The Economic Lot Scheduling Problem with Finite Backorder Costs

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We are concerned with the problem of scheduling m items, facing constant demand rates, on a single facility to minimize the long-run average holding, backorder, and setup costs. The inventory holding and backlogging costs are charged at a linear time weighted rate. We develop a lower bound on the cost of all feasible schedules and extend recent developments in the economic lot scheduling problem, via time-varying lot sizes, to find optimal or near-optimal cyclic schedules. The resulting schedules are used elsewhere as target schedules when demands are random. © 1992 John Wiley & Sons, Inc.

1. INTRODUCTION

Economies of scale often dictate the choice of a single high-speed facility capable of producing a given set of items over the choice of one dedicated machine for each item in the set. These high-speed facilities are only capable of producing one item at a time, requiring a setup each time a different item is scheduled for production. Minimizing the long-run average holding and setup cost under the assumptions of known constant demand and production rates is known in the literature as the economic lot scheduling problem (ELSP).

We consider an extension of the ELSP where all of the demand must be met, but it need not be met immediately; demand can be backordered. Backorder costs are charged at a linear time weighted rate. We refer to our model as the extended economic lot scheduling problem (EELSP).

We were motivated to study the EELSP by two limitations of the ELSP. The first of these is economical, the second one is related to the practical problem of scheduling production in real time when demands are random or the facility is subject to sudden failures.

The ELSP does not allow stockouts; imposing this constraint in the presence of finite backorder costs is not economical. In many situations customers do not need immediate delivery and considerable savings can be achieved by backordering. In manufacturing systems where the items are used to feed production lines the cost of a temporary stockout is finite, however large, and lost production at the line is often recovered either by working faster or longer.

Naval Research Logistics, Vol. 39, pp. 729–739 (1992)
 CCC 0894-069X/92/050729-11$04.00

To sense the savings of allowing finite backorder cost rates, suppose that the backorder cost rate is k times larger than the holding cost rate. Then it is possible to find feasible schedules with an average cost not larger than $100\sqrt{k/(k+1)}\%$ of the average cost of a schedule with infinite backorder cost rate.

Backorder costs are difficult to measure; however, it is known (see Gallego [5]), that "the proportion of time an item is in stock in an optimal cyclic schedule is the ratio of backorder to holding plus backorder cost rates." Consequently, once a manager has decided on the service level, say 99%, he has implicitly imputed a backorder cost rate 99 times larger than the holding cost rate.

The second and most compelling motivation for studying the EELSP is as a fundamental step in the development of a practical real-time scheduling tool to manage a multiitem single-facility production system in the presence of random demands and machine failures. See [5] for the development of such a tool under random demands.

The ELSP literature assumes that a cyclic schedule can be repeated ad infinitum. This is possible only if (i) the initial inventories are in agreement with the schedule, (ii) the facility is perfectly reliable, (iii) setups actually consume a constant amount of time, and (iv) the demand rates are constant. At best (i)–(iv) represent an idealized situation. Any significant amount of randomness in the system will eventually force deviations from the cyclic schedule and will make stockouts inevitable.

There is a clear need for a control mechanism that monitors disruptions and takes corrective measures to dampen their effect. In order to find corrective measures that are optimal in some economic sense, we must first model the cost associated with stockouts.

To see that such a control mechanism is nontrivial, imagine that in a real factory, a cyclic schedule has been computed using one of the algorithms in the ELSP literature. We begin to follow the cyclic schedule. All goes well until the first disruption occurs, e.g., machine failure, lack of raw materials, delays in a setup, etc. After the disruption the inventory levels of all of the items have been significantly depleted, and we are in danger of not meeting all the demands. Our initial reaction may be to produce smaller than normal batch sizes, so that we can manufacture at least a small amount of each item in the very near future and avoid stockouts in the short term. However, that will only make matters worse. The smaller batch sizes would cause us to consume more time than normal in setting up the machine, with the result that less time than normal is devoted to actual production. We have made a bad situation worse.

