Optimization and Engineering, 6, 85–115, 2005

Pareto Frontier Based Concept Selection Under Uncertainty, with Visualization

CHRISTOPHER A. MATTSON
ACHILLE MESSAC
Department of Mechanical, Aerospace, and Nuclear Engineering, Rensselaer Polytechnic Institute, Multidisciplinary Design and Optimization Laboratory, Troy, New York, 12180, USA
email: messac@rpi.edu

Received December 12, 2002; Revised August 27, 2003

Abstract. In a recent publication, we presented a new multiobjective decision-making tool for use in conceptual engineering design. In the present paper, we provide important developments that support the next phase in the evolution of the tool. These developments, together with those of our previous work, provide a concept selection approach that capitalizes on the benefits of computational optimization. Specifically, the new approach uses the efficiency and effectiveness of optimization to rapidly compare numerous designs, and characterize the tradeoff properties within the multiobjective design space. As such, the new approach differs significantly from traditional (non-optimization based) concept selection approaches where, comparatively speaking, significant time is often spent evaluating only a few points in the design space. Under the new approach, design concepts are evaluated using a so-called *s-Pareto frontier*; this frontier originates from the Pareto frontiers of various concepts, and is the Pareto frontier for the *set* of design concepts. An important characteristic of the s-Pareto frontier is that it provides a foundation for analyzing tradeoffs between design objectives *and* the tradeoffs between design concepts. The new developments presented in this paper include; (i) the notion of *minimally representing* the s-Pareto frontier, (ii) the quantification of concept goodness using s-Pareto frontiers, (iii) the development of an interactive design space exploration approach that can be used to visualize *n*-dimensional s-Pareto frontiers, and (iv) s-Pareto frontier-based approaches for considering uncertainty in concept selection. Simple structural examples are presented that illustrate representative applications of the proposed method.

Keywords: concept selection, concept evaluation, multiobjective optimization, Pareto frontier, s-Pareto frontier

1. Introduction and literature review

Successful engineering design generally requires the resolution of various conflicting design objectives. In order to identify an optimal design, conflicting objectives are typically considered simultaneously. One of the most powerful tools for resolving such objectives, in a computational setting, is multiobjective optimization. Generally speaking, there are many potential solutions to multiobjective optimization problems; a particular set of optimal solutions is referred to as the Pareto optimal set or Pareto frontier. By definition, Pareto solutions are considered optimal because there are no other designs that are superior in all objectives (Steuer, 1986; Belegundu and Chandrupatla, 1999; Miettinen, 1999; Pareto, 1964).

In a recent publication (Mattson and Messac, 2003a), we presented a new Pareto frontier-based approach to concept selection in conceptual engineering design. The new approach, termed *s-Pareto frontier-based concept selection*, seeks optimal decision-making early in the design process where the impact of decision-making is still significantly high. In the present paper, we provide key developments that support the next phase in the evolution of the concept selection tool. These developments, together with those of our previous work, provide a concept selection approach that capitalizes on the benefits of computational optimization. Specifically, the new approach uses the efficiency and effectiveness of optimization to rapidly compare numerous designs, and characterize the tradeoff properties within the multiobjective design space. As such, the new approach differs significantly from traditional (non-optimization based) concept selection approaches where, comparatively speaking, significant time is often spent evaluating only a few points in the design space.

Under the first phase of developments, which were presented in Mattson and Messac (2003a), concept goodness was assessed by qualitatively examining two-dimensional Pareto frontiers. While this was useful for conveying important details about the approach, few particulars were provided for assessing concept goodness for problems of n-dimension. This paper importantly develops a quantitative measure of concept goodness, and subsequently uses the measure to explore possibilities in visualization and uncertainty handling in concept selection.

There are two fundamental developments upon which the quantitative measure of concept goodness is based; (i) the notion of an *s-Pareto frontier*, and (ii) the notion of a *minimal* (or *smart*) *representation* of multiobjective design space. The former is a frontier originating from the Pareto frontiers of one or more design concepts, and is the Pareto frontier for a set of concepts. The latter is a strategy for identifying (through filtering) the minimal set of Pareto points needed to adequately represent the tradeoff properties of the design. Such a set takes us from the current state-of-the-art (uniformly spaced points on the Pareto frontier) to a minimal set, which is shown to play a critical role in the concept selection process. The motivation for the current work is illustrated in the next section, which is followed by a review of pertinent literature.

1.1. Motivation

The motivation for the current work stems from challenges encountered during our initial development of the s-Pareto frontier-based concept selection framework, and from challenges observed in the works of others. An illustrative example is the conceptual design of a 50-seat commuter aircraft by Crossley et al. (2001). In the design, they consider two basic aircraft configurations (one with a turbofan engine type and one with a turboprop), and use a Genetic Algorithm (GA) to obtain approximate Pareto frontiers for each aircraft concept. Using the frontiers, Crossley et al. identify the strengths and the weaknesses of each concept early in the design process, where it is well-known that decisions significantly impact design success (Ishii, 1995; Homan and Thornton, 1998).

From the design of the commuter aircraft by Crossley et al., we can make three observations that are representative of common challenges in Pareto frontier-based design. The first observation is that the distribution of points in the Pareto set plays an important role in representing the Pareto frontier. In the study by Crossley et al., Pareto points are haphazardly clustered together in some regions of the design space, while they are extremely sparse in others. Haphazard clustering leaves some regions of the Pareto frontier unintentionally under-represented, while cluster regions are left over-represented. The state-of-the art of *uniformly spaced* points remedies this problem. We submit, however, that uniformly spaced points are not always desirable in concept selection (this point is elaborated on in Sections 3 and 4). Therefore, we seek to identify distribution properties that support the concept selection method presented in this paper.

The second observation is that the strengths and weaknesses of the aircraft concepts are assessed by qualitatively examining two-dimensional frontiers. To carry Pareto frontier-based design to a generically implementable level, approaches must be developed for visualizing Pareto frontiers for problems of more than two objectives. Additionally, when multiple concepts are being evaluated using Pareto frontiers we must develop measures that quantify the concepts' strengths and weaknesses. The latter is essential to bringing the usefulness of Pareto frontiers to conceptual design where multiple disparate concepts are evaluated. Therefore, we wish to quantitatively identify concept strengths and weaknesses for problems of n-dimension.

The third observation is that the effect of uncertainty is not considered when drawing conclusions and characterizing aircraft strengths and weaknesses. We note that the decisions made in conceptual design are among the most important and that these decisions are being made when the greatest uncertainty exists. Therefore, we must develop non-deterministic tools that can be used in conceptual design to perform concept selection.

1.2. Literature review

In this section, pertinent literature in the following areas is briefly reviewed; (i) concept selection, (ii) handling uncertainty, and (iii) visualization.

1.2.1. Concept selection. The various concept selection methods found in the literature can be divided into two groups; (i) non-numerical approaches, and (ii) numerical approaches. Each is briefly discussed below. We also discuss the role of optimization in concept selection by reviewing developments in Pareto frontier-based design.

Non-numerical concept selection is qualitative and void of a mathematical basis. Some argue that mathematics is of little assistance—early in the design process—and in fact hinders decision-making (Pugh, 1996). As a result, various non-numerical methods for concept selection have been developed, including the following: (1) Selection approaches based on individual arbitrary preference such as external or internal decision makers, and voting (Ulrich and Eppinger, 2000). (2) Selection approaches based on methodical/structured preference such as feasibility judgement/intuition, go/no-go screening, pros and cons, prototype and test, and technology-readiness assessment (Otto, 1995; Ullman, 1992; Ulrich and Eppinger, 2000). (3) Selection methods based on decision matrices such as concept screening and the Pugh concept selection method (Pugh, 1996; Ulrich and Eppinger, 2000).

Numerical concept selection is quantitative in nature and includes the following methods; Axiomatic Design (Suh, 1990), Decision Matrices (Dieter, 1991; Magrab, 1997; Pahl and Beitz, 1996; Ullman, 1992; Ulrich and Eppinger, 2000), Fuzzy Approaches (Wang, 2001), Knowledge Based Systems (Dieter, 1991; Chin and Wong, 1996), Utility Function Method, and Quality Function Deployment (Otto, 1995; Magrab, 1997). One argument for using numerical concept selection approaches is that they can facilitate uncertainty analyses during the decision-making process.

The decision matrix approach is perhaps the most popular numerical concept selection approach. Recently, the pitfalls of the decision matrix approach have been discussed in Mullur et al. (2003). The authors bring to the fore well-known drawbacks of weighted sum methods and present them in the context of concept selection. Importantly, the authors show that other summation approaches, such as compromise programming, do not readily avoid the pitfalls of decision matrix approaches.

Multiobjective optimization has been used by some researchers to perform concept selection by posing the design as an optimization problem, then choosing the designs that satisfy the optimization conditions. More specifically, Pareto frontiers, which depict the tradeoffs between objectives, have been used to select the best design alternative. Pareto based methods found in the literature include works by Balling (2000), Chankong and Haimes (1983), Charnes and Cooper (1961), Crossley et al. (2001), Das (1999), Eschenauer et al. (1990), Gass and Saaty (1955), Hwang and Masud (1979), Kasprzak and Lewis (2000), Li et al. (1998), Mattson and Messac (2003a), Palli et al. (1998), Steuer (1983), Tappeta et al. (2000), and Wierzbicki (1980).

Difficulties with Pareto frontier-based selection methods include (i) visualizing the Pareto frontier for n-dimensional problems, (ii) obtaining a representative set of discrete solutions to represent the frontier, and (iii) handling uncertainty associated with the frontiers.

1.2.2. Handling uncertainty. Uncertainties caused by variations in material properties, loading conditions, manufacturing precision, modeling assumptions, and other sources can—and should—significantly affect the decision-making process.

A typical approach for handling uncertainties in engineering design is to use safety factors. Although popular, the safety-factor approach generally results in overly conservative designs (Koch, 2002), and has other serious deficiencies as described by Rao (1992). Alternatively, various non-deterministic methods for engineering design have been developed. Two noteworthy approaches are: (i) *reliability-based methods* (Thanedar and Kodiyalam, 1991; Melchers, 1999; Cornell, 1969) and (ii) *robust design based methods* (Parkinson et al., 1993; Chen et al., 1999, 2000; Su and Renaud, 1997; Taguchi, 1993; Messac and Ismail-Yahaya, 2002).

Both approaches focus on the probability of design failure, where failure is said to occur when design constraints are violated. To minimize the likelihood of design failure, both approaches seek to reduce the area under the probability density function (PDF) that lies outside the constraint boundaries.

Under the first approach (reliability-based design) the area under the PDF that lies outside the constraint boundaries is reduced by shifting the mean value away from constraint limits. This assumes the shape of the PDF does not change during shifting. Probabilistic design constraints are typically added to optimization formulations to perform this shifting. Note that under the reliability-based approach, the mean performance is optimized, however the variation thereof is not minimized.

The second approach (robust design) is different from reliability-based design in that the area under the PDF that lies outside the constraint boundaries is assumed reduced by optimizing the mean performance *and* minimizing its variation. Again, feasibility is maintained with probabilistic constraints. A robust design is defined as one whose performance remains relatively unchanged and remains feasible in the presence of uncertainty.

