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ORIGINAL-COMPARATIVE COMPUTATIONAL STUDY

Alternate second order conic program reformulations for hub location under stochastic demand and congestion

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Abstract
In this paper, we study the single allocation hub location problem with capacity selection in the presence of congestion at hubs. Accounting for congestion at hubs leads to a non-linear mixed integer program, for which we propose 18 alternate mixed integer second order conic program (MISOCP) based reformulations. Based on our computational studies, we identify the best MISOCP-based reformulation, which turns out to be 20–60 times faster than the state-of-the-art. Using the best MISOCP-based reformulation, we are able to exactly solve instances up to 50 nodes in less than half-an-hour. We also theoretically examine the dimensionality of the second order cones associated with different formulations, based on which their computational performances can be predicted. Our computational results corroborate our theoretical findings. Such insights can be helpful in the generation of efficient MISOCPs for similar classes of problems.

Keywords Hub-and-spoke network · Congestion · Capacity selection · Stochastic demand · Single allocation · Second order conic programming

1 Introduction

Hub-and-spoke is a widely studied network structure, which finds applications in supply chain networks, airline networks, telecommunications, postal deliveries, etc. Several studies have established that the hub-and-spoke network model offers increased profitability (Toh and Higgins 1985), cost savings (McShan and Windle 1989), and competitive advantage (Oum et al. 1995; Bania et al. 1998; Martín and Román 2003). The key idea behind a hub-and-spoke network is to route all the flows through intermediate facilities, called hubs, where they are aggregated before being sent to their respective destinations. Hubs serve as centres to collect, sort, break-bulk or switch modes of travel while transferring flows. The main cost advantage in a hub-and-spoke network comes from the economies of scale in

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inter-hub transfers achieved due to aggregation of flows. After the seminal work by O'kelly (1986), hub-and-spoke network designs have gained significant attention from a modelling perspective. Review papers by Klincewicz (1998), Alumur and Kara (2008), Campbell and O'Kelly (2012) and Farahani et al. (2013) discuss the basics and the various extensions of a hub-and-spoke network.

A major drawback faced by hub-and-spoke networks is the issue of congestion. Since hubs deal in large volumes of goods, any variation in the demand or service rate leads to congestion at the hubs that adversely affects the service quality (Elhedhli and Hu 2005; Elhedhli and Wu 2010). One of the ways often sought to alleviate congestion for an existing network is capacity expansion at the hubs. However, expansion of capacities are often expensive infrastructure projects that may even be infeasible in some cases. Hence, including capacity decisions while considering congestion issues during the design phase of the network is often a better alternative.

In this paper, we study a hub location problem with capacity selection in the presence of congestion. This includes finding the location and capacities of the hubs through which all the flows (between origin-destination pairs) shall be routed at the minimum cost. In light of the expected congestion, the model also considers the trade-off between installing a higher service capacity at the hub, hence minimizing the effect of congestion, or installing a lower capacity, and bearing the high cost due to congestion.

We model the hub-and-spoke network as spatially distributed M/G/1 queues, whose locations and capacities need to be selected in order to minimize the total cost. M/G/1 queues are single server queuing models where the arrival of items (customers, packages etc.) is assumed to follow a Poisson distribution under a first-come-first-serve queue discipline and the service times are assumed to follow a general distribution such that these uncertainties result in queue formation at the hubs. The total cost for this problem consists of the capacity installation cost, the transportation cost, and the congestion cost.

The congestion term introduces non-linearity in the objective function, which makes the resulting hub location problem with capacity selection under congestion a mixed integer non linear program (MINLP). HLPs (Hub location problems), even without capacity selection decision and congestion, is known to be NP-hard (Kara and Tansel 2000). Due to the strategic nature of the problem, there has been significant effort on finding the exact solutions for HLPs including the work of Ernst and Krishnamoorthy (1998), de Camargo et al. (2008), Contreras et al. (2011), Contreras et al. (2012) among many others. Moreover, capacity selection decision along with the non-linearity introduced due to congestion makes the problem even more challenging. The objective of this paper is to solve the resulting problem exactly and efficiently. To this end, we present several alternate MISOCP-based reformulations of the problem, which are solved directly using the state-of-the-art solvers, and compare their computational performances against the Mixed Integer Linear Programming (MILP) based reformulations, obtained using Outer-Approximations (OA).

Through this paper, we make the following contributions to the literature on hub location problems. First, we propose two new MINLP-based formulations for the hub location problem with capacity selection under congestion. Our new formulations are built on the basic model (without capacity selection and congestion) proposed by Ernst and Krishnamoorthy (1996). We refer to these models as EK-based models. We compare our proposed formulations with two other MINLP-based formulations from the literature (Elhedhli and Wu 2010; Azizi et al. 2018), which are based on the well-known model proposed by Skorin-Kapov et al. (1996). We refer to these models as SK-based models. We subsequently show, through computational experiments, that the models proposed by us significantly outperform the latter two formulations. Second, we present nine different MISOCP-based reformulations for each of the SK-based and EK-based MINLPs. From our extensive computational experiments using two of the well-known datasets, namely the Civil Aeronautics Board (CAB) dataset and the Australian Post (AP) dataset, we suggest the overall best formulation of the problem. The best reformulation, which is one of the EK-based MISOCPs, solves the problem 20–60 times faster as compared to the existing formulation/method in the literature. We further provide insights about the reformulations based on the computational results and the properties of the second order cones. These insights should be useful as a general guideline for the selection of a given MISOCP from among several alternatives.

The rest of the paper is organized as follows. In Sect. 2, we present a review of the literature on the hub location problem and its variants. The problem description, followed by its different MINLP-based formulations are presented in Sect. 3. In Sect. 4, we present our alternate MISOCP-based reformulations of the problem, followed by extensive computational results in Sect. 5. Finally, the conclusions and directions for future research are presented in Sect. 6.

2 Literature review

HLPs have been broadly categorized as p-hub median, hub location with fixed costs, p-hub center and hub covering problems depending upon (i) whether the number of hubs, p, are intrinsically determined (hub covering) or are passed as extrinsic arguments (p-hub median or center), or (ii) whether the objective function is to minimize the transportation cost (p-hub median), minimize the maximum transportation cost between any two pairs (p-hub center), or maximize the coverage of the demand (hub covering). Another classification is based on the allocation decisions of the non-hub nodes to the hub nodes, which can be single or multiple. *Single allocation* refers to the case where all the flows originating at a given node always traverses through the same first hub, irrespective of its final destination. Single allocation hub location problems are relevant to situations in which sorting at the source is not possible (or too costly), so that all shipments are transported from the origin as a whole to the allocated hub, as is typically done in postal or parcel networks (Meier and Clausen 2017). The alternate to this strategy is *multiple allocation* where flows from a node can be routed via multiple immediate next hubs.

By their very own design, hubs face huge influx of demand. Uncapacitated HLPs often result in an unbalanced distribution of flows with a few hubs that receive very large volumes while others remain less utilized. Capacitated version of the problem addresses this issue by limiting the flows to the hubs through capacity constraints for both single allocation (Aykin 1994; Ernst and Krishnamoorthy 1999; da Graça et al. 2008; Contreras et al. 2009) and multiple allocation models (Ebery et al. 2000; Boland et al. 2004; Rodríguez-Martín and Salazar-González 2008). However, despite offering a balanced flow in the network, capacity restrictions do not account for the exponential increase in service delay at the hubs when their incoming flows hit their capacity limits, especially in the presence of variability in the service demand. Congestion-based models explicitly account for such exponential increase in the delays while designing the hub-and-spoke network.

Table 1 summarizes the literature on congestion-based hub-and-spoke network design. For the main classification, we divide the literature based on: (i) congestion penalty imposed in the constraint (Cons.) or in the objective function (Obj.); (ii) modelling congestion as either a power-law function that is convex and increasing in the total flow at a hub or as a queuing based system where hubs are modeled as queues; (iii) assuming hubs as uncapacitated

Table 1 Literature review

	Model											Method	
	Modelling congestion				Capacity			Allocation		Formulation		Exact	Heuristics
	Power-law	Queue	Obj.	Cons.	UnCap.	Endo.	Exo.	Single	Multiple	SK-based	EK-based		
Marianov et al. (1999)		M/D/c		✓	✓				✓	✓			✓
Jayaswal and Vidyarthi (2013)		M/M/1		✓		✓			✓	✓		✓	
Elhedhli and Hu (2005)	✓		✓		✓			✓		✓			✓
de Camargo et al. (2009b)	✓		✓		✓				✓	✓		✓	
Elhedhli and Wu (2010)		M/M/1	✓			✓		✓		✓			✓
de Camargo et al. (2011)	✓	M/M/1	✓		✓				✓	✓		✓	
Kian and Kargar (2016)	✓				✓			✓			✓	✓	
Azizi et al. (2018)		M/G/1	✓			✓		✓		✓		✓	✓
Alkaabneh et al. (2019)	✓		✓					✓		✓			✓
Hasanzadeh et al. (2018)		M/M/c	✓				✓	✓		✓			✓
This work		M/G/1	✓			✓		✓		✓	✓	✓	

(UnCap.), capacitated with capacities being exogenous (Exo.) to the model or as a decision variable (Endo.); (iv) the allocation scheme, single or multiple, and (v) the formulation as either a SK-based formulation or as an EK-based formulation. We refer to a formulation as a SK-based formulation if a four-subscripted path based variable defined as x_{ijkm} is used to model the flows going from i to j through hubs k and m, in that order. An EK-based formulation is defined as one where a three-subscripted flow based variable x_{ikm} is used to model flows originating from i is going through hubs k and m, in that order.

Congestion penalties were first imposed in the constraint by Marianov and Serra (2003) who addressed the issue of service delays by assuming peak hour arrivals and departures at airport hubs leading to M/D/c queues. They imposed a probabilistic constraint on the number of planes waiting for landing, and used Tabu search heuristic as a solution method. Jayaswal and Vidyarthi (2013) proposed an extension of the above model by considering heterogeneity among customers, thereby imposing a different service level constraint for each customer class. Most of the other papers on congestion-based hub-and-spoke network design use a penalty for congestion in the objective function, as opposed to imposing a strict limit on delays at the hubs through constraints [for example, see Elhedhli and Hu (2005)].

Congestion in the hub-and-spoke literature has mostly been modelled in the objective function using the following two forms: (i) power-law function; (ii) queuing based function. When capacities at the hubs are sufficiently high such that they can be treated as uncapacitated systems, a power-law function is commonly used to model congestion at the hubs. Power-law functions capture the non-linear increase in congestion as flows pass through a hub. Elhedhli and Hu (2005), Kian and Kargar (2016) and Alkaabneh et al. (2019) use a power-law function in a hub-and-spoke network under congestion for single allocation based models, whereas de Camargo et al. (2009a) use a similar function for a multiple allocation based model. The solution methods for the power-law based congestion function includes piece-wise linearization of the congestion function along with Lagrangian heuristic (Elhedhli and Hu 2005), second order conic programming based reformulations strengthened with perspective cuts (Kian and Kargar 2016), and generalized Benders decomposition (de Camargo et al. 2009a). Another approach to model congestion is to account for uncertainties related to the arrivals and transit times at the hubs by modelling the hubs as queues. Queuing based congestion functions also consider the relative difference between the demand and the capacity at the hubs such that the exponential behaviour is realized when the demand approaches the capacity limits (Elhedhli and Wu 2010). Solution methods for the queuing based congestion function include cutting plane method with Lagrangean heuristics (Elhedhli and Wu 2010; Azizi et al. 2018), genetic algorithm based heuristics (Azizi et al. 2018; Hasanzadeh et al. 2018), and hybrid of outer approximation and Benders decomposition (de Camargo et al. 2011).

In this work, we use the more general M/G/1 system as it also includes the cases for M/M/1 and M/D/1 queues. We also consider a single server system because multiple server systems can be modelled as single server system: (i) if the utilization levels are extremely high (as is true in the case of hubs) such that t parallel servers with capacity levels as μ behave as a single server system with capacity level as $t\mu$, (ii) if the server can be modelled as a flexible capacity system where the capacity (or service rate) can be varied in continuous or discrete steps (Elhedhli and Wu 2010; Jayaswal and Vidyarthi 2013; Azizi et al. 2018). Note that by considering capacities as discrete choices, additional computational difficulties are introduced which is exactly our case, and (iii) single server system leads to cleaner analytical results that is of much practical significance to the planner (Dan and Marcotte 2019).

From the literature review it can be seen that the solution methods for more complex queuing systems that include multiple servers have largely remained heuristic (Marianov and Serra 2003; Alkaabneh et al. 2019). However, as already stated, modelling hubs as

M/G/1 queues with discrete flexible capacity levels is a reasonable approximation for a more complex queuing system. Hence, our main contribution lies in proposing efficient second order conic programming based reformulation for the existing SK-based formulation and the new EK-based formulation proposed in this work for the problem on hub location with capacity selection and congestion.

3 Model formulation

Consider a complete graph $G = (N, A)$, where $N = \{1, \ldots, |N|\}$ represents the set of nodes corresponding to origins/destinations for some traffic/flow W_{ij} between each pair of origin $i \in N$ and destination $j \in N$. It is required that flows must be transported at minimum cost. Since a direct link between each origin-destination (O-D) pair increases the cost in the network, a subset of nodes $H \in N$ is chosen as hubs through which all the flows are routed. We assume that all the flows originating at a given node always traverses through the same first hub, irrespective of its final destination, which makes it a single allocation problem.

To encourage flows between hubs, a discount, due to economies of scale, is offered for inter-hub transfers, which helps reduce the overall cost. However, high flows and variability in demand and service rates result in congestion at hubs, which affects its service quality. Poor service quality is modelled as increase in cost for every extra unit of flow entering the hub. One way to address this issue is to install higher capacities that can accommodate larger flows and reduce congestion. Since there is a fixed cost for every capacity level, a trade-off exists between the congestion cost and the capacity installation cost. Therefore, while locating hubs, a decision on capacity level also needs to be considered with an overall objective to minimize the sum total of the flow costs, the hub setup costs, and the congestion costs. We refer to the resulting problem as single allocation hub location problem with multiple capacity levels and congestion (SHLPCC).

We use the following standard modelling assumptions: (i) distances follow triangle inequality; (ii) discount is offered only on inter-hub flows; (iii) inter-hub discount is the same for every pair of hubs; (iv) inter-hub discount is constant, independent of the volume of flows; (v) arcs do not have any capacity restrictions, and do not require any set up cost; (vi) demand originates at the nodes according to a Poisson process, and service times at the hubs follow a general distribution.

Sets and indices

i	: Index for origin nodes, $i \in N$;
j	: Index for destination nodes, $j \in N$;
k, m	: Index for potential hubs, $k, m \in N$;
l	: Index for capacity levels, $l \in L$;

Parameters

p	: Number of hubs to be opened;
W_{ij}	: Demand rate from origin i to destination j;
d_{ij}	: Distance between nodes;
χ	: Collection cost - cost per unit flow per unit distance from non-hub node to hub node;
δ	: Distribution cost - cost per unit flow per unit distance from hub node to non hub node;
α	: Transfer cost/Discount - cost per unit flow per unit distance between two hubs. $0 \leq \alpha \leq 1, \alpha < \chi, \alpha < \delta$;

γ_k^l	: Capacity (service rate) at hub k with capacity level l;
c_{kl}	: Coefficient of variation of service times of a hub k with capacity levels l;
Q_k^l	: Fixed cost for installing hub at node k with capacity level l;
$O_i = \sum_j W_{ij}$	: Total flow originating at i;
$D_i = \sum_j W_{ji}$	: Total flow reaching i;
λ_k	: Total incoming flow at hub k due to collection;
θ	: Congestion cost per unit user at hub k;

The assumption of arrivals following a Poisson process with rate W_{ij} and of service times following a general distribution results in the following expression, using the Pollaczek-Khintchine formula, for the expected number of units of flow ($E[N_{kl}]$) at hub k with selected capacity level l. Note that the total incoming flow, λ_k, at hub k also follows a Poisson distribution (due to the superposition property of Poisson distributions).

$$E[N_{kl}] = \left(\frac{1+c_{kl}^2}{2}\right)\frac{\lambda_k^2}{\gamma_k^l(\gamma_k^l - \lambda_k)} + \frac{\lambda_k}{\gamma_k^l} = \frac{1}{2}\left\{\left(1+c_{kl}^2\right)\frac{\lambda_k}{\left(\gamma_k^l - \lambda_k\right)} + \left(1-c_{kl}^2\right)\frac{\lambda_k}{\gamma_k^l}\right\} \tag{1}$$

We use two different ways to assign capacity levels to hubs, similar to Correia et al. (2011), for Ernst and Krishnamoorthy (1999) based model. In the first scheme, we use one set of binary variables for the assignment of non hub nodes to hubs, and another set for the assignment of capacity levels to hubs. We refer to this scheme as two-subscripted capacity allocation variable scheme using which the complete MINLP formulation is presented in Sect. 3.1. In the second scheme, we use a single set of three-subscripted variables for assignment of non hub nodes to hubs, along with the selection of capacity levels. We refer to this scheme as three-subscripted capacity allocation variable scheme using which the complete MINLP formulation is described in Sect. 3.2.

3.1 SHLPCC model based on two-subscripted capacity allocation variable

For SHLPCC with two-subscripted capacity allocation variable for EK-based models, we define the following decision variables.

$$x_{ikm} = \text{Amount of flow originating at } i \text{ and flowing through hubs } k \text{ and } m, \text{ in that order.}$$

$$z_{ik} = \begin{cases} 1, & \text{if node } i \text{ is assigned to hub } k \\ 0, & \text{otherwise.} \end{cases}$$

$$y_{kl} = \begin{cases} 1, & \text{if hub } k \text{ is assigned capacity level } l \\ 0, & \text{otherwise.} \end{cases}$$

Note, z_{kk} assigns a hub k to itself, thus eliminating the use of a separate variable for locating hubs. A hub can have only one capacity level hence $\sum_l y_{kl} = 1$. Therefore, the service rate (μ_k) and the coefficient of variation of service times (c_k), which depend on capacity level, are related to y_{kl} as follows

$$\mu_k = \sum_l \gamma_k^l y_{kl}, \qquad c_k^2 = \sum_l c_{kl}^2 y_{kl} \qquad \forall k \in N.$$

The expected flow at hub k is given by

$$\lambda_k = \sum_i \sum_m x_{ikm} = \sum_i O_i z_{ik} \qquad \forall k \in N.$$

Note that λ_k in the above expression captures only the flows directly entering hub k from the origin nodes, but not the flows entering hub k via some other hub. This is appropriate in situations where the flows require processing (e.g., collecting, sorting, batching, etc.) only at the first hub in their path from their origin to their destination, but do not require further processing at the second hub (Ebery et al. 2000; Jayaswal and Vidyarthi 2013). However, in situations where the flows need further processing before distribution, the expected flow at a hub k is given by

$$\lambda_k = \sum_i \sum_m x_{ikm} + \sum_i \sum_{m \neq k} x_{imk}.$$

Here, the first term includes the flows for which node k is the first hub, while the second term includes the flows that are routed through some other hub m before entering hub k (Marín 2005; de Camargo et al. 2009a).

