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REVIEW

Kinematic Modeling of Robotic Manipulators

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Abstract This paper presents the exhaustive review on the kinematic modeling of robotic manipulators in a systematic manner. A lot of techniques are available in the literature for kinematic modeling of robotic manipulators, however the ambiguity lies with the user to select the most appropriate method. This paper presents the comparative study of different kinematic modeling techniques in terms of complexity, applicability of the method to a particular class of robots and number of parameters or variables required to define the robot. Determination of the correct kinematic parameters, required to develop accurate kinematic models, using different methods has been demonstrated by considering the case study of a five degrees-of-freedom (DOFs) articulated manipulator. Moreover, a seven-DOFs manipulator is considered to highlight and address the inconsistencies of popular methods, while dealing with spatial hybrid manipulators. In this article, a review of 100 research papers is presented to investigate the kinematic study of robotic manipulators with a variety of modeling techniques, which are evolved or refined during the last sixty-odd years (1955–2016).

Keywords Robotic manipulators · D–H parameters · Kinematic modeling · Open- and closed-loop chains

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1 Introduction

In the last three decades, the field of robotics has widened its range of applications, due to recent developments in the major domains of robotics like kinematics, dynamics and control, which leads to the sudden growth of robotic applications in areas such as manufacturing, medical surgeries, defense, space vehicles, under-water explorations etc. Out of listed applications, except first, all other applications are advanced applications of robotics. This is primarily because of relatively complex mechanical structure of these manipulators. Based on the mechanical structure, robotic manipulators can be classified as shown in Fig. 1.

To use robotic manipulators in real-life applications, the first step is to obtain the accurate kinematic model. In this context, a lot of research has been carried out in the literature, which leads to the evolution of new modeling schemes along with the refinement of existing methodologies describing the kinematics of robotic manipulators. Based on the exhaustive literature review, the robot kinematics can be broadly classified as shown in (Fig. 2).

2 Literature Review

The methods available in literature for kinematic modeling, are shown in Fig. 2, These are discussed in this section to highlight the evolution, refinement, merits and demerits of each method.

2.1 Denavit and Hartenberg (D–H) Method

The first pioneer effort for kinematic modeling of robotic manipulators was made by Denavit and Hartenberg [1] in

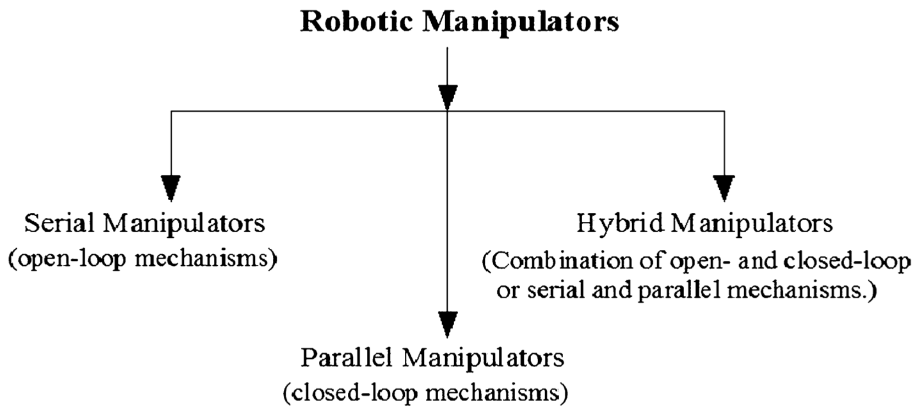


Fig. 1 Classification of robotic manipulators

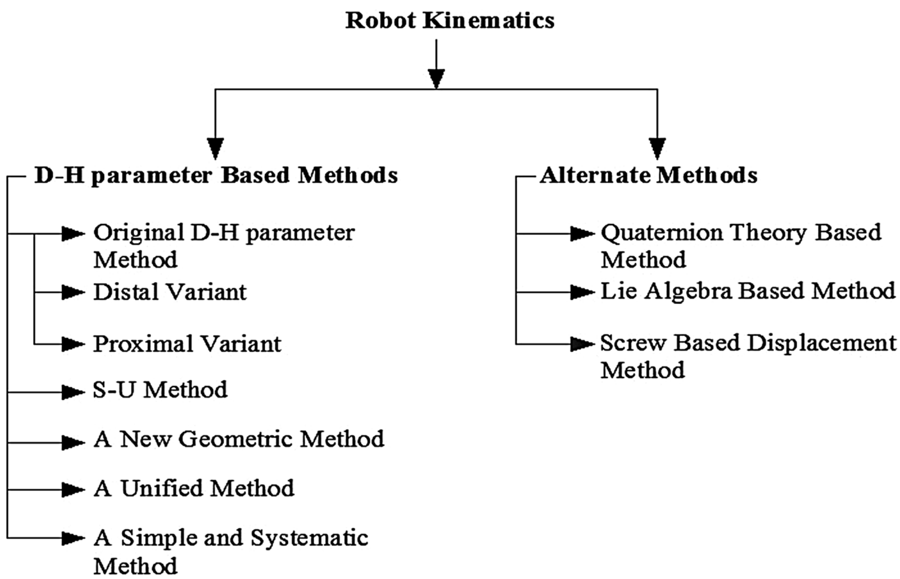


Fig. 2 Classification of kinematic modeling techniques of robotic manipulators

introducing a consistent and concise method to assign reference coordinate frames to serial manipulators.

In this method, four parameters, popularly known as *D–H parameters,* are defined to provide the geometric description to serial mechanisms. Out of the four, two are known as *link parameters,* which describe the relative location of two attached axes in space. These link parameters are: (1) link length (a_i) which is measured from z_i to z_{i+1} along x_{i+1}; (2) link twist (α_i) which is measured from z_i to z_{i+1} about x_{i+1}.

The remaining two parameters are described as *joint parameters,* which describe the connection of any link to its neighboring link [2]. These are: (3) joint offset (d_i) which is measured from x_i to x_{i+1} along z_i; (4) joint angle (θ_i) which is measured from x_i to x_{i+1} about z_i. Figure 3

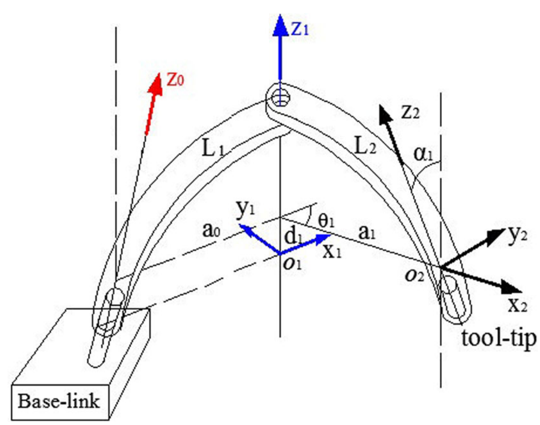


Fig. 3 Schematic representation of original D–H parameter method [6]

describes the schematic representation of original D–H parameter method with the help of two spatial links forming a lower pair (surface to surface contact) where the link parameters a_1, α_1 are measured along x_2, and the joint parameters d_1 and θ_1 are measured along z_1.

Two modifications of original D–H parameter method have been reported in literature—first by Kahn and Roth [3] and second by Featherstone [4]. These modifications are given below:

2.1.1 Distal Variant

The first modified method is known as *distal variant* [3] of D–H parameter method. The aim of this modification was to make the parameter identification simpler [5]. The distal variant is widely used for the kinematic description of serial robotic manipulators, as reported by Lipkin [6]. The term *distal* refers to the location of the link-frame relative to manipulator base and according to this variant, a frame is attached to farther (distal) end point of link, while travelling from manipulator base towards the end effector.