Performing uncertainty analyses generally involves obtaining the variations in the response (design metric) given variations in the input variables. Various approaches are typically used to obtain this output variation. Among others, these methods include Monte Carlo simulations, and sensitivity-based estimations (Koch, 2002).

The Monte Carlo simulation is known to appropriately represent the behavior of the response variable—when an adequately large number of samples is used. For computationally expensive problems, the number of samples needed to obtain the output variation may render the Monte Carlo approach prohibitive. In this case, other reduced-sampling methods exist that can provide an adequate representation of the response function variation. These methods include Latin Hyper-cube sampling, Taguchi's orthogonal arrays, and importance sampling (Du and Chen, 2000).

Sensitivity based methods focus on predicting the change in output performance, given variations in input parameters. This is typically done by using Taylor's series expansions. The mean and standard deviation of the response are approximated using these expansions. These approximations should be used carefully as their misuse can result in erroneous solutions. The main reason for this behavior is that sensitivities are fundamentally of local validity.

1.2.3. Visualization. Visualization in engineering design optimization has recently become a significant area of research. Various approaches exist for visualizing optimization *results* (Goel, 1999; Winer and Bloebaum, 1997; Mattson and Messac, 2002); other approaches exist for visualizing the optimization *process* (Messac and Chen, 2000). The latter may incorporate interactive features that allow the designer to terminate the optimization process if it progresses in an unfavorable way. In the context of Pareto frontier based concept selection, it is particularly challenging to visualize n-dimensional Pareto frontiers.

One element that will lead to successful visualization of the n-dimensional design space is the ability to quantify Pareto set properties. Wu and Azarm (2001) present Pareto set measures that characterize the goodness of a Pareto set as obtained by one or many multiobjective optimization methods. The measures are primarily used to compare how well different optimization approaches generate Pareto sets (Van Veldhuizen, 1998; Wu and Azarm, 2001; Zitzler and Thiele, 1998). Their proposed measures include: Pareto hypersurface area, Pareto spread, number of distinct Pareto choices, and Pareto cluster, which can be appropriately applied to continuous Pareto frontiers.

1.2.4. Observations from literature review. In summary, we can make the following observations from the literature survey. (i) Relatively few concept selection methods use computational optimization techniques to evaluate concepts. As a result, few concept selection methods capitalize on desired benefits of computational optimization. Namely, the rapid exploration and characterization of tradeoffs in multiobjective design spaces. (ii) Pareto frontier-based design practitioners face challenges in identifying an appropriate set of discrete solutions to represent the frontiers. They also face challenges in visualizing n-dimensional design spaces. Finally, (iii) the influence of uncertainty should not be omitted from the decision-making process. The developments of this paper seek to support the visualization/exploration of n-dimensional design spaces and account for uncertainty under the s-Pareto frontier-based concept selection framework.

The remainder of the paper is organized as follows. In Section 2, we briefly review the basic s-Pareto frontier-based concept selection framework. We then introduce a new strategy for representing the multiobjective design space (smart s-Pareto frontiers) in Section 3. In Section 4, we then present analytical developments for quantifying concept goodness using s-Pareto frontiers. In Section 5, an interactive design space exploration approach is presented that facilitates visualizing smartly distributed n-dimensional s-Pareto frontiers. Uncertainty is then considered in Section 6 where two non-deterministic approaches for performing concept selection using s-Pareto frontiers are presented. Finally, in Section 7, we present a simple truss example that shows the usefulness of the proposed methods.

2. The s-Pareto frontier-based concept selection framework

In this section, we briefly describe *s-Pareto frontier-based concept selection*. While only a basic definition of the approach is provided here, important details and optimization problem formulations for generating the s-Pareto frontier are presented in full in Mattson and Messac (2003a). We begin this section by presenting a generic deterministic multiobjective optimization problem (MOP). Pareto optimality is then defined, followed by a brief description of the s-Pareto frontier and decision-making based thereon.

2.1. Multiobjective optimization and Pareto optimality

Under the developments of this paper, conflicting objectives encountered during concept selection are resolved using multiobjective optimization. The generic deterministic multiobjective optimization problem (MOP) formulation is given as follows.

$$\min_{x}\{\mu_1(x, p), \quad \mu_2(x, p), \dots, \mu_n(x, p)\} \quad (n \geq 2) \tag{1}$$

subject to

$$g_q(x, p) \leq 0 \quad q = 1, \dots, r \tag{2}$$

$$h_j(x, p) = 0 \quad j = 1, \dots, v \tag{3}$$

$$x_{il} \leq x_i \leq x_{iu} \quad i = 1, \dots, n_x \tag{4}$$

where μ_i denotes the i-th generic design objective; g is a vector of inequality constraints; h is a vector of equality constraints; x is a vector of design variables; and p is a vector of design parameters. The design variables are bound by their lower (l) and upper (u) limits as shown in (4).

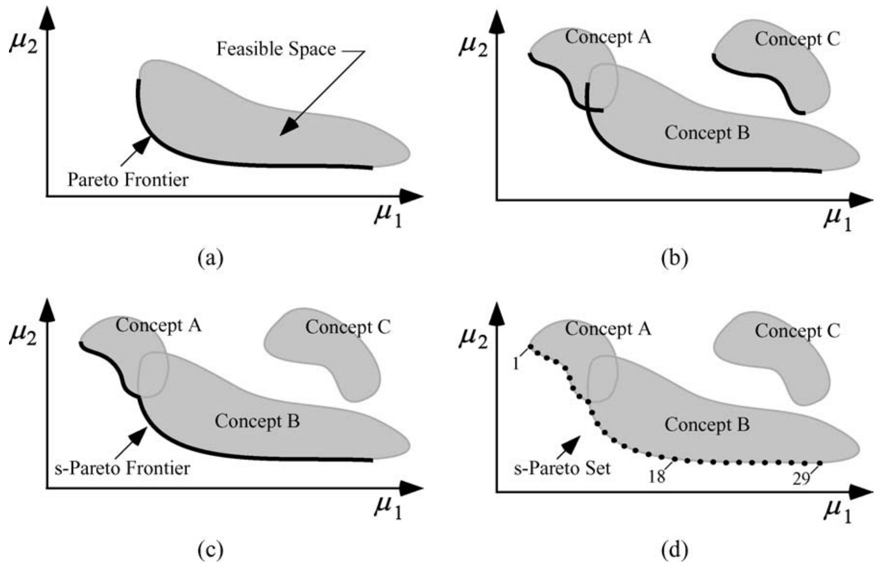

Figure 1. (a) Pareto frontier for one concept. (b) Pareto frontiers for various concepts. (c) The s-Pareto frontier. (d) An s-Pareto set.

An important class of solutions to the MOP are said to belong to the Pareto frontier. The heavy line of Figure 1(a) represents the Pareto frontier for a bi-objective minimization problem, where μ_1 and μ_2 are the design objectives, and the shaded region represents the feasible space. Each solution comprising the frontier is said to be *Pareto optimal*, which means there are no other designs for which *all* objectives are better. Pareto optimality is more specifically defined below.

Pareto optimality: A design objective vector μ^* is Pareto optimal if there does not exist another design objective vector μ in the feasible design space such that $\mu_i \leq \mu_i^*$ for all $i \in \{1, 2, \ldots, n\}$, and $\mu_j < \mu_j^*$ for at least one index of j, $j \in \{1, 2, \ldots, n\}$.

We generally seek Pareto solutions because obtaining them means the objectives have been improved as much as possible without having to give up anything in exchange.

There are two general approaches to Pareto based decision-making. Under the first, the objectives of the MOP given in (1) are mathematically combined (scalarized) to form an Aggregate Objective Function (AOF), which yields a single optimal solution to the MOP—a member of the Pareto frontier. This approach has been called Integrated Generating-Choosing (IGC) (Messac and Mattson, 2002). When the AOF is properly constructed, it is expected to fully capture all design preferences, yielding what is understood to be the preferred design. Unfortunately, properly constructing the AOF is extremely difficult. Few methods exist that facilitate proper construction of the AOF in a timely and logical manner. Physical programming is one of them (Messac, 1996).

The second Pareto based decision-making approach involves generating many potentially preferable Pareto optimal designs, followed by choosing the most attractive one. This approach has been called Generate First—Choose Later (GF-CL) (Deb, 2001; Messac and Mattson, 2002). In this paper we focus only on the GF-CL approach. An important benefit of the GF-CL approach is that by generating numerous Pareto solutions a designer can compare a range of design alternatives, where the range covers differing levels of tradeoff between objectives. One drawback to GF-CL is that generating many Pareto solutions may be too computationally expensive to justify.

2.2. *The s-Pareto frontier*

In this section, we briefly describe the meaning of *s-Pareto frontier*, and show how it can play a meaningful role in concept selection. Before describing the s-Pareto frontier, we define the following terms as they are used in the definition of the s-Pareto frontier.

A design concept is an idea that has evolved to the point where there is a parametric model, however rudimentary, that represents the performance of the family of design alternatives that belong to that concept's definition.

A design alternative is a specific design resulting from specified unique parameter values used in the parametric model of a concept.

We note that selecting a design concept is generally the focus of conceptual design, while selecting a design alternative is typically the focus of detailed design. We assume that each design concept has its own Pareto frontier (consisting of numerous design alternatives), which is defined by the concept's parametric model.

To introduce the notion of an s-Pareto frontier, we consider Figure 1(b) and the accompanying design space for the design objectives μ_1 and μ_2. Candidate design concepts are represented by the shaded areas labeled Concept A, B, and C. Each concept is unique in that it has its own parametric model that defines the space it occupies. The model quantifies the performance of a family of design alternatives, which belong to that concept's definition. The set of potentially optimal alternatives comprise the individual Pareto frontiers (black curves) of each concept. The s-Pareto frontier originates from the Pareto frontiers of one or more of the design concepts and is the Pareto frontier for the set of concepts. The s-Pareto frontier is illustrated by the heavy line in Figure 1(c). Each solution comprising the s-Pareto frontier is said to be *s-Pareto optimal*, which means there are no other designs—from the same or any other concept—for which all objectives are better. Formally, we define s-Pareto optimality as follows.

s-Pareto optimality: A design objective vector μ^{s*} is s-Pareto optimal if there does not exist another design objective vector μ^k in the feasible design space of concept k such that $\mu_i^k \leq \mu_i^{s*}$ for all $i \in \{1, 2, \ldots, n\}$ and all concepts k, where $k \in \{1, 2, \ldots, n_c\}$; and $\mu_j^k < \mu_j^{s*}$ for at least one index of j, $j \in \{1, 2, \ldots, n\}$ for any concept k, $k \in \{1, 2, \ldots, n_c\}$. The number of design concepts is denoted by n_c.

The "s" in s-Pareto frontier indicates that the frontier is for the *set* of concepts. Similar to other Pareto frontiers, the s-Pareto frontier can be used to determine the tradeoffs between design objectives. However, unlike other Pareto frontiers, the s-Pareto frontier can be used to characterize the tradeoffs between design concepts. This property of the s-Pareto frontier is what makes it extremely useful for decision-making in conceptual design. For example, it can be seen through examining the s-Pareto frontier (Figure 1(c)) that Concept C is inferior to Concepts A and B, both of which are partially-dominant. Furthermore, it can be seen that Concept A is superior to Concept B when low values of μ_1 are preferred, and that Concept B is superior for low values of μ_2. Importantly, the point at which one concept becomes superior to another—for any objective—can also be identified by examining the s-Pareto frontier.

