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# Manipulator Performance Measures - A Comprehensive Literature Survey

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**Abstract** Performance measures are quintessential to the design, synthesis, study and application of robotic manipulators. Numerous performance measures have been defined to study the performance and behavior of manipulators since the early days of robotics; some more widely accepted than others, but their real significance and limitations have not always been well understood. The aim of this survey is to review the definition, classification, scope, and limitations of some of the widely used performance measures. This work provides an extensive bibliography that can be of help to researchers interested in studying and evaluating the performance and behavior of robotic manipulators. Finally, a few recommendations are proposed based on the review so that the most commonly noticed limitations can be avoided when new performance measures are proposed.

**Keywords** Performance index · Manipulator metrics · Local · Global · Kinematic · Dynamic · Intrinsic · Extrinsic

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## 1 Introduction

### 1.1 Goals

This survey paper does not put forth any novel research ideas, approaches or results. The goal of this work is to document, classify and, discuss the scope and the merits and demerits of various performance indices used to quantify the behavior and performance of robotic manipulators. The contribution of this paper derives from consolidating and summarizing the great volume of published research in this domain by means of an extensive bibliography, conducted at various institutions over an extended period of time.

There have been very few other such surveys in this area. For example [1], by Klein and Blaho and [2] by Tanev and Stoyanov are limited study of local performance indices based on the Jacobian [3], by Merlet that is a study of performance indices applied to parallel manipulators, and the very recently published [4] by Moreno, et al. This paper, however, is more elaborate, up-to-date and reflects the current state-of-the-art in this field. This paper provides an elaborate discussion of the indices, their classification, scope and their inherent limitations. Various proposed improvements to overcome few of the limitations of these indices have also been reviewed and cited in this survey. This survey can be a useful guide to any robotic researcher involved in designing, performance evaluation, and application of robotic manipulators by providing a better understanding of the indices.

### 1.2 Definition

Doel and Pai [5] defined performance measures as a *"field defined on the configuration manifold, i.e. the space of all postures of the manipulator, that measures some general property of the manipulator"*.

Performance indices are metrics designed to measure and quantify the different performance characteristics of a robotic manipulator in its workspace. Performance indices help researchers study, evaluate and optimize manipulator designs and application. Performance indices are also needed to compare the architectures and performance of two manipulators applied to the same task.

### 1.3 Motivation

In [6] Chang notes, *"a quantitative measure provides us with a rational basis upon which we can, without having to rely on experience and intuition alone, analyze, design, and control the systems"*. Performance indices or parameters are commonly used metrics in the design, synthesis, task planning, and performance evaluation of robotic manipulators. These metrics help specify task requirements in order to optimize manipulator structures. They also help in task planning and optimal task placement. In case of redundant manipulators that have multiple inverse kinematic solutions, performance metrics help choose the best solution based on a specific criterion, that is, pick the best posture to perform a given task. The design of optimal task-oriented manipulators is based on minimizing or maximizing such performance indices.

Numerous performance indices have been defined since the early days of robotics; some have been more widely accepted than others. The design and development of highly dexterous robots have fostered the formulation of these performance indices but their real significance, scope and limitations have not always been well understood.

Most indices have some inherent limitations. Hence, a better understanding of these indices is needed so that they are applied in the right context and their value significance is properly understood. This literature and bibliography can serve as a guide for deciding the right performance parameter to study and to evaluate the performance of a manipulator.

### 1.4 Paper Organization

Although it is tempting to organize the indices by their scope or kinematic property like few of the previous survey on this subject, this paper follows a "mathematical derivation" approach in discussing three of the most important metrics – manipulability index, minimum singular value, and condition number. Such an approach helps in better understanding of the parameters as the both local and global versions of the same metric and their variations can be discussed together.

The rest of this paper is organized as follows: First, a broad classification of indices is presented in Section 2 based on three different criteria. Section 3 discusses the manipulator workspace as a performance measure. Sections 4 to 9 discuss joint performance indices, such as service angle, dexterity index, etc. Section 10 discusses the Jacobain matrix, its importance and its dependencies. Next in Section 11, one the most widely used performance measure manipulability index is discussed in great detail with its limitations and suggested improvements and variations. In Sections 14 and 15 minimum singular value and relative minimum singular value are discussed. Next in Sections 16 to 19, another important metric the condition number is discussed with its limitations and other suggested variations. In Sections 20 to 24, more measures for measuring manipulator Isotropy are presented. Next, manipulator redundancy indices are discussed in Sections 25 to 28, followed by task based performance indices in Sections 29 and 30. A few other interesting performance indices are present in Sections 31 to 37. Finally, a unified approach for defining performance measures is discussed.

Next, based on the literature review, a few recommendations are made so that the commonly noticed limitations of indices are avoided while defining new performance metrics. Finally, the conclusion summarizes the survey.

In some robotic research literature the words 'dexterity' and 'manipulability' have been used ambiguously [6], however, in this paper dexterity strictly refers to all possible orientations of the end-effector about a point in the manipulator workspace, and manipulability refers to the ability of the manipulator to move and apply forces in arbitrary directions.

## 2 Classification of Performance Indices

Performance indices have been broadly classified either based on their scope (local, global), performance characteristic of the manipulator (kinematic, dynamic, neither) or application (intrinsic, extrinsic). Every performance index can be categorized based on the following three groups.

### 2.1 Local vs. Global Indices

Local indices are performance metrics that are dependent on the posture of the manipulator, and are also known as posture-dependent indices. The scope of such local indices is confined to a particular manipulator posture or position alone in the workspace i.e., they demonstrate a local property of the manipulator. The value of these local indices varies from point-to-point or from posture-to-posture. These posture dependent indices play a very important in manipulator control applications. Most Jacobian based performance indices are local indices as the Jacobian matrix depends on the posture of the manipulator at a given point in the workspace, for example, manipulability index, condition number, etc.

Global indices are posture independent indices. They represent a global characteristic of the manipulator's workspace. Global indices are needed to compare the structure and behavior of two manipulators that perform the same task. Unlike local indices, global performance indices have a single value for a given manipulator workspace. Some global performance indices have been formulated by extending the definition of local indices. Local indices can be adapted to global scale by integrating the local measures over the region of the configuration space [7, 8], for example, the Global Conditioning Index (GCI). Such global indices measure the overall performance of the manipulator in some average sense. It is important to note here that, high values of a local performance index do not always translate into high values for its global version [9], and the converse is also true [10].

Kline and Blaho [1] have commented that local performance indices are more important than global indices because the performance of the manipulator's end-effector at the task point(s) is more important than its performance over a trajectory, region or the entire workspace.

### 2.2 Kinematic vs. Dynamic Indices

Kinematic indices are metrics that quantify the kinematic behavior of the manipulator. Kinematic performance indices are based on the Jacobian matrix. There is a close relationship between the kinematic performance and the manipulator structure [9, 11], hence, kinematic performance indices are structure-dependent.

In [12], authors introduced the term kinestatics to mean *"the dualistic properties and relations between the first order kinematics and statics of a rigid body"*. The study of kinetostatic behavior of manipulators had been the focus of extensive research for the last two decades of the last century; during this period many kinematic performance measures were defined [13].

Metrics that evaluate the dynamic performance of the manipulator are classified as dynamic performance indices. The dynamic performance of a robotic manipulator strongly depends on its inertial characteristics [14].

There are other indices that are neither kinematic nor dynamic, i.e., they are simple indices, such as dexterity index, service angle, etc.

### 2.3 Intrinsic vs. Extrinsic Indices

Performance indices that are unrelated to the manipulator's task or application are intrinsic indices [15]. Intrinsic indices convey the inherent characteristics of the manipulator. Intrinsic indices are independent of manipulator task specifications. Metrics such as dexterity index, manipulability, and condition number are examples of intrinsic indices.

Extrinsic indices measure the ability of the manipulator to perform a specific task. Extrinsic performance indices are directly related to the manipulator's task [15], for example the robot-task conformance index, power manipulability index, etc.

## 3 Manipulator Workspace as a Performance Index

Early works on quantifying a manipulator's performance were focused on the study of its workspace and singularities [16–20], and continues to be a topic of great interest to researchers [9, 21–25]. Many analytical and numerical methods were formulated to determine the workspace of manipulators since the mid 1970's; however, a satisfactory, computationally cost effective and generalized method is yet to be formulated [26]. The workspace of a manipulator has also been referred to in literature as work-volume or work-envelope [9].

Determining the workspace boundaries, its singularities, voids, holes, contours of dexterity and manipulability are essential for the design and performance analysis of robotic manipulators [21]. Depending on the number of degrees of freedom, their relative alignment in the manipulator structure, and length of the links, the workspace of a manipulator can either be flat, cylindrical or sometimes very complex, consisting of voids and holes, for example a four DoF revolute (4R) manipulator [27].

The study of workspace is essential for the optimal placement of the work piece and achieving high manipulator dexterity [28]. The workspace is also an important criterion for comparing manipulator structures [9]; it can be thought of as a global measure. Many researchers have used the manipulator workspace as an important performance index [9, 29] to optimize manipulator structure.

