Brad Paden*
Department of Mechanical and Environmental Engineering and the NSF Center for Robotic Systems in Microelectronics
University of California, Santa Barbara
Santa Barbara, California 93106

Shankar Sastry
Department of Electrical Engineering and Computer Science
University of California, Berkeley
Berkeley, California 94720

Optimal Kinematic Design of 6R Manipulators

Abstract

A fundamental theorem for the kinematic design of robot manipulators is formulated and proved. Roughly speaking, the theorem states that a manipulator having six revolute joints is optimal if and only if the manipulator or its kinematic inverse is an elbow manipulator. By "optimal" we mean a manipulator that has the properties of (i) maximal work-volume subject to a constraint on its length and (ii) well-connected workspace—that is, the ability to reach all positions in its workspace in each configuration. The notion of work-volume we use is that derived from the translation-invariant volume form on the group of rigid motions. This notion of volume is intermediate between those of "reachable" and "dextrous" workspace and appears to be more natural in that it leads to simple analytical results.

1. Introduction

This paper develops a tight relationship between the kinematic performance and design of manipulators having six revolute joints. In order to state precisely this relationship, we develop the notions of length of a manipulator, work-volume, maximal work-volume, well-connected workspace, and kinematic inverse. With these notions defined, a relationship between the design and performance of a manipulator having six revolute joints (6R) is summarized in the theorem: *A 6R manipulator has maximal work-volume and a well-connected workspace if and only if the manipulator or its kinematic inverse is an elbow manipulator.* We begin our discussion of previous work with a review of various notions of manipulator workspaces.

A robot manipulator defines a map from its configuration space J (called *jointspace*) to the configuration space of its hand, which is typically the group G of proper rigid motions in $\mathbb{R}^3$ (our nomenclature is listed after the appendix). For example, the manipulator sketched in Fig. 1 defines a map from $\mathbb{R}^1 \times T^5$ to G; for each set of joints displacements (a linear variable and five angular variables) we can specify the configuration of the hand. We write $f: J \rightarrow G$ for the forward kinematic map of a manipulator. For a given manipulator, this map depends on which point in jointspace we call zero, a detail which we resolve later. The image of J under f, $f(J) \subset G$, is called the *workspace* of the manipulator.

The thrust of the recent studies of manipulator workspaces has been on the generation of projections of $f(J)$ onto $\mathbb{R}^3$ and characterizing these projections. There are two important projections of $f(J)$. They are called the *reachable workspace* and *dextrous workspace* by Kumar and Waldron (1981). These two projections are obtained as follows. Each $g \in G$ can be written as a rotation about a point P *followed* by a translation. That is, there is a natural diffeomorphism, ϕ_P, of G onto $G_P \times \mathbb{R}^3$, where G_P is the subgroup of G whose elements leave P fixed. Let $\pi_{\mathbb{R}^3}$ be the natural

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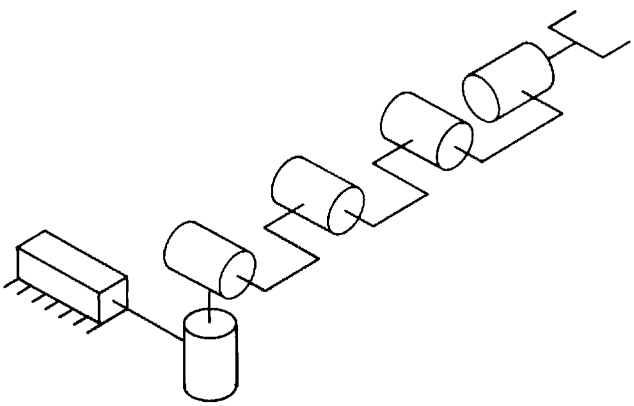
Fig. 1. Schematic diagram of a P5R manipulator.



projection of $G_P \times \mathbb{R}^3$ onto its $\mathbb{R}^3$ component. In terms of $\pi_{\mathbb{R}^3}$ and ϕ_P,

$$W_R(P) \triangleq \pi_{\mathbb{R}^3} \circ \phi_P \circ f(J) \quad (1)$$

is called the *reachable workspace* (of P), and

$$W_D(P) \triangleq [\pi_{\mathbb{R}^3} \circ \phi_P([f(J)]^C)]^C \quad (2)$$

is called the *dextrous workspace* (of P). The point P represents some significant point attached to the hand of the manipulator. Often P is chosen to be a point between the fingers of a manipulator hand or the point of intersection of the wrist axes. In words, the reachable workspace is the set of points that can be reached by P, and the dextrous workspace is the set of points that can be reached by P with arbitrary orientation of the hand.

Bounds on $W_R(P)$ were obtained numerically by Kumar and Waldron (1981). They observed that for manipulators with only revolute joints the boundary of $W_R(P)$ consists of critical values[1] of $\pi_{\mathbb{R}^3} \circ \phi_P \circ f$. By generating a plot of these critical values, Kumar and Waldron were able to obtain graphical bounds on $W_R(P)$. Additional work on the shape of $W_D(P)$ and $W_R(P)$ is contained in Gupta and Roth (1982), where they classify holes and voids in the workspace projections and give conditions for the existence of holes and voids. The basic difference between a hole and void can be demonstrated with a bakery doughnut. If the surface of a doughnut represents the reachable workspace of a manipulator, then the doughnut hole is a hole in this workspace and the doughnut dough is a void. Gupta and Roth also discuss the behavior of $W_D(P)$ and $W_R(P)$ as functions of P. For a manipulator having its final three axes intersecting at a wrist point P_w, the reachable workspace increases and the dextrous workspace decreases with increasing $\|P - P_w\|$. Further analytical work by Freudenstein and Primrose (1984) in this area develop precise relationships between kinematic parameters and workspace parameters. This work was followed by an application to workspace optimization for three-joint manipulators (Lin and Freudenstein 1986).

Numerical studies of manipulator workspaces have been carried out by Yang and Lee (1983), Tsai and Soni (1985), and Hansen, Gupta, and Kazerounian (1983). These studies rely on numerical methods to generate projections of the manipulator workspace. The advantage of these schemes over analytical approaches is that mechanical constraints can be easily included; a disadvantage is that general design principles are hard to obtain.

Work that is closely related to ours is that of Vijaykumar, Tsai, and Waldron (1986). These researchers develop design criteria for optimal manipulator performance, which is measured primarily in terms of dextrous workspace. Vijaykumar, Tsai, and Waldron argue as we do that the elbow manipulator is an optimal design. Our work extends these results by eliminating the a priori assumptions that the manipulator has a positioning component and an orientation component and that the Hartenberg-Denavit parameters satisfy certain constraints. We also extend these results to show that the kinematic inverse of an elbow manipulator is an optimal design as well. For a clear comparison of our theorem (stated above and rigorously developed in Section 4) with the recent theory developed by Vijaykumar, Tsai, and Waldron, we state their results in the form of a theorem: *If a 6R manipulator M has maximal dextrous work-volume and satisfies the assumptions that* (1) *the fourth, fifth, and sixth joint axes intersect,* (2) *the Hartenberg-Denavit twist*

1. Critical points of a map f are those points in its domain where Tf fails to be surjective. Critical values are values in the range of f that are the images of critical points. Regular points are noncritical points, and regular values are noncritical values. See Abraham, Marsden, and Ratiu (1983).

angles are 0 *or* 90° *for all links, and* (3) *either the link length* a_i *or the joint offset* r_i *is zero for each link i, then M is an elbow manipulator.*

The approach taken in this paper to derive relationships between design and performance of 6R manipulators differs in several ways from those mentioned above: (1) Rather than projecting $f(J)$ onto $\mathbb{R}^3$, we consider the volume[2] of $f(J)$ as a subset of the group of rigid motions as a performance measure of a kinematic design. (2) We find *all* designs that optimize our performance measures. Our techniques differ as well. In our proof we find it essential to focus on the intrinsic geometric objects that affect the kinematic performance of a 6R manipulator. Thus, we avoid, as much as possible, particular parameterizations of a manipulator's geometry. For example, we do not use Hartenberg-Denavit parameters. Although they provide a good parameterization of a manipulator's geometry, they are by no means canonical. The important geometric objects are the joint axes, especially the positions of the joints' axes in the "zero configuration" (see Gupta 1986). We make the significance of the joint axes as clear as possible by expressing the forward kinematic map in terms of the elements in the Lie algebra of the group of rigid motions that generate the rotations about the joint axes. These generators are commonly known as infinitesimal (or differential) twists about a screw. In addition to the benefit of using a geometric notation, we are inclined to take advantage of the Lie group structure of the group of rigid motions; for example, we can express rotations as exponentials of twists, and use the unique translation-invariant volume form on the group of rigid motions to define the work-volume of a robot.

The format of this paper is as follows: Section 2 describes the notation we use for elements in G and their expression as exponentials of elements of the Lie algebra of G. Section 3 contains the mathematical framework we use for analyzing 6R manipulators. Here we define the notions of length, work-volume, maximal work-volume, and well-connected workspace. The definition of the kinematic inverse of a manipulator is also formally stated. Section 4 centers on the design theorem and its proof. Our conclusions are made in Section 5.

2. This notion of volume is directly related to Roth's service coefficient (Roth 1976) for manipulators whose last axis is revolute.

2. Proper Rigid Motions

The group of proper rigid motions on $\mathbb{R}^3$ plays a central role in the study of rigid-link robot manipulators. It is the configuration space of the links of such manipulators, the most important of which is the final link or hand. This section describes the representation of rigid motions as exponentials of differential twists about a screw (Brockett 1983; Paden 1985). This is a special case of the more general notion of an exponential map from a Lie algebra into a Lie group (Boothby 1975). We review the representation of proper rigid motions as a subgroup (homogeneous transformations) of the group of invertible matrices. First, the notation used for points and vectors is introduced.

Identify points in physical three-space with points in $\mathbb{R}^3$ via an orthonormal right-handed coordinate system as usual. Thus, a point P can be represented by an element of $\mathbb{R}^3$ written

$$P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{pmatrix}, \tag{3}$$

where the 1 in the last row allows for the matrix representation of rigid motions by homogeneous transformations. If O is the origin of the coordinate system, then

$$O = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}.$$

Identifying a point with its coordinates in Eq. (3) should not cause any confusion, since only one coordinate system, fixed relative to the base of the manipulator, will be used in developing manipulator kinematics. The reader will note that it is not necessary to specify the coordinate system. This follows from the fact that objects we consider are intrinsic to three-space, and anything said about them is independent of the choice of coordinate system.

Vectors have the form

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{pmatrix}. \tag{4}$$

For points P and Q, the displacement from Q to P, $P - Q$, is a vector since the fourth component "1's" in P and Q cancel. Also, if v is a vector, $Q + v$ is the point Q translated by v. Both points and vectors will be considered as elements in $\mathbb{R}^3$; their physical interpretations are completely different, however. The 0 and 1 in the fourth component of vectors and points, respectively, will remind us of this and enforce a few rules of syntax. For example, it is meaningless geometrically to add two points.