Various approaches may be used to obtain a set of points that discretely represents the s-Pareto frontier. This set, referred to as the s-Pareto set, is illustrated in Figure 1(d). Approaches for obtaining the s-Pareto set include the following: (i) Obtaining Pareto sets for each concept (using any Pareto set generator, be it gradient-based or not), and filtering out any solution that does not satisfy the definition of s-Pareto optimality. (ii) Directly obtaining the s-Pareto set using a single optimization problem statement. The latter is discussed in detail in Mattson and Messac (2003a). We note that the method presented in Mattson and Messac (2003a) yields a set of evenly distributed points along the s-Pareto frontier, not unlike the distribution of points illustrated in Figure 1(d). An even distribution of points is one where the separations between any two neighboring points along each given objective are nearly equal.

As mentioned in Section 1, the developments of this paper support the next phase in the evolution of the s-Pareto frontier-based concept selection framework. Specifically the new developments allow us to (i) quantify concept goodness using the s-Pareto frontier, (ii) interactively explore n-dimensional s-Pareto frontiers, and (iii) consider the effect of uncertainty on concept selection.

3. Minimal representation of multiobjective design space

In this section, we present a requisite development that supports the proposed method for quantifying concept goodness using s-Pareto frontiers. Specifically, a new philosophy for discretely representing the s-Pareto frontier (or any Pareto frontier) is presented in this section. The emphasis of the new philosophy is to keep the number of s-Pareto solutions (design alternatives) sufficiently small while meaningfully representing the tradeoffs in the design space.

Until recently, the emphasis in Pareto frontier representation in the community has been to obtain a comprehensive set of Pareto solutions. The most common approach is to systematically vary generic parameters/weights in an AOF. The weighted sum AOF is one such option, where the pre-multiplying factors of each objective are varied, yielding different Pareto solutions. The pitfalls of this and other weighting approaches are well-documented in the literature (Messac and Ismail-Yahaya, 2001; Das and Dennis, 1997; Koski, 1985; Mullur et al., 2003). One of the greatest drawbacks to weighting methods is that the distribution of Pareto solutions is generally haphazard and not easily under the designer's control.

Successful attempts to circumvent haphazard distributions of Pareto points have led to the current state-of-the-art in Pareto frontier generation—even distributions of Pareto points. An even distribution of points is one where the separations between any two neighboring points along each given objective are nearly equal. Specifically, given two neighboring Pareto points μ^p and μ^q, and two other neighboring Pareto points μ^r and μ^t, then we must have $|\mu_k^p - \mu_k^q| \approx |\mu_k^r - \mu_k^t|$ for all k, where the subscript k in μ_k^i denotes the k-th entry of the vector μ^i.

Three methods have been distinguished for their ability to generate even distributions. They are the Normal Boundary Intersection method (Das and Dennis, 1998), the Physical Programming method (Messac and Mattson, 2002), and the Normal Constraint method (Messac et al., 2003). The strengths and weaknesses of these three methods, as well as those of other popular approaches, are discussed in Messac et al. (2003). Another useful approach for generating a Pareto frontier is to use a Genetic Algorithm (GA). Although they do not typically yield even distributions, the GAs are particularly attractive when handling discrete variables such as the number of engines on an aircraft. We note that the Normal Constraint and Physical Programming methods are also applicable to discrete variables.

Although useful in many Pareto-based design applications, even distributions of Pareto points are not beneficial to the s-Pareto frontier-based concept selection framework in some very practical cases. Let us elaborate. Consider the s-Pareto frontier for the design objectives μ_1 and μ_2, shown in Figure 1(d). The solid points are discrete s-Pareto solutions that comprise the s-Pareto set. For convenience, some of the s-Pareto points have been numbered. To illustrate that even distributions may hinder the selection process, consider the s-Pareto frontier region between points 18 and 29 in Figure 1(d). Although technically s-Pareto optimal, these points exhibit little tradeoff between the objectives. As a result, the designer may wish to declare these regions to be less useful than regions where significant tradeoff is observed.

The usefulness of an s-Pareto frontier region can be represented for the bi-objective case by the following expression.

$$\Upsilon = \frac{d\mu_2}{d\mu_1} \tag{5}$$

For regions of the s-Pareto frontier where $\Upsilon \rightarrow 0$ or $\Upsilon \rightarrow -\infty$ there is little tradeoff between the objectives μ_1 and μ_2. Consequently, these regions are considered less useful than others, and are removed or deemphasized under the new philosophy. From a practical perspective, we presume that a designer is willing to give up an insignificant amount in one objective to gain significantly in another. Therefore it is reasonable to assume that Point 18 is preferred over any s-Pareto point between 18 and 29. To keep the set of candidate design alternatives manageably small, we remove or deemphasize the region between such points as shown in Figure 3(a). For regions where $\Upsilon \simeq -1$, there is nearly exact tradeoff between objectives, therefore we retain these regions and those with similar Υ values.

We note that obtaining a manageably small set of alternatives is desired over large sets because one cannot generally assess more than say 10 to 20 alternatives in one's own mind. In addition, identifying the smart s-Pareto set is critical to the measure of concept goodness developed in the next section. We parenthetically note that the visualization/exploration technique presented in the Section 5 will facilitate our handling a relatively large number of s-Pareto points. In addition, we note that appropriate scaling of the design objectives may be required. For the sake of presentation simplicity, we will assume here that the objectives are of similar magnitude. Important scaling issues related to the generation of Pareto points are discussed in detail in Messac et al. (2003).

Importantly we note that for problems of n objectives, Υ is a function of the off-diagonal terms in A below.

$$A_{n\times n} = \begin{bmatrix} 1 & \frac{\partial\mu_1}{\partial\mu_2} & \cdots & \frac{\partial\mu_1}{\partial\mu_n} \\ \frac{\partial\mu_2}{\partial\mu_1} & 1 & \cdots & \frac{\partial\mu_2}{\partial\mu_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\mu_n}{\partial\mu_1} & \frac{\partial\mu_n}{\partial\mu_2} & \cdots & 1 \end{bmatrix} \tag{6}$$

Regions of the s-Pareto frontier where any of the off-diagonal elements in A approach 0 (negligible tradeoff) are deemed less useful when compared to one where all off-diagonal elements in A approach 1 (perfect tradeoff).

After removing or deemphasizing less useful regions of the frontier, the new frontier is said to be *smart* in that it consists of only useful parts of the frontier. Under the new approach the designer controls the reduction process, as will be shown shortly. A filtering process can be used to eliminate less useful points in the s-Pareto set, and consequently identify the smart s-Pareto set. When eliminating such points, we must also remove clusters of points—even if they exhibit good tradeoff properties. This is part of the strategy to keep the number of design alternatives manageably small. Simply stated, the strategy is to remove any design that is not significantly different from another, thus reducing the set to only those designs that meaningfully characterize the tradeoff properties, and from which the designer can make a reasonable assessment.

The filtering approach, termed *smart filtering*, can be understood graphically by considering Figure 2(a). Let the s-Pareto point μ^i be located at the origin of the axes. The axes

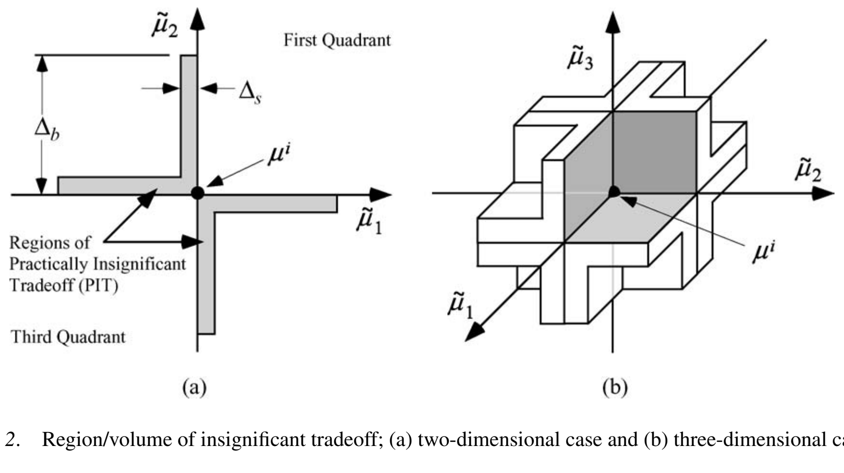

Figure 2. Region/volume of insignificant tradeoff; (a) two-dimensional case and (b) three-dimensional case.

$\tilde{\mu}_1$ and $\tilde{\mu}_2$ represent deviations of the respective objectives from the point μ^i. The shaded region of Figure 2(a) is called the region of Practically Insignificant Tradeoff (PIT). Any design alternative (s-Pareto point) that lies within the PIT region of μ^i is considered to be not materially different from μ^i, and is therefore removed by the filter. The same procedure is carried out for all points μ^i, $i \in \{1, 2, \ldots, n_p\}$ that have not already been removed from the set. The resulting points constitute the smart s-Pareto set. We importantly note that under the developments of this paper, the smart s-Pareto set is obtained only after (i) generating many s-Pareto points, and (ii) removing all points except only those that are needed to meaningfully represent the design space. A significantly number of s-Pareto solutions is potentially needed to accurately represent the s-Pareto frontier in a discrete way.

The designer controls the filtering process by specifying two control variables; Δ_s, and Δ_b. By prescribing values for these two control variables, the designer implicitly makes two dependent statements: (i) Do not keep two s-Pareto points for which the difference between two corresponding objectives is less than Δ_s. This difference is considered to entail *practically insignificant tradeoff*. Second, (ii) given that the difference between the corresponding objectives of two s-Pareto points is less than Δ_s, if the difference between any other corresponding objectives is greater than Δ_b, then both can be retained in the smart s-Pareto set.

Specifically, the filter performs pairwise comparisons of two s-Pareto points by evaluating the absolute vector difference between the two points as

$$\nu = \mathrm{abs}(\mu^j - \mu^i) \tag{7}$$

The minimum and maximum vector components of ν are denoted by $\nu_{\min}$ and $\nu_{\max}$, respectively. Mathematically, μ^j is removed, when compared to μ^i, if the following conditions are true

$$\nu_{\min} \leq \Delta_s \quad \text{and} \quad \nu_{\max} \leq \Delta_b \tag{8}$$

For problems of three dimensions, the PIT volume is shown in Figure 2(b). The shaded surfaces represent the two octants where no tradeoff occurs with respect to μ^i (analogous to the first and third quadrants in two-dimensional space). The PIT region for n-dimensional design spaces is a hyper-volume.

To successfully implement the smart filter, the data to be smartly filtered must be globally s-Pareto optimal. This is particularly important because the smart filtering process begins only after a single initial point is chosen and declared *smart*; this initial point must be s-Pareto optimal. Additionally, the data to be smartly filtered, must adequately represent the entire s-Pareto frontier; without under representing any particular part of it. We note that the problem of over-representation can be solved by using a smart filter, which removes clusters. Detailed descriptions of a global Pareto and smart Pareto filters are presented in Mattson et al. (2002).

