgruen and Zheng [5] to compute optimal (Q, r) policies, by Federgruen and Zipkin [6] and Zheng and Federgruen [14] to compute optimal (s, S) policies, and by Zheng and Chen [13] to compute optimal (nQ, r) policies. (If the inventory position at a review time is at or below r, order nQ where n is the largest integer such that the inventory position after order placement is less than or equal to $r + Q$.) These properties of the loss function also play a central role in analyzing many multi-location inventory models, see, e.g., Federgruen [3] for a comprehensive description of these models. Consequently, in this paper, we focus on the property of the loss function under the general cost structure.

It is well known that the loss function is convex under the prevalent assumption of linear inventory holding and backorder costs. However, when the fixed backorder costs are included, the convexity no longer holds in general. Therefore, the above results based on convex loss functions are not applicable under the general cost structure. On the other hand, it is not clear what cost structure would lead to a quasiconvex loss function, and hence the usefulness of the existing results based on quasiconvex loss functions has not materialized. In this study, we establish quasiconvexity of the loss function under the general cost structure.

For single-item, single-location, stationary inventory models under the general cost structure, we identify a necessary and sufficient condition for the quasiconvexity of the loss function. This condition is satisfied by a wide spectrum of demand distributions, namely, gamma, Poisson, binomial and negative binomial distributions. We also show that the loss function is quasiconvex under a commonly used normal approximation scheme for demand distribution. Therefore, when the demand sizes can be modelled with the above spectrum of distributions, many important results in inventory theory hold under the general cost structure – e.g., (s, S) policies are still optimal and the existing algorithms to compute various optimal policies (e.g., (s, S) or (nQ, r) policies) are still applicable.

Although the spectrum of distributions listed above is enough for many applications, it certainly does not exhaust all possible demand distributions. For other demand distributions, the property of the loss function becomes more complex. In fact, there are cases where the loss function is not quasiconvex (i.e., the loss function has multiple local minima) – one of these cases is reported in this paper. This suggests that we should not blindly apply existing results that are based on quasiconvex loss functions to models with the general cost structure.

Section 2 considers periodic review models and establishes a necessary and sufficient condition for the quasiconvexity of the loss function. This condition is satisfied by gamma demand distribution. Section 3 is parallel to Section 2 and is concerned with continuous review models. In Section 4, we consider models with discrete demand and show that the loss function is quasiconvex when the demand distribution is Poisson, binomial or negative binomial; we then construct an example where the loss function is not quasiconvex. In Section 5, we discuss normal approximation of demand sizes and show that the loss function is quasiconvex under a commonly used approximation scheme. Section 6 summarizes the paper.

2. Periodic review model

After deriving the loss function under the general cost structure, we identify a necessary and sufficient condition for the quasiconvexity of the loss function. This condition is satisfied by gamma distribution, or any other demand distribution that leads to a monotone likelihood ratio family (to be defined later). We first describe our model in detail.

We use D_t to denote the demand of period t, and assume D_t's are nonnegative iid (independent, identically distributed) random variables. Write $\phi(\cdot)$ and $\Phi(\cdot)$, respectively, for the density and cumulative distribution functions of D_1. We use D^m, where m is a nonnegative integer, to denote the aggregate demand during m consecutive periods (with $D^0 = 0$). Write $\phi^m(\cdot)$ and $\Phi^m(\cdot)$, respectively, for the density and cumulative distribution functions of D^m. Ordering decisions are made at the beginning of each period, and each order takes L (fixed) periods to arrive, i.e. the order placed at the beginning of period t will arrive at the beginning of period $t+L$. Demand occurs after order placement, and is met on a first-come first-serve basis, with any excess demand backordered. For each period, the holding (variable backorder) costs are proportional to the period ending inventory (backorder) level at unit rate h (p). In addition, a fixed penalty cost π is charged for each unit of backorder, independent of the backorder duration. Write G_t for the holding and backorder costs incurred in period t. Let α be the discount factor.

Let IP_t be the inventory position (on-hand inventory plus outstanding orders minus backorders) immediately after order placement (if any) at period t. Given $\mathrm{IP}_t = y$, since all and only those orders placed by period t will arrive by period $t+L$, the distributions of both the beginning and ending inventory levels of period $t+L$ are determined; and hence the expected holding and backorder costs in period $t+L$, $E(G_{t+L} \mid \mathrm{IP}_t = y)$, are also determined. Consequently, it is natural to charge the costs in period $t+L$ to the decision epoch at period t, and, after discounting, we have the loss function $G(y) = \alpha^L E(G_{t+L} \mid \mathrm{IP}_t = y)$ (clearly this expected value does not depend on t). Since a positive, multiplicative factor does not change the quasiconvexity of the loss function, we will hereafter simply assume that $\alpha = 1$.