The vector cross and dot products are computed in a natural way. If $v = (v_1, v_2, v_3, 0)^{\mathrm{T}}$ and $u = (u_1, u_2, u_3, 0)^{\mathrm{T}}$, then

$$u \cdot v \triangleq u^{\mathrm{T}} v \quad \text{and} \quad u \times v \triangleq \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \\ 0 \end{pmatrix}. \tag{5}$$

It is well known that when points are represented as in (3), rigid motions can be identified with a 4×4 homogeneous transformation of the form

$$g = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}, \tag{6}$$

where $A \in SO(3)$, the group of 3×3 orthogonal matrices with determinant 1, or simply the rotation group, $b \in \mathbb{R}^3$, and 0 is the zero row-vector in $\mathbb{R}^3$. Let G be the group (under matrix multiplication) of all 4×4 matrices of the form (6). G is called the group of proper rigid motions in $\mathbb{R}^3$. By Chasle's theorem, any element in G can be decomposed into a rotation about an axis and a translation along the same axis; that is, any $g \in G$ can be thought of as a helical or "screw" motion, which is why an element of G is called a *twist about a screw*.

If we identify the identity element $I \in G$ with some nominal configuration of a rigid body, then any other configuration of the rigid body is identified with the rigid motion that translates the body from the identity configuration to the given configuration. With this identification, a curve in G defines a trajectory of a rigid body such as the moving hand of a manipulator.

If $g(t)$ is a C^1 curve in G representing the trajectory of a rigid body, then the generalized velocity of the rigid body is given by $dg(t)/dt \in T_{g(t)}G$, where $T_h G$ is the tangent space of G at h (Abraham, Marsden, and Ratiu, 1983). The trajectory $g(t)$ can be written

$$g(t) = \left[\begin{array}{c:c} A(t) & P(t) \\ \hdashline 0 & \end{array}\right], \tag{7}$$

where $P(t)$ is a C^1 curve of points and $A(t)$ is a C^1 curve in $SO(3)$ that necessarily satisfies

$$A(t)A(t)^{\mathrm{T}} = I \quad \forall t \in \mathbb{R}. \tag{8}$$

Evaluating the derivatives of (7) and (8) with respect to t at $t = 0$ for an arbitrary curve $g(t)$ through

$$h = \left[\begin{array}{c:c} A & P \\ \hdashline 0 & \end{array}\right] \qquad (\text{at } t = 0)$$

yields

$$T_h G = \left\{ \left[\begin{array}{c:c} B & v \\ \hdashline 0 & \end{array}\right] \middle| \, v \text{ is a vector}, \; BA^{\mathrm{T}} + AB^{\mathrm{T}} = 0 \right\}. \tag{9}$$

There is an isomorphism of $T_g G$ onto $T_I G$ given by right multiplication by g^{-1}. That is, $\eta g^{-1} \in T_I G \ \forall \, \eta \in T_g G$. Thus, the velocity of a rigid body can be identified with an element of $T_I G$. The advantage of doing this is that the element of $T_I G$ that represents the velocity of a rigid body is independent of the choice of the identity position of the rigid body.

Let $\omega = (\omega_1, \omega_2, \omega_3, 0)^{\mathrm{T}}$ and define the cross-product operator constructed from ω by

$$S(\omega) \triangleq \begin{bmatrix} 0 & -\omega_3 & \omega_2 & 0 \\ \omega_3 & 0 & -\omega_1 & 0 \\ -\omega_2 & \omega_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (10)$$

(Observe that $S(\omega)b = \omega \times b \; \forall$ vectors ω and b.)
With this, $\xi \in T_I G$ can be expressed as

$$\xi = S(\omega) + \left[\begin{array}{c:c} 0 & \\ \hdashline 0 & v \end{array}\right] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 & v_1 \\ \omega_3 & 0 & -\omega_1 & v_2 \\ -\omega_2 & \omega_1 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (11)$$

for some ω and v. This follows from Eq. (9) with A set equal to the 3×3 identity matrix.

The vector space $T_I G$ is the Lie algebra of G and has the same dimension (=six) as the manifold G. Elements in $T_I G$ are called (differential or infinitesimal) twists about a screw or simply *twists*. (Ball 1900; Roth 1984). This definition is motivated by the differential version of Chasle's theorem: the linearized motion of a rigid body is the linearization of some helical or "screw motion." The position and the pitch of the twist changes with time, of course. It is common practice to identify $T_I G$ with $\mathbb{R}^6$. The six components in $\mathbb{R}^6$ are called twist coordinates and are defined by the linear invertible map $tc: T_I G \to \mathbb{R}^6$.

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 & v_1 \\ \omega_3 & 0 & -\omega_1 & v_2 \\ -\omega_2 & \omega_1 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{tc} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}. \quad (12)$$

For notational convenience a map that extracts the angular velocity and the linear velocity *vectors* from a twist is defined similarly.

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 & v_1 \\ \omega_3 & 0 & -\omega_1 & v_2 \\ -\omega_2 & \omega_1 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\underline{tc}} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ 0 \\ v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix} = \begin{bmatrix} \omega \\ v \end{bmatrix}. \quad (13)$$

(Here is a case where we pay a notational price for using the four component representation of points and vectors.)

A twist is given the attributes of a pitch, an amplitude, and a screw axis, where the notions of pitch and axis derive their meaning from an ordinary machine screw. They are defined as follows. Let

$$\xi = \underline{tc}^{-1} \left[\frac{\omega}{v}\right].$$

Then the *pitch* of ξ is defined by

$$\text{pitch of } \xi \triangleq \begin{cases} \omega^{\mathrm{T}} v / \|\omega\|^2 & \text{if } \omega \neq 0, \\ \infty & \text{otherwise.} \end{cases} \quad (14)$$

The *amplitude*[3] of ξ is defined by

$$\text{amplitude of } \xi \triangleq \begin{cases} \|\omega\| & \text{if } \omega \neq 0, \\ \|v\| & \text{otherwise.} \end{cases} \quad (15)$$

The amplitude of ξ is not a norm on $T_I G$ but is useful in describing zero and infinite-pitch differential twists particularly. We define a *screw axis*[4] of $\xi \in T_I G$ to be a directed line l_u (line l with nonzero direction vector u) having the properties that (i) $u \times \xi Q = 0$ for all Q on l_u and (ii) u has the same direction[5] as ω (v resp.) if $\omega \neq 0$ ($\omega = 0$ resp.). It is well known that (1) every

3. Our definition of the amplitude of a differential twist about a screw is unconventional in that we define an amplitude of an infinite-pitch differential twist about a screw.

4. Some authors define a screw axis to be an undirected line and then by abuse of notation draw screw axes with a direction. Our definition includes this directedness. This is useful, since we like to specify the sense of positive rotation.

5. The zero vector has arbitrary direction.

twist $\xi = \underline{tc}^{-1}[{}^{\omega}_{v}]$ has an axis (if $\omega \neq 0$, then the axis is unique), and (2) for each axis, pitch $h \in (-\infty, \infty]$, and amplitude $M \in (-\infty, \infty)$ there exists a unique twist having these attributes. A *screw* is a pair consisting of a line (called the screw axis[4]) together with an element of $(-\infty, \infty]$ (called the pitch). Thus, there is nearly a one-to-one relationship between twists and screws, and this is why screws are used to graphically represent twists.

The following proposition allows us to relate the rigid motion of an element of $T_I G$ and the change in its axis.

Proposition 1 (Rigid motion of a differential twist about a screw) Let $\xi \in T_I G$ with pitch h, amplitude M, and axis l_u. If $g \in G$, then $g\xi g^{-1} \in T_I G$ with pitch h, amplitude M, and axis $(gl)_{gu}$.

Proof: Let $\xi = \underline{tc}^{-1}\begin{bmatrix}\omega \\ v\end{bmatrix}$. Then verify that

$$g\xi g^{-1} = \underline{tc}^{-1}\begin{bmatrix} g\omega \\ gv + gS(\omega)g^{-1}O \end{bmatrix}.$$

It follows that $g\xi g^{-1}$ has the same amplitude and pitch as ξ. Now $u \times \xi Q = 0\ \forall Q \in l_u \Rightarrow gu \times g\xi g^{-1} R = 0\ \forall R \in (gl)_{gu}$. Finally, u has direction ω (v resp.) if $\omega \neq 0$ ($\omega = 0$ resp.) implies gu has direction $g\omega$ (gv resp.) if $g\omega \neq 0$ ($g\omega = 0$ resp.).■

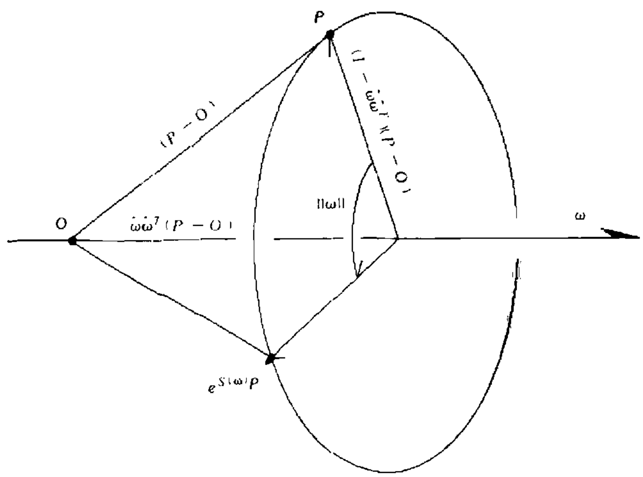
In the next section we will see that exponentials of differential twists are very useful in the study of manipulator kinematics. Elements of $T_I G$ are square matrices, so their exponential is well defined. There is also an important geometric interpretation given by the following propositions and discussion. The first proposition is well known.

Proposition 2 Let ω be a vector and define $\hat{\omega} \triangleq \omega / \|\omega\|$ when $\omega \neq 0$. Then

$$e^{S(\omega)} = \begin{cases} \hat{\omega}\hat{\omega}^{\mathrm{T}} + \sin(\|\omega\|)S(\hat{\omega}) + \cos(\|\omega\|)(I - \hat{\omega}\hat{\omega}^{\mathrm{T}}) & \text{if } \omega \neq 0, \\ I & \text{if } \omega = 0. \end{cases} \quad (16)$$

Figure 2 gives the geometric interpretation of $e^{S(\omega)}$ as a rotation by $\|\omega\|$ about the axis through the origin

Fig. 2. Interpretation of $e^{S(\omega)}$.



O having direction ω. The following proposition tells us how to compute the exponential of a general twist.

Proposition 3 Let $\xi = \underline{tc}^{-1}\begin{bmatrix}\omega \\ v\end{bmatrix}$, Q be a point on an axis of ξ, and P be an arbitrary point. Then

$$e^{\xi}P = \begin{cases} Q + e^{S(\omega)}(P - Q) + \dfrac{\omega\omega^{\mathrm{T}}}{\|\omega\|^2} v & \text{if } \omega \neq 0, \\ P + v & \text{if } \omega = 0. \end{cases} \quad (17)$$

Proof: This follows from the preceding proposition (see Paden 1985 for a complete proof).■

Figure 3 gives the geometric interpretation of e^{ξ} as a rotation about the axis of ξ by $\|\omega\|$ followed by a translation along the axis by the pitch times ω. Note that when the pitch of ξ is zero, e^{ξ} is a pure rotation about the axis of ξ by $\|\omega\|$. When the pitch of ξ is infinity, e^{ξ} is a pure translation by v.