Figure 4, describes the schematic representation of the distal variant. The coordinate frame having components x_1, y_1 and z_1 is attached at father endpoint of first moving link L_1. Later on, this variant was popularized by Paul [7] and Shilling [8] and has been applied to different kinds of kinematic problems [6, 9–12]. It differs from original D–H parameter method in the sense of parameters measurement. In original D–H parameter method, link parameters, a_i and α_i, are measured from z_i to z_{i+1} about x_{i+1}, whereas in this variant, link parameters are measured from z_{i-1} to z_i about x_i. Also, joint parameters θ_i and d_i are measured from x_{i-1} to x_i about z_{i-1} rather than measuring from x_i to x_{i+1} about z_i as in original D–H parameter method. From Fig. 2, it can be seen that a_2 and α_2 were measured about x_2, whereas d_2 and θ_2 are measured about z_1.

2.1.2 Proximal Variant

The second modified method is known as the *proximal variant* [4] of D–H parameter method. Again, the reason behind this modification was same as in case of distal variant i.e. to make the parameters identification simpler. In this method, the term *proximal* refers location of frame, being rigidly attached to link, relative to manipulator base.

According to this variant, a coordinate frame is placed at that end of the link which is closer to manipulator base or which comes first, when travelling from base towards end effector. Later on, this variant, is popularized by Craig [2] and followed by others [13, 14] in their respective work. It differs from original D–H parameter method in the sense that link parameters a_i and α_i are measured from z_i to z_{i+1} about x_i rather than measuring from z_i to z_{i+1} about x_{i+1} as in case of original method. Also, joint parameters θ_i and d_i are measured from x_{i-1} to x_i about z_i rather than measuring from x_i to x_{i+1} about z_i. With reference to Fig. 5, it can be seen that link coordinate frame having coordinate axes x_1, y_1, and z_1 is attached at closer endpoint of first moving link L_1 and link parameters a_2 and α_2 were measured about x_2, whereas d_2 and θ_2 were measured about z_2.

2.1.3 Limitations of the Original D–H Parameter Method and Its Variants

From above discussion of original D–H parameter method and its counterparts, it is concluded that this method is a powerful tool for dealing with robot kinematics problems, however there are couple of limitations, as reported by some of the researchers in the past:

1. Two limitations of D–H parameter method has been reported by Sheth and Uicker [15], these are:
(a) In the case of mechanisms containing links with more than one joint, there is a problem, known as *multi-*

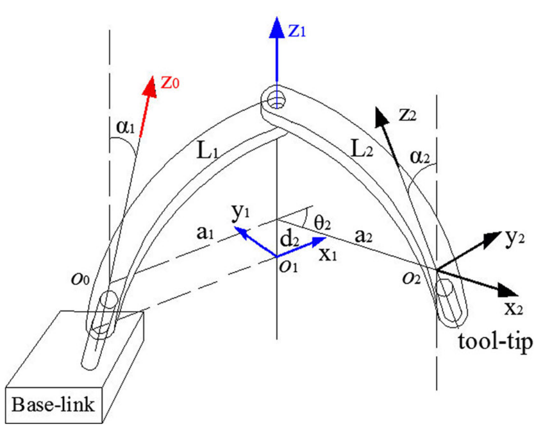


Fig. 4 Schematic representation of distal variant of original D–H parameter method [6]

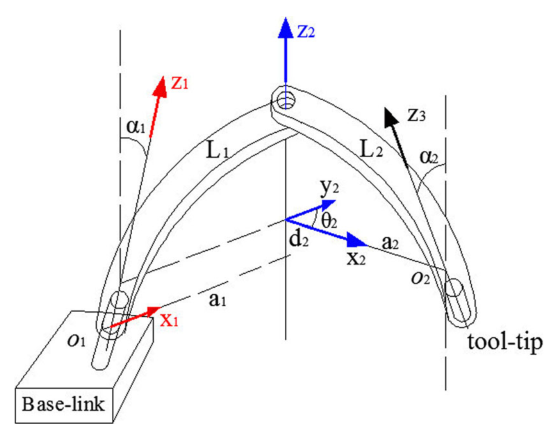


Fig. 5 Schematic representation of proximal variant of D–H parameter method [6]

parameters sets. Now, if there is a binary joint between two links, one binary and other ternary, then according to D–H parameter method there will be two parameters set for describing this joint with respect to ternary link (previous link). This problem arises because D–H parameters for link under consideration not only depends on its shape, but also on the shape of previous link in a serial kinematic chain.

(b) Evaluating D–H parameters for some spatial linkages become quite cumbersome. This is due to the strong requirement of the common perpendicular between two skew lines in space.

2. Parallel and hybrid manipulators contains joints, like spherical or cardan joints, with more than one degree of freedom, describing these joints by using D–H parameter method results non-uniqueness representation as reported by Thomas et al. [16]. Also, problem of multiple parameters set arises.
3. D–H parameter method is applicable to serial mechanisms only, as pointed by Thomas et al. [16] and Khalil and Kleinfinger [17].
4. D–H parameter method defines only *z-axis* for base coordinate frame. However, it does not give unique representation of direction of *x*-axis. Also, for tool-tip frame, *n*th in case of distal variant and $(n + 1)$th in case of proximal variant, directions of *x*- and *z*-axes are not defined uniquely rather these are chosen arbitrarily as reported by Siciliano et al. [18].
5. Homogeneous link coordinate transformation matrix (4×4), on which this method is based, is redundant as it uses twelve parameters to represent six degree of freedom, as reported by Sahu et al. [19].

2.2 Sheth and Uicker (S–U) Method

A new method, describing the manipulator kinematics, has been proposed by Sheth and Uicker [15]. The proposed method is quite general and has a wide range of applicability to all types of rigid link mechanisms. These mechanisms include open-loop, closed-loop, hybrid or tree-like structured robots. Also, the problem of multi-parameters sets, as reported in limitations of D–H parameter method, has been successfully solved by this method. Later, based on this method, many researchers [20–28] have reported significant contributions. The general purpose mechanism analysis computer program has been modified by Vandervaart and Cipra [20], kinematic equations of motion of wheeled mobile robots has been developed by Muir and Neuman [21], two methods, first, of kinematic modeling and second, of forward kinematics computation has been described. Thereafter, the same has been implemented to a new kinematic modeling software, named as CAD-2-SIM by Bongardt [22], an efficient approach to the inverse kinematic analysis of redundant moving base robot has been presented by Dutta and Wong [23], a simple radial distance linear transducer has been used by Goswami et al. [24] for identification of robot kinematic parameters, method of structural synthesis, direct and inverse kinematic modeling has been presented by Baigunchekov et al. [25], Megahed [26] modeled human hand as a tree structure and proposed a new notation [27], Acaccia et al. [28] developed kinematics of mobile robots using non-holonomic path planning.

In this method, each link of a serial chain has been assigned with two coordinate frames, as shown in Fig. 6. The frame which is at proximal end has components *u, v* and *w,* whereas the frame at the distal end has components as *x, y* and *z.*, following procedure has been given by the Sheth and Uicker to apply this method to a spatial mechanism.

For better understanding of S–U parameters, it is important to first know that for any two links, under consideration, first link frame has components as u_i, v_i, w_i at proximal end and second frame has x_j, y_j, z_j components at its distal end. Next, second link has two frames u_j, v_j, w_j and x_k, y_k, z_k at respective ends.

