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Manipulability of Robotic Mechanisms

Abstract

This paper discusses the manipulating ability of robotic mechanisms in positioning and orienting end-effectors and proposes a measure of manipulability. Some properties of this measure are obtained, the best postures of various types of manipulators are given, and a four-degree-of-freedom finger is considered from the viewpoint of the measure. The postures somewhat resemble those of human arms and fingers.

1. Introduction

Determination of the mechanism and size of a robot manipulator at the design stage and determination of the posture of the manipulator in the workspace (Gupta and Roth 1982) for performing a given task at the operation stage have been done largely on the basis of experience and intuition. One of various measures used for these determinations seems to be the ease of changing arbitrarily the position and orientation of the end-effector at the tip of the manipulator. It will be beneficial for design and control of robots and for task planning to have a quantitative measure of manipulating ability of robot arms in positioning and orienting the end-effectors. The concept "manipulability measure" has been proposed in a previous paper (Yoshikawa 1983) as one such measure.

In this paper, some properties of this measure and its utilization for determining the best postures of various types of manipulators and of an articulated robot finger are discussed.

2. Manipulability Measure

We consider a manipulator with n degrees of freedom whose joint variables are denoted by θ_i, $i = 1, 2, \ldots, n$. We assume that a class of tasks we are interested in can be described by m variables r_j, $j = 1, 2, \ldots, m$ $(m \leqq n)$ and that the relation between θ_i and r_i is given by

$$\mathbf{r} = f(\theta), \tag{1}$$

where $\theta = [\theta_1, \theta_2, \ldots, \theta_n]^T$ is the joint vector, $\mathbf{r} = [r_1, r_2, \ldots, r_m]^T$ is the manipulation vector, and the superscript T denotes the transpose. The manipulation velocity $\dot{\mathbf{r}}$ is related to the joint velocity $\dot{\theta}$ by

$$\dot{\mathbf{r}} = J(\theta)\dot{\theta}, \tag{2}$$

where $\dot{\mathbf{r}} \in R^m$ (m-dimensional Euclidean space), $\dot{\theta} = d\theta/dt \in R^n$, and $J(\theta) \in R^{m \times n}$ (the set of all $m \times n$ real matrices). The matrix $J(\theta)$ is called the Jacobian.

We assume that the following condition is satisfied:

$$\max_{\theta} \text{rank } J(\theta) = m. \tag{3}$$

Failing to satisfy this condition usually means that the selection of manipulation variables is redundant and that the number of these variables m can be reduced. When condition (3) is satisfied, we say that the degree of redundancy of this manipulator is $(n - m)$. More detailed discussion on the degree of redundancy and a related concept of redundant space is given in Hanafusa, Yoshikawa, and Nakamura (1981).

If, for some θ^*,

$$\text{rank } J(\theta^*) < m, \tag{4}$$

then we say that the manipulator is in a singular state. This state θ^* is not desirable because the manipulation vector $\mathbf{r}$ cannot move in a certain direction, meaning that the manipulability is seriously deteriorated.

To analyze this problem, Yoshikawa (1983) proposed the following quantitative measure of manipulability.

Definition: A scalar value w given by

$$w = \sqrt{\det J(\theta)J^T(\theta)} \tag{5}$$

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is called the manipulability measure at state θ with respect to manipulation vector $\mathbf{r}$.

The following three facts have been established in Yoshikawa (1983).

1. Let the singular value decomposition (Klema and Laub (1980) of J be

$$J = U \sum V^T, \tag{6}$$

where $U \in R^{m\times m}$ and $V \in R^{n\times n}$ are orthogonal matrices and

$$\sum = \left[\begin{array}{ccccc|c} \sigma_1 & & & & 0 & \\ & \sigma_2 & & & & \\ & & \cdot & & & 0 \\ & & & \cdot & & \\ 0 & & & & \sigma_m & \end{array}\right] \epsilon R^{m\times n}, \tag{7}$$

with

$$\sigma_1 \geqq \sigma_2 \geqq \cdots \geqq \sigma_m \geqq 0. \tag{8}$$

Then the measure w can be expressed as the product of the singular values $\sigma_1, \sigma_2, \ldots, \sigma_m$:

$$w = \sigma_1\sigma_2 \cdots \sigma_m. \tag{9}$$

2. We can show that the subset S_v of the realizable velocity $\dot{r}$ in the space R^m using joint velocity $\dot{\theta}$ such that $\| \dot{\theta} \|^2 = \dot{\theta}_1^2 + \dot{\theta}_2^2 + \cdots + \dot{\theta}_n^2 \leqq 1$ is an ellipsoid with principal axes $\sigma_1\mathbf{u}_1, \sigma_2\mathbf{u}_2, \ldots, \sigma_m\mathbf{u}_m$, where $\mathbf{u}_i \in R^m$ is the ith column vector of U, that is, $[\mathbf{u}_1\mathbf{u}_2 \cdots \mathbf{u}_m] = U$. This ellipsoid can be called the manipulability ellipsoid and could be a good means for the analysis, design, and control of robot manipulators. The volume of this ellipsoid is given by

$$d\sigma_1\sigma_2 \cdots \sigma_m, \tag{10}$$

where the constant d is given by

$$d = \begin{cases} (2\pi)^{m/2}/(2 \cdot 4 \cdot 6 \cdots (m-2) \cdot m), & \text{when } m \text{ is even}, \\ 2(2\pi)^{(m-1)/2}/(1 \cdot 3 \cdot 5 \cdots (m-2) \cdot m), & \text{when } m \text{ is odd}. \end{cases} \tag{11}$$

Therefore, w is equal to the volume of the manipulability ellipsoid except for the constant coefficient d.

3. When $m = n$, that is, when we consider nonredundant manipulators, the measure w reduces to

$$w = |\det J(\theta)|. \tag{12}$$

This type of measure has been used by Paul and Stevenson (1983) for analysis of robot wrists.

Now we establish several new properties of the measure.

When $m = n$, the measure has the following physical interpretation as well as that of number 2 above. The subset of the realizable velocity $\dot{\mathbf{r}}$ such that

$$|\dot{\theta}_i| \leqq 1, \qquad i = 1, 2, \ldots, m \tag{13}$$

is a parallelepiped in R^m, and its volume is $2^m w$. In other words, the measure w is proportional to the volume of the parallelpiped. This result can easily be obtained from a property of determinants.

Next we discuss the relation between the measure w and the maximum velocities of joints. So far, we have implicitly assumed that the maximum velocities of all joints are the same. When this assumption does not hold, the velocities of joints should be normalized. After fixing a set of units for distance, angle, and time (for example, m, rad, s), we denote the maximum (angular) velocity of joint i by $\dot{\theta}_{i0}$. We also select the desirable maximum (angular) velocity of each manipulation variable $\dot{r}_{j0}$ taking into consideration the class of tasks the manipulator is supposed to perform. Then, letting

$$\dot{\hat{\theta}} = [\dot{\hat{\theta}}_1, \dot{\hat{\theta}}_2, \ldots, \dot{\hat{\theta}}_n]^T, \quad \dot{\hat{\theta}}_i = \dot{\theta}_i/\dot{\theta}_{i0}, \tag{14}$$

$$\dot{\hat{\mathbf{r}}} = [\dot{\hat{r}}_1, \dot{\hat{r}}_2, \ldots, \dot{\hat{r}}_m]^T, \quad \dot{\hat{r}}_j = \dot{r}_j/\dot{r}_{j0}, \tag{15}$$

we obtain

$$\dot{\hat{\mathbf{r}}} = \hat{J}(\theta)\dot{\hat{\theta}}, \tag{16}$$

where

$$\hat{J}(\theta) = T_r J(\theta) T_\theta^{-1}, \tag{17}$$

$$T_r = \text{diag}\,[1/\dot{r}_{j0}] \in \mathbf{R}^{\mathbf{m}\times\mathbf{m}}, \tag{18}$$

$$T_\theta = \mathrm{diag}\,[1/\dot{\theta}_{i0}] \in \mathrm{R}^{n\times n}, \tag{19}$$

Since $|\hat{\dot{\theta}}_i| \leqq 1$ and $|\hat{\dot{r}}_j| \leqq 1$, we can define the manipulability measure w using the normalized Jacobian $\hat{J}(\theta)$.