3. Mathematical Framework

A formal theory of manipulator kinematics requires, first of all, a mathematical representation of manipulators. The amount of information contained in the representation depends on what we are trying to ac-

Fig. 3. Interpretation of e^{ξ}.

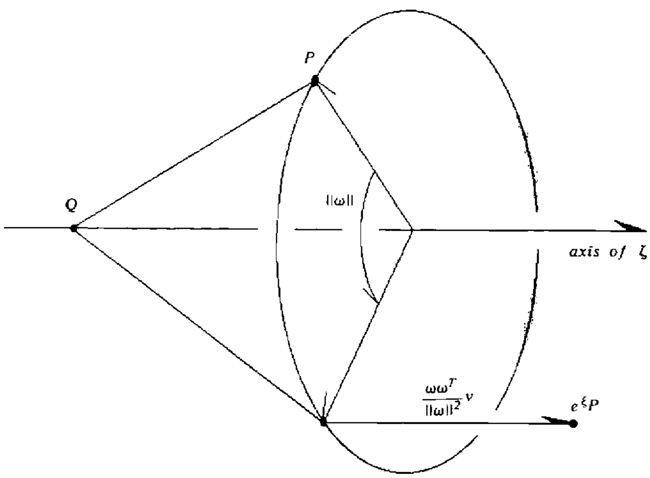
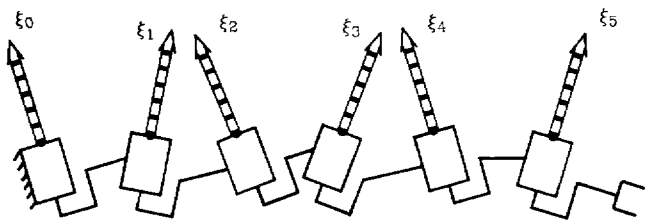


Fig. 4. Representing a 6R manipulator by an ordered set of twists.



complish. In dynamics, for example, a representation must contain information on inertias, etc. For the purposes of this paper, the only significant objects on 6R manipulators are the joint axes. Thus we will represent a 6R manipulator by an ordered set (ordered from base to gripper) of six zero-pitch unit-amplitude twists whose screw axes are coincident with the manipulator axes for some configuration of the manipulator.[6] Figure 4 depicts a manipulator with its representative screws. The ordered set of twists $\bar{\xi} = \{\xi_0, \ldots, \xi_5\}$, $\xi_i \in T_I G$, is called a representative of the manipulator M. Since there are many representatives corresponding to different "zero configurations" and different senses of positive rotation of the joint axes, we say that $\bar{\zeta}$ is equivalent to $\bar{\xi}$, and write $\bar{\zeta} \sim \bar{\xi}$, if $\bar{\zeta}$ and $\bar{\xi}$ are representatives of the same manipulator. Formally, $\bar{\zeta} \sim \bar{\xi}$ if there exists $\bar{\phi} \in \mathbf{T}^5$ such that

$$\begin{aligned}\xi_i = \pm(e^{\phi_0\zeta_0}e^{\phi_1\zeta_1} \cdots e^{\phi_{i-1}\zeta_{i-1}})\zeta_i \\ \times (e^{\phi_0\zeta_0}e^{\phi_1\zeta_1} \cdots e^{\phi_{i-1}\zeta_{i-1}})^{-1}, \qquad i \in \{0, \ldots, 5\}.\end{aligned} \tag{18}$$

From Proposition 1 we have that $g\zeta g^{-1}$, $g \in G$, has the same pitch and amplitude as ζ, and that the axis of $g\zeta g^{-1}$ is the axis of ζ translated by the rigid motion g. This gives the interpretation of "$\sim$" as $\bar{\zeta} \sim \bar{\xi}$ if the axis of each ζ_i can be rotated successively about the previous axes $\zeta_{i-1}, \ldots, \zeta_0$ such that its axis is coincident (possibly antiparallel) with that of ξ_i. A manipulator is identified with an equivalence class generated by $\sim$. If $\bar{\zeta}$ is a representative of M, we write $M = [\bar{\zeta}]$, and the brackets $[\cdot]$ are read "the equivalence class containing."

The length of a manipulator is important for obtaining an upper bound on its work-volume. Presently, we develop a notion of length for 6R manipulators that has operational significance. Define the (nonempty) set of linking curves of $\bar{\xi}$ as the following set of curves that map the interval [0, 5] into $\mathbb{R}^3$.

Definition 1.

$C_{\bar{\xi}} \triangleq \{c: [0, 5] \to \mathbb{R}^3 | c$ is continuous, linear on $[i, i+1]$,
$i \in \{0, 1, 2, 3, 4\}$, and $c_i \cap \xi_i \neq \emptyset\}$

is called the set of *linking curves* of $\bar{\xi}$. (We use the notation c_t to denote c evaluated at t and refer to the line segments $c_{[i,i+1]}$, $i \in \{0, 1, 2, 3, 4\}$, as *links* of $\bar{\xi}$.)

Thus, a linking curve is simply a curve that passes through the axes of the manipulator *in order.* Using the set of linking curves, we define the length of a representative $\bar{\xi}$ as the minimum of the lengths of all linking curves.

Definition 2.

Define the *length* of $\bar{\xi}$, $l_{\bar{\xi}}$ by

$$l_{\bar{\xi}} \triangleq \min_{c \in C_{\bar{\xi}}} \int_0^5 \left\| \frac{d}{dt} c_t \right\| dt = \min_{c \in C_{\bar{\xi}}} \sum_{i=1}^{5} \|c_i - c_{i-1}\|. \tag{19}$$

On a physical manipulator, rotating the joints clearly does not affect the lengths of the links; likewise, choosing a different representative of a manipulator

6. Since this paper deals exclusively with 6R manipulators, we will often drop the 6R and simply write manipulator.

does not affect its length as defined in Definition 2. This is expressed formally in the following:

PROPOSITION 4 If $\bar{\xi} \sim \bar{\zeta}$, then $l_{\bar{\xi}} = l_{\bar{\zeta}}$.

Proof: $\bar{\xi} \sim \bar{\zeta} \Rightarrow \exists\, \bar{\Phi} \in T^5$ such that

$$\xi_i = \pm(e^{\phi_0\zeta_0} \cdots e^{\phi_{i-1}\zeta_{i-1}})\zeta_i(e^{\phi_0\zeta_0} \cdots e^{\phi_{i-1}\zeta_{i-1}})^{-1}, \quad i \in \{0, \ldots, 5\}.$$

Let $c \in C_{\bar{\zeta}}$ and define the curve d by

$$d_t = e^{\phi_0\zeta_0} \cdots e^{\phi_i\zeta_i}c_t \quad \text{for } t \in [i, i+1], \quad i \in \{0, 1, 2, 3, 4\}. \tag{20}$$

Observe that d_t is continuous and piecewise linear on [0, 5]. Also $c_i \in \zeta_i$ (i.e., c_i is on the axis of ζ_i), so $d_i = e^{\phi_0\zeta_0} \cdots e^{\phi_{i-1}\zeta_{i-1}}c_i \in \xi_i$. It follows that $d \in C_{\bar{\xi}}$. Since rigid motions preserve length, it is easy to see that

$$\int_0^5 \left\| \frac{d}{dt} c_t \right\| dt = \int_0^5 \left\| \frac{d}{dt} d_t \right\| dt. \tag{21}$$

Thus, for every linking curve of $\bar{\zeta}$, there exists a linking curve of $\bar{\xi}$ having the same length. By the symmetry of the equivalence relation, ~, the opposite is also true. Thus, $l_{\bar{\xi}} = l_{\bar{\zeta}}$.■

Proposition 4 tells us that each representative of a manipulator has the same length. Thus, the length of a manipulator is well defined in terms of its representatives.

Definition 3.

Let $M = [\bar{\xi}]$. Then its *length* l_M is

$$l_M \leq l_{\bar{\xi}}. \tag{22}$$

It is often the case that a property of a manipulator is invariant under the reversal of the axis order. Reversing the order of the axes is equivalent to using the gripper end of the robot as the base and the base end as a gripper (clearly, this is not practical unless we replace the final link with a gripping device and the first link with a mounting flange)! This is an important symmetry that is made precise by defining formally the kinematic inverse of a manipulator.

Definition 4.

If $M = [(\xi_0, \ldots, \xi_5)]$, then $M^{-1} \triangleq [(\xi_5, \ldots, \xi_0)]$ is called the *kinematic inverse* of M. The representative of M^{-1} is denoted $\bar{\xi}^{-1}$.

One property of a manipulator that is invariant under kinematic inversion is its length:

PROPOSITION 5 Let M be a manipulator, then $l_M = l_{M^{-1}}$.

Proof: If c_t is a linking curve for $\bar{\xi}$, a representative of M, then c_{5-t} is a linking curve for $\bar{\xi}^{-1}$, and vice versa.■

For manipulation we express the motion of the hand attached after the last joint in terms of the joint angles. If $\bar{\xi}$ represents the configuration of the manipulator axes when all the joint angles are set to zero, and if we define the resulting hand configuration to be the identity configuration, then the configuration of the hand as a function of the joint angles is

$$R = e^{\theta_0\xi_0}e^{\theta_1\xi_1} \cdots e^{\theta_5\xi_5}. \tag{23}$$

This rigid motion of the hand relative to its identity configuration is a rotation about ξ_5 by θ_5 followed by a rotation about ξ_4 by θ_4, etc. The map $f_{\bar{\xi}}$ defined by $f_{\bar{\xi}}: \bar{\theta} \mapsto e^{\theta_0\xi_0}e^{\theta_1\xi_1} \cdots e^{\theta_5\xi_5}$ is called the *forward kinematic map* for $[\bar{\xi}]$ associated with $\bar{\xi}$. Thus, the forward kinematic map is determined by the zero configuration of the manipulator's axes (Gupta 1986).

It would be nice to be able to define a unique forward kinematic map; however, there does not seem to be a natural way to do this. The following proposition demonstrates the relationship between forward kinematic maps associated with different representatives of the same manipulator.

PROPOSITION 6 If $\bar{\xi} \sim \bar{\zeta}$, then $\exists\, \bar{\Phi} \in \mathbf{T}^5$ such that

$$f_{\bar{\xi}}(\theta_0, \ldots, \theta_5) = f_{\bar{\zeta}}((\pm\theta_0 + \phi_0), \ldots, (\pm\theta_4 + \phi_4), (\pm\theta_5))e^{-\phi_4\zeta_4} \cdots e^{-\phi_0\zeta_0} \tag{24}$$

for some choice of signs in the RHS of (24).