The S–U parameters are defined as follows:

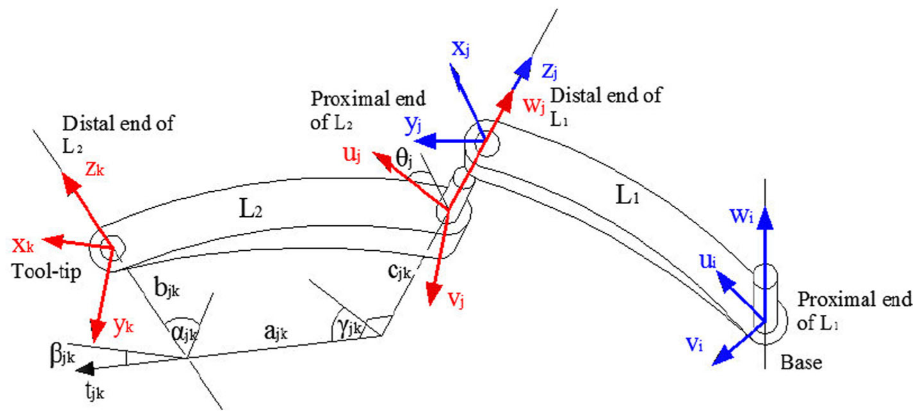
a_{jk} = distance measured from w_j to z_k along t_{jk}.
α_{jk} = angle measured from w_j to z_k about t_{jk}.
b_{jk} = distance measured from t_{jk} to x_k along z_k.
β_{jk} = angle measured from t_{jk} to x_k about z_k.
c_{jk} = distance measured from u_j to t_{jk} along w_j.
γ_{jk} = angle measured from u_j to t_{jk} about w_j.
θ_j = angle measured from u_j to x_j about common joint axis z_j, w_j.

Next, in order to perform the coordinate mapping (transformation), the proposed method is used directly with existing matrix method of coordinate transformation. However, to reduce the task of data collection, a numerical scheme has been presented [15].

In the proposed method, each transformation matrix consists of two parts, which are as follows:

1. *Shape matrix* A constant part specifying the shape of a link by using six parameters. The shape of link is specified by relative orientation of a coordinate frame at the beginning of link and one which is at following end of link. To determine the constant shape parameters for a link, the common perpendicular is found between two axes and assigned with arbitrary positive direction.
2. *Joint matrix* A homogeneous link coordinate transformation matrix which is a function of joint variable (revolute or prismatic). Moreover, it is a distinct variable part representing the joint motion.

Fig. 6 Schematic representation of the S–U method



2.2.1 *Advantages of S–U Method over the D–H Parameter Method*

1. Applicable to all rigid link mechanisms, which include open, close, hybrid and tree like structures.
2. It solves the problem of multi-parameters sets for the definition of a single joint, which is encountered in case of a mechanism containing ternary or higher order links in addition to binary links, when D–H parameter method is used.
3. Two frames, one for each link, at the point where two links connect to form a joint, are chosen independently, which provide more flexibility. The same is not true in case of D–H parameter method and its counterparts, where a frame is defined with respect to common perpendicular between two consecutive joint axes.

2.2.2 *Limitations of S–U Method*

Though S–U method is a powerful and generalized method, it still have some limitations, which are

1. It uses non-minimal parameter representation (seven parameters) to kinematically describe a link shape and joint motion with respect to previous link, as compared to four parameters by D–H parameter method.
2. Due to the complexity of S–U method, it is more suitable to closed-loop chains [29, 30] only. It involves high degree of complicacy because of more number of coordinate frames, as reported by Reichenbach and Kovacic [31].

2.3 A New Geometric Method

The problem of multiple parameter set is encountered, when D–H parameter method is applied to a multi-chain mechanisms containing ternary or higher-order links in addition to binary links. This problem is solved by using a new method, which is an *extension* of well-known D–H parameter method, proposed by Khalil and Kleinfinger [17]. The proposed method is applicable to an open-loop, closed-loop or tree-like robot structures and uses four parameters. In case of links with 2 joints, four parameters are used to describe the link shape and joint motion with respect to the previous joint. However, these parameters become six, if the links contain more than two joints. The objective of the new method was to describe a close-loop robot kinematically without any ambiguity.

The proposed method has been described for three classes of robot structures, namely open-loop, closed-loop and tree like structured robots. However, in this article, this method is presented for open-loop robots only, as shown in Fig. 7. An open-loop robot is considered to have $(n+1)$ links. The base link and terminal link are abbreviated as link (0) and link (n) respectively. Further, joint (i) connects link $(i-1)$ to link (i). The x_i is defined along common perpendicular from z_i to z_{i+1}.

Parameters required to define frame (i) w.r.t. $(i-1)$ are as follow:

α_i = angle measured from z_{i-1} to z_i about x_{i-1}.
d_i = distance measured from O_{i-1} to z_i along x_{i-1}.
r_i = distance measured from O_{i-1} to x_{i-1} along z_i.
θ_i = angle measured from x_{i-1} to x_i about z_i.

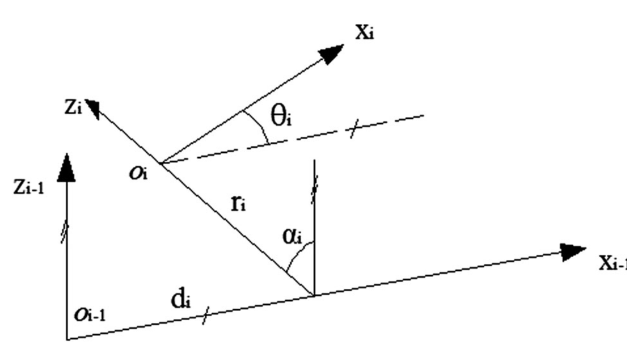


Fig. 7 Khalil and Kleinfinger method [17] for binary Links

The proposed method is similar to D–H method as the parameters (θ_i,r_i, d_{i+1} and α_{i+1}) of new method are similar to θ_i, r_i, d_i, α_i respectively of D–H parameter method.

2.3.1 Applications of New Geometric Method in Robot Kinematics

This method has been followed by many researchers [32–55] in the past. A virtual reality based system for rapid design, prototyping and simulation, known as CINEGEN, has been developed by Flückiger et al. [32, 33], an efficient algorithm [34] has been developed for inverse kinematics as well as for the inverse dynamic control of robots in Cartesian space, a direct method [35] is proposed to determine the minimum number of inertial parameters of a robot. Moreover, the closed-form solutions for the inverse and direct dynamic model [36] are obtained for Gough–Stewart platform, a general method [37] is developed to compute the direct and inverse dynamic models of parallel robots, a novel solution [38] is presented for kinematic and dynamic modeling of 3-PRS parallel robot, the experimental identification of dynamic parameters of orthoglide is carried out [39], two universal numerical methods [40] were developed for open-loop as well as graph structured robots, a comparison study [41] is carried out on the geometric parameter calibration methods of robots (examples are Puma and Stanford manipulators), to determine the set of base parameters, a symbolic method [42] is proposed to find the minimum set of inertial parameters of robots containing parallelogram closed-loop, a new method [43] for calibration of geometric parameters of links and tool of serial robots is developed, two methods [44] are proposed to calculate the direct dynamic model of walking robots and a comparison for each method, two methods [45] are developed for calibration of geometric parameters of serial robots. Thereafter, direct and inverse dynamic modeling is performed, using recursive Newton–Euler formulism, of tree, parallel and closed-loop robot structures [46], a new method [47] is developed for calculation of filtered dynamic model of robots, human-body is modeled kinematically as a series of revolute joints [48], an algorithm based on generalized Newton–Euler models of flexible manipulators [49], the general and unified presentation is presented for dynamic modeling and identification issues of car [50], an efficient method [51] for the calculation of inverse dynamic models of tree structure robots, four methods are presented for identification of inertial parameters of the load of a manipulator [52], dynamic modeling is given for multi-body systems [53], modeling, simulation and control [54] of high speed machine tools is carried out using robotics formulism has been presented by Khalil et al. [34–54], Pham et al. [55] used Khalil and Kleinfinger method [17] to identify the joint stiffness of a robot using band pass filter.