Defining the measure w for $J(\theta)T_\theta^{-1}$ as $\hat{w}_\theta$, and the measure for $\hat{J}(\theta) = T_r J(\theta) T_\theta^{-1}$ as $\hat{w}$, we have

$$\hat{w} = \left[\prod_{j=1}^{m} (1/\dot{r}_{j0})\right] \hat{w}_\theta, \tag{20}$$

and, especially when $n = m$, we have

$$\hat{w} = \left[\prod_{i=1}^{n} (\dot{\theta}_{i0}/\dot{r}_{i0})\right] \hat{w}, \tag{21}$$

Hence the transformation (18) has only the effect of multiplying the scalar value $\Pi_{j=1}^{m} (1/\dot{r}_{j0})$. The relative shape of w as a function of θ is independent of the transformation T_r. Furthermore, when $n = m$, the relative shape of w is independent of both T_r and T_θ.

3. Best Postures of Various Robotic Mechanisms from the Viewpoint of Manipulability

3.1. Two-Joint Link Mechanism

In this section, we calculate the manipulability measure for various robotic mechanisms and determine the best postures and the best points in the workspace of these mechanisms from the viewpoint of manipulability. These will be called the *optimal postures* and the *optimal working positions.*

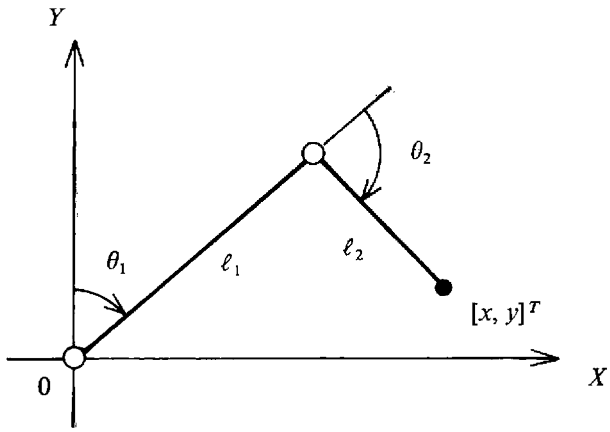
Consider the two-joint link mechanism shown in Fig. 1, which is the simplest case of multijoint manipulators. When the hand position $[x, y]^T$ is taken as the manipulation vector $\mathbf{r}$, the Jacobian matrix is given by

$$J(\theta) = \begin{bmatrix} \ell_1 c_1 + \ell_2 c_{12} & \ell_2 c_{12} \\ -\ell_1 s_1 - \ell_2 s_{12} & -\ell_2 s_{12} \end{bmatrix}, \tag{22}$$

where $c_1 = \cos\theta_1$, $c_{12} = \cos(\theta_1 + \theta_2)$, $s_1 = \sin\theta_1$, $s_{12} = \sin(\theta_1 + \theta_2)$. Hence the manipulability measure w is

$$w = |\det J(\theta)| = \ell_1 \ell_2 |\sin\theta_2| \tag{23}$$

Fig. 1. Two-joint link mechanism.



Therefore, the manipulator takes its best posture when $\theta_2 = \pm 90°$, for any given values of ℓ_1, ℓ_2, and θ_1. If the lengths ℓ_1 and ℓ_2 can be specified under the condition of constant total length, that is, $\ell_1 + \ell_2 =$ constant, the manipulability measure attains its maximum when $\ell_1 = \ell_2$ for any given θ_1 and θ_2.

When the human arm is regarded as a two-joint link mechanism by neglecting the degree of freedom of sideward direction at the shoulder and the degree of freedom of the wrist, it approximately satisfies the relation $\ell_1 = \ell_2$. Moreover, when we handle some object with our hands, the elbow angle is usually in the neighborhood of 90°. Hence it could be said that people are unconsciously taking the best arm postures from the viewpoint of manipulability.

3.2. SCARA-Type Robot Manipulators

Consider the SCARA-type manipulator with four degrees of freedom shown in Fig. 2. Let $\mathbf{r} = [x, y, z, \alpha]^T$, where $[x, y, z]^T$ is the hand position and α is the rotational angle of the hand about Z-axis. The Jacobian matrix for this case is given by

$$J(\theta) = \begin{bmatrix} \ell_1 c_1 + \ell_2 c_{1_2} & \ell_2 c_{12} & 0 & 0 \\ -\ell_1 s_1 - \ell_2 s_{1_2} & -\ell_2 s_{12} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}. \tag{24}$$

Fig. 2. SCARA-type robot.