Proof: Since $\bar{\xi} \sim \bar{\zeta}$, there exists $\bar{\Phi} \in \mathbf{T}^5$ such that

$$\xi_i = \pm(e^{\theta_0\zeta_0}e^{\theta_1\zeta_1} \cdots e^{\theta_{i-1}\zeta_{i-1}})\zeta_i(e^{\theta_0\zeta_0}e^{\theta_1\zeta_1} \cdots e^{\theta_{i-1}\zeta_{i-1}})^{-1}.$$

Thus,

$$
\begin{aligned}
f_{\bar{\xi}}(\theta_0, \ldots, \theta_5) &= e^{\theta_0\xi_0} \cdots e^{\theta_5\xi_5} \\
&= e^{\pm\theta_0\zeta_0}\exp(\pm\theta_1 e^{\phi_0\zeta_0}\zeta_1 e^{-\phi_0\zeta_0}) \cdots \exp(\pm\theta_5(e^{\phi_0\zeta_0} \cdots e^{\phi_4\zeta_4})\zeta_5(e^{\phi_0\zeta_0} \cdots e^{\phi_4\zeta_4})^{-1}) \\
&= e^{\pm\theta_0\zeta_0}[e^{\phi_0\zeta_0}e^{\pm\theta_1\zeta_1}e^{-\phi_0\zeta_0}] \cdots \exp(\pm\theta_5(e^{\phi_0\zeta_0} \cdots e^{\phi_4\zeta_4})\zeta_5(e^{\phi_0\zeta_0} \cdots e^{\phi_4\zeta_4})^{-1}) \\
&= e^{(\pm\theta_0+\phi_0)\zeta_0}e^{(\pm\theta_1+\phi_1)\zeta_1} \cdots e^{\pm\theta_5\zeta_5}e^{-\phi_4\zeta_4} \cdots e^{-\phi_0\zeta_0} = \text{RHS of (24).}
\end{aligned}
\tag{25}
$$

■

Thus, $f_{\bar{\xi}} = R_g f_{\bar{\zeta}} \circ h$, where h is an automorphism of $\mathbf{T}^6$, and R_g is a right translation by some $g \in G$. It follows that $f_{\bar{\xi}}(\mathbf{T}^6) = R_g f_{\bar{\zeta}}(\mathbf{T}^6)$ (i.e., a translated version of $f_{\bar{\zeta}}(\mathbf{T}^6)$). If we define volume in G such that it is invariant under translations by group elements, then we can associate a unique volume to each manipulator. Let w be the unique translation invariant volume form on $SO(3)$ (Boothby 1975) such that $\int_{SO(3)} w = 8\pi^2$. This normalization gives the units of orientation volume in radians cubed. Define a volume form on $\mathbb{R}^3$ by $dx \wedge dy \wedge dz$, as usual. Then we construct the volume form $\Omega \triangleq dx \wedge dy \wedge dz \wedge w$ on $\mathbb{R}^3 \times SO(3)$. From Abraham, Marsden, and Ratiu (1983), page 399, we have that $dx \wedge dy \wedge dz$ induces a measure μ_1 (Lebesgue measure) on $\mathbb{R}^3$ and w induces a measure μ_2 on $SO(3)$. The volume form Ω induces a measure on $\mathbb{R}^3 \times SO(3)$, which is simply the product measure $\mu_1 \times \mu_2$. This observation will simplify our calculation of volumes in $\mathbb{R}^3 \times SO(3)$, since we only calculate volumes of rectangles of the form $A \times SO(3)$, $A \subset \mathbb{R}^3$. We make the identification of G with $\mathbb{R}^3 \times SO(3)$ and note that Ω is a translation-invariant volume form on G.

Intuitively, this notion of volume is quite simple. For example, consider an aircraft restricted to a cube of airspace that is 1 km on a side. At each point in the airspace the aircraft can point itself anywhere in a 4π solid angle and roll 2π about the direction it is pointing. Thus the orientation freedom or *orientation volume* at this point is $4\pi \times 2\pi = 8\pi^2$ rad^3. Multiplying by the positional volume, we obtain $8\pi^2$ rad^{3}km^3 for the volume of the free configuration space of the aircraft. This is the notion of work-volume we use for manipulators. It has the advantage of being able to trade off orientation freedom for positional freedom smoothly, unlike the popular notion of dextrous workspace.

We define the work-volume of a manipulator as the volume of the image of its jointspace under the forward kinematic map.

Definition 5.

The work-volume V_M of a manipulator $M = [\bar{\xi}]$ is given by

$$V_M \triangleq \int_{f_{\bar{\xi}}(\mathbf{T}^6)} \Omega. \tag{26}$$

Remark The translation invariance of Ω in (26) guarantees that V_M is independent of the choice of representative $\bar{\xi}$. This property also tells us that the volume is independent of hand size and depends only on the positions of the joint axes! This is a useful measure for the robot manufacturer who is building a general-purpose robot and not designing the robot's hand. With the definitions above it is simple to obtain an upper bound on the work-volume of a manipulator.

PROPOSITION 7 For any manipulator M,

$$V_M \leq 4/3\pi(l_M)^3 \cdot 8\pi^2.$$

Proof: Let ξ be a representative of M, and let $c \in C_{\bar{\xi}}$ with length l_M. Then $\|f_{\bar{\xi}}(\bar{\theta})c_5 - c_0\| \leq l_M \; \forall \; \bar{\theta} \in \mathbf{T}^6$, since we can construct a curve from c_0 to $f_{\bar{\xi}}(\bar{\theta})c_5$ of length l_M for all $\bar{\theta}$, as was done in Proposition 4. Thus,

$$f_{\bar{\xi}}(\bar{\theta})c_5 - c_0 \in \bar{B}(0, l_M), \tag{27}$$

since c_5 is always "tethered" to c_0 by a curve of length l_M. (The other c_i are also tethered together by shorter curves.) It follows that

$$
\begin{aligned}
f_{\bar{\xi}}(\mathbf{T}^6) \subset \{g_1 g_2 | g_1 \in G &\text{ is a translation by} \\
v \in \bar{B}(c_0 - c_5, l_M) &\text{ and } g_2 \in G_{c_5}\} \\
\cong_V \bar{B}(0, l_M) &\times SO(3).
\end{aligned}
\tag{28}
$$

Thus,

$$V_M \leq \tfrac{4}{3}\pi(l_M)^3 \cdot 8\pi^2. \tag{29}$$

■

This proposition says that the *most* we can expect from a manipulator in terms of work-volume is the upper bound given. Since large work-volume is a desirable property in general, we make the following:

Definition 6.

M has *maximal work-volume* (MWV) if $V_M = \tfrac{4}{3}\pi(l_M)^3 \cdot 8\pi^2$.

In addition to large work-volume, we would like the manipulator to be able to reach all points in its workspace in *each* of its configurations. This guarantees that the manipulator can move between any two points in its workspace without changing its configuration. We call this property the well-connected workspace property. To be precise, we define $\mathbf{C}_{\bar{\xi}}$ to be the set of critical points (Abraham, Marsden, and Ratiu 1983) of $f_{\bar{\xi}}$, and we define well-connected workspace as follows.

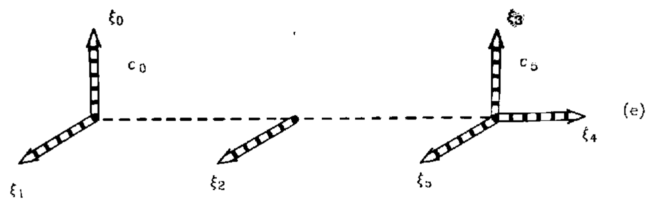
Definition 7.

The manipulator $[\xi]$ has *well-connected workspace* if $f_{\bar{\xi}}(B) = f_{\bar{\xi}}(\mathbf{T}^6) \backslash f_{\bar{\xi}}(\mathbf{C}_{\bar{\xi}})$ for all connected components B (commonly called configurations) of $\mathbf{T}^6 \backslash \mathbf{C}_{\bar{\xi}}$. (Since $f_{\bar{\xi}}$ and $f_{\bar{\zeta}}$ are related, for $\bar{\xi} \sim \bar{\zeta}$, by compositions with diffeomorphisms this property is well defined for a manipulator.)

A manipulator having the well-connected-workspace property has the ability (modulo obstacles and mechanical constraints) to move its gripper from one regular value (Abraham, Marsden, and Ratiu 1983) to another without passing through a critical value (singularity). This is a "nice" property of manipulators, and it guarantees that the manipulator need not change configurations (e.g., from elbow up to elbow down) in order to move from one regular value of its forward kinematic map to another.

Kinematic inversion has no effect on the MWV or on the well-connected-workspace properties.

Fig. 5. Representative of an elbow manipulator.



PROPOSITION 8 Let M be a manipulator. Then $V_M = V_{M^{-1}}$.

Proof: Let $\bar{\xi}$ be a representative of M. Then

$$\begin{aligned} f_{\bar{\xi}}(\mathbf{T}^6) &= \{e^{\theta_0\xi_0} \cdots e^{\theta_5\xi_5} \mid \bar{\theta} \in \mathbf{T}^6\} \\ &= \{(e^{-\theta_5\xi_5} \cdots e^{-\theta_0\xi_0})^{-1} \mid \bar{\theta} \in \mathbf{T}^6\} \\ &= \{(e^{\theta_5\xi_5} \cdots e^{\theta_0\xi_0})^{-1} \mid \bar{\theta} \in \mathbf{T}^6\} = \mathbf{I} f_{\bar{\xi}^{-1}}(\mathbf{T}^6), \end{aligned} \tag{30}$$

where $\mathbf{I}$ is the inversion map in the group G. That $\mathbf{I}$ is volume preserving for translation-invariant volume elements implies $V_M = V_{M^{-1}}$.■

COROLLARY M has MWV if and only if M^{-1} has MWV.

Using the fact that $\mathbf{I}$ is a diffeomorphism we can show the following.

PROPOSITION 9 M has well-connected workspace if and only if M^{-1} has a well-connected workspace.

Proposition 8, its corollary, and Proposition 9 tell us that the notion of a kinematic inverse is a fundamental symmetry in the analysis of manipulators. We will use this symmetry to simplify our proof of Theorem 1. When we exploit this symmetry, we say "by kinematic inversion. . . ."

We will see shortly that elbow manipulators are very special 6R manipulators. First, a formal definition of an elbow manipulator is given.

Definition 8.

M is an *elbow manipulator* if it has a representative $\bar{\xi}$ satisfying the following (see Fig. 5): There exists a line segment $\overline{c_0c_5}$ with the properties that

1. $\xi_0 \cap \xi_1 = c_0$.
2. $\xi_0 \perp \xi_1$.
3. $\xi_1 \perp \overline{c_0c_5}$.
4. $\xi_1 \| \xi_2$.

5. ξ_2 is a perpendicular bisector of $\overline{c_0c_5}$.
6. $\xi_3 \cap \xi_4 \cap \xi_5 = c_5$.
7. $\xi_3 \perp \xi_4$.
8. $\xi_4 \perp \xi_5$.

(The line segment $\overline{c_0c_5}$ is introduced in anticipation of a linking curve with endpoints c_0 and c_5.)

PROPOSITION 10 Let M be an elbow manipulator and $\overline{c_0c_5}$ be as in the definition of the elbow manipulator. Then $l_M = \|c_5 - c_0\|$.

Proof: Let $\bar{\xi}$ be a representative of M that satisfies Definition 8. Since the last three axes of $\bar{\xi}$ intersect consecutively orthogonally at c_5, we can point joint axis 5 arbitrarily. Thus we can rotate ξ_5 and ξ_4 and then about ξ_3 to a new position ξ_5' with $\xi_5' \| \xi_0$. Then $\overline{c_0c_5}$ is a mutual perpendicular of ξ_0 and ξ_5'. There is no curve shorter than $\|c_5 - c_0\|$ connecting ξ_0 and ξ_5', and therefore $l_M \geq \|c_5 - c_0\|$. Now it is easy to construct a linking curve c_t with $c_{[0,5]} = \overline{c_0c_5}$ and whose length is $\|c_5 - c_0\|$. Therefore $l_M = \|c_5 - c_0\|$.■

This concludes the mathematical setup. In the next section a basic theorem is proved relating the concepts of this section.

