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Optimal price and order size under partial backordering incorporating shortage, backorder and lost sale costs

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Abstract

In this paper, we consider the pricing and lot-sizing problem for a product subject to general rate of deterioration and partial backordering. We use impatience functions to model backlogging of demand. We show that even when lost sale and backorder costs are present, the problem is well posed in the reduced space. We provide an iterative procedure for solving the overall problem. We describe structural properties of the solution for the new model and comment on the recent work incorporating backorder cost. We illustrate the solution procedure for the new model with examples.

Keywords: Order size; Pricing; Backordering; Perishable good

1. Introduction

The idea of an "impatient" customer in backlogging situation was proposed in Abad (1996). It was suggested that customers do not like to wait and therefore that the fraction of customers who choose to place backorders be a decreasing function of waiting time. Two specific examples of functions for backlogging of demand were given. If τ is the waiting time (i.e., time till the new supply becomes available and the backorder is filled), the fraction of customers backordering can be modeled as $B(\tau) = k_0 e^{-k\tau_1}$ or $B(\tau) = k_0/(1+k_1\tau)$, k_0, k_1 being parameters. These two functions—the exponential rate and the hyperbolic rate with respect to waiting time—have been used to model backordering in several recent studies.

San Jose et al. (2006) consider the problem of determining the lot size and the backorder level when demand is backlogged at the exponential rate. Demand is assumed to be constant and the order and lost sale costs are included. The authors coined the term "impatience function" to refer to the functions for modeling backlogging of demand. They have proposed a continuous two-piece function [San Jose et al. (2005a)] as well as a discontinuous step function [San Jose et al. (2005b)] for modeling backlogged demand. In the step function, backlogging rate is equal to 1 (i.e., full backordering) when waiting time is less than the specified fixed period and 0 (i.e., complete lost sale) when the waiting time is more than the fixed period. Skouri and Papachristos (2003) formulate a production lot size model where production rate, demand rate and deterioration rate are exogenous, time-varying functions and demand is backlogged with the hyperbolic rate.

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There also have been studies in which demand is considered to be price sensitive and the selling price is an endogenous variable. Abad (2001) considers the pricing and lot-sizing problem for infinite horizon. However, he ignores the shortage cost, the lost sale and the backorder cost in his analysis. Papachristos and Skouri (2003) consider the pricing and lot-sizing problem for infinite horizon when the supplier offers a lot size-based quantity discount on materials, time to deterioration is described by Weibull function and demand is backlogged at the hyperbolic rate. They include the lost sale cost and the shortage cost (i.e., cost per unit short) in their analysis. They assume that the demand rate is a convex decreasing function of selling price and that the inventory cycle time is fixed. Dye (2007) considers the pricing and lot-sizing problem for infinite planning horizon assuming the demand rate to be convex, decreasing function of selling price and the revenue to be a concave function of selling price. He assumes that the deterioration rate is time varying and that backlogging occurs at the hyperbolic rate. He includes only the lost sale cost and the cost of carrying backorders but excludes the shortage cost in his analysis. Dye et al. (2007) consider a similar problem except that backlogging occurs at the exponential rate. They again exclude the shortage cost in their analysis (see also Chang et al., 2006).

In this paper we consider the pricing and lot-sizing problem for an infinite planning horizon in a general framework. We assume that the demand rate is a decreasing function of price and that the marginal revenue is an increasing function of price. We include all three costs—the lost sale cost, the cost of carrying backorders and the shortage cost in our analysis. Furthermore, we use a general (continuous and smooth) impatience function to model the backlogging phenomenon. We provide an algorithm to determine the optimal solution. We also highlight some structural properties of the optimal solution.

2. Model formulation

The assumptions underlying the model are

1. The planning horizon is infinite.
2. The entire order quantity is received at the same time.
3. The good decays at a general rate; i.e., the decay rate is any differentiable function of time.
4. Demand is represented by a general function; i.e., the demand function can be any twice differentiable function of price subject to two conditions (described later in this section).
5. There is shortage cost, backorder cost as well as the lost sale cost.