This article represents the first step in an effort that addresses this problem—that of introducing backorders into the traditional ELSP. We show how to compute an optimal or near-optimal cyclic schedule, called the target schedule, when backorders are allowed, demands are known and constant, and the facility is perfectly reliable. The second step is to solve an optimal control problem to adjust the length of the production runs of the target schedule in response to disruptions in expected inventories. The third step determines appropriate levels of safety stocks to achieve the serviceability yardstick: "the average probability of having an item in stock is the ratio of backorder to holding plus backorder cost rates"; see [5].

For the above reasons the results of this article are a key to the development of a real-time scheduling tool capable of managing a multiitem single-facility system under more realistic assumptions than those of the traditional ELSP.

Although we are dissatisfied with the limitations of the ELSP, we find the ELSP literature rich in insights and techniques. The ELSP has been widely studied during the last 30 years. There are two main approaches to heuristic algorithms. One is the basic period approach; see Elmaghraby [4] for a review. Under this approach, it is NP hard to find a feasible solution given the number of production runs per cycle for each of the items; see Hsu [7]. The second and most successful approach allows time-varying lot sizes; i.e., an item produced several times in the cycle need not be made in constant lot sizes. See Maxwell [10] and Delporte and Thomas [2] for the fundamental ideas. Dobson [3] developed a time-varying lot-size heuristic with guaranteed feasibility. Near-optimal schedules can be obtained by combining Dobson's heuristic with Zipkin's [12] algorithm. See Gunter [6] for an alternative time-varying heuristic.

The basic idea of the time-varying approach is to decompose the problem into a combinatorial and a continuous part. In the combinatorial part, one first determines production frequencies; see Jackson, Maxwell, and Muckstad [8] for multimachine systems. Then the production frequencies are rounded to powers of two by an algorithm proposed by Roundy [11]. Finally the items are bin packed with respect to frequencies and average loads, resulting in a production sequence. The continuous part takes the production sequence as given and consists of finding actual production run and idle times [12].

The rest of the article is organized as follows: In Section 2 we introduce notation and develop an easily computable lower bound on the average cost over all cyclic schedules. The information gained in solving the lower bound is used to determine a production sequence f. In Section 3, the f given, we focus on the continuous part of the problem; that of finding actual production run and idle times. We show that given enough capacity there exists a feasible schedule that allows time-varying lot sizes. Finally we present computational results and an example in Section 4.

2. NOTATION AND A LOWER BOUND

The data for the problem are

$i = 1, \ldots, m$ index for the items
p_i'' constant production rate of item i
d_i'' constant demand rate of item i
h_i'' inventory holding cost of item i
b_i'' backlogging cost of item i
s_i' setup time of item i
a_i' setup cost of item i

Without loss of generality we redefine the item units so all the demand rates are one. This is accomplished by setting $d_i' \equiv d_i''/d_i'' = 1$, $p_i' = p_i''/d_i''$, $h_i' = h_i''/d_i''$, and $b_i' = b_i''/d_i''$. The transformation maps many equivalent problems to one that is easier to manipulate. Let $p' = (p_1', \ldots, p_m')$, etc.; also let e' denote the vector of ones in m space. For convenience define $\rho_i' = 1/p_i' = d_i''/p_i''$. Note that ρ_i' is the long-term proportion of time, exclusive of setups, that the facility is engaged in the production of item i. We can also interpret ρ_i' as the unit processing time, i.e., the time required to produce enough of item i to satisfy its demand for one unit of time. Let $\kappa = 1 - \Sigma_i \rho_i'$. Note that κ is the proportion of the facility's time available for setups and idling. We assume that $p' > e'$, $a' \geq 0$, $s' \geq 0$, $a' + s' > 0$, and $\kappa > 0$. The reason for the primed notation will become clear in the following section.