A well conditioned and dexterous workspace is a desirable characteristic for all manipulators. As the manipulator's performance is not uniform over its entire workspace, depending on its behavior, researchers have classified workspace volumes into different types. Broadly, the workspace of a manipulator is classified into two types: the reachable workspace and dexterous workspace. Before applying a manipulator to a specific task it is important that the task points are encompassed by the reachable workspace of the manipulator, and possibly even the dexterous workspace. The following are some of the definitions of workspaces based on the manipulator's behavior:

3.1. **Reachable Position Workspace:** The reachable workspace, as defined by Gupta and Roth [17], is the set of points that can be reached by a reference point on a manipulator with at least one orientation and does not include singular points where the manipulator loses one or more degrees of freedom [9].

3.2. **Dexterous Workspace / Full Orientation Angle Workspace:** The fully dexterous workspace or the full orientation angle workspace is defined as a space in which a point is approached in all directions. For these points, the range of approach angles is 360 degrees [16]. For any point in the dexterous workspace the manipulator's end-effector can be *"completely rotated about any (every) axis through that point"* [30]. The dexterous workspace is a subset of the reachable workspace.

3.3. **Orientation Angle Workspace:** The orientation angle workspace is the set of angle ranges with which the end-effector can reach with a certain orientation for any point in the reachable position workspace.

3.4. **Partial Orientation Angle Workspace:** The partial orientation angle workspace is defined as a space in which a point can be approached by a range of angles that is less than 360 degrees.

3.5. **Operating Volume:** The operating volume is the total volume of space that the manipulator and its links occupy while reaching every point in the workspace.

3.6. **Workspace Index:** The Workspace Index (WSI) quantifies the points in the workspace that can be attained by the manipulator without exceeding any physical limitations. In the discrete form the WSI is given as [10]:

$$WSI = \frac{n_{ws}}{n_G} \in [0, 1]$$

where $n_G$ are the nodes in the objective space and $n_{ws}$ are the nodes that are feasible. The WSI is a bounded global index that can have values between zero and one. The WSI represents the percentage of the objective workspace that is physically reachable by the manipulator.

**Joint Performance Indices**

## 4 Service Angle and Service Sphere

One of the simplest non-kinematic performance measures is Service Angle [2] or Service Region [31]. Introduced by Vinogradov [32], Service Angle/Region is defined as *"the range of the approach angle of the manipulator around a given point in the workspace"* [32]. Yang et al. [18] improved upon the concept of Service Angle and proposed the concept of Service Spheres. The service sphere is defined as a sphere about a given point in the workspace that *"may be*

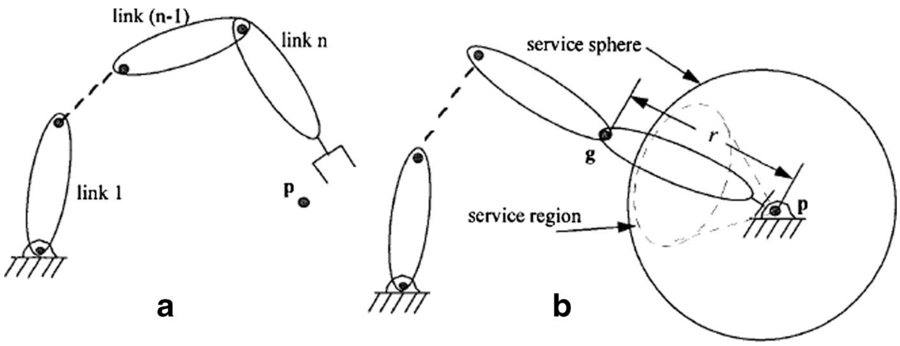


**Fig. 1** **a** Serial n-degree of freedom manipulator; **b** Service sphere and Service region [31]. (© [1999] Mechanism and Machine Theory, Elsevier). (© [1999] Mechanism and Machine Theory, Elsevier) Reproduced from [31] with permission from Dr. K. A. Abdel-Malek

*used to detect all possible penetrations of the end-effector through it"* [31]. Figure 1 shows the (a) Serial n-degree of freedom manipulator and (b) Service region and service sphere for an n-DoF serial manipulator about a point 'p'. Both Service Angle and Service Sphere are local performance measures.

At any given point in the workspace the service sphere can have multiple service regions. To determine the dexterity of the manipulator at a point it is essential to know all the service regions/angles about that point. In [31] the authors proposed an effective method for determining service regions.

## 5 Dexterity Index

Dexterity Index was proposed by Kumar and Waldron [16] as a performance measure. They defined dexterous workspace as *"the volume within which every point can be reached by the manipulator end-effector with any desired orientation"* [2]. The dexterity index of a manipulator at a point in the workspace can be also defined as "*a measure of a manipulator to achieve varying orientations at that point*."

The orientation of a manipulator at any given point in the workspace can be represented in terms of the yaw ($\alpha$), pitch ($\beta$) and roll ($\gamma$) angles as:

$$R_{xyz} = R_{x,\gamma} R_{y,\beta} R_{z,\alpha} \tag{1}$$

All three of the angles have a range 0–$2\pi$ to provide all possible orientations. The dexterity index can be defined as the summation of the dexterity indices about each of the axes [16] given by:

$$D = \frac{1}{3}\left(d_x + d_y + d_z\right) \tag{2}$$

$$D = \frac{1}{3}\left(\frac{\Delta\gamma}{2\pi} + \frac{\Delta\beta}{2\pi} + \frac{\Delta\alpha}{2\pi}\right) \tag{3}$$

where $d_x$, $d_y$, and $d_z$ are X, Y and Z dexterity indices, and $\Delta\alpha$, $\Delta\beta$, and $\Delta\gamma$ are the range of possible yaw, pitch and roll angles about a point Therefore, points in the workspace with multiple inverse kinematic solutions will have a higher dexterity index when compared to points with unique solutions. The manipulator is said to be fully dexterous at a given point if the dexterity at that point is equal to unity. Dexterity is a desirable characteristic of manipulators whether redundant or not [33].

The mean dexterity index ($D_{Mean}$) of a manipulator over a given region of the workspace or trajectory with $N$ points can be defined as [16]:

$$D_{Mean} = \frac{\sum D}{N} \tag{4}$$

Similarly, an area or point in the workspace can be said to be completely X-dexterous or Y-dexterous if $d_x$ or $d_y$ is equal to unity. In the case of a planar manipulator operating in the XY – plane.

$$d_x = d_y = 0 \tag{5}$$

As seen in the Eq. (3), the dexterity index can vary between a minimum of 0 to a maximum of 1, and hence, is a dimensionless and bounded metric. Figure 2 shows the distribution of the dexterity index for a SCARA manipulator.

The dexterity index depends on the manipulator structure. For a serial planar manipulator with revolute joints, the manipulator has maximum dexterity when the last link in the chain is also the shortest link [34].

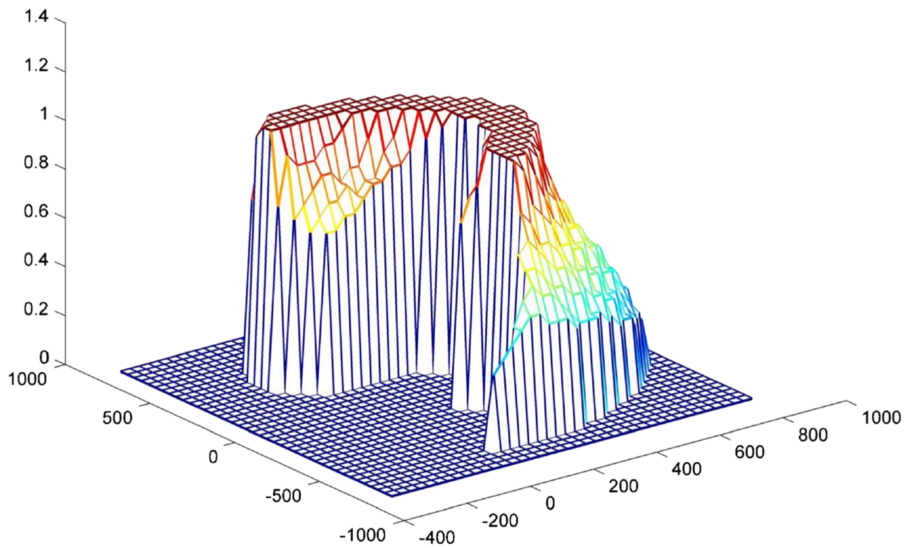


**Fig. 2** Shows the distribution of the Dexterity index for a SCARA manipulator [2]. (© [2000] PECR). (©[2000] Problems of Engineering, Cybernetics & Robotics (PECR), Bulgarian Academy of Sciences), Reproduced from [2] with permission from Dr. Tanio Tanev

## 6 Joint Range Availability

Leigeois [35] proposed a simple measure to determine if a joint would reach a stop, known as the Joint Range Availability (JRA) index, given as:

$$JRA = \sum (\theta_i - \theta_{ci})^2 \tag{6}$$

Where as in [36] authors use the sum of the squares.

$$JRA = \sum \left( \frac{\theta_i - \theta_{ci}}{\theta_{i,\max}} \right)^2 \tag{7}$$

Where $\theta_i$ is the current joint angle, $\theta_{ci}$ is the center of range of travel and $\theta_{i,max}$ is the maximum joint extrusion

The Joint Range Availability (JRA) index depicts a relation between the joint displacement and maximum joint displacement. JRA index tracks the deviation of the joint angles from their mid range [37]. This measure is used to study the naturalness or evenness of the joint range distribution [1].