2.3.2 Advantages of Khalil and Kleinfinger Method over S–U Method

As reported by Flückiger et al. [32, 33], the advantages of present method are

1. Degree of simplicity is similar to D–H parameter method in case of serial robotic manipulators.
2. Though the mechanism under consideration is complex, this method uses lesser number of parameters to define shape of link and joint motion.

2.4 A Unified Method

This method, developed by Thomas et al. [16], provides a unique representation of spherical or cardan joints involving in a parallel or hybrid manipulator. However, this is not true in the case, when D–H parameter method is applied to same joints, as reported earlier in limitations of D–H parameter method. This method is based on previous work of Belfirore and Benedetto [56], who have used graph to represent serial, parallel and hybrid redundant kinematic structures, where each joint has one degree of freedom. The proposed method is an extension of, most widely used, D–H parameter method (Fig. 8).

This method is based on following conventions, for *uniqueness* of extended D–H parameter method:

1. Link (i) has origin of coordinate system located at joint ($i+1$).
2. Axis of joint ($i+1$) is z-axis. For spherical joint z-axis is assigned in the direction given by cross product of joint normal vectors of succeeding and preceding joints.
3. x-axis of link (i) is collinear to the normal of joint and points toward higher indexed joint.
4. Right hand thumb rule is used to obtain direction of y-axis.
5. For single DOF, θ_i will be joint variable. However, if joint has two DOFs, joint variables become θ_i, and β_i respectively. Moreover, if DOFs are three, in that case, θ_i, β_i and α_i are the joint variables.

Thereafter, proposed method has been applied to spherical and cardan joints having DOFs more than one. With this method, a spherical joint is represented by five parameters only instead of twelve as in case of conventional D–H method.

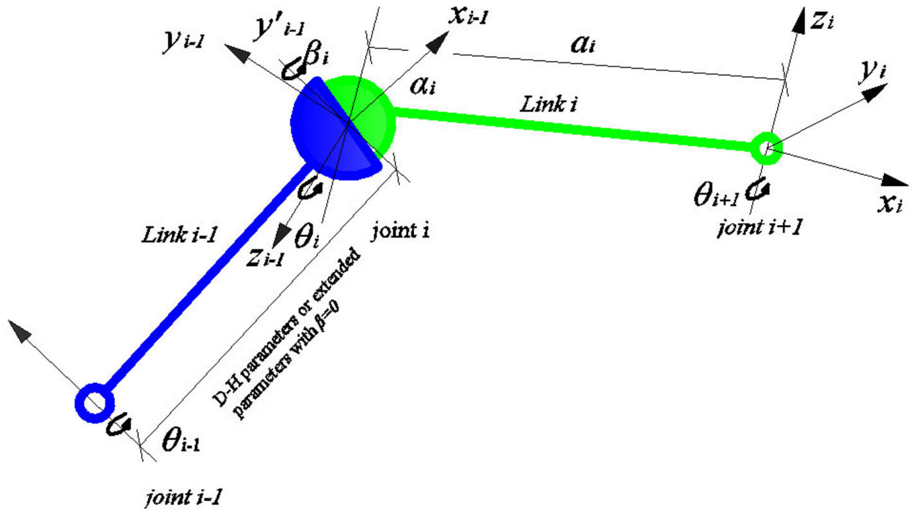


Fig. 8 D–H parameters of spherical joint according to Unified method

2.4.1 Advantages of the Unified Method

1. Proposed method has wide range of applicability.
2. It provides unique representation of higher degree-of-freedom joints like spherical or cardan joints.
3. This method uses only five parameters to represent a spherical joint, whereas D–H parameter method uses twelve parameters for same.

2.5 A Simple and Systematic Method

In all of the methods discussed above, the axis of rotation is of great concern, i.e. it cannot be chosen arbitrarily. However, a simple and systematic approach for assigning D–H parameters to a kinematic chain was proposed by Corke [57]. It is a two-step approach, according to which rotations and translations is taken about an axis of interest. Later, this approach is followed by Minas et al. [58] to assign D–H parameters. The procedure of application of this approach is as follows:

1. Moving an imagined right handed coordinate base frame, which is chosen arbitrarily, from manipulator base to end effector executing series of fundamental translations and rotations. This step is also known as *walk through.* Also, end effector coordinate frame is chosen arbitrarily.
2. Applying set of algebraic rules to transform these fundamental transformations into form that is factorized as link transform represented into standard or modified D–H form. These transformation rules are as follows:

(a) Push the constant terms as far, towards right, as possible using commutative rule which is:

$$T_1(r_1)T_2(r_2) = T_2(r_2)T_1(r_1) \tag{1}$$

(b) To transform series of fundamental transformations into standard D–H form, rotations about x- or y-axes are transformed to be about z-axis and translations about y-axis are transformed to be about x- or z-axes as per requirement. The transformation rules are given as:

$$R_x(q_i) = R_y R_z(q_i) R'_y \tag{2}$$

$$R_y(q_i) = R'_x R_z(q_i) R_x \tag{3}$$

$$T_x(q_i) = R_y T_z(q_i) R'_y \tag{4}$$

$$T_y(q_i) = R'_x T_z(q_i) R_x \tag{5}$$

(c) To eliminate undesirable transformations about y-axis, use following transformations:

$$R_y(q_i) \equiv R_z R_x(q_i) \quad R'_z \equiv R'_x R_z(q_i) R_x. \tag{6}$$

$$T_y(q_i) \equiv R_z T_z(q_i) \quad R'_z \equiv R'_x T_z(q_i) R_x. \tag{7}$$

(d) Joint angle offset, if required, is introduced automatically to have by the chosen zero-angle configuration.

Thereafter, an algorithm has been presented by the Corke [57] for automatic symbolic manipulation, which is applicable to serial manipulators only and has been applied to PUMA robot.

2.6 Quaternion Algebra Based Method

The quaternions [59–61] have been discovered and described by Hamilton in order to broaden the scope of vector algebra (3D) to include division and multiplication.

With quaternion algebra, the spatial representation of finite rotations becomes simple, straightforward and powerful. A quaternion consists of an ordered-set of four real numbers, having scalar and vector components. These real numbers are *a, b, c, d* and has units *i*, *j*, *k*, and +1 respectively.