We now compute a lower bound on the average cost of all cyclic schedules. Let T be the cycle length of a cyclic schedule where item i is produced n_i times per cycle. It is well known that an item's holding and backorder cost is minimized by producing at constant intervals of size $T_i = T/n_i$. It is also well known that in each such interval the inventory is positive $100b_i/(b_i + h_i)\%$ of the time. Thus the average holding, backorder and setup cost is bounded below by $\Sigma_i (H_i' T_i + a_i'/T_i)$, where $H_i' = 0.5h_i'b_i'/(b_i' + h_i')(1 - \rho_i)$ is a convenient factor that expresses the per-unit-time cost of holding inventory and backlogging demand. Minimizing $\Sigma_i (H_i' T_i + a_i'/T_i)$ yields the lower bound known as the independent solution (IS). A better lower bound is obtained by explicitly considering the capacity constraint stating that enough time must be made available for setups. Since the long-run average time spent on setups is $\Sigma_i s_i'/T_i$, and the available time is κ; a lower bound over all cyclic schedules is given by the value of the following nonlinear program.

$$\text{Problem LB: } C^{\text{LB}} = \min \sum_{i=1}^{m} \left(H_i' \text{i} T_i + \frac{a_i'}{T_i} \right),$$

$$\text{s.t. } \sum_{i=1}^{m} \frac{s_i'}{T_i} \leq \kappa,$$

$$T_i \geq 0, \qquad i = 1, \ldots, m. \tag{1}$$

A similar lower bound was first proposed by Bomberger [1] for the ELSP. Roundy developed a heuristic algorithm to solve problem LB where the reorder intervals are restricted to be powers-of-two times a base planning period $\beta > 0$. For the ELSP, the base planning period can be day or a shift. The cost of Roundy's solution is guaranteed to be within 6% of C^{LB}. Let T_i^* denote Roundy's solution; define

$$T^* = \max T_i^* : i = 1, \ldots, m$$

and

$$n_i = T^*/T_i^*, \qquad i = 1, \ldots, m.$$

By construction, the n_i's are nonnegative integers which are powers of 2 and represent the number of times an item is produced in a cycle of length T^*. Since each item is produced at least once in the sequence, it follows that $n \equiv \Sigma_i n_i \geq m$. A rotation schedule (RS), also called a common cycle schedule (CC) is characterized by $n = m$.

Given the n_i's and the tentative cycle length T^*, the next step is to construct the sequence $f = (f_1, \ldots, f_n)$, in which the items are to be produced in each cycle; here f_j represents the item produced in the jth position of f and $n = \Sigma_i n_i$ denotes the number of positions in f. The main thrust in creating f is to spread the production runs of each item as evenly as possible. For this step [3] uses a bin-packing heuristic that applies verbatim to the EELSP. An alternative way of determining a production sequence is provided by [6].

3. OPTIMAL *f*-CYCLIC SCHEDULES

As explained in Section 2, the sequence f is assumed to be the outcome of the combinatorial part of a time-varying heuristic, namely, the computation of the production frequencies n_i followed by bin-packing the loads. A cyclic schedule based on the sequence f will be called an f-cyclic schedule.

For convenience we define the $n \times m$ dimensional matrix F whose (j, i) entry $F_{ji} = 1$ if $f_j = i$ and 0 otherwise. We use F to transform the data from m-dimensional space to n-dimensional space. Define $e = Fe'$, $p = Fp'$, $\rho = F\rho'$, $h = Fh'$, $b = Fb'$, $s = Fs'$, and $a = Fa'$. Note that all of these vectors are in n space. Clearly p_j is the production rate of the item produced in position j. The vectors ρ, a, s, h, and b have similar interpretations. For instance, if $m = 3$, and $f = (1, 2, 1, 3)$, then $p = (p'_1, p'_2, p'_1, p'_3)$, etc.

The following variables serve to describe an f-cyclic schedule:

u_j idle time at position j
t_j production run time at position j
y_j inventory of item f_j at the end of production run j.

Let $u = (u_j)_{j=1}^n$; similarly define t and y. Clearly $u \geq 0$ and $t \geq 0$.