In [37] authors presented a normalized form of the joint range availability given as:

$$\psi_{JRA} = \sum_{i=1}^{n} \frac{\left(\theta_i - \theta_i'\right)^2}{\theta_{i,\max}} \tag{8}$$

where $\theta_i, \theta_i', \theta_{i,\max}$ are the joint displacement, mid-range displacement, and the displacement at joint limit.

## 7 Indices for Joint Mid-Range Proximity

In most serial robotic manipulators the motion of the joint is often limited either due to mechanical constraints or space constraints. The motion of the joints is often range bounded as:

$$\theta_{\min} \leq \theta \leq \theta_{\max} \tag{9}$$

Few measures have been defined for the avoidance of joint limits and to maintain the joint displacement as close to mid-range as possible. These joint range parameters have special importance in automated motion control applications where the motion algorithm has to take into consideration the fact that the joint's motion is limited to various degrees.

In [38, 39] Baron defined the following objective function to maintain the manipulator joints close to the mid-range position:

$$z = \frac{1}{2} \left(\theta - \bar{\theta}\right)^T W \left(\theta - \bar{\theta}\right) \tag{10}$$

where $W$ is a positive-definite weighing matrix, $\theta$ is the current joint position, $\bar{\theta}$ is the mid-range joint position computed as:

$$\bar{\theta} = \frac{1}{2} (\theta_{\min} + \theta_{\max}) \tag{11}$$

The weighing matrix $W$ is a diagonal matrix that represents allowable deviations from the mid-range joint position. In [38] Baron applied this formulation to control the motion of arc-welding robots.

Another similar approach is the parameterization of joint angles. Joint angle parameterization is often implemented in manipulator optimization and motion algorithms [31, 40].

$$\theta = a + bSin(\lambda) \tag{12}$$

where

$$a = \frac{1}{2}(\theta_{\min} + \theta_{\max}) \text{ and } b = \frac{1}{2}(\theta_{\min} - \theta_{\max})$$

For an n-DOF manipulator its joint position vector $q = [q_1, q_2 \ldots q_n]$ can be represented in parameterized forms as $\lambda = [\lambda_1, \lambda_2, \ldots \lambda_n]$. $\lambda$ is also known as slack variables.

## 8 Joint Velocity Measure

The Joint Velocity Measure (JVM) represents the magnitude of joint velocity by computing the displacement to be traveled, given by [37]:

$$\psi_{JVM} = \sum_{i=1}^{n} (\theta_i - \theta_{i,curr})^2 \tag{13}$$

where $\theta_i, \theta_{i,curr}$ are the desired and current joint displacement respectively.

## 9 Manipulator Velocity Ratio

Velocity ratio is a commonly used criterion in the kinematic performance evaluation of single-input-single-output (SISO) mechanisms, such as four-bar chains. This concept of velocity ratio can be extended to study the kinematic behavior of a robotic manipulator's multiple-input-multiple-output (MIMO) mechanisms [41].

In [42], Dubey and Luh proposed Manipulator Velocity Ratio (MVR) as a performance measure for kinematic evaluation of manipulators. MVR is defined as the ratio of transformed end-effector velocity norm to the joint velocity vector norm, given as:

$$r_v = \sqrt{\frac{\dot{x}_v^T \dot{x}_v}{\dot{\theta}_v^T \dot{\theta}_v}} \tag{14}$$

such that

$$\dot{x}_v = \sqrt{W_x}\dot{x} \text{ and } \dot{\theta}_v = \sqrt{W_\theta}\dot{\theta}$$

where $W_x$ and $W_\theta$ are weighing matrices.

The value of MVR ($r_v$) depends on the manipulator configuration, the two weighing matrices and also the direction of the end-effector velocity vector. MVR is a posture dependent local performance metric.

**Manipulability Measures**

## 10 The Jacobian Matrix

The Jacobian matrix an indispensable matrix in understanding the motion of the end-effector, hence most kinematic performance measures are based on the Jacobian and its evaluations. Since most metrics are based on the Jacobian, and its evaluations such as its determinant, Eigenvalues, Singular values, determinant etc., it is important at this point to make a note of the inherent limitations of the Jacobian, in order to avoid repetition later in this paper. Most Jacobian-based performance indices suffer from a few, very significant limitations, like scale dependence, non-homogeneity of the Jacobian and unbounded nature of the metric [2, 11, 43].

10.1. **Scale Dependency:** The manipulability index is scale or units dependent. The result of various Jacobian evaluations heavily depends on the choice of the physical units used, therefore the manipulability index will have different values for different units used to represent the link lengths and joint angles [2]. In [44], authors showed that the absolute of the determinant of the Jacobian is not a robust measure of invertibility, because the determinant can sometimes have very large values.

10.2. **Dimensional dependency:** It is simple to calculate the determinant of the Jacobian when it is homogeneous, i.e. when the units are the same. But in the case of complex manipulator structures consisting of both prismatic and revolute joints, the Jacobian becomes non-homogeneous [7] due to the different units used for translating and rotating degrees of freedom. In such cases, the evaluation of the Jacobian's determinant, its Eigenvalues, singular values, etc. becomes physically inconsistent [7, 45] and non-commensurable [45]. In [45] the authors claim that the robotics research community is largely unaware of the physical inconsistency of the Jacobian.

Hence, performance indices based on the Jacobian are most accurate when the manipulator consists of the same type of degrees of freedom either prismatic or revolute, but not a combination of both [7, 12]. Further, even if the joints are the same, translational and rotational velocities should not be combined in the same performance metric [46, 47]. The use of a non-homogeneous Jacobian in manipulator control applications can be problematic [48].

To resolve the problem of dimensional non-homogeneity of the Jacobian matrix, the concept of 'characteristic length' was applied by Angeles [49–51]. The characteristic length is a normalizing length. The Jacobian's bottom three rows that represent the position are divided by the characteristic length to obtain a dimensionless homogeneous Jacobian. The singular values of this Jacobian are dimensionless also. The characteristic length is derived such that it minimizes the singular values of the normalized Jacobian [49, 52, 53]. The characteristic length that produces the best performance measure was called the *'Natural Length'* [54]. The method for calculating the characteristic length for different types of manipulators can be found in [55]. A major drawback of the concept of characteristic length is that it lacks a geometrical interpretation [7].

10.3. **Frame Dependency:** The Jacobian matrix is not invariant with respect to change in the reference frame [56, 57]. Hence measures based on the Jacobian and its evaluations, such as singular values, Eigenvalues, etc., with the exception of the determinant of the Jacobian [57], are also not invariant to the changes in the reference frame.

Most methods proposed to make the Jacobian invariant to the changes in units, scale, and frame involve arbitrary assumptions that make Jacobian evaluations unsuitable for characterizing the performance the manipulator.

## 11 Manipulability Index

The manipulability index was proposed as a kinematic performance measure by Yoshikawa [58]. The Yoshikawa manipulability index happens to be the most widely accepted and used measure for kinematic manipulability [57]. Like most kinematic indices the manipulability index is based on the manipulator's Jacobian matrix.

For a redundant manipulator the manipulability index is defined as the square root of the determinant of the product of the Jacobian matrix and its transpose.

$$\mu = \sqrt{\det\left(J.J^T\right)} \tag{15}$$

The Jacobian manipulability can also be expressed as:

$$\mu = \sqrt{\lambda_1\lambda_2\lambda_3\ldots\lambda_m} = \sigma_1\sigma_2\sigma_3\ldots\sigma_m \tag{16}$$

where $\lambda_i$ is the Eigenvalue of $\left(J.J^T\right)$ matrix and $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots \geq \lambda_m$

$\sigma_i$ is the singular value of $(J)$ matrix

In case of non-redundant manipulators the Jacobian is a square matrix and the manipulability index is equal to the absolute of the determinant of the Jacobian [2, 11]

$$\mu = |\det(J)| \tag{17}$$

In [59], Paul and Stevenson, had also used the absolute of the Jacobian determinant to study the performance of spherical wrists. The determinant of the Jacobian being equal to zero is a necessary and sufficient condition for the existence of a singularity [60]. At the same time, the manipulability index "does not represent a measure of distance from a singularity" [57]. The manipulability index is equal to zero if the Jacobian is not full rank [61]. Another disadvantage of using the manipulator determinant is that once the Jacobian loses its full rank, it does not distinguish between one type of singularity from the other, as both determinants are zero [6].

There is a debate about whether or not the manipulability is a measure of the degree of ill-conditioning of the workspace. Authors in [11] state that the manipulability measure defines the degree of conditioning of the workspace, but in [57, 62, 63] authors have cautioned that, even though the determinant diminishes in value in the proximity of a singularity, it cannot be considered as a good measure for the degree of ill-conditioning of the manipulator.

In [64] the authors have showed that the manipulability index is base invariant, that is, it is independent of the manipulator's first DoF and depends solely on the relative positioning of the links and geometry of the manipulator structure. It is also independent of the location of the operation point [13, 64], where the operation point is defined as the point on the end-effector up which the Jacobian is based [13]. This is another drawback of the manipubility index as it fails to distinguish between a long end-effector and a short one. The manipulability index is independent of the task-space coordinates [64].

