Lot Sizes under Continuous Demand: The Backorder Case

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We study a deterministic lot-size problem, in which the demand rate is a (piecewise) continuous function of time and shortages are backordered. The problem is to find the order points and order quantities to minimize the total costs over a finite planning horizon. We show that the optimal order points have an interleaving property, and when the orders are optimally placed, the objective function is convex in the number of orders. By exploiting these properties, an algorithm is developed which solves the problem efficiently. For problems with increasing (decreasing) demand rates and decreasing (increasing) cost rates, monotonicity properties of the optimal order quantities and order intervals are derived.

INTRODUCTION

We are concerned with a deterministic single-product lot-size problem, in which the demand rate can be any continuous or piecewise continuous function of time. The planning horizon is given and finite, replenishments are instantaneous (i.e., zero lead time) and shortages are backordered. The order (or setup) cost is an increasing convex function of the number of orders placed. For each unit backordered or held as inventory, there is a cost per unit of time (to be called a 'cost rate"). Both the backorder and the holding-cost rates can be functions of time. The problem is to find the order points (time for placing orders) and order quantities which minimize the sum of the order, holding, and backorder costs over the horizon.

The above, with backorders prohibited, was studied by Friedman [6]. Earlier, the no-backorder case had been worked on under restrictive conditions on the demand rate, by Carr and Howe [3] for a constant demand rate, by Resh, Friedman, and Barbosa [7] for a linear demand rate, and by Barbosa and Friedman [1, 2] for a demand rate in the form of a power function.

As an application of one of their lattice programming results, Denardo, Huberman, and Rothblum [4] demonstrated that the no-backorder model has the following interesting properties. Suppose n orders are placed over the planning horizon, and $\mathbf{y}(n) = (y_i(n))_{i=1}^{n}$ is a vector of optimally placed order points. Then (i) $\mathbf{y}(n)$ and $\mathbf{y}(n+1)$ are "interleaved," i.e., $y_i(n) \leq y_{i+1}(n+1) \leq y_{i+1}(n)$, for all $i = 1, \ldots, n-1$; and (ii) the total holding cost (over the entire horizon) is a convex function of n. (These properties were also independently

Naval Research Logistics, Vol. 36, pp. 615–624 (1989)
 CCC 0028-1441/89/050615-10$04.00

derived in Friedman [6] via a more ad hoc approach.) These structural properties greatly facilitate the identification of the optimal number of orders and the order points.

In this article, making use of the lattice programming result of Denardo et al., we show that the above properties are also present in the backorder case.

There are practical reasons for studying the backorder model. First, the backorder case often arises in government organizations (e.g., the military) and in wholesale-retail systems (e.g., exclusive dealership); see, e.g., Silver and Peterson [9, p. 253]. Second, the backorder model includes the more restrictive no-backorder model as a special case (see the discussions that follow Theorem 2 in Section 3). Third, the backorder model is often a close approximation to the "lost sales" case, which is usually less tractable; see, e.g., Silver [8]. Finally, the backorder model has an interesting "dual" (Theorem 3), which enables us to show that, in addition to the interleaving property of the optimal order points, the optimal regeneration points (i.e., times at which the inventory is zero) are also interleaved. Based on these properties, we show that the total backorder and holding costs are convex in the number of orders, provided the order points, and hence the regeneration points, are optimally placed. Exploiting these properties, an algorithm is developed which solves the problem efficiently. For problems with increasing or decreasing demand rates and decreasing or increasing cost rates, monotonicity properties of the optimal order quantities and order intervals are derived. A numerical example is given in the last section.