4. Optimality Theorem

It is generally accepted that "elbow" manipulators have large work-volumes. This is made precise, and an interesting converse is proved in the following theorem.

Theorem 1: A 6R manipulator M has well-connected workspace and maximal work-volume if and only if M or M^{-1} is an elbow manipulator.

The formal proof of this theorem contains many technical (yet important) details, so a heuristic proof is given first to give an overview of the basic geometry that motivated the complete proof. Readers should situate themselves such that one arm is free to do some of the "exercises."

Heuristic Proof: ($\Leftarrow$) This direction of the proof is relatively easy. Elbow manipulators are modeled after the human arm, which is capable, modulo mechanical constraints, to position and orient the hand anywhere in a ball of radius arm's length. Thus, the elbow manipulator has MWV. This is also true for the kinematic inverse of an elbow manipulator by the kinematic inversion properties. The well-connected-workspace property is left to the formal proof.

($\Rightarrow$) The basic idea of this part of the proof is to construct a set of "exercises" that only a manipulator with maximal work-volume and well-connected workspace could do and then successively conclude the geometric features of such a manipulator.

Exercise 1 (Shoulder and Wrist Swirl) A manipulator with MWV can reach any point on a sphere of arm's length radius and any orientation on that sphere. In other words, the manipulator has at least five degrees of freedom remaining out of six at full reach. Since joints that do not intersect the shoulder or the wrist are frozen at full reach, there must be only *one such joint* (called the elbow).

Exercise 2 (Armpit Scratcher) A manipulator with MWV must be able to reach its shoulder with its wrist. Therefore, the upper arm and forearm have the *same length,* and the axis of the elbow joint must be *perpendicular* to both the forearm and the upper arm.

Exercise 3 (Elbow Swirl) A manipulator with MWV must be able to place its elbow anywhere on a sphere of radius half the arm's length. For suppose there is a region on this sphere that the elbow cannot reach. Then project this region onto the larger sphere of radius arm's length, and the wrist cannot reach this region, in contradiction to MWV. Since the elbow must move with 2 DOF, there must be at least two joints intersecting at the shoulder. By kinematic inversion there are at least two joints intersecting the wrist. So the manipulator must have two joints at one end and three at the other. We can assume without loss of generality (by kinematic inversion) that the end with only two joints is the *shoulder.* It is well known that these two joints must intersect *orthogonally* and must be *orthogonal* to the upper arm for the elbow to reach any point on a sphere.

Exercise 4 (Hand Raiser) Assume that the first shoulder joint is vertical. Then by MWV the arm must reach a configuration where the wrist is on the first shoulder axis and above the shoulder by half the arm's length. In this configuration the elbow is 120° from full extension. Now the upper arm and lower arm *and* the first shoulder axis lie in a plane. The elbow axis is perpendicular to both arm segments and is therefore perpendicular to the plane. Also, the second shoulder joint axis is perpendicular to the first joint axis and perpendicular to the upper arm, so the second shoulder axis is perpendicular to the plane. It follows that the second joint axis and the elbow axis are *parallel.*

Exercise 5 (Elbow-Down Wrist Swirl) Choose a configuration with the elbow pointing down. By MWV the hand should reach all orientations about the current position of the wrist. If the three wrist axes are consecutively orthogonal, then this is possible and we have an elbow manipulator or its kinematic inverse. If not, there is some orientation that is not possible in the present elbow-down configuration. By MWV this orientation is attainable by changing the elbow configuration to another, say elbow-up, but this requires that the manipulator move through a singularity (full extension or full retraction of the elbow), violating the well-connected-workspace assumption. Therefore, the only possibility is that the wrist axes are consecutively orthogonal.

Thus, we have narrowed the possible manipulators to the elbow manipulators and their kinematic inverses. The formal proof now follows.

Proof: ($\Leftarrow$) By kinematic inversion, it is sufficient to prove this implication for M an elbow manipulator. Let $\bar{\xi}$ be a representative of an elbow manipulator satisfying the properties in the definition and the additional (nonrestrictive) properties $\xi_3 = \xi_5$ and $z_0^T(c_5 - c_0) = \|c_5 - c_0\|$, where z_0 is a unit vector parallel to ξ_0. (These added conditions allow us to write a closed-form inverse kinematic solution for $f_{\bar{\xi}}$.) We prove the well-connectedness and maximal work-volume of M by examining the solution to the kinematic equation[7]

$$e^{\theta_0\xi_0}e^{(\theta_1-\theta_2/2)\xi_1}e^{\theta_2\xi_2}e^{\theta_3\xi_3}e^{\theta_4\xi_4}e^{\theta_5\xi_5} = R, \quad (31)$$

7. Note that we make the minor change in joint coordinates $\theta_1 \to \theta_1 - \theta_2/2$. This is for convenience and has no effect on work-volume or the well-connected-workspace property.

where $R \in G$ is a desired rigid motion. Let z_i be a unit vector parallel to ξ_i, $i \in \{0, \ldots, 5\}$. Also, c_5 and c_0 are as in Definition 8, and $l_M = \|c_5 - c_0\|$ is the length of the manipulator. The solution of Eq. (31) is well understood, and a straightforward calculation yields that $\bar{\theta}$ is a solution to (31) if and only if $\bar{\theta}$ satisfies (32–35):

$$\begin{aligned} &\theta_2 = \pm\cos^{-1}[2\|Rc_5 - c_0\|^2/(l_m)^2 - 1], \\ &u_1 \triangleq Rc_5 - c_0, \end{aligned} \quad (32)$$

$$\theta_1 = \pm \text{ATAN}_2[\sqrt{\|u_1\|^2 - (z_0^T u_1)^2}/z_0^T u_1], \quad (33a)$$

$$\theta_0 = \text{ATAN}_2\,[\pm z_1^T u_1/\pm(z_0 \times z_1)^T u_1], \quad (33b)$$

(the upper and lower signs are matched in (33a) and (33b))

$$\begin{aligned} &R_1 \triangleq [e^{\theta_0\xi_0}e^{(\theta_1-\theta_2/2)\xi_1}e^{\theta_2\xi_2}]^{-1}R, \\ &u_2 \triangleq R_1 z_3, \\ &\theta_4 = \pm \text{ATAN}_2[\sqrt{1 - (z_3^T u_2)^2}/z_3^T u_2], \end{aligned} \quad (34a)$$

$$\begin{aligned} &\theta_5 = \text{ATAN}_2[\mp z_4^T u_2/\pm(z_3 \times z_4)^T u_2], \\ &R_2 \triangleq [e^{\theta_3\xi_3}e^{\theta_4\xi_4}]^{-1}R_1, \qquad u_3 \triangleq R_2 z_4, \end{aligned} \quad (34b)$$

(the upper and lower signs are matched in (34a) and (34b))

$$\theta_5 = \text{ATAN}_2[(z_5 \times z_4)^T u_3/z_4^T u_3]. \quad (35)$$

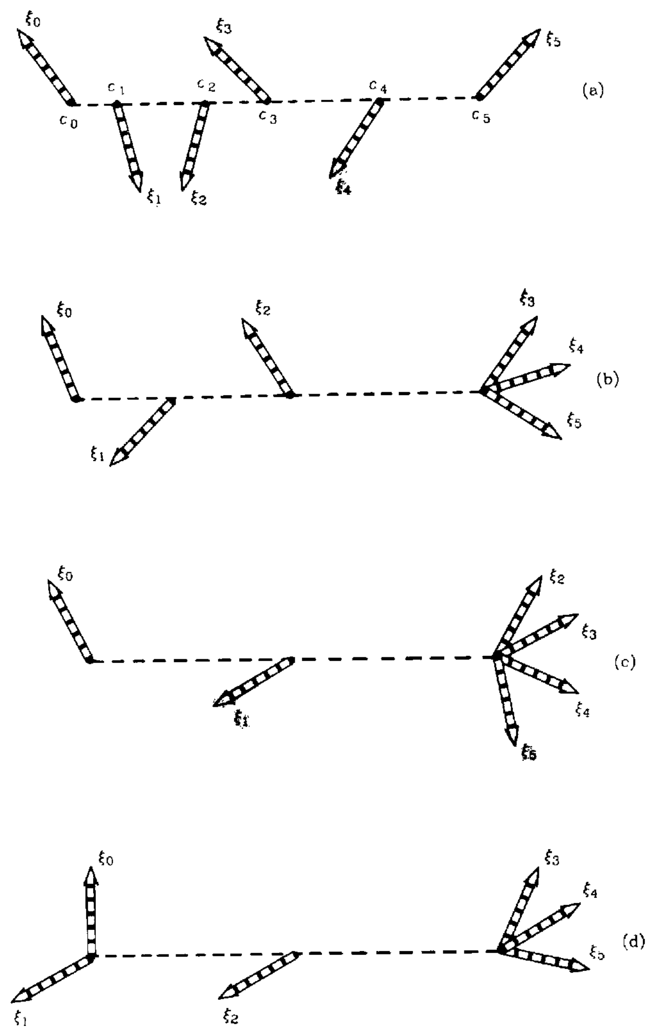
Note that all quantities under the radical in (33–34) are nonnegative and that the domain of the ATAN_2 function is the entire plane. (The numerator and denominator are two distinct arguments of the ATAN_2 function. When they are both zero, the function is set-valued and the θ_i may take any value in $[0, 2\pi]$.) The only restrictions we have on R are given by (32). That is, there is a solution for R if and only if

$$\begin{aligned} R &\in \{g \in G \mid (2\|gc_5 - c_0\|^2/(l_M)^2 - 1) \in [-1, 1]\} \\ &= \{g \in G \mid \|gc_5 - c_0\| \leq l_M\} \\ &\cong_V \bar{B}(0, l_M) \times SO(3). \end{aligned} \quad (36)$$

Thus the work-volume of the elbow manipulator is $4/3\, \pi(l_M)^3 \cdot 8\pi^2$ and is therefore maximal.

The singular configurations of the elbow manipulator are those where the elbow is fully extended or re-

Fig. 6. *Pictorial outline of the proof of the "only if" part of theorem 1.*



tracted, joints 3 and 5 are coincident, or the intersection of the last three joints lies on joint axis zero. A calculation of the Jacobian determinant of $f_{\bar{\xi}}$ (see Paul 1981 for this type of calculation) yields that the set of critical points for $f_{\bar{\xi}}$ in (31) is

$$\mathbf{C}_{\bar{\xi}} = \{\bar{\theta} | \theta_1 \in \{0, \pi\} \text{ or } \theta_2 \in \{0, \pi\} \text{ or } \theta_4 \in \{0, \pi\}\}. \quad (37)$$

It is easy to see that $\mathbf{T}^6 \backslash \mathbf{C}_{\bar{\xi}}$ has eight connected components. We must verify that

$$f_{\bar{\xi}}(B) = f_{\bar{\xi}}(\mathbf{T}^6) \backslash f_{\bar{\xi}}(\mathbf{C}_{\bar{\xi}}) \quad \mathbf{(38)}$$

for all connected components B of $\mathbf{T}^6 \backslash \mathbf{C}_{\bar{\xi}}$. Let $R \in f_{\bar{\xi}}(\mathbf{T}^6) \backslash f_{\bar{\xi}}(\mathbf{C}_{\bar{\xi}})$. Then when we apply the solution procedure (32–35) with such an R there must be no choice of signs in (32–35) such that $\theta_i \in \{0, \pi\}$, $i \in \{1, 2, 4\}$ (otherwise $R \in \mathbf{C}_{\bar{\xi}}$). Thus, whenever we choose a solution branch for θ_1, θ_2, and θ_4 (32, 33a, and 34a) there is a choice of $\theta_i \in (0, \pi)$ or $\theta_i \in (-\pi, 0)$, $i \in \{1, 2, 4\}$. It follows that we can find a solution in each connected component of $\mathbf{T}^6 \backslash \mathbf{C}_{\bar{\xi}} \Rightarrow (38)$ holds. This completes the proof of the "if" part of the theorem.