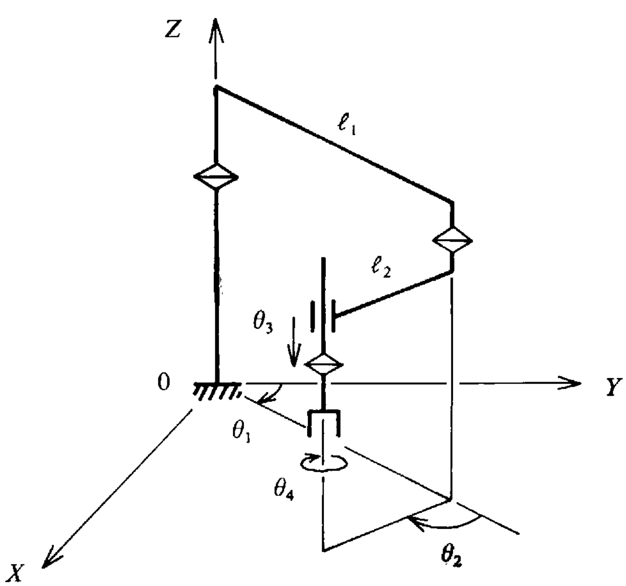
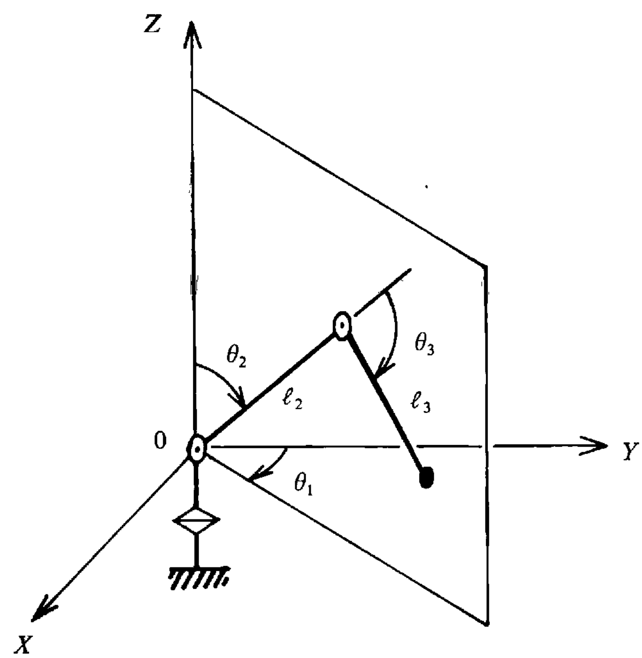


Fig. 3. PUMA-type robot.



Hence

$$w = \ell_1 \ell_2 |\sin \theta_2|.$$

Therefore, as in the case of two-joint link mechanism of Fig. 1, the best posture is given by $\theta_2 = \pm 90°$, for any given values of ℓ_1, ℓ_2, θ_1, and θ_3, and θ_4. Also under the constraint of $\ell_1 + \ell_2 =$ constant, the manipulability measure attains its maximum when $\ell_1 = \ell_2$, $\theta_2 = \pm 90°$. Notice that there are many commercial SCARA-type manipulators satisfying $\ell_1 = \ell_2$.

3.3. PUMA-Type Robot Manipulators

Most of the commercially available PUMA-type robot manipulators have five or six degrees of freedom. Many of them have links with some displacements in the direction of joint axes. However, we consider only the main three joints shown in Fig. 3, neglecting the degrees of freedom placed at the wrist and neglecting the displacements in the direction of joint axes. The joint vector is $\boldsymbol{\theta} = [\theta_1, \theta_2, \theta_3]^T$. The manipulation vector is taken to be $\mathbf{r} = [x, y, z]^T$. Then the Jacobian matrix is

