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WORKSPACE SYNTHESIS OF 3*R*, 4*R*, 5*R* AND 6*R* ROBOTS

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Abstract—In a previous study [Y. C. Tsai and A. H. Soni, *Mech. Mach. Theory* **19**, 9–16 (1984)], the regional and orientational structures of a general 6*R* robot were analyzed as separate 3*R* assemblies and optimum kinematic parameters for these structures were obtained. This paper utilizes the above results to develop design methodologies for 3*R*, 4*R*, 5*R* and 6*R* industrial robots. For a given set of design conditions involving full description of the accessible region and a type of robot, the paper presents in a stepwise manner a procedure to obtain the design data involving link-lengths, relative orientations of the axes of the revolute pairs and the range of angular displacements at each of the joints. The method is illustrated by an example of a synthesis of 6*R* robots.

1. INTRODUCTION

The regional and the orientational structures, each consisting of a 3*R* assembly, were analyzed, and optimum kinematic parameters for these structures were obtained[1]. The synthesis problem in each case arrives at a set of kinematic parameters (time and nontime dependent) of a robot to satisfy a given set of input-data that describes an accessible region for the robot.

The following section illustrates a general procedure to synthesize a 3*R* robot. The results of this synthesis procedure are utilized in the subsequent sections to synthesize 4*R*, 5*R* and 6*R* robots.

2. SYNTHESIS OF 3*R* ROBOTS

From [1], a 3*R* robot (Fig. 1) with its kinematic parameters $a_1 = 0$, $a_2 = a_3$, $s_2 = s_3 = 0$, $\alpha_1 = \pm 90°$ and $\alpha_2 = 0°$ and $180°$, is found to be capable of having a maximum working space. These data may be directly used for a synthesis problem, where it is required to enclose a prescribed accessible region or volume by the volume or accessible region of the 3*R* robot and to arrive at the specific base location, link-length values and the rotational displacemeents at each of the revolute joints of the 3*R* robot.

Let $(x_e, y_e, z_e)_{j,i}$ represent the coordinates of the given *i*th position of the end point of the robot arm in reference frame $\{X_j, Y_j, Z_j\}$ which is attached to joint *j*. In Fig. 1 for a given location of robot end $(X_e, Y_e, Z_e)_{1,i}$, the joint displacements $(\theta_1')_i$, $(\theta_2')_i$ and $(\theta_3')_i$ can be calculated from

$$(\theta_1')_i = \tan^{-1}(y_3/x_e)_{1,i}, \tag{1}$$

$$(\theta_2')_i = \cos^{-1}\frac{(z_e)_{1,i} - s_1}{l_i} - \cos^{-1}\frac{l_i^2 + (a_2^2 - a_3^2)}{2a_2\, l_i}, \tag{2}$$

$$(\theta_3') = \cos^{-1}\frac{l_i^2 - (a_2^2 + a_3^2)}{2a_2 a_3}, \tag{3}$$

where

$$l_i = \{(x_e)^2_{1,i} + (y_e)^2_{1,i} + [(z_e)_{1,i} - s_1]^2\}^{1/2}.$$

Following are the steps to synthesize a desired 3*R* robot. The procedure, however, is iterative since it is based on an optimization technique.

(1) Assign arbitrary values to the coordinates x_b, y_b, z_b (assume $s_1 = 0$) describing the location of the base-joint of a 3*R* robot.

(2) Calculate the coordinates of robot end in the reference frame $X_1Y_1Z_1$, which is attached to joint 1, using

$$\begin{aligned}(x_e)_{1,i} &= (x_e)_i - x_b,\\ (y_e)_{1,i} &= (y_e)_i - y_b,\\ (z_e)_{1,i} &= (z_e)_i - z_b.\end{aligned} \tag{4}$$

(3) Obtain polar projection of each $(x_e y_e z_e)_{1,i}$ on the X_1Z_1 plane, and get $(x_e^*, 0, z_e)_{1,i}$, where $x_e^* = (x_e^2 + y_e^2)$. Then, let $a_2 = a_3 = \frac{1}{2}(l_i)_{max}$, where $l_i = (x_e^*)^2_{1,i} + (z_e)^2_{1,i}$.

(4) Calculate $(\theta_1')_{min}$, $(\theta_1')_{max}$, $(\theta_2')_{min}$, $(\theta_2')_{max}$, $(\theta_3')_{min}$ and $(\theta_3')_{max}$ using eqns (1)–(3).

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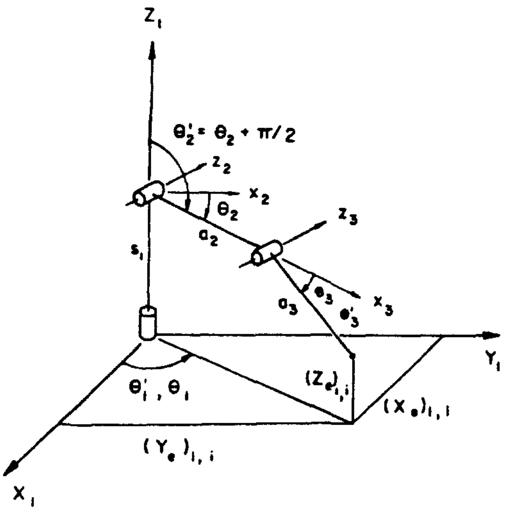


Fig. 1. The popular 3*R* industrial robot.