The manipulability index is one of the most commonly used performance indices [43, 57, 58]. In [65] authors have argued that the manipulability index is a better indicator of dexterity than condition number or minimum singular value. Because the manipulability index considers the motion of the end-effector in all directions while the minimum singular value and condition number consider motion in only one and two directions. And unlike the condition number and minimum singular value, the manipulability index is independent of any changes in the reference frame [57].

The manipulability index is widely used for manipulator synthesis, workspace optimization, task planning, motion control etc. The manipulability index helps estimate the overall manipulator sensitivity to actuator displacement, which is an important design criteria [47]. Since the manipulability index depends on the structure and the posture of the manipulator at a given point, it is a local [11, 66] and intrinsic performance measure [15].

It is important to mention here that Jacobian based performance indices are also extensively used to study the performance and behavior of parallel manipulators. For example the Quality Index for parallel manipulators that is very similar to the Manipulabilty Index in the case of serial manipulators.[1] In [67, 68] authors Lee et al. proposed the Quality Index as a dimensionless bounded performance metric for an octahedral manipulator and a modified Quality Index was suggested in [69].

[1]Clarified via correspondences with the author (Dr. Tanio Tanev)

11.1. **Limitations**: Few of the significant limitations of the Yoshikawa manipulability index are discussed below:

11.1.1 **Scale (Units) Dependency:** The manipulability index is scale or units dependent. The value of the Jacobian determinant depends on the choice of the physical units used; therefore, the manipulability index will have different values for different units used to represent the link lengths and joint angles, due to this the actual value of the determinant cannot be considered as a measure of the degree of ill-conditioning of the Jacobian [63] or a measure of the distance from a singularity [57]. In [63], authors cautioned that the absolute of the determinant of the Jacobian is not a robust measure of invertibility, because the determinant can sometimes have very large values.

11.1.2. **Dimensional Dependency:** Due to the non-homogeneity of the Jacobian in case of a combination of both translating and rotating joints, the manipulability index does not accurately represent the degree of ill-conditioning of the manipulator's Jacobian [63].

11.1.3. **Unbounded Index:** The manipulability index is not a bounded index. Therefore it only serves as a relative measure of the degree of conditioning of the manipulator at a given point when compared to any other point in the workspace. It is important to know the maximum manipulability index in the workspace in order to appreciate the value significance of the manipulability at any other point in the workspace.

11.1.4. **Order Dependency:** The manipulability index has an order dependency as well. As each Eigenvalue has the dimension of $length^2$, for an n-DoF manipulator, the order of the manipulability index is given as [70]:

$$order\,(\mu) = (eignvalue)^{n/2} = \left(length^2\right)^{n/2} = length^n \quad (18)$$

11.2. **Improvements:** Some of the proposed improvements to overcome the limitations of the manipulability index are discussed below:

11.2.1. **Order independent manipulability:** Kim and Khosla [70] solved the problem of dimensional dependency of the manipulability index by taking the geometric mean of the manipulability index ($\mu$). The dimension independent manipulability ($\mu_O$) for n-link manipulator is given as:

$$\mu_O = \sqrt[n]{\mu} = \sqrt[n]{\det(J.J^T)} \quad (19)$$

11.2.2. **Relative Manipulability:** Relative manipulability ($\mu_r$) was proposed by Kim and Khosla [70] to make the manipulability scale and order independent. The relative manipulability index is given as:

$$\mu_r = \frac{\mu_O}{f_M} \quad (20)$$

where $f_M$ is a function with dimensions of $\left[\text{length}\right]^n$ and $n$ is the number of links in the manipulator.

11.2.3. **Normalized Manipulability Index:** The concept of normalized manipulability was introduced in order to make the manipulability index a bounded parameter. Normalized manipulability is defined as

$$\mu_n = \frac{\mu_\mathrm{i}}{\max(\mu_1\mu_2\mu_3\ldots\mu_\mathrm{n})} \quad (21)$$

where $\mu_\mathrm{i}$ is the manipulability index at a given point and $\max(\mu_1\mu_2\mu_3\ldots\mu_\mathrm{n})$ is the maximum manipulability in the entire workspace. Normalization makes the manipulability index invariant to changes in scale, units and reference frame [57]. Figure 3 shows the distribution of the normalized manipulability ($\mu_n$) index over the workspace of a SCARA manipulator.

The biggest advantage of the Yoshikawa manipulability index is that it has an analytical expression in terms of the joint angles that makes it easily computable in real time applications. The manipulability index is a good measure of kinematic dexterity as it considers the end-effector motion in all directions, unlike condition number and MSV. The normalized

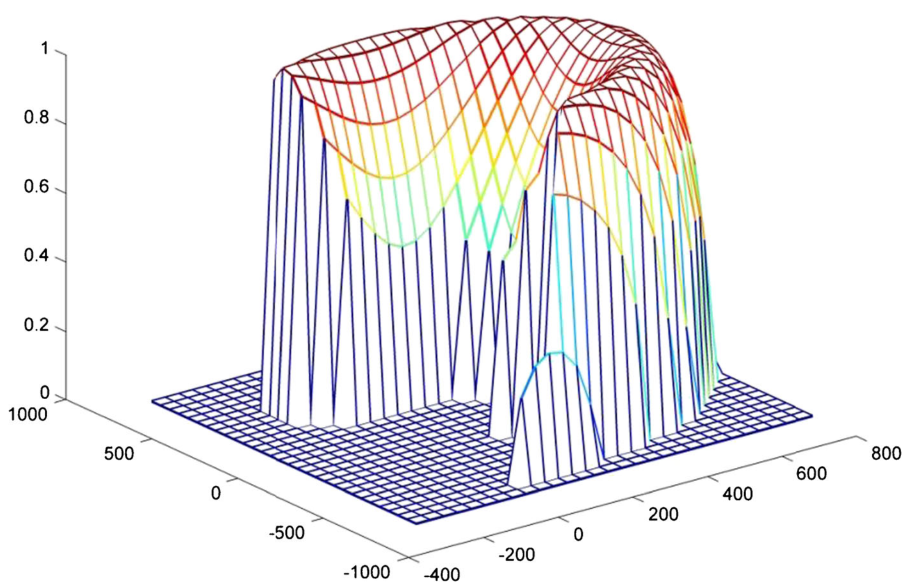


**Fig. 3** Shows the distribution of normalized manipulability from a SCARA manipulator [2]. (© [2000] PECR). (©[2000] Problems of Engineering, Cybernetics & Robotics (PECR), Bulgarian Academy of Sciences), Reproduced from [2] with permission from Dr. Tanio Tanev

manipulability index is recommended measure for studying the workspace of a manipulator, so that the value significance can be easily understood.

## 12 Global Manipulability Index

The Global Manipulability Index is based on the manipulability index. The global manipulability index (GMI) is defined as the integral of a manipulability index over the whole manipulator workspace, given as:

$$GMI = \frac{\mathrm{A}}{\mathrm{B}} \tag{22}$$

where A and B are given as

$$A = \int_W (\mu)\, dW \ and \ B = \int_W dW$$

where W is a specific point in the manipulator workspace, $\mu$ is the manipulability at that point in the workspace, and B is the workspace volume. A GMI closer to zero demonstrates poor handleability.

## 13 Dynamic Manipulability

The concept of a dynamic manipulability measure quantifies the manipulating ability of a robot's end-effector with consideration of its arm dynamics. This measure is an extension of the simple manipulability index ($\mu$), which is a kinematic performance metric introduced by Yoshikawa [58]. Dynamic manipulability index measures the ability of the manipulator to generate acceleration based on a joint driving force [56, 58, 71]. The dynamic manipulability index is defined as:

$$\mu_d = \sqrt{\det\left[J\left(M.M^T\right)^{-1} J^T\right]} \tag{23}$$

For a non-redundant manipulator the equation reduces to [71]:

$$\mu_d = \left|\frac{\det(J)}{\det(M)}\right| \tag{24}$$

where M is the inertia matrix.

Like manipulability index, the dynamic manipulability index ($\mu_d$) is also a posture-dependent local performance metric.

## 14 Minimum Singular Value

Using the Singular Value Decomposition (SVD) theorem, the Jacobian matrix can be represented as a product of three matrices:

$$J = U\Sigma V^T \tag{25}$$

where U is a mxm orthogonal matrix; V is a nxn orthogonal matrix; and $\Sigma$ is a mxn diagonal matrix. The diagonal matrix $\Sigma$ consists of elements $a_{i,j}$ such that: $a_{i,j} = 0$ if $i \neq j$ and $a_{i,j} = \sigma_i$ if $i = j$.

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_m \end{bmatrix} \tag{26}$$

The elements of $\sigma_i$ (scalars) are singular values of matrix $\Sigma$, such that:

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \tag{27}$$

Singular values of Jacobian ($J$) are defined as the non-negative square roots of non-zero Eigenvalues of the square matrices ($J.J^T$) and ($J^T.J$). Extensive work on the calculation of SVD and it applications can be found in [72].

$$\sigma_{\min} = \min(\sigma_1, \sigma_2, \ldots \sigma_m) \tag{28}$$

The Minimum Singular Value (MSV) as a performance index was introduced by Klein and Blaho [1]. The minimum singular value represents the minimum transmission ratio, the maximum force transmission, and maximum accuracy [70]. MSV represents the direction in which it is most difficult for the manipulator's end-effector to move, ignoring all other directions [65]. Klien [73] and Yoshikawa [71] have interpreted MSV as an upper bound for the velocity with which the manipulator can move in all directions. Another interpretation of MSV is the minimum change in end-effector velocity produced due to a unit change in joint velocity [73].