Let $\mathrm{IP}_t = y$. The net inventory level (on-hand inventory minus backorders) at the end of period $t+L$ is $y - D^{L+1}$. Thus the holding and variable backorder costs incurred in period $t+L$ are $h(y - D^{L+1})^+ + p(y - D^{L+1})^-$ (define $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$). Notice that, for period $t+L$, the beginning (after delivery if any) backorder level is $(y - D^L)^-$ and the ending backorder level is $(y - D^{L+1})^-$; their difference, $(y - D^{L+1})^- - (y - D^L)^-$, is the number of backorders incurred in period $t+L$ alone. Therefore,

$$(G_{t+L} \mid \mathrm{IP}_t = y) = h(y - D^{L+1})^+ + (p+\pi)(y - D^{L+1})^- - \pi(y - D^L)^- .$$

By taking expectation and the usual algebra,

$$G(y)=(h+p+\pi)\int_0^y \Phi^{L+1}(x)\,\mathrm{d}x-\pi\int_0^y \Phi^L(x)\,\mathrm{d}x-py+\pi\mu+p(L+1)\mu \tag{1}$$

where $\mu=ED_1$. By taking derivatives with respect to y on both sides of (1), we have

$$G'(y)=-p+(h+p+\pi)\Phi^{L+1}(y)-\pi\Phi^L(y). \tag{2}$$

Write $R(y)=\Phi^{L+1}(y)/\Phi^L(y)$, and set $R(y)$ equal to zero when both its numerator and denominator are zero. Notice that D^{L+1} is stochastically larger than D^L, and thus $R(y)\le 1$ for all y. The next lemma identifies a necessary and sufficient condition for the quasiconvexity of $G(\cdot)$. (In the sequel, when we say 'positive values of h, p and π', we mean nonnegative values of h, p and π such that $h+p+\pi>0$.)

Lemma 1. *$G(\cdot)$ is quasiconvex for all positive values of h, p and π if and only if $R(\cdot)$ is nondecreasing.*

Proof. We first show the necessary condition. Assume $G(\cdot)$ is quasiconvex for all positive values of h, p and π. To show that $R(\cdot)$ is nondecreasing, we suppose, on the contrary, that there exist $y_1<y_2$ such that $R(y_1)>R(y_2)$. Let $\epsilon=R(y_1)-R(y_2)(>0)$. Choose the values of h, p and π such that

$$h+p+\pi=1 \text{ and } \pi=R(y_2)+\xi$$

where $0\le\xi<\frac{1}{2}\epsilon$ and

$$p=\min\{\tfrac{1}{2}(\Phi^{L+1}(y_1)-\pi\Phi^L(y_1)),1-\pi\}. \tag{3}$$

Obviously π is nonnegative. To verify that p is positive, we first notice that

$$\pi=R(y_2)+\xi<R(y_2)+\epsilon=R(y_1),$$

and thus the first term inside the curly brackets in (3) is positive. Note further that

$$\pi+\tfrac{1}{2}\xi=R(y_2)+\tfrac{3}{2}\xi<R(y_2)+\epsilon=R(y_1)\le 1,$$

and thus the second term in the curly brackets in (3) is also positive. Consequently, p is positive. Finally, h is nonnegative because, from (3), $p+\pi\le 1$. For such values of h, p and π, from (2), we have

$$G'(y_1)=-p+\Phi^{L+1}(y_1)-\pi\Phi^L(y_1)\ge\tfrac{1}{2}(\Phi^{L+1}(y_1)-\pi\Phi^L(y_1))>0$$

where the first inequality follows by replacing p with the first term in the curly brackets in (3), and the last follows since $\pi<R(y_1)$ as noted above. On the other hand, we have

$$G'(y_2)=-p+\Phi^{L+1}(y_2)-\pi\Phi^L(y_2)<\Phi^{L+1}(y_2)-\pi\Phi^L(y_2)\le 0$$

where the first inequality is clear and the last inequality follows since $\pi\ge R(y_2)$ due to the definition of π. This contradicts our hypothesis that $G(\cdot)$ is quasiconvex for all positive values of h, p and π.

We now show the sufficient condition. Take any positive h, p and π, and assume $R(\cdot)$ is nondecreasing. Let us rewrite $G'(y)$ from (2) as

$$G'(y)=-p+(h+p+\pi)\Phi^L(y)[R(y)-\theta] \tag{4}$$

where $\theta=\pi/(h+p+\pi)$. Since $R(y)$ is nondecreasing and tends to 1 as $y\to\infty$, there are only two possible cases:

Case 1: $R(y)\ge\theta$ for all y. In this case, from (4), we see that $G'(\cdot)$ is nondecreasing, or $G(\cdot)$ is convex (thus quasiconvex).

Case 2: There exists a point w such that $R(y)\ge\theta$ for all $y\ge w$ and $R(y)\le\theta$ for all $y<w$. Consequently, from (4), we see that $G'(\cdot)$ is nondecreasing over $[w,\infty)$, or $G(\cdot)$ is convex in $[w,\infty)$.