Mathematically, a quaternion [62, 63] is defined as,

$$q = S + V \tag{8}$$

$$S = d \tag{9}$$

$$V = ai + bj + ck \tag{10}$$

$$q = d + ai + bj + ck \tag{11}$$

where, S is scalar part of quaternion and V is vector part of quaternion. The last unit possesses the properties of real number one, whereas the properties of first three units are,

$$i^2 = j^2 = k^2 = 1 \tag{12}$$

$$ij = k, \quad ji = -k \tag{13}$$

2.6.1 Applications of Quaternion in Robot Kinematics

An extension of Hamilton's quaternions, known as *dual quaternion*, was made by Clifford [64, 65]. Later, Pervin and Webb [66] explored the general properties of quaternion as rotational operators. Further, quaternion has been used by Taylor [67] to plan kinematic path and make its comparison with homogeneous transformations. A set of conversion routines between matrix, Euler angles and quaternion representations of rotations has been proposed by Shoemake [60]. An approach based on quaternions, to find the relative orientation between link-mounted sensor and a link, has been presented by Jack and Kamel [59]. The presented approach provides with a unique solution of kinematic equation. A four bar spatial mechanism has been considered by Yang and Freudenstein [62], for demonstrating the application of quaternion algebra and dual numbers. Dooley and McCarthy [68] used the dual quaternion to analyze kinematic properties of spatial mechanisms, which accounts for the dynamic forces. A similar work has been done by Yang and Freudenstein [62], however, it was limited to static forces only. Azariadis and Aspragathos [69] used dual unit vectors and dual unit quaternion to propose a representational model, for description and transformation of 3D geometric elements. A dual quaternion synthesis methodology for kinematic synthesis of constrained serial robotic manipulators has been introduced by Perez and McCarthy [70]. The design of strapdown inertial navigation system (INS) algorithms has been presented by Yuanxin et al. [71]. A brief analysis of quaternion in terms of computational efficiency and an application of complex quaternion, representing Lorentz transformations in the theory of relativity, has been presented by Salamin [72]. An algorithm based on dual number quaternion, for determination of position and orientation of an object by minimizing the single cost function, has been presented by Walker and Shao [73]. The use of dual quaternion is introduced by Konstantinos [74] to represent a line transformation.

The kinematic properties of dual quaternion have been studied and presented by Bottema and Roth [75], Yang and Freudenstein [62] and Study [76]. A comparison, in terms of storage capacity and computational efficiency, of homogeneous transformations and quaternion pairs has been made by Funda et al. [77]. This leads to the fact that quaternion approach is more efficient, compact and elegant. The quaternion pairs are used widely to represent the rotations. This is because of their good performance [75, 78] in terms of storage efficiency and being mathematical powerful.

In recent past, in the field of robotics and computer graphics, the dual quaternion has got a significant recognition in describing rigid-body motion [69]. While deriving the kinematic equations of a 3D mechanism, Shoemake [60] reported that dual quaternions are useful for screw displacements. It has been reported [60, 61, 67, 68, 77] that dual quaternion is a simple, economic, crisp and systematic way to represent rotation and translation. Quaternions have been used extensively to describe the position and orientation of an object in terms of parameters [74, 79, 80]. Unit quaternions has been used by Horn [81] to represent rotations. Gan et al. [82] used dual quaternion to construct the 7-link, *7R* mechanism. A PID control scheme, based on unit quaternion, has been proposed by Wang and Yu [83]. A quaternion algebra based method, for kinematic representation of link shape and joint motion, has been proposed by Sahu et al. [19]. This method is faster because it requires less number of parameters, as compared to method based on homogeneous transformation matrix, to define six degrees of freedom. Also, less complications are there in understanding the representations derived with this method. It has been reported by Sahu et al. [19] that quaternion algebra has been used to represent rigid body translations and rotations. However, it has not been used yet for practical applications of robotic manipulators. So, objective of authors was to develop a novel method, based on quaternion algebra, to represent robot kinematics.

2.7 Lie Algebra

Lie groups were originally introduced by Lie [84, 85], as a tool to solve or simplify ordinary and partial differential equations. Lie got inspired from Galois's study of algebraic equations and symmetries and attempted with *group theory* to solve or at least simplify the ordinary differential equations. Further, Murray et al. [86] described Lie groups and Lie algebra and illustrated well with appropriate examples.

A group is defined as a set of operations $\{g_1, g_2, g_3, \ldots\}$, known as *group elements or operations,* along with combinatorial operation (called group multiplication) construct a group G.

$$G = \{g_1, g_2, g_3, \ldots\} \tag{14}$$

A Lie group is defined by four axioms i.e. group elements along with group multiplication will form a group if all of these four axioms [84] are satisfied. These axioms are called as *closure*, *associativity*, *identity* and *inverse*. Lie groups appear in two basic types named as simple Lie groups and solvable Lie groups. Both types play a significant role in the study of differential equations as all other Lie groups evolve from these two basic forms. Under rearrangement, the former groups recreate themselves, whereas the later ones do not. Solvable groups contain subgroups, each of which is an invariant subgroup of its predecessor.

Figure 9, describes the subgroups of a *symmetric group* (S_3). The *symmetric* (S_n) and *permutation* (P_n) *groups* are the synonyms of Galois group G [84], which allows the rearrangement of the roots of a general polynomial Eqs. (15) and (16). The subscript n of S_n or P_n represents the number of roots of the prescribed equation. The group S_n has $n!$ group operations, where each operation is the rearrangement of some of the roots of the given polynomial equation.

$$(z - z_1)(z - z_1)(z - z_1)\ldots(z - z_n) = 0 \tag{15}$$

$$\left(z^n - I_1 z^{n-1} + I_2 z^{n-2} + \cdots + (-1)^n I_n\right) = 0 \tag{16}$$

Here, for instance, S_3 is elaborated in Fig. 9. Now, as $n = 3$, it implies that number of group operations become $3! = 6$ and number of roots are equal to n i.e. 3. The symmetric group S_3 is appropriately represented in terms of 3×3 matrices as

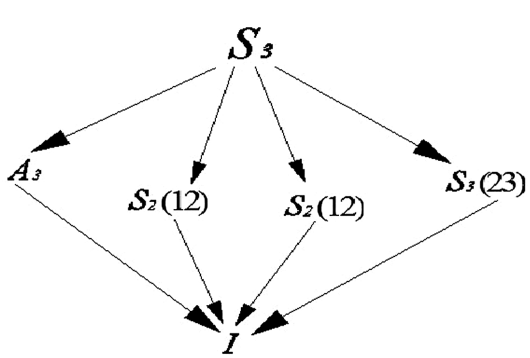


Fig. 9 Subgroups of S_3

$$\begin{array}{ll}
I \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & (123) \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
(321) \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & (12) \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
(23) \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & (13) \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\end{array} \tag{17}$$

The notation (123) represents the operation under which a proceeding root is replaced by its succeeding root. For example, z_1 is replaced by z_2, z_2 by z_3 and z_3 by z_1, respectively. The permutation matrix corresponding to this operation have identical permutations

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \overset{123}{\Longrightarrow} \begin{bmatrix} z_2 \\ z_3 \\ z_1 \end{bmatrix} \tag{18}$$

$$\begin{bmatrix} z_2 \\ z_3 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{19}$$

Also, Fig. 9, describes A_3, known as *alternating group*, represents the subgroup of S_3. It is evolved as subset of operations I, (123) and (321) and composed of those operations in S_3 whose determinants, in permutation representation, are $+1$.

$$S_2(12) = \{I, (12)\} \tag{20}$$

$$S_2(23) = \{I, (23)\} \tag{21}$$

$$S_2(13) = \{I, (13)\} \tag{22}$$

Furthermore, Lie algebra is a linear vector space and is obtained by linearizing the Lie group in the neighborhood of any of its points. Linearization [84] significantly simplifies the study of Lie groups. The expression given below, briefly describes the relation between Lie group and Lie algebra as

$$\text{Lie Group} \underset{\text{Exponential mapping}}{\overset{\text{Linearization}}{\leftrightarrows}} \text{Lie Algebra}$$

2.7.1 Applications of Lie Groups and Lie Algebra in Robot Kinematics

Based on Lie groups and Lie algebra, a lots of robot kinematics methods [87–91] have been proposed in the literature. Herve [89], proposed a new method, to describe the rigid-body positions, based on group theory. Further, the work of Herve has been used, as a theoretical foundation, by Thomas and Torras [87] to develop a generalized method, using theory of continuous groups. Their proposed method was more general as compared to other methods proposed in the past. Three Lie group integrator, *commutator free method*, *Crouch-Grossman method* and *Munthe-Kaas*, have been developed by Jonghoon and Chung [88], for mathematical modeling of articulated multi-body systems. These methods were characterized by singularity-free integration. A tool, based on Lie groups, for the synthesis of new parallel structure robots has been presented by Herve and Sparacino [90]. A first order Lie group variational integrators, using Lagrange–d'Alembert principle, for the discretization of equations of motion of rigid-body in a Lie or special Euclidean group $SE(3)$, have been developed by Nordkvist and Sanyal [91]. A computationally efficient recursive algorithm, based on Lie groups concepts, description and Riemannian geometry, for the inverse dynamics of an open-chain manipulator, has been presented by Park and Bobrow [92]. A detailed analysis, based on the original conceptions of kinematics and Lie algebra has been presented and used to explore a unified representation method for both position and orientation by Gu [93].