MSV varies more radically near singularities than the other singular values [1, 2, 6]. MSV can be seen as an efficient indicator of whether the determinant is close to zero [63], in other words, the MSV is better indicator of closeness to singularities than the manipulability index or the condition number. Figure 4 depicts the MSV distribution for the SCARA manipulator workspace. As seen in the figure MSV varies

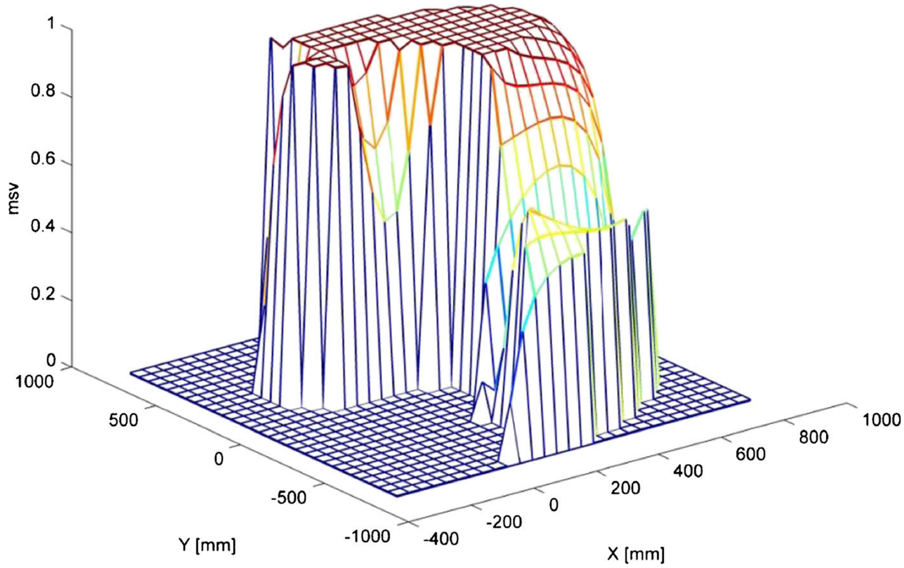


**Fig. 4** Shows the distribution of MSV for a SCARA manipulator [2]. (© [2000] PECR). (©[2000] Problems of Engineering, Cybernetics & Robotics (PECR), Bulgarian Academy of Sciences), Reproduced from [2] with permission from Dr. Tanio Tanev

more drastically as the manipulator approaches singular positions when compared to the manipulability index in Fig. 3.

Eigenvalues of the Jacobian and performance indices based on these eigenvalues are not meaningful due to the non-homogeneity of the very Jacobian that they are derived from [57]. The MSV also suffers from frame dependency [57], and unlike the manipulability index depends on the location of the operation point. In [45], authors demonstrated that the SVD of a Jacobian matrix combining different physical units is invalid and physically inconsistent. Gosselin tried to overcome this limitation by proposing the formulation and use of a homogeneous Jacobian [62].

Yoshikawa also proposed geometric mean (see Section 10.2.2) and harmonic mean of singular values as additional measures for manipulator dexterity [55, 74].

## 15 Relative Minimum Singular Value

To non-dimensionalize the minimum singular value, the relative minimum singular value was introduced in [70]. The relative minimum singular value is given as:

$$\kappa = \frac{\sigma_{\min}}{f_\sigma} \tag{29}$$

where $f_\sigma$ is a non-dimensionalizing function with the exact same dimensions as the singular values.

## 16 Condition Number

Salisbury and Craig introduced the condition number as a kinematic performance measure [43]. The condition number is a measure of the degree of independence of the columns of the manipulator's Jacobian matrix. The condition number of a Jacobian of full rank is defined as the ratio of the maximum and minimum singular values of the Jacobian

$$\kappa = \frac{\sigma_{\max}}{\sigma_{\min}} \tag{30}$$

The computation of the above formulation of condition number is not simple [49] since singular values depend on the Eigenvalues that do not have an easy analytical expression. A computationally simpler form for calculating the condition number for a homogeneous Jacobian is given as:

$$\kappa = || J || \, || J^{-1} || \tag{31}$$

where is the matrix norm. Using Frobenius norm the equation can be written as:

$$\kappa = \frac{1}{n}\sqrt{tr(JNJ^T)tr(J^{-1}NJ^{-T})} \tag{32}$$

where tr is the trace and matrix $N$ is given as:

$$N = \frac{1}{n} I_{nxn} \tag{33}$$

where n is the of the dimension of the square matrix and I is the identity matrix

It has been proven by numerical analysts that the condition number is better measure of the degree of ill-conditioning of the manipulator than the manipulability index. A condition number close to unity means a well conditioned Jacobian at that point; this happens when the Jacobian has similar singular values. Manipulator configurations for which the condition number is unity are known as *Isotropic* configurations [62].

When the Jacobian loses its full rank, the minimum singular value $\sigma_{\min}$ is equal to zero and the condition number becomes infinity. In other words, the condition number is a measure of kinematic isotropy of the Jacobian [70, 75, 76]. Asada and Granitio [41] showed that isotropic configurations can be achieved by minimizing the condition number.

The condition number does not have an upper bound.

$\kappa \in [1, \infty]$

The condition number is a good measure of the manipulator's distance singularity and kinematic accuracy [13, 62, 70]. Unlike the minimum singular value (MSV), the conditioning number only considers two directions of motion for the end-effector, the most difficult and the most easiest direction of motion, ignoring the rest [65]. The condition number is also a good measure of invertibility of the Jacobian matrix [77]. Yoshikawa [71], interpreted the condition number as a measure of directional uniformity of the velocity ellipsoid. The condition number has also been interpreted as a measure of the accuracy with which the manipulator can generate output forces from input torques, and workspace velocity from joint velocity [6, 66]. However, the condition number is not completely devoid of the drawbacks of the manipulability index since the determinant of the Jacobian is still a component of the condition index [9].

There is a difference of opinion in the research community regarding the claim that the condition number is also a measure of force and velocity error amplification. Salisbury and Craig utilized the condition number as measure of force amplification [43], but Chiu, in his work [75], has questioned this interpretation of the condition number. Also, at singular points in the workspace the condition number index "fails and yields uncontrollable values" [9], because at singular points $\sigma_{\min}$ is equal to zero.

The condition number is a local kinematic conditioning index due to the posture-dependence of the Jacobian matrix. The condition number is not invariant to scaling of the manipulator dimensions [62] and also suffers from frame dependency [57]. Unlike the manipulability index the condition number depends on the location of the operation point, and does not have a clear analytical expression as a function of the joint angles. Like all other performance indices based on the Eigenvalues of the Jacobian, the condition number also suffers from the limitations of non-homogeneity of the Jacobian [57].

To address the problem of scaling, authors in [47] propose two distinct metrics, maximum joint rotation sensitivity and maximum point-displacement sensitivity. In [62], Gosselin suggested two new dexterity indices based on the condition number to overcome the problem of scaling. He defined the two indices, one based on a redundant formulation of velocity equations ($\nu\prime$), and the second based on a minimum number of parameters ($\upsilon'\prime$). They are given as follows:

$$\left.\begin{array}{l} \upsilon' = \kappa\left(J'\right) \\ \upsilon'' = \kappa\left(J''\right) \end{array}\right\} \tag{34}$$

where $J'$ and $J''$ are the newly defined Jacobian matrices invariant to scaling of the manipulator.

## 17 Local Conditioning Index

As the condition number does not have an upper bound, the reciprocal of the condition number, known as the Local Conditioning Index (LCI) is more commonly used. To avoid computational problems due to the condition number ($\kappa$) becoming infinity when the Jacobian is not full rank, the reciprocal of LCI is used. LCI is bounded between zero and unity.

$$LCI = \frac{1}{\kappa} \tag{35}$$

The reciprocal condition number is a local performance metric that is both bounded, $LCI \in [0, 1]$ and scale independent. Figure 5 show the distribution of the local conditioning index for a SCARA manipulator.