Notice that the second term on the right-hand-side of (4) is ≤ 0 over $(-\infty, w)$, we see that $G'(y) \leq 0$ for $y < w$, or $G(\cdot)$ is nonincreasing over $(-\infty, w)$. Thus $G(\cdot)$ is quasiconvex. □

Note. From the above proof of the sufficient condition in Lemma 1, we notice that the loss function has additional properties when $R(\cdot)$ is nondecreasing: There exists a point y_0 such that $G(y)$ decreases for $y \leq y_0$ and is convex for $y > y_0$. (We may, for example, choose $y_0 = 0$ (resp., $= w$) if *Case 1* (resp., *Case 2*) in the proof of Lemma 1 holds.)

Consider first the special case with $L = 0$. Since $R(y) = \Phi(y)$ for all y and $\Phi(y)$ is nondecreasing, we know, from Lemma 1, that $G(\cdot)$ is quasiconvex for all positive values of h, p and π. Further notice that $G(\cdot)$ is not convex since $G'(0^-) = -p$ and $G'(0^+) = -p - \pi$. We summarize these observations in the following corollary:

Corollary 1. *In the periodic review model with zero leadtime, $G(\cdot)$ is quasiconvex (but not convex) for all positive values of h, p and π.*

Although the above corollary is distribution-free, i.e. $G(\cdot)$ is quasiconvex for any demand distributions when the leadtime is zero, this is not the case if the leadtime is positive. In Section 4, we construct an example in which $R(\cdot)$ is not monotone, so, by Lemma 1, there exist some cost parameters (h, p and π) such that the corresponding loss function is not quasiconvex (our constructive proof of the necessary condition in Lemma 1 facilitates the selection of these cost parameters). This suggests that caution should be taken when applying existing results to models with the general cost structure. The rest of this section is only concerned with positive leadtimes.

We now proceed to identify distribution families that guarantee quasiconvex loss functions. Define $r(y) = \phi^{L+1}(y)/\phi^L(y)$, i.e. the likelihood ratio of D^{L+1} and D^L, and set $r(y)$ equal to zero when both its numerator and denominator are zero.

Lemma 2. *If $r(\cdot)$ is nondecreasing, so is $R(\cdot)$.*

Proof. Assume $r(\cdot)$ is nondecreasing. Since $R(y) = 0$ for all $y < 0$, we only need to show that $R(y)$ is nondecreasing for $y \geq 0$, or $R'(y) \geq 0$ for all $y > 0$ (since $R(y)$ is continuous to the left). Take any $y > 0$. Notice that

$$R'(y) = \frac{\phi^{L+1}(y)\Phi^L(y) - \Phi^{L+1}(y)\phi^L(y)}{(\Phi^L(y))^2}$$

and that

$$\text{numerator of } R'(y) = \int_0^y \left[\phi^{L+1}(y)\phi^L(x) - \phi^{L+1}(x)\phi^L(y)\right] \mathrm{d}x.$$

Since $r(\cdot)$ is nondecreasing as assumed, the integrand in the above integral is nonnegative, or $R'(y) \geq 0$. □

Note. The above results suggest a new type of ordering of random variables, which we may call 'distribution ratio ordering'. Let U (resp., V) be a nonnegative random variable whose density and distribution functions are, respectively, $f_U(\cdot)$ and $F_U(\cdot)$ (resp., $f_V(\cdot)$ and $F_V(\cdot)$). Define that U is larger than V in the sense of distribution ratio, written as $U \geq_{\mathrm{DR}} V$, if $F_U(x)/F_V(x)$ is nondecreasing in x. Recall that U is larger than V in the sense of likelihood ratio, written as $U \geq_{\mathrm{LR}} V$, if $f_U(x)/f_V(x)$ is nondecreasing in x (see, e.g., Ross [9]). A straightforward modification of the proof of Lemma 2 shows that $U \geq_{\mathrm{LR}} V$ implies $U \geq_{\mathrm{DR}} V$. Now suppose $U \geq_{\mathrm{DR}} V$, i.e. $F_U(x)/F_V(x)$ is nondecreasing in x. Since $F_U(x)/F_V(x) \to 1$ as $x \to \infty$, we know that $F_U(x)/F_V(x) \leq 1$ for all x, or U is stochastically larger than

V. Therefore, 'distribution ratio ordering' is stronger than 'stochastic ordering', and is weaker than 'likelihood ratio ordering'. This notion might be useful for other statistical problems.