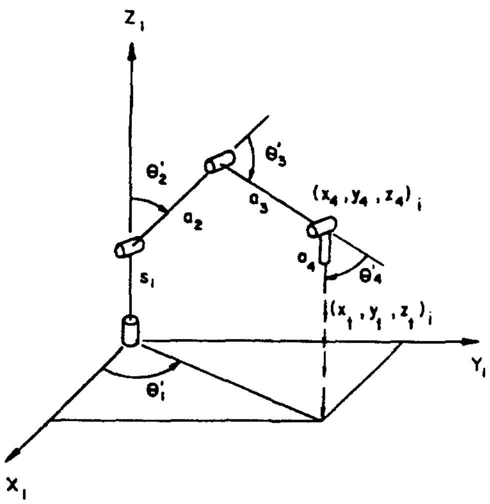


Fig. 2. The 4*R* industrial robot.

(5) Calculate on the X_1Z_1 plane, the cross-section area A and the solid of revolution volume V (generated by A) of the workspace.

$$A = [\cos(\theta_3')_{\min} - \cos(\theta_3')_{\max}](\Delta\theta_2')(a_2)^2, \quad (5)$$

$$\begin{aligned} X_{cg} = {} & \Delta\theta_2'(a_2)^3\,[\sin\theta_2^*(\cos(\theta_3')_{\min} - \cos(\theta_3')_{\max}], \\ & - \sin\theta_2^*[\sin^2(\theta_3')_{\min} - \sin^2(\theta_3')_{\max}]/2, \\ & - \cos\theta_2^*[\sin 2(\theta_3')_{\min} - \sin 2(\theta_3')_{\max} \\ & - \sin 2(\theta_3')_{\min} + \sin 2(\theta_3')_{\max})]/A, \quad (6) \end{aligned}$$

$$V = (\Delta\theta_1')X_{cg}A, \quad (7)$$

where

$$\begin{aligned} \Delta\theta_1' &= (\theta_1')_{\max} - (\theta_1')_{\min}\ (\text{rad}), \\ \Delta\theta_2' &= (\theta_2')_{\max} - (\theta_2')_{\min}\ (\text{rad}), \\ \theta_2^* &= [(\theta_2')_{\max} + (\theta_2')_{\min}]/2. \end{aligned}$$

(6) Use suitable optimization method (e.g., grid search) and repeat step (1) to step (5) to find the optimal values for x_b, y_b and z_b such that V is minimum. These optimal values of x_b, y_b and z_b and the corresponding values of a_2, a_3, $(\theta_1')_{\min}$, $(\theta_1')_{\max}$, $(\theta_2')_{\min}$, $(\theta_2')_{\max}$, $(\theta_3')_{\min}$ and $(\theta_3')_{\max}$ provide the necessary design data for the synthesized 3*R* robot.

This 3*R* robot synthesis procedure will be used in the following sections, to synthesize 4*R*, 5*R* and 6*R* industrial robots.

3. SYNTHESIS OF 4*R* ROBOTS

If we add one more link to the 3*R* regional structure, it then becomes a 4*R* industrial robot as shown in Fig. 2. This kind is suitable for jobs which require one to locate the robot hand (the tool) in a three-diimensional space and in a direction parallel to the axis of the first joint of the robot at the same time. A 4*R* robot may be used to drill vertical holes in a machine part.

If some locations along each working process (say initial position, final position and some positions in between) are specified, the corresponding location of the joint $4(x_4, y_4, z_4)_i$ for each given location of the tip of robot hand $(x_t, y_t, z_t)_i$ can be calculated by

$$\begin{aligned} (x_4)_i &= (x_t)_i, \\ (y_4)_i &= (y_t)_i, \\ (z_4)_i &= (z_t)_i + a_4, \\ i &= 1, 2, \ldots, n, \end{aligned} \quad (8)$$

where a_4 is determined by the machine tool to be used.

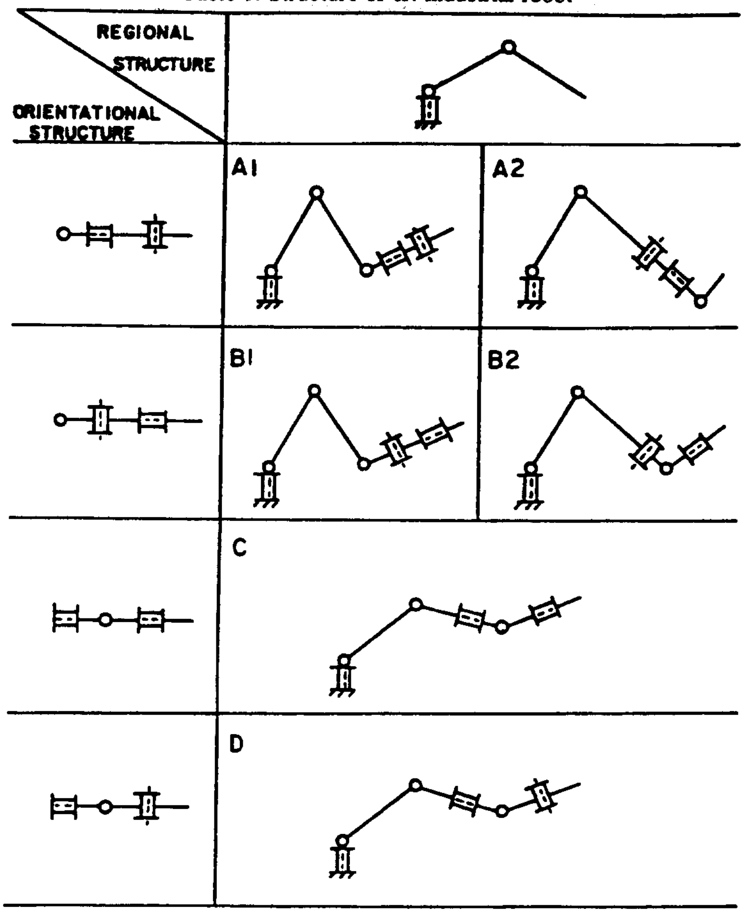
The coordinates $(x_4, y_4, z_4)_i$ described here are the same as $(x_e, y_e, z_e)_i$ described in Section 2. Now it becomes a problem of synthesizing a 3*R* robot for a prescribed workspace $(x_e, y_e, z_e)_i$, $i = 1, 2, \ldots, n$.