Substituting λ_k, μ_k and c_k in Eq. (1) with the appropriate decision variables z_{ik} and y_{kl}, the congestion term modifies to:

$$E[N_k(y,z)] = \frac{1}{2}\left\{\left(1+\sum_l c_{kl}^2 y_{kl}\right)\frac{\sum_i \sum_j W_{ij} z_{ik}}{\left(\sum_l \gamma_k^l y_{kl} - \sum_i \sum_j W_{ij} z_{ik}\right)} + \left(1-\sum_l c_{kl}^2 y_{kl}\right)\frac{\sum_i \sum_j W_{ij} z_{ik}}{\sum_l \gamma_k^l y_{kl}}\right\}$$

The SHLPCC formulation with two-subscripted capacity allocation variables is given as:

$$\min \sum_i \sum_k d_{ik}(\chi O_i + \delta D_i) z_{ik} + \sum_i \sum_k \sum_m \alpha d_{km} x_{ikm} + \sum_k \sum_l Q_k^l y_{kl} + \sum_k \theta E[N_k(y,z)] \tag{2}$$

$$\text{s.t.} \sum_k z_{ik} = 1 \qquad \forall i \in N \tag{3}$$

$$z_{ik} \leq z_{kk} \qquad \forall i, k \in N \tag{4}$$

$$\sum_k z_{kk} = p \tag{5}$$

$$\sum_m x_{ikm} - \sum_m x_{imk} = O_i z_{ik} - \sum_j W_{ij} z_{jk} \qquad \forall i, k \in N \tag{6}$$

$$\sum_m x_{ikm} \leq O_i z_{ik} \qquad \forall i, k \in N \tag{7}$$

$$\sum_i \sum_j W_{ij} z_{ik} \leq \sum_l \gamma_k^l y_{kl} \qquad \forall k \in N \tag{8}$$

$$\sum_l y_{kl} = z_{kk} \qquad \forall k \in N \tag{9}$$

$$y_{kl} \in \{0, 1\} \qquad \forall k \in N, l \in L \qquad (10)$$

$$x_{ikm} \geq 0 \qquad \forall i, k, m \in N \qquad (11)$$

$$z_{ik} \in \{0, 1\} \qquad \forall i, k \in N \qquad (12)$$

The first component of the objective function (2) includes the cost of transferring flows from non hub nodes to hub nodes. The second component is the cost of inter-hub transfer, and the third component is the cost of locating hubs at certain capacity levels. Constraint set (3) ensures single-allocation of non hub nodes to hubs. Constraint set (4) prevents assignment of a node to another node unless that latter is a hub. Constraint (5) is to enforce the selection of only p-hubs, and constraint set (6) represents the flow balance constraints. Constraint set (7) prevents any traffic originating at node i from flowing via hub k unless the node i is allocated to hub k. Constraint set (8) ensures that the total flow at hub k does not exceed its installed capacity, which is required for the stability of the queueing system at the open hubs. Note that the queue stability constraint ideally requires a strict inequality in the constraint set (8). However, we exploit the knowledge that the constraint set (8) can never be binding at optimality (since that will make the congestion term in the objective function tend to infinity, which can never be optimal), and retain the $\leq$ sign in constraint set (8). Constraint set (9) ensures that a hub k, given by variable z_{kk}, can have only one capacity allocation.

3.1.1 Partial linearization

Equations (2)–(12) is a MINLP, which is challenging to solve. Next, we suggest its partial linearization to convert it into a form with a linear objective function and one set of non-linear constraints, which can be further reformulated using outer-approximation and MISOCPs. To this end, we introduce ρ_k and s_k as additional sets of variables, which are defined using the following relations:

$$\rho_k = \frac{\lambda_k}{\mu_k} = \frac{\sum_i \sum_j W_{ij} z_{ik}}{\sum_l \gamma_k^l y_{kl}} \qquad \forall k \in N$$

$$s_k = \frac{\sum_i \sum_j W_{ij} z_{ik}}{\left(\sum_l \gamma_k^l y_{kl} - \sum_i \sum_j W_{ij} z_{ik}\right)} \qquad \forall k \in N \qquad (13)$$

$$0 \leq \rho_k \leq 1, s_k \geq 0 \qquad \forall k \in N. \qquad (14)$$

$E[N_k(y, z)]$ in (2) can be rewritten as:

$$E[N_k(y)] = 1/2\left\{s_k + \rho_k + \sum_l c_{kl}^2 y_{kl}\left(s_k - \rho_k\right)\right\}.$$

To linearize the above expression, we introduce auxiliary variables L_{kl} and V_{kl} $\forall k, l$, defined as $L_{kl} = \rho_k y_{kl}$ and $V_{kl} = s_k y_{kl}$. $E[N_k(y)]$ can be further restated as:

$$E[N_k] = 1/2\left\{s_k + \rho_k + \sum_l c_{kl}^2\left(V_{kl} - L_{kl}\right)\right\},$$

where L_{kl} and V_{kl} are non-linear, for which we use the following standard linearization technique using Big-M method, where M is a large value:

$$\sum_l V_{kl} = s_k \qquad \forall k \in N \qquad (15)$$

$$V_{kl} \leq M y_{kl} \qquad \forall k \in N, l \in L \tag{16}$$
$$V_{kl} \geq 0 \qquad \forall k \in N, l \in L \tag{17}$$
$$\sum_l L_{kl} = \rho_k \qquad \forall k \in N \tag{18}$$
$$L_{kl} \leq y_{kl} \qquad \forall k \in N, l \in L \tag{19}$$
$$0 \leq L_{kl} \leq 1 \qquad \forall k \in N, l \in L. \tag{20}$$

Since $\rho_k = \frac{\lambda_k}{\mu_k} = \frac{\sum_i \sum_j W_{ij} z_{ik}}{\sum_l \gamma_k^l y_{kl}}$, we have:

$$\sum_i \sum_j W_{ij} z_{ik} = \rho_k \sum_l \gamma_k^l y_{kl} = \sum_l \gamma_k^l y_{kl} \rho_k = \sum_l \gamma_k^l L_{kl} \quad \forall k \in N. \tag{21}$$

Using the above transformations, Eqs. (2)–(12) can be written as follows:

$$\textbf{[EK-2s]}\min \sum_i \sum_k d_{ik}(\chi O_i + \delta D_i) z_{ik} + \sum_i \sum_k \sum_m \alpha d_{km} x_{ikm}$$
$$+ \sum_k \sum_l Q_k^l y_{kl} + \theta/2 \sum_k \left\{ s_k + \rho_k + \sum_l c_{kl}^2 \left(V_{kl} - L_{kl}\right) \right\}$$
$$\text{s.t.}(3)-(21).$$

The only non-linearity in EK-2s appears in (13), which can be handled using the well-known OA-based method, as detailed in "Appendix B.1". We refer to this OA-based method for the two-subscripted capacity allocation based variable as EK-OA-2s. In Sect. 4.1, we propose five alternate MISOCP-based reformulations of EK-2s.

3.2 SHLPCC model based on three-subscripted capacity allocation variable

For SHLPCC with three-subscripted capacity allocation variable for the EK-based model, we define the following decision variables.

x_{ikm} = Amount of flow with origin at i that goes through hubs k and m

$$t_{ik}^l = \begin{cases} 1, & \text{if node } i \text{ is assigned to hub } k \text{ which has capacity level } l. \\ 0, & \text{otherwise.} \end{cases}$$

The new variable t_{ik}^l is related to the previous variables y_{kl} and z_{ik} as follows:

$$z_{ik} = \sum_l t_{ik}^l \quad \forall i, k. \qquad y_{kl} = t_{kk}^l \quad \forall k, l \qquad z_{kk} = \sum_l t_{kk}^l \tag{22}$$

Since we do not have variable y_{kl}, as in the case with two-subscripted capacity based allocation scheme, we do not encounter complexities arising from $\gamma_k^l y_{kl}$. Therefore, we work with the following two forms of the expected number of users, $E[N_{kl}(t)]$.

$$E[N_{kl}(t)] = \frac{(1 + c_{kl}^2)(\sum_i \sum_j W_{ij} t_{ik}^l)^2}{2(\gamma_k^l)(\gamma_k^l - \sum_i \sum_j W_{ij} t_{ik}^l)} + \frac{\sum_i \sum_j W_{ij} t_{ik}^l}{\gamma_k^l} \tag{23}$$

$$E[N_{kl}(t)] = 1/2 \left\{ \left(1 + c_{kl}^2\right) \frac{\sum_i \sum_j W_{ij} t_{ik}^l}{\left(\gamma_k^l - \sum_i \sum_j W_{ij} t_{ik}^l\right)} + \left(1 - c_{kl}^2\right) \frac{\sum_i \sum_j W_{ij} t_{ik}^l}{\gamma_k^l} \right\} \tag{24}$$

The SHLPCC formulation with three-subscripted capacity allocation variable is given as:

[EK-3s]

$$\min \sum_i \sum_k d_{ik}(\chi O_i + \delta D_i) \sum_l t_{ik}^l + \sum_i \sum_k \sum_m \alpha d_{km} x_{ikm} + \sum_k \sum_l Q_k^l t_{kk}^l + \sum_k \sum_l \theta E[N_{kl}(t)] \quad (25)$$

$$\text{s.t.} \sum_k \sum_l t_{ik}^l = 1 \qquad \forall i, j \in N \quad (26)$$

$$t_{ik}^l \leq t_{kk}^l \qquad \forall i, k \in N, l \in L \quad (27)$$

$$\sum_k \sum_l t_{kk}^l = p \quad (28)$$

$$\sum_m x_{ikm} - \sum_m x_{imk} = O_i \sum_l t_{ik}^l - \sum_j W_{ij} \sum_l t_{jk}^l \qquad \forall i, k \in N \quad (29)$$

$$\sum_m x_{ikm} \leq O_i \sum_l t_{ik}^l \qquad \forall i, k \in N \quad (30)$$

$$\sum_i \sum_j W_{ij} t_{ik}^l \leq \gamma_k^l \qquad \forall k \in N, l \in L \quad (31)$$

$$\sum_l t_{kk}^l \leq 1 \qquad \forall k \in N \quad (32)$$

$$x_{ikm} \geq 0 \qquad \forall i, k, m \in N \quad (33)$$

$$t_{ik}^l, t_{kk}^l \in \{0, 1\} \qquad \forall i, k \in N, l \in L \quad (34)$$

Constraints (26)–(34) are straightforward conversions of constraints (3)–(12) by using the relationships shown in (22). Note that constraint (27) is a disaggregated version of constraint (4), as $z_{ik} \leq z_{kk} = \sum_l t_{ik}^l \leq \sum_l t_{kk}^l$, which can be disaggregated $\forall\, l \in L$. For the non-linear terms, $\frac{(\sum_i \sum_j W_{ij} t_{ik}^l)^2}{\gamma_k^l - \sum_i \sum_j W_{ij} t_{ik}^l}$ and $\frac{\sum_i \sum_j W_{ij} t_{ik}^l}{\gamma_k^l - \sum_i \sum_j W_{ij} t_{ik}^l}$ in (23) and (24), we propose MISOCP-based reformulations in Sect. 4.2. The OA-based method for EK-3s, which we refer to as EK-OA-3s, is discussed in "Appendix B.2".

We do a similar study on SHLPCC with the two-subscripted and the three-subscripted capacity allocation variables for the SK-based model, and propose MISOCP-based reformulations in "Appendix A". In the following sections, we discuss our MISOCP-based reformulations and analysis for EK-2s and EK-3s models.

4 MISOCP-based reformulations

In this section, we briefly describe second order conic programs. In Sect. 4.1, we propose MISOCP-based reformulations for EK-2s, followed by MISOCP-based reformulations for EK-3s in Sect. 4.2.

A Second Order Conic Program is a convex optimization problem of the following form:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f^T x \\ \text{s.t.} \quad & \left\| A_i x + b_i \right\| \leq c_i^T x + d_i, \quad \forall i = 1 \ldots m \end{aligned} \quad (35)$$

Each of the constraints in (35) is a second-order cone constraint of dimension k_i. Here, $f \in \mathbb{R}^n, A_i \in \mathbb{R}^{(k_i-1)n}, b_i \in \mathbb{R}^{k_i-1}, c_i \in \mathbb{R}^n, d_i \in \mathbb{R}$. Note that when $A_i = 0, \forall i = 1, \ldots, m$, the above SOCP reduces to an LP, while $c_i = 0, \forall i = 1, \ldots m$ reduces it to a Quadratically Constrained Quadratic Programming (QCQP).

CPLEX accepts second order conic constraints in the following two forms:

$$x^T Qx \le y^2 \qquad y \ge 0 \tag{Form-1}$$

$$x^T Qx \le yz \qquad y, z \ge 0 \tag{Form-2}$$

Form-2 constraints can be written as SOCs by the following two transformations:

$$2x^T Qx + y^2 + z^2 \le (y+z)^2 \qquad y, z \ge 0 \tag{Form-2.1}$$

$$4x^T Qx + (y-z)^2 \le (y+z)^2 \qquad y, z \ge 0 \tag{Form-2.2}$$

From our initial experiments, and as also illustrated by Ahmadi-Javid and Hoseinpour (2017), constraints of the (Form-2.1) perform the best. We have, therefore, represented all the constraint of Form-2 as (Form-2.1) in this paper.

4.1 MISOCP-based reformulations for EK-2s

In the model EK-2s (based on two-subscripted capacity allocation variables), the non-linear term (13) includes two decision variables y_{kl} and z_{ik}. In this section, we propose MISOCP-based reformulations based on both the variables. The first SOC is based on z_{ik} variables, the next three SOCs are based on y_{kl} variables, while the fifth SOC is based on the relationship between the traffic intensity ρ_k and s_k.

EK-MISOCP1: For our first reformulation, since $s_k \ge \dfrac{\sum_i \sum_j W_{ij} z_{ik}}{\sum_l \gamma_k^l y_{kl} - \sum_i \sum_j W_{ij} z_{ik}}$, and z_{ik} is binary, we have $\sum_i \left(\sum_j W_{ij}\right) z_{ik}^2 \le s_k (\sum_l \gamma_k^l y_{kl} - \sum_i \sum_j W_{ij} z_{ik})$. We introduce variable t_k such that $(\sum_l \gamma_k^l y_{kl} - \sum_i \sum_j W_{ij} z_{ik}) \ge t_k$. The constraint holds with inequality for the minimization problem. On substituting, we get the following set of constraints,

$$\sum_i \sum_j W_{ij} z_{ik}^2 \le s_k t_k \quad \forall k \in N, \tag{36}$$

$$\sum_l \gamma_k^l y_{kl} - \sum_i \sum_j W_{ij} z_{ik} \ge t_k \quad \forall k \in N, \tag{37}$$

$$t_k \ge 0 \qquad \forall k \in N. \tag{38}$$

Note that (37) dominates (8), hence (8) is eliminated. Since (36) are hyperbolic constraint, it is transformed to (Form-2.1) as:

$$2 \sum_i \sum_j W_{ij} z_{ik}^2 + s_k^2 + t_k^2 + \le (s_k + t_k)^2 \quad \forall k \in N. \tag{39}$$

Our first EK-MISOCP-based reformulation is then given as follows:

[EK-MISOCP1]

$$\min \sum_i \sum_k d_{ik}(\chi O_i + \delta D_i) z_{ik} + \sum_i \sum_k \sum_m \alpha d_{km} x_{ikm}$$

$$+\sum_k \sum_l Q_k^l y_{kl} + \frac{\theta}{2}\sum_k \big(s_k + \rho_k + \sum_l c_{kl}^2 (V_{kl} - L_{kl})\big)$$
$$\text{s.t.} \quad (3)-(7), (9)-(12), (14)-(21), (37)-(39).$$

EK-MISOCP1 has $2N$ additional constraints and N additional variables, out of which N constraints are SOCs, each of dimension $N+3$.

EK-MISOCP2: We next propose SOC constraints based on y_{kl} variables. Similar to the previous formulation, we have variable s_k such that,

$$\frac{\sum_i \sum_j W_{ij} z_{ik}}{\sum_l \gamma_k^l y_{kl} - \sum_i \sum_j W_{ij} z_{ik}} \le s_k = \sum_i \sum_j W_{ij} z_{ik} \le s_k (\sum_l \gamma_k^l y_{kl} - \sum_i \sum_j W_{ij} z_{ik}).$$

Adding $(\sum_l \gamma_k^l y_{kl} - \sum_i \sum_j W_{ij} z_{ik})$ on both sides we get $\sum_l \gamma_k^l y_{kl} \le (1+s_k)(\sum_l \gamma_k^l y_{kl} - \sum_i \sum_j W_{ij} z_{ik})$. We introduce auxiliary variables t_k and τ_k such that $(\sum_l \gamma_k^l y_{kl} - \sum_i \sum_j W_{ij} z_{ik}) \ge t_k$ and $\tau_k = 1 + s_k$. Also, using the fact that y_{kl} is binary, we have the following constraints

$$\sum_l \gamma_k^l y_{kl}^2 \le \tau_k t_k \qquad \forall k \in N, \tag{40}$$
$$\sum_l \gamma_k^l y_{kl} - \sum_i \sum_j W_{ij} z_{ik} \ge t_k \qquad \forall k \in N, \tag{41}$$
$$\tau_k = 1 + s_k \qquad \forall k \in N, \tag{42}$$
$$t_k, \tau_k \ge 0 \qquad \forall k \in N. \tag{43}$$

Hyperbolic constraint (40) are transformed to

$$2\sum_l \gamma_k^l y_{kl}^2 + \tau_k^2 + t_k^2 \le (\tau_k + t_k)^2 \qquad \forall k \in N. \tag{44}$$

The second EK-MISOCP-based reformulation is, therefore:

[EK-MISOCP2]

$$\min \quad \sum_i \sum_k d_{ik}(\chi O_i + \delta D_i) z_{ik} + \sum_i \sum_k \sum_m \alpha d_{km} x_{ikm}$$
$$+\sum_k \sum_l Q_k^l y_{kl} + \frac{\theta}{2}\sum_k \big(s_k + \rho_k + \sum_l c_{kl}^2 (V_{kl} - L_{kl})\big)$$
$$\text{s.t.} \quad (3)-(7), (9)-(12), (14)-(21), (41)-(44).$$

EK-MISOCP2 has $3N$ additional constraints and $2N$ additional variables, including N SOCs, each of dimension L+3.