The pattern of variation within the inventory cycle for our case is shown in Fig. 1. Let

$I(t)$	net stock (on hand-backorders) level at time t
T	the length of the duration over which net stock is positive (see Fig. 1)
Ψ	the length of the duration over which net stock is less than or equal to zero (see Fig. 1)
h	inventory carrying cost for the vendor (\$/unit/period)
$\sigma(t)$	a coefficient representing instantaneous decay rate. Assumed to be non-negative and bounded
$\sigma(t)\ I(t)$	wastage rate at time t (units/period)
K	order cost
v	unit purchase cost for the reseller
p	selling price within the inventory cycle
$D(p)$	demand rate (units/period).

In addition, let

c_1	shortage cost per unit
c_2	backorder cost per unit per unit time
c_3	lost sale cost per unit.

The following assumptions are made concerning the demand function:

(i) $D' = \dfrac{\mathrm{d}D(p)}{\mathrm{d}p} < 0 \quad \text{for all} \quad p \in (0, \infty)$

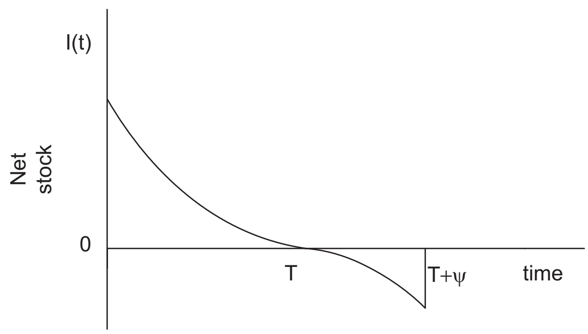

Fig. 1. The pattern of net stock.

(ii) The marginal revenue $\mathrm{d}\{pD(p)\}/\mathrm{d}D = p + (D(p)/D')$ is a strictly increasing function of p. This also implies that $1/D(p)$ is a strictly convex function of p.

Note, we will use D and $D(p)$ interchangeably in the paper. Assumption (ii) above is weaker than the assumptions in Dye (2007) and Dye et al. (2007). During $t \in [0,T)$, net stock level $I(t)$ is given by

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -D(p) - \sigma(t)I(t), \quad I(T) = 0, \tag{1}$$

and for $t \in [0,T]$, net stock level $I(t)$ is given by [Rajan et al. (1992)]

$$I(t) = \int_t^T D(p)\,\mathrm{e}^{\int_t^r \sigma(s)\,\mathrm{d}s}\,\mathrm{d}r. \tag{2}$$

Also,

$$I(0) = \int_0^T D(p)\,\mathrm{e}^{\int_0^r \sigma(s)\,\mathrm{d}s}\,\mathrm{d}r. \tag{3}$$

During $t \in [T, T+\psi]$, the fraction of demand backlogged is $B(\tau)$ where τ is the amount of time the customer waits before receiving the item. $B(\tau)$ is a decreasing function of τ; i.e., customers are impatient. For $t \in [T, T+\psi]$,

$$\begin{aligned} I(t) &= -D(p)\int_T^t B(T+\psi-r)\,\mathrm{d}r \\ &= -D(p)\int_{T+\psi-t}^{\psi} B(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{4}$$

Also

$$I_\psi = I(T+\psi) = -D(p)\int_0^\psi B(\tau)\,\mathrm{d}\tau = -D(p)M_b(\psi). \tag{5}$$

Note that $\mathrm{d}M_b(\psi)/\mathrm{d}\psi = B(\psi)$ and $\mathrm{d}^2M_b(\psi)/\mathrm{d}^2\psi = \mathrm{d}B(\psi)/\mathrm{d}\psi$. Given that $B(\tau)$ is a decreasing function of τ, $\mathrm{d}^2M_b(\psi)/\mathrm{d}^2\psi = \mathrm{d}B(\psi)/\mathrm{d}\psi < 0$. Hence $M_b(\psi)$ is concave.