A family of random variables $\{D^m,\ m = 1, 2, \ldots\}$ is said to be a *monotone likelihood ratio* (MLR) family (see, e.g., Ross [9]) if D^m is larger than D^n in the sense of likelihood ratio for all positive integers $m \geq n$, i.e. $\phi^m(x)/\phi^n(x)$ is nondecreasing in x. Thus, by Lemmas 1 and 2, if $\{D^m,\ m = 1, 2, \ldots\}$ is an MLR family, then $G(\cdot)$ is quasiconvex.

Theorem 1. *In the periodic review model, $G(\cdot)$ is quasiconvex if D_1 is a gamma random variable.*

Proof. Assume that D_1 is a gamma random variable, i.e.

$$\phi(x) = \frac{b^a x^{a-1}\,\mathrm{e}^{-bx}}{\Gamma(a)}, \quad x \geq 0,$$

with $b, a > 0$. By Lemmas 1 and 2, we only need to show that $\{D^m,\ m = 1, 2, \ldots\}$ is an MLR family. Take any positive integers $m > n$. Since, for any positive integer j,

$$\phi^j(x) = \frac{b^{ja} x^{ja-1}\,\mathrm{e}^{-bx}}{\Gamma(ja)}, \quad x \geq 0,$$

we have

$$\frac{\phi^m(x)}{\phi^n(x)} = \frac{\Gamma(an)b^{a(m-n)}}{\Gamma(am)} x^{a(m-n)} \quad \text{for } x \geq 0,$$

which is nondecreasing in x. This completes the proof. □

3. Continuous review model

This section is parallel to the previous section and is concerned with continuous review models. Whenever possible, we will refer the reader to the previous section for proofs if a considerable amount of imitation is involved. We first describe our model in detail.

We assume that the demand process is a compound Poisson process $\{X_n, D_n\}_{n \geq 1}$, i.e., X_k's are iid exponential random variables with rate λ and are independent of D_i's, which are iid, nonnegative random variables. The sequence $X_1, X_2, \ldots$ represents the customer interarrival times. Let $S_n = \sum_{i=1}^n X_i$ with $S_0 = 0$ be the arrival epoch of the n-th customer whose demand size is D_n. Write $\phi(\cdot)$ and $\Phi(\cdot)$, respectively, for the density and cumulative distribution functions of D_1. Write D^m, where m is a positive integer, for the total demand of m customers, and $\phi^m(\cdot)$ and $\Phi^m(\cdot)$, respectively, for the density and cumulative distribution functions of D^m. We use $D(T)$ to denote the total demand over a period of length T. Demands are satisfied on a first-come first-serve basis with any excess demand backordered. Ordering decisions are made only at customer arrival epochs, and each order takes T (fixed) units of time to arrive. Again, holding and variable backorder costs are linear with per unit per unit time rates h and p, respectively; and a fixed penalty cost, π, is incurred for each unit of backorder, independent of backorder duration. The interest rate is β, and continous discounting is used. (As the reader may have noticed, some notation is used in both the periodic and continuous review models. The exact meaning of these terms will be clear from the context.)

Let IP_n be the inventory position immediately after the n-th demand arrival epoch (after order placement if any). Given $\mathrm{IP}_n = y$, since all and only those orders placed by time S_n will arrive by time $S_n + T$ and there will not be any deliveries thereafter until $S_{n+1} + T$, the distribution of net inventory level and thus the expected holding and backorder costs incurred in $[S_n + T, S_{n+1} + T)$ are determined.

Therefore it is natural to charge the costs incurred in $[S_n + T, S_{n+1} + T)$ to the decision epoch S_n. Write G_n for the holding and backorder costs in $[S_n + T, S_{n+1} + T)$ discounted to S_n, and we have the loss function $G(y) = E(G_n \mid \mathrm{IP}_n = y)$. For any $z \geqslant 0$, let $D_z(T)$ be the total demand in interval $(z, z+T]$. From Federgruen and Schechner [4], we have

$$G(y) = \int_0^\infty e^{-\beta(T+z)} P(X_1 > z) \times E\Big\{h[y - D_z(T)]^+ + p[y - D_z(T)]^- + \lambda\pi\big[D - (y - D_z(T))^+\big]^+ \mid X_1 > z\Big\}\, dz$$

where D is identically distributed as D_1, and is independent of $D_z(T)$ and $\{X_1 > z\}$. Given $X_1 > z$, due to the memoryless property of a Poisson process, we know that the demand process after time z is just the same (statistically) as if the demand process started at time z. Therefore, $(D_z(T) \mid X_1 > z)$ is identically distributed as $D(T)$ (whose distribution is independent of z) and the above equation becomes

$$G(y) = \int_0^\infty e^{-\beta(T+z)} P(X_1 > z)\, dz \times E\Big\{h[y - D(T)]^+ + p[y - D(T)]^- + \lambda\pi\big[D - (y - D(T))^+\big]^+\Big\}.$$