4. SYNTHESIS OF 5*R* ROBOTS

A lot of work involving welding, flame cutting, spray painting, drilling, assembling, etc., can be done by 5*R* industrial robots. All these jobs mentioned above require the robots to hold the tool at some specified locations, in particular directions. In other words, these jobs require five degrees of freedom, three translations and two rotations. Such a motion characteristic may be accomplished by using a robot with five joints.

In order to get the maximum working space and maximum dexterity of the same time, the kinematic parameters of the 5*R* robots should be properly selected. From [1], the optimal values of the link pa-

Table 1. Structure of 6*R* industrial robot



rameters for 5*R* robots have been arrived as

(1) Regional structure:

$$a_1 = 0,\ a_2 = a_3,\ s_2 = s_3 = 0,$$

$$\alpha_1 = \pm 90°,\ \alpha_2 = \alpha_3 = 0°,\ 180°.$$

(2) Orientational structure:

$$\alpha_5 = \pm 90°,\ a_5 \neq 0.$$

Table 1 shows the best combinations of 5*R* industrial robots with link parameters mentioned above.

If we put an imaginary revolute joint along the last link of the 5*R* type-A and type-B robot as shown in Figs. 3 and 4, then they become a 6*R* type-C and type-B1, respectively. One can use the synthesis procedures of 6*R* type-C and type-B1 robots to synthesize the 5*R* type-A and type-B robots, respec-

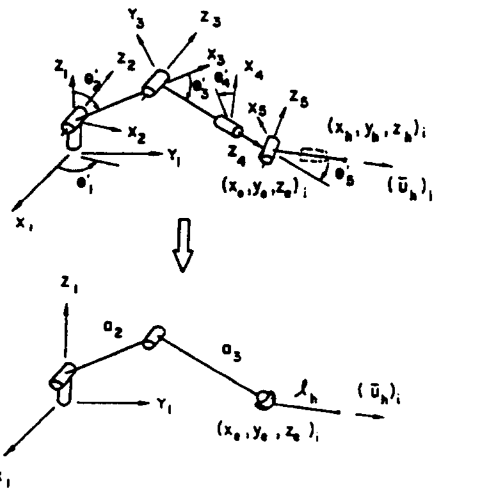


Fig. 3. The 5*R* type-A robot with one imaginary *R* joint.

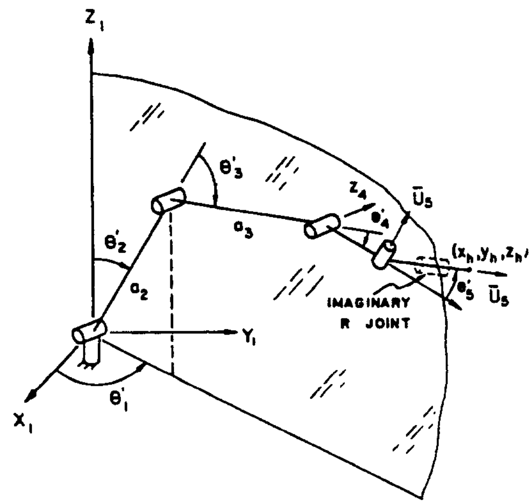


Fig. 4. The 5*R* type-B robot with one imaginary *R* joint.

tively. 6*R* type-D and 5*R* type-C robots cannot be directly degenerated to a problem of synthesizing a 3*R* robot. The same procedure as that of synthesizing the 6*R* type-D robot should be used to synthesize the 5*R* type-C robot. The synthesis procedures for the different types of 6*R* robots are given in Section 5 of this paper.

5. SYNTHESIS OF 6*R* ROBOTS

There are 18 time-independent kinematic parameters (a_i, α_i, s_i, $i = 1, 6$) and six time-dependent parameters (θ_i, $i = 1, 6$) for a 6*R* robot. The problem of synthesizing a 6*R* robot for a prescribed accessible region or volume requires one to arrive at an optimum set of values for these 24 parameters. This problem of synthesis is treated in two parts: synthesis of regional structure and synthesis of orientational structure. In Table 2 six types of 6*R* robots are shown, and these may be constructed using different orientation and regional structures. These six robots may be classified into three groups:

Group 1: For the type-C in Table 2, an equivalent spherical joint can be found at the junction of the regional structure and the orientational structure. Because a spherical joint can rotate in any direction in space, the compatibility condition of assembling the region structure and orientation structure at the junction will be satisfied at all times. The locations of the junction point can be determined independent of the link parameters and the location of the region structure. Hence the original synthesis problem is degenerated into a problem of synthesizing 3*R* robots. The robot MA-23 and Unimation Puma 250 series robots belong to this group.