The lot size is given by

$$Q = I(0) - I_\psi. \tag{6}$$

2.1. Objective function

Assume that there is no salvage value for deteriorated items. Then

Revenue $= pD(p)T + pD(p)M_b(\psi)$,

Purchase cost $= vD(p)T + vD(p)M_b(\psi)$.

The wastage cost (purchase cost of deteriorated units) plus inventory carrying cost [Rajan et al. (1992), pp. 256–257] is

$$\int_0^T [v\sigma(t)I(t) + hI(t)]\,\mathrm{d}t = D(p)\int_0^T c(t)\,\mathrm{d}t - vD(p)T, \tag{7a}$$

where

$$c(t) = v\mathrm{e}^{\int_0^t \sigma(s)\,\mathrm{d}s} + h\int_0^t \mathrm{e}^{\int_r^t \sigma(s)\,\mathrm{d}s}\,\mathrm{d}r. \tag{7b}$$

Given (4), the cost of carrying backorders during time-span $t \in [T, T+\psi]$ is

$$\begin{aligned} -c_2\int_T^{T+\psi} I(t)\,\mathrm{d}t &= c_2D(p)\int_T^{T+\psi}\int_{T+\psi-t}^{\psi} B(\tau)\,\mathrm{d}\tau\,\mathrm{d}t \\ &= c_2D(p)\int_0^\psi B(\tau)\int_{T+\psi-\tau}^{T+\psi}\mathrm{d}t\,\mathrm{d}\tau \\ &= c_2D(p)\int_0^\psi \tau B(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{8}$$

The shortage cost plus the lost sale cost is

$$\begin{aligned} &c_1D(p)M_b(\psi) + c_3D(p)[\psi - M_b(\psi)] \\ &\quad = D(p)\{(c_1 - c_3)M_b(\psi) + c_3\psi\}. \end{aligned} \tag{9}$$

Profit during time-span $[0,T+\psi]$ is

$$\begin{aligned} F(p,T,\psi) = {} & pD(p)T - D(p)\int_0^T c(t)\,\mathrm{d}t \\ & + (p - v - c_1 + c_3)D(p)M_b(\psi) - c_3D(p)\psi \\ & - c_2D(p)\int_0^\psi \tau B(\tau)\,\mathrm{d}\tau - K. \end{aligned} \tag{10}$$

Thus average profit per time period is

$$\Pi(p,T,\psi) = \frac{F(p,T,\psi)}{T+\psi}. \tag{11}$$

The optimization problem faced by the reseller is

$$\text{(P1)} \quad \text{Max} \quad \Pi(p,T,\psi) \tag{12a}$$

$$T \geqslant 0 \tag{12b}$$

$$\psi \geqslant 0. \tag{12c}$$

Let $F(T,\psi|p)$ denote $F(p,T,\psi)$ for a fixed value of p, and $\Pi(T,\psi|p)$ denote $\Pi(p,T,\psi)$ for a fixed value of p. For a fixed p, projection of problem (P1) in (T, ψ) space is

$$\text{(P2)} \quad \text{Max} \quad \Pi(T,\psi|p) \tag{13a}$$

$$T \geqslant 0 \tag{13b}$$

$$\psi \geqslant 0. \tag{13c}$$

3. Solution procedure

Assumption 1. . The set $G=\{T, \psi|T>0, \psi\geqslant 0, F(T, \psi|p)>0\}$ is not a null set.

Proposition 1. . *Let T^* and ψ^* represent the optimal values of T and ψ, respectively, for problem (P2). Then $T^*>0$.*

Proposition 2. . *For $p\geqslant v+c_1-c_3$, $F(T, \psi|p)$ is a strictly concave function on $\{T>0, \psi\geqslant 0\}$*

Proposition 3. . *For $p\geqslant v+c_1-c_3$, $\Pi(T, \psi|p)$ is a strictly pseudo-concave function on $G=\{T, \psi|T>0, \psi\geqslant 0, F(T, \psi|p)>0\}$.*

Proposition 4. . *For $p<v+c_1-c_3$, $\psi^*=0$.*

Assumption 2. . The set

$$H=\left\{p|pT-\int_0^T c(t)dt+(p-v-c_1+c_3)M_b(\psi)-c_3\psi-c_2\int_0^{\psi}\tau B(\tau)\,d\tau>0,\quad D(p)>0\right\}$$

is not a null set.