$$J(\theta) = \begin{bmatrix} -s_1(\ell_2 s_2 + \ell_3 s_{23}) & c_1(\ell_2 c_2 + \ell_3 c_{23}) & c_1 \ell_3 c_{23} \\ c_1(\ell_2 s_2 + \ell_3 s_{23}) & s_1(\ell_2 c_2 + \ell_3 c_{23}) & s_1 \ell_3 c_{23} \\ 0 & -(\ell_2 s_2 + \ell_3 s_{23}) & -\ell_3 s_{23} \end{bmatrix}, \tag{25}$$

and the manipulability measure is

$$w = \ell_2 \ell_3 |(\ell_2 s_2 + \ell_3 s_{23}) s_3|. \tag{26}$$

The best posture for given ℓ_2 and ℓ_3 is obtained as follows. First, θ_1 is not related to w and can take any value. Second, from $\partial w / \partial \theta_2 = 0$ we have

$$\tan \theta_2 = \frac{\ell_2 + \ell_3 c_3}{\ell_2 s_3}. \tag{27}$$

This means that the tip of the arm should be on the X, Y-plane, that is, at the same height as the second joint. This can further be interpreted as maximizing the

Fig. 4. Best arm posture for PUMA-type robot.

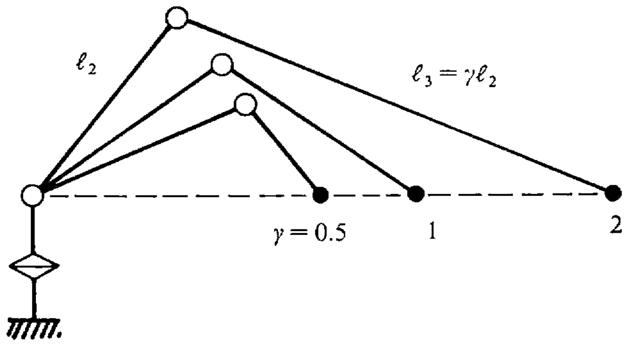
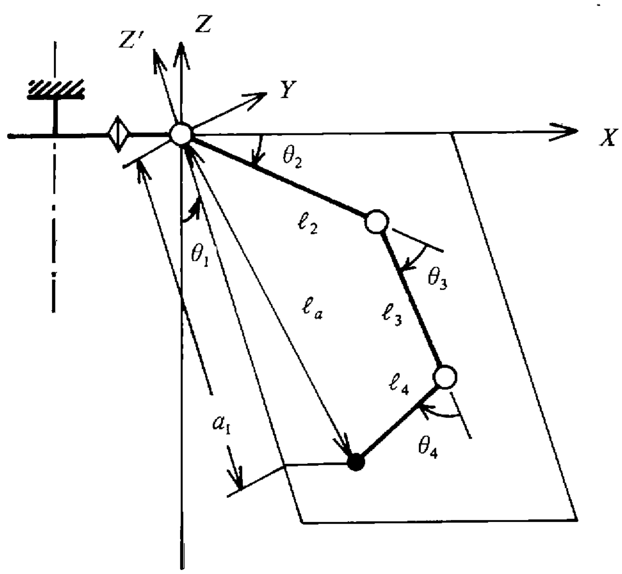


Fig. 5. Finger with four degrees of freedom.



contribution of the angular velocity of the first joint to the manipulability measure.

Substituting Eq. (27) into Eq. (26) yields

$$w = \ell_2\ell_3\sqrt{\ell_2^2 + \ell_3^2 + 2\ell_2\ell_3 c_3}\,|s_3|. \tag{28}$$

The value of θ_3 that maximizes w is

$$\cos\theta_3 = \frac{\sqrt{(\ell_2^2+\ell_3^2)^2 + 12\ell_2^2\ell_3^2} - (\ell_2^2 + \ell_3^2}{6\ell_2\ell_3}. \tag{29}$$

Figure 4 shows the best postures for the cases $\ell_3 = \gamma\ell_2$, $\gamma = 0.5$, 1, 2 (only those satisfying $0° \leqslant \theta_2 \leqslant 90°$ are shown in the figure). If the manipulator is regarded as a two-joint mechanism consisting of θ_2 and θ_3, the optimal angle for θ_3 is 90°, from the discussion in Section 3.1. In the present case, however, the optimal θ_3 is smaller than 90° because the contribution of $\dot{\theta}_1$ to w can be made larger by placing the tip of the arm farther from the first joint axis.