PROBLEM FORMULATION

The planning horizon is a continuous-time interval $[0, T]$, where T is given, and $0 < T < \infty$. The demand rate $r(t)$ is a continuous or piecewise continuous function of time with, at most, a finite number of discontinuities of the first kind (i.e., jump discontinuities). Hence the cumulative demand (up to time t), $R(t) = \int_0^t r(u)\, du$, is continuous and smooth or piecewise smooth. We assume that $R(t)$ is strictly increasing (i.e., $r(t) > 0$ for all $t \in [0, T]$). Let n be the number of orders placed over the horizon. Following previous studies (e.g., Friedman [6]) the order cost, $A(n)$, is an increasing convex function of n. (Unless otherwise stated, we use the terms "increasing/decreasing" and "convex/concave" is the nonstrict sense in the sequel.) Let $h(t)$ and $b(t)$ be the cost rates at time t for, respectively, holding one unit of inventory and backordering one unit; assume that both $h(t)$ and $b(t)$ are positive and finite for all $t \in [0, T]$. Let $H(t) = \int_0^t h(u)\, du$ and $B(t) = \int_0^t b(u)\, du$.

It is clear that the optimal solution to this problem, under the conditions specified, has a structure similar to its discrete-time counterpart (see Zangwill [10]). Specifically, an order should *not* be placed when stock is on hand, and between two regeneration points there is exactly one order point. Hence the order quantity equals the total demand between the two successive regeneration points. Falkner [5] has shown that this type of property characterizes optimal solutions under more general conditions.

Let $\mathbf{x}(n) = (x_i(n))_{i=0}^n$ and $\mathbf{y}(n) = (y_i(n))_{i=1}^n$ be, respectively, the regeneration and the order points:

$$0 = x_0(n) \le y_1(n) \le x_1(n) \le \cdots \le x_{n-1}(n) \le y_n(n) \le x_n(n) = T; \tag{1}$$

see Figure 1. The problem is then to find the optimal n, $\mathbf{x}(n)$, and $\mathbf{y}(n)$, such that the sum of ordering, holding, and backorder costs will be minimized. [The order quantity placed at $y_i(n)$ is then the cumulative demand between $x_{i-1}(n)$ and $x_i(n)$; $i = 1, \ldots, n$.] Let

$$C(\mathbf{x}(n), \mathbf{y}(n)) = \sum_{i=1}^{n} c(x_{i-1}(n), y_i(n), x_i(n)),$$

where

$$c(x_{i-1}(n), y_i(n), x_i(n)) = \int_{x_{i-1}(n)}^{y_i(n)} b(t)[R(t) - R(x_{i-1}(n))]\,dt + \int_{y_i(n)}^{x_i(n)} h(t)[R(x_i(n)) - R(t)]\,dt \tag{3}$$