Lie algebra of the special Euclidean group has been used for direct kinematics of a serial robot by Selig [94], whereas Rico et al. [95] used it for the mobility analysis of kinematic chains. An illustration and application of Lie algebra, to control nonlinear systems (mobile robots), has been presented by Coelho and Nunes [96]. An application of Lie group theory, to the structural design of parallel robotic manipulators, has been presented by Sparacino and Herve [97]. The Lie algebra of Lie group of Euclidean motions has been explained as the vector space by Dekret and Baska [98]. They reported algebra and spaces are orthogonal and preferred the algebra of vector couples to dual number and dual quaternion technique, because of cleaner geometrical and mechanical interpretation. Finally, it has been reported [88] that as compared to quaternion approach, algorithms based on Lie groups do not involve any algebraic constraint.

2.8 Screw Based Displacement Method

The fundamental transformations (rotation and translation) of a rigid-body are described by a geometric entity, which is a *screw*. It consists of an axis, about which these transformations are defined, and a scalar *pitch,* which establishes the relationship between rotation and translation. According to this method [5], the motion of a link with respect to its previous link is represented by a screw ($\$_i$). The kinematic model of a manipulator is developed by following the procedure given below:

1. Select a fixed coordinate system with respect to which screws are defined.
2. Choose an appropriate pose (home position) of the manipulator so as to determine screw parameters s (screw axis) and s_0 (position vector of a point in screw axis) for each joint.
3. Determine Rodrigues parameters (s, s_0, θ, t) and joint variable, for each joint.
4. Calculate homogeneous matrices for each joint.

The position and orientation of a given link is obtained by using homogeneous transformation matrix $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{R}(\theta) & | & \boldsymbol{p}(t) \\ - & | & - \\ 0 & | & 1 \end{bmatrix} \tag{23}$$

where

$$\begin{aligned} \boldsymbol{R}(\theta) &= \begin{bmatrix} c_\theta + s_x^2(1-c_\theta) & s_y s_x(1-c_\theta) - s_z s_\theta & s_z s_x(1-c_\theta) - s_y s_\theta \\ s_y s_x(1-c_\theta) - s_z s_\theta & c_\theta + s_y^2(1-c_\theta) & s_y s_z(1-c_\theta) - s_x s_\theta \\ s_z s_x(1-c_\theta) - s_y s_\theta & s_y s_z(1-c_\theta) - s_x s_\theta & c_\theta + s_z^2(1-c_\theta) \end{bmatrix} \\ \boldsymbol{p}(t) &= t\boldsymbol{s} + [\mathbf{I} - \mathbf{R}(\theta)]\mathbf{s}_0 \end{aligned} \tag{24}$$

Using Eq. (25), obtain concatenated homogeneous transformation matrix by pre-multiplication of matrices obtained in step 4. The position and orientation of last link of a kinematic chain consisting of n links is obtained by pre-multiplication of transformation matrices, given the joints displacements:

$${}^0_n\boldsymbol{A} = {}^0_1\boldsymbol{A}{}^1_2\boldsymbol{A}{}^2_3\boldsymbol{A}\ldots{}^{n-1}_{n}\boldsymbol{A}. \tag{25}$$

2.8.1 Advantages of Screw Based Displacement Method

1. For kinematic description of robotic manipulators, screw based methods uses only two frames for entire chain. However, this is not true in the case of methods like D–H parameter method, Sheth and Uicker method, Khalil and Kleinfinger method etc.

3 Applications of Methods Describing Manipulator Kinematics

This section presents a case study of five-axis articulated manipulator, to demonstrate a couple of methods discussed in Sect. 2, to describe its kinematic configuration. The line diagrams and corresponding parameters tables for this manipulator are obtained according to each method.

(a) *Distal variant*: Fig. 10 shows the kinematic configuration of a five axis articulated robot, which resembles very closely to the human arm anatomy, is five-axis articulated robot.

It consists of one fixed link (marked as L_0 and four moving links (L_1, L_2, L_3 and L_4). All joints are of revolute type. At last link, usually known as end-effector, two motions (pitch and roll) are observed. Two examples of this

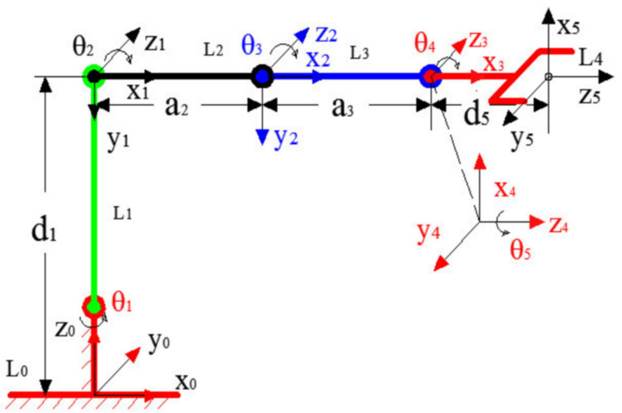


Fig. 10 Description of five-axis articulated robot according to distal variant

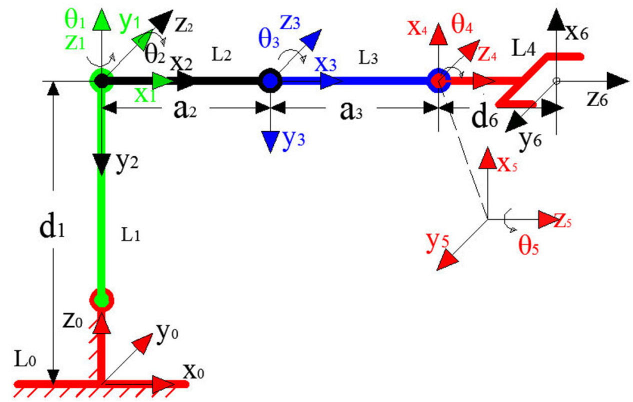


Fig. 11 Description of five-axis articulated robot according to proximal variant

Table 1 Distal variant parameters of five-axis articulated arm

S. no.	Link length (a_i) (mm)	Link twist (α_i) (deg.)	Joint offset (d_i) (mm)	Joint angle (θ_i) (deg.)	Home position
1.	$a_1 = 0$	$\alpha_1 = -90$	d_1	θ_1	0
2.	a_2	$\alpha_2 = 0$	$d_2 = 0$	θ_2	0
3.	a_3	$\alpha_3 = 0$	$d_3 = 0$	θ_3	0
4.	$a_4 = 0$	$\alpha_4 = -90$	$d_4 = 0$	θ_4	−90
5.	$a_5 = 0$	$\alpha_5 = 0$	d_5	θ_5	0

Table 2 Proximal variant D–H parameters of articulated arm

S. no.	Link length (a_{i-1}) (mm)	Link twist (α_{i-1}) (deg.)	Joint offset (d_i) (mm)	Joint angle (θ_i) (deg.)	Home position
1.	$a_0 = 0$	$\alpha_0 = 0$	d_1	θ_1	0
2.	$a_1 = 0$	$\alpha_1 = -90$	$d_2 = 0$	θ_2	0
3.	a_2	$\alpha_2 = 0$	$d_3 = 0$	θ_3	0
4.	a_3	$\alpha_3 = 0$	$d_4 = 0$	θ_4	−90
5.	$a_4 = 0$	$\alpha_4 = -90$	$d_5 = 0$	θ_5	0
6.	$a_5 = 0$	$\alpha_5 = 0$	d_6	–	–

robot are reported, these are—(a) Microbot Alpha II; (b) Rhino XR-3 [8]. Both of these manipulators have same kinematic configuration or mechanical structure. The line diagram of five axis articulated manipulator is shown in Fig. 10. Using the distal variant of D–H parameter method, all the parameters are obtained and tabulated in Table 1.