There are three events corresponding to each position in f; these are the setup for item f_j, the production run of length t_j and the idel time u_j. Here we are assuming without loss of generality that the idle time if any occurs between the end of a production run and the start of a setup. Note that no idle time can be inserted in the middle of a production run and that any or all of the idle times can be zero.

Consider the inventory level of an item over a cycle; see Figure 1. The production run times t_j can be split into two parts t_j^h and t_j^b by letting $t_j^h = y_j/(p_j - 1)$ and $t_j^b = t_j - t_j^h$. Let $t^h = (t_j^h)_{j=1}^n$ and $t^b = (t_j^b)_{j=1}^n$. Virtually all f-cyclic schedules based on well-spread sequences satisfy $t^b \geq 0$ and $t^h \geq 0$. In this case t_j^b is the portion of the run time spent satisfying backorders, while t_j^h is the portion of the run time spent accumulating inventory. This property can be thought of as an extension of the zero switch rule (ZSR) for the ELSP, and is henceforth called the extended zero switch rule (EZSR). From now on we consider only f-cyclic schedules satisfying the EZSR. See [10] for an example where the ZSR fails to hold for the ELSP.

The holding and backorder cost incurred in a cycle is the sum of the corresponding costs through all the positions of the cycle, namely,

$$0.5 \sum_j p_j(p_j - 1)\{h_j(t_j^h)^2 + b_j(t_j^b)^2\}.$$

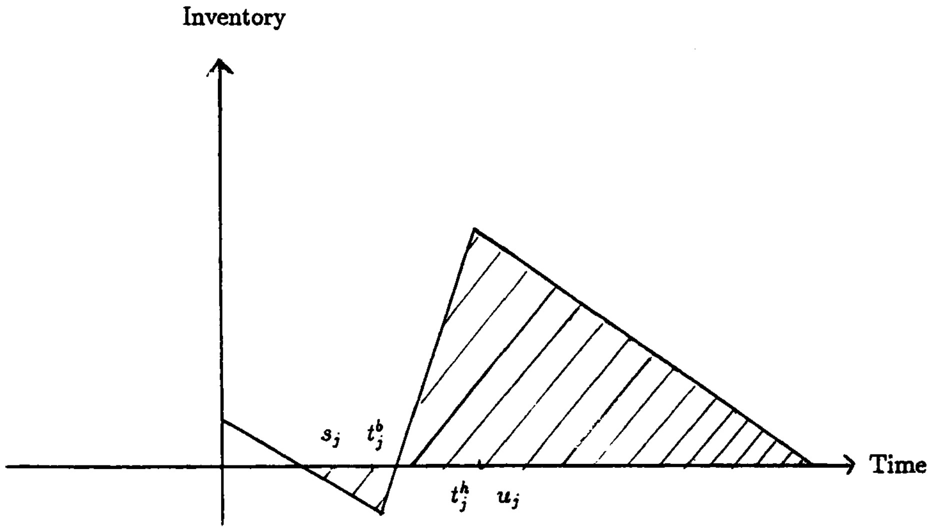

Figure 1. Cost associated with position j.

The holding and backorder cost corresponding to position j is the weighted area in Figure 1. The total holding, backorder, and setup costs can be written as a function of f, t^b and t^h,

$$C(f, t^b, t^h, u) = \sum_{j=1}^{n} [a_j + 0.5p_j(p_j - 1)\{b_j(t_j^b)^2 + h_j(t_j^h)^2\}]. \tag{2}$$

The total cycle length T is simply the sum of the time of all the events that occur over the cycle,

$$T = \sum_{j=1}^{n} (s_j + t_j^b + t_j^h + u_j). \tag{3}$$

Some additional notation is required to write the synchronization constraints that must be satisfied by an f-cyclic schedule (these constraints state that no two items can be scheduled for production at the same time). To this end let j^+ denote the first position in f after j where item f_j is produced. The word "next" is interpreted in the circular sense; that is, if item f_j does not appear after position j then j^+ is the first position in the sequence where item f_j appears. For $f = (1, 2, 1, 3)$ we have $1^+ = 3$, $2^+ = 2$, $3^+ = 1$, and $4^+ = 4$.