is the backorder plus holding costs between the regeneration pointx $x_{i-1}(n)$

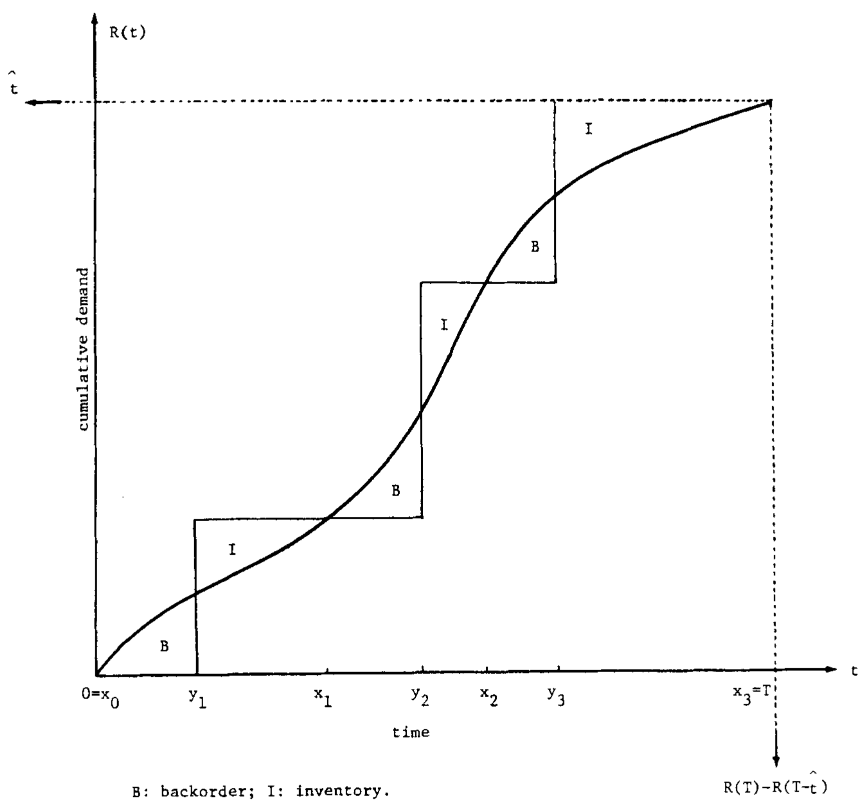

Figure 1. The inventory system with backorder ($n = 3$).

and $x_i(n)$. The optimization problem, here designated as problem P, is

$$P\colon \min[C(\mathbf{x}(n), \mathbf{y}(n)) + A(n)], \tag{4}$$

with $\mathbf{x}(n)$ and $\mathbf{y}(n)$ satisfying (1).

PROPERTIES OF THE OPTIMAL SOLUTION

The first-order necessary conditions for the optimal solution are $(\partial/\partial \mathbf{x}_i(n))C(\mathbf{x}(n), \mathbf{y}(n)) = 0$ for $i = 1, \ldots, n-1$, and $(\partial/\partial y_i(n))C(\mathbf{x}(n), \mathbf{y}(n) = 0$ for $i = 1, \ldots, n$. From (2) and (3), we then have

$$H(x_i(n)) + B(x_i(n)) = H(y_i(n)) + B(y_{i+1}(n)), \tag{5a}$$

$$R(y_i(n)) = R(x_i(n)) - [R(x_i(n)) - R(x_{i-1}(n))]\theta(y_i(n)), \tag{5b}$$

where

$$\theta(y_i(n)) = b(y_i(n))/[b(y_i(n)) + h(y_i(n))]. \tag{6}$$

Therefore, by letting $y_i(n)$ be implicitly defined through (5b) as a function of $x_{i-1}(n)$ and $x_i(n)$, we can suppress it in the cost function c, and write the latter simply as $c(x_{i-1}(n), x_i(n))$. Accordingly, we shall write $C(\mathbf{x}(n)) = \sum_{i=1}^{n} e(x_{i-1}(n), x_i(n))$. Now consider problem P':

$$P'\colon C(n) = \min_{\mathbf{x}(n) \in S_n} C(\mathbf{x}(n)), \tag{7}$$

where $S_n = \{\mathbf{x}(n) = (x_i(n))_{i=0}^{n}\colon 0 = x_0(n) \le x_1(n) \le \cdots \le x_n(n) = T\}$.

The following lemma is a paraphrase of Theorem 1 of Denardo et al.

LEMMA 1: Let $\mathbf{x}(n)$ and $\mathbf{x}(n+1)$ be the vectors that solve problem P' for n and $n+1$, respectively. If $c(\cdot, \cdot)$ is lower semicontinuous and submodular, then (i) there exist $\mathbf{x}(n)$ and $\mathbf{x}(n+1)$ that are interleaved, i.e., $x_i(n) \le x_{i+1}(n+1) \le x_{i+1}(n)$, for all $i = 0, 1, \ldots, n-1$; and (ii) $C(n)$ is convex in n [if in addition $c(x, x) \le 0$ for some x, then $C(n)$ is also decreasing in n].

THEOREM 1: The function $c(\cdot, \cdot)$ in the objective of P' satisfies the conditions of Lemma 1; hence the conclusions (i) and (ii) hold.