We wish to add a brief commentary on the nature of the control parameters Δ_s and Δ_b. Δ_s is primarily a *design objective tradeoff* parameter, and Δ_b is primarily a *distribution/representation* parameter. For each of these parameters, their values are generally different for each design objective, especially when large differences in magnitude exist between objectives. For simplicity of presentation we have kept Δ_s and Δ_b equal in all dimensions for our discussions.

In the following section, we show how the smart s-Pareto set facilitates the quantification of concept goodness.

4. Quantifying concept goodness using smart s-Pareto frontiers

This section provides a theoretical development for quantifying the goodness of candidate design concepts using smart and non-smart s-Pareto frontiers. Importantly, the proposed quantification is required for the interactive design space exploration approach presented in the next section; the exploration approach is used to visualize tradeoff properties within n-dimensional design spaces.

After a smart s-Pareto set has been identified, the designer can use it to quantify the goodness of each concept. In evaluating concept goodness, we examine Pareto frontier surface areas and assume that concepts whose Pareto frontiers have larger surface areas potentially offer more design *flexibility* than those with smaller Pareto surfaces. More flexible concepts are assumed to be preferred because they provide more design freedom for detailed design.

Under the proposed approach, the designer evaluates the concepts within a particular region of the design space that is of interest to him or her. We call this a *Region of Interest*. By exploring various regions of interest, the designer can start to collect information about the design space (i.e., which concepts occupy which parts of the design space); this information can be used to identify the most attractive design.

To define a region of interest, the designer specifies a single point in the design space. The space that dominates the specified point is the region of interest. Consider Figure 3(b). Here, the region of interest is defined by the Point RI_1, and is represented by the shaded area. We call this *Region of Interest 1* (RI_1). When the designer selects a region of interest, the design space is effectively reduced to that region and only the s-Pareto frontier within that region is considered during the evaluation.

During this evaluation phase, the designer may choose to examine many different regions of interest to determine which concepts are superior in which regions. By so doing, the design space is explored and an understanding of the goodness of each concept is gained.

The goodness of each concept is quantified by determining the intersection of a concept's Pareto frontier with the s-Pareto frontier. Mathematically, we express the goodness of the

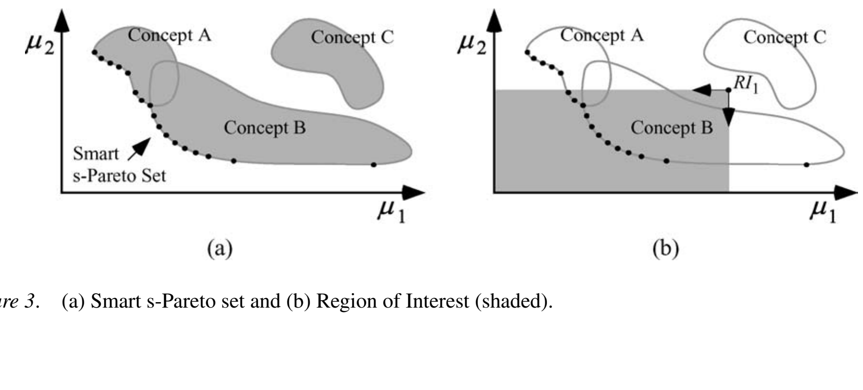

Figure 3. (a) Smart s-Pareto set and (b) Region of Interest (shaded).

i-th concept as

$$\Gamma_i = \frac{\int_{S_p \cap S_{pi}} dS_p}{\int_{S_p} dS_p} \tag{9}$$

where S_p is the s-Pareto frontier, and S_{pi} is the Pareto frontier for the i-th concept. The numerator and denominator in (9) are $n-1$ dimensional integrals. Equation (9) denotes the fraction of the s-Pareto frontier that originates from the i-th concept.

Under the likely scenario that only a discrete representation of the s-Pareto frontier is available, the following development can be used to approximate (9). Let n_s be the total number of s-Pareto solutions and n_{si} be the number of s-Pareto solutions originating from the i-th concept. When n_s is evenly distributed, it can be used as a measure of the s-Pareto frontier surface area. That is,

$$\int_{S_p} dS_p \propto n_s \tag{10}$$

Likewise, the area of the intersection between the i-th frontier and the s-Pareto frontier is proportional to the number of s-Pareto solutions originating from the i-th Pareto frontier (i-th concept). That is,

$$\int_{S_p \cap S_{pi}} dS_p \propto n_{si} \tag{11}$$

Therefore, the goodness of the i-th concept can be approximated as

$$\Gamma_i \approx \frac{n_{si}}{n_s} \tag{12}$$

As the overall density of s-Pareto solutions increases, this approximation of the measure of goodness becomes more accurate.

We now consider evaluating the goodness of each concept when using the smart s-Pareto frontier, which does not generally comprise evenly distributed points. A weight function, $w(\Upsilon(S_p))$, is used to deemphasize less useful regions of the s-Pareto frontier. The weight function is such that $w(\Upsilon(S_p)) \to 0$ when $\Upsilon \to -\infty$ or $\Upsilon \to 0$; and $w(\Upsilon(S_p)) \to 1$ when $\Upsilon \to -1$. Given an s-Pareto frontier with less useful regions that are deemphasized by the weight function, we evaluate the goodness of concept i as

$$\Gamma_i^s = \frac{\int_{S_p \cap S_{p_i}} w(\Upsilon(S_p))\, dS_p}{\int_{S_p} w(\Upsilon(S_p))\, dS_p} \tag{13}$$

The superscript s on Γ_i^s denotes that it is based on the *smart* s-Pareto frontier. Through discretization, it can be shown that

$$\Gamma_i^s \approx \frac{n_{ssi}}{n_{ss}} \tag{14}$$

where n_{ssi} is the number of smart s-Pareto solutions originating from concept i, and n_{ss} is the total number of smart s-Pareto solutions.

The previous developments are particularly useful since, in general, we can more easily obtain a discrete representation of the s-Pareto frontier (s-Pareto set) than the frontier itself. In the following section, we show how these measures of concept goodness can be used to explore the multiobjective design space.

5. Characterizing the design space through interactive exploration

Visualizing n-dimensional Pareto surfaces poses a significant challenge, which has in many cases been a key factor in deciding to not use Generate-First–Choose-Later (GF–CL) approaches. In this section, we introduce an interactive visualization approach that proves useful in exploring the n-dimensional design space, and selecting the most desirable design from among the set of proposed concepts.

In this section, we develop a visualization tool that can be used to understand how each concept occupies regions of interest within the n-dimensional design space. The tool is interactive in that it allows the user to change the region of interest, and consequently see the resulting measure of goodness for each concept as evaluated in that region of interest, according to (12) and (14).

The interactive exploration of the design space requires that the user provide candidate Pareto sets for each concept, or the s-Pareto set itself. To begin the exploration process the user specifies filtering options using the Graphical User Interface (GUI) shown in Figure 4. The designer may select a global Pareto filter and/or a smart Pareto filter. When selecting a smart Pareto filter, the control parameters Δ_b and Δ_s are also specified. The results of the filtering process are shown on the right side of the window. The user then explores the design space by interactively changing the region of interest using the GUI shown in Figure 5. The top left portion of the window shows information relating to the number of solutions that are considered at any one time. The top right side shows a bar plot indicating the percentage of s-Pareto solutions in the set that originate from each concept. In the bottom half of the GUI, the user specifies a region of interest by moving the sliders, between "best" and "worst" for each design objective. By positioning the slider, a point between the best (utopia) value and the worst (nadir) value for that metric is specified. Doing this for all design objectives

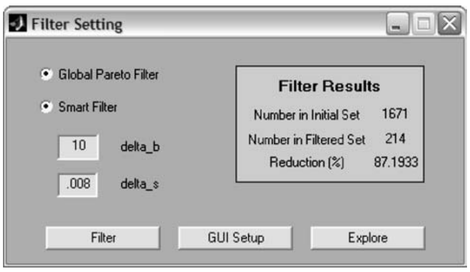

Figure 4. Graphical user interface for filter setup.

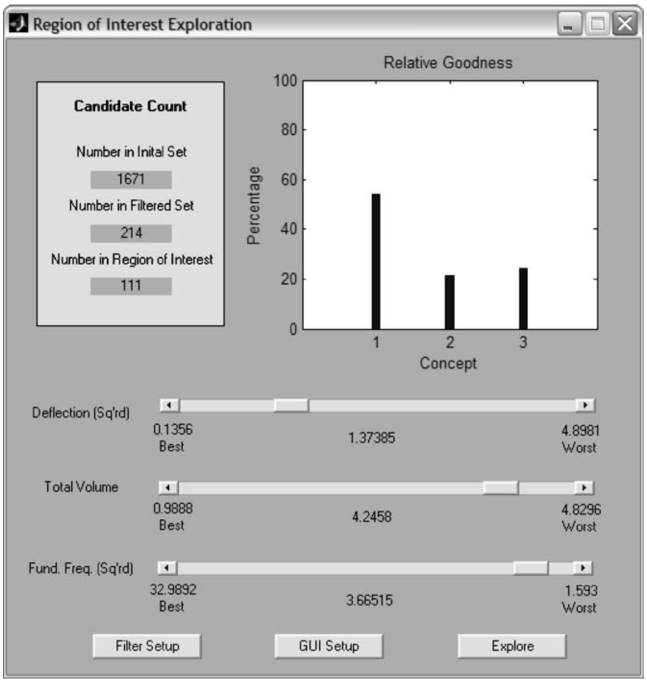

Figure 5. Graphical user interface for design space exploration.

results in an n-dimensional point that defines the region of interest as described earlier in the paper. Each change that the user makes to the sliders, results in an automatic update of the results.

Two important measures are output in the GUI: (i) the bar graph shows the relative goodness of each concept and (ii) the "Number (of points) in Region of Interest" displayed as the last entry of the box in the upper left portion of the GUI. The first measure indicates the dominance disposition of each concept. Concepts with large values of relative goodness occupy more of the region of interest than those with lower values. The second measure is simply an indication of how much of the s-Pareto set is actually being evaluated within a given region of interest. Rather than actually visualizing an n-dimensional frontier, we visualize the characteristics of the concepts in terms of their goodness through the graphical user interface.

Although any region of interest can be chosen by the user, we suggest that at least the following be considered. (i) The region of interest defined by the worst (nadir) point. That is, all sliders would be moved to the far right position; and all design points that dominate the nadir point would be within the region of interest. This region of interest gives an overall view of the design space. (ii) The regions of interest that minimize each objective. That is, all sliders except the i-th slider are moved to the worst position, and the i-th slider is moved to the best position. This tells the user which concept minimizes the i-th objective. (iii) Any region that is of particular interest to the user. For example, additional constraints such as cost, package size, or availability of historical data may make some parts of the s-Pareto frontier interesting to the designer. As the user chooses a region of interest, s/he is warned when s/he has gone outside the feasible space.