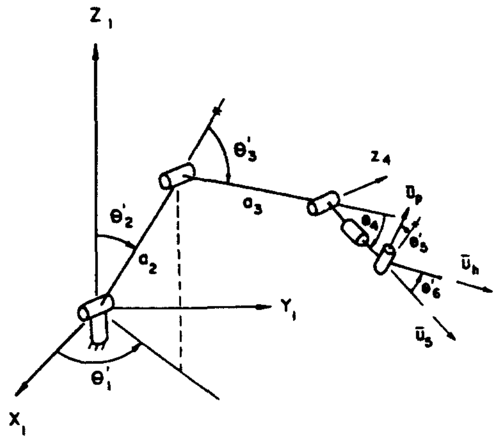
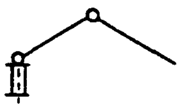
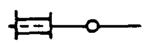
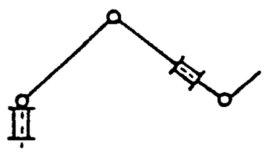
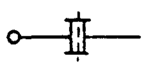
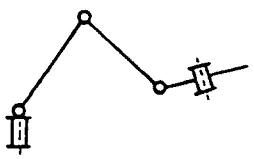
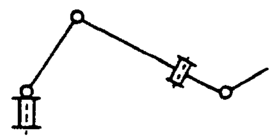


Fig. 5. The 6*R* type-A1 robot.

Group 2: Type-A1 and -B1 in Table 2 belong to this group. The direction of the junction joint axis is independent of link length a_2 and a_3 of the regional structure. Assume that a robot is located at (x_b, y_b). Then use the compatibility conditions of assembly (compatibilities of both direction and location) to determine the locations of junction joints. Thus the problem of synthesizing a 6*R* robot is degenerated to a problem of synthesizing a 3*R* robot. The Cincinnati Milacron T^3 and the Polar 6000 robots belong to type-B1 and ASEA IRS6 and Nordson robots belong to type-A1.

Group 3: In this group, the direction of the junc-

Table 2. Structure of 5*R* industrial robots

REGIONAL STRUCTURE / ORIENTATIONAL STRUCTURE	
	A 
	B 
	C 

tion joint axis depends on all the link parameters and the location of the base of the regional structure. Synthesizing a robot of this group cannot be directly degenerated to a problem of synthesizing $3R$ robots. The type-B2, -A2 and -D in Table 2 belong to this group.

5.1 *Synthesis of the 6R, type-A1, robot*

Figure 5 shows the structure of the $6R$, type-A1, robot. For the given robot hand length l_h (depending on the length of tool to be used), hand position $(x_h, y_h, z_h)_i$, and the direction vector of the robot hand $[(\bar{U}_h)_i = \{(L_h)_i, (M_h)_i, (N_h)_i\}$ and $(\bar{U}_p)_i = \{(L_p)_i, (M_p)_i, (N_p)_i\}$, $i = 1, 2, \ldots, n]$, location of joint 6 is defined by

$$\begin{aligned} (x_6)_{1,i} &= (x_h)_i - l_h(L_h)_i - x_b, \\ (y_6)_{1,i} &= (y_h)_i - l_h(M_h)_i - y_b, \\ (z_6)_{1,i} &= (z_h)_i - l_h(N_h)_i - z_b. \end{aligned} \tag{9}$$

The Plucker line coordinates of axis Z_5 and Z_1 are $\{L_5, M_5, N_5, P_5, Q_5, R_5\}$ and $\{0, 0, 1, 0, 0, 0\}$, respectively. From the geometric configuration and the compatibility condition, the unit vector $(\bar{U}_5)_i$ of joint axis Z_5 must pass through $(x_6, y_6, z_6)_{1,i}$ and be coplanar with axis Z_1. The above condition yields

$$(R_5)_i = (x_6)_{1,i}(M_5)_i - (y_6)_i(L_5)_i = 0. \tag{10}$$

But $(\bar{U}_5)_i$ is perpendicular to $(\bar{U}_p)_i$. Hence

$$(L_5)_i(L_p)_i + (M_5)_i(M_p)_i + (N_5)_i(N_p)_i = 0 \tag{11}$$

and

$$(L_5)_i^2 + (M_5)_i^2 + (N_5)_i^2 = 1. \tag{12}$$

Solving eqns (10)), (11) and (12) for $(L_5)_i$, $(M_5)_i$ and $(N_5)_i$,

$$\begin{aligned} (L_5)_i &= \pm\left(\frac{1}{1 + E^2 + F^2}\right)^{1/2}, \\ (M_5)_i &= E(L_5)_i, \\ (N_5)_i &= -F(L_5)_i, \end{aligned} \tag{13}$$

where

$$E = (y_6)_{1,i}/(x_6)_{1,i},$$

$$F = \frac{(L_p)_{1,i}}{(N_p)_{1,i}} + \frac{(M_p)_i}{(N_p)_i} \cdot E.$$

Then the coordinates of joint 4 will be

$$\begin{aligned} (x_4)_{1,i} &= (x_6)_{1,i} - (L_5)_i \cdot s_5, \\ (y_4)_{1,i} &= (y_6)_{1,i} - (M_6)_i \cdot s_5, \\ (z_4)_{1,i} &= (z_6)_{1,i} - (N_6)_i \cdot s_5. \end{aligned} \tag{14}$$

In this way, for the given $(x_h, y_h, z_h)_i$, and $(\bar{U}_h)_i$ and $(\bar{U}_p)_i$, one may get the corresponding location of joint 4 $(x_4, y_4, z_4)_{1,i}$. The coordinates $(x_4, y_4, z_4)_{1,i}$ here are the same as $(x_e, y_e, z_e)_{1,i}$ given in Section 2. Now, the problem is reduced to a synthesis problem of $3R$ robots. Use the procedure described in Section 2 to obtain the location of robot (x_b, y_b, z_b), the link length a_2, a_3 and the joint motion limits $(\theta_1')_{min}$, $(\theta_1')_{max}$, $(\theta_2')_{min}$, $(\theta_2')_{max}$, $(\theta_3')_{min}$ and $(\theta_3')_{max}$.