(b) *Proximal variant*: In Fig. 11, Link diagram of a five-axis articulated arm is shown in the assumed home position. In this case, link frame assignment and D–H parameters are described using proximal variant of D–H parameter method. The corresponding D–H parameters are given in Table 2.

(c) S–U *method*: Fig. 12 represents the application of S–U method to five-axis articulated manipulator. The link-frame assignment and parameters (given in Table 3) are as shown.

4 D–H Method Augmented with Dummy Frames

The standard D–H parameter method leads to ambiguities, when applied to a spatial hybrid manipulator. Singh et al. [99, 100] have observed discrepancies in tool-tip position obtained from kinematic model as compared to one obtained from physical prototype of a seven-DOF spatial hybrid

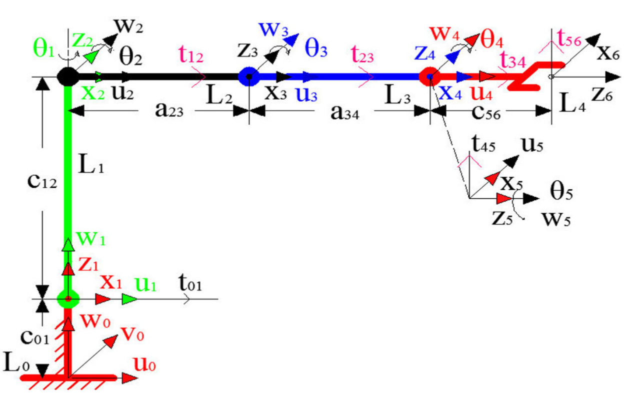


Fig. 12 Description of five-axis articulated robot according to S–U method

manipulator. This geometrical inconsistency arises because all the segments of spatial links L_4, L_5, L_6, and L_7 do not get accounted for into corresponding D–H parameters (Table 4) as is seen in Fig. 13a. This leads to *recognizable deficiency* of D–H notation in case of spatial links, where two consecutive joint axes at right angles to each other.

However, the reported deficiency is successfully eliminated (see Fig. 13b) by introducing a new concept. Also, new D–H parameters augmented with dummy frames are given in Table 4 of Singh et al. [99]. The introduced concept is coined as

Table 3 S–U parameters of five-axis articulated arm

S. No.	(a_{jk}) (mm)	(α_{jk}) (deg.)	(b_{jk}) (mm)	(β_{jk}) (deg.)	(c_{jk}) (mm)	(γ_{jk}) (deg.)	(θ_i) (deg.)	Home position
1.	$a_{01} = 0$	$\alpha_{01} = 0$	$b_{01} = 0$	$\beta_{01} = 0$	c_{01}	$\gamma_{01} = 0$	–	–
2.	$a_{12} = 0$	$\alpha_{12} = -90$	$b_{12} = 0$	$\beta_{12} = 0$	c_{12}	$\gamma_{12} = 0$	θ_1	0
3.	a_{23}	$\alpha_{23} = 0$	$b_{23} = 0$	$\beta_{23} = 0$	$c_{23} = 0$	$\gamma_{23} = 0$	θ_2	0
4.	a_{34}	$\alpha_{34} = 0$	$b_{34} = 0$	$\beta_{34} = 0$	$c_{34} = 0$	$\gamma_{34} = 0$	θ_3	0
5.	$a_{45} = 0$	$\alpha_{45} = -90$	$b_{45} = 0$	$\beta_{45} = -90$	$c_{45} = 0$	$\gamma_{45} = -90$	θ_4	0
6.	$a_{56} = 0$	$\alpha_{56} = 0$	$b_{56} = 0$	$\beta_{56} = -90$	c_{56}	$\gamma_{56} = 90$	θ_5	0

Table 4 D–H parameters of a seven-DOF spatial hybrid manipulator in home position

S. No.	Link length (a_{i-1}) (mm)	Joint distance (d_i) (mm)	Link twist (α_{i-1}) (deg.)	Joint variable (θ_i) (deg.)	Home position
1.	$a_0 = 0$	$d_1 = 137.77$	$\alpha_0 = 0$	θ_1	0
2.	$a_1 = 0$	$d_2 = 0$	$\alpha_1 = -90$	θ_2	−90
3.	$a_2 = 187.46$	$d_3 = 0$	$\alpha_2 = 0$	θ_3	90
4.	$a_3 = 160.76$	$d_4 = 0$	$\alpha_3 = 0$	θ_4	0
5.	$a_4 = 0$	$d_5 = 67.82$	$\alpha_4 = 90$	θ_5	0
6.	$a_5 = 0$	$d_6 = -35.07$	$\alpha_5 = -90$	θ_6	−90
7.	$a_6 = 0$	$d_7 = -38.40$	$\alpha_6 = -90$	θ_7	90
8.	$a_7 = 0$	$d_8 = 63.79$	$\alpha_7 = 90$	–	–