The process of examining regions of interest is iterative. If the designer finds that no concept is desirable after exploring a chosen region of interest, then another region of interest should be explored. If there are no other regions in which the designer is interested, and still no concept is desirable, then the designer should return to the beginning of the design process and generate more concepts.

In summary, the designer gains information about the design space by using the proposed interactive exploration tool. He or she then uses the new information to make informed decisions regarding which concept (or concepts) to proceed to detailed design with.

6. Considering uncertainty in concept selection

In this section, we introduce two non-deterministic approaches to concept selection using the s-Pareto frontier. Both approaches take into account two of the most common sources of uncertainty; (i) uncertainty caused by stochastic design parameters, and (ii) uncertainty associated with the engineering design model. The latter is particularly significant during conceptual design, and may be much more consequential than the former (Alexandrov et al., 2000a, b).

We account for uncertainty caused by *stochastic design parameters* by obtaining transmitted variations in response variables. We may do this by using methods such as Monte Carlo simulations or Taylor's series expansions. By examining the variations in output response, we can estimate the amount by which Pareto solutions may deviate from their expected values. In this paper, we make the conservative assumption that the deviations cause the expected values to worsen (e.g., northeast of a bi-objective Pareto frontier). Although we do not discuss them here, we note that interesting phenomena occur when we instead allow the expected values to improve.

We also account for uncertainty caused by the *fidelity of the engineering model* (or lack thereof) according to the following development. The engineering design model is given as

$$\mu_m = \mu_m(x, p) \tag{15}$$

and the actual (true) performance, modeled by μ_m, is given as

$$\mu_a = \mu_a(x, p) \tag{16}$$

A general non-deterministic prediction of the actual design response is assumed to take the convenient form

$$\mu = \mu_a = \rho(x, p)\mu_m(x, p) + \tau(x, p) \tag{17}$$

where ρ and τ respectively denote multiplicative and additive uncertainties. We shall refer to (17) as the Multiplicative Additive Uncertainty Model, or MAU model. A simplified version of the MAU model can be obtained by letting

$$\rho(x, p) = \eta \tag{18}$$

$$\tau(x, p) = \beta \tag{19}$$

where η and β are stochastic variables that represent the non-deterministic nature of the engineering model. This simplified assumption leads to

$$\mu = \eta\mu_m(x, p) + \beta \tag{20}$$

The variables η and β are called Model-Fidelity Uncertainty-Parameters (MFUP). These parameters can be estimated by (i) reviewing historical accuracies of the chosen modeling approaches, or (ii) by assigning them according to an intentional variable model-fidelity approach, should one be used (Alexandrov et al., 2000a, b). For example, these parameters can be chosen by attempting to match results from higher fidelity models. As can be seen in (20), μ_m is modified by the MFUP to account for uncertainty in the engineering model. Although more complex models could be devised, we believe that this MAU model offers a pragmatic balance between modeling fidelity and computational efficiency. Unlike parameter uncertainty, we may allow the uncertainty caused by the engineering model to shift expected Pareto solutions to more optimal regions (e.g., southwest of a bi-objective Pareto frontier).

We now present the last phase of our analytical development, which leads to two approaches for performing non-deterministic concept selection.

Throughout the remainder of this development, we refer to design decisions as having a certain level of reliability. For example, a "zero-sigma" decision is one where decisions are made based on the expected (mean) values of the design metrics. We denote such decisions as "$\kappa\sigma$-decisions", where for example $\kappa = 0$ for deterministic, less reliable, decisions; and $\kappa = 6$ for highly reliable decisions ("six-sigma" decisions). The variable κ is used as any positive number.

Approach 1—Minimize the Mean: Under this approach, the Pareto frontier for each design concept is obtained by minimizing the mean values of the design metrics, subject to probabilistic constraints. The standard deviation for each metric is determined at each point along the Pareto frontier. The Pareto solutions are then shifted by $\kappa\sigma_\mu$, and by using the corresponding MFUP (17). The shifted solutions are used in the comparison and selection of concepts. Approach 1 poses the n-dimensional multiobjective problem associated with that of (1) according to the following problem statement.

$$\min_x\{\hat{\mu}_1(x, p), \hat{\mu}_2(x, p), \ldots, \hat{\mu}_n(x, p)\} \quad (n \geq 2) \tag{21}$$

subject to

$$g_q(x, p) + \kappa\sigma_{gq} \leq 0 \quad q = 1, \ldots, r \tag{22}$$

$$x_{li} + \kappa\sigma_{xi} \leq x_i \leq x_{ui} - \kappa\sigma_{xi} \quad i = 1, \ldots, n_x \tag{23}$$

where $\hat{\mu}_n$ is the mean of μ_n. The standard deviations of g_q and x_i are respectively denoted as σ_{gq} and σ_x. The terms that have been added to (22) and (23) maintain feasibility under probabilistic conditions. Interestingly, we note that the equality constraint in the deterministic problem (see (3)) is not present in the non-deterministic formula given by (21)–(23). To avoid an unduly long discussion regarding the handling of equality constraints under uncertainty, we show how Approaches 1 and 2 are carried out for inequality constrained problems. In the likely event that equality constraints are present, they may be handled according to the developments presented in Messac and Ismail-Yahaya (2002) and Mattson and Messac (2003b). The following steps are used in Approach 1.

1. *Obtain Pareto frontiers*: Obtain the Pareto frontiers for the multiobjective problem given in (21). Do this for each concept being evaluated.
2. *Obtain standard deviations*: Given a random input variables x (or random parameters, p), with means of $\hat{x}$ and $\hat{p}$; and standard deviations σ_x and σ_p, determine the standard deviation of the response variables, σ_μ.
3. *Consider parameter uncertainty*: Shift each Pareto solution by $\kappa\sigma_\mu$ where κ is the number of standard deviations by which each solution is shifted. With this shift, each Pareto solution obtained in Step 1 becomes $\hat{\mu} + \kappa\sigma_\mu$.
4. *Consider model uncertainty*: Shift the frontiers using MFUP (model-fidelity uncertainty-parameters). We note that the convenient functional forms presented in (15–20) may be used.
5. *Select concepts*: Consider the frontier obtained by shifting the Pareto solutions (note that it is generally not a Pareto frontier in $\hat{\mu}$ space), and assess the concepts based on the expected Pareto frontier and the knowledge gained from the shifted Pareto frontier.

For a bi-objective problem the result of shifting the Pareto solutions is depicted in Figure 6. An important observation relating to this approach is that the frontiers, labeled shifted frontier in Figure 6, are not necessarily Pareto frontiers in the $\hat{\mu}$ design space. The frontiers simply represent the location of the expected Pareto frontier when shifted, in a worst case sense, by $\kappa\sigma_\mu$ and through the MFUP. Furthermore, as depicted in Figure 6, a shifted solution may not even be a feasible one.

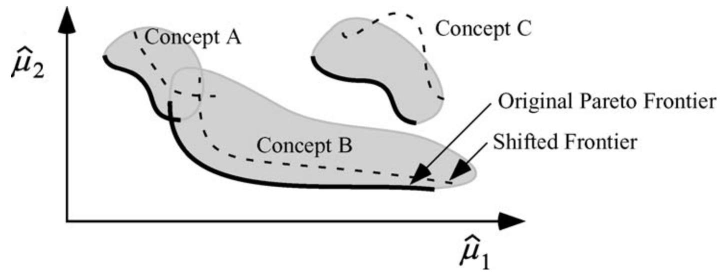

Figure 6. Pareto frontiers (solid curves) for three concepts, and frontiers (dashed curves) shifted by $\kappa\sigma_{\mu n}$ and by model-fidelity uncertainty-parameters.

Approach 2—Minimize the mean plus κ standard deviations: In this approach, the mean values of the design metrics plus their standard deviations ($\hat{\mu} + \kappa\sigma_\mu$) are minimized as indicated below in (24). The main difference between this and Approach 1 is the design space in which the concepts are evaluated. In Approach 1, the $\hat{\mu}$ space was considered; while in Approach 2, the $\hat{\mu} + \kappa\sigma_\mu$ space is considered. The advantage of Approach 2 is that s-Pareto frontiers (in the $\hat{\mu} + \kappa\sigma_\mu$ space) can be obtained for various values of κ. These s-Pareto frontiers can then be used to evaluate concepts according to the s-Pareto approach. In contrast, the advantage of Approach 1 is the design space, which is directly linked to the design objectives of the original multiobjective problem (see (1)). We note that both approaches lend themselves to making $\kappa\sigma_\mu$-decisions that reflect the desired level of decision reliability.

Approach 2 poses the n-dimensional multiobjective problem associated with that of (1) according to the following problem statement.

$$\min_x\{\hat{\mu}_1 + \kappa\sigma_{\mu 1}, \ldots, \hat{\mu}_n + \kappa\sigma_{\mu n}\} \quad (n \geq 2) \tag{24}$$

subject to

$$g_q(x, p) + \kappa\sigma_{gq} \leq 0 \quad q = 1, \ldots, r \tag{25}$$

$$x_{li} + \kappa\sigma_{xi} \leq x_i \leq x_{ui} - \kappa\sigma_{xi} \quad i = 1, \ldots, n_x \tag{26}$$

where $\hat{\mu}$ and σ_μ are respectively the mean and standard deviation of μ; and the standard deviations of g_q and x_i are respectively denoted as σ_{gq} and σ_x. The following steps are used in this approach.

1. *Obtain pareto frontiers for different levels of reliability*: Obtain Pareto frontiers for different values of κ (see (24)). This step only accounts for parameter uncertainty. We do this for each concept being evaluated.
2. *Consider model uncertainty*: Shift the frontiers using MFUP (model-fidelity uncertainty-parameters). We note that the convenient functional forms presented in (15–20) may be used.
3. *Select concepts*: Consider the s-Pareto frontier at the desired $\kappa\sigma_{\mu n}$ and model uncertainty shift, and use the design space exploration methods described earlier in the paper to characterize the tradeoffs within the space.

Approaches 1 and 2 offer two distinct strategies for managing the effect of uncertainties under the s-Pareto frontier-based concept selection framework. As shown in the next section, considering uncertainty using the proposed approaches provides the designer important information that is likely to impact the decision-making process.

In summary, we have presented four distinct developments that support the general applicability of the s-Pareto frontier-based concept selection framework. These developments are (i) the notion of a smart s-Pareto frontier, (ii) the quantification of concept goodness with smart and non-smart s-Pareto frontiers, (iii) the interactive design space exploration approach, and (iv) the two approaches for managing the effect of uncertainty in concept selection. The next section provides two examples that illustrate the usefulness of the new developments.

7. Simple structural examples

In this section, we present two examples. The first is a two dimensional truss optimization problem; the second is a three dimensional extension of the first. The purpose of these examples is to demonstrate (i) the usefulness of the s-Pareto frontier, (ii) the value of smart Pareto distributions, (iii) the utility of the interactive design space exploration approach, and (iv) two approaches for considering uncertainty in the context of concept selection. The following section describes the generic deterministic truss design problem; the description applies to both examples, except where noted.

7.1. Description of simple truss design problem

The multiobjective truss design problem used in both examples is described as follows: Design a truss to support applied loads, minimize nodal displacement, minimize structural volume, and maximize the fundamental frequency (the latter is for Example 2 only)—subject to size and stress constraints. Given this design problem, the concepts shown in Figure 7 have been generated. A truss concept must now be chosen.