3.4. Orthogonal, Cylindrical, and Polar Coordinate Manipulators

Only the main three degrees of freedom of manipulators and the hand position are considered in this section, as in the previous section.

The manipulability measure w of orthogonal coordinate manipulators is 1 everywhere in the workspace. The best posture of cylindrical coordinate manipulators is attained when the arm is fully stretched out. Similarly, the best posture of polar coordinate manipulators is attained when the arm is fully stretched out in the horizontal direction. Hence, for cylindrical and polar coordinate manipulators, the best posture is attained on the boundary of their workspace. Although this is inconvenient, it is true for all robotic mechanisms that have a prismatic joint whose axis is in the radial direction of a rotational joint. This is also intuitively understandable. For these manipulators, considerations other than the manipulability measure will be necessary to determine the best working position.

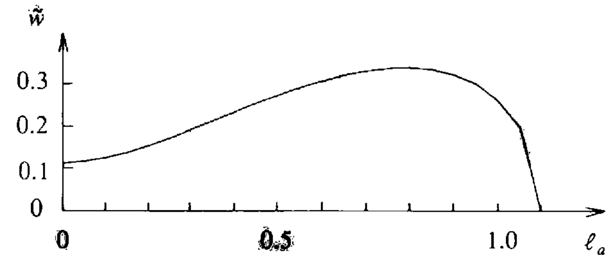
3.5. Four-Joint Robotic Finger

Various robotic hands with multi-articulated fingers have been developed (Okada 1980; Salisbury and Craig 1982; Hanafusa, Yoshikawa, and Nakamura 1983) to realize the dexterity and flexibility of human hands in handling and assembling jobs. In this section the four-joint finger shown in Fig. 5 is considered from the viewpoint of the manipulability measure.

The Jacobian matrix relating $\theta = [\theta_1, \theta_2, \theta_3, \theta_4]^T$ to $\mathbf{r} = [x, y, z]^T$ is

$$J(\theta) = \begin{bmatrix} 0 & -a_1 & -a_2 & -a_3 \\ c_1 a_1 & s_1 b_1 & s_1 b_2 & s_1 b_3 \\ s_1 a_1 & -c_1 b_1 & -c_1 b_2 & -c_1 b_3 \end{bmatrix}, \tag{30}$$

Fig. 6. Maximum value of $\tilde{w}$ as a function of ℓ_a.



where

$$
\begin{aligned}
a_1 &= \ell_2 s_2 + \ell_3 s_{23} + \ell_4 s_{234} \\
a_2 &= \ell_3 s_{23} + \ell_4 s_{234} \\
a_3 &= \ell_4 s_{234} \\
b_1 &= \ell_2 c_2 + \ell_3 c_{23} + \ell_4 c_{234} \\
b_2 &= \ell_3 c_{23} + \ell_4 c_{234} \\
b_3 &= \ell_4 c_{234}
\end{aligned}
$$

The manipulability measure is calculated as

$$w = |a_1| \tilde{w}(\theta_2, \theta_3, \theta_4), \tag{31}$$

where

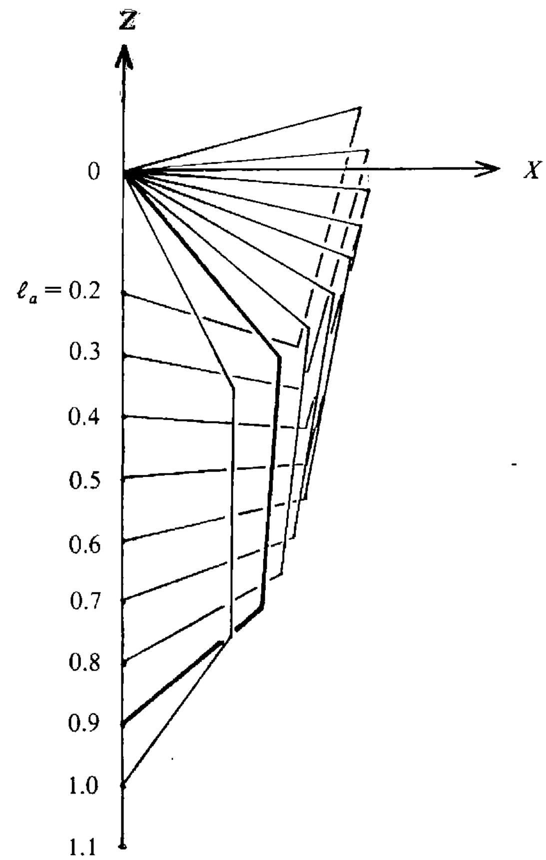
$$\tilde{w}(\theta_2, \theta_3, \theta_4) = \sqrt{\det \tilde{J}\tilde{J}^T}, \tag{32}$$