Proposition 5. . *Let $F(p|T, \psi)$ denote $F(p, T, \psi)$ for a fixed values of T and ψ. Similarly, let $\Pi(p|T, \psi)$ denote $\Pi(p, T, \psi)$ for a fixed values of T and ψ . Then* (1) *$F(p|T, \psi)$ is strictly pseudo-concave on H*; *and* (2) *the global maximum of $F(p|T, \psi)$ is characterized by*

$$p+\frac{D(p)}{D'(p)}=\phi(T,\psi) \tag{14}$$

where

$$\phi(T,\psi)=\frac{\int_0^T c(t)dt+(v+c_1-c_3)M_b(\psi)+c_3\psi+c_2\int_0^{\psi}\tau B(\tau)\,d\tau}{T+M_b(\psi)}. \tag{15}$$

Proofs of above propositions are given in Appendix A. Given the above results, the following algorithm given in Abad (2001) applies.

1. Let $p=p_r$, where p_r is some starting value for p.
2. For the current p, solve problem (P2) and let the optimal solution to (P2) be denoted as T_p^*, ψ_p^*.
3. Let $T=T_p^*$ and $\psi=\psi_p^*$ and solve (14). Let the solution to (14) be the current p and go to step 1.

Note that problem (P2) is well posed since the objective function is pseudo-concave and constraints are linear. Given Propositions 3 and 4, there should be a unique local maximum T^*, ψ^* for a given p, which will be the global maximum. Since the objective function $\Pi(p, T, \psi)$ is improving in steps 2 and 3, the procedure would converge to a local maximum. The search is simplified since one would need to provide a starting value in just one-dimensional, p, space. Thus, by using different starting values of p, one can determine the global maximum for $\Pi(p, T, \psi)$.

4. Structural properties of the solution

Proposition 6. . *Suppose $c_1=0$. Also suppose impatience function $B(\tau)$ is such that $B(0)=1$. Then $\psi^*>0$.*

Proposition 7. . *Suppose that $c_1>0$. Let p_0 and T_0 denote the optimal values of p and T when we set $\psi=0$ and solve the resultant problem (P1). It is easy to show that (p_0, T_0) would satisfy the first-order necessary conditions (A.1) and (14) when we set $\lambda=0$ and $\psi=0$. Then $\psi^*=0$; i.e., $(p_0, T_0, 0)$ is a local maximum of $\Pi(p, T, \psi)$ if c_1 is greater than $\frac{c(T_0)-vB(0)-(p_0+c_3)(1-B(0))}{B(0)}$.*

Proofs of Propositions 6 and 7 are given in Appendix A. The propositions are analogous to the results for the EOQ model when backordering is allowed [Hadley and Whitin (1963), pp. 45–46]. Dye (2007) considers the formulation alluded in Proposition 6 for the case in which the backlogging function is hyperbolic. He assumes that $T^*>0$ and $\psi^*>0$ and considers the unconstrained version of problem (P1). He shows that the first-order conditions for the unconstrained problem (i.e., conditions (A.1) and (A.2) in Appendix A wherein the Lagrange multipliers λ and β are set to zero) yield $T^*>0$ and $\psi^*>0$. He also proves that for a given price, there is a unique T^*, ψ^* satisfying the necessary conditions. Since the solution procedure is based upon the properties of the first-order conditions, it cannot be generalized to other demand or backlogging functions or the case in which the shortage cost is not excluded. Dye et al. (2007) provide a similar analysis for the case in which the backlogging function is exponential. Our formulation is a generalization in the following way: (1) the demand function $D(p)$ is not restrictive; (2) the impatience function $B(\tau)$ is general; and (3) the model incorporates not only the cost of carrying backorders and the lost sale cost but also the shortage cost. Propositions 3 and 4 prove the uniqueness of the solution for the general case. Proposition 6 proves that $\psi^*>0$ when $c_1=0$ and $B(0)=1$. Finally, Proposition 7 shows that when c_1

is more than the threshold value, $\psi^* = 0$; i.e., backordering becomes uneconomic for the reseller.