After synthesizing the $3R$ robot, the unit vector along link a_3 is found.

$$\begin{aligned} (\bar{U}_3)_i &= (L_3)_i\bar{I} + (M_3)_i\bar{J} + (N_3)_i\bar{K}, \\ (L_3)_i &= \sin(\theta_{2i}' + \theta_{3i}') \cos \theta_{1i}', \\ (M_3)_i &= \sin(\theta_{2i}' + \theta_{3i}') \sin \theta_{1i}', \\ (N_3)_i &= \cos(\theta_{2i}' + \theta_{3i}'). \end{aligned} \tag{15}$$

The unit vector along the axis of joint 4 is given by

$$(\bar{U}_4)_i = -\sin \theta_{1i}'\bar{I} + \cos \theta_{1i}'\bar{J}. \tag{16}$$

From Fig. 5, we note that

$$(\theta_4')_i = \pm\cos^{-1}[(\bar{U}_3)_i \cdot (\bar{U}_5)_i], \tag{17}$$

where the positive sign holds when

$$(\bar{U}_3)_i \times (\bar{U}_5)_i \cdot (\bar{U}_4)_i \geq 0$$

and the directions $(\bar{U}_5)_i$ are determined by eqn (12).

$$(\theta_5')_i = \pm\cos^{-1}[(U_4)_i \cdot (U_p)_i], \tag{18}$$

where the positive sign holds when

$$(\bar{U}_4)_i \times (\bar{U}_p)_i \cdot (\bar{U}_5)_i \geq 0,$$

$$(\theta_6')_i = \pm\cos^{-1}[(\bar{U}_5)_i \cdot (\bar{U}_h)_i], \tag{19}$$

where the positive sign holds when

$$(\bar{U}_5)_i \times (\bar{U}_h)_i \cdot (\bar{U}_p)_i \geq 0.$$

The above synthesis procedure may be summarized in the following steps:

(1) Given the following information:
 (a) Robot hand position $(x_h, y_h, z_h)_i$, direction $(\bar{U}_h)_i$ and $(\bar{U}_p)_i$, $i = 1, 2, 3, \ldots, n$.
 (b) Robot hand length l_h depending on the length of the tool to be used.
 (c) The link length s_4 depending on the driving motor to be used for joint 4.
(2) Assume the location (x_b, y_b) of the robot base (the first joint).
(3) Use eqn (14) to find the corresponding location $(x_4, y_4, z_4)_{1,i}$, $i = 1, 2, 3, \ldots, n$..
(4) Use the procedure described in Section 2 to synthesize the $3R$ robot which can access all $(x_4, y_4, z_4)_{1,i}$, $i = 1, 2, 3, \ldots, n$.

(5) Find the motion limits for joints 4, 5 and 6 by using eqns (17), (18) and (19).

5.2 *Synthesis of the 6R, type-B1, robot*

Using the procedure developed in the previous section for the given work stations $(x_h, y_h, z_h)_i$, $(\bar{U}_h)_i$ and $(\bar{U}_p)_i$ (see Fig. 6) the location of joint 5 can be calculated by the following equations:

$$\begin{aligned}(x_5) &= (x_h)_{1,i} - s_6\cdot(L_h)_i - x_b,\\ (y_5)_i &= (y_h)_{1,i} - s_6\cdot(M_h)_i - y_b,\\ (z_5)_i &= (z_h)_{1,i} - s_6\cdot(N_h)_i - z_b.\end{aligned} \tag{20}$$

Because $\alpha_5 = 90°$, i.e. $(\bar{U}_5)_{1,i}$ is perpendicular to $(\bar{U}_h)_{1,i}$, it yields

$$(\bar{U}_5)_{1,i}\cdot(\bar{U}_h)_{1,i} = 0$$

or

$$(L_5)_i(L_h)_i + (M_5)_i(M_h)_i + (N_5)_i(N_h)_i = 0 \tag{21}$$

and

$$(L_5)_i^2 + (M_5)_i^2 + (N_5)_i^2 = 1. \tag{22}$$

Let $\{L_5, M_5, N_5, P_5, Q_5, R_5\}$ and $\{0, 0, 1, 0, 0, 0\}$ represent the Plucker coordinates of the axes of joints 5 and 1, respectively. From the conditions of assembly, axis of joint 5 and axis of joint 1 are coplanar. The coplanar condition for these two lines presented by Plucker coordinate yields

$$(R_5)_i = 0. \tag{23}$$

By definition

$$(R_5)_i = (x_5)_{1,i}(M_5)_i - (y_5)_{1,i}(L_5)_i = 0. \tag{24}$$

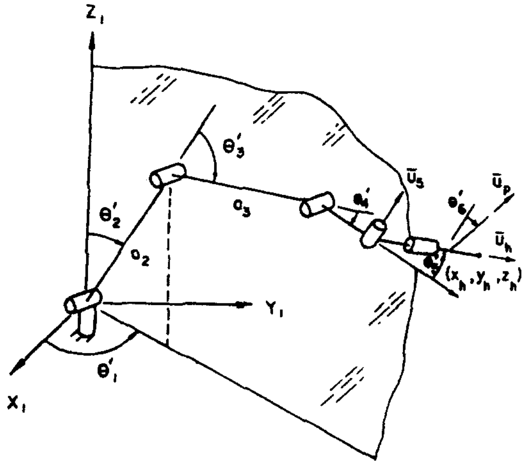


Fig. 6. The 6*R* type-B1 robot.