($\Rightarrow$) Let $M = [\bar{\zeta}]$ and $c' \in C_{\bar{\zeta}}$ with length l_M and define $l' \triangleq \max_{\bar{\theta} \in \mathbf{T}^6} \| e^{\theta_0 \zeta_0} \cdots e^{\theta_5 \zeta_5} c'_5 - c'_0 \|$. If $l' < l_M$, then $V_M \leq 4/3\, \pi(l')^3 \cdot 8\pi^2 < 4/3\, \pi(l_M)^3 \cdot 8\pi^2$ and M does not have MWV. Thus there must exist $\bar{\theta}$ such that $\| e^{\theta_0 \zeta_0} \cdots e^{\theta_5 \zeta_5} c_5 - c_0 \| = l_M$. Let $\bar{\xi}$ be a representative corresponding to this configuration. As in Proposition 5, we can construct $c \in C_{\bar{\xi}}$ from c' such that c has length l_M, $c_0 = c_0'$, and $c_5 = e^{\theta_0 \zeta_0} \cdots e^{\theta_5 \zeta_5} c_5'$. Thus, there exists c, a linking curve for $\bar{\xi}$ with length $l_M = \| c_5 - c_0 \|$. So $c_{[0,5]} = \overline{c_0 c_5}$, a line segment. Since c is a linking curve, all ξ_i intersect $\overline{c_0 c_5}$ and the c_i are ordered on $\overline{c_0 c_5}$. (Figure 6A shows a representative consistent with our knowledge of M at this point in the proof.) Let $f_{\bar{\xi}}: \mathbf{T}^6 \to G$ be the forward kinematic map for M associated with $\bar{\xi}$. Now $f_{\bar{\xi}}(\mathbf{T}^6)$ is compact implies that for any $x \in f_{\bar{\xi}}(\mathbf{T}^6)^C$, x has a neighborhood of nonzero volume. Thus

$$\text{MWV} \Rightarrow f_{\bar{\xi}}(\mathbf{T}^6) = \{g_1 g_2 | g_1 \in G \text{ is a translation by } v \in \bar{B}(c_0 - c_5, l_M),\ g_2 \in G_{c_5}\}$$

(see proof of Proposition 8). Thus $f_{\bar{\xi}}(T^6)$ contains $\mathbf{O}$, defined by

$$\mathbf{O} \triangleq \{g_1 g_2 | g_1 \text{ is a translation by } v \in [l_M S^2 + (c_0 - c_5)],\ g_2 \in G_{c_5}\} \cong S^2 \times SO(3), \quad (39)$$

and $\mathbf{O}$ is a 5-manifold.

Next, define

$$\begin{aligned} \min &= \min\{i | \xi_i \cap c_0 = \emptyset\}, \\ \max &= \max\{i | \xi_i \cap c_5 = \emptyset\}. \end{aligned} \quad (40)$$

By definition of a linking curve, min > 0 and max < 5. Also, max ≥ min; otherwise there would be only one link with nonzero length and MWV would not hold.

From (39), we have that

$$\|e^{\theta_0\xi_0} \cdots e^{\theta_5\xi_5}c_5 - c_0\| = l_M \quad \forall \bar{\theta} \text{ in the inverse image } f_{\bar{\xi}}^{-1}(\mathbf{O}). \quad (41)$$

Since $c_0 \cap \xi_i \neq \emptyset\ \forall i < \min$ and $c_5 \cap \xi_i \neq \emptyset\ \forall i > \max$, (41) becomes

$$\|e^{\theta_{\min}\xi_{\min}} \cdots e^{\theta_{\max}\xi_{\max}} c_5 - c_0\| = l_M \quad \forall \bar{\theta} \in f_{\bar{\xi}}^{-1}(\mathbf{O}), \quad (42)$$

$$\|(e^{\theta_{\min}\xi_{\min}} \cdots e^{\theta_{\max}\xi_{\max}}c_5 - c_{\min}) + (c_{\min} - c_0)\| = l_M\ \forall \bar{\theta} \in f_{\bar{\xi}}^{-1}(\mathbf{O}). \quad (43)$$

Since $\int_0^5 \|\frac{d}{dt} c_t\| dt = \|c_5 - c_0\|$, we have $\int_a^b \|\frac{d}{dt} c_t\| dt = \|c_b - c_a\|$, $\forall a, b \in [0, 5]$. Let $a = \min$, $b = 5$. Then by a tethering argument similar to the first part of the proof of Proposition 8,

$$\|e^{\theta_{\min}\xi_{\min}} \cdots e^{\theta_{\max}\xi_{\max}}c_5 - c_{\min}\| \le \|c_5 - c_{\min}\| \quad \forall \bar{\theta} \in f_{\bar{\xi}}^{-1}(\mathbf{O}). \quad (44)$$

Since $\|c_{\min} - c_0\| + \|c_5 - c_{\min}\| = l_M$, the two terms in the norm in (43) have the same direction and the first term attains the bound in (44):

$$\begin{aligned} e^{\theta_{\min}\xi_{\min}} \cdots e^{\theta_{\max}\xi_{\max}}c_5 - c_{\min} &= \frac{c_{\min} - c_0}{\|c_{\min} - c_0\|} \|c_5 - c_{\min}\| \\ &= c_5 - c_{\min} \qquad (45) \\ &\Rightarrow e^{\theta_{\min}\xi_{\min}} \cdots e^{\theta_{\max}\xi_{\max}}c_5 = c_5 \end{aligned}$$

or, in other words,

$$e^{\theta_{\min}\xi_{\min}} \cdots e^{\theta_{\max}\xi_{\max}} \in G_{c_5} \quad \forall \bar{\theta} \in f_{\bar{\xi}}^{-1}(\mathbf{O}). \quad (46)$$

A similar argument with $c_{\max}$ (kinematic inversion) yields

$$(e^{\theta_{\min}\xi_{\min}} \cdots e^{\theta_{\max}\xi_{\max}}) \in G_{c_0} \quad \forall \bar{\theta} \in f_{\bar{\xi}}^{-1}(\mathbf{O}). \quad (47)$$

Let η be a zero-pitch unit twist whose axis is coincident with $\overline{c_0c_5}$. Then from (46) and (47) we have

$$\{e^{\theta_{\min}\xi_{\min}} \cdots e^{\theta_{\max}\xi_{\max}} | \bar{\theta} \in f_{\bar{\xi}}^{-1}(\mathbf{O})\} \subset G_{c_5} \cap G_{c_0} = \{e^{\phi\eta} | \phi \in S^1\}. \quad (48)$$

$$\begin{aligned} \mathbf{O} = f(f^{-1}(\mathbf{O})) &= \{e^{\theta_0\xi_0} \cdots e^{\theta_5\xi_5} | \bar{\theta} \in f^{-1}(\mathbf{O})\} \\ &\subset \{e^{\theta_0\xi_0} \cdots e^{\theta_{\min-1}\xi_{\min-1}} e^{\phi\eta} e^{\theta_{\max+1}\xi_{\max+1}} \cdots e^{\theta_5\xi_5} | \\ &\quad \theta_i \in S^1, i \notin \{\min, \ldots, \max\}, \phi \in S^1\}. \qquad (49) \end{aligned}$$

The containment in (49) follows from (48). Since $\mathbf{O}$ is a 5-manifold, the order of $\{\theta_0, \ldots, \theta_{\min-1}, \phi, \theta_{\max+1}, \ldots, \theta_5\}$ must be 5 by Proposition I.4 of the appendix. (Intuitively, we need 5 DOF to sweep out a 5-manifold.) Thus, $\max - \min \le 1$. (Figure 6B represents approximately our knowledge of M at this point in the proof—at most two of the ξ_i do not intersect c_0 or c_5). Suppose $\max - \min = 1$ and that $\{e^{\theta_{\min}\xi_{\min}} e^{\theta_{\max}\xi_{\max}} | \bar{\theta} \in f_{\bar{\xi}}^{-1}(\mathbf{O})\}$ is a finite set. Then from (48) we have that there exists a finite set $\{\phi_1, \ldots, \phi_n\}$ such that

$$\{e^{\theta_{\min}\xi_{\min}} e^{\theta_{\max}\xi_{\max}} | \bar{\theta} \in f_{\bar{\xi}}^{-1}(\mathbf{O})\} = e^{\phi\eta} | \phi \in \{\phi_1, \ldots, \phi_n\}\} \quad (50)$$

and

$$\mathbf{O} \subset \bigcup_{k=1}^{n} \{e^{\theta_0\xi_0} \cdots e^{\theta_{\min}\xi_{\min-1}} e^{\phi_k} e^{\theta_{\max+1}\xi_{\max+1}} \cdots e^{\theta_5\xi_5} | \theta_i \in S^1\}.$$

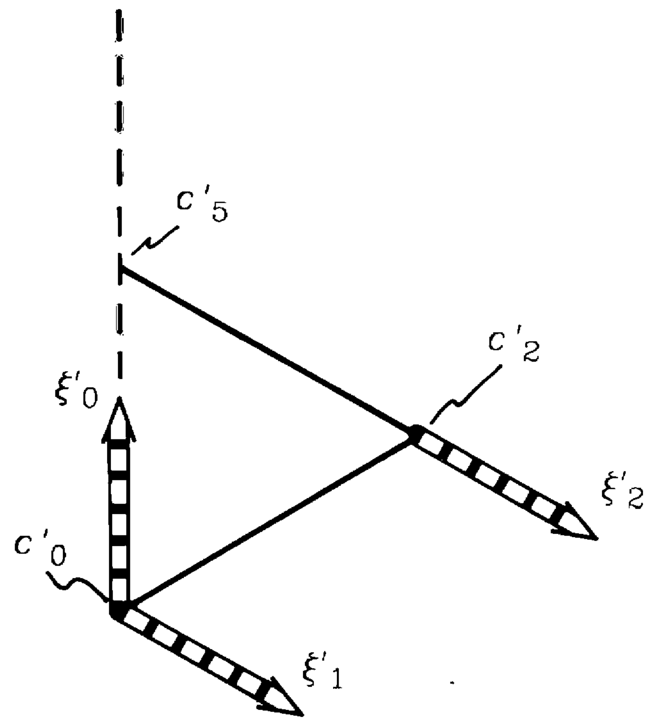
Since the union is finite and each set in the union is compact, it follows (Baire category theorem (Royden 1968)) that one of the sets in the union has nonempty interior in $\mathbf{O}$. Now $\mathbf{O}$ is a 5-manifold, so the order of $\{\theta_0, \ldots, \theta_{\min-1}, \theta_{\max+1}, \ldots, \theta_5\}$ must be 5 by Proposition I.4, but the order is at most 4 when $\max - \min = 1$. Thus we must have that $\{e^{\theta_{\min}\xi_{\min}} e^{\theta_{\max}} e^{\xi_{\max}} | \bar{\theta} \in f\bar{\xi}^{-1}(\theta)\}$ is infinite when $\max - \min = 1$. This and (46) imply that $e^{\theta_{\min}\xi_{\min}} e^{\theta_{\max}\xi_{\max}} c_5 = c_5$ has an infinite number of solutions. This, in turn, implies that (see Paden 1985, page 24) $\xi_{\max} \cap c_5 \neq \emptyset$ or $\xi_{\min} \cap c_5 \neq \emptyset$. The definition of max then implies that $\xi_{\max} \cap c_5 = \emptyset$. By kinematic inversion $\xi_{\min} \cap c_0 = \emptyset$ also. This is a contradiction, so $\max - \min = 1$ is impossible and $\max = \min$. Thus, there is exactly one twist, $\xi_{\min}$, which intersects neither c_0 nor c_5. Next, we show that $\xi_{\min}$ is a perpendicular bisector of $\overline{c_0c_5}$.