Let $L_j \equiv \{j, j + 1, \ldots, j^+ - 1\}$ be the set of positions from j up to, but not including, j^+. $L_1 = \{1, 2\}$, $L_2 = \{2, 3, 4, 1\}$, $L_3 = \{3, 4\}$, $L_4 = \{4, 1, 2, 3\}$. The following equation (see Figure 2) balances the production of item f_j with the demand accrued by items with positions in L_j:

$$p_j(t_{j^+}^b + t_j^h) = \sum_{k \in L_j} (s_k + t_{k+1}^b + t_k^h + u_k), \qquad \forall j. \tag{4}$$

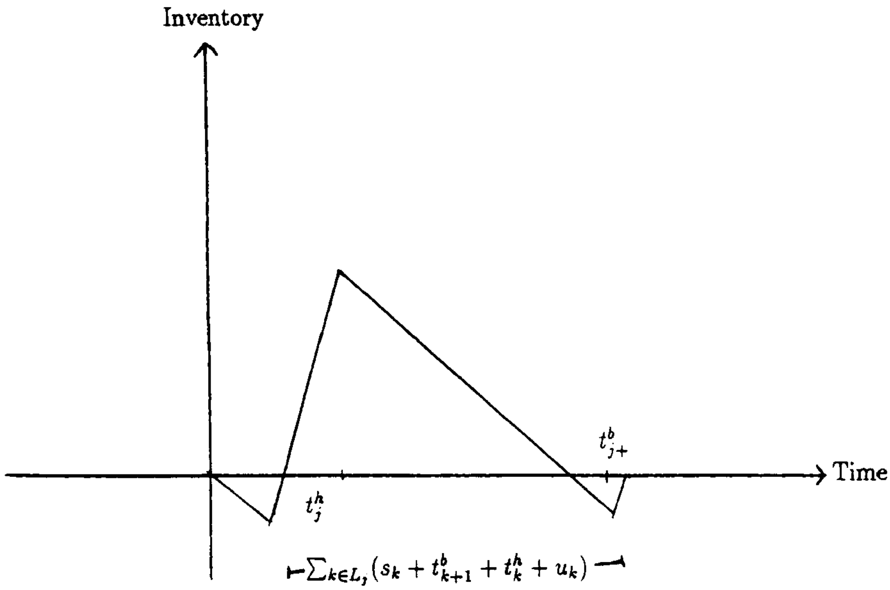

Figure 2. Balance of production and demand.

Let $J_i = \{j: f_j = i\}$ be the set of positions where item i is produced. Adding Eq. (4) over J_i and using (3) results in $\Sigma_{j \in J_i} t_j = \rho_i' T$. Consequently $\Sigma_j t_j = (\Sigma_i \rho_i')T = (1 - \kappa)T$. Substituting into (3) yields

$$\sum_{j=1}^{n} (u_j + s_j) = \kappa T. \tag{5}$$

Recalling the definition of $C(f, t^b, t^h, u)$ we formulate the problem of finding an optimal f-cyclic schedule as the following nonlinear program:

$$\min \frac{1}{T} C(f, t^b, t^h, u),$$

$$\text{s.t. } (4), (5), t^b, t^h, u \geq 0. \tag{6}$$

4. FEASIBILITY

We continue to assume that the production sequence is given. We will show that a feasible f-cyclic schedule satisfying (6) exists provided that the cycle length T is long enough. To do so we introduce additional notation to handle the formulation more efficiently.