Since the above integral term is a constant and we are only interested in the quasivexity of $G(y)$, we will hereafter just assume, without loss of generality, that $\beta = 0$. In this way, the integral is equal to $EX_1 = 1/\lambda$, and we have

$$\lambda G(y) = E\Big\{h[y - D(T)]^+ + p[y - D(T)]^- + \lambda\pi\big[D - (y - D(T))^+\big]^+\Big\}.$$

It can be easily verified that

$$\big[D - (y - D(T))^+\big]^+ = [y - D(T) - D]^- - [y - D(T)]^-.$$

Thus

$$\lambda G(y) = E\big\{h[y - D(T)]^+ + (p - \lambda\pi)[y - D(T)]^- + \lambda\pi[y - D(T) - D]^-\big\}.$$

Write $\Psi_1(\cdot)$ and $\Psi_2(\cdot)$, respectively, for the cumulative distribution functions of $D(T)$ and $D(T) + D$. By spelling out the above expectations, we have

$$\lambda G(y) = \lambda\pi \int_0^y \Psi_2(x)\, dx - (\lambda\pi - h - p) \int_0^y \Psi_1(x)\, dx - p(y - \lambda T\mu) + \lambda\pi\mu \tag{5}$$

where $\mu = ED_1$. By taking derivatives with respect to y, we have

$$\lambda G'(y) = -p + \lambda\pi\Psi_2(y) - (\lambda\pi - h - p)\Psi_1(y). \tag{6}$$

Note. Notice that the loss function here has a different form compared to the periodic review model. This is solely due to the convention that the holding and proportional backorder costs are assessed based on *period-ending* inventory levels in periodic review models. If we assessed holding and proportional backorder costs based on *period-beginning* inventory levels in periodic review models, a similar loss function would result.

From (6), observe that if $h + p - \lambda\pi \geq 0$, then $G'(y)$ is nondecreasing. Therefore we have

Theorem 2. *If $h + p - \lambda\pi \geq 0$, then $G(\cdot)$ is convex.*

Hereafter, for continous review models, we will only be concerned with situations where $h+p-\lambda\pi<0$. Let p_n, $n=0, 1,\dots$, be the probability that there are exactly n customer arrivals during $(0, T]$, i.e.

$$p_n=\frac{(\lambda T)^n\,\mathrm{e}^{-\lambda T}}{n!},\quad n=0, 1,\dots$$

Define

$$\psi_1(x)=\sum_{n=0}^{\infty}p_n\phi^n(x)\ \text{ and }\ \psi_2(x)=\sum_{n=0}^{\infty}p_n\phi^{n+1}(x)\quad\text{for all } x$$

where $\phi^0(x)=\delta(x)$, the impulse function. Thus we have

$$\Psi_1(y)=\int_0^y\psi_1(x)\,\mathrm{d}x\ \text{ and }\ \Psi_2(y)=\int_0^y\psi_2(x)\,\mathrm{d}x\quad\text{for all } y\ge 0.$$

As before, define $\hat{R}(y)=\Psi_2(y)/\Psi_1(y)$ and $\hat{r}(y)=\psi_2(y)/\psi_1(y)$.

Lemma 3. *$G(\cdot)$ is quasiconvex for all positive values of h, p, π and λ if and only if $\hat{R}(\cdot)$ is nondecreasing.*

Proof. Write π' for $\lambda\pi-h-p$ (>0), and thus $\lambda\pi=h+p+\pi'$. Now (6) reduces to the form of (2), therefore the proof of Lemma 1 applies here. □

Lemma 4. *If $\hat{r}(x)$ is nondecreasing for $x>0$, then $\hat{R}(y)$ is nondecreasing for all y.*

Proof. Assume $\hat{r}(x)$ is nondecreasing for $x>0$. Since $\hat{R}(y)=0$ for all $y<0$, we only need to show that $\hat{R}(y)$ is nondecreasing for $y\ge 0$, or $\hat{R}'(y)\ge 0$ for $y>0$ (since $\hat{R}(y)$ is continous to the left). Take any $y>0$. Since $\Psi_1'(y)=\psi_1(y)$ and $\Psi_2'(y)=\psi_2(y)$, we have

$$\hat{R}'(y)=\frac{\psi_2(y)\Psi_1(y)-\psi_1(y)\Psi_2(y)}{(\Psi_1(y))^2}.$$

Notice that $\Psi_1(0)=p_0$ and $\Psi_2(0)=0$, we have

$$\text{numerator of } \hat{R}'(y)=p_0\psi_2(y)+\int_{0^+}^{y}[\psi_2(y)\psi_1(x)-\psi_1(y)\psi_2(x)]\,\mathrm{d}x.$$

Since $\hat{r}(x)$ is nondecreasing for $x>0$, the above integrand is nonnegative, or $\hat{R}'(y)\ge 0$. □

Lemma 5. *If $\{D^m, m=1, 2,\dots\}$ is an MLR family, then $\hat{r}(x)$ is nondecreasing for all $x>0$.*