## 18 Kinematic Conditioning Index

The Kinematic Conditioning Index (KCI) is another posture-independent performance measure based on the condition number proposed by Angeles and Lopez

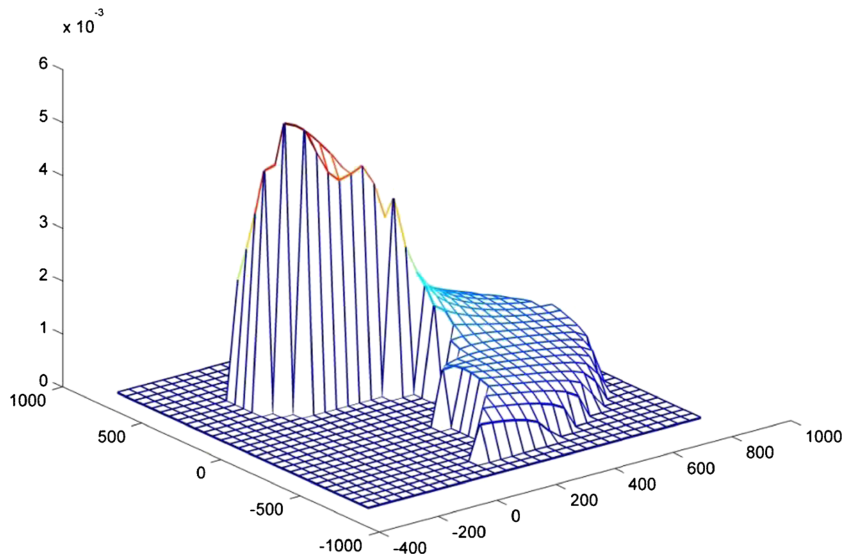


**Fig. 5** Shows the distribution of reciprocal of the LCI for a SCARA manipulator [2]. (© [2000] PECR). (©[2000] Problems of Engineering, Cybernetics & Robotics (PECR), Bulgarian Academy of Sciences), Reproduced from [2] with permission from Dr. Tanio Tanev

[76]. The kinematic conditioning index is defined as:

$$KCI = \frac{1}{\kappa_{\min}} * 100 \in (0, 100\ \%] \tag{36}$$

where $\kappa_{\min}$ is the minimum of the condition number the entire manipulator workspace.

As discussed above, the computation of the condition number is very computationally intensive and finding the minimum condition number for the entire workspace can be even more cumbersome [49]. It is therefore recommended to use the norm representation for calculating the condition number.

KCI is a global index that demonstrates the worst possible performance of the manipulator. The kinematic conditioning index is upper-bound at 100 %. A manipulator with a KCI of 100 % is an isotropic manipulator [10, 49] i.e., all the singular values of the Jacobian are similar at the condition of minimum singular value.

## 19 Global Conditioning Index

The Global Conditioning Index (GCI) is based on the condition number but unlike the conditioning number is not a local metric. The GCI proposed by Gosselin and Angeles [8], demonstrates the distribution of the conditioning number over the entire workspace. The Global Conditioning Index ($\eta$) is given as:

$$\eta = \frac{A}{B} \in (0, 1) \tag{37}$$

where A and B in the Cartesian space are given as

$$A = \int\limits_W \left(\frac{1}{\kappa}\right) dW \tag{38}$$

$$B = \int\limits_W dW \tag{39}$$

where W is a specific point in the manipulator workspace, $\kappa$ is the condition number at that point in the workspace, and B is the workspace volume. In joint same A and B can be represented in joint space as:

$$A = \int\limits_R \left| \left(\frac{1}{\kappa}\right)^2 \Delta \right| d\theta_n..d\theta_2 d\theta_1 \tag{40}$$

$$B = \int\limits_R |\Delta| d\theta_n..d\theta_2 d\theta_1 \tag{41}$$

where R is the workspace of the manipulator in the joint space and $\Delta$ is the determinant of the Jacobian matrix.

Given the difficulties in computing the integral, a simpler discrete formulation is given as [10]:

$$\eta = \frac{1}{n_{WS}} \sum\nolimits_{j=1}^{n_{WS}} \frac{1}{\kappa} \tag{42}$$

where $n_{ws}$ are the manipulator workspace nodes, and $\kappa$ is the condition number.

GCI is a bounded global index that assumes values between zero and one. As the GCI approaches zero the manipulator is said to have a bad GCI and as it approaches unity the workspace is said to have good GCI.

## Measures of Manipulator Isotropy

### 20 Isotropic Index

Kim and Khosla [70] proposed a measure to quantify the isotropy of the manipulator ellipsoid. The Isotropic Index ($\Delta$) is defined as the ratio of the geometric mean to the arithmetic mean of the Eigenvalues.

$$\Delta = \frac{\mu_O}{\psi} \tag{43}$$

where is $\psi$ the arithmetic mean of the Eigenvalues and $\mu_O$ is the order independent manipulability index.

$\Delta$ is upper bound at one. A larger isotropic index means a more isotropic ellipsoid. When all the Eigenvalues are the same the Isotropic Index is one, and the manipulator ellipsoid is completely isotropic. The Isotropic Index is a local performance metric.

### 21 Layout Conditioning Index

In [13], authors defined another measure for manipulator conditioning based on the measure of isotropy discussed above (also see [70]) called the Layout Conditioning Index. The layout conditioning index was proposed to help optimize the manipulator design for a given task, by minimizing the index. For a given manipulator in a given layout ($\mathcal{L}$), the Layout Conditioning Index is defined as:

$$\kappa_{\mathcal{L}} = \sqrt{\frac{tr^m\left(\bar{J}.\bar{J}^T\right)}{m^m \det\left(\bar{J}.\bar{J}^T\right)}} \tag{44}$$

where $\bar{J}$ is the normalized m $\times$ m Jacobian matrix given by:

$$\bar{J} = \begin{bmatrix} E \\ \frac{1}{l_{\mathcal{L}}} R \end{bmatrix} = \begin{bmatrix} e_1 & \dots & e_m \\ \frac{1}{l_{\mathcal{L}}} e_1 \times r_1 & \dots & \frac{1}{l_{\mathcal{L}}} e_m \times r_m \end{bmatrix} \tag{45}$$

where $l_{\mathcal{L}}$ is layout length is a normalizing length defined as the rms value of all distances of the axes from the layout center. For detailed definitions and determination of manipulator layout, layout center and layout length, please refer to [13]. The layout conditioning index is a transformation invariant metric that is independent of the choice of operation point on the manipulator end-effector.

### 22 Inertia Matrix as a Measure of Manipulator Isotropy

Asada [78] introduced the concept of Generalized Inertia Ellipsoid (GIE) to study the mass properties and dynamic behavior of the manipulator arm. The Generalized Inertia Matrix is defined as the Hessian matrix of its kinetic energy with respect to its generalized speeds.

$$I = I\left(q\right) \equiv \frac{\partial^2 K}{\partial \dot{q}^2} \tag{46}$$

where K is the kinetic energy of the manipulator

$q$ is the n dimensional vector generalized co-ordinates
$\dot{q}$ is the n dimensional vector generalized speeds

The manipulator is said to be dynamically isotropic when the general inertia matrix can take the form:

$$I = \sigma I_{nxn} \tag{47}$$

where $I_{nxn}$ is the $nxn$ identity matrix

GIE is a posture-dependent and therefore is not a global measure of the dynamic isotropy of the manipulator [53]. When the manipulator is in an isotropic posture the non-linear forces are minimal [79].

### 23 Global Isotropy Index

The Global Isotropy Index (GII) proposed by Stocco [80], is defined as the ratio of the minimum to the maximum singular values for the entire workspace.

$$GII = \frac{\sigma_{\min}}{\sigma_{\max}} \tag{48}$$

Unlike the condition number, in this case $\sigma_{\min}$ and $\sigma_{\max}$ reflect the minimum and maximum singular values for the entire workspace. GII is a measure of the manipulator's worst performance.

### 24 Dynamic Conditioning Index

Ma and Angeles [53, 81] proposed the concept of Dynamic Conditioning Index (DCI) as a measure of the degree of isotropy of the moment of inertia of the manipulator. DCI is defined as "*the Frobenius norm between the generalized inertia matrix and an* *isotropic matrix"* [53]. The Dynamic Conditioning Index can be analytically represented as:

$$\mu \equiv \frac{1}{2} d^T w_d \tag{49}$$

where w is a diagonal weighting matrix.

d is the upper triangular vector of the difference matrix. d is a $\frac{n(n+1)}{2}$ dimensional vector of the form

$$d \equiv \left[ I_{11} - \sigma \cdots I_{nn} - \sigma \quad I_{12} \cdots I_{1n} \quad I_{n-1} \cdots I_n \right]^T \tag{50}$$

The matrix D is the difference between the generalized inertia matrix and it's nearest isotropic matrix.

$$D \equiv I - \sigma I_{nxn} \tag{51}$$

$I$ is the generalized inertia n x n matrix, $I_{nxn}$ is the $nxn$ identity matrix.

The DCI is a local (posture-dependent) dynamic performance measure that represents how far the manipulator is from a dynamically isotropic posture. To ensure the best dynamic performance, the manipulator's trajectory should be such that the associated generalized inertia matrix is close to isotropy.

## Manipulator Redundancy Indices

## 25 Degree of Redundancy

Redundancy is the ability of a manipulator to reconfigure itself with the end-effector remaining in a fixed position [82]. The simplest measure of a manipulator's redundancy is the Degree of Redundancy (DoR). It is defined as the number of degrees of freedom (DoF) of the manipulator less the minimum dimensions required to perform the task [83]. The degree of redundancy ($r$) is equal to the number of degrees of manipulator freedom ($n$) less the rank of the workspace ($m$), given as [6]:

$$r = n - m \tag{52}$$

For example the DoR for a three-DoF planar manipulator is 1. However, DoR is not a good measure of a manipulator's redundancy because the manipulator might have redundancy in some postures and lose it redundancy in other postures. In other words, the redundancy is a posture-dependent characteristic of the manipulator, and therefore to give a manipulator workspace a global value for the degree of redundancy is inaccurate as well as misleading.