PROOF: (Here we omit the argument n.) The required lower semicontinuity of c in Lemma 1 is satisfied, since the function is continuous; also, notice that $c(x_{i-1}, x_i) = 0$ if $x_{i-1} = x_i$. It then remains to show that $c(x_{i-1}, x_i)$ is submodular; i.e., we want to show $(\partial^2/\partial x_{i-1}\partial x_i)c(x_{i-1}, x_i) \le 0$. In the following, R' denotes derivative. If x is a discontinuity of $r(t)$, then define $R'(x) = r(x+0)$. From (3), we have

$$(\partial^2 \partial x_{i-1}\partial x_i)c(x_{i-1}, x_i) = \{[b(y_i) + h(y_i)]R(y_i) - [b(y_i)R(x_{i-1}) + h(y_i)R(x_i)]\} \cdot (\partial y_i/\partial x_{i-1}) - R'(x_{i-1})[B(y_i) - B(x_{i-1})]. \tag{8}$$

Substituting (5b) and (6) into (8) yields

$$(\partial/\partial x_{i-1})c(x_{i-1}, x_i) = -R'(x_{i-1})[B(y_i) - B(x_{i-1})], \tag{9}$$

and hence

$$(\partial^2/\partial x_{i-1}\partial x_i)c(x_{i-1}, x_i) = -R'(x_{i-1})b(y_i)(\partial y_i/\partial x_i). \tag{10}$$

From (5a) we have

$$\partial y_i/\partial x_i = [h(x_i) + b(x_i)]/h(y_i). \tag{11}$$

Since both $h(\cdot)$ and $b(\cdot)$ are positive, we have $\partial y_i/\partial x_i > 0$, and $(\partial^2/\partial x_{i-1}\,\partial x_i)$ $c(x_{i-1}, x_i) < 0$ from (10).

Similar to the above proof, we can also establish the following:

COROLLARY 1.1: Let $\mathbf{y}(n)$ be the vector of order points that solve P' for n. Then $\mathbf{y}(n)$ and $\mathbf{y}(n+1)$ are interleaved.

Note that Corollary 1.1 above also follows directly from the "duality" result in Theorem 3 (Section 3), which reveals an interesting "dual" of the backorder model considered here.

THE ALGORITHM

The solution to problem P of (4) can be derived in two steps: first solve problem

$$P1\colon\ C(n) = \min C(\mathbf{x}(n), \mathbf{y}(n))$$

with $\mathbf{x}(n)$ and $\mathbf{y}(n)$ satisfying (1); and then solve problem

$$P2\colon\ \min[C(n) + A(n)].$$

From Theorem 1, we know that the objective function of $P2$ is convex. Therefore, problem P can be solved through the following simple procedure. Start from $n = 1$, increase n one unit at a time; for each n, solve $P1$ to obtain $C(n)$. Repeat this process until the decrease in $C(n)$ fails to offset the increase in $A(n)$. To solve $P1$, from (5a) and (5b) we have

$$x_i = R^{-1}\{R(x_{i+1}) - [R(x_{i+1}) - R(y_{i+1})]/\theta(y_{i+1})\} \qquad (i = n-1, \ldots, 1, 0), \tag{12}$$

$$y_i = H^{-1}\{H(x_i) + B(x_i) - B(y_{i+1})\} \qquad (i = n-1, \ldots, 2, 1), \tag{13}$$

where $R^{-1}\{\cdot\}$ and $H^{-1}\{\cdot\}$ denote inverse functions (recall that both R and H are strictly increasing). Therefore, for each n there is only one unknown, $z(n) \equiv y_n(n)$, to be derived; all other x_i's and y_i's can then be recursively determined through (12) and (13). In view of the interleaving property of the order points (Corollary 1.1), $z(n)$ can be derived by searching the interval $[z(n-1), T]$ (note that as n increases, the length of the search interval decreases). Initially (i.e., when $n = 1$), $z(1)$ can be derived from

$$R(z(1)) + R(T)\theta(z(1)) = R(T) \tag{14}$$

[cf. (5b)], through a bisection search on the interval $[0, T]$, for instance.

The algorithm can be summarized as follows.

Step 0. [Initialization]
Derive $z(1)$ from (14). Computer OBJ' $= C(1) + A(1)$.
Set $n = 2$, LB $= z(1)$, UB $= T$; set ϵ (error bound).