Proof. Assume $\{D^m, m=1, 2,\dots\}$ is an MLR family. Take any $x>0$. Define $\psi(x)=\psi_2'(x)\psi_1(x)-\psi_2(x)\psi_1'(x)$. Since $\hat{r}'(x)=\psi(x)/[\psi_1(x)]^2$, we only need to show that $\psi(x)\ge 0$. From the definitions of $\psi_1(x)$ and $\psi_2(x)$, we have

$$\psi(x)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}p_np_m\left[\phi^m(x)\frac{\mathrm{d}}{\mathrm{d}x}\phi^{n+1}(x)-\phi^{n+1}(x)\frac{\mathrm{d}}{\mathrm{d}x}\phi^m(x)\right],$$

which becomes

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}p_{n-1}p_m\left[\phi^m(x)\frac{\mathrm{d}}{\mathrm{d}x}\phi^n(x)-\phi^n(x)\frac{\mathrm{d}}{\mathrm{d}x}\phi^m(x)\right]$$

because $\phi^0(x)=0$ (recall that $\phi^0(\cdot)$ is an impulse function). Notice that $p_{n-1}=np_n/(\lambda T)$ and that the term under summation is zero when $m=n$. The above expression becomes

$$\frac{1}{\lambda T}\sum_{n=1}^{\infty}\sum_{\substack{m=1\\ m\neq n}}^{\infty} np_n p_m\left[\phi^m(x)\frac{\mathrm{d}}{\mathrm{d}x}\phi^n(x)-\phi^n(x)\frac{\mathrm{d}}{\mathrm{d}x}\phi^m(x)\right].$$

After combining symmetric (with respect to the diagonal $m=n$) terms and some algebra, we have

$$\psi(x)=\frac{1}{\lambda T}\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} p_n p_m \phi^n(x)\phi^m(x)\left\{\log\left[\frac{\phi^m(x)}{\phi^n(x)}\right]^{m-n}\right\}'.$$

Since $\{D^m, m=1, 2,\dots\}$ is an MLR family as assumed, we know that $\phi^m(y)/\phi^n(y)$, and thus $\log[\phi^m(y)/\phi^n(y)]^{m-n}$, is nondecreasing in y for all $m>n$, or, in particular, $(\log[\phi^m(x)/\phi^n(x)]^{m-n})'\geq 0$. From the above equation, we see that $\psi(x)\geq 0$. This completes the proof. □

Theorem 3. *In the continuous review model with compound Poisson demand, $G(\cdot)$ is quasiconvex if D_1 is a gamma random variable.*

Proof. Assume that D_1 is a gamma random variable. From the previous section we know that $\{D^m, m=1, 2,\dots\}$ is an MLR family. Thus, the theorem follows by using Lemmas 3, 4 and 5. □

4. Discrete demand models

This section is concerned with discrete demand models. We assume that D_1 takes nonnegative integer values. It is clear that (1) and (5) take the following forms: for periodic review models,

$$G(y)=(h+p+\pi)\sum_{x=0}^{y-1}\Phi^{L+1}(x)-\pi\sum_{x=0}^{y-1}\Phi^{L}(x)-py+\pi\mu+p(L+1)\mu,$$

and, for continuous review models,

$$\lambda G(y)=\lambda\pi\sum_{x=0}^{y-1}\Psi_2(x)-(\lambda\pi-h-p)\sum_{x=0}^{y-1}\Psi_1(x)-p(y-\lambda T\mu)+\lambda\pi\mu.$$

It is straightforward to show that all the lemmas in Sections 2 and 3 apply here. In particular, we know that if $\{D^m, m=1, 2,\dots\}$ is an MLR family, then the loss function of the periodic or continuous review model is quasiconvex. It can be easily verified that if

$$\phi(x)=\frac{\mathrm{e}^{-a}a^x}{x!},\quad x=0, 1,\dots,$$

(Poisson distribution) or

$$\phi(x)=\binom{n}{x}q^x(1-q)^{n-x},\quad x=0, 1,\dots,n,$$

(binomial distribution) or

$$\phi(x)=\binom{x+a-1}{x}q^a(1-q)^x,\quad x=0, 1,\dots,$$

(negative binomial distribution) where $a>0$, n is a positive integer and $0<q<1$, then $\{D^m, m=1, 2,\dots\}$ is an MLR family. (For binomial distributions, the condition for being an MLR family should read: if $\phi^k(x)/\phi^l(x)$ is nondecreasing in x *over the support* of $\phi^l(\cdot)$, i.e. $x=0, 1,\dots,ln$, for all positive integers

Table 1
Periodic review. $L = 1$

i	$\phi(i)$	$R(i)$
0	0.1	0.10
1	0.4	0.18
2	0.0	0.50
3	0.0	0.50
4	0.1	0.45
5	0.4	0.43