EK-MISOCP3: Disaggregating (40) results in higher number of SOCs of smaller dimensions, which gives us a new set of constraints as follows:

$$\gamma_k^l y_{kl}^2 \le \tau_k t_k \qquad \forall k \in N, l \in L. \tag{45}$$

Transforming to (Form-2.1), we have

$$2\gamma_k^l y_{kl}^2 + \tau_k^2 + t_k^2 \le (\tau_k + t_k)^2 \qquad \forall k \in N, l \in L. \tag{46}$$

Our third EK-MISOCP-based reformulation is, therefore,

[EK-MISOCP3]

$$\begin{aligned}\min \quad & \sum_i \sum_k d_{ik}(\chi O_i + \delta D_i) z_{ik} + \sum_i \sum_k \sum_m \alpha d_{km} x_{ikm} \\ & + \sum_k \sum_l Q_k^l y_{kl} + \frac{\theta}{2} \sum_k \Big(s_k + \rho_k + \sum_l c_{kl}^2 (V_{kl} - L_{kl}) \Big) \\ \text{s.t.} \quad & (3)-(7), (9)-(12), (14)-(21), (41)-(43), (46).\end{aligned}$$

EK-MISOCP3 has $2N + NL$ additional constraints and $2N$ additional variables, out of which NL constraints are SOCs, each of dimension 4.

EK-MISOCP4: Another slight variation results in SOCs of smaller dimension as compared to (40). We define p_k, such that,

$$p_k^2 = \sum_l \gamma_k^l y_{kl}^2 \iff p_k = \sum_l \sqrt{\gamma_k^l} y_{kl} \qquad \forall k \in N. \tag{47}$$

Substituting in (40) we get,

$$p_k^2 \le \tau_k t_k \qquad \forall k \in N, \tag{48}$$

$$p_k \ge 0 \qquad \forall k \in N. \tag{49}$$

Transformation to (Form-2.1) results in the following constraint

$$2p_k^2 + \tau_k^2 + t_k^2 \le (\tau_k + t_k)^2 \qquad \forall k \in N. \tag{50}$$

Our fourth EK-MISOCP-based reformulation is as follows:

[EK-MISOCP4]

$$\begin{aligned}\min \quad & \sum_i \sum_k d_{ik}(\chi O_i + \delta D_i) z_{ik} + \sum_i \sum_k \sum_m \alpha d_{km} x_{ikm} \\ & + \sum_k \sum_l Q_k^l y_{kl} + \frac{\theta}{2} \sum_k \Big(s_k + \rho_k + \sum_l c_{kl}^2 (V_{kl} - L_{kl}) \Big) \\ \text{s.t.} \quad & (3)-(7), (9)-(12), (14)-(21), (41)-(43), (47), (49), (50).\end{aligned}$$

EK-MISOCP4 has $4N$ additional constraints and $3N$ additional variables, respectively, out of which there are N SOCs, each of dimension 4 .

EK-MISOCP5: From the definition of s_k and ρ_k, as used earlier, we can write $\frac{\rho_k}{1-\rho k} \le s_k \iff \rho_k \le s_k(1-\rho_k)$. Adding $(1-\rho_k)$ on both the sides we get the following hyperbolic constraint

$$1 \le (1 + s_k)(1 - \rho_k) \qquad \forall k \in N. \tag{51}$$

We use the following substitution

$$\tau_k = 1 + s_k \qquad \forall k \in N, \tag{52}$$

$$\psi_k = 1 - \rho_k \qquad \forall k \in N, \tag{53}$$

$$\tau_k, \psi_k \ge 0 \qquad \forall k \in N, \tag{54}$$

which results in the following hyperbolic constraint

$$1 \le \tau_k \psi_k \qquad \forall k \in N. \tag{55}$$

The above hyperbolic constraint is converted to (Form-2.1) as

$$2 + \tau_k^2 + \psi_k^2 \le (\tau_k + \psi_k)^2 \qquad \forall k \in N. \tag{56}$$

Our fifth EK-MISOCP-based reformulation is, therefore,

[EK-MISOCP5]

$$\begin{aligned} \min \quad & \sum_i \sum_k d_{ik}(\chi O_i + \delta D_i) z_{ik} + \sum_i \sum_k \sum_m \alpha d_{km} x_{ikm} \\ & + \sum_k \sum_l Q_k^l y_{kl} + \frac{\theta}{2} \sum_k \Big(s_k + \rho_k + \sum_l c_{kl}^2 (V_{kl} - L_{kl})\Big) \\ \text{s.t.} \quad & (3)-(12),\ (52)-(54),\ (56). \end{aligned}$$

EK-MISOCP5 has $3N$ additional constraints and $2N$ additional variables with N SOC constraints, each of dimension 3.

4.2 MISOCP-based reformulations for EK-3s

In the model EK-3s (based on three-subscripted capacity allocation variables), the non-linear term in the objective function can be expressed in two alternate forms, given by (23) and (24), using which we obtain alternate MISOCP-based reformulations.

EK-MISOCP6: From (23), we have

$$E[N_{kl}(t)] = \frac{(1 + c_{kl}^2)(\sum_i \sum_j W_{ij} t_{ik}^l)^2}{2(\gamma_k^l)(\gamma_k^l - \sum_i \sum_j W_{ij} t_{ik}^l)} + \frac{\sum_i \sum_j W_{ij} t_{ik}^l}{\gamma_k^l}.$$

We introduce variable $r_{kl}, t_{kl} \ge 0$, such that

$$\frac{(\sum_i \sum_j W_{ij} t_{ik}^l)^2}{\gamma_k^l - \sum_i \sum_j W_{ij} t_{ik}^l} \le r_{kl} = \left(\sum_i \sum_j W_{ij} t_{ik}^l\right)^2 \le r_{kl}(\gamma_k^l - \sum_i \sum_j W_{ij} t_{ik}^l) \quad \forall k, l.$$

Also, $\gamma_k^l - \sum_i \sum_j W_{ij} t_{ik}^l \ge t_{kl} \quad \forall k, l$, therefore we have the following set of constraints

$$\left(\sum_i \sum_j W_{ij} t_{ik}^l\right)^2 \le r_{kl} t_{kl} \qquad \forall k \in N, l \in L, \tag{57}$$

$$\gamma_k^l - \sum_i \sum_j W_{ij} t_{ik}^l \ge t_{kl} \qquad \forall k \in N, l \in L, \tag{58}$$

$$t_{kl}, r_{kl} \ge 0 \qquad \forall k \in N, l \in L. \tag{59}$$

Constraint set (57), which is hyperbolic, is transformed to (Form-2.1) as:

$$2\left(\sum_i \sum_j W_{ij} t_{ik}^l\right)^2 + r_{kl}^2 + t_{kl}^2 \le (r_{kl} + t_{kl})^2 \qquad \forall k \in N, l \in L. \tag{60}$$

Notice that because of minimization objective, both r_{kl} and t_{kl} hold with equality at the optimum. Also (58) dominates (31), hence (31) is eliminated. Our sixth EK-MISOCP-based formulation is as follows:

[EK-MISOCP6]

$$\begin{aligned}\min \quad & \sum_i \sum_k d_{ik}(\chi O_i + \delta D_i) \sum_l t_{ik}^l + \sum_i \sum_k \sum_m \alpha d_{km} x_{ikm} \\ & + \sum_k \sum_l Q_k^l t_{kk}^l + \theta \sum_k \sum_l \Big(\frac{(1+c_{kl}^2)}{2\gamma_k^l} r_{kl} + \frac{\sum_i \sum_j W_{ij} t_{ik}^l}{\gamma_k^l} \Big) \\ \text{s.t.} \quad & (26)-(30),\ (32)-(34),\ (58)-(60).\end{aligned}$$

EK-MISOCP6 has $2NL$ additional constraints and $2NL$ additional variables, out of which there are NL number of SOCs of dimension 4.

EK-MISOCP7: From (24) we have,

$$E[N_{kl}(t)] = 1/2 \left\{ \left(1+c_{kl}^2\right) \frac{\sum_i \sum_j W_{ij} t_{ik}^l}{\left(\gamma_k^l - \sum_i \sum_j W_{ij} t_{ik}^l\right)} + \left(1-c_{kl}^2\right) \frac{\sum_i \sum_j W_{ij} t_{ik}^l}{\gamma_k^l} \right\}.$$

Introducing variable s_{kl} such that, $\dfrac{(\sum_i \sum_j W_{ij} t_{ik}^l)}{\gamma_k^l - \sum_i \sum_j W_{ij} t_{ik}^l} \leq s_{kl}$ and $(\gamma_k^l - \sum_i \sum_j W_{ij} t_{ik}^l) \geq t_{kl}$, we have the following constraints

$$\sum_i \sum_j W_{ij} (t_{ik}^l)^2 \leq s_{kl} t_{kl} \qquad \forall k \in N, l \in L, \tag{61}$$

$$\gamma_k^l - \sum_i \sum_j W_{ij} t_{ik}^l \geq t_{kl} \qquad \forall k \in N, l \in L, \tag{62}$$

$$t_{kl}, s_{kl} \geq 0 \qquad \forall k \in N, l \in L. \tag{63}$$

Hyperbolic constraint (61) is transformed to Form-2.1 as

$$2 \sum_i \sum_j W_{ij} (t_{ik}^l)^2 + s_{kl}^2 + t_{kl}^2 \leq (s_{kl} + t_{kl})^2 \qquad \forall k \in N, l \in L. \tag{64}$$

Seventh EK-MISOCP-based reformulation is as follows:

[EK-MISOCP7]

$$\begin{aligned}\min \quad & \sum_i \sum_k d_{ik}(\chi O_i + \delta D_i) \sum_l t_{ik}^l + \sum_i \sum_k \sum_m \alpha d_{km} x_{ikm} \\ & + \sum_k \sum_l Q_k^l t_{kk}^l + \frac{\theta}{2} \sum_k \sum_l \left((1+c_{kl}^2) s_{kl} + (1-c_{kl}^2) \frac{\sum_i \sum_j W_{ij} t_{ik}^l}{\gamma_k^l} \right) \\ \text{s.t.} \quad & (26)-(30),\ (32)-(34),\ (62)-(64).\end{aligned}$$

EK-MISOCP7 has $2NL$ additional constraints and $2NL$ additional variables, out of which there are NL SOCs, each of dimension $N+3$.

EK-MISOCP8: We propose another reformulation by using s_k and introducing variable u^l_{ijk} such that,

$$\sum_i \sum_j u^l_{ijk} \le s_{kl} \qquad \forall k \in N, l \in L \quad \text{, and} \quad \frac{W_{ij} t^l_{ik}}{\gamma^l_k - \sum_i \sum_j W_{ij} t^l_{ik}} \le u^l_{ijk}.$$

Substituting $\gamma^l_k - \sum_i \sum_j W_{ij} t^l_{ik} \ge t_{kl}$, we get the following set of constraints

$$W_{ij}(t^l_{ik})^2 \le u^l_{ijk} t_{kl} \qquad \forall i, j, k \in N, l \in L, \tag{65}$$

$$\sum_i \sum_j u^l_{ijk} \le s_{kl} \qquad \forall k \in N, l \in L, \tag{66}$$

$$\gamma^l_k - \sum_i \sum_j W_{ij} t^l_{ik} \ge t_{kl} \qquad \forall k \in N, l \in L, \tag{67}$$

$$u^l_{ijk} \ge 0 \qquad \forall i, j, k \in N, l \in L, \tag{68}$$

$$t_{kl}, s_{kl} \ge 0 \qquad \forall k \in N, l \in L. \tag{69}$$

Transformation of hyperbolic constraint (65) to Form-2.1 gives us,

$$2W_{ij}(t^l_{ik})^2 + (u^l_{ijk})^2 + t^2_{kl} \le (u^l_{ijk} + t_{kl})^2 \forall i, j, k \in N, l \in L. \tag{70}$$

Our eighth EK-MISOCP-based reformulation is as follows:

[EK-MISOCP8]

$$\min \sum_i \sum_k d_{ik}(\chi O_i + \delta D_i) \sum_l t^l_{ik} + \sum_i \sum_k \sum_m \alpha d_{km} x_{ikm}$$
$$+ \sum_k \sum_l Q^l_k t^l_{kk} + \theta/2 \sum_k \sum_l \left((1 + c^2_{kl}) s_{kl} + (1 - c^2_{kl}) \frac{\sum_i \sum_j W_{ij} t^l_{ik}}{\gamma^l_k} \right)$$
$$\text{s.t.} \quad (26)-(30), (32)-(34), (66)-(70).$$

EK-MISOCP8 has $N^3 L + 2NL$ additional constraints and $N^3 L + 2NL$ additional variables, out of which there are $N^3 L$ SOCs, each of dimension 4.

EK-MISOCP9: Multiplying both sides with γ^l_k, and adding $(\sum_i \sum_j W_{ij} t^l_{ik})^2$ to the constraint $(\sum_i \sum_j W_{ij} t^l_{ik}) \le s_{kl}(\gamma^l_k - \sum_i \sum_j W_{ij} t^l_{ik}) \quad \forall k, l$, we have $(\sum_i \sum_j W_{ij} t^l_{ik})^2 \le (s_{kl}\gamma^l_k - \sum_i \sum_j W_{ij} t^l_{ik})(\gamma^l_k - \sum_i \sum_j W_{ij} t^l_{ik}) \quad \forall k, l.$
Define t_{kl} and v_{kl}, such that,

$$v_{kl} = s_{kl}\gamma^l_k - \sum_i \sum_j W_{ij} t^l_{ik}, \qquad \forall k \in N, l \in L, \tag{71}$$

$$t_{kl} = \gamma^l_k - \sum_i \sum_j W_{ij} t^l_{ik} \qquad \forall k \in N, l \in L, \tag{72}$$

$$t_{kl}, v_{kl}, s_{kl} \ge 0 \qquad \forall k \in N, l \in L. \tag{73}$$

The constraint becomes

$$\left(\sum_i \sum_j W_{ij} t^l_{ik} \right)^2 \le t_{kl} v_{kl} \qquad \forall k, l, \tag{74}$$

which is transformed to Form-2.1 as:

$$2\left(\sum_i \sum_j W_{ij} t^l_{ik}\right)^2 + v^2_{kl} + t^2_{kl} \le (v_{kl} + t_{kl})^2 \qquad \forall k, l. \tag{75}$$

Our ninth EK-MISOCP-based formulation is, therefore:

[EK-MISOCP9]

$$\begin{aligned}
\min \quad & \sum_i \sum_k d_{ik}(\chi O_i + \delta D_i) \sum_l t^l_{ik} + \sum_i \sum_k \sum_m \alpha d_{km} x_{ikm} \\
& + \sum_k \sum_l Q^l_k t^l_{kk} + \theta/2 \sum_k \sum_l \left((1 + c^2_{kl}) s_{kl} + (1 - c^2_{kl}) \frac{\sum_i \sum_j W_{ij} t^l_{ik}}{\gamma^l_k} \right) \\
\text{s.t.} \quad & (26)-(30), (32)-(34), (71)-(73), (75).
\end{aligned}$$

EK-MISOCP9 introduces $3NL$ additional constraints and $3NL$ additional variables including NL SOCs, each of dimension 4.

Table 2 presents a summary of all MISOCPs including number (#) of constraints, variables, and SOCs along with their dimensions.

5 Computational experiments

For our computational experiments, we use two well-known datasets, CAB (O'kelly 1987) and AP (Ernst and Krishnamoorthy 1996). In Sect. 5.1, we perform experiments on instances generated from the CAB dataset for all MISOCP-based reformulations (SK-2s, SK-3s, EK-2s and EK-3s based reformulations), which are solved directly using the solver. We compare their performances against the two MILP reformulations, which are obtained using outer-approximation (OA). Outer-approximations are known to solve MINLPs within ϵ tolerance with finite convergence (Vidyarthi and Jayaswal 2014; Vidyarthi et al. 2013; Jayaswal et al. 2017; Tiwari et al. 2020; Jayaswal et al. 2011; Jayaswal and Jewkes 2016). All the computational experiments are run on a workstation with a 2.20 GHz Intel Xeon E5-2630 processor and 64 GB RAM. All MISOCP-based reformulations are solved directly using CPLEX 12.7.1. For the two OA-based methods, the MILP at each iteration is solved using CPLEX 12.7.1.

CPLEX uses one of the following two alternate parameter settings to solve MISOCPs: *miqcpstrat 1* and *miqcpstrat 2*. In *miqcpstrat 1*, it uses an SOCP based branch and bound algorithm, where at each node the continuous relaxation is solved using an interior point algorithm, specifically designed for SOCPs. In *miqcpstrat 2*, it uses the fact that the SOCPs are NLPs, which can be approximated using outer-approximations. To allow CPLEX to choose the best strategy from among the two, we set the parameter to *miqcpstrat 0*.

5.1 Experiments based on CAB dataset

Using CAB dataset, we run our experiments on instances with the number of nodes ($|N|$) as 10, 15, 20 and 25, and the number of hubs (p) as 3 and 4. We use two values of the inter-hub discount factor (α) as 0.4 and 0.8, and two values of congestion cost(θ) as 20 and 50. The coefficient of variation (c_k) of the service times at hub k is varied as 0, 1 and 2 to model (M/D/1), (M/M/1) and (M/G/1) queues, respectively. This gives us a total of 96 instances for our computational experiments. For each of these instances, the three capacity levels

Table 2 Summary of MISOCP-based reformulations

EK	SOCs			Model constraints		Binary var.	Conti. var.
MISOCP	Eqn #	#	Size	Eqn #	#		
1	39	N	N+3	3–7, 9, 15–16, 18–19, 21, 37, 39	$3N^2 + 7N + 2NL + 1$	$N^2 + NL$	$N^3 + 3N + 2NL$
2	44	N	L+3	3–7, 9, 15–16, 18–19, 21, 41–42, 44	$3N^2 + 8N + 2NL + 1$	$N^2 + NL$	$N^3 + 4N + 2NL$
3	46	NL	4	3–7, 9, 15–16, 18–19, 21, 41–42, 46	$3N^2 + 7N + 3NL + 1$	$N^2 + NL$	$N^3 + 4N + 2NL$
4	50	N	4	3–7, 9, 15–16, 18–19, 21, 41–42, 47, 50	$3N^2 + 9N + 2NL + 1$	$N^2 + NL$	$N^3 + 5N + 2NL$
5	56	N	4	3–9, 15–16, 18–19, 21, 52–53, 56	$3N^2 + 9N + 2NL + 1$	$N^2 + NL$	$N^3 + 4N + 2NL$
6	60	NL	4	26–30, 32, 58, 60	$3N^2 + N^2L + 2NL + N + 1$	N^2L	$N^3 + 2NL$
7	64	NL	N+3	26–30, 32, 62, 64	$3N^2 + N^2L + 2NL + N + 1$	N^2L	$N^3 + 2NL$
8	70	N^3L	4	26–30, 32, 66–67, 70	$3N^2 + N^3L + N^2L + 2NL + N + 1$	N^2L	$N^3 + N^3L + 2NL$
9	75	NL	4	26–30, 32, 71–72, 75	$3N^2 + N^2L + 3NL + N + 1$	N^2L	$N^3 + 3NL$

($l = 1, 2, 3$) at any hub are set as $\frac{\sum_i \sum_j W_{ij}}{p} + \beta A_l \sum_i \sum_j W_{ij}$, where A_l = -1, 0, 1 for $l = 1, 2, 3$, respectively, and β is set to 0.21, 0.22, 0.23 and 0.24 for $|N|$= 10, 15, 20 and 25. Fixed cost Q_k^l is set as 150, 200 and 250 for the three capacity levels. A similar scheme for setting the capacity levels and the corresponding costs is used by Azizi et al. (2018). The optimality tolerance (ϵ) for OA is set to 10^{-6}. For each instance, a maximum CPU time limit of 14,400 seconds (4 hours) is used.