The position of node P is characterized by the dimension b and may vary from $\frac{1}{2}L$ to $\frac{3}{2}L$, where L is 60 ft. The cross-sectional areas of each bar may also vary from 0.8 in^2 to 3.0 in^2. The multiobjective optimization formulation for both examples is given below. Additional conditions for each example are given in the subsections below.

$$\min_{a,b} \mu(a, b, E) \tag{27}$$

subject to

$$S_i \leq S_{\max} \quad i = 1, 2, 3 \tag{28}$$

$$0.8\,\text{in}^2 \leq a_i \leq 3\,\text{in}^2 \quad i = 1, 2, 3 \tag{29}$$

$$L/2 \leq b \leq 3L/2 \tag{30}$$

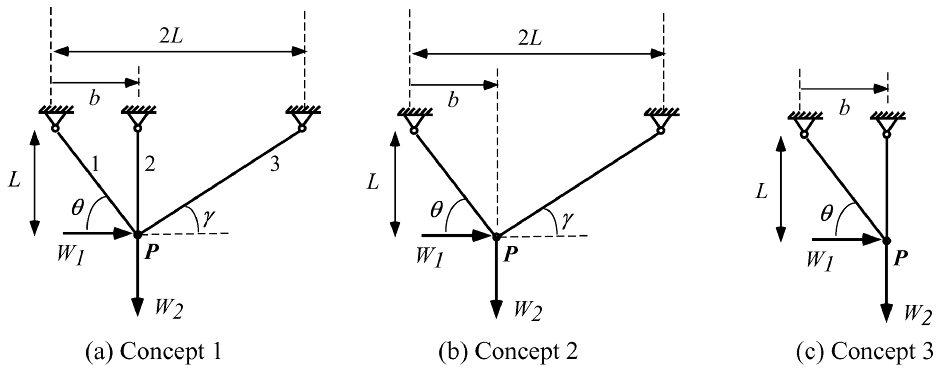

Figure 7. Candidate truss design concepts for structural example.

where $\mu(a, b, E)$ is a vector of design objectives. As indicated by (28), the stress in each bar must be lower than the maximum allowable stress. The left-most bar in Figure 7(a) is Bar 1, the vertical bar is Bar 2, and the bar on the right is Bar 3. The cross-sectional area of each bar, a_i, is constrained by (29), and the horizontal location of node P is constrained by (30). The fixed parameters for these examples are defined as follows. The modulus of elasticity, E, is 29×10^3 ksi; the truss dimension L is 60 ft; the maximum allowable stress, $S_{\max}$, is 550 ksi; and the loads W_1 and W_2 are 100 and 1000 kips, respectively.

7.2. *Example 1: Bi-objective truss design*

In this bi-objective example, we consider three cases. In Case 1, the truss is optimized using a deterministic approach. In Cases 2 and 3, non-deterministic approaches are used to evaluate the truss.

Case 1—deterministic approach: For this deterministic bi-objective case, the two objectives are given as

$$\mu(a, b, E) = [\mu_1(a, b, E) \quad \mu_2(a, b)]^T \tag{31}$$

where $\mu_1(a, b, E)$ is the square of the nodal displacement at node P and $\mu_2(a, b)$ is the total structural volume.

In Figure 8(a), the Pareto frontiers for each concept are shown as solid curves. The solid points are each concept's Pareto solutions, which were obtained using the Normal Constraint method (Messac et al., 2003) and the developments presented in Mattson and

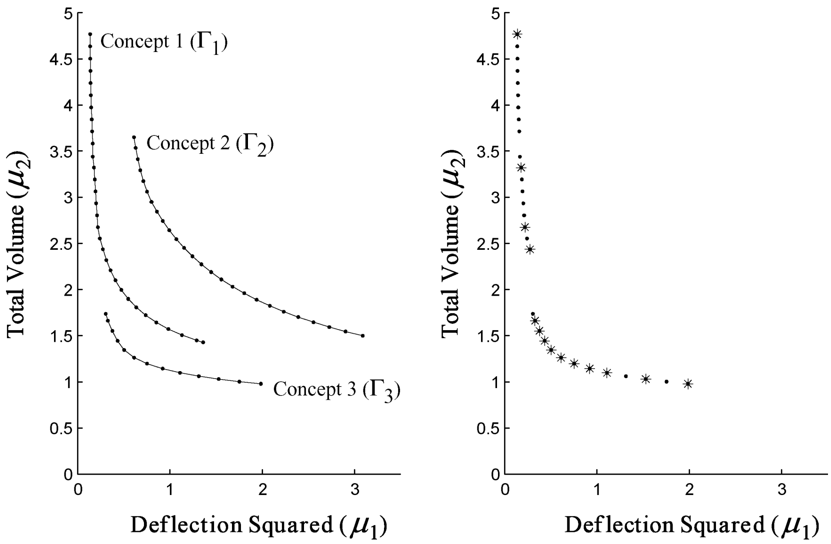

Figure 8. Example 1 Case 1 (a) Pareto frontier and Pareto set, (b) s-Pareto set (·) and smart s-Pareto set (∗).

Messac (2003a). An s-Pareto set is obtained for the three truss concepts and is shown in Figure 8(b) as "·". The smart s-Pareto set is indicated by the set of "$*$" in the figure; the smart set is obtained using smart filter control parameter values of $\Delta_s = 0.04$ and $\Delta_b = 10.00$. The control parameters (Δ_s and Δ_b) are chosen according to the designer's preference. Recall that by specifying a value of say 0.04 for Δ_s, the designer declares that given an initial s-Pareto point he or she is willing to (i) eliminate any other s-Pareto point that is within 0.04 in^2 of it along the deflection squared axis, and (ii) eliminate any other Pareto point that is within 0.04 in^3 of it along the volume axis. A relatively large value for Δ_b indicates that we wish to eliminate all points within Δ_s in one objective, regardless of the distance in any other objective.

Figure 8 explicitly shows the meaning of the s-Pareto frontier, which captures the tradeoff properties between concepts *and* the tradeoffs between objectives; and *is* the Pareto frontier for the set of concepts. The smart s-Pareto set comprises only the s-Pareto points that the designer deems useful. For this two dimensional case, it can be seen that s-Pareto points in regions where $\Upsilon \to 0$ and $\Upsilon \to -\infty$ are not considered smart (see (5)).

For bi-objective cases, the exploration of the design space can be accomplished by visually examining the two dimensional s-Pareto frontier—or by using the interactive exploration tool developed earlier in the paper. Various regions of interest were examined using the interactive exploration tool, and the results are tabulated in Table 1. Five different regions of interest (RI_i) were evaluated; each region is defined by specifying a single point in the design space using the GUI of Figure 5. Recall that the region of interest is that which dominates the specified point. The five regions of interest are described by the values of μ_1 (deflection squared) and μ_2 (volume) in the associated row of the table. The relative goodness of each concept, Γ_i^s, is also listed. The columns labeled 0σ apply to Case 1, while the others apply to Case 2.

Consider the three boldface entries in Table 1. These entries correspond to RI_1 and 0σ, and show that 71% of the smart s-Pareto solutions originate from the Pareto frontier of Concept 3. Similarly, 29% and 0% originate from Concepts 1 and 2, respectively. The results of Table 1 can be compared to the smart s-Pareto set shown in Figure 8(b). Note that if the smart filter were not used to eliminate less useful s-Pareto points, Concept 1 would likely be misrepresented in the measure of goodness. Numerous s-Pareto solutions originate from Concept 1, which initially indicates that it offers good flexibility in the design space. However, most of the s-Pareto solutions would be eliminated if the designer declared that

Table 1. Example 1—Cases 1 and 2: Exploration results.

			Γ_1^s (Concept 1)			Γ_2^s (Concept 2)			Γ_3^s (Concept 3)		
	μ_1	μ_2	0σ	3σ	6σ	0σ	3σ	6σ	0σ	3σ	6σ
RI_1	1.984	4.635	**0.29**	0.38	0.41	**0.00**	0.00	0.00	**0.71**	0.62	0.59
RI_2	0.136	4.635	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
RI_3	1.984	0.979	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00
RI_4	1.376	3.103	0.20	0.36	0.40	0.00	0.00	0.00	0.80	0.64	0.60
RI_5	0.376	3.286	0.50	1.00	1.00	0.00	0.00	0.00	0.50	0.00	0.00

he or she would give up a small amount in the deflection objective to gain significantly in the volume objective.

For Case 1 we can specifically conclude that (i) Concept 1 is superior if low nodal displacement is desired, (ii) Concept 2 is inferior and can be eliminated, and (iii) Concept 3 is superior if low structural volume is desired. Importantly, when a compromise between displacement and volume is desired, the interactive exploration approach allows the designer to draw conclusions about each concept's strengths and weaknesses.

Case 2—Non-deterministic Approach 1: In this case, we assume that there is a known variation (standard deviation of 2×10^3 ksi) in the design parameter E. We consider the effect of uncertainty by formulating the optimization problem according to (21–23) where $\hat{\mu}_n$ is the mean of μ_n and is approximated as

$$\hat{\mu}_n = \mu_n|_{E=\hat{E}} \tag{32}$$

Under this approach, the expected s-Pareto frontier is shifted—in a worst case sense—through the design space by multiples of the standard deviation of μ, $(\kappa\sigma_\mu)$, as shown in Figure 9. Under this approach we do not minimize $\kappa\sigma_\mu$. The model fidelity uncertainty parameters used in this example are $\eta = [1.1 \quad 1.1]$ and $\beta = [0.2 \quad 0]$. It is worth noting that (32) assumes a small standard deviation of E.

The results listed under the 3σ and 6σ columns in Table 1 pertain to Case 2. For RI_5, the results indicate that considering the uncertainty significantly impacts the concept selection

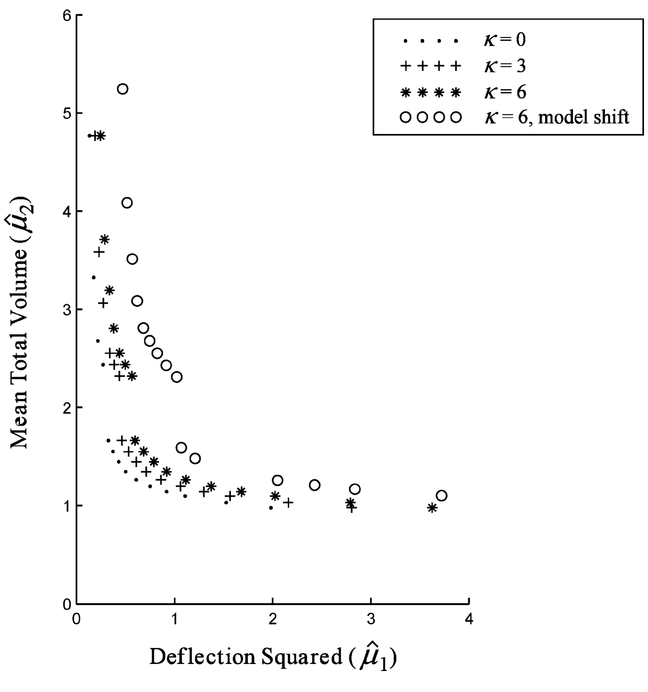

Figure 9. Example 1—Case 2: Shifted s-Pareto frontiers, smart filtered.