5. Examples

To illustrate the new model and the solution procedure, we consider four examples.

Example 1. . The first example is the same as in Dye (2007). The parameter values in the example are $K = \$250$, $v = \$40$ per unit, $h = \$1.50$/unit/year, $c_2 = \$5$/unit/year, $c_3 = \$5$/unit, $D(p) = 16\times 10^7 p^{-3.21}$, $\sigma(t) = 0.1t$ and $B(\tau) = 1/(1+0.5\tau)$. The solution found is $p^* = 59.12$, $T^* = 0.6368$ and $\psi^* = 0.1110$. The solution is the same as in Dye (2007). With this solution, $\Pi^* = 5695.88$. Note that in this example, conditions of Proposition 6 are met (i.e., $c_1 = 0$) and we see that in this example, $T^* > 0$ and $\psi^* > 0$.

Example 2. . We let the parameter values be the same as in Example 1 except now $c_1 = 6$, $c_2 = 0$ and $c_3 = 5$.

The solution is $p^* = 59.29$, $T^* = 0.6757$ and $\psi^* = 0$. With this solution, $\Pi^* = 5647.07$.

Note in this example, the conditions of Proposition 7 are met. In fact, $p_0 = 59.29$, $T_0 = 0.6757$, $c(T_0) = 41.953$ and $B(0) = 1.0$. And hence,

$$\frac{c(T_0) - vB(0) - (p_0 + c_3)(1 - B(0))}{B(0)} = \frac{41.9526 - 40 - (59.29 + 5)(1 - 1)}{1} = 1.953.$$

Clearly $c_1 = 6$ satisfies the condition of Proposition 7. Hence ψ^* is predicted to be zero by Proposition 7.

Example 3. . The third example is the same as in Dye et al. (2007). The parameter values in the example are $K = \$250$, $v = \$8$ per unit, $h = \$0.50$/unit/year, $c_2 = \$2$/unit/year, $c_3 = \$2$/unit, $D(p) = 25-0.5p$, $\sigma(t) = 0.075t^{0.5}$ and $B(\tau) = e^{-0.2\tau}$. The solution is $p^* = 30.08$, $T^* = 4.959$ and $\psi^* = 1.1319$. With this solution, $\Pi^* = 153.348$.

Example 4. . In this example, parameter values are the same as in Example 1 but $c_1 = 0.5$, $c_2 = 5$ and $c_3 = 5$.

Note in this example we also have shortage cost c_1 and hence solution procedures in Dye (2007) or Dye et al. (2007) are not applicable. The solution is $p^* = 59.24$, $T^* = 0.6552$ and $\psi^* = 0.0843$. With this solution, $\Pi^* = 5674.91$.

6. Conclusions

In this paper, a general model is formulated for determining the optimal selling price and order size for a perishable product subject to partial backordering. We include not only the cost of carrying backorders and the lost sale cost but also the shortage cost. Convexity properties are proved and an iterative procedure is provided for solving the problem, which guarantees a local maximum. We have used a general impatience function to model backlogging of demand. The model assumes a general deterioration function and is not restrictive in terms of demand function. We have also highlighted some structural properties of the solution.