We have our unknowns $(L_5)_i$, $(M_5)_i$, $(N_5)_i$, $(R_5)_i$ and four independent equations (21) through (24). Solving for $(L_5)_i$, $(M_5)_i$ and $(N_5)_i$,

$$\begin{aligned}(L_5)_i &= \pm\left(\frac{1}{1 + A^2 + B^2}\right)^{1/2},\\ (M_5)_i &= A(L_5)_i,\\ (N_5)_i &= -B(L_5)_i,\end{aligned} \tag{25}$$

where

$$A = (y_5)_{1,i}/(x_5)_{1,i},$$

$$B = \frac{(L_h)_i}{(N_h)_i} + \frac{(M_h)_i}{(N_h)_i}\cdot A.$$

Similarly, the location and direction of joint 4 can be calculated.

$$\begin{aligned}(L_4)_i &= \pm\left(\frac{1}{1 + C^2 + D^2}\right)^{1/2},\\ (M_4)_i &= C(L_4)_i,\\ (N_4)_i &= -D(L_4)_i,\end{aligned} \tag{26}$$

where

$$C = (y_5)_{1,i}/(x_5)_{1,i},$$

$$D = \frac{(L_5)_i}{(N_5)_i} + \frac{(M_5)_i}{(N_5)_i}\cdot C.$$

After solving $(L_4)_i$, $(M_4)_i$ and $(N_4)_i$, the location of joint 4 can be calculated as

$$\begin{aligned}(x_4)_{1,i} &= (x_5)_{1,i} + (L_4)_i\cdot a_4,\\ (y_4)_{1,i} &= (y_5)_{1,i} + (M_4)_i\cdot a_4,\\ (z_4)_{1,i} &= (z_5)_{1,i} + (N_4)_i\cdot a_4.\end{aligned} \tag{27}$$

The problem of synthesizing a 6*R* robot is again reduced to synthesizing a 3*R* robot.

Use the procedure described in Section 2 to synthesize the 3*R* robot, and obtain the link lengths a_2, a_3, the location of robot base x_b, y_b and z_b, and ranges of joint motion $(\theta_1')_{min}$, $(\theta_1')_{max}$, $(\theta_2')_{min}$, $(\theta_2')_{max}$ and $(\theta_3')_{max}$.

After obtaining the 3*R* regional structure, the motion ranges of θ_4', θ_5', θ_6', are calculated from

$$\begin{aligned}(\theta_4')_i &= \cos^{-1}(L_3L_4 + M_3M_4 + N_3N_4)_i,\\ (\theta_5')_i &= \cos^{-1}(L_4L_h + M_4M_h + N_4N_h)_i,\\ (\theta_6')_i &= \pm\cos^{-1}[(\bar{U}_5)_i\cdot(\bar{U}_p)_i],\end{aligned} \tag{28}$$

where the positive sign holds when

$$(\bar{U}_5)_i \times (\bar{U}_p)_i\cdot(\bar{U}_h)_i \geq 0.$$

$(L_4, M_4, N_4)_i$ are calculated by using eqns (26) and $(L_3, M_3, N_3)_i$ by using eqn (29).

$$(L_3)_i = \frac{(x_4)_{1,i} - (x_3)_{1,i}}{a_3},$$
$$(M_3)_i = \frac{(y_4)_{1,i} - (y_3)_{1,i}}{a_3}, \quad (29)$$
$$(N_3)_i = \frac{(z_4)_{1,i} - (z_3)_{1,i}}{a_3},$$

where

$$(x_3)_{1,i} = a_2 \sin(\theta_2')_i \cos(\theta_1')_i,$$
$$(y_3)_{1,i} = a_2 \sin(\theta_2')_i \sin(\theta_1')_i,$$
$$(z_3)_{1,i} = a_2 \cos(\theta_2')_i,$$
$$i = 1, 2, 3, \ldots, n.$$

The synthesis procedure may be summarized in the following steps:

Step 1: Choose proper values for a_4 and s_6. These values depend upon the type of driving motor to be used. Let a_4 and s_6 be as small as possible.

Step 2: Assume the robot base (first joint) to locate x_b, y_b.

Step 3: Calculate the location and direction of joint 5 corresponding to each given hand location.

Step 4: Calculate the location and direction of joint 4 corresponding to each given hand location.

Step 5: Repeat Step 3 and 4 until we get all $(x_4)_{1,i}$, $(y_4)_{1,i}$ and $(z_4)_{1,i}$, $i = 1, 2, \ldots, n$ corresponding to each given working station. The above steps have the synthesis problem reduced to a 3*R* robot synthesis problem. Use the procedure stated in Section 2 paper to synthesize the 3*R* robot.

Step 6: Calculate the joint-motion limits.

5.3 *Synthesis of the* 6R, *type-C, robot*

The synthesis procedure of the 6*R*, type-C, robot is easier than that of type-A1 and -B1, because the last three joint axes intersect at one point.

The unit vector of z_4 axis, $(\overline{U}_4)_i$, can be written as

$$(\overline{U}_4)_i = (L_4)_i\overline{I} + (M_4)_i\overline{J} + (N_4)_i\overline{K},$$
$$(L_4)_i = \sin(\theta_2' + \theta_3')_i \cos(\theta_1')_i, \quad (30)$$
$$(M_4)_i = \sin(\theta_2' + \theta_3')_i \sin(\theta_1')_i,$$
$$(N_4)_i = \cos(\theta_2' + \theta_3')_i.$$

From Fig. 3 (a 5*R* case similar to the case of 6*R* under consideration) we note

$$(\theta_5')_i = \cos^{-1}[(\overline{U}_4)_i \cdot (\overline{U}_h)_i]. \quad (31)$$