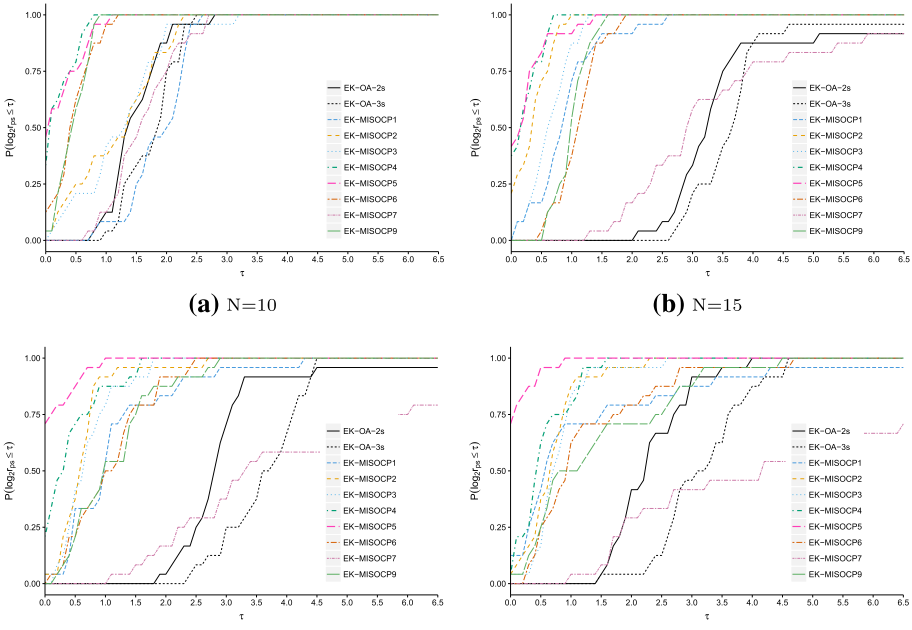
5.1.1 Results and analysis

From our initial experiments, as reported in Tables 4 in "Appendix A", we find that the results from the SK-based models are far inferior to those based on EK-based models. Hence, in the remaining experiments, we restrict our analysis to only the EK-based models. We compare the performances of all the proposed EK-based MISOCP reformulations against EK-OA-2s, EK-OA-3s, obtained using OA, and strengthened using perspective reformulation (Günlük and Linderoth 2012). Reformulations using OA is presented in "Appendix B". Our initial computational experiments clearly suggested EK-MISOCP8 to be the worst among all formulations, and is, therefore, excluded from further analysis. We discuss the reasons for the inferior performance of EK-MISOCP8 in Sect.5.3. Tables 5, 6, 7, and 8 in the "Appendix" present the computational performances (in terms of CPU time in seconds) of eight of the nine MISOCPs (excluding EK-MISOCP8), and the two OA-based reformulations for $|N|$=10, 15, 20 and 25. In each of these tables, the column "Hub" reports the set of p hubs opened in the resulting solution, while columns "Cap" and "Intensity" report the corresponding capacity level and the traffic intensity at each of the open hubs, respectively. The percentage contribution of flow cost, location (capacity installation) cost, congestion cost is indicated in the columns "FC", "LC", "CC". Total cost,"TC", refers to the actual objective function value for the instance. Table 3 presents the number of instances for which each of the formulations performs the best, based on which EK-MISOCP4 and EK-MISOCP5 seem to be the best two formulations, exhibiting the best performances for 23 and 55 (out of a total of 96) instances, respectively.

Comparison among the rest of the formulations is not so obvious from the tables. Further, for the remaining 19 instances where neither EK-MISOCP4 nor EK-MISOCP5 is the best, it is not immediately obvious from the tables how well they perform vis-à-vis the rest. Hence, for a better comparative analysis, we use performance profiles (Dolan and Moré 2002). For this, let n_s and n_p be the number of formulations and the number of test instances, respectively, and S and P be their respective sets. Let $t_{p,s}$ be the CPU time taken to solve the instance

Table 3 No. of best performing instances

N	EK-OA-2s	EK-OA-3s	MISOCPs							
			1	2	3	4	5	6	7	9
10	0	0	0	1	0	8	11	3	0	1
15	0	0	0	5	0	9	10	0	0	0
20	0	0	1	1	0	5	17	0	0	0
25	0	0	3	1	1	1	17	0	0	1
Total	0	0	4	8	1	23	55	3	0	2



(a) N=10 **(b)** N=15

(c) N=20 **(d)** N=25

Fig. 1 Performance profile of EK-MISOCPs and OA-based method for N = 10,15,20,25 for CAB datset

$p \in P$ using formulation $s \in S$. Then, performance ratio $r_{p,s}$ is calculated as:

$$r_{p,s} = \frac{t_{p,s}}{\min_{s \in S}\{t_{p,s}\}}$$

Treating $t_{p,s}$ as a random variable, a performance profile $\rho_s(\tau)$ gives its cumulative probability distribution at τ. In other words, it gives the probability with which the CPU time taken by a given formulation s does not exceed 2^τ times the CPU time required by the best among all the formulations under study. Mathematically, it can be stated as:

$$\rho_s(\tau) = \frac{1}{n_p}\left\{p \in P : \log_2(r_{p,s}) \leq \tau\right\}.$$

Specifically, $\rho_s(\tau = 0)$ gives the probability that s is the best among all the formulations.

Figure 1 presents four different performance profiles of the different EK-MISOCP and OA-based reformulations, one corresponding to each value of $|N|$. For $|N| = 10$, Fig. 1a shows that EK-MISOCP5 and EK-MISOCP4 perform the best for around 46% (11 out of 24) and 33% (8 out of 24) of the instances, respectively. This is something that is already obvious from Table 5. However, what is less obvious from that table is the fact that EK-MISOCP5 never takes more than twice the CPU time taken by the best formulation for any of the 24 instances, which is only revealed by the performance profile. Even less obvious is another fact that at $\tau = 0.8$, EK-MISOCP4 is able to solve all the 24 instances, whereas EK-MISOCP5 can solve only 96% of the instances. Further, from Fig. 1b–d, we see that EK-MISOCP5 and EK-MISOCP4 are the best two formulations across all problem instances. We do similar analyses for all the formulations with respect to the congestion factor θ and the coefficient of variation (c), the results for which are presented in Figs. 2 and 3 in "Appendix C". Once again, we observe that all MISOCPs dominate the OA-based reformulations, with EK-MISOCP5 being the best, followed by EK-MISOCP4. Moreover, EK-MISOCP5, which is the best formulation, solves all the instances 20-60 times faster as compared to SK-OA-2s, which is the existing best formulation/method in the literature.

In the absence of congestion, hub location model with three-subscripted capacity allocation variable is known to dominate the model with two-subscripted capacity allocation variable in terms of LP relaxation (Correia et al. 2010). For CAB dataset, we observe from our computational results that this dominance of EK-3s does not necessarily translate into an advantage from a computational point of view owing to its larger model size. Specifically, all the formulations based on EK-2s, except EK-MISOCP1, perform better than EK-3s. However, in the presence of congestion, this conclusion may not be generalized to other datasets. Hence, for the next set of computational experiments, based on AP dataset, we proceed with EK-MISOCP4 and EK-MISOCP5 (the best two among EK-2s) and EK-MISOCP6 and EK-MISOCP9 (the best two among EK-3s).

5.2 Experiments based on AP dataset

For the AP dataset, we limit our computational experiments only to EK-MISOCP4, EK-MISOCP5, EK-MISOCP6 and EK-MISOCP9 for $|N|$=25 and 50. The AP dataset specifies two different possible values for the capacity at each hub, referred to as tight (T) and loose (L). Following the scheme used by Contreras et al. (2012), we set the capacity (γ_k^l) and the fixed cost (Q_k^l) for level l at the potential hub node k as $\gamma_k^L = \Gamma_k^L$, $\gamma_k^l = 0.7 \times \gamma_k^{l+1}$, $Q_k^L = \Gamma_L^k$, and $Q_k^l = 0.9 \times Q_k^{l+1}$ $\forall l = 1 \ldots L-1$, where Γ_k^L is the capacity for hub k provided in the dataset. Similar to experiments using CAB dataset, corresponding to each value of $|N|$, we run our experiments with the number of hubs (p) as 3 and 4, inter-hub discount factor (α) as 0.4 and 0.8, and the coefficient of variation (c_k) of the service times at hub k as 0, 1 and 2. In addition, we set the congestion cost(θ) as 50,100 and 200.

5.2.1 Results and analysis

Tables 9 and 10 present the computational results for our experiments with the AP dataset. The column names in these tables are the same as the ones used for CAB dataset. For EK-MISOCP5, the column "CPU time (s)/% Gap" reports the CPU time in seconds taken to solve the instance to optimality. For the other MISOCPs (4, 6, and 9), the maximum CPU time limit is set as the CPU time required to solve EK-MISOCP5 to optimality. For the instances that could not be solved to optimality within the time limit, we provide their optimality gap as reported by CPLEX. For a few instances that were solved within the time limit, we report the corresponding CPU times. For many of the instances, CPLEX was not able to find even a single integer feasible solution, which we denote in the table using an asterisk (*). As obvious from Tables 9 and 10, EK-MISOCP5 continues to outperform all other MISOCPs, followed by EK-MISOCP4. However, for smaller instances ($|N|$=25) with loose capacity (L), EK-3s models (EK-MISOCP6 and EK-MISOCP9) are able to beat EK-2s models (EK-MISOCP5 and EK-MISOCP4) in a few cases. This clearly suggests that for smaller instances continuous relaxation may play a role; however, for larger instances the performance is primarily dictated by the model size.

5.3 Observations

The difference in performances among the alternate MISOCPs can be partly understood from Table 2, which summarizes their properties. The number of SOC constraints ($|N|$) and their dimensions (4) are the smallest for EK-MISOCP5. It also has the least number of binary variables ($|N|^2+|N|\times|L|$) and continuous variables ($|N|^3+4|N|+2|N|\times|L|$). This explains the superiority of EK-MISOCP5 over all the other MISOCPs. EK-MISOCP4 also has the same number and dimension of SOC constraints, and the number of binary variables. However, its performance is slightly worse overall, as compared to EK-MISOCP5, due to a slightly higher number of continuous variables. Understandably, EK-MISOCP8 performs the worst since it has the largest number of SOC constraints ($|N|^3|L|$) and also the largest number of binary variables ($|N|^2|L|$) and continuous variables ($|N|^3 + |N|^3|L| + 2|N| \times |L|$). Though the dimension of the SOCs in EK-MISOCP8 is the smallest (4), its computational performance gets dictated largely by the other attributes, which are comparatively far worse. For similar reasons, the performance of EK-MISOCP7 is also very poor, only next to EK-MISOCP8. EK-MISOCP1, EK-MISOCP2, EK-MISOCP3, EK-MISOCP6, and EK-MISOCP9 are intermediate performers. Their performances can be explained based on the attributes discussed above. We hope the insights presented in this section will be useful in selecting an MISOCP from among several alternatives in other problem contexts as well.

6 Conclusions and future research directions

In this paper, we have proposed several MISOCP-based reformulations for the hub location problem with capacity selection under congestion. The hub location problem is computationally challenging to solve; accounting for congestion at hubs adds another layer of difficulty by introducing non-linearity in the resulting model. The contribution of the paper lies in proposing several MISOCP-based reformulations for the MINLP problem, which can be efficiently handled by the existing mixed integer programming solvers. All our MISOCP-based reformulations outperform the reformulations based on outer approximation on all the 96 instances generated from the CAB dataset. Further, based on our computational studies, we have identified the best MISOCP-based reformulation (EK-MISOCP5), which turns out to be several times faster than the existing best formulation/method in the literature. In particular, EK-MISOCP5 has allowed us to exactly solve instances from the AP dataset of the size of up to 50 nodes in less than half-an-hour. The paper also provides insights into the properties of MISOCPs that determine their computational efficiency, which would be important in making the best selection from among various alternate formulations.

Based on current findings, we foresee applications of reformulations of MILPs into second order cone programs in other classes of hub location problems, where non-linearities may arise. For example, one potential application can be in the area of competitive hub location (Marianov et al. 1999; Lüer-Villagra and Marianov 2013), and more broadly in competitive facility location, where the non-linearity results from the market share function. On the methodology side, decomposition based techniques, like Lagrangean decomposition, Dantzig-Wolfe decomposition, and Benders decomposition, can be explored to solve the resulting MISOCP-based reformulation even more efficiently. Further, valid inequalities like polymatroid cuts (Atamtürk et al. 2012), which characterize the convex hull corresponding to the mixed integer second order conic constraint, can be used in a branch-and-cut framework.

Appendix

A SHLPCC for the SK-based model

For SK-based model, the flow variables x_{ikm} are replaced by path variables x_{ijkm}, where $x_{ijkm} = \begin{cases} 1, & \text{if flows from } i \text{ to } j \text{ are routed via hub } k \text{ and } m \\ 0, & \text{otherwise.} \end{cases}$

Definition of other variables and parameters remain same.

A.1 Two-subscripted capacity allocation variable

$$\textbf{[SK-2s]} \quad \min \sum_i \sum_j \sum_k \sum_m F_{ijkm} x_{ijkm} + \sum_k \sum_l Q_k^l y_{kl} + \theta \sum_k 1/2E[N_k(y,z)]$$

$$\text{s.t.} \quad (3)-(5), (7)-(10), (12), (14)-(21)$$

$$\sum_m x_{ijkm} = z_{ik} \qquad \forall i, j, k \tag{76}$$

$$\sum_k x_{ijkm} = z_{jm} \qquad \forall i, j, m \tag{77}$$

$$x_{ijkm} \in \{0, 1\} \qquad \forall i, j, k, m, l \tag{78}$$

Here, $F_{ijkm} = W_{ij}(\chi d_{ik} + \alpha d_{km} + \delta d_{mj})$ is the total flow through path $i - j - k - m$. (76) and (77) connect the assignment variables and path variables.

[SK-MISOCP1] (3)–(5), (7), (9)–(10), (12), (14)-(21), (37)–(39), (76)–(78)

[SK-MISOCP2] (3)–(5), (7), (9)–(10), (12), (14)–(21), (41)–(44), (76)–(78).

[SK-MISOCP3] (3)–(5), (7), (9)–(10), (12), (14)–(21), (41)–(43), (46), (76)–(78).

[SK-MISOCP4] (3)–(5), (7), (9)–(10), (12), (14)–(21), (41)–(43), (47), (49), (50), (76)–(78).

[SK-MISOCP5] (3)–(5), (7)–(10), (12), (14)-(21), (52)–(54), (56), (76)–(78).

A.2 Three-subscripted capacity allocation variable

$$\textbf{[SK-3s]} \quad \min \sum_i \sum_j \sum_k \sum_m F_{ijkm} x_{ijkm} + \sum_k \sum_l Q_k^l t_{kk}^l + \theta E[N_k]$$

$$\text{s.t.} \quad (26) - (28), (30), (32), (34)$$

$$\sum_m x_{ijkm} = \sum_l t_{ik}^l \qquad \forall i, j, k \tag{79}$$

$$\sum_k x_{ijkm} = \sum_l t_{jm}^l \qquad \forall i, j, m \tag{80}$$

$$x_{ijkm} \in \{0, 1\} \qquad \forall i, j, k, m, l \tag{81}$$

[SK-MISOCP6] (26)–(28), (30), (32),(34), (58)–(60), (79)–(81).

Table 4 Computation time for N = 25 for CAB dataset

c	θ	Hub located	Cap level	Costs		CPU time (s)										
				% Contribution	TC	SK	SK	SK-MISOCP								
				FC, LC, CC		OA-2s	OA-3s	1	2	3	4	5	6	7	8	9
$p = 4, \alpha = 0.4$																
0	20	1,4,12,17	2,3,2,3	0.4 ,0.5 ,0.1	1704	1299	1334	1727	2952	2334	1601	3647	2290	2344	*	1660
	50	1,4,12,17	3,3,2,3	0.4 ,0.5 ,0.1	1963	2531	3001	4203	7094	5006	4020	5525	4254	9471	*	7612
1	20	1,4,12,17	2,3,2,3	0.4 ,0.5 ,0.1	1857	3000	3531	3816	3771	4101	3002	3366	3457	4993	*	2294
	50	1,4,12,17	3,3,3,3	0.4 ,0.5 ,0.1	2050	4474	7000	8092	7735	11477	10445	8077	*	*	*	*
2	20	1,4,12,17	3,3,3,3	0.4 ,0.5 ,0.1	1988	4691	6531	6539	*	*	11061	2623	1988	*	*	*
	50	1,4,12,17	3,3,3,3	0.4 ,0.4 ,0.2	2244	5855	9001	*	*	*	*	2856	2244	*	*	*

SK-OA-2s and SK-OA-3s represents outer-approximation results for the twosubscripted and three-subscripted model based on SK
*Represents no solution in the given time limit

[SK-MISOCP7] (26)–(28), (30), (32),(34), (62)–(64), (79)–(81).
[SK-MISOCP8] (26)–(28), (30), (32),(34),(66)–(70), (79)–(81).
[SK-MISOCP9] (26)–(28), (30), (32),(34), (71)–(73), (75), (79)–(81).