## 26 Redundancy Index

In [82], Chen et al. defined Redundancy Index (RI) as the normalized distance between the joint rate solution point and the hyper plane constraint boundaries specified in the joint-rate space. Redundancy Index is given as:

$$RI = \frac{2 * \min(d_l, d_u)}{(d_l + d_u)} \tag{53}$$

where $d_l, d_u$ are the distances between the joint rate solution and the two hyper planes constraint boundaries. For the calculation of distances $d_l, d_u$ please see [82].

## 27 Relative Manipulability Index

Even though redundancy may seem wasteful, it is now an established fact that redundant manipulators have several advantages over their non-redundant counterparts. Redundant manipulators offer great potential for singularity avoidance, high dexterity, obstacle avoidance, torque minimization, and importantly, fault tolerance [84].

In [84], authors Roberts and Maciejewski proposed a local measure for quantifying the fault tolerance in redundant manipulators. Using the Yoshikawa manipulability measure ($\mu$) and the basis, the authors investigated the reduced performance of the manipulator under fault conditions.

The Jacobian matrix for n-DoF manipulator can be expressed as:

$$J = \left[ J_1 \ldots J_{i-1} J_i J_{i+1} \ldots J_n \right] \tag{54}$$

Under fault conditions, in which the $i$th joint is locked, the reduced Jacobian determines the behavior of the manipulator. The reduced Jacobian is expressed as:

$$^iJ = \left[ J_1 \ldots J_{i-1} J_{i+1} \ldots J_n \right] \tag{55}$$

Roberts and Maciejewski defined the *i*th relative manipulability index to be the ratio of the manipulability of the reduced Jacobian to the manipulability of the original Jacobian.

$$r_i(\theta) = \frac{\mu\left({}^iJ\right)}{\mu(J)} \in [0, 1] \text{ } and \text{ } \mu(J) \neq 0 \tag{56}$$

The relative manipulability index $(r_i)$ determines if the manipulator with become singular in a given posture if the *i*th joint fails. The relative manipulability index is a local and bounded index that assumes values between zero and unity. If $r_i = 0$, it means that the manipulator is fault intolerant with respect to the *i*th link and a failure of this link will make the manipulator singular. $r_i = 1$, demonstrates fault tolerance, that is, the failure of the *i*th will not result in any loss of manipulability. The authors suggested that manipulators can be optimized for fault tolerance by maximizing the minimum value of the relative manipulability index $(r_i)$. Figure 6 depicts the optimal postures for a RRR planar manipulator that can tolerate a fault in the first link.

## 28 Minor Measure

Chang [85] introduced the minor measure based on the concept of aspect. Borrel and Liegeois [86] proposed that the manipulator workspace can be divided into volumes corresponding to different classes of configurations called aspects. The minor measure is given as:

$$H_{minor} = \left|\prod\nolimits_{i=1}^{p} \delta_i\right| \frac{1}{p} \tag{57}$$

where $\delta_i$ are minors of rank *m* of the Jacobian

$p = {}^nC_m$ a combination of m taken out of n

The absolute product of all the minors represents the distance from a kinematic singularity. The number of minors of rank m determines the number of distinct combinations of m linearly independent column vectors [33].

In case of a planar redundant manipulator there is an increase in the number of m rank minors at the rate of ${}^nC_m$ as the degrees of freedom increases. This causes a computation burden in calculating the minor measure.

To overcome this burden, Chung et al. [33] proposed the effective minor measure. They defined the *effective minor* as "second order minors composed of

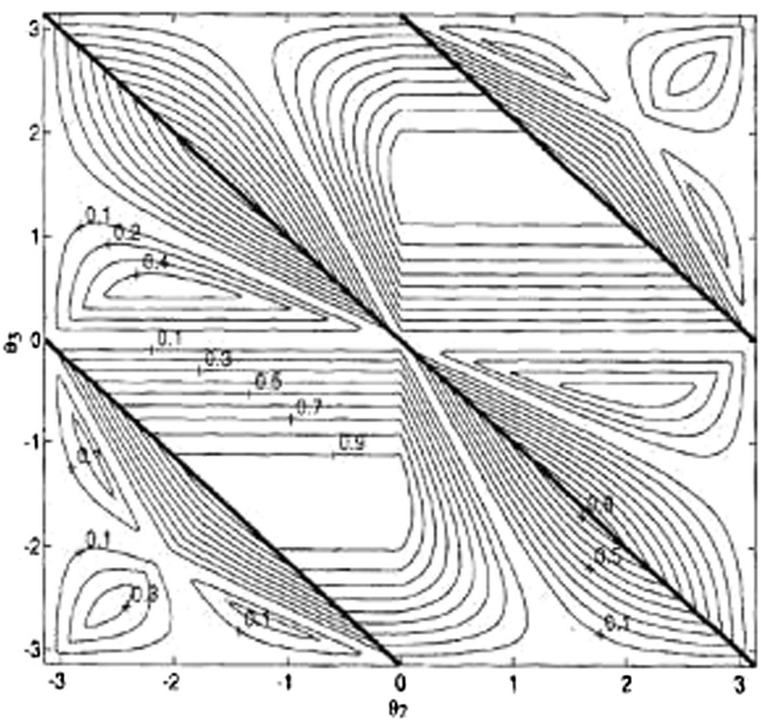


**Fig. 6** Contour of the minimum reduced manipulability index $\mu\left({}^iJ\right)$ a RRR planar manipulator. The bold lines show configurations that optimal value for the minimum relative manipulability index [84]. (© [1996] IEEE). (©[1996] IEEE) Reproduced from [74] with the permission of Dr. Rodney G. Roberts

adjacent columns of the Jacobian matrix" [33]. The number of effective minors for a n-DoF manipulator is n-1. The effective minor measure is defined as:

$$H_{effective,\ minor} = \prod_{i=1}^{n-1} \delta^{*}_{i,i+1} \tag{58}$$

where $\delta^{*}_{i,i+1}$ is the effective minor composed of $i$th and $i+1$th column vectors of the Jacobian

## Manipulator Task Based Performance Indices

### 29 Task Dependent Performance Index

Often manipulator tasks require exerting a determined amount of force along a specified direction, for example in pick and place tasks. In order to quantify the manipulator's ability to perform such tasks, Chiu [32] proposed the Task Dependent Performance Index. The Task Dependent Performance Index is defined as the weighted sum of the squares of the deviation between the actual and desired force transmission ratios. This can be interpreted as the difference between the velocity and task ellipsoid.

$$T_M = \sum_{i=1}^{m} w_i \left(\frac{T_{f,i} - T_{d,i}}{T_{d,i}}\right)^2 \tag{59}$$

where $u_i = 1, 2 \ldots m$ are the unit task direction vectors

$T_{f,i}$, $T_{d,i}$ are the force transmission ratios along $u_i$

$w_i$ is a weighting factor that indicate the relative importance of the task along $u_i$

Since this performance measure is related to the task specification it is an extrinsic performance index.

### 30 Robot Task-Conformance Index

Cloutier et al [15] defined a new task conformance index based for four ellipsoids – the robot ellipsoid $\xi_r$, task ellipsoid $\xi_t$, largest contained ellipsoid $\xi_{\subseteq}$, and the smallest containing ellipsoid $\xi_{\supseteq}$, as seen in Fig. 7.

The task conformance index is defined as the ratio of volumes of the task ellipsoid and the smallest containing ellipsoid or the ratio of the largest contained ellipsoid and the robot ellipsoid.

$$C_l = \frac{V_t}{V_{\supseteq}} = \frac{V_{\subseteq}}{V_r} \tag{60}$$

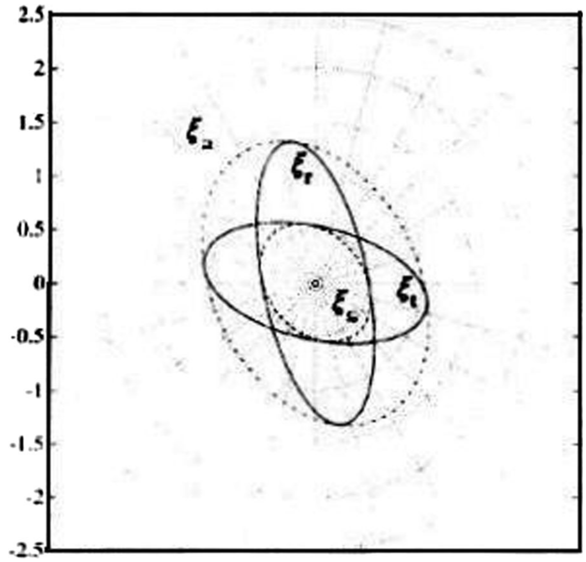


**Fig. 7** Basics of conformance index: robot $(\xi_r)$, task $(\xi_t)$, largest contained $(\xi_{\subseteq})$, and the smallest containing $(\xi_{\supseteq})$ ellipsoids [15]. (© [1994] Robotics & Autonomous Systems, ELSEVIER). (©[1994] Robotics & Autonomous Systems, ELSEVIER) Reproduced from [14]

In its simplest form the task-referenced conformance index in the diagonalized space $C_{lt}^{(d)}$ is given as:

$$C_{lt}^{(d)} = \sqrt{\prod_{j=1}^{m} \lambda_j} \tag{61}$$

where $\lambda_j$ are the Eigenvalues for the smallest containing ellipsoid $\xi_{\supseteq}$

The task conformance index is a well bounded [0, 1] extrinsic index. The index is also well defined in and outside singularities and is independent of any dimensional dependencies due to a mix of translational and rotational units. This index can used as an optimization critierion to help the designer in modeling the manipulator and deciding its optimal placement with respect to the task space.