The unit vector along Z_5 axis can be obtained by

$$(\overline{U}_5)_i = \frac{-(\overline{U}_4)_i \times (\overline{U}_h)_i}{(\overline{U}_4)_i \times (\overline{U}_h)_i},$$
$$(\theta_6')_i = \pm\cos^{-1}[(\overline{U}_5)_i \cdot (\overline{U}_p)_i], \quad (32)$$

where positive sign holds when

$$(\overline{U}_5)_i \times (\overline{U}_p)_i \cdot (\overline{U}_h)_i \geq 0$$

and the unit vector along Z_3 is

$$(\overline{U}_3)_i = -\sin(\theta_1')_i\overline{I} + \cos(\theta_1')_i\overline{J},$$

then

$$(\theta_4')_i = \cos^{-1}[(\overline{U}_3)_i \cdot (\overline{U}_5)_i]. \quad (33)$$

The synthesizing procedure is summarized below:

(1) For a given length of robot hand l_h, specified working positions $(x_h, y_h, z_h)_i$ and approaching directions $(L_h, M_h, N_h)_i$, the corresponding locations of the equivalent spherical joint $(x_e, y_e, z_e)_i$ (intersection of axes 4, 5 and 6) can be obtained.

$$(x_e)_i = (x_h)_i - l_h(L_h)_i,$$
$$(y_e)_i = (y_h)_i - l_h(M_h)_i, \quad (34)$$
$$(z_e)_i = (z_h)_i - l_h(N_h)_i,$$
$$i = 1, 2, 3, \ldots, n.$$

(2) After getting the location (x_e, y_e, z_e), it becomes a problem of synthesizing a 3*R* robot with a prescribed working space. The same procedure developed in Section 2 can be applied here to get the optimal location of the robot base (first joint) x_b, y_b, z_b, link parameters a_2, a_3 and motion ranges of joints $(\theta_1')_{min}$, $(\theta_1')_{max}$, $(\theta_2')_{min}$, $(\theta_2')_{max}$, $(\theta_3')_{min}$ and $(\theta_3')_{max}$.

(3) Using eqns (31), (32) and (33), find the ranges of motion for joints 4, 5 and 6.

5.4 *The* 6R *type-A2, -B2 and -D robots*

Because the direction of joint axis 4 depends on the location base (x_b, y_b, z_b) and the link length a_2 and a_3 the 6*R*, type-A2, -B2 and -D robots cannot be directly decomposed into two parts and degenerated to a problem of synthesis of 3*R* robots.

However, for a given robot-hand position $(x_h, y_h, z_h)_i$ and approaching direction $(\overline{U}_h)_i$, the location of joint 5 $(x_5, y_5, z_5)_{1,i}$ can be calculated.

$$(x_5)_{1,i} = (x_h)_{1,i} - (L_h)_i a_5,$$
$$(y_5)_{1,i} = (y_h)_{1,i} - (M_h)_i a_5, \quad (35)$$
$$(z_5)_{1,i} = (z_h)_{1,i} - (N_h)_i a_5.$$

The joint 4 must be located somewhere in a sphere which has a radius of a_4 and center-location

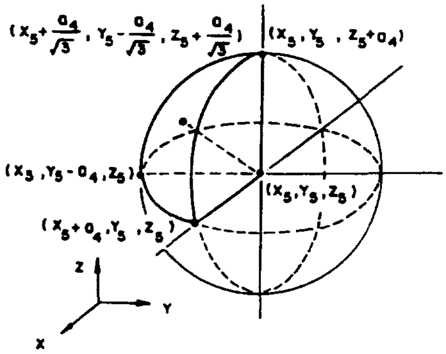


Fig. 7. Possible location of joint 4.

$(x_5, y_5, z_5)_{1,i}$. If the regional structure is synthesized in such a way that its accessible region can cover every spherical region corresponding to every working station $(x_5, y_5, z_5)_{1,i}$, then it is possible to degenerate the original problem to a problem of synthesizing $3R$ robots.

For each given working position of robot hand $(x_h, y_h, z_h)_{1,i}$, let the following 14 points describe the spherical region (see Fig. 7).

$$(x_5 \pm a_4, 0, 0)_{1,i},$$
$$(0, y_5 \pm a_4, 0)_{1,i},$$
$$(0, 0, a_5 \pm a_4)_{1,i},$$
$$(x_5 \pm a_4/\sqrt{3}, y_5 \pm a_4/\sqrt{3}, z_5 \pm a_4/\sqrt{3})_{1,i}.$$

Then synthesize a $3R$ robot such that its accessible region can cover all of the points described above. After the $3R$ robot has been synthesized, determine the motion ranges of joints 4, 5 and 6. This is achieved by performing a joint displacement analysis of a $6R$ robot. For this type of robots, it is not easy to express the joint displacements in simple closed-form solutions. Either the method by Duffy[2] or the numerical method developed by Pieper and Roth[3] can be used to solve this problem. It may take relatively longer time to compute the joint displacements. However, this kind of structure is seldom preferred for the popular industrial robots.

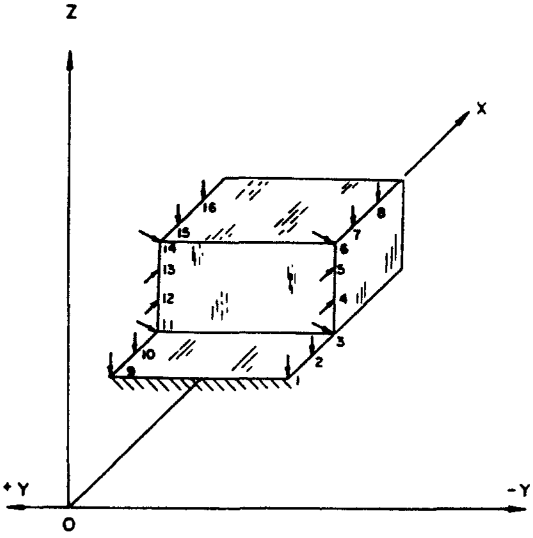


Fig. 8. Working stations for a specified job.