M has maximal work-volume implies that the map $\bar{f}: \bar{\theta} \mapsto \|e^{\theta_0\xi_0} \cdots e^{\theta_5\xi_5}c_5 - c_0\|$ is onto $[0, l_M]$. Since $\xi_{\min}$ is the only axis not intersecting c_5 or c_0, $\bar{f}(\bar{\theta}) = \|e^{\theta_{\min}\xi_{\min}}c_5 - c_0\|$, and by Proposition I.3 we have that

$$\xi_{\min} \text{ is a perpendicular bisector of } \overline{c_0c_5}. \quad (51)$$

So we have that part 5 of Definition 8 is satisfied if

Fig. 7. Showing that $\xi_1 \parallel \xi_2$.



min = 2. (We will see this shortly.) (Figure 6C is a representative consistent with our knowledge of M at this point in the proof. The drawings are isometric.)

Next, we claim that MWV implies that $d: \bar{\theta} \mapsto e^{\theta_0 \xi_0} \cdots e^{\theta_{\min-1} \xi_{\min-1}} c_{\min} - c_0$ is onto $\|c_{\min} - c_0\| S^2$. Suppose not. Then since $d(\mathbf{T}^6) \subset \|c_{\min} - c_0\| S^2$, there exists a unit vector u such that

$$\begin{aligned} & u^{\mathrm{T}} d(\bar{\theta}) < \|c_{\min} - c_0\| \; \forall \bar{\theta} \in \mathbf{T}^6 \\ & \quad \Rightarrow u^{\mathrm{T}}(f_{\bar{\xi}}(\bar{\theta}) c_5 - c_0) < \|c_5 - c_0\| \quad \forall \bar{\theta} \in \mathbf{T}^6, \end{aligned}$$

which is a contradiction to MWV since $f_{\bar{\xi}}(\mathbf{T}^6)$ closed (compact). Thus the claim is true.

Now d is a smooth function onto a 2-manifold, so d is a function of two or more of the θ_i by Proposition I.4. Thus, min $\geqslant 2$. By kinematic inversion, min = max $\leqslant (5-2) = 3$. Thus, there are two axes intersecting one end of $\overline{c_0 c_5}$ and three the other. In other words, M or M^{-1} has min = 2. Since we are only trying to prove M or M^{-1} is an elbow manipulator, there is no loss of generality in assuming M has the property min = 2. It follows from Proposition A.1 of the appendix and the fact that d is onto $\|c_{\min} - c_0\| S^2$ that

$$\xi_0 \perp \xi_1 \qquad \text{and} \qquad \xi_1 \perp \overline{c_0 c_5}. \tag{52}$$

At this point, parts 1, 2, 3, and 5 of Definition 8 are satisfied. Next we show that $\xi_1 \parallel \xi_2$.

Recall that z_0 is the unit vector parallel to ξ_0. Then since M has MWV, there exists $\bar{\theta} \in \mathbf{T}^6$ such that $f_{\bar{\xi}}(\bar{\theta}) c_5 = c_0 + (l_M/2) z_0$ (this point lies in $\bar{B}(c_0, l_M)$). Let $\bar{\xi}'$ be a representative corresponding to this configuration, and let $c' \in C_{\bar{\xi}'}$ be the linking curve derived from c (see the proof of Proposition 5). Figure 7 shows the configuration of the manipulator represented by $\bar{\xi}'$. Note that c_0', c_2', and c_5' are not collinear and consider the plane containing them. Since ξ_0' and $\overline{c_0' c_2'}$ are in the plane, we have by (52) that ξ_1' is perpendicular to the plane. Now $\overline{c_0' c_2'}$ and $\overline{c_2' c_5'}$ are in the plane imply that ξ_2' is perpendicular to the plane by (51), and so $\xi_2' \parallel \xi_1'$. This, in turn, implies[8] that $\xi_2 \parallel \xi_1$ and part 4 is satisfied. (Figure 6D represents approximately the information we have at this point in the proof.)

8. Angles between successive joints, and between joints and adjacent links are independent of choice of representative when the linking curves are related as in the proof of Proposition 5.

Since min = max = 2, the last three twist axes intersect at c_5, satisfying part 6 of the definition of the elbow manipulator. To show that these axes are consecutively orthogonal (7 and 8 of Definition 8), we write

$$f_{\bar{\xi}}(\bar{\theta}) = h(\theta_0, \theta_1, \theta_2) w(\theta_3, \theta_4, \theta_5), \tag{53}$$

where $h(\theta_0, \theta_1, \theta_2) \triangleq e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} e^{\theta_2 \xi_2}$ and $w(\theta_3, \theta_4, \theta_5) \triangleq e^{\theta_3 \xi_3} e^{\theta_4 \xi_4} e^{\theta_5 \xi_5}$.

Since the wrist axes of M are intersecting, the singularities of the manipulator decouple. That is, $\bar{\theta}$ is a critical point of $f_{\bar{\xi}}$ if and only if $(\theta_0, \theta_1, \theta_2)$ is a critical point of h': $(\theta_0, \theta_1, \theta_2) \mapsto h(\theta_0, \theta_1, \theta_2) c_5$ or $(\theta_3, \theta_4, \theta_5)$ is a critical point of $w: \mathbf{T}^3 \to G_{c_5}$. More concisely,

$$\mathbf{T}^6 \backslash \mathbf{C}_{\bar{\xi}} = \mathbf{T}^3 \backslash \mathbf{C}_{h'} \times \mathbf{T}^3 \backslash \mathbf{C}_w. \tag{54}$$

Since we know the relationship among ξ_0, ξ_1, and ξ_2, it is straightforward to verify that h' has the property that each connected component, B_1, of $\mathbf{T}^3 \backslash \mathbf{C}_{h'}$ satisfies $h'(B_1) = h'(\mathbf{T}^3) \backslash h'(\mathbf{C}_{h'})$ and $\mathbf{T}^3 \backslash \mathbf{C}_{h'}$ has four connected components. Also h' is 1-1 when restricted to a connected component of $\mathbf{T}^3 \backslash \mathbf{C}_{h'}$.

Suppose that ξ_3, ξ_4, and ξ_5 are not consecutively orthogonal. Then w is not onto G_{c_5} by Proposition I.2. Then $w(\mathbf{T}^3)$ is a compact, proper subset of G_{c_5}, and $w(\mathbf{T}^3)^C$ is a nonempty open set in G_{c_5}. In addition, let $\bar{\theta}$ be a regular point of $f_{\bar{\xi}}$, and let $(\theta_{0i}, \theta_{1i}, \theta_{2i})$, $i \in \{1, 2, 3, 4\}$, be the four solutions that satisfy $h'(\theta_{0i}, \theta_{1i}, \theta_{2i}) = h'\theta_0, \theta_1, \theta_2)$. Since the critical values of w have measure zero in G_{c_5} (the measure on G_{c_5} is that induced by a translation-invariant volume form),

$$D_w \triangleq \bigcup_{i=1}^{4} [h(\theta_0, \theta_1, \theta_2)]^{-1} h(\theta_{0i}, \theta_{1i}, \theta_{2i}) w(\mathbf{C}_w) \quad (55)$$

has measure zero in G_{c_5}. Therefore $w(\mathbf{T}^3)^C \setminus D_w$ is nonempty. Let

$$R^1 \in w(\mathbf{T}^3)^C \setminus D_w. \quad (56)$$

It follows that $h(\theta_0, \theta_1, \theta_2)R^1$ is a regular value of $f_{\bar{\xi}}$ and by the choice of $R^1, f_{\bar{\xi}}(\bar{\theta}) \neq h(\theta_0, \theta_1, \theta_2)R^1$. The manipulator has maximal work-volume, so there exists $\bar{\theta}' \in \mathbf{T}^6$ such that

$$f_{\bar{\xi}}(\bar{\theta}') = h(\theta_0, \theta_1, \theta_2)R^1, \quad (57)$$

since $h(\theta_0, \theta_1, \theta_2)R^1 \in \{g \in G \mid \|gc_5 - c_0\| \leqslant l_M\}$. Also, by (56), $(\theta_0', \theta_1', \theta_2') \neq (\theta_0, \theta_1, \theta_2)$. Now h' is 1-1 on connected components of $\mathbf{T}^3 \setminus \mathbf{C}_{h'}$ implies that $\bar{\theta}'$ and $\bar{\theta}$ are not in the same connected component of $\mathbf{T}^6 \setminus \mathbf{C}_{\bar{\xi}} = \mathbf{T}^3 \setminus \mathbf{C}_{h'} \times \mathbf{T}^3 \setminus \mathbf{C}_w$, thereby contradicting the well-connected-workspace assumption. Thus

$$\xi_3 \perp \xi_4 \quad \text{and} \quad \xi_4 \perp \xi_5. \quad (58)$$

In other words, parts 7 and 8 are satisfied, and we have that M or M^{-1} is an elbow manipulator.

Theorem 1 reinforces the generally accepted idea that elbow manipulators are good kinematic designs and are optimal with respect to work-volume. It is somewhat surprising that the elbow manipulators and their kinematic inverses are the *only* designs that meet the optimality criteria of maximal work-volume/well-connected workspace. This theorem should encourage special consideration of elbow manipulators and their kinematic inverses in the study of path planning and collision avoidance problems.

5. Conclusions

We have developed a formalism for describing 6R manipulators that allows us to define quantitatively the properties of well-connected workspace and maximal work-volume. The result of this formalism is a fundamental theorem relating the design of 6R manipulators to their performance.