Step 1. [Solving P1]
Step 1.0. Set $z = (\text{LB} + \text{UB})/2$, $z' = z$; set $x_n = T$, $y_n = z$.
Step 1.1. For $i = n - 1$ to 0 (step -1), compute x_i and y_i following (12) and (13) (if $i = 0$, compute x_0 only).
If $x_0 > 0$ then UB $= z$; if $x_0 < 0$ then LB $= z$; else go to Step 2.
Set $z = (\text{LB} + \text{UB})/2$.
If $|z - z'| > \epsilon$ then $z' = z$, go to Step 1.1; else go to Step 2.

Step 2. [Solving $P2$]
Compute OBJ $= C(n) + A(n)$.
If OBJ $<$ OBJ' then set OBJ' $=$ OBJ, $n = n + 1$, LB $= z$, UB $= T$, go to Step 1.0; else output, end.

THEOREM 2: The above algorithm yields the optimal solution to problem P.

PROOF: It suffices to show that Step 1 of the algorithm yields the optimal solution to problem $P1$. Since $P1$ is to minimize a continuous function $C(\mathbf{x}(n), \mathbf{y}(n))$ on a compact set defined in (1), the minimum must be attained. Since both $b(t)$ and $h(t)$ are positive and finite, the minimum clearly cannot be located at the boundary. On the other hand, the recursive relations in (12) and (13) yield a unique solution to $\mathbf{x}(n)$ and $\mathbf{y}(n)$, which satisfy the necessary conditions in (5a) and (5b), since the functions involved, R, H, and B, are all strictly increasing. Therefore, Step 1 does generate the optimal solution to $P1$. □

For the no-backorder case, let $b(t) \to \infty$ in (5b) and (6) we then have $y_i = x_{i-1}$ for all $i = 1, \ldots, n$. Substituting these into (2) and (3), we have the following relation:

$$R'(x_i)[H(x_i) - H(x_{i-1})] = h(x_i)[R(x_{i+1}) - R(x_i)]; \tag{15}$$

hence, (12) should be replaced by

$$x_i = H^{-1}\{H(x_{i+1}) - h(x_{i+1})[R(x_{i+2}) - R(x_{i+1})]/R'(x_{i+1})\}, \tag{16}$$

for $i = n - 2, \ldots, 1, 0$. For each n, $x_{n+1}(n) \equiv T$; also, let $z(n) = x_n(n)$, then the other x_i's can be recursively obtained through (16) and Step 1 of the algorithm searches the solution to $z(n)$.

Before we close this section, we point out the following fact. Suppose we denote the lot-size problem under discussion by a four-tuple, $\{T, r(t), b(t), h(t)\}$, where the four components denote, respectively, the planning horizon, the demand rate, the backorder cost rate and the inventory holding cost rate. Let $(n, \mathbf{x}, \mathbf{y})$ be the optimal solution to the problem, with n, $\mathbf{x}$, and $\mathbf{y}$ denoting, respectively, the number of orders, the regeneration and the order points. The following result is readily verified from (5a) and (5b).

THEOREM 3: Suppose $(n, \mathbf{x}, \mathbf{y})$ is the optimal solution to problem P: $\{T, r(t), b(t), h(t)\}$, then the optimal solution to problem D: $\{T, r(T-t), h(T-t), b(T-t)\}$ is $(n, \mathbf{u}, \mathbf{v})$, with $u_i = T - x_{n-i}$ $(i = 0, 1, \ldots, n)$ and $v_i = T - y_{n+1-i}$ $(i = 1, \ldots, n)$ being its regeneration and order points.

Note that the optimal solution P remains optimal for D: the only change is to reverse time and the numbering of the regeneration and the order points (see Figure 1). What Theorem 3 says is that the effect of time reversal is equivalent to the effect of exchanging the backorder and the holding cost rates.

INCREASING/DECREASING DEMAND RATES

Here we are concerned with the monotonicity of the optimal lot sizes and order intervals when the demand rate is increasing or decreasing and the cost rates are decreasing or increasing.