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Appendix A

We use D and $D(p)$ interchangeably. Let λ and β be the Lagrange multiplier associated with constraints (13b) and (13c), respectively. The Kuhn–Tucker conditions for problem (P2) are [Bazarra et al. (2006)]

$$\frac{1}{T+\psi}[\Pi(T,\psi|p) - pD + Dc(T)] - \lambda = 0, \tag{A.1}$$

$$\frac{1}{T+\psi}[\Pi(T,\psi|p) - (p - v - c_1 + c_3 - c_2\psi)DB(\psi) + c_3 D] - \beta = 0, \tag{A.2}$$

$$\lambda \geqslant 0, \tag{A.3}$$

$$\beta \geqslant 0, \tag{A.4}$$

$$\lambda T = 0, \tag{A.5}$$

$$\beta\psi = 0, \tag{A.6}$$

Proof of Proposition 1. From (10) and (11), $\Pi(0,0|p) = -\infty$. Hence both T^* and ψ^* cannot be equal to zero at the same time. Now suppose $T^* = 0$. Hence $\psi^* > 0$ and $\beta = 0$. From (7b), $c(0) = v$. Thus, from (A.1),

$$\Pi(0,\psi|p) \geqslant (p - v)D. \tag{A.7}$$

Also from (A.2),

$$\Pi(0,\psi|p) = (p - v - c_1 - c_2\psi)DB(\psi) - c_3D(1 - B(\psi)). \tag{A.8}$$

Since $0 \leqslant B(\psi) \leqslant 1$ for $\psi \geqslant 0$, (A.8) and (A.9) are contradictory; i.e., $T^* > 0$. □

Proof of Proposition 2. From (10),

$$\frac{\partial F}{\partial T} = pD - Dc(T) \quad \text{and} \quad \frac{\partial^2 F}{\partial^2 T} = -D\frac{\mathrm{d}c(T)}{\mathrm{d}T}. \tag{A.9}$$

From (7b),

$$-\frac{\mathrm{d}c(T)}{\mathrm{d}T} = -v\sigma(T)\mathrm{e}^{\int_0^T \sigma(s)\,\mathrm{d}s} - h\sigma(\mathrm{T})\int_0^T \mathrm{e}^{\int_r^T \sigma(s)\,\mathrm{d}s}\,\mathrm{d}r - h$$
$$= -\sigma(T)c(T) - h < 0. \tag{A.10}$$

Given (A.9) and (A.10), $(\partial^2 F/\partial^2 T) < 0$ for $T \geqslant 0$. Similarly from (10),

$$\frac{\partial F}{\partial \psi} = (p - v - c_1 + c_3)DB(\psi) - c_3D - c_2D\psi B(\psi) \tag{A.11}$$

and

$$\frac{\partial^2 F}{\partial^2 \psi} = (p - v - c_1 + c_3 - c_2\psi)D\frac{\mathrm{d}B(\psi)}{\mathrm{d}\psi} - c_2DB(\psi). \tag{A.12}$$

Note $B(\psi) > 0$ and $\left(\mathrm{d}B(\psi)/\mathrm{d}\psi\right) < 0$ for $\psi > 0$. Also, $p \geqslant v + c_1 - c_3 > v + c_1 - c_3 - c_2\psi$. Hence $(\partial^2 F/\partial^2\psi^2) < 0$ for $\psi \geqslant 0$. Also, $\left(\partial^2 F/\partial T\partial\psi\right) = 0$. Thus the Hessian of $F(T, \psi|p)$ with respect to T and ψ is negative-definite. This completes the proof. □

Proof of Proposition 3. Proof of Proposition 3 follows from the fact that for $p \geqslant v + c_1 - c_3$, $F(T, \psi|p)$ is a strictly concave function on $\{T > 0, \psi \geqslant 0\}$. See Abad (2001). □

Proof of Proposition 4. Note $B(\psi) \geqslant 0$ for $\psi \geqslant 0$. We see from (A.2) that for $p < v + c_1 - c_3 < v + c_1 - c_3 + c_2\psi$, (A.2) can hold only if $\beta > 0$. Hence given (A.6), $\psi^* = 0$. □