Table 2. Example 1—Case 3: Exploration results.

			Γ_1^s (Concept 1)			Γ_2^s (Concept 2)			Γ_3^s (Concept 3)		
	μ_1	μ_2	0σ	3σ	6σ	0σ	3σ	6σ	0σ	3σ	6σ
RI_1	1.984	4.768	0.29	0.38	0.38	0.00	0.00	0.00	0.71	0.62	0.62
RI_2	0.136	4.768	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
RI_3	1.984	0.979	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00
RI_4	1.376	3.103	0.27	0.30	0.30	0.00	0.00	0.00	0.73	0.70	0.70
RI_5	0.376	3.286	0.67	1.00	1.00	0.00	0.00	0.00	0.33	0.00	0.00

decision. Using the deterministic analysis, Concept 1 and 3 each contribute to half of the s-Pareto frontier (RI_5). In contrast, Concept 1 dominates when uncertainty is considered.

Case 3—Non-deterministic Approach 2: In this case, we again assume that there is a known variation (standard deviation of 2×10^3 ksi) in Young's Modulus, E. We consider the effect of uncertainty by formulating the optimization problem as according to (24–26) where $\hat{\mu}_n$ and σ_{μ_n} are respectively the mean and standard deviation of μ_n. The mean objective, $\hat{\mu}_n$, is evaluated as described in (32) and σ_{μ_n} is evaluated as

$$\sigma_{\mu_n} = \sqrt{\left(\frac{\partial \mu_n}{\partial E}\right)^2 \sigma_E^2} \tag{33}$$

Under this approach, the mean values of the objectives and the variation thereof are minimized—according to the desired level of reliability (i.e. κ value). As the designer chooses different multiples of the standard deviation, a different smart s-Pareto frontier is obtained. Various regions of interest were explored and the results are given in Table 2.

The data for RI_2 and RI_3 indicate that minimizing the variation in the design objectives comes at the cost of objective performance. For the 3σ columns of RI_2, it can be seen that no concept contributes to the s-Pareto frontier. This is because the s-Pareto frontier has moved out of this region of interest as a result of minimizing the design objective variation. Note that in this case we do not consider uncertainty associated with model fidelity, as we have already shown its implementation and effect in Case 2 above.

An important difference between Approach 1 (Case 2) and Approach 2 (Case 3) is that under Approach 1, we can choose to only evaluate the standard deviation of μ for entries in the smart s-Pareto set. Under Approach 2, however, we must evaluate the standard deviation of μ in order to obtain the s-Pareto set—which is later filtered to obtain the smart s-Pareto set. In this sense, Approach 1 offers a computational advantage.

7.3. Example 2: Tri-objective truss design

In this example we consider the same general cases of the previous; that is, a deterministic approach and two non-deterministic approaches. The same stochastic parameter is used in this example.

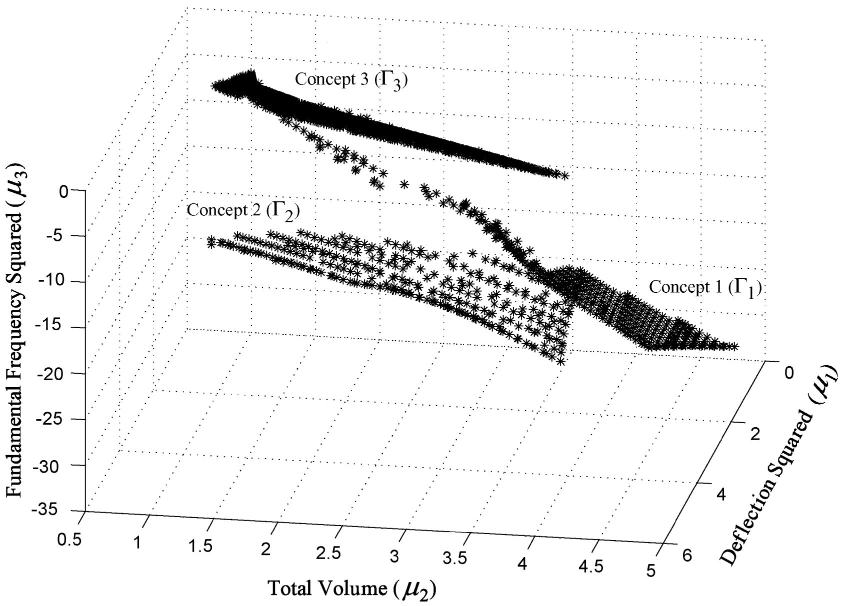

Figure 10. Example 2: Pareto surfaces for each concept.

Case 1—Deterministic approach: For this deterministic tri-objective case, the three objectives are given as

$$\mu(a, b, E) = [\mu_1(a, b, E) \quad \mu_2(a, b) \quad -\mu_3(a, b, E)]^T \tag{34}$$

where $\mu_1(a, b, E)$ is the square of the nodal displacement at node P, $\mu_2(a, b)$ is the total structural volume, and $\mu_3(a, b, E)$ is the fundamental frequency of the structure. A lumped mass is concentrated at node P, such that it applies a load W_2 to the undamped massless structure of stiffness K.

Pareto surfaces (frontiers) for each concept are obtained using the Normal Constraint method (Messac et al., 2003) and are shown in Figure 10. As can be seen from the figure, it is extremely difficult to visualize the design space—even for this simple tri-objective case. It is difficult to see which concepts occupy which regions of the design space and particularly difficult is the determination of how the concepts compare to one another. Visualization is therefore a critical element of Pareto frontier based concept selection approach. We visualize the n-dimensional design space by using the interactive exploration method described earlier in the paper. The graphical user interface shown in Figure 5 shows the exploration interface, for this example.

The s-Pareto frontier was obtained using the Normal Constraint method (Messac et al., 2003) and the developments presented in Mattson and Messac (2003a), and the set was reduced using the smart filtering approach with $\Delta_b = 10$ and $\Delta_s = 0.008$. The exploration results are obtained and tabulated in Table 3 under the 0σ column. A discussion of the results is given below in Case 2.

Table 3. Example 2—Cases 1 and 2: Exploration results.

				Γ_1^s (Concept 1)			Γ_2^s (Concept 2)			Γ_3^s (Concept 3)		
	μ_1	μ_2	μ_3	0σ	3σ	6σ	0σ	3σ	6σ	0σ	3σ	6σ
RI_1	5.195	4.786	1.645	0.30	0.31	0.32	0.40	0.41	0.39	0.30	0.28	0.29
RI_2	0.173	4.786	1.645	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
RI_3	5.195	1.007	1.645	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00
RI_4	5.195	4.786	32.99	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
RI_5	2.231	3.550	3.713	0.23	0.24	0.25	0.36	0.34	0.32	0.41	0.42	0.43
RI_6	1.390	3.320	15.62	0.56	0.87	1.00	0.44	0.13	0.00	0.00	0.00	0.00
RI_7	0.895	3.705	11.54	0.43	0.86	1.00	0.24	0.00	0.00	0.33	0.14	0.00

The visualization/exploration approach is used to understand which concepts minimize what objectives and how the concepts occupy the design space in general. The approach is not designed to rely on tables such as those shown in this example; rather the interactive nature of the visualization/exploration approach lends itself to acquiring information about the design space in a quick and intuitive way.

Cases 2 and 3—Non-deterministic approaches: The two approaches presented in this paper for considering uncertainty are used in this example. Results from Approach 1 are given in Table 3, and those from Approach 2 are given in Table 4. From the RI_1 row in Table 3, we can see that Concepts 1, 2, and 3 contribute to the s-Pareto frontier relatively equally (0.30, 0.40, and 0.30). When uncertainties are considered, the concepts are still well-represented in the design space. From this, we can initially conclude that none of the concepts should be eliminated and that other regions of interest should be explored to determine which concepts minimize what objectives, and to obtain sufficient information to select a final design. The other regions of interest in Table 3 show respective exploration results. Table 4 shows the results from Approach 2 where the variation of the design objectives is also minimized. Note that for some regions of interest, the results are significantly different from the results given in Table 3 (Approach 1).

Table 4. Example 2—Case 3: Exploration results.

				Γ_1^s (Concept 1)			Γ_2^s (Concept 2)			Γ_3^s (Concept 3)		
	μ_1	μ_2	μ_3	0σ	3σ	6σ	0σ	3σ	6σ	0σ	3σ	6σ
RI_1	4.859	4.786	1.877	0.34	0.23	0.38	0.33	0.52	0.40	0.33	0.25	0.27
RI_2	0.173	4.786	1.877	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
RI_3	4.859	1.007	1.877	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00
RI_4	4.859	4.786	32.99	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
RI_5	2.231	3.550	3.713	0.27	0.16	0.19	0.28	0.46	0.32	0.45	0.38	0.49
RI_6	1.390	3.320	15.62	0.61	0.71	1.00	0.39	0.29	0.00	0.00	0.00	0.00
RI_7	0.895	3.705	11.54	0.75	0.86	1.00	0.03	0.00	0.00	0.22	0.14	0.00

The two examples presented in this section show that the s-Pareto frontier-based concept selection framework can play a meaningful role in the concept selection process. Specifically, the s-Pareto frontier allows the designer to consider a range of behaviors of the concepts objectives when evaluating each concept—which is a marked departure from traditional (non-optimization based) methods.

8. Concluding remarks

The important task of concept selection in engineering design has been the subject of this paper. We have presented four important developments that support a new phase in the evolution of the s-Pareto frontier-based concept selection framework. Specifically, the four developments are (i) the notion of a smart s-Pareto frontier, (ii) the quantification of concept goodness with smart and non-smart s-Pareto frontiers, (iii) an interactive design space exploration approach, and (iv) two approaches for managing the effect of uncertainty in concept selection. Under the new developments, the s-Pareto frontier-based concept selection framework is used to quantitatively assess the goodness of candidate concepts—for problems of n-dimension. The quantitative measure is integral to this paper's interactive visualization/exploration approach, which aids the designer in characterizing the design tradeoffs. Finally, the effects of uncertainty can also be considered under the new developments.

Acknowledgments

Support of the National Science Foundation through Grant DMI-0196243 is gratefully acknowledged.

References

N. M. Alexandrov, R. M. Lewis, C. R. Gumbert, L. L. Green, and P. A. Newman, "Optimization with variable-fidelity models applied to wing design," in *Proceedings of the 38th AIAA Aerospace Sciences Meeting and Exhibit*, AIAA Paper 2000–0841, 2000a.

N. M. Alexandrov, E. J. Nielsen, R. M. Lewis, and W. K. Anderson, "First-order model management with variable-fidelity physics applied to multi-element airfoil optimization," in *Proceedings of the 8th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization*, AIAA Paper 2000–4886, 2000b.

R. J. Balling "Pareto sets in decision-based design," *Engineering Valuation and Cost Analysis* vol. 3, pp. 189–198, 2000.