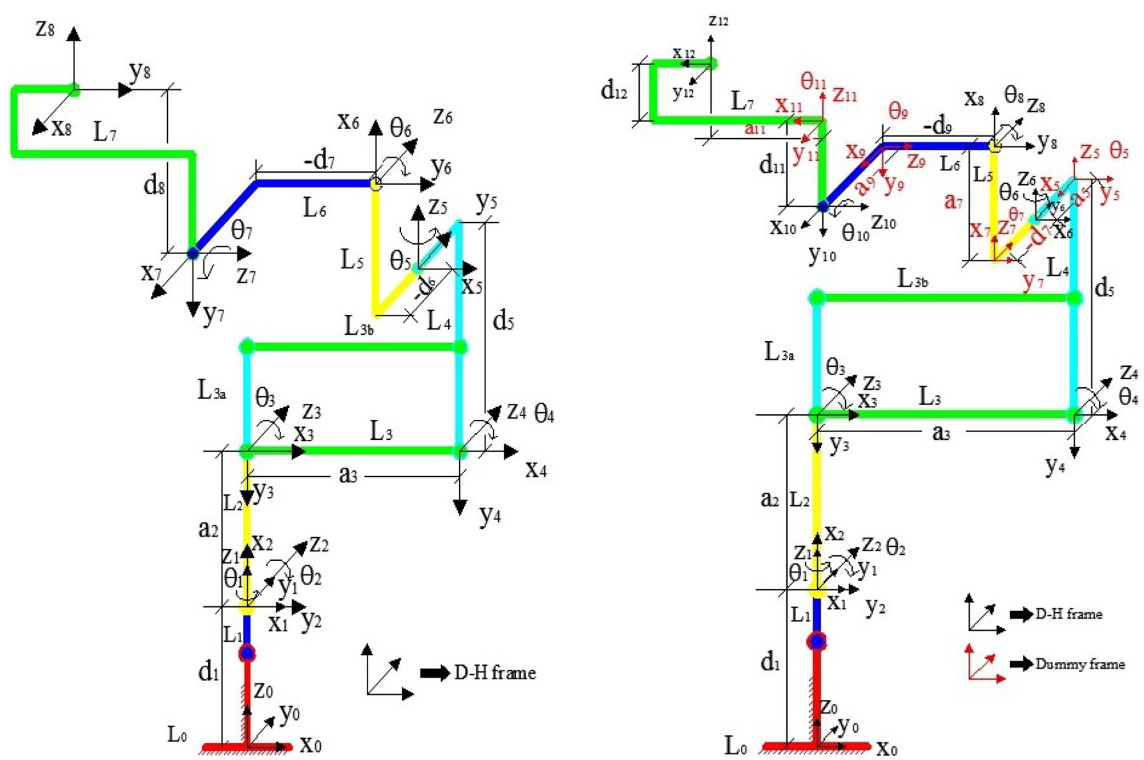


Fig. 13 **a** The line diagram of a seven-DOF spatial hybrid manipulator with D–H parameters marked, with attached reference frames according to proximal variant of D–H convention in assumed home position. **b** The D–H parameters augmented with dummy frames

the concept of ***dummy frames.*** Dummy frame (shown in dark color) is different from D–H frame in the sense that it is placed at a point where there is no motion, whereas a D–H frame is placed at a point where motion is taking place. Absence of motion at dummy frame does not imply that joint angle corresponding to dummy frame vanishes rather it becomes a constant, whose value is determined (as angle from x_{i-1} to x_i about z_i) according to D–H notation. The detailed discussion of this method has been given by Singh et al. [100].

4.1 Advantages of Dummy Frame Method

1. Simple and easy to apply.
2. There is no any fixed rule to follow for the assignment of a dummy frame. It is oriented in an appropriate direction as per requirement.
3. The proposed method is applicable to hybrid manipulators.

5 Comparative Study of Different Methods

A comparison is made between D–H parameter method and its counterparts. Thereafter, this comparison is extended to other methods discussed in Sect. 3.

From above discussion of original D–H parameter method and its variants, it is observed that parameter identification is easiest in proximal variant than distal and original D–H parameter method. This is because the subscript index of axes (x_i and z_i), about which link and joint parameters are being measured, are same whereas this is not true in the case of original and its distal variant. This creates a confusion in remembering the definition of these parameter measurement. However, distal variant is more widely used as compared to original and proximal D–H parameter method. This comparison is summarized in Table 5.

The following observations are drawn from above illustrated schematic diagrams of standard manipulators:

Table 5 Comparison of original D–H parameter method and its variants in terms of parameter representation

Parameter (s)	Original D–H parameter method	Distal variant	Proximal variant
$\cos\alpha_1$	$z_1 \cdot z_2$	$z_0 \cdot z_2$	$z_1 \cdot z_2$
$\sin\alpha_1$	$z_1 \times z_2 \cdot x_2$	$z_0 \times z_1 \cdot x_1$	$z_1 \times z_2 \cdot x_1$
$\cos\theta_1$	$x_1 \cdot x_2$	$x_0 \times x_1$	$x_0 \times x_1$
$\sin\theta_1$	$x_1 \times x_2 \cdot z_1$	$x_0 \times x_1 \cdot z_0$	$x_0 \times x_1 \cdot z_1$
a_1	$vec\ (O_1O_2) \cdot x_2$	$vec\ (O_1O_2) \cdot x_1$	$vec\ (O_1O_2) \cdot x_1$
d_1	$vec\ (O_1O_2) \cdot z_1$	$vec\ (O_0O_2) \cdot z_0$	$vec\ (O_1O_2) \cdot z_1$

1. Both, proximal and distal variants of D–H parameter method use one frame per link. However, S–U method uses two frames per link.
2. To provide geometric description to a manipulator having n-DOFs, the number of coordinate frames required according to distal, proximal variants and S–U method are n, $n+1$ and $(n \times 2 + 2)$ respectively. This analysis justifies the first advantage (Sect. 2.2.1) and second limitation (refer Sect. 2.2.2) of the S–U method, which is more general (applicable to open, close, hybrid and tree like structures), and more complex as it uses $(n \times 2 + 2)$ frames as compared to the n frames used by the distal variant.
3. A contradiction between two statements has been observed in case of distal variant when being applied to PUMA 560 manipulator. However, both statements are satisfied in case of proximal variant of D–H notation. These statements are as follow [2]:

   (a) The point on body, whose position needs to be described, can be chosen arbitrarily. However, for our convenience, point on body whose position need to be described is chosen as a body-attached frame.
   (b) Find the common perpendicular between ith and $(i+1)$ th joint axes, or their point of intersection. At the point of intersection or at the point where common perpendicular meets the ith axis, assign the link-frame origin.

4. Both, proximal and distal variants of D–H parameter method use four parameters to describe link-shape and joint motion in three dimensional Cartesian space (R^3). However, S–U method uses seven parameters to describe the same.
5. A dummy frame is attached to a link when it becomes a spatial link [99, 100]. For dummy frame method, it is stated that total number of frames (F), with distal and proximal variants, will be $\{n + (s^* d)\}$ and $\{n + (s^* d) + 1\}$ respectively. Here, s is the number of spatial links in the manipulator and d is the number of dummy frames per spatial link. For instance, in the case of a seven-DOF spatial hybrid manipulator there are total four spatial links ($s = 4$) and one dummy frame per link ($d = 1$). Therefore, total number of frames become 13 (see Fig. 13b) which were initially 9 (see Fig. 13a).

6 Conclusions

Based on the review presented in this paper, it has been observed that active research in the area of kinematic modeling has been started in the middle of twentieth century, when in 1955, Denavit and Hartenberg have made a

Table 6 Comparison of different robot kinematics methods

Parameter	Method					
	Distal variant	Proximal variant	S–U method [15]	K–K method [17]	Screw based method [5]	D–H method augmented with dummy frames [100]
F	n	$(n+1)$	$(n \times 2 + 2)$	–	2	$n + (s \times d) + 1$
P	4	4	7	4 or 6	7	4
R	Serial	Serial	Any	Any	–	Any
C	Less	Less	More	Less	More	Less

F total number of frames, *P* number of parameters required to describe the link shape and joint motion, *R* range of application, *C* complexity of method

significant contribution in the form of a proposed method, known as D–H parameter method. Thereafter, a lot of methods have been reported in the literature for the kinematic modeling of robotic manipulators. These methods have been broadly classified into two categories: D–H parameter based and non-D–H parameter based (alternate) methods. Through this review, an attempt has been made to categorize the methods based on applicability to a certain class of robots and complexity of the method. It has been observed that the D–H parameter method and its variants use the minimal number of parameters to define a robot, however, they are limited to open, serial chains only. There are other methods like S–U (Sheth and Uicker), K–K (Khalil and Kleinfinger) and unified method, which are applicable to open, closed, hybrid and tree like structure robots. However, these methods are more complex to apply as compared to D–H parameter based methods. The present article provides the comparison of all of these methods in terms of complexity of method, number of frames required for a particular method, number of parameters required, and range of application. Moreover, the inconsistencies of D–H parameter method and its variants, while dealing with spatial hybrid manipulators, have been highlighted and addressed in this work through a case study of seven-DOFs hybrid manipulator. Through the comparative study, it has been concluded that a lot of modeling techniques are available for the kinematic study of robotic manipulators, however the ambiguity lies in the selection of most appropriate method. The authors recommend the end-user to use the comparison table, given in this work as a reference, to select the appropriate method based on robot type and the computational cost to be borne (Table 6).

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