Let N and L be $n \times n$ matrices with the nonzero (j, k) entries defined by $N_{jk} = 1$ if $k = j^+$, and $L_{jk} = 1$ if $k \in L_j$. Let $P = \text{diag}(p)$, $B = \text{diag}(b_j p_j (p_j - 1))$, $H = \text{diag}(h_j p_j (p_j - 1))$ and set

$$A = PL$$

and

$$R = I + (I - P)(N - I).$$

The feasibility of (6) is equivalent to the feasibility of the system of linear equations

$$(R - A)t^b + (I - A)t^h - Au = As, \tag{7}$$

$$e'u = \kappa T - e's, \tag{8}$$

$$t^b \geq 0, \qquad t^h \geq 0, \qquad u \geq 0. \tag{9}$$

Dobson shows that $A \geq 0$ and that $e'A = (1 - \kappa)e'$. Consequently the spectral radius of A, $\rho(A) \leq \|A\|_\infty = 1 - \kappa < 1$. An elementary theorem on the theory of nonnegative matrices guarantees that

$$(I - A)^{-1} \geq 0.$$

Dobson uses this fact to show that if the idle times are nonnegative then there exists a feasible f-cyclic schedule for the ELSP. Conversely, he shows that if $\kappa = 0$ and some setup times are positive then there is no feasible f-cyclic schedule. The reason is clear. If $\kappa = 0$ then there is no time available for setups. We now show the following lemma.

LEMMA 1: $(R - A)$ has a nonnegative inverse.

PROOF: $N'(R - A) = I - N'P(L + N - I)$, $N'P(L + N - I) \geq 0$ and $e'NP(L + N - I) = e'A + e'P(N - I) = e'A = (1 - \kappa)e'$. Hence the spectral radius $\rho(N'P(L + N - I)) < 1$. By an elementary theorem on nonnegative matrices $(N'(R - A))^{-1} = (R - A)^{-1}N \geq 0$. Consequently $(R - A)^{-1} \geq 0$. □

We now show the following proposition.

PROPOSITION 2: The system (7), (8), and (9) has a feasible solution if and only if $\kappa T \geq e's$.

PROOF: Assume that (t^b, t^h, u) is a feasible solution to the system (7), (8), and (9). Then (8) and (9) imply that $\kappa T \geq e's$. Conversely if $\kappa T \geq e's$, then select any nonnegative u that satisfies (7) and let

$$t^b = (R - A)^{-1}H(B + H)^{-1}A(s + u),$$

$$t^h = (I - A)^{-1}B(B + H)^{-1}A(s + u);$$

then, by the lemma, the fact that $(I - A)^{-1} \geq 0$, and the fact that B and H are diagonal matrices with positive diagonal elements, we conclude that (t^b, t^h, u) satisfies (7), (8), and (9). □

Note that as the backorder costs tend to infinity t^b tends to the zero vector and t^h tends to $(I - A)^{-1}A(s + u)$. Similarly, if the holding costs tend to infinity t^h tends to zero and t^b tends to $(R - A)^{-1}A(s + u)$.

5. FINDING AN OPTIMAL f-CYCLIC SCHEDULE

To solve the nonlinear program we note that for fixed $T \geq T^0 \equiv e's/\kappa$, the problem reduces to the following quadratic programming problem subject to linear constraints:

$$c(T) = \min \tfrac{1}{2}\{t^{b\prime}Bt^b + t^{h\prime}Ht^h\}, \qquad \text{subject to (7), (8) and (9).}$$

Note that by our choice of T, there exists a feasible solution to constraints (7), (8), and (9) so $c(T)$ is well defined. Given $c(T)$ we obtain the optimal cycle length T^* by minimizing the long-run average cost:

$$C(T) = \frac{1}{T}(c(T) + e'a),$$

$$\text{s.t. } T \geq T^0.$$

This problem can be solved parametrically in T by an algorithm developed by Markowitz [9]. See [12] for a specialized parametric algorithm for the ELSP.