A. Belegundu and T. Chandrupatla, *Optimization Concepts and Applications in Engineering*, Prentice Hall: New Jersey, 1999.

V. Chankong and Y. Y. Haimes, *Multiobjective Decision Making: Theory and Methodology*, Elsevier Science Publishing Co., Inc.: New York, 1983.

A. Charnes and W. W. Cooper, *Management Models and Industrial Applications of Linear Programming*, John Wiley & Sons, Inc.: New York, vol. 1, 1961.

W. Chen, A. Sahai, A. Messac, and G. J. Sundararaj, "Exploration of the effectiveness of physical programming in robust design," *ASME Journal of Mechanical Design* vol. 122, 2000.

W. Chen, M. M. Wiecek, and J. Zhang, "Quality Utility—A Compromise Programming Approach to Robust Design," *ASME Journal of Mechanical Design* vol. 121, 1999.

K.-S. Chin and T. Wong, "Knowledge-based evaluation for the conceptual design development of injection molding parts," *Engineering Applications in Artificial Intelligence* vol. 9, no. 4, pp. 359–376, 1996.

C. A. Cornell, "A probability-based structural code," *Journal of American Concrete Institute* vol. 66, no. 12, pp. 974–985, 1969.

W. A. Crossley, E. T. Martin, and D. W. Fanjoy, "Conceptual aircraft design environment: Case study evaluation of computing architecture technologies," 1st AIAA Aircraft, Technology, Integration, and Operations Forum, Paper Number AIAA 2001–5247, 2001.

I. Das, "A preference ordering among various Pareto optimal alternatives," *Structural Optimization* vol. 18, no. 1, pp. 30–35, 1999.

I. Das and J. Dennis, "A closer look at drawbacks of minimizing weighted sums of objective for Pareto set generation in multicriteria optimization problems," *Structural Optimization* vol. 14, pp. 63–69, 1997.

I. Das and J. Dennis, "Normal-boundary intersection: A new method for generating Pareto optimal points in multicriteria optimization problems," *SIAM Journal on Optimization* vol. 8, no. 3, pp. 631–657, 1998.

K. Deb, *Multi-Objective Optimization Using Evolutionary Algorithms*, Chapt. 1, John Wiley & Sons, 1st edition, pp. 1–12, 2001.

G. E. Dieter, *Engineering Design: A Materials and Processing Approach*, 2nd edition, McGraw-Hill, Inc., 1991.

X. Du and W. Chen, "Methodology for managing the effect of uncertainty in simulation-based design," *AIAA Journal* vol. 38, no. 8, 2000.

H. Eschenauer, J. Koski, and A. Osyczka, *Multicriteria Design Optimization*, Springer-Verlag, Inc.: New York, 1990.

S. Gass and T. Saaty, "The computational algorithm for the parametric objective function," *Naval Research Logistics Quarterly* vol. 2, pp. 39–45, 1955.

A. Goel, "Visualization in problem solving environments," Master's thesis, Virginia Polytechnic Institute and State University, Blacksburg, VA, 1999.

B. S. Homan and A. C. Thornton, "Precision machine design assistant: A constraint-based tool for the design and evaluation of precision machine tool concepts," *Artificial Intelligence for Engineering Design, Analysis and Manufacturing: AIEDAM* vol. 12, no. 5, pp. 419–429, 1998.

C. L. Hwang and A. S. M. Masud, *Multiple Objective Decision Making-Methods and Applications: A State of the Art Survey*, Springer-Verlag, Inc.: Berlin, 1979.

K. Ishii, "Life-cycle engineering design," *Transactions of the ASME, Special 50th Anniversary Design Issue* vol. 117, pp. 42–47, 1995.

E. M. Kasprzak and K. E. Lewis, "An approach to facilitate decision tradeoffs in Pareto solution sets," *Engineering Valuation and Cost Analysis* vol. 3, pp. 173–187, 2000.

P. N. Koch, "Probabilistic design: Optimizing for six sigma quality," *AIAA 43rd AIAA/ASME/ASCE/AHS Structures, Structural Dynamics, and Materials Conference*, Paper Number AIAA–2002–1471, 2002.

J. Koski, "Defectiveness of weighting methods in multicriterion optimization of structures," *Communications in Applied Numerical Methods* vol. 1, pp. 333–337, 1985.

Y. Li, G. M. Fadel, and M. M. Wiecek, "Approximating Pareto curves using the hyper-ellipse," in *Proceedings of the 7th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization*, AIAA Paper No. AIAA-98-4961, 1998.

E. B. Magrab, *Integrated Product and Process Design and Development: The Product Realization Process*, 1st edition, Chapt. 6, CRC Press: New York, pp. 129–136, 1997.

C. A. Mattson and A. Messac, "Concept selection in n-dimension using s-Pareto frontiers and visualization," *AIAA Multidiscipinary Analysis and Optimization Symposia*, Paper Number AIAA–2002–5418, 2002.

C. A. Mattson and A. Messac, "Concept selection using s-Pareto frontiers," *AIAA Journal* vol. 41, no. 6, pp. 1190–1204, 2003a.

C. A. Mattson and A. Messac, "Handling equality constraints in robust design optimization," *AIAA 44rd AIAA/ASME/ASCE/AHS Structures, Structural Dynamics, and Materials Conference*, Paper Number AIAA–2002–1780, 2003b.

C. A. Mattson, A. A. Mullur, and A. Messac, “Minimal representation of multiobjective design space using a smart Pareto filter,” *AIAA Multidiscipinary Analysis and Optimization Symposia*, Paper Number AIAA–2002–5458, 2002.

R. E. Melchers, *Structural Reliability: Analysis and Prediction*, Ellis Horwood Series in Civil Engineering. John Wiley & Sons: New York, 1999.

A. Messac, “Physical programming: Effective optimization for computational design,” *AIAA Journal* vol. 34, no. 1, pp. 149–158, 1996.

A. Messac and X. Chen, “Visualizing the optimization process in real-time using physical programming,” *Engineering Optimization* vol. 32, pp. 721–747, 2000.

A. Messac and A. Ismail-Yahaya, “Required relationship between objective function and Pareto frontier orders: Practical implications,” *AIAA Journal* vol. 39, no. 11, pp. 2168–2174, 2001.

A. Messac and A. Ismail-Yahaya, “Multiobjective robust design using physical programming,” *Structural and Multidisciplinary Optimization* vol. 23, no. 5, pp. 357–371, 2002.

A. Messac, A. Ismail-Yahaya, and C. A. Mattson, “The normalized normal constraint method for generating the Pareto frontier,” *Structural and Multidisciplinary Optimization* vol. 25, no. 2, pp. 86–98, 2003.

A. Messac and C. A. Mattson, “Generating well-distributed sets of Pareto points for engineering design using physical programming,” *Optimization and Engineering* vol. 3, pp. 431–450, 2002, Kluwer Publisher.

K. M. Miettinen, *Nonlinear Multiobjective Optimization*, International Series in Operations Research & Management Science. Kluwer Academic Publishers, 1999.

A. A. Mullur, C. A. Mattson, and A. Messac, “Pitfalls of the typical construction of decision matrices,” *AIAA 41th Aerospace Sciences Meeting and Exhibit*, Paper Number AIAA–2003–0466, 2003.

K. N. Otto, “Measurement analysis of product design methods,” *Research in Engineering Design* vol. 7, no. 2, pp. 86–101, 1995.

G. Pahl and W. Beitz, *Engineering Design: A Systematic Approach*, 2nd edition, Chapt. 4, Springer-Verlag: London, pp. 99–117, 1996.

N. Palli, S. Azarm, P. McCluskey, and R. Sundararajan, “An interactive multistage ϵ-inequality constraint method for multiple objectives decision making,” *ASME Journal of Mechanical Design* vol. 120, pp. 678–686, 1998.

V. Pareto, *Cour d’Economie Politique*, Geneve: Librarie Droz. the 1st edition in 1896, 1964.

A. Parkinson, C. Sorensen, and N. Pourhassan, “A general approach for robust optimal design,” *ASME Journal of Mechanical Design* vol. 115, pp. 74–80, 1993.

S. Pugh, *Creating Innovative Products Using Total Design: The Living Legacy of Stuart Pugh*, 1st edition, Chapt. 14 and 15, Reading, MA: Addison-Wesley, with contributions by Don Clausing and Ron Andrade, pp. 167–182, 1996.

S. S. Rao, *Reliability-Based Design*, 1st edition, Chapt. 1, McGraw-Hill, Inc., pp. 1–24, 1992.

R. E. Steuer, “An interactive weighted tchebycheff procedure for multiobjective programming,” *Mathematical Programming* vol. 26, no. 3, pp. 326–344, 1983.

R. E. Steuer, *Multiple Criteria Optimization, Theory Computations and Applications*, John Wiley & Sons, Inc.: New York, 1986.

J. Su and J. E. Renaud, “Automatic differentiation in robust optimization,” *AIAA Journal* vol. 35, no. 6, pp. 1072–1079, 1997.

N. P. Suh, *The Principles of Design*, Oxford University Press: New York, 1990.

G. Taguchi, *Taguchi on Robust Technology Development: Bringing Quality Engineering Upstream*, ASME Press: New York, 1993.

R. V. Tappeta, J. E. Renaud, A. Messac, and G. J. Sundararaj, “Interactive physical programming: Tradeoff analysis and decision making in multicriteria optimization,” *AIAA Journal* vol. 38, no. 5, pp. 917–926, 2000.

P. B. Thanedar and S. Kodiyalam, “Structural optimization using probabilistic constraints,” *AIAA/ASME/ASCE/AHS Structures, Structural Dynamics, and Materials Conference*, Paper Number AIAA-91-0922-CP, 1991.

D. G. Ullman, *The Mechanical Design Process*, Chapt. 9, McGraw-Hill, Inc., pp. 168–184, 1992.

K. T. Ulrich and S. D. Eppinger, *Product Design and Development*, 2nd edition, Chapt. 7, McGraw-Hill, Inc.: Boston, pp. 138–159, 2000.

D. A. Van Veldhuizen, “Multiobjective evolutionary algorithm: Classifications, analysis, and new innovations,” Ph.D. thesis, Air Force Institute of Technology, 1998.

J. Wang, "Ranking engineering design concepts using a fuzzy outranking preference model," *Fuzzy Sets and Systems* vol. 119, pp. 161–170, 2001.

A. P. Wierzbicki, "The use of reference objectives in multiobjective optimization," *Multiple Criteria Decision Making and Applications*, 1980, pp. 468–486, edited by Fandel, G., Lecture Notes in Economics and Mathematical Systems 177, Springer-Verlag, Inc.: Berlin.

E. Winer and C. L. Bloebaum, "Design visualization by graph morphing for multidisciplinary design optimization," *First International Conference on Engineering Design and Automation*, Bangkok, Thailand, 1997.

J. Wu and S. Azarm, "Metrics for quality assessment of a multiobjective design optimization solution set," *ASME Journal of Mechanical Design* vol. 123, pp. 18–25, 2001.

E. Zitzler and L. Thiele, "Multiobjective optimization using evolutionary algorithms—A comparative case study," *Parallel Problem Solving from Nature*, Springer Verlag: Amsterdam, The Netherlands, pp. 292–301, 1998.