Proof of Proposition 5. Note

$$\begin{aligned} F(p|T,\psi) &= pD(p)T - D(p)\int_0^T c(t)\,\mathrm{d}t + (p - v - c_1 + c_3) \\ &\quad \times D(p)M_b(\psi) - c_3D(p)\psi \\ &\quad - c_2D(p)\int_0^\psi \tau B(\tau)\,\mathrm{d}\tau - K \\ &= \frac{pT - \int_0^T c(t)\mathrm{d}t + (p - v - c_1 + c_3)M_b(\psi) - c_3\psi - c_2\int_0^\psi \tau B(\tau)\mathrm{d}\tau}{1/D(p)} - K \\ &= \frac{f(p)}{1/D(p)} - K. \end{aligned} \tag{A.13}$$

Now suppose for each distinct $p_2, p_1 \in H$

$$\frac{f(p_2)}{1/D(p_2)} - K \geqslant \frac{f(p_1)}{1/D(p_1)} - K. \tag{A.14}$$

Since $f(p) > 0$ and $1/D(p) > 0$ on H, we rewrite (A.14) as

$$\frac{f(p_2)}{f(p_1)} \geqslant \frac{1/D(p_2)}{1/D(p_1)} \tag{A.15}$$

or

$$\frac{f(p_2) - f(p_1)}{f(p_1)} \geqslant \frac{1/D(p_2) - 1/D(p_1)}{1/D(p_1)}. \tag{A.16}$$

Since $f(p)$ is a linear function of p,

$$f(p_2) = f(p_1) + \left.\frac{\mathrm{d}f}{\mathrm{d}p}\right|_{p=p_1}(p_2 - p_1). \tag{A.17}$$

Similarly, because $1/D(p)$ is strictly convex on H,

$$1/D(p_2) - 1/D(p_1) > -\frac{\left.\frac{\mathrm{d}D}{\mathrm{d}p}\right|_{p=p_1}}{[D(p_1)]^2}(p_2 - p_1). \tag{A.18}$$

Combining (A.16), (A.17) and (A.18), we have

$$\frac{\left.\frac{\mathrm{d}f}{\mathrm{d}p}\right|_{p=p_1}(p_2 - p_1)}{f(p_1)} \geqslant \frac{1/D(p_2) - 1/D(p_1)}{1/D(p_1)} > -\frac{\left.\frac{\mathrm{d}D}{\mathrm{d}p}\right|_{p=p_1}}{D(p_1)}(p_2 - p_1). \tag{A.19}$$

Multiplying by $\frac{f(p_1)}{1/D(p_1)}$, we have from (A.19)

$$\left\{\frac{\left.\frac{\mathrm{d}f}{\mathrm{d}p}\right|_{p=p_1}}{1/D(p_1)}+f(p_1)\left.\frac{\mathrm{d}D}{\mathrm{d}p}\right|_{p=p_1}\right\}(p_2-p_1)>0 \tag{A.20}$$

or

$$\left.\frac{\mathrm{d}\left\{\frac{f(p)}{1/D(p)}-K\right\}}{\mathrm{d}p}\right|_{p=p_1}(p_2-p_1)>0. \tag{A.21}$$

We have proven that (A.14) implies (A.21) [see Bazarra et al. (2006), p. 142]. This completes the proof of part (1).

Part (2) is simply the first-order necessary and sufficient condition for the maximization of $F(p|T,\psi)$. Also note that $\Pi(p|T,\psi)$ is proportional to $F(p|T,\psi)$. □

Proof of Proposition 6. . Proof is by contradiction. Suppose $\psi^*=0$ and hence $\beta>0$. From Proposition 1 and (A.1),

$$\Pi(T,0|p)=pD-Dc(T). \tag{A.22}$$

Also given (A.2) and given that $\beta>0$ and $B(0)=1$,

$$\Pi(T,0\big|p)\geqslant(p-v+c_3)DB(0)-c_3D\equiv(p-v)D. \tag{A.23}$$

Now from (7b), $c(T)>v$. Clearly (A.22) and (A.23) are contradictory; i.e., $\psi^*>0$. □