B OA method

B.1 EK-OA-2s: OA-based method for EK-2s

The auxillary variable L_{kl} and ρ_k which were defined as $L_{kl} = \rho_k y_{kl}$ and $\rho_k = \frac{s_k}{1+s_k}$, imply

$$L_{kl} = \begin{cases} 0, & \text{if } y_{kl} = 0 \\ \frac{s_k}{1+s_k}, & \text{if } y_{kl} = 1. \end{cases} \tag{82}$$

Also, earlier results, $\sum_l L_{kl} = \rho_k \quad \forall k$, $L_{kl} \le y_{kl} \quad \forall k, l$ and $\sum_i \sum_j W_{ij} z_{ik} = \sum_l \gamma_k^l L_{kl}$, remain. The function, $L_{kl} = \frac{s_k}{(1+s_k)}$ is a concave function which can be approximated with piecewise linear functions that are tangent to the function L_{kl} . The method chooses the minimum of these tangents at points $s_k^h \quad \forall h \in H, k \in N, l \in L$. The cuts are given by

$$L_{kl} = \min_{h \in H} \left\{ \frac{1}{(1+s_k^h)^2} s_k + \left(\frac{s_k^h}{1+s_k^h} \right)^2 \right\}$$
$$\iff L_{kl} \le \frac{1}{(1+s_k^h)^2} s_k + \left(\frac{s_k^h}{1+s_k^h} \right)^2 \quad \forall k \in N, l \in L, h \in H \tag{83}$$

In the OA-based method proposed by Elhedhli and Hu (2005), for every $k - l$ pair, the non linear congestion term is approximated with tangents (cuts), given by (82), at points s_k^h (set at h_0 initially). At every iteration a relaxed mixed integer linear problem is solved. The solution of which not only gives the lower bound but also supplies information for the next cut. Also, this solution is feasible for the main problem thus giving the upper bound. The algorithm terminates when both upper and lower bounds are ϵ (or less) away from each other where $\epsilon \ge 0$. Günlük and Linderoth (2012) proposed perspective counterpart for (83) as

$$L_{kl} = \frac{s_k}{1 + s_k / y_{kl}} k \in N, \; l \in L$$

and the corresponding perspective cut at s_k^h as

$$L_{kl} \le \frac{1}{(1+s_k^h)^2} s_k + \left(\frac{s_k^h}{1+s_k^h} \right)^2 y_{kl}, \quad \forall k, l \tag{84}$$

The formulation for the EK-OA-2s is as follows:

[EK-OA-2s]

$$\begin{aligned} \min \quad & \sum_i \sum_k C_{ik} (\chi O_i + \delta D_i) z_{ik} + \sum_i \sum_k \sum_m \alpha C_{km} x_{ikm} \\ & + \sum_k \sum_l Q_k^l y_{kl} + \theta/2 \sum_k \left\{ s_k + \rho_k + \sum_l c_{kl}^2 \big(V_{kl} - L_{kl} \big) \right\} \\ \text{s.t.} \quad & (3){-}(21), (84) \end{aligned}$$

Algorithm for the above discussed OA-based method is as follows:

1: Set $UB^{\tau-1} \to \infty$, $LB^{\tau-1} \to -\infty$, $\tau \to 1$
2: Choose initial cuts at points $s_k^{h_0}$ where $h_0 \in H$
3: **while** $\frac{UB^{\tau-1}-LB^{\tau-1}}{UB^{\tau-1}} \geq \epsilon$ **do**
4: Find LB^{τ} by solving $(EK-OA-2s)^{\tau}$ and obtain the optimal solution $(x^{\tau}, z^{\tau}, y^{\tau}, \rho^{\tau}, s^{\tau}, L^{\tau}, V^{\tau})$.
5: Find UB^{τ} by substituting $(x^{\tau}, z^{\tau}, y^{\tau})$ in the objective function as

$$
\begin{aligned}
UB^{\tau} = \min \Bigg\{ & UB^{\tau-1}, \sum_i \sum_k C_{ik}(\chi O_i + \delta D_i) z_{ik}^{\tau} + \sum_i \sum_k \sum_m \alpha C_{km} x_{ikm}^{\tau} \\
& + \sum_k \sum_l Q_k^l y_{kl}^{\tau} + \frac{\theta}{2} \Bigg\{ \Big(1 + \sum_l c_{kl}^2 y_{kl}^{\tau}\Big) \frac{\sum_i \sum_j W_{ij} z_{ik}^{\tau}}{\Big(\sum_l \gamma_k^l y_{kl}^{\tau} - \sum_i \sum_j W_{ij} z_{ik}^{\tau}\Big)} \\
& + \Big(1 - \sum_l c_{kl}^2 y_{kl}^{\tau}\Big) \frac{\sum_i \sum_j W_{ij} z_{ik}^{\tau}}{\sum_l \gamma_k^l y_{kl}^{\tau}} \Bigg\} \Bigg\}
\end{aligned}
$$

6: Update new point $s_k^{h_{new}} = \dfrac{\sum_i \sum_j W_{ij} z_{ik}^{\tau}}{\sum_l \gamma_k^l y_{kl}^{\tau} - \sum_i \sum_j W_{ij} z_{ik}^{\tau}}$ with the current solution $(x^{\tau}, z^{\tau}, y^{\tau})$.
7: Generate new cut: $L_{kl} \leq \dfrac{1}{(1+s_k^{h_{new}})^2} s_k + \left(\dfrac{s_k^{h_{new}}}{1+s_k^{h_{new}}}\right)^2 y_{kl}, \quad \forall k, l$
8: Add new cut: $H^{\tau+1} \to H^{\tau} + h_{new}$
9: $\tau \to \tau + 1$
10: **end while**

B.2 EK-OA-3s: OA-based method for EK-3s

For formulations with t_{ik}^l, we had objective function as

$$
\begin{aligned}
\min \quad & \Bigg(\sum_i \sum_k C_{ik}(\chi O_i + \delta D_i) \sum_l t_{ik}^l \Bigg) + \sum_i \sum_k \sum_m \alpha C_{km} x_{ikm} + \sum_k \sum_l Q_k^l t_{kk}^l \\
& + \theta \sum_k \sum_l 1/2 \left\{ (1 + c_{kl}^2) \frac{\sum_i \sum_j W_{ij} t_{ik}^l}{\Big(\gamma_k^l - \sum_i \sum_j W_{ij} t_{ik}^l\Big)} + (1 - c_{kl}^2) \frac{\sum_i \sum_j W_{ij} t_{ik}^l}{\gamma_k^l} \right\}
\end{aligned}
$$

We introduce variable ρ_{kl} and s_{kl} such that

$$
\frac{\sum_i \sum_j W_{ij} t_{ik}^l}{\gamma_k^l} \leq \rho_{kl} \qquad \forall k, l \tag{85}
$$

$$
\frac{\sum_i \sum_j W_{ij} t_{ik}^l}{\gamma_k^l - \sum_i \sum_j W_{ij} t_{ik}^l} \leq s_{kl} \quad \forall k, l \tag{86}
$$

ρ_{kl} and s_{kl} are related as

$$
\rho_{kl} = \frac{s_{kl}}{1 + s_{kl}} \quad \forall k, l
$$

which is non-linear concave function and can be approximated using tangent hyperplanes as discussed in the previous section. For perspective reformulation, we introduce the following constraint

$$\rho_{kl} \leq t_{kk}^{l} \tag{87}$$

to the following equivalent perspective form

$$\rho_{kl} = \frac{s_{kl}}{1 + (s_{kl}/t_{kk}^{l})} \tag{88}$$

By following logic similar to (83), we have perspective cuts as

$$\rho_{kl} \leq \frac{1}{(1+s_{kl}^{h})^2} s_{kl} + \frac{(s_{kl}^{h})^2}{(1+s_{kl}^{h})^2} t_{kk}^{l}, \quad \forall k, l \tag{89}$$

The overall formulation for the approximation method is as follows:

[EK-OA-3s]

$$\begin{aligned}
\min & \sum_{i}\sum_{k} C_{ik}(\chi O_i + \delta D_i)\sum_{l} t_{ik}^{l}) + \sum_{i}\sum_{k}\sum_{m} \alpha C_{km} x_{ikm} + \sum_{k}\sum_{l} Q_{k}^{l} t_{kk}^{l} \\
& + \theta \sum_{k}\sum_{l} 1/2 \left\{ \left(1 + c_{kl}^{2}\right) s_{kl} + \left(1 - c_{kl}^{2}\right) \rho_{kl} \right\} \\
\text{s.t.} \quad & (26) - (34), (87), (89)
\end{aligned}$$

Algorithm for the above discussed OA-based method is as follows:

1: Set $UB^{\tau-1} \rightarrow \infty$, $LB^{\tau-1} \rightarrow -\infty$, $\tau \rightarrow 1$
2: Choose initial cuts at points $s_{k}^{h_0}$ where $h_0 \in H$
3: **while** $\frac{UB^{\tau-1} - LB^{\tau-1}}{UB^{\tau-1}} \geq \epsilon$ **do**
4: Find LB^{τ} by solving $(EK - OA - 3s)^{\tau}$ and obtain the optimal solution $(x^{\tau}, t^{\tau}, \rho^{\tau}, s^{\tau})$
5: Find UB^{τ} by substituting (x^{τ}, t^{τ}) in the objective function as

$$\begin{aligned}
UB^{\tau} = \min \Bigg\{ & UB^{\tau-1}, \sum_{i}\sum_{k} C_{ik}(\chi O_i + \delta D_i)\sum_{l} t_{ik}^{l(\tau)}) + \sum_{i}\sum_{k}\sum_{m} \alpha C_{km} x_{ikm}^{\tau} \\
& + \sum_{k}\sum_{l} Q_{k}^{l} t_{kk}^{l(\tau)} + \frac{\theta}{2}\Bigg\{ \left(1 + c_{kl}^{2}\right) \frac{\sum_i \sum_j W_{ij} t_{ik}^{l(\tau)}}{\left(\gamma_{k}^{l} - \sum_i \sum_j W_{ij} t_{ik}^{l(\tau)}\right)} \\
& + \left(1 - c_{kl}^{2}\right) \frac{\sum_i \sum_j W_{ij} t_{ik}^{l(\tau)}}{\gamma_{k}^{l}} \Bigg\}
\end{aligned}$$

6: Update new point $s_{k}^{h_{new}} = \dfrac{\sum_i \sum_j W_{ij} t_{ik}^{l(\tau)}}{\sum_l \gamma_{k}^{l} - \sum_i \sum_j W_{ij} t_{ik}^{l(\tau)}}$ with the current solution (x^{τ}, t^{τ}).
7: Generate new cut: $\rho_{kl} \leq \dfrac{1}{(1+s_{kl}^{h_{new}})^2} s_{kl} + \dfrac{(s_{kl}^{h_{new}})^2}{(1+s_{kl}^{h_{new}})^2} t_{kk}^{l}, \quad \forall k, l$
8: Add new cut: $H^{\tau+1} \rightarrow H^{\tau} + h_{new}$
9: $\tau \rightarrow \tau + 1$
10: **end while**

Table 5 Comparison of EK-MISOCPs against EK-OA-2s and EK-OA-3s for N = 10 (CAB dataset)

c	θ	Hub Located	Cap level	Intensity	Costs		CPU time (s)									
					% Contribution	TC	EK	EK	MISOCPs							
					FC, LC, CC		OA-2s	OA-3s	1	2	3	4	5	6	7	9
p = 3, $\alpha = 0.4$																
0	20	4,6,7	3,3,2	0.7, 0.7, 0.8	0.4 ,0.5 ,0.1	1373	2.5	1.8	2.4	2.3	1.0	1.0	**1.0**	1.9	2.3	1.7
	50	4,6,7	3,3,3	0.7, 0.7, 0.5	0.4 ,0.5 ,0.1	1503	1.7	1.6	3.8	3.1	5.9	1.1	**0.7**	1.1	2.8	0.9
1	20	4,6,7	3,3,2	0.7, 0.7, 0.8	0.4 ,0.5 ,0.1	1427	2.9	2.1	3.9	3.2	3.7	**0.9**	1.2	1.8	2.7	1.1
	50	4,6,7	3,3,3	0.7, 0.7, 0.5	0.4 ,0.5 ,0.2	1588	1.7	**4**	4.1	1.0	2.6	1.3	**0.9**	1.2	3.7	0.9
2	20	4,6,7	3,3,3	0.7, 0.7, 0.5	0.4 ,0.5 ,0.1	1533	2	3.5	4.5	1.1	2.6	**1.0**	2.0	1.3	3.3	1.4
	50	4,6,7	3,3,3	0.7, 0.7, 0.5	0.3 ,0.4 ,0.3	1845	3.6	5.2	5.5	2.9	4.3	1.6	**1.1**	1.7	6.8	1.4
p = 3, $\alpha = 0.8$																
0	20	4,7,9	3,2,3	0.7, 0.6, 0.8	0.5 ,0.5 ,0.1	1513	2.0	2.8	4.7	2.7	1.2	**1.2**	1.2	1.5	1.8	1.6
	50	4,7,9	3,3,3	0.7, 0.5, 0.7	0.4 ,0.5 ,0.1	1649	2.2	2.4	5.0	4.4	1.3	1.2	1.6	**1.0**	1.8	1.3
1	20	4,7,9	3,2,3	0.7, 0.6, 0.8	0.5 ,0.4 ,0.1	1567	3.0	2.5	4.3	2.4	1.9	1.0	**0.9**	1.2	2.3	1.4
	50	4,7,9	3,3,3	0.7, 0.5, 0.7	0.4 ,0.4 ,0.2	1734	3.2	3.5	2.6	2.9	1.7	**0.9**	1.4	1.9	3.6	1.5
2	20	4,7,9	3,3,3	0.7, 0.5, 0.7	0.4 ,0.4 ,0.1	1679	3.5	3.8	3.0	1.4	1.9	**1.0**	1.6	1.6	3.4	1.6
	50	4,7,9	3,3,3	0.7, 0.5, 0.7	0.4 ,0.4 ,0.3	1991	4.1	6.8	4.0	4.0	1.8	1.6	**1.4**	1.8	5.4	1.5

Table 5 continued

c	θ	Hub Located	Cap level	Intensity	Costs		CPU time (s)									
					% Contribution	TC	EK	EK	MISOCPs							
					FC, LC, CC		OA-2s	OA-3s	1	2	3	4	5	6	7	9
p = 4, $\alpha = 0.4$																
0	20	3,4,6,7	2,3,3,3	0.4, 0.7, 0.7, 0.6	0.3 ,0.6 ,0.1	1539	1.7	2.2	2.6	2.8	1.7	0.9	1.5	**0.9**	2.2	1.5
	50	3,4,6,7	2,3,3,3	0.4, 0.7, 0.7, 0.6	0.3 ,0.6 ,0.1	1672	2.6	2.8	2.0	1.3	2.1	1.2	1.1	1.4	2.5	**1.1**
1	20	3,4,6,7	2,3,3,3	0.4, 0.7, 0.7, 0.6	0.3 ,0.6 ,0.1	1581	2.6	2.4	4.0	3.4	1.4	**0.8**	1.2	1.1	1.6	1.4
	50	2,4,7,9	3,3,3,3	0.5, 0.5, 0.6, 0.6	0.3 ,0.6 ,0.1	1768	3.0	3.8	3.1	3.1	1.5	**1.1**	1.3	1.2	2.8	1.2
2	20	2,4,7,9	3,3,3,3	0.5, 0.5, 0.6, 0.6	0.3 ,0.6 ,0.1	1703	2.6	4.5	3.5	1.8	4.2	1.2	**1.1**	1.5	3.5	1.9
	50	2,4,7,9	3,3,3,3	0.5, 0.5, 0.6, 0.6	0.3 ,0.5 ,0.2	1963	2.6	6.6	2.0	2.5	3.4	1.4	**1.2**	1.2	7.2	1.4
p = 4, $\alpha = 0.8$																
0	20	4,5,6,7	3,2,3,2	0.7, 0.5, 0.8, 0.8	0.4 ,0.5 ,0.1	1705	2.0	3	4.3	1.4	3.3	**0.9**	1.1	1.0	2.4	1.2
	50	3,4,7,9	2,3,3,3	0.4, 0.7, 0.6, 0.7	0.4 ,0.5 ,0.1	1865	2.6	5	3.5	**1.1**	2.6	1.4	1.1	1.8	2.0	1.3
1	20	3,4,7,9	2,3,3,3	0.4, 0.7, 0.6, 0.7	0.4 ,0.5 ,0.1	1773	2.2	5	3.5	1.1	2.6	1.4	**1.0**	1.4	3.7	1.3
	50	4,5,6,7	3,3,3,3	0.5, 0.5, 0.6, 0.6	0.4 ,0.5 ,0.1	1942	4.6	3.8	4.9	1.9	3.5	1.2	**1.1**	1.4	2.6	1.2
2	20	4,5,6,7	3,3,3,3	0.5, 0.5, 0.6, 0.6	0.4 ,0.5 ,0.1	1877	7.1	4.2	5.0	4.6	3.5	1.7	**1.1**	1.6	3.2	1.8
	50	4,5,6,7	3,3,3,3	0.5, 0.5, 0.6, 0.6	0.3 ,0.5 ,0.2	2137	5.4	5	5.8	4.2	3.2	1.9	1.5	**1.3**	6.2	1.8
No. of best performing instances							0	0	0	1	0	8	11	3	0	1

Bold indicate best CPU time for a given instance

Table 6 Comparison of EK-MISOCPs against EK-OA-2s and EK-OA-3s for N = 15 (CAB dataset)

c	θ	Hub located	Cap level	Intensity	Costs		CPU time (s)									
					% Contribution	TC	EK	EK	MISOCPs							
					FC, LC, CC		OA-2s	OA-3s	1	2	3	4	5	6	7	9
p = 3, α = 0.4																
0	20	4,12,13	3,2,3	0.9, 0.4, 0.7	0.5 ,0.4 ,0.1	1743	22.2	20.9	5.0	**2.7**	4.1	4.0	3.4	6.0	9.8	6.0
	50	4,12,13	3,2,3	0.8, 0.5, 0.7	0.5 ,0.4 ,0.1	1900	29.7	41.4	3.1	**2.9**	4.4	3.1	3.0	7.6	14.8	8.3
1	20	4,12,13	3,2,3	0.8, 0.4, 0.8	0.5 ,0.4 ,0.1	1809	16.8	33.8	3.5	**2.3**	3.0	3.4	5.8	5.9	18.8	5.6
	50	4,6,7	3,3,3	0.6, 0.6, 0.6	0.5 ,0.4 ,0.1	1990	39.8	45.4	4.2	**3.5**	3.7	4.5	4.0	7.5	27.4	7.4
2	20	4,6,7	3,3,3	0.6, 0.6, 0.6	0.5 ,0.4 ,0.1	1936	124.9	59.2	4.4	4.4	3.9	4.0	**3.8**	9.3	53.3	10.4
	50	4,6,7	3,3,3	0.6, 0.6, 0.6	0.5 ,0.3 ,0.2	2198	30.6	53.3	8.8	6.1	9.0	5.9	4.9	14.5	195.6	10.2
p = 3, α = 0.8																
0	20	4,5,11	2,3,3	0.7, 0.7, 0.6	0.6 ,0.4 ,0	1932	24.0	34.1	5.6	3.5	4.9	**3.4**	5.1	6.8	8.1	6.4
	50	4,5,11	3,3,3	0.6, 0.6, 0.6	0.6 ,0.4 ,0.1	2047	32.6	43.5	5.7	4.1	4.4	**2.7**	3.2	3.8	20.5	4.6
1	20	4,5,11	3,3,3	0.6, 0.6, 0.6	0.6 ,0.4 ,0	1979	25	48.0	3.8	3.8	4.4	**3.6**	7.3	9.2	17.6	6.2
	50	4,5,11	3,3,3	0.6, 0.6, 0.6	0.5 ,0.4 ,0.1	2116	29.1	41.7	5.8	4.3	5.9	4.0	**3.2**	4.9	34.5	6.4
2	20	4,5,11	3,3,3	0.6, 0.6, 0.6	0.6 ,0.4 ,0.1	2062	29.0	44.8	5.1	4.4	6.0	4.9	**3.4**	8.1	25.1	5.3
	50	4,5,11	3,3,3	0.6, 0.6, 0.6	0.5 ,0.3 ,0.2	2324	46	589.6	12.1	7.0	9.7	6.9	**4.5**	15.3	621.3	9.5