THEOREM 4: Suppose that the ratio $b(t)/h(t) \equiv \rho$, where ρ is a constant, and for a given number of orders n, the regeneration and the order points are optimally placed. Then the order quantity, $Q_i = R(x_i) - R(x_{i-1})$, of the ith order (placed at y_i) is increasing (decreasing) in i $(i = 1, \ldots, n)$, if the demand rate $r(t)$ is increasing (decreasing) over time, and the inventory holding cost rate $h(t)$ [and hence the backorder cost rate $b(t)$] is decreasing (increasing) over time.

PROOF: Suppose that $r(t)$ is decreasing and $h(t)$ is increasing [then $R(t)$ is concave and $H(t)$ is convex]. First, consider $n = 2$. We want to show that $Q_1 = R(x_1) \geq Q_2 = R(T) - R(x_1)$, or $R(x_1) \geq R(T)/2$. Since $b(t) = \rho h(t)$, we have $B(t) = \rho H(t)$, and hence from (13),

$$H(x_1) = [H(y_1) + \rho H(y_2)]/(1+\rho) \geq H[(y_1 + \rho y_2)/(1+\rho)], \tag{17}$$

where the inequality above is due to the convexity of $H(t)$. Since both $H(t)$ and $R(t)$ are increasing, and $R(t)$ is concave, we have

$$R(x_1) \geq R[(y_1 + \rho y_2)/(1+\rho)] \geq [R(y_1) + \rho R(y_2)]/(1+\rho). \tag{18}$$

From (5b) we have

$$R(y_1) = R(x_1)/(1+\rho) \quad \text{and} \quad R(y_2) = [R(T) + \rho R(x_1)]/(1+\rho). \tag{19}$$

Substituting (19) into (18) and collecting terms yields

$$[(1+\rho)^2 - (1+\rho^2)]R(x_1) \geq \rho R(T), \tag{20}$$

which further simplifies to the desired result, $R(x_1) \geq R(T)/2$. The proof for $n > 2$ is then completed through induction and use of the regeneration property of an optimal solution.

The case of $r(t)$ increasing and $h(t)$ decreasing can be proved similarly. It can also be proved directly through considering the time reversal problem and using Theorem 3 [in the time reversal problem, the demand rate $r(T-t)$ is decreasing and the holding cost rate $b(T-t)$ increasing].

COROLLARY 4.1: Under the conditions of Theorem 4, the order interval $d_i = y_{i+1} - y_i$, is decreasing (increasing) in i $(i = 1, \ldots, n-1)$, if the demand rate $r(t)$ is increasing (decreasing) and the holding cost rate $h(t)$ decreasing (increasing) over time.

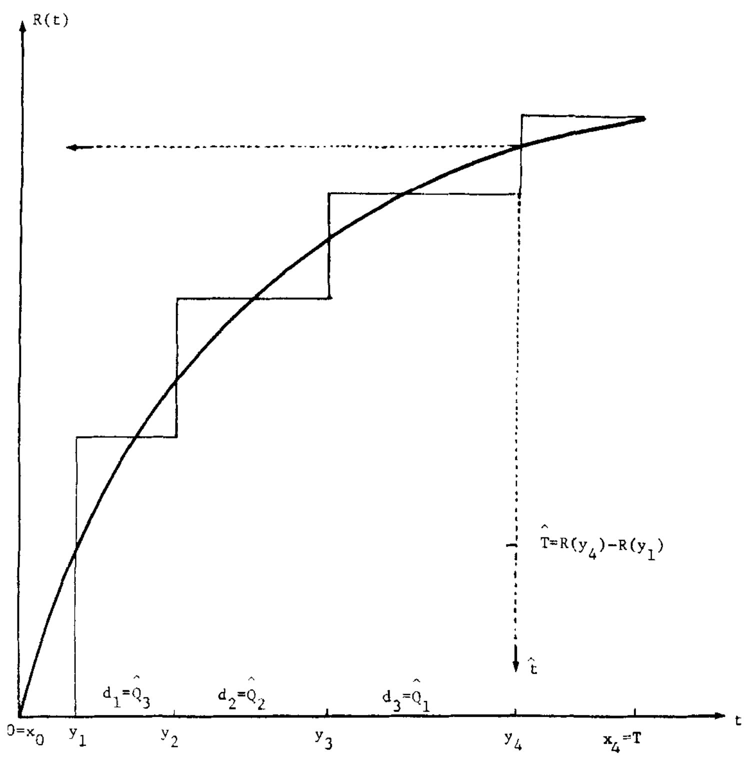

Figure 2. Proof of Corollary 4.1 ($n = 4$).