## Other Performance Indices

### 31 Product of the Manipulability Index and Condition Number

Both the manipulability and condition number, despite their limitation, are widely accepted and used local performance measures. To overcome their combined limitations, Kucuk et al. [9] proposed a new performance index ($\rho$) that is simply a product of the manipulability index ($\mu$) and the condition number ($\kappa$)

$$\rho = \mu.\kappa \tag{62}$$

The aim of this new index was to make it independent of the drawbacks of the determinant of the Jacobian. For a non-redundant manipulator the above equation can be formulated as:

$$\rho = \mu.\kappa = |\det(J)| \, . \frac{\sqrt{m}}{|\det(J)|} = \sqrt{m} \tag{63}$$

where $m$ is the numerator value of the product of the Jacobian matrix and its transpose with the inverse Jacobian matrix and its transpose.

$\rho$ being the square root of $m$ is always positive. This performance index completely eliminates the dependency on the determinant of the Jacobian [9]. The authors extended this local parameter to the global scale by integrating it over the entire workspace.

## 32 The Level and Distribution Index

Puglisi et al. [10] noted that sometimes GCI may overrate the performance of a manipulator by ignoring small regions of poor manipulator performance, which may sometimes be unacceptable in practice.

In order to study the degree of uniformity in the manipulator's performance on a global scale, Moreno[2] [4] proposed the ratio of level and distribution as a new performance metric to quantify the distribution of GCI over a manipulator's workspace, called the Level and Distribution Index ($R_{CI}$), it is given as:

$$R_{CI} = \frac{\eta}{\alpha_4 \rho_{CI} + 1} \in (0, 1] \tag{64}$$

where $\eta$ is the Global Conditioning Index, $\rho_{CI}$ is the standard deviation of the condition number over the workspace, and $\alpha_4$ is the ponderation fraction.

[2]Clarified via correspondences with the author (Dr. Hector A. Moreno)

## 33 Harmonic Mean Manipulability Index

Hashimoto noted that even though a singular value approaches zero, there is no noticeable decrease in the manipulability if the other singular values increase in value [65]. Hashimoto proposed the Harmonic Mean Manipulability Index (HMMI) that is given as [87]:

$$HMMI = \sqrt{\frac{1}{tr\left[\left(J.J^T\right)^{-1}\right]}} \tag{65}$$

where $tr$ is the trace.

High values of HMMI are representative of high manipulator dexterity. In the neighborhood of any singular point, the HMMI becomes zero. HMMI is a posture-dependent local index.

## 34 Stochastic Manipulability Index

Hashimoto proposed another performance measure based on the stochastic model for the movement of the manipulator arm in a specific direction known as the Stochastic Manipulability index ($w_d$). The Stochastic Manipulability Index is given as [87]:

$$w_d = \begin{cases} \left(\frac{n}{\iint_{s_d} P(ds_d)\dot{x}_d (JJ^T)^{-1}\dot{x}_d ds_d}\right)^{\frac{1}{2}} & if \det\left(JJ^T\right) \neq 0 \\ 0 & if \det\left(JJ^T\right) = 0 \end{cases} \tag{66}$$

Where $n$ is the degrees of freedom of the manipulator, $P(ds_d)$ is the probability density function of the manipulator arm corresponding to motion in the direction range $ds_d$, and $\dot{x}_d$ is the velocity vector.

If the probability density function $P(ds_d)$ is not known, assuming uniform probability in all directions of manipulator motion, the above formulation for the stochastic manipulability function can be reduced and expressed in terms of the Harmonic Mean Manipulability Index (HMMI) as:

$$w_d = \begin{cases} \left(\frac{nt}{tr\left[(JJ^T)^{-1}\right]}\right)^{\frac{1}{2}} = \sqrt{nt}(HMMI) & if \det\left(JJ^T\right) \neq 0 \\ 0 & if \det\left(JJ^T\right) = 0 \end{cases} \tag{67}$$

where t is the number of degrees of freedom of the task space

## 35 Configuration Index

For an n-DoF planar manipulator with large redundancy that can be decomposed into $s$ sub-arms, the Jacobian matrix is given as:

$$J = \left[ J_1 \vdots J_2 \vdots \ldots \vdots J_s \right] \tag{68}$$

Chung et al. [88] measure called the configuration index. The configuration index ($C$) is given as:

$$C = \prod_{i=1}^{s} \Delta_i \tag{69}$$

where $\Delta_i = \begin{cases} \det(J_i) & for\, J_i \in R^{2x2} \\ productofminorsofrank2 & for\, J_i \in R^{2x3} \end{cases}$

The authors claim that the configuration index can help determine if a manipulator is suitable for a given task or not and help improve its manipulability [88]. An optimal manipulator performance can be attained by maximizing the configuration index [88].

## 36 Structural Length Index

The Structural Length Index is a global index that is defined as the ratio of the sum of the length of the links to the cube root of the reachable workspace volume. The Structural Length Index (Q) is represented as:

$$Q = \frac{L}{\sqrt{3}V} \tag{70}$$

where $V$ is the volume of the reachable workspace and $L$ is the length sum of the manipulator given by:

$$L = \sum_{i=1}^{n} a_{i-1} + d_{i-1} \tag{71}$$

where $a_{i-1}$ is the link length

$d_{i-1}$ is the link offset

## 37 Distortion Density

In [66], authors Park and Brockett proposed the distortion density to quantify the amount of distortion that is produced due to the forward kinematic mapping $f : \mathcal{N} \to \mathcal{M}$ i.e. the mapping of the joint space $\mathcal{N}$ into the manipulator workspace $\mathcal{M}$.

The distortion index is given as:

$$d(f) = \frac{1}{2}\left(J^T G J H^{-1}\right) \tag{72}$$

where matrices $G$ and $H$ are Riemannian metrics on $\mathcal{N}$ and $\mathcal{M}$, and $J$ is the manipulator Jacobian matrix.

The distortion index is invariant to base-coordinate changes but depends on changes in the end-effector coordinates [13]. The distortion density being a function of the Jacobian matrix is a posture dependent local performance measure. A global integration of the distortion density over the entire manipulator workspace gives an indication of the degree of distortion or 'non-flat' nature of the manipulator workspace [66].

## 38 Unified Approach for Defining Performance Measures

Doel and Pai [5, 83, 89] proposed a unified framework based on differential geometry for the defining local manipulator performance measures. They showed that both existing and new performance measures could be defined based on this unified framework.

The performance metrics were defined in this new framework as the distance metric tensor between two manipulator configurations in the configuration space. This distance metric tensor, calculated using differential geometry, is interpreted as a performance measure.

The distance tensor can be interpreted as a kinematic or dynamic performance measure depending on the choice of the metric of configuration space [5, 89]. The authors showed that using the Euclidean metric in the configuration space leads to kinematic measures, while using an inertia matrix as the metric tensor yields dynamic measures. In [5] the authors defined the existing performance measures based on this unified approach and defined two new metrics – non-linearity measure and redundancy measure [89].

## 39 Future Recommendations

Advancements in designing, simulation, and manufacturing, have made it possible to produce highly dexterous and task optimized robotic manipulators. Performance parameters play an important part in both formalizing design specifications and task planning of these manipulators. With such advancements and manipulators being applied to a wide range of tasks, from simple material handling to highly critical and complex tasks like tele-robotic surgery, more and more performance measures are being proposed to study and quantify their behavior and performance.

In summary, it can be safely concluded that any performance indices defined in the future should be devoid of few of the very commonly noticed limitations. From this exercise of surveying the literature and discussing the merits and demerits of performance indices, the following inferences can be drawn that can help researchers in formulating and defining new performance metrics:

1. The performance metric should be independent of any dependencies of scale, dimensions, coordinates (frame) or order.
2. Combining both position and orientation terms in a single scalar measure makes it physically inconsistent [7].
3. The metric should be well bounded i.e., it should have well defined upper and lower bounds so that the value significance of the metric is easily understood.
4. The performance measure should have an analytical expression, preferably as a function of the joint angles, so that the value of the metric can easily be calculated in real-time applications.
5. The performance measure should be well defined and computable in and outside singularities.

## 40 Note

There are many other performance indices that are not mentioned in this paper, for example the Average Service Coefficient (ASC) [90], Dexterity Effective Coefficient (DEC) [90], Acceleration Radius [91], etc. due to the their very limited use and in the interest of the length of this paper. Also, indices dealing with the performance of parallel manipulators have been omitted from the discussion, since in most cases they are modified adaptations of the serial version.

## 41 Conclusion

Although numerous performance metrics have been proposed till date, there is a lack of consensus in the robotics community on any of them due to their inherent limitations of scale, dimensions, order, and bounds. These dependencies limit the use of the indices in manipulator design and optimization applications. This work discusses, with proper bibliographic references, a wide range of performance metrics that have been defined to quantify the performance characteristics of robotic manipulators, with a focus on understanding their scope, characteristic and application. This work has also discussed some of the modifications made to overcome a few of the limitations. A unified approach for defining new performance parameters has been reviewed based on which both existing and new performance indices can be defined. Finally, based on the study, a few recommendations have been proposed to aid researchers in defining new performance indices.

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