Table 6 continued

c	θ	Hub located	Cap level	Intensity	Costs		CPU time (s)									
					% Contribution	TC	EK	EK	MISOCPs							
					FC, LC, CC		OA-2s	OA-3s	1	2	3	4	5	6	7	9
p = 4, $\alpha = 0.4$																
0	20	4,6,7,12	3,3,2,2	0.7, 0.8, 0.7, 0.5	0.4 ,0.5 ,0.1	1835	12.9	16.4	4.8	3.3	4.7	**2.4**	3.2	4.3	9.9	3.7
	50	4,6,12,13	3,3,2,3	0.6, 0.6, 0.5, 0.7	0.4 ,0.5 ,0.1	1991	28.2	33.6	5.5	3.4	3.7	2.8	**2.6**	4.9	15.5	5.5
1	20	4,5,7,12	3,3,2,2	0.8, 0.8, 0.7, 0.5	0.4 ,0.5 ,0.1	1902	30.1	45.2	4.9	3.7	3.8	**3.1**	3.8	6.9	11.0	4.6
	50	4,6,12,13	3,3,2,3	0.6, 0.6, 0.5, 0.7	0.4 ,0.5 ,0.1	2084	31.1	70.7	7.3	5.1	4.5	3.3	**3.0**	6.0	22.2	5.9
2	20	4,6,12,13	3,3,2,3	0.6, 0.6, 0.5, 0.7	0.4 ,0.5 ,0.1	2013	44.1	36.6	6.0	4.3	6.8	**3.5**	3.7	7.8	80.8	6.8
	50	4,6,12,13	3,3,3,3	0.6, 0.6, 0.4, 0.6	0.4 ,0.4 ,0.2	2304	22.0	36.6	21.6	6.5	11.6	8.0	**5.2**	10.3	1704.1	11.8
p = 4, $\alpha = 0.8$																
0	20	4,5,7,8	3,3,2,2	0.8, 0.8, 0.5, 0.7	0.5 ,0.4 ,0.1	2064	26.7	22.8	4.1	**2.8**	3.7	3.4	3.4	4.5	8.8	4.2
	50	1,4,6,7	2,3,3,3	0.5, 0.7, 0.6, 0.6	0.5 ,0.4 ,0.1	2233	27.3	34.1	5.5	4.5	3.7	**2.9**	3.3	6.6	20.2	7.4
1	20	4,5,7,8	3,3,2,2	0.8, 0.8, 0.5, 0.7	0.5 ,0.4 ,0.1	2130	36.2	37.9	4.9	3.4	3.8	**2.7**	4.0	4.8	14.3	5.2
	50	4,8,9,13	3,3,3,3	0.6, 0.4, 0.6, 0.6	0.5 ,0.4 ,0.1	2306	27.9	27.2	6.7	6.8	5.4	**4.2**	4.9	6.1	52.2	7.5
2	20	4,8,9,13	3,3,3,3	0.6, 0.4, 0.6, 0.6	0.5 ,0.4 ,0.1	2243	28.3	38.7	9.3	6.2	6.2	4.1	**3.6**	6.8	56.8	7.0
	50	4,8,9,13	3,3,3,3	0.6, 0.4, 0.6, 0.6	0.4 ,0.4 ,0.2	2513	40.3	45.9	33.8	11.3	13.6	7.4	**5.8**	20.3	346.7	13.5
No. of best performing instances							0	0	0	5	0	9	0	0	0	0

Bold indicate best CPU time for a given instance

Table 7 Comparison of EK-MISOCPs against EK-OA-2s and EK-OA-3s for N = 20 (CAB dataset)

c	θ	Hub located	Cap level	Intensity	Costs		CPU time (s)									
					% Contribution	TC	EK	EK	MISOCPs							
					FC, LC, CC		OA-2s	OA-3s	1	2	3	4	5	6	7	9
p = 3, α = 0.4																
0	20	4,12,17	3,2,3	0.8, 0.3, 0.8	0.5 ,0.4 ,0.1	1652	34.1	49.9	12.3	10.8	14.2	**9.2**	13.4	11.2	19.3	10.0
	50	4,12,17	3,2,3	0.8, 0.4, 0.8	0.5 ,0.4 ,0.1	1803	80.3	122.8	**8.6**	13.6	13.1	10.9	10.9	16.6	27.4	12.8
1	20	4,12,17	3,2,3	0.8, 0.4, 0.8	0.5 ,0.4 ,0.1	1717	66.7	155.9	14.7	12.4	10.1	**7.9**	15.6	13.7	34.4	21.7
	50	5,7,17	3,3,3	0.7, 0.5, 0.7	0.5 ,0.4 ,0.1	1894	99.2	215.1	19.5	17.8	17.3	17.4	**14.0**	17.5	118.5	19.6
2	20	5,7,17	3,3,3	0.7, 0.5, 0.7	0.5 ,0.4 ,0.1	1841	83.1	257.0	19.9	18.5	19.1	18.3	**15.0**	34.8	730.7	29.2
	50	4,7,17	3,3,3	0.6, 0.6, 0.6	0.4 ,0.4 ,0.2	2099	84.9	227.5	40.7	33.6	34.7	36.5	**19.9**	70.5	6484.1	60.1
p = 3, α = 0.8																
0	20	1,4,17	2,3,3	0.6, 0.7, 0.7	0.6 ,0.4 ,0	1882	85.4	231.9	18.7	**15.0**	17.5	15.9	15.4	18.9	74.6	22.4
	50	1,4,17	2,3,3	0.6, 0.7, 0.7	0.5 ,0.3 ,0.1	2010	109.7	301.4	27.7	18.7	19.2	15.2	**14.0**	32.6	164.4	34.6
1	20	1,4,17	2,3,3	0.6, 0.7, 0.7	0.6 ,0.4 ,0.1	1927	97.7	252.6	21.9	18.0	17.9	**15.6**	17.5	22.3	131.1	22.8
	50	6,11,17	3,3,3	0.7, 0.5, 0.6	0.5 ,0.4 ,0.1	2080	110.3	288.9	32.6	20.9	29.5	23.0	**15.8**	42.2	565.8	39.9
2	20	6,11,17	3,3,3	0.7, 0.5, 0.6	0.5 ,0.4 ,0.1	2027	143.7	243.3	41.1	36.1	28.7	24.0	**21.3**	51.1	1143.5	54.8
	50	4,17,20	3,3,3	0.6, 0.6, 0.6	0.5 ,0.3 ,0.2	2286	512.1	316.5	106.0	32.9	74.9	58.7	**22.8**	123.1	*	168.5

Table 7 continued

c	θ	Hub located	Cap level	Intensity	Costs		CPU time (s)									
					% Contribution	TC	EK	EK	MISOCPs							
					FC, LC, CC		OA-2s	OA-3s	1	2	3	4	5	6	7	9
p = 4, $\alpha = 0.4$																
0	20	1,4,12,17	2,3,2,3	0.7, 0.7, 0.4, 0.8	0.4 ,0.5 ,0.1	1752	49.7	57.3	20.0	16.5	12.1	**7.6**	10.2	9.6	27.2	10.5
	50	1,4,12,17	3,3,2,3	0.5, 0.7, 0.4, 0.7	0.4 ,0.5 ,0.1	1906	89.3	88.4	15.1	15.4	15.8	12.0	**11.7**	12.1	32.5	14.0
1	20	1,4,12,17	3,3,2,3	0.4, 0.7, 0.4, 0.8	0.4 ,0.5 ,0.1	1815	86.2	104.1	40.7	33.6	34.7	36.5	**19.9**	70.5	6484.1	60.1
	50	4,12,13,17	3,2,3,3	0.6, 0.4, 0.6, 0.7	0.4 ,0.5 ,0.1	1998	80.0	138.0	16.7	21.2	23.7	14.7	**11.8**	23.4	85.3	22.7
2	20	4,12,13,17	3,2,3,3	0.6, 0.4, 0.6, 0.7	0.4 ,0.5 ,0.1	1929	90.1	134.6	27.1	23.1	20.3	15.3	**14.4**	22.2	149.5	26.3
	50	4,7,17,20	3,3,3,3	0.5, 0.5, 0.6, 0.5	0.4 ,0.4 ,0.2	2234	112.2	158.8	150.8	35.5	55.0	60.6	**20.9**	103.0	7720.8	130.0
p = 4, $\alpha = 0.8$																
0	20	1,4,8,17	2,3,2,3	0.6, 0.7, 0.5, 0.8	0.5 ,0.4 ,0.1	2030	14400	268.7	22.0	14.2	21.1	**12.1**	19.6	26.4	53.6	23.1
	50	4,8,17,20	3,2,3,3	0.6, 0.5, 0.7, 0.6	0.5 ,0.4 ,0.1	2177	89.8	329.7	38.4	27.3	20.4	21.1	**18.7**	27.2	179.2	23.4
1	20	4,8,17,20	3,2,3,3	0.6, 0.5, 0.7, 0.6	0.5 ,0.5 ,0.1	2090	166.4	359.1	40.8	25.0	27.3	23.6	**17.5**	31.7	129.6	29.8
	50	4,7,17,20	3,3,3,3	0.4, 0.5, 0.6, 0.6	0.5 ,0.4 ,0.1	2261	166.0	232.7	83.1	33.8	33.2	29.7	**22.9**	57.6	1094.6	75.3
2	20	4,8,17,20	3,2,3,3	0.6, 0.5, 0.7, 0.6	0.5 ,0.4 ,0.1	2192	193.9	232.5	95.4	35.5	43.8	36.7	**21.6**	56.1	1461.9	54.8
	50	4,11,17,20	3,3,3,3	0.5, 0.5, 0.6, 0.5	0.4 ,0.4 ,0.2	2435	222.1	165.3	499.7	167.5	87.5	78.8	**26.3**	93.1	9901.5	115.9
No. of best performing instances							0	0	1	1	0	5	17	0	0	0

Bold indicate best CPU time for a given instance

*Represents optimal solution not found in the given CPU time limit of 4 h

Table 8 Comparison of EK-MISOCPs against EK-OA-2s and EK-OA-3s for N = 25 (CAB dataset)

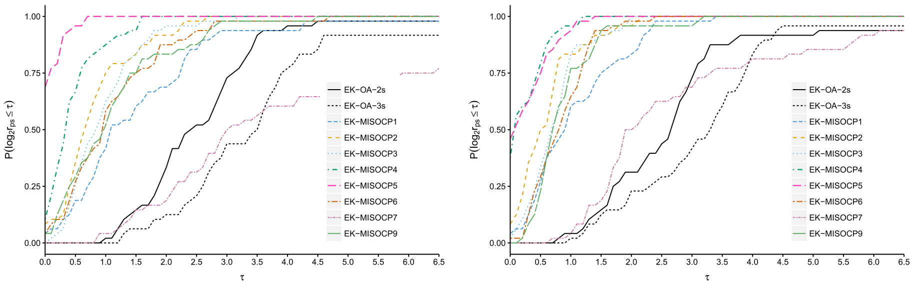
c	θ	Hub located	Cap level	Intensity	Costs		CPU time (s)									
					% Contribution	TC	EK	EK	MISOCPs							
					FC, LC, CC		OA-2s	OA-3s	1	2	3	4	5	6	7	9
p = 3, $\alpha = 0.4$																
0	20	4,12,18	3,2,3	0.7, 0.5, 0.8	0.5 ,0.4 ,0.1	1687	96.8	131.9	28.5	30.7	**19.9**	44.8	35.2	42.5	61.1	31.5
	50	4,12,17	3,2,3	0.7, 0.5, 0.7	0.5 ,0.4 ,0.1	1813	144.5	182.2	34.2	40.0	37.3	31.5	31.0	41.1	82.6	**29.6**
1	20	4,12,17	3,2,3	0.7, 0.5, 0.7	0.5 ,0.4 ,0.1	1732	155.4	198.4	29.0	37.3	42.2	32.6	**24.2**	48.0	85.9	38.8
	50	4,12,17	3,2,3	0.7, 0.6, 0.7	0.5 ,0.4 ,0.2	1925	210.6	343.7	46.0	53.0	64.7	36.3	**27.9**	49.4	486.2	66.5
2	20	4,12,17	3,2,3	0.7, 0.6, 0.7	0.5 ,0.4 ,0.1	1866	222.1	379.2	50.3	53.8	61.3	52.3	**33.4**	51.8	994.7	83.7
	50	5,8,17	3,3,3	0.6, 0.5, 0.6	0.5 ,0.3 ,0.2	2182	314.7	512.6	263.0	**88.7**	131.7	124.1	102.3	385.0	*	581.8
p = 3, $\alpha = 0.8$																
0	20	2,4,12	3,3,2	0.8, 0.7, 0.5	0.6 ,0.4 ,0	1951	290.6	482.6	88.1	78.3	101.5	90.4	**65.5**	86.7	201.2	88.3
	50	4,12,18	3,2,3	0.7, 0.5, 0.7	0.6 ,0.3 ,0.1	2078	245.0	663.0	119.6	96.4	100.3	79.9	**65.0**	128.8	283.0	144.5
1	20	4,12,18	3,2,3	0.7, 0.5, 0.7	0.6 ,0.4 ,0.1	1997	214.0	698.9	81.9	91.7	102.4	91.0	**70.6**	88.0	443.6	86.5
	50	4,12,18	3,2,3	0.7, 0.5, 0.7	0.5 ,0.3 ,0.1	2190	554.2	1250.9	148.3	228.5	178.6	141.5	**110.3**	208.3	13787.1	332.5
2	20	4,12,18	3,2,3	0.7, 0.5, 0.7	0.5 ,0.3 ,0.1	2132	726.2	1045.4	243.2	166.7	165.6	134.2	**92.2**	291.8	*	757.6
	50	5,11,18	3,3,3	0.6, 0.6, 0.6	0.5 ,0.3 ,0.2	2391	632.9	1079.6	812.7	479.5	382.7	354.5	**159.9**	1064.9	*	1038.2

Table 8 continued

c	θ	Hub located	Cap level	Intensity	Costs		CPU time (s)									
					% Contribution	TC	EK	EK	MISOCPs							
					FC, LC, CC		OA-2s	OA-3s	1	2	3	4	5	6	7	9
p = 4, $\alpha = 0.4$																
0	20	1,4,12,17	2,3,2,3	0.6, 0.7, 0.6, 0.8	0.4 ,0.5 ,0.1	1704	115.3	116.4	**42.5**	57.1	63.0	55.2	56.7	63.3	82.2	54.8
	50	1,4,12,17	3,3,2,3	0.4, 0.6, 0.6, 0.7	0.4 ,0.5 ,0.1	1963	184.1	322.5	**54.3**	75.8	93.4	80.0	**75.8**	71.8	350.9	84.4
1	20	1,4,12,17	2,3,2,3	0.6, 0.7, 0.6, 0.8	0.4 ,0.5 ,0.1	1857	214.9	272.2	54.9	73.2	84.0	**44.4**	47.8	52.8	140.7	63.8
	50	1,4,12,17	3,3,3,3	0.4, 0.6, 0.4, 0.7	0.4 ,0.5 ,0.1	2050	281.4	511.7	115.7	149.9	141.1	102.7	**75.4**	130.2	2370.5	108.3
2	20	1,4,12,17	3,3,3,3	0.4, 0.6, 0.4, 0.7	0.4 ,0.5 ,0.1	1988	292.4	465.9	125.2	130.3	139.9	122.5	**95.3**	112.8	4437.5	129.5
	50	1,4,12,17	3,3,3,3	0.5, 0.5, 0.4, 0.7	0.4 ,0.4 ,0.2	2244	391.7	527.1	2027.1	191.6	224.8	212.6	**105.2**	686.3	*	615.9
p = 4, $\alpha = 0.8$																
0	20	1,4,12,18	2,3,2,3	0.6, 0.7, 0.6, 0.8	0.5 ,0.4 ,0.1	2100	259.6	795.4	**83.7**	89.9	139.8	85.9	88.7	140.5	311.6	102.1
	50	1,4,12,18	2,3,2,3	0.6, 0.7, 0.6, 0.8	0.5 ,0.4 ,0.1	2262	649.7	3258.8	**215.9**	136.5	190.3	140.2	**134.8**	266.8	2344.9	226.4
1	20	1,4,12,18	2,3,2,3	0.6, 0.7, 0.6, 0.8	0.5 ,0.4 ,0.1	2156	690.8	1349.7	124.8	137.4	112.3	90.1	**87.9**	189.7	865.1	244.1
	50	4,12,17,20	3,3,3,3	0.6, 0.3, 0.5, 0.6	0.5 ,0.4 ,0.1	2339	1977.0	2848.1	834.4	224.9	186.6	232.0	**123.8**	455.9	*	628.5
2	20	4,12,17,20	3,3,3,3	0.6, 0.3, 0.5, 0.6	0.5 ,0.4 ,0.1	2277	950.7	2558.5	1487.8	254.8	296.7	322.0	**148.7**	761.2	*	1297.7
	50	4,8,17,20	3,3,3,3	0.6, 0.4, 0.5, 0.6	0.4 ,0.4 ,0.2	2522	1641.3	2165.8	*	755.6	919.1	450.9	**153.6**	3911.4	*	3349.6
No. of best performing instances							0	0	4	1	1	1	18	0	0	1