$k > l$. It is easy to show that, for binomial distributions, both $R(x)$ and $\hat{R}(x)$ are nondecreasing for all x). Therefore we have

Theorem 4. *For the periodic or continous review models, $G(\cdot)$ is quasiconvex if D_1 is a Poisson, or binomial, or negative binomial random variable.*

Note. If $D_1 \equiv 1$ in the continuous review model, i.e. simple Poisson demand, then $G(\cdot)$ is quasiconvex (Federgruen and Zheng [5]). This is also a direct result of Lemmas 3 and 4. Notice that $\psi_2(x) = p_{x-1}$ for $x = 1, 2, \ldots$ and $\psi_2(0) = 0$, and that $\psi_1(x) = p_x$ for $x = 0, 1, \ldots$ Observe that $\hat{r}(0) = 0$ and $\hat{r}(x) = p_{x-1}/p_x = x/(\lambda T)$ for $x = 1, 2, \ldots,$ thus $\hat{r}(\cdot)$ is nondecreasing. By Lemmas 3 and 4, $G(\cdot)$ is quasiconvex.

As noted previously, the quasiconvexity of the loss function under the general cost structure is not a distribution-free property. To illustrate this, we construct an example where $R(\cdot)$ is not monotone (see Table 1); thus, by Lemma 1, there exist some cost parameters (h, p and π) such that $G(\cdot)$ is not quasiconvex. Notice that our proof of the necessary condition in Lemma 1 is constructive and provides a procedure to choose those cost parameters such that the corresponding loss function is not quasiconvex. In particular, by setting $y_1 = 3$, $y_2 = 4$, and $\xi = 0$ in the proof of the necessary condition of Lemma 1, we obtain: $h = 0.5375$, $p = 0.0125$ and $\pi = 0.45$. The corresponding loss function is depicted in Figure 1. Clearly, this loss function has multiple local minima.

From the numerical examples we have studied, it appears that $R(\cdot)$ tends to be nonmonotonic if the underlying demand (size) density function is multi-modal. On the other hand, the demand densities identified so far that guarantee quasiconvex loss functions are all unimodal. An interesting question remains: Is the unimodality of a demand density sufficient for the quasiconvexity of the loss function?