Table 1. Solutions to the numerical example.

n		$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$C(n)+A(n)$
	x_i	2.288	3.719	5.000				
3	y_i	1.168	2.873	4.220				58.535
	Q_i	0.886	3.135	10.721				
	d_i	1.706	1.346					
	gi	1.810	3.018	4.018	5.000			
4	y_i	0.823	2.272	3.375	4.367			51.170
	Q_i	0.511	1.434	3.515	9.283			
	d_i	1.448	1.104	0.991				
	x_i	1.502	2.556	3.419	4.201	5.000		
5	y_i	0.627	1.885	2.851	3.678	4.468		49.914
	Q_i	0.349	0.840	1.765	3.618	8.169		
	d_i	1.258	0.965	0.828	0.789			
	x_i	1.286	2.225	2.996	3.678	4.324	5.000	
6	y_i	0.504	1.615	2.480	3.214	3.882	4.540	51.987
	Q_i	0.262	0.564	1.075	1.955	3.596	7.290	
	d_i	1.111	0.865	0.734	0.668	0.658		

PROOF: We only give a proof for $r(t)$ decreasing and $h(t)$ increasing. Consider the following related problem (see Figure 2). Let the point $(y_n, R(y_n))$ be the new origin, and let $\hat{T} = R(y_n) - R(y_1)$ be the new planning horizon. Then, clearly the optimal order points of the original problem become the optimal regeneration points of this new problem (subject to a renumbering of the points); otherwise the optimal solution to the original problem could have been improved upon. Hence, $\hat{Q}_i = d_{n-i}$, where $\hat{Q}_i$ denotes the order quantity of the new problem ($i = 1, \ldots, n-1$). Since in the new problem the demand rate is also decreasing and the holding cost rate increasing (which are easily verified), from Theorem 4, we know that $\hat{Q}_i$ is decreasing in i, and hence d_i increasing in i. The proof for $r(t)$ increasing and $h(t)$ decreasing can be obtained in a similar fashion. □

In the no-backorder case, replace (17) by the following, which is from (15),

$$R'(x_1)H(x_1) = h(x_1)[R(T) - R(x_1)]. \tag{21}$$

Since R is concave and H convex, we have $x_1R'(x_1) \le R(x_1)$ and $x_1h(x_1) \ge H(x_1)$. Substituting these into (21), we have

$$R(T) - R(x_1) = R'(x_1)H(x_1)/h(x_1) \le x_1R'(x_1) \le R(x_1); \tag{22}$$

that is, $R(x_1) \ge R(T)/2$.

A NUMERICAL EXAMPLE

As a numerical example, consider a problem with the following data: the planning horizon is $T = 5$, the demand rate $r(t) = 0.01e^t$, the holding cost rate

$h(t) = (t - T - 1)^2$, the backorder cost rate $b(t) = 3h(t)$, and the order cost function $A(n) = 2n^{1.5}$. Using the algorithm of Section 3, we obtain the optimal number of orderings $n = 5$, and summarize the optimal order points, regeneration points, order quantities and order intervals in Table 1, where results corresponding to $n = 3$, 4 and 6 are also listed for comparison. From these results, the interleaving property of the order and the regeneration points is easily observed. Also, note that in this example we have increasing demand rate and decreasing cost rates, which satisfy the conditions of Theorem 4. Hence, the order quantities are increasing and the order intervals decreasing, as is evident from Table 1.

ACKNOWLEDGEMENT

David Yao is supported in part by NSF under grant DMC-850396 and by ONR under contract N00014-84-K-0465.

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Received September 19, 1986
Revised manuscript received September 1988
Accepted September 27, 1988