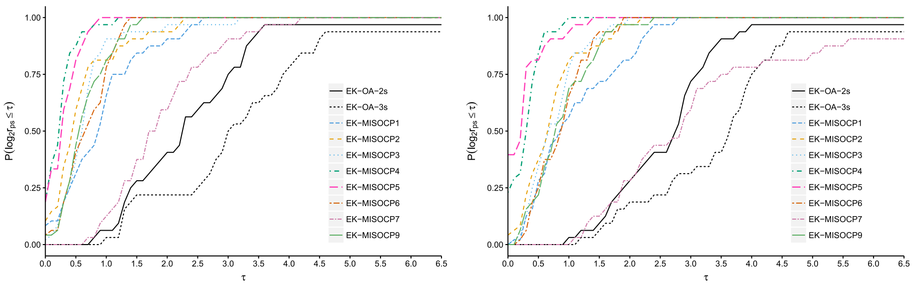
Bold indicate best CPU time for a given instance
*Represents optimal solution not found in the given CPU time limit of 4 h

C Performance profile for coefficient of variation (c), and unit congestion cost θ for CAB dataset

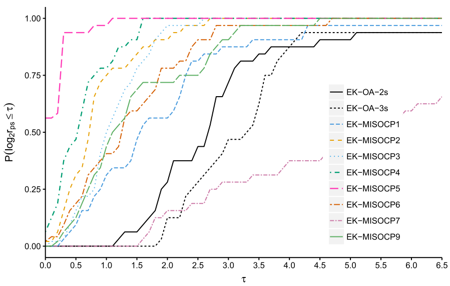


(a) $\theta = 20$ **(b)** $\theta = 50$

Fig. 2 Performance profile of EK-MISOCPs and EK-OA-based method for θ= {20, 50}



(a) c=0 **(b)** c=1



(c) c=2

Fig. 3 Performance profile of EK-MISOCPs and EK-OA-based method for $\{c = 0, 1, 2\}$

Table 9 Comparison among EK-MISOCP4, 5, 6 and 9 for $|\mathbf{N}| = \mathbf{25}$ (AP dataset)

c	θ	Hub	Cap	Intensity	Costs		CPU time (s)			
					FC, LC, CC	TC	MISOCPs			
							4	5	6	9
N = 25L										
p = 3, $\alpha = 0.4$										
0	50	2,13,18	3,3,3	0.9, 0.8, 0.9	0.93,0.06,0.01	73475.9	1.5%	**12.872**	5.5%	5.5%
	100	2,13,18	3,3,3	0.9, 0.8, 0.9	0.92,0.06,0.02	74158.5	**24.734**	25.52	7.8%	7.8%
	200	7,14,18	1,1,3	0.6, 0.4, 0.9	0.88,0.09,0.02	75346.6	10.4%	**14.268**	*	5.7%
1	50	2,13,18	3,3,3	0.9, 0.8, 0.9	0.92,0.06,0.02	74028.6	*	**9.624**	*	*
	100	7,14,18	1,1,3	0.6, 0.4, 0.9	0.89,0.09,0.02	75155.9	2%	**33.06**	12.5%	12.5%
	200	8,17,18	2,1,3	0.6, 0.5, 0.8	0.89,0.1,0.01	76711.6	48.435	62.64	49.231	**46.591**
2	50	7,14,18	1,1,3	0.6, 0.4, 0.9	0.88,0.09,0.03	75423	2.2%	**37.048**	9.4%	6.5%
	100	8,17,18	2,1,3	0.6, 0.5, 0.8	0.89,0.1,0.01	76716.6	3.8%	**34.7**	24.1%	24.1%
	200	8,17,18	3,1,3	0.5, 0.4, 0.8	0.88,0.1,0.02	77689.1	11%	**42.232**	17.6%	17.4%
p = 3, $\alpha = 0.8$										
0	50	2,13,18	3,3,3	0.9, 0.8, 1	0.94,0.05,0.01	84981.2	2.7%	**36.672**	4.6%	5.3%
	100	8,17,18	1,1,3	0.8, 0.5, 0.8	0.91,0.08,0.01	85481.6	3%	**37.608**	5.3%	5.3%
	200	8,17,18	1,1,3	0.8, 0.5, 0.8	0.91,0.08,0.01	86060.1	5.1%	**16.312**	*	*
1	50	8,17,18	1,1,3	0.8, 0.5, 0.8	0.92,0.08,0.01	85377.4	3.2%	**44.14**	4.9%	*
	100	8,17,18	1,1,3	0.8, 0.5, 0.8	0.91,0.08,0.01	85851.8	6.1%	**16.044**	*	*
	200	8,17,18	1,1,3	0.8, 0.5, 0.8	0.9,0.08,0.02	86800.5	41.5%	**11.844**	*	*
2	50	8,17,18	1,1,3	0.8, 0.5, 0.8	0.91,0.08,0.01	85932.7	3%	**52.296**	3.7%	5.8%
	100	8,17,18	2,1,3	0.6, 0.5, 0.8	0.9,0.09,0.01	86913	6.1%	**14.792**	*	*
	200	8,17,18	3,1,3	0.4, 0.5, 0.8	0.89,0.09,0.02	88008.2	54.512	65.824	**51.902**	17%

Table 9 continued

c	θ	Hub	Cap	Intensity	Costs		CPU time (s)			
					FC, LC, CC	TC	MISOCPs			
							4	5	6	9
p = 4, $\alpha = 0.4$										
0	50	2,8,16,18	3,1,1,3	0.9, 0.6, 0.6, 0.9	0.91,0.09,0.01	67818	1.2%	**17.968**	4.2%	4.2%
	100	2,8,16,18	3,1,1,3	0.9, 0.6, 0.6, 0.9	0.9,0.08,0.02	68348.3	2%	**13.924**	21.7%	*
	200	2,8,17,18	3,1,1,3	0.9, 0.6, 0.5, 0.8	0.88,0.1,0.02	69400.6	3.7%	**18.5**	5.8%	6%
1	50	2,8,16,18	3,1,1,3	0.9, 0.6, 0.6, 0.9	0.9,0.08,0.01	68197.1	3.5%	**18.06**	3.5%	4.8%
	100	2,8,16,18	3,1,1,3	0.9, 0.6, 0.6, 0.9	0.89,0.08,0.03	69106.4	4%	**23.996**	6%	6.1%
	200	2,8,17,18	3,1,1,3	0.9, 0.6, 0.5, 0.8	0.86,0.1,0.03	70320.9	6.4%	**40.188**	7.8%	8.5%
2	50	2,8,17,18	3,1,1,3	0.9, 0.6, 0.5, 0.8	0.88,0.1,0.02	69224.2	2.6%	**36.576**	6.5%	6.6%
	100	7,14,17,18	1,1,1,3	0.5, 0.4, 0.4, 0.7	0.86,0.13,0.01	70275.8	8%	**34.652**	9.8%	6.1%
	200	7,14,17,18	1,1,1,3	0.5, 0.4, 0.4, 0.7	0.85,0.13,0.02	71062.4	61.668	69.608	**53.996**	56.637
p = 4, $\alpha = 0.8$										
0	50	2,8,17,18	3,1,1,3	0.9, 0.5, 0.5, 0.8	0.91,0.09,0	81685.2	1.8%	**53.9**	3.5%	3.7%
	100	2,8,17,18	3,1,1,3	0.9, 0.5, 0.5, 0.8	0.9,0.09,0.01	82061.7	2.7%	**41.792**	5.1%	8%
	200	7,14,17,18	1,1,1,3	0.5, 0.4, 0.4, 0.7	0.88,0.11,0.01	82668.9	2%	**53.464**	4.3%	4.5%
1	50	2,8,17,18	3,1,1,3	0.9, 0.5, 0.5, 0.8	0.9,0.09,0.01	81928.1	2.7%	**47.76**	4.1%	7.3%
	100	7,14,17,18	1,1,1,3	0.5, 0.4, 0.4, 0.7	0.88,0.11,0.01	82472.2	2.8%	**39.248**	4.8%	7.5%
	200	7,14,17,18	1,1,1,3	0.5, 0.4, 0.4, 0.7	0.88,0.11,0.01	82904.8	18.1%	**15.096**	42%	19.3%
2	50	7,14,17,18	1,1,1,3	0.5, 0.4, 0.4, 0.7	0.88,0.11,0	82432.9	3.7%	**47.904**	13.7%	4.9%
	100	7,14,17,18	1,1,1,3	0.5, 0.4, 0.4, 0.7	0.88,0.11,0.01	82826.2	2%	**68.824**	4.8%	6%
	200	7,14,17,18	1,1,1,3	0.5, 0.4, 0.4, 0.7	0.87,0.11,0.02	83612.8	2.8%	**50.936**	4.9%	17.9%
No. of best performing instances							1	32	2	1

Table 9 continued

c	θ	Hub	Cap	Intensity	Costs		CPU time (s)			
					FC, LC, CC	TC	MISOCPs			
							4	5	6	9
N = 25T										
p = 3, $\alpha = 0.4$										
0	50	12,14,19	3,3,3	1, 0.7, 1	0.93,0.05,0.02	89407.7	2.8%	**19.512**	*	*
	100	9,12,19	3,3,3	0.9, 0.8, 1	0.93,0.05,0.02	90731.5	*	**8.14**	*	*
	200	9,12,25	3,3,3	0.7, 0.8, 0.9	0.93,0.05,0.02	91740.7	3.1%	**20.96**	*	*
1	50	9,12,19	3,3,3	0.9, 0.8, 1	0.93,0.05,0.02	90595.3	*	**8.068**	*	*
	100	9,12,25	3,3,3	0.7, 0.8, 0.9	0.93,0.05,0.02	91503.5	4.1%	**19.326**	*	*
	200	9,12,25	3,3,3	0.7, 0.8, 0.9	0.91,0.05,0.03	93086.9	2.3%	**18.256**	*	*
2	50	9,12,25	3,3,3	0.7, 0.8, 0.9	0.93,0.05,0.02	91721.4	2.9%	**21.396**	*	*
	100	9,12,25	3,3,3	0.7, 0.8, 0.9	0.91,0.05,0.04	93522.8	2.1%	**19.764**	*	*
	200	9,12,25	3,3,3	0.8, 0.8, 0.8	0.89,0.05,0.06	96947.7	2%	**24.568**	*	*
p = 3, $\alpha = 0.8$										
0	50	12,14,19	3,3,3	1, 0.7, 1	0.94,0.05,0.01	98899.4	**40.638**	41.11	40.2%	20.2%
	100	12,14,19	3,3,3	1, 0.7, 1	0.93,0.05,0.03	100276	*	**24.216**	*	*
	200	12,14,19	3,3,3	0.9, 0.8, 1	0.92,0.04,0.04	102434	4.2%	**34.826**	*	*
1	50	12,14,19	3,3,3	1, 0.7, 1	0.93,0.05,0.03	100144	*	**30.952**	*	*
	100	12,14,19	3,3,3	0.9, 0.8, 1	0.92,0.04,0.03	102170	4.3%	**30.578**	*	*
	200	12,14,19	3,3,3	0.9, 0.8, 0.9	0.91,0.04,0.04	105076	6.8%	**37.612**	*	*
2	50	12,14,19	3,3,3	0.9, 0.8, 1	0.91,0.04,0.04	102860	5.1%	**25.23**	*	*
	100	12,14,19	3,3,3	0.9, 0.8, 0.9	0.91,0.04,0.05	105792	6.4%	**38.08**	*	*
	200	12,14,25	3,3,3	0.8, 0.6, 0.8	0.91,0.05,0.05	110863	6.2%	**40.008**	*	*

Table 9 continued

c	θ	Hub	Cap	Intensity	Costs		CPU time (s)			
					FC, LC, CC	TC	MISOCPs			
							4	5	6	9
p = 4, $\alpha = 0.4$										
0	50	2,12,14,19	2,3,2,3	0.7, 0.8, 0.9, 1	0.92,0.06,0.01	78054.8	**31.451**	36.94	*	11.5%
	100	2,12,14,19	3,3,3,3	0.5, 0.8, 0.6, 1	0.91,0.07,0.02	78933.7	2.6%	**27.284**	*	*
	200	2,12,14,19	3,3,3,3	0.5, 0.8, 0.6, 1	0.89,0.07,0.04	80588.9	3.1%	**31.452**	*	21%
1	50	2,12,14,19	3,3,3,3	0.5, 0.8, 0.6, 1	0.91,0.07,0.02	78787.7	6.1%	**19.97**	*	*
	100	2,12,14,19	3,3,3,3	0.5, 0.8, 0.6, 1	0.9,0.07,0.04	80296.9	3.4%	**27.364**	*	*
	200	2,12,14,19	3,3,3,3	0.5, 0.8, 0.7, 0.9	0.9,0.06,0.04	82452.7	30.3%	**36.924**	23.6%	23.9%
2	50	2,12,14,19	3,3,3,3	0.5, 0.8, 0.6, 1	0.89,0.07,0.04	80832.5	6.2%	**30.398**	*	*
	100	2,12,14,19	3,3,3,3	0.5, 0.8, 0.7, 0.9	0.89,0.06,0.04	82786.7	5.4%	**29.784**	*	*
	200	2,12,14,25	3,3,3,3	0.5, 0.7, 0.5, 0.8	0.88,0.07,0.05	85939.7	6.9%	**29.778**	*	28.3%
p = 4, $\alpha = 0.8$										
0	50	2,12,13,19	2,3,3,3	0.7, 0.8, 0.9, 1	0.94,0.05,0.01	91489.3	5.1%	**52.408**	11.6%	18.8%
	100	2,12,13,19	3,3,3,3	0.5, 0.8, 0.9, 1	0.93,0.05,0.02	92556.1	32.7%	**43.4**	11.9%	9.1%
	200	12,13,19,23	3,3,3,3	0.9, 0.8, 0.9, 0.6	0.92,0.05,0.02	93854.5	4.6%	**47.118**	15.4%	14.3%
1	50	2,12,13,19	3,3,3,3	0.5, 0.8, 0.9, 1	0.93,0.05,0.02	92395.4	4.4%	**48.376**	15.7%	8.6%
	100	12,13,19,23	3,3,3,3	0.9, 0.8, 0.9, 0.6	0.93,0.05,0.02	93538.7	27.7%	**29.52**	*	*
	200	12,13,19,23	3,3,3,3	0.9, 0.8, 0.9, 0.7	0.91,0.05,0.04	95539.1	53.8%	**38.78**	18.1%	21%
2	50	12,13,19,23	3,3,3,3	0.9, 0.8, 0.9, 0.6	0.92,0.05,0.02	93807.1	53.9%	**26.634**	*	*
	100	12,14,19,23	3,3,3,3	0.9, 0.5, 0.9, 0.7	0.91,0.06,0.04	95969.7	6.8%	**51.56**	11.5%	13.2%
	200	12,14,19,23	3,3,3,3	0.9, 0.5, 0.9, 0.7	0.87,0.05,0.07	99633.4	0.1%	**94.404**	10.5%	8%
No. of best performing instances							2	34	0	0

Bold indicate best CPU time for a given instance
*Represents no solution in the given time limit

Table 10 Comparison among EK-MISOCP4, 5, 6 and 9 for $|\mathbf{N}| = \mathbf{50}$ (AP dataset)

c	θ	Hub	Cap	Intensity	Costs		CPU time (s)			
					FC, LC, CC	TC	MISOCPs			
							4	5	6	9
N = 50L										
p = 3, $\alpha = 0.4$										
0	50	15,27,35	2,1,3	0.8, 0.8, 0.8	0.92,0.08,0	74763.4	3%	**116.774**	*	*
	100	15,27,35	2,1,3	0.8, 0.8, 0.8	0.92,0.08,0.01	75086.8	3.2%	**338.784**	*	*
	200	15,28,35	2,1,3	0.7, 0.5, 0.8	0.9,0.08,0.01	75579.8	2.2%	**431.446**	28.9%	*
1	50	15,27,35	2,1,3	0.8, 0.8, 0.8	0.92,0.08,0.01	74969.9	2.4%	**370.762**	*	*
	100	15,28,35	2,1,3	0.7, 0.5, 0.8	0.91,0.08,0.01	75380.4	2.6%	**385.506**	*	*
	200	15,28,35	3,1,3	0.5, 0.5, 0.8	0.9,0.09,0.01	75979.3	8.9%	**448.654**	*	9.4%
2	50	15,28,35	2,1,3	0.7, 0.5, 0.8	0.91,0.08,0.01	75409.5	3.7%	**340.874**	70%	*
	100	15,28,35	3,1,3	0.5, 0.5, 0.8	0.9,0.09,0.01	75992.2	3.9%	**446.93**	*	*
	200	15,28,35	3,1,3	0.5, 0.5, 0.8	0.89,0.08,0.03	77123.6	18.5%	**493.07**	10.5%	6.4%
p = 3, $\alpha = 0.8$										
0	50	15,27,35	1,2,3	0.9, 0.7, 0.8	0.93,0.06,0.01	85447.6	2.7%	**537.106**	*	*
	100	15,27,35	2,1,3	0.7, 0.8, 0.8	0.93,0.07,0.01	85827	3%	**280.716**	*	*
	200	15,27,35	2,2,3	0.7, 0.6, 0.8	0.92,0.07,0.01	86349.1	3.6%	**452.032**	65.4%	*
1	50	15,27,35	2,1,3	0.7, 0.8, 0.8	0.93,0.07,0.01	85709.7	3.7%	**258.748**	*	*
	100	15,27,35	2,2,3	0.7, 0.6, 0.8	0.92,0.07,0.01	86136.9	3.1%	**649.472**	26.5%	*
	200	15,27,35	3,3,3	0.5, 0.4, 0.8	0.91,0.07,0.01	86796.6	8.9%	**528.212**	*	*
2	50	15,27,35	2,2,3	0.7, 0.6, 0.8	0.92,0.07,0.01	86174.8	3.4%	**367.834**	*	*
	100	15,27,35	3,3,3	0.5, 0.4, 0.8	0.91,0.07,0.01	86820.2	7.2%	**755.904**	6.7%	*
	200	15,27,35	3,3,3	0.5, 0.4, 0.8	0.9,0.07,0.03	87972.6	5.3%	**685.908**	25.8%	7.8%