$$\tilde{J} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}. \tag{33}$$

The maximum value of $\tilde{w}(\theta_2, \theta_3, \theta_4)$ for a given distance ℓ_a between joint 2 and the tip of the finger in Fig. 5 is shown in Fig. 6 for the case $\ell_2 = \ell_3 = 0.4$, $\ell_4 = 0.3$. The corresponding finger posture is shown in Fig. 7. Notice that these postures are independent of the angle θ_2. Figure 8 shows the maximum value of w as a function of the finger tip position in the X,Z'-plane (only the lower half-portion is shown since the value in the upper half is symmetric with respect to the X-axis). The best finger posture is also shown in Fig. 8 by a broken line.

Notice that the finger postures given in Fig. 7 are quite similar to those taken by human fingers during the manipulation of small objects. Hence, we can expect that these postures would be useful in determining the grasping posture of a robotic hand with several four-degree-of-freedom fingers.

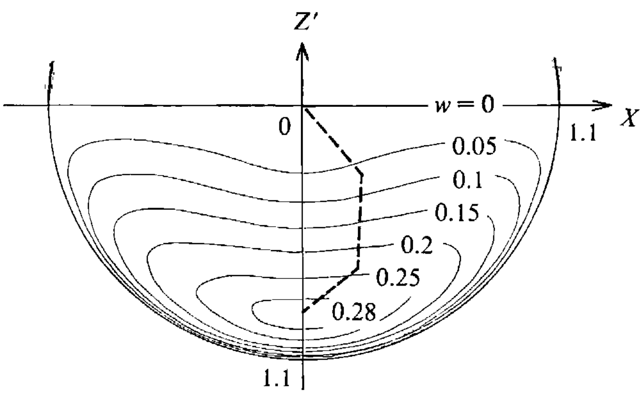
Fig. 7. Best finger posture.

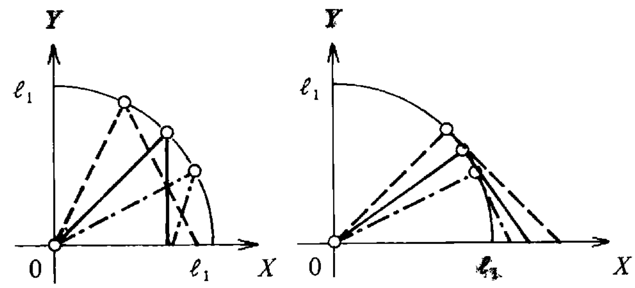


Salisbury and Craig (1982) have used the condition number $c(J^T(\theta))$ of the transpose of the Jacobian matrix $J(\theta)$ as a measure of workspace quality. This is a measure of the accuracy with which forces can be exerted. Hence, the manipulability measure w discussed in this paper is quite different from the condition number. For example, Fig. 9 compares the best postures of a simple two-joint link mechanism for manipulability measure and for condition number. The figure clearly shows the difference between the two measures. Note, however, that the condition number can also be interpreted as a measure of directional uniformity of the manipulability illipsoid. Further study is necessary on this aspect of the condition number.

Fig. 8. *Maximum value of w as a function of the finger tip position.*

Fig. 9. *Comparison of the best postures of two-link mechanism.* (*a*) $c(J^T)$; (*b*) *w*.





4. Conclusion

Properties of the manipulability measure, which was proposed in a previous paper (Yoshikawa 1983) as a measure of manipulating ability of robotic arms in positioning and orienting end-effectors, have been studied. Utilization of the manipulability measure for determining the best postures of various types of manipulators and an articulated robot finger has been discussed. The best postures obtained have some resemblance to those taken by human arms and fingers. These postures will be useful for planning the working position of robots for various tasks.

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