Alternatively, a near-optimal solution can be found by ignoring the synchronization constraint in the computation of $c(T)$. In that case, the optimal choice for t^b and t^h is constant across the positions corresponding to the same item. That is, $t_j^b = (h_i'/b_i' + h_i')\rho_i'T/n_i$ where $t_j^h = (b_i'/b_i' + h_i')\rho_i'T/n_i$, for all $j \in J_i$ where n_i is the cardinality of J_i. Substituting into $c(T)$ and then into $C(T)$, we obtain

$$C(T) = \sum_{i=1}^{m} \left(\frac{H_i'(1 - \rho_i')T}{n_i} + \frac{n_i a_i}{T}\right).$$

But then, minimizing $C(T)$ subject to $T \geq T^0$ is a trivial exercise resulting in

$$\overline{T} = \max\left(T^0, \sqrt{\frac{\Sigma_i n_i a_i'}{\Sigma_i H_i'(1 - \rho_i')/n_i}}\right).$$

Clearly $\overline{T}$ can be used as the initial solution in a parametric algorithm that takes into account the synchronization constraint. Of course, the synchronization constraint can always be met at $\overline{T}$, but $\overline{T}$ may not be optimal. Zipkin [12] proposed a similar advanced solution for the ELSP and reports that "in most cases T^* lies within 1% of $\overline{T}$." Although Zipkin's algorithm is faster than standard nonlinear codes for the resulting program, because of the flatness of $C(T)$ and the effort required to solve for T^* we recommend using $\overline{T}$.

6. COMPUTATIONAL EXPERIENCE

In order to get an idea of how close the lower bound and the cost of the cyclic schedule are likely to be in practice, we replicated Dobson's experiments with an inventory holding cost rate of 20%. In two sets of experiments the backorder costs were randomly selected between 9 and 99 times the holding cost, yielding

Table 1. Data.

Parameters	Sets 1 and 3	Sets 2 and 4
Setup times (hours)	[1, 4]	[1, 8]
Setup cost ($)	[50, 100]	[10, 350]
Production rate (units/day)	[4000, 20000]	[1500, 30000]
Demand rate (units/day)	[1000, 2000]	[500, 2000]
Holding cost ($/$day)	[0.2, 1]	[0.001, 1.4]

service levels between 90% and 99%. We then repeated the experiments with the backorder randomly selected between 99 and 999 times the holding costs, yielding service levels between 99% and 99.9%. The results are very similar to Dobson's, in particular for the case of high backorder costs.

The heuristic was tried on four sets of problems (40 problems in each set). Sets 1 and 3 (resp., Sets 2 and 4) differ only on the backorder cost as just outlined. The data sets were generated randomly from uniform distributions on the given intervals.

The problems were generated by adding parts until the $\Sigma\, d_i/p_i''$ exceeded 0.8. As in Dobson, the problems were solved several times, starting with the frequencies suggested by the lower bound; then every item's frequency exceeding 1 was cut in half until all the items were produced only once per cycle. Tables 1 and 2 list the best solution for each problem. The rationale for doing this, as explained by Dobson, is that there may be a troublesome item produced infrequently with large lots affecting other items produced more frequently. The idea is to produce the troublesome item relatively more frequently in smaller lots. We report the mean and the maximum ratio of the cost of the solution to the lower bound.

7. CONCLUSIONS

The ELSP has been extended to allow backorders in an effort to model effectively a frequently encountered scheduling problem. We have extended earlier methods for the ELSP to the EELSP. The resulting algorithm finds a good production sequence f and an optimal f-cyclic schedule. This research is a first step in an effort to obtain an algorithm that schedules several items on a single machine in the presence of unforeseen disruptions.

ACKNOWLEDGMENT

The authors thank the referees for suggesting many ways of improving the exposition of the article. Thanks are also due to the Associate Editor in charge of the article. Gallego's research was supported in part by NSF Grant No. DDM-

Table 2. Ratio of cost to lower bound.

	Set 1	Set 2	Set 3	Set 4
Mean	1.002	1.010	1.003	1.041
Maximum	1.029	1.062	1.022	1.123

9109636 and by an IBM manufacturing research grant. Roundy's research was supported in part by NSF Grant No. DMC-8451984, by ATT, and by Dupont Corporation.

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Manuscript received July 17, 1991
Revised manuscript received October, 1991
Accepted February 24, 1992