Table 10 continued

c	θ	Hub	Cap	Intensity	Costs		CPU time (s)			
					FC, LC, CC	TC	MISOCPs			
							4	5	6	9
p = 4, α = 0.4										
0	50	6,27,32,35	1,2,3,2	0.7, 0.8, 0.6, 0.8	0.9,0.09,0.01	69738.1	*	**484.038**	*	*
	100	6,27,32,35	1,3,3,2	0.7, 0.6, 0.6, 0.8	0.9,0.09,0.01	70087.8	6%	**955.1**	3.7%	3.1%
	200	6,27,32,35	1,3,3,3	0.7, 0.6, 0.6, 0.6	0.89,0.1,0.01	70662	10.5%	**281.668**	*	*
1	50	6,27,32,35	1,3,3,2	0.7, 0.6, 0.6, 0.8	0.9,0.09,0.01	69951.7	5.1%	**780.996**	5.3%	8.3%
	100	6,27,32,35	1,3,3,3	0.7, 0.6, 0.6, 0.6	0.89,0.1,0.01	70414.7	5.6%	**476.92**	45.9%	*
	200	6,27,32,35	1,3,3,3	0.7, 0.6, 0.6, 0.6	0.88,0.1,0.02	71079.5	8.8%	**652.208**	51.1%	*
2	50	6,27,32,35	1,3,3,3	0.7, 0.6, 0.6, 0.6	0.89,0.1,0.01	70395.4	*	**348.712**	*	*
	100	6,27,32,35	1,3,3,3	0.7, 0.6, 0.6, 0.6	0.89,0.1,0.02	71041	8.3%	**500.232**	*	*
	200	15,28,32,35	3,1,3,3	0.5, 0.5, 0.5, 0.6	0.87,0.11,0.02	71955.1	*	**523.654**	*	*
p = 4, α = 0.8										
0	50	15,27,33,35	2,1,1,2	0.7, 0.8, 0.7, 0.8	0.91,0.08,0	82134.2	3.6%	**1440.12**	4.6%	4.7%
	100	15,27,33,35	2,1,1,2	0.7, 0.8, 0.7, 0.8	0.91,0.08,0.01	82511.8	3.2%	**1127.72**	7.3%	4.9%
	200	15,27,33,35	2,2,1,2	0.7, 0.6, 0.7, 0.8	0.9,0.09,0.01	83244.3	2.9%	**1609.54**	5.1%	3.5%
1	50	15,27,33,35	2,1,1,2	0.7, 0.8, 0.7, 0.8	0.91,0.08,0.01	82361.8	2.9%	**1208.93**	5.6%	6.1%
	100	15,27,33,35	2,2,1,2	0.7, 0.6, 0.7, 0.8	0.9,0.09,0.01	82964.2	4.4%	**1086.15**	4.9%	7.1%
	200	15,27,33,35	3,3,1,3	0.5, 0.4, 0.7, 0.5	0.89,0.09,0.01	83715.8	4.9%	**938.108**	9.2%	7.7%
2	50	15,27,33,35	2,2,1,2	0.7, 0.6, 0.7, 0.8	0.9,0.09,0.01	82994.9	3.7%	**1104.28**	5.5%	10.3%
	100	15,27,33,35	3,3,1,3	0.5, 0.4, 0.7, 0.5	0.89,0.09,0.01	83660.3	4%	**1067.78**	9%	9.2%
	200	15,27,32,35	3,3,3,3	0.5, 0.4, 0.5, 0.6	0.89,0.09,0.02	84417.4	4.2%	**1224.02**	8.1%	6.7%
No. of best performing instances							0	36	0	0

Table 10 continued

c	θ	Hub	Cap	Intensity	Costs		CPU time (s)			
					FC, LC, CC	TC	MISOCPs			
							4	5	6	9
N = 50T										
p = 3, $\alpha = 0.4$										
0	50	14,32,46	3,3,3	1, 1, 1	0.92,0.05,0.03	85655.5	3.3%	**210.299**	*	*
	100	25,32,46	3,3,3	0.9, 0.9, 1	0.92,0.05,0.03	86986.3	28.7%	**77.007**	*	*
	200	25,32,46	3,3,3	0.9, 0.9, 1	0.9,0.05,0.05	89232	23.9%	**120.364**	*	*
1	50	25,32,46	3,3,3	0.9, 0.9, 1	0.93,0.05,0.02	86847.5	4.5%	**253.749**	*	*
	100	25,32,46	3,3,3	0.9, 0.9, 1	0.91,0.05,0.05	88954.4	24.2%	**179.458**	*	*
	200	25,32,46	3,3,3	0.9, 0.9, 0.9	0.88,0.05,0.08	92614.4	*	**149.601**	*	*
2	50	25,32,46	3,3,3	0.9, 0.9, 0.9	0.9,0.05,0.05	89698.6	18.6%	**239.085**	*	*
	100	25,32,46	3,3,3	0.9, 0.9, 0.9	0.86,0.05,0.09	94004.7	39.5%	**404.728**	*	*
	200	25,32,46	3,3,3	0.9, 0.9, 0.9	0.79,0.04,0.17	102617	5.9%	**708.027**	*	*
p = 3, $\alpha = 0.8$										
0	50	25,32,46	3,3,3	0.9, 0.9, 1	0.94,0.04,0.01	95990.9	*	**215.879**	*	*
	100	25,32,46	3,3,3	0.9, 0.9, 1	0.93,0.04,0.02	97130.5	*	**226.852**	*	*
	200	25,32,46	3,3,3	0.9, 0.9, 1	0.91,0.04,0.05	99409.7	*	**460.623**	*	*
1	50	25,32,46	3,3,3	0.9, 0.9, 1	0.93,0.04,0.02	96991.6	6.5%	**550.565**	*	*
	100	25,32,46	3,3,3	0.9, 0.9, 1	0.91,0.04,0.04	99131.9	2.5%	**320.46**	*	*
	200	25,32,46	3,3,3	0.9, 0.9, 0.9	0.89,0.04,0.07	102782	1.2%	**845.82**	*	*
2	50	25,32,46	3,3,3	0.9, 0.9, 0.9	0.91,0.04,0.04	99866.1	4%	**211.736**	*	*
	100	25,32,46	3,3,3	0.9, 0.9, 0.9	0.88,0.04,0.08	104172	**895.252**	1167.57	*	*
	200	25,32,46	3,3,3	0.9, 0.9, 0.9	0.81,0.04,0.15	112785	1.5%	**1219.12**	*	*

Table 10 continued

c	θ	Hub	Cap	Intensity	Costs		CPU time (s)			
					FC, LC, CC	TC	MISOCPs			
							4	5	6	9
p = 4, $\alpha = 0.4$										
0	50	14,32,35,38	3,2,3,3	0.9, 0.8, 0.9, 0.9	0.92,0.07,0.01	72031.6	*	**102.578**	*	*
	100	14,32,35,38	3,2,3,3	0.9, 0.8, 0.9, 0.9	0.91,0.07,0.03	73059.9	1.3%	**188.701**	*	*
	200	14,32,35,38	3,3,3,3	0.9, 0.6, 0.9, 0.9	0.89,0.07,0.04	74710.3	1.8%	**474.116**	*	*
1	50	14,32,35,38	3,2,3,3	0.9, 0.8, 0.9, 0.9	0.91,0.07,0.02	72884.4	1.2%	**320.526**	*	*
	100	14,32,35,38	3,3,3,3	0.9, 0.6, 0.9, 0.9	0.89,0.07,0.04	74383.7	1.5%	**378.696**	*	*
	200	14,32,38,46	3,3,3,3	0.8, 0.6, 0.8, 0.8	0.89,0.07,0.04	76258.5	3.9%	**571.256**	*	*
2	50	14,32,38,46	3,3,3,3	0.8, 0.5, 0.9, 0.9	0.9,0.07,0.03	74727.3	1.3%	**535.173**	*	*
	100	14,26,32,46	3,3,3,3	0.8, 0.8, 0.5, 0.8	0.89,0.07,0.04	76523	7.3%	**441.907**	*	*
	200	14,26,32,46	3,3,3,3	0.7, 0.7, 0.6, 0.8	0.86,0.07,0.07	79231.3	3.7%	**358.919**	*	*
p = 4, $\alpha = 0.8$										
0	50	14,34,35,38	3,3,3,3	0.9, 0.9, 0.9, 1	0.93,0.05,0.02	84130.7	2.4%	**511.334**	*	*
	100	14,32,35,38	3,2,3,3	0.9, 0.8, 0.9, 0.9	0.92,0.06,0.03	85214.3	2.5%	**1000.13**	*	*
	200	14,32,35,38	3,3,3,3	0.8, 0.6, 0.9, 0.9	0.91,0.06,0.04	87023.4	6.6%	**927.888**	53.3%	*
1	50	14,32,35,38	3,2,3,3	0.9, 0.8, 0.9, 0.9	0.92,0.06,0.02	85038.2	4.1%	**617.19**	*	*
	100	14,32,35,38	3,3,3,3	0.8, 0.6, 0.9, 0.9	0.91,0.06,0.03	86697.2	5.1%	**778.654**	*	*
	200	14,32,38,46	3,3,3,3	0.8, 0.6, 0.8, 0.8	0.9,0.06,0.04	88759.3	4.2%	**716.398**	*	*
2	50	14,32,35,38	3,3,3,3	0.8, 0.7, 0.9, 0.8	0.92,0.06,0.03	87125	3.4%	**870.678**	*	*
	100	14,32,38,46	3,3,3,3	0.8, 0.6, 0.8, 0.8	0.9,0.06,0.04	89112	6.1%	**939.039**	*	*
	200	14,26,32,46	3,3,3,3	0.7, 0.7, 0.6, 0.8	0.88,0.06,0.06	92041.5	4.4%	**1578.46**	2.1%	4.1%
No. of best performing instances							1	35	0	0

Bold indicate best CPU time for a given instance
*Represents no solution in the given time limit

References

Ahmadi-Javid, A., & Hoseinpour, P. (2017). Convexification of queueing formulas by mixed-integer second-order cone programming: An application to a discrete location problem with congestion. arXiv preprint arXiv:1710.05794.

Alkaabneh, F., Diabat, A., & Elhedhli, S. (2019). A Lagrangian heuristic and grasp for the hub-and-spoke network system with economies-of-scale and congestion. *Transportation Research Part C: Emerging Technologies*, *102*, 249–273.

Alumur, S., & Kara, B. Y. (2008). Network hub location problems: The state of the art. *European Journal of Operational Research*, *190*(1), 1–21.

Atamtürk, A., Berenguer, G., & Shen, Z. J. (2012). A conic integer programming approach to stochastic joint location-inventory problems. *Operations Research*, *60*(2), 366–381.

Aykin, T. (1994). Lagrangian relaxation based approaches to capacitated hub-and-spoke network design problem. *European Journal of Operational Research*, *79*(3), 501–523.

Azizi, N., Vidyarthi, N., & Chauhan, S. S. (2018). Modelling and analysis of hub-and-spoke networks under stochastic demand and congestion. *Annals of Operations Research*, *264*(1–2), 1–40.

Bania, N., Bauer, P. W., & Zlatoper, T. J. (1998). Us air passenger service: A taxonomy of route networks, hub locations, and competition. *Transportation Research Part E: Logistics and Transportation Review*, *34*(1), 53–74.

Boland, N., Krishnamoorthy, M., Ernst, A. T., & Ebery, J. (2004). Preprocessing and cutting for multiple allocation hub location problems. *European Journal of Operational Research*, *155*(3), 638–653.

Campbell, J. F., & O'Kelly, M. E. (2012). Twenty-five years of hub location research. *Transportation Science*, *46*(2), 153–169.

Contreras, I., Cordeau, J. F., & Laporte, G. (2012). Exact solution of large-scale hub location problems with multiple capacity levels. *Transportation Science*, *46*(4), 439–459.

Contreras, I., Díaz, J. A., & Fernández, E. (2009). Lagrangean relaxation for the capacitated hub location problem with single assignment. *OR Spectrum*, *31*(3), 483–505.

Contreras, I., Díaz, J. A., & Fernández, E. (2011). Branch and price for large-scale capacitated hub location problems with single assignment. *INFORMS Journal on Computing*, *23*(1), 41–55.

Correia, I., Nickel, S., & Saldanha-da Gama, F. (2010). Single-assignment hub location problems with multiple capacity levels. *Transportation Research Part B: Methodological*, *44*(8–9), 1047–1066.

Correia, I., Nickel, S., & Saldanha-da Gama, F. (2011). Hub and spoke network design with single-assignment, capacity decisions and balancing requirements. *Applied Mathematical Modelling*, *35*(10), 4841–4851.

da Graça, C. M., Captivo, M. E., & Clímaco, J. (2008). Capacitated single allocation hub location problema bi-criteria approach. *Computers & Operations Research*, *35*(11), 3671–3695.

Dan, T., & Marcotte, P. (2019). Competitive facility location with selfish users and queues. *Operations Research*, *67*(2), 479–497.

de Camargo, R. S., de Miranda, J. G., & Ferreira, R. P. (2011). A hybrid outer-approximation/benders decomposition algorithm for the single allocation hub location problem under congestion. *Operations Research Letters*, *39*(5), 329–337.

de Camargo, R. S., Miranda, G, Jr., Ferreira, R. P. M., & Luna, H. (2009a). Multiple allocation hub-and-spoke network design under hub congestion. *Computers & Operations Research*, *36*(12), 3097–3106.

de Camargo, R. S., Miranda, G, Jr., & Luna, H. (2008). Benders decomposition for the uncapacitated multiple allocation hub location problem. *Computers & Operations Research*, *35*(4), 1047–1064.

de Camargo, R. S., de Miranda, J. G., & Luna, H. P. L. (2009b). Benders decomposition for hub location problems with economies of scale. *Transportation Science*, *43*(1), 86–97.

Dolan, E. D., & Moré, J. J. (2002). Benchmarking optimization software with performance profiles. *Mathematical Programming*, *91*(2), 201–213.

Ebery, J., Krishnamoorthy, M., Ernst, A., & Boland, N. (2000). The capacitated multiple allocation hub location problem: Formulations and algorithms. *European Journal of Operational Research*, *120*(3), 614–631.

Elhedhli, S., & Hu, F. X. (2005). Hub-and-spoke network design with congestion. *Computers & Operations Research*, *32*(6), 1615–1632.

Elhedhli, S., & Wu, H. (2010). A lagrangean heuristic for hub-and-spoke system design with capacity selection and congestion. *INFORMS Journal on Computing*, *22*(2), 282–296.

Ernst, A. T., & Krishnamoorthy, M. (1996). Efficient algorithms for the uncapacitated single allocation p-hub median problem. *Location Science*, *4*(3), 139–154.

Ernst, A. T., & Krishnamoorthy, M. (1998). An exact solution approach based on shortest-paths for p-hub median problems. *INFORMS Journal on Computing*, *10*(2), 149–162.

Ernst, A. T., & Krishnamoorthy, M. (1999). Solution algorithms for the capacitated single allocation hub location problem. *Annals of Operations Research*, *86*, 141–159.

Farahani, R. Z., Hekmatfar, M., Arabani, A. B., & Nikbakhsh, E. (2013). Hub location problems: A review of models, classification, solution techniques, and applications. *Computers & Industrial Engineering*, *64*(4), 1096–1109.
Günlük, O., & Linderoth, J. (2012). Perspective reformulation and applications. In *Mixed integer nonlinear programming*, Springer, pp. 61–89.
Hasanzadeh, H., Bashiri, M., & Amiri, A. (2018). A new approach to optimize a hub covering location problem with a queue estimation component using genetic programming. *Soft Computing*, *22*(3), 949–961.
Jayaswal, S., & Vidyarthi, N. (2013). Capacitated multiple allocation hub location with service level constraints for multiple consignment classes. Technical report, Indian Institute of Management Ahmedabad, Research and Publication Department.
Jayaswal, S., & Jewkes, E. M. (2016). Price and lead time differentiation, capacity strategy and market competition. *International Journal of Production Research*, *54*(9), 2791–2806.
Jayaswal, S., Jewkes, E., & Ray, S. (2011). Product differentiation and operations strategy in a capacitated environment. *European Journal of Operational Research*, *210*(3), 716–728.
Jayaswal, S., Vidyarthi, N., & Das, S. (2017). A cutting plane approach to combinatorial bandwidth packing problem with queuing delays. *Optimization Letters*, *11*(1), 225–239.
Kara, B. Y., & Tansel, B. C. (2000). On the single-assignment p-hub center problem. *European Journal of Operational Research*, *125*(3), 648–655.
Kian, R., & Kargar, K. (2016). Comparison of the formulations for a hub-and-spoke network design problem under congestion. *Computers & Industrial Engineering*, *101*, 504–512.
Klincewicz, J. G. (1998). Hub location in backbone/tributary network design: A review. *Location Science*, *6*(1–4), 307–335.
Lüer-Villagra, A., & Marianov, V. (2013). A competitive hub location and pricing problem. *European Journal of Operational Research*, *231*(3), 734–744.
Marianov, V., & Serra, D. (2003). Location models for airline hubs behaving as m/d/c queues. *Computers & Operations Research*, *30*(7), 983–1003.
Marianov, V., Serra, D., & ReVelle, C. (1999). Location of hubs in a competitive environment. *European Journal of Operational Research*, *114*(2), 363–371.
Marín, A. (2005). Formulating and solving splittable capacitated multiple allocation hub location problems. *Computers & Operations Research*, *32*(12), 3093–3109.
Martín, J. C., & Román, C. (2003). Hub location in the south-Atlantic airline market: A spatial competition game. *Transportation Research Part A: Policy and Practice*, *37*(10), 865–888.
McShan, S., & Windle, R. (1989). The implications of hub-and-spoke routing for airline costs. *Logistics and Transportation Review*, *25*(3), 209.
Meier, J. F., & Clausen, U. (2017). Solving single allocation hub location problems on Euclidean data. *Transportation Science*, *52*(5), 1141–1155.
O'kelly, M. E. (1986). The location of interacting hub facilities. *Transportation Science*, *20*(2), 92–106.
O'kelly, M. E. (1987). A quadratic integer program for the location of interacting hub facilities. *European Journal of Operational Research*, *32*(3), 393–404.
Oum, T. H., Zhang, A., & Zhang, Y. (1995). Airline network rivalry. *Canadian Journal of Economics*, *28*, 836–857.
Rodríguez-Martín, I., & Salazar-González, J. J. (2008). Solving a capacitated hub location problem. *European Journal of Operational Research*, *184*(2), 468–479.
Skorin-Kapov, D., Skorin-Kapov, J., & O'Kelly, M. (1996). Tight linear programming relaxations of uncapacitated p-hub median problems. *European Journal of Operational Research*, *94*(3), 582–593.
Tiwari, R., Jayaswal, S., & Sinha, A. (2020). Alternate solution approaches for competitive hub location problems. *European Journal of Operational Research*, *290*, 68–80.
Toh, R. S., & Higgins, R. G. (1985). The impact of hub and spoke network centralization and route monopoly on domestic airline profitability. *Transportation Journal*, *24*, 16–27.
Vidyarthi, N., Jayaswal, S., Chetty, V. B. T., et al. (2013). Exact solution to bandwidth packing problem with queuing delays. Indian Institute of Management.
Vidyarthi, N., & Jayaswal, S. (2014). Efficient solution of a class of location-allocation problems with stochastic demand and congestion. *Computers & Operations Research*, *48*, 20–30.