Proof of Proposition 7. . First note that given Proposition 1, $T+\psi>0$. If we replace notation $\Pi(T,\psi|p)$ by notation $\Pi(p,T,\psi)$ in (A.1) and (A.2), then conditions (A.1)–(A.6) along with condition (14) would constitute Kuhn–Tucker conditions for the extended problem (P1). Now (p_0,T_0) by definition satisfies condition (A.1) and (14) for $\lambda=0$ and $\psi=0$. Hence from (A.1),

$$\Pi(T_0,0|p_0)=p_0D_0-D_0c(T_0). \tag{A.24}$$

Also the condition in the Proposition is

$$c_1>\frac{c(T_0)-vB(0)-(p_0+c_3)(1-B(0))}{B(0)} \tag{A.25}$$

or

$$c_1B(0)>c(T_0)-vB(0)-(p_0+c_3)(1-B(0))$$

or

$$p_0D_0-D_0c(T_0)>(p_0-v-c_1+c_3)D_0B(0)-c_3D_0. \tag{A.26}$$

Combining (A.24) and (A.26), we have

$$\Pi(T_0,0|p_0)>(p_0-v-c_1+c_3)D_0B(0)-c_3D_0. \tag{A.27}$$

Thus given (A.27), there exists a $\beta\geqslant0$ so that

$$\Pi(T_0,0|p_0)=(p_0-v-c_1+c_3)D_0B(0)-c_3D_0+\beta T_0. \tag{A.28}$$

Conditions (A.28) and (A.2) are identical for $\psi=0$. Also conditions (A.3–A.6) hold and (p_0,T_0) by definition satisfies necessary conditions (A.1) and (14). Thus when (A.25) holds, $(p_0,T_0,0)$ satisfies Kuhn–Tucker conditions associated with the extended problem (P1). Hence $(p_0,T_0,0)$ is a local maximum for problem (P1). □

References

Abad, P.L., 1996. Optimal pricing and lot-sizing under conditions of perishability and partial backordering. Management Science 42, 1093–1104.

Abad, P.L., 2001. Optimal price and order size for a reseller under partial backordering. Computers and Operations Research 28, 53–65.

Bazarra, M., Sherali, H., Shetty, C.M., 2006. Nonlinear Programming, Theory and Applications, third ed. Wiley, New York.

Chang, H.J., Teng, J.T., Ouyang, L.Y., Dye, C.Y., 2006. Retailer's optimal pricing and lot-sizing policies for deteriorating items with partial backlogging. European Journal of Operational Research 168, 51–64.

Dye, C.Y., 2007. Joint pricing and ordering for a deteriorating inventory with partial backlogging. Omega, The International Journal of Management Science 35, 184–189.

Dye, C.Y., Hsieh, T.P., Quyang, L.Y., 2007. Determining optimal selling price and lot size with a varying rate of deterioration and exponential partial backlogging. European Journal of Operational research 181, 668–678.

Hadley, G., Whitin, T., 1963. Analysis of Inventory Systems. Prentice Hall, Englewood Cliffs.

Papachristos, S., Skouri, K., 2003. An inventory model with deteriorating items, quantity discount, pricing and time-dependent partial backlogging. International Journal of Production Economics 83, 247–256.

Rajan, A., Rakesh, Steinberg, R., 1992. Dynamic pricing and ordering decisions by a monopolist. Management Science 38, 240–262.

San Jose, L., Sicilia, J., Garcia-Laguna, J., 2005a. An inventory system with partial backlogging modeled according to a linear function. Asia-Pacific Journal of Operational Research 22, 189–209.

San Jose, L., Sicilia, J., Garcia-Laguna, J., 2005b. The lot size-reorder level inventory system with customer impatience functions. Computers and Industrial Engineering 49, 349–362.

San Jose, L., Sicilia, J., Garcia-Laguna, J., 2006. Analysis of an inventory system with exponential backordering. International Journal of Production Economics 100, 76–86.

Skouri, K., Papachristos, S., 2003. Optimal stopping and restarting production times for an EOQ model with deteriorating items and time-dependent partial backlogging. International Journal of Production Economics 81–82, 525–531.