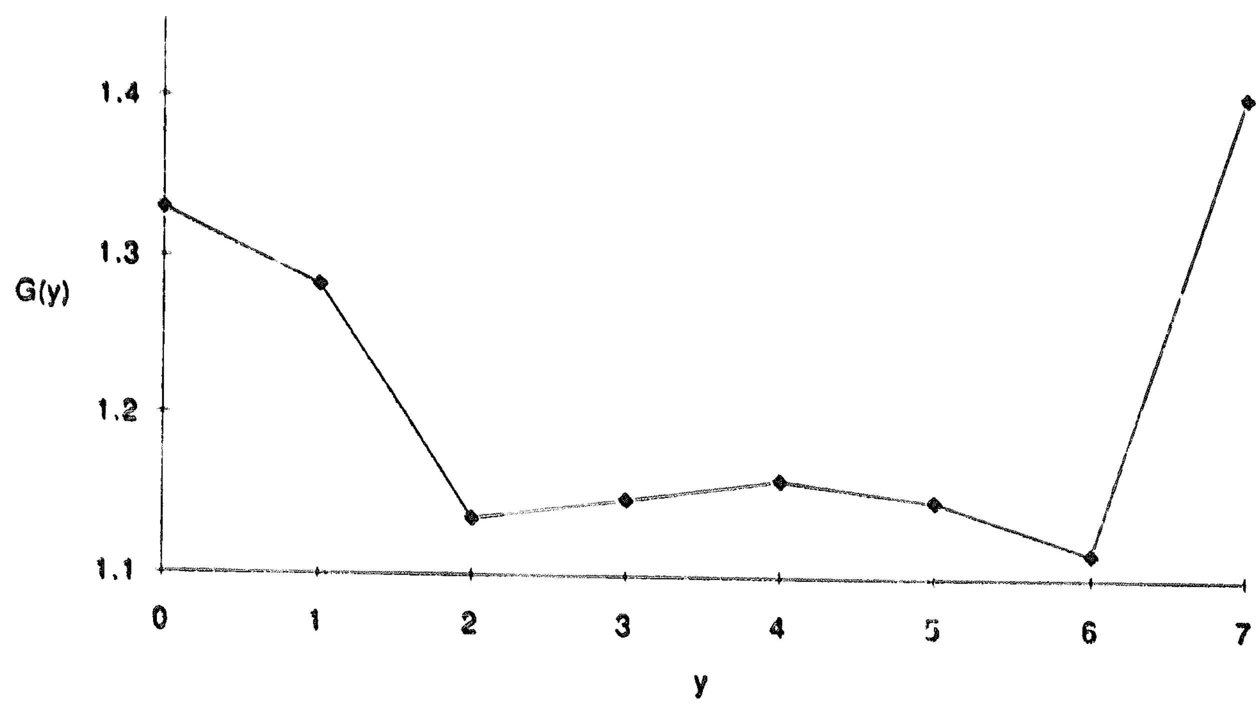

Figure 1. A non-quasiconvex loss function

5. Normal approximation

Normal distributions are usually used to model demand sizes. Since normal random variables take negative values, some kind of truncation is necessary. If we use a truncated normal distribution to model D_1, then the density or distribution functions of D^k for $k = 1, 2, \ldots$ are convolutions of truncated normals, which do not have closed forms. Therefore, it is a common practice, especially in designing computational studies, that truncation is done after convolution. More specifically, for $k = 1, 2, \ldots$, let $f^k(\cdot)$ be the density function of a normal random variable (without truncation) with mean $k\mu$ and variance $k\sigma^2$, and the density function of D^k is assumed to be

$$\phi^k(x) = \begin{cases} f^k(x)/A_k & \text{if } x \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

where $A_k = \int_0^\infty f^k(x)\,\mathrm{d}x$ (notice that the superscript k in $\phi^k(x)$ no longer means the number of times of convolutions). The above procedure is approximate because, strictly speaking, $\phi^k(\cdot)$ for $k = 1, 2, \ldots$ are not consistent in the sense that they are not convolutions of a common density function (which they should). But this approximation would be 'almost' consistent (thus accurate) if the negative tail of $f^1(\cdot)$ is negligible, or its truncation does not significantly change the density function. When we use a normal distribution to model demand sizes, the normal density usually has a negligible tail; and thus the above approximation is reasonable. By using the above expressions in (1) and (5) of Sections 2 and 3, we obtain the (approximate) loss functions for both the periodic and continuous review models.

It is clear that all the Lemmas of Sections 2 and 3 apply here. Notice that $\phi^k(x)/\phi^l(x)$ is nondecreasing in x for all positive integers $k > l$ since

$$\frac{\phi^k(x)}{\phi^l(x)} = \frac{A_l}{A_k}\sqrt{\frac{l}{k}}\,\exp\left(\left[\frac{1}{l} - \frac{1}{k}\right]\frac{x^2}{2\sigma^2} - (k-l)\frac{\mu^2}{2\sigma^2}\right).$$

By using the lemmas in the previous sections, we have

Theorem 5. For the periodic or continuous review models where normal distribution is used in the above fashion to model demand sizes, $G(\cdot)$ is quasiconvex.

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References

[1] Archibald, B., and Silver, E., "(s, S) policies under continuous review and discrete compound Poisson demands", *Management Science* 24 (1978) 899–908.

[2] Beckmann, M., "An inventory model for arbitrary interval and quantity distributions of demands", *Management Science* 8 (1961) 35–57.

[3] Federgruen, A., "Centralized planning models for multi-echelon inventory systems under uncertainty", *Handbook in Operations Research and Management Science*, Volume 4: Logistics of Productions and Inventory, Chapter 4, North-Holland, Amsterdam, 1992.

[4] Federgruen, A., and Schechner, Z., "Cost formulas for continuous review inventory models with fixed delivery lags", *Operations Research* 31 (1983) 957–965.

[5] Federgruen, A., and Zheng, Y.S., "An efficient algorithm for computing an optimal (r, Q) policy in continuous review stochastic inventory systems", to appear in *Operations Research*.

[6] Federgruen, A., and Zipkin, P., "An efficient algorithm for computing optimal (s, S) policies", *Operations Research* 32 (1984) 1268–1285.

[7] Hadley, G., and Whitin, T.M., *Analysis of Inventory Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1963.

[8] Iglehart, D., "Optimality of (s, S) inventory policies in the infinite horizon dynamic inventory problem", *Management Science* 9 (1963) 259–267.
[9] Ross, S.M., *Stochastic Processes*, Wiley, New York, 1983.
[10] Scarf, H.E., "The optimality of (s, S) policies in the dynamic inventory problem", in: K. Arrow, S. Karlin and P. Suppes (eds.), *Mathematical Methods in the Social Sciences*, Stanford University Press, Stanford, CA, 1960.
[11] Veinott, A., "On the optimality of (s, S) inventory policies: New conditions and a new proof", *SIAM Journal on Applied Mathematics* 14 (1966) 1067–1083.
[12] Veinott, A., and Wagner, H., "Computing optimal (s, S) inventory policies", *Management Science* 11 (1965) 525–552.
[13] Zheng, Y.S., and Chen, F., "Inventory policies with quantized ordering", *Naval Research Logistics* 39 (1992) 285–305..
[14] Zheng, Y.S., and Federgruen, A., "Finding optimal (s, S) policies is about as simple as evaluating a single policy", *Operations Research* 39 (1991) 654–665.