6. ILLUSTRATING EXAMPLE

A specified job needs a robot to work on 18 working stations as shown in Fig. 8. The location

Table 3. Locations and orientations of working stations

No.	X	Y	Z	L_h	M_h	N_h	L_p	M_p	N_p
1	100	-25	0	0.000	0.000	-1.000	0.000	1.000	0.000
2	110	-25	0	0.000	0.000	-1.000	0.000	1.000	0.000
3	120	-25	0	0.707	0.000	-0.707	0.000	1.000	0.000
4	120	-25	15	1.000	0.000	0.000	0.000	1.000	0.000
5	120	-25	30	1.000	0.000	0.000	0.000	1.000	0.000
6	120	-25	45	0.707	0.000	-0.707	0.000	1.000	0.000
7	130	-25	45	0.000	0.000	-1.000	0.000	1.000	0.000
8	140	-25	45	0.000	0.000	-1.000	0.000	1.000	0.000
9	100	25	0	0.000	0.000	-1.000	0.000	1.000	0.000
10	110	25	0	0.000	0.000	-1.000	0.000	1.000	0.000
11	120	25	0	0.707	0.000	-0.707	0.000	1.000	0.000
12	120	25	15	1.000	0.000	0.000	0.000	1.000	0.000
13	120	25	30	1.000	0.000	0.000	0.000	1.000	0.000
14	120	25	45	0.707	0.00	-0.707	0.000	1.000	0.000
15	130	25	45	0.000	0.000	-1.000	0.000	1.000	0.000
16	140	25	45	0.000	0.000	-1.000	0.000	1.000	0.000

of and orientation at each of the working stations are listed in Table 3.

Obviously, this job requires rigid-body motion of the robot hand. In other words, a 6*R* robot is required to perform this job. Assume that the minimum required space for mounting the driving motor is 5 in.

The procedures developed in previous sections were computerized. Following are the designs for different types of robots. Each of the designs required about 5 s of CPU time on IBM 370/168.

(1) Type-A1 robot.
 (a) Base location: (75.0, 0.0, 45.).
 (b) Link lengths: $a_2 = a_3 = 35.18$.
 (c) Joint displacements:

$$\begin{aligned} \theta_1 &= -45^\circ \sim 45^\circ, \\ \theta_2 &= 35^\circ \sim 97^\circ, \\ \theta_3 &= 0^\circ \sim 94^\circ, \\ \theta_4 &= 1^\circ \sim 99^\circ, \\ \theta_5 &= -45^\circ \sim 45^\circ, \\ \theta_6 &= -90^\circ \sim 0^\circ. \end{aligned}$$

(2) Type-B1 robot.
 (a) Base location: (92.1, 0.0, 22.5).
 (b) Link lengths: $a_2 = a_3 = 31.52$.
 (c) Joint displacements:

$$\begin{aligned} \theta_1 &= -73^\circ \sim 73^\circ, \\ \theta_2 &= 0^\circ \sim 59^\circ, \\ \theta_3 &= 0^\circ \sim 126^\circ, \\ \theta_4 &= -122^\circ \sim 122^\circ, \\ \theta_5 &= -48^\circ \sim 48^\circ, \\ \theta_6 &= -48^\circ \sim 48^\circ, \\ \theta_6 &= -163^\circ \sim -18^\circ. \end{aligned}$$

(3) Type-C robot.
 (a) Base location: (75.0, 0.0, 42.5).
 (b) Link lengths: $a_2 = a_3 = 35.03$.
 (c) Joint displacements:

$$\begin{aligned} \theta_1 &= -45^\circ \sim 45^\circ, \\ \theta_2 &= 38^\circ \sim 102^\circ, \\ \theta_3 &= 0^\circ \sim 92^\circ, \\ \theta_4 &= -65^\circ \sim 65^\circ, \\ \theta_5 &= -73^\circ \sim 97^\circ, \\ \theta_6 &= -86^\circ \sim 86^\circ. \end{aligned}$$

7. SUMMARY AND CONCLUSION

This paper developed the methodology to design a 3*R* industrial robot. This design procedure was then extended to 4*R* robots. It was also shown that the design of 5*R* robots (type-A and -B) and 6*R* robots (type-A1, -B1 and -C) could be broken down to a problem of synthesizing 3*R* robots. These robots are easy to control also, because the joint displacements of these robots can be expressed in closed-form equations. Because of this reason, these robot structures have been widely used. The 6*R* robots of type-A2, -B2, and -D are more complex and appear to be seldom in use in the popular industrial robots.

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SYNTHESE DES ROBOTS 3R, 4R, 5R ET 6R POUR UNE REGION DE TRAVAIL DONNEE

Résumé—Au cours d'une étude précédente, les auteurs ont analysé les structures régionales et d'orientation d'un robot général 6R comme des ensembles séparés 3R; les paramètres cinématiques optimaux ont aussi été obtenus. L'article présent utilise les résultats précédents pour développer des méthodes de conception des robots industriels 3R, 4R, 5R et 6R. Pour un groupe de conditions de conception donné, comprenant une description complète de la région accessible et du type de robot, l'article présente pas à pas un procédé permettant d'obtenir les données de conception concernant les tiges d'assemblage. les orientations relatives des axes des paires de révolution et la série des déplacements angulaires des articulations. La méthode est illustrée par un exemple de synthèse du robot 6R.