Admittedly, the class of 6R manipulators is limited. However, 6 DOF is the minimum number required for a manipulator's work-volume to be nonzero, and the extension of the theorem to 6 DOF manipulators with both R and P (prismatic) joints is feasible with an extension of the definition of the length of a manipulator. The length of a manipulator having both R and P joints is defined to be the length of the shortest curve passing through the axes of the revolute joints (minimize over configurations also) in order *plus* the sum of the travel distances of the prismatic joints. When P joints are allowed, the class of optimal manipulators enlarges to include the Stanford manipulator with zero shoulder offset and its kinematic inverse. Also note that the definitions of work-volume and length of a manipulator extend directly to redundant nR manipulators. One can conclude from arguments in the proof of Theorem 1 that nR redundant manipulators having maximal work-volume have their first two and last two joint axes intersecting. Because redundancy supposedly increases the "connectivity" of the workspace, it will be interesting to see if there is a useful "very"-well-connectedness measure for redundant manipulators.

Appendix

Collected here for convenience are several subproblems that appear in the proof of Theorem 1.

Proposition I.1: Let ξ_0 and ξ_1 be zero-pitch unit twists, $O, P \in \mathbb{R}^3$, $O \neq P$, and $\xi_0 \cap \xi_1 = O$. If f: $(\theta_0, \theta_1) \mapsto e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} P$ is onto $O + \|P - O\| S^2$ (the sphere of radius $\|P - O\|$ centered at O), then

$$\xi_0 \perp \xi_1 \quad \text{and} \quad \xi_1 \perp \overline{OP}. \quad (\text{I.1})$$

Proof: Let z_0 and z_1 be unit vectors parallel to ξ_0 and ξ_1, respectively. Now

$$f(\mathrm{T}^2) = O + \|P - O\| S^2 \tag{I.2}$$

$$\Rightarrow z_0^{\mathrm{T}}(f(\mathrm{T}^2) - O) = [-\|P - O\|, \|P - O\|] \tag{I.3}$$

$$\Rightarrow \exists\ \theta_1, \theta_1' \text{ such that}$$

$$\begin{aligned} e^{\theta_1 \xi_1}(P - O) &= \|P - O\| z_0, \\ e^{\theta_1' \xi_1}(P - O) &= -\|P - O\| z_0 \end{aligned} \tag{I.4}$$

$$\Rightarrow e^{\theta_1 \xi_1}(P - O) = -e^{\theta_1' \xi_1}(P - O). \tag{I.5}$$

Premultiplying both sides by z_1^{T}

$$\Rightarrow z_1^{\mathrm{T}}(P - O) = -z_1^{\mathrm{T}}(P - O) \Rightarrow z_1^{\mathrm{T}}(P - O) = 0. \tag{I.6}$$

Thus

$$\xi_1 \perp \overline{OP}. \tag{I.7}$$

Next from (I.4) we have

$$z_0 = e^{\theta_1 \xi_1} \frac{P - O}{\|P - O\|} \tag{I.8}$$

$$\Rightarrow z_1^{\mathrm{T}} z_0 = z_1^{\mathrm{T}} e^{\theta_1 \xi_1} \frac{P - O}{\|P - O\|} = z_1^{\mathrm{T}} \frac{P - O}{\|P = O\|} = 0 \tag{I.9}$$

$$\Rightarrow \xi_1 \perp \xi_0. \tag{I.10}$$

■

Proposition I.2: Let $O \in \mathbb{R}^3$ and ξ_0, ξ_1, ξ_2 be zero-pitch unit twists with $\xi_0 \cap \xi_1 \cap \xi_2 = O$. If

$$f\colon (\theta_0, \theta_1, \theta_2) \mapsto e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} e^{\theta_2 \xi_2} \text{ is onto } \mathrm{G}_O, \tag{I.11}$$

then $\xi_0 \perp \xi_1$ and $\xi_1 \perp \xi_2$.

Proof: f onto G_O implies that

$$\forall R \in \mathrm{G}_0, \exists(\theta_0, \theta_1, \theta_2) \text{ such that } e^{\theta_0 \xi_0} e^{\theta_1 \xi_1} e^{\theta_2 \xi_2} = R. \tag{I.12}$$

Let $P \neq O$ be a point on ξ_2, and let both sides of (I.12) act on P. Then (I.12) implies

$$\forall R \in \mathrm{G}_O, \exists(\theta_1, \theta_2) \text{ such that } e^{\theta_1 \xi_0} e^{\theta_1 \xi_1} P = RP,$$

but $\mathrm{G}_O P = O + \|P - O\| S^2$, so the result follows from Proposition I.1.

Proposition I.3: Let ξ be a zero-pitch unit twist and $O, P \in \mathbb{R}^3$ with $O \neq P$ and $\xi \cap \overline{OP} \neq \varnothing$. If $f\colon \theta \mapsto \|e^{\theta \xi} P - O\|$ is onto $[0, \|P - O\|]$, then ξ is a perpendicular bisector of $\overline{OP}$.

Proof: Let $Q \in \xi \cap \overline{OP}$. Then

$$\begin{aligned} \|e^{\theta \xi} P - O\| &= \|e^{\theta \xi}(P - Q) - (O - Q)\| \\ &\geq |\|P - Q\| - \|O - Q\||, \end{aligned} \tag{I.13}$$

but $\|e^{\phi \xi} P - O\| = 0$ for some ϕ implies that $\|P - Q\| = \|O - Q\|$, which implies that

$$\xi \cap \overline{OP} \text{ contains the midpoint of } \overline{OP}. \tag{I.14}$$

Now, let z be a unit vector parallel to ξ. Then

$$z^{\mathrm{T}} e^{\phi \xi}(P - Q) = z^{\mathrm{T}}(O - Q) \tag{I.15}$$

$$\Rightarrow z^{\mathrm{T}}(P - Q) = z^{\mathrm{T}}(O - Q) \tag{I.16}$$

$$\Rightarrow z^{\mathrm{T}}(P - O) = 0. \tag{I.17}$$

This together with (I.14) yields the result.■

Proposition I.4: Let M and N be m- and n-manifolds, respectively, and let $f\colon M \to N$ be smooth. If $f(M)$ has nonempty interior then $m \geq n$.

Proof: Since the set of regular values of f is dense in N, there exists a regular value $y \in$ interior of $f(M)$. Thus, there exists $x \in M$ such that $Tf\colon TM \to TN$ is surjective implies that $m \geq n$.■

Nomenclature

$\mathbb{R}$	Real numbers
$\|\cdot\|$	Euclidean norm in $\mathbb{R}^n$
$\overline{B}(x, \delta)$	The closed ball of radius δ centered at x

$SO(3)$	The group of proper rotations in $\mathbb{R}^3$ (the group of 3×3 orthogonal matrices with determinant one)
G	The group of proper rigid motions in $\mathbb{R}^3$ (the group of 4×4 homogeneous transformations)
G_O	The isotropy group of G at $Q \in \mathbb{R}^3$ (those rigid motions that have the point Q as a fixed point)
S^1	The unit circle (i.e., $[0, 2\pi]$ with the endpoints identified)
S^2	The unit 2-sphere
$\mathbf{T}^n$	The n-torus
TM	Tangent bundle of the manifold M
T_xM	Tangent space of M at x
Tf	Tangent map of f
$\wedge$	Tensor wedge product
$\sim$	Equivalent to
$\cong$	Diffeomorphic to
$\cong_V$	Diffeomorphic to via a volume-preserving diffeomorphism
$\parallel$	Parallel to
O, P	Points in $\mathbb{R}^3$
$\overline{OP}$	The line segment connecting O and P
$\perp$	Perpendicular to
$\bar{\theta}, \bar{\zeta}$	An ordered set or list of indexed objects (e.g., $\bar{\theta} = (\theta_0, \ldots, \theta_5)$)
$\bar{\zeta}^{-1}$	The ordered set or list of objects whose elements are those of $\bar{\zeta}$ with their order reversed
S^C	Complement of the set S
$A \backslash B$	The set difference A minus B
$\emptyset$	Empty set
C^r	Continuously differentiable r times
$C_{\bar{\xi}}$	The set of linking curves of $\bar{\xi}$
C_f	The set of critical points of the function f
ATAN_2	Two-argument arctangent function
min	The smallest element in a set
max	The largest element in a set

For economy we make the following abuse of notation: A differential twist about a screw and its screw axis are often identified so that for differential twists about screws ξ_0 and ξ_1, $P \in \xi_0 \cap \xi_1$ means that the point P is in the intersection of the axes of ξ_0 and ξ_1. Also, a differential twist about a screw will sometimes be referred to as simply a twist.

References

Abraham, R., Marsden, J. E., and Ratiu T. 1983. *Manifolds, Tensor Analysis, and Applications*, Reading, Mass: Addison-Wesley.

Ball, R. S. 1900. *A Treatise on the Theory of Screws*. Cambridge: Cambridge University Press.

Boothby, W. M. 1975. *An Introduction to Differentiable Manifolds and Riemannian Geometry*. New York: Academic Press.

Brockett, R. W. 1983 (Beer Sheba, Israel). Robotic manipulators and the product of exponentials formula. *Proc. MTNS-83 Int. Symp.*, pp. 120–129.

Freudenstein, F., and Primrose, E. 1984. On the analysis and synthesis of the workspace of a three-link, turning-pair connected robot arm. *ASME J. Mechanisms, Transmissions, and Automation in Design* 106:365–370.

Gupta, K. C., and Roth, B. 1982. Design considerations for manipulator workspace. *ASME J. Mechanical Design* 104:704–711.

Gupta, K. C. 1986. Kinematic analysis of manipulators using the zero reference position description. *Int. J. Robotics Res.* 5(2):5–13.

Hansen, J. A., Gupta, K. C., and Kazerounian, S. M. K. 1983. Generation and evaluation of the workspace of a manipulator. *Int. J. Robotics Res.* 2(3):22–31.

Kumar, A., and Waldron, K. J. 1981. The workspaces of a mechanical manipulator. *ASME J. Mechanical Design* 103:665–672.

Lin, C.-C. D., and Freudenstein, F. 1986. Optimization of the workspace of a three-link turning-pair connected robot arm. *Int. J. Robotics Res.* 5(2):91–103.

Paden, B. E. 1985. Kinematics and control of robot manipulators. Ph.D. thesis, University of California, Berkeley, Department of Electrical Engineering and Computer Sciences.

Paul, R. P. 1981. *Robot Manipulators: Mathematics, Programming, and Control*. Cambridge, Mass.: MIT Press.

Roth, B. 1976. Performance evaluation of manipulators from a kinematic viewpoint. National Bureau of Standards, NBS Special Publication 495, pp. 39–61.

Roth, B. 1984. Screws, motors, and wrenches that cannot be bought in a hardware store. In *Robotics Research,* eds. M. Brady and R. Paul, Cambridge, Mass.: MIT Press, pp. 679–693.

Royden, H. L. 1968. *Real Analysis.* New York: Macmillan.

Tsai, Y. C., and Soni, A. H. 1985. An algorithm for the workspace of a general n-R robot. *ASME J. Mechanisms, Transmissions, and Automation in Design* 105:52–57.

Vijaykumar, R., Tsai, M.J., and Waldron, K. J. 1986. Geometric optimization of serial chain manipultor structures for working volume and dexterity. *Int. J. Robotics Res.* 5(2):91–103.

Yang, D. C. H., and Lee, T. W. 1983. On the workspace of mechanical manipulators. *ASME J. Mechanisms, Transmissions, and Automation in Design* 105:62–69.