An Brief introduction to complex number

(From http://www.purplemath.com/modules/complex.htm)

1.Invention of a imaginary number

In junior high school, you might be told that you can't take the square root of a negative number. That's because you had no numbers which were negative after you'd squared them (so you couldn't "go backwards" by taking the square root). Every number was positive after you squared it. So you couldn't very well square-root a negative and expect to come up with anything sensible.

Now, however, you can take the square root of a negative number, but it involves using a new number to do it. This new number was invented (discovered?) around the time of the Reformation. At that time, nobody believed that any "real world" use would be found for this new number, other than easing the computations involved in solving certain equations, so the new number was viewed as being a pretend number invented for convenience sake.

(But then, when you think about it, aren't *all* numbers inventions? It's not like numbers grow on trees! They live in our heads. We made them *all* up! Why not invent a new one, as long as it works okay with what we already have?).

Anyway, this new number was called "i", standing for "imaginary", because "everybody knew" that i wasn't "real". (That's why you couldn't take the square root of a negative number before: you only had "real" numbers; that is, numbers without the "i" in them.) The imaginary is defined to be:

$$i = \sqrt{-1}$$

Then:

$$i^2 = \left(\sqrt{-1}\right)^2 = -1$$

Now, you may think you can do this:

$$i^2 = \left(\sqrt{-1}\right)^2 = \sqrt{(-1)^2} = \sqrt{1} = 1$$

But this doesn't make any sense! You already *have* two numbers that square to 1; namely -1 and $+1$. And i already squares to -1. So it's not reasonable that i would also square to 1. This points out an important detail: When dealing with imaginaries, you gain something (the ability to deal with negatives inside square roots), but you also lose something (some of the flexibility and convenient rules you used to have when dealing with square roots). In particular, YOU MUST ALWAYS DO THE i-PART FIRST!

- **Simplify *sqrt*(–9).**

$$\sqrt{-9} = \sqrt{9 \cdot (-1)} = \sqrt{9}\sqrt{-1} = \sqrt{9} \cdot i = 3i$$

(Warning: The step that goes through the third "equals" sign is " $\sqrt{-1} = i$", not "$\sqrt{-1} = \sqrt{i}$". The i is *outside* the radical.)

- **Simplify *sqrt*(–25).**

$$\sqrt{-25} = \sqrt{25\cdot(-1)} = \sqrt{25}\sqrt{-1} = 5i$$

- **Simplify *sqrt*(–18).**

$$\sqrt{-18} = \sqrt{9\cdot 2\cdot(-1)} = \sqrt{9}\sqrt{2}\sqrt{-1} = 3\sqrt{2}i$$

- **Simplify –*sqrt*(–6).**

$$-\sqrt{-6} = -\sqrt{6\cdot(-1)} = -\sqrt{6}\sqrt{-1} = -\sqrt{6}i$$

In your computations, you will deal with i just as you would with x, except for the fact that x^2 is just x^2, but i^2 is -1:

- **Simplify $2i + 3i$.**

$2i + 3i = (2 + 3)i = \mathbf{5i}$

- **Simplify $16i - 5i$.**

$16i - 5i = (16 - 5)i = \mathbf{11i}$

- **Multiply and simplify $(3i)(4i)$.**

$(3i)(4i) = (3\cdot 4)(i\cdot i) = (12)(i^2) = (12)(-1) = \mathbf{-12}$

- **Multiply and simplify $(i)(2i)(-3i)$.**

$(i)(2i)(-3i) = (2\cdot -3)(i\cdot i\cdot i) = (-6)(i^2\cdot i)$

$=(-6)(-1\cdot i) = (-6)(-i) = \mathbf{6i}$

Note this last problem. Within it, you can see that $i^3 = -i$, because $i^2 = -1$. Continuing, we get:

$$i^4 = i^2\cdot i^2 = (-1)(-1) = 1$$

This pattern of powers, signs, 1's, and i's is a cycle:

$$i^1 = i$$
$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$
$$i^5 = i^1 = i$$
$$i^6 = i^2 = -1$$
$$i^7 = i^3 = -i$$
$$i^8 = i^4 = 1$$

In other words, to calculate any high power of i, you can convert it to a lower power by taking the closest multiple of 4 that's no bigger than the exponent and subtracting this multiple from the exponent. For example, a common trick question on tests is something along the lines of "Simplify i^{99}", the idea being that you'll try to multiply i ninety-nine times and you'll run out of time, and the teachers will get a good giggle at your expense in the faculty lounge. Here's how the shortcut works:

$$i^{99} = i^{96+3} = i^{(4\times 24)+3} = i^3 = -i$$

That is, $i^{99} = i^3$, because you can just lop off the i^{96}. (Ninety-six is a multiple of four, so i^{96} is just 1, which you can ignore.) In other words, you can divide the exponent by 4 (using long division), discard the answer, and use only the remainder. This will give you the part of the exponent that you care about. Here are a few more examples:

- **Simplify i^{17}.**

  $i^{17} = i^{16+1} = i^{4\cdot 4+1} = i^1 = \boldsymbol{i}$

- **Simplify i^{120}.**

  $i^{120} = i^{4\cdot 30} = i^{4\cdot 30+0} = i^0 = \mathbf{1}$

- **Simplify $i^{64,002}$.**

  $i^{64,002} = i^{64,000+2} = i^{4\cdot 16,000+2} = i^2 = \mathbf{-1}$

Now you've seen how imaginaries work; it's time to move on to complex numbers.

2. Complex numbers

"Complex" numbers have two parts, a "real" part (being any "real" number that you're used to dealing with) and an "imaginary" part (being any number with an "i" in it). The "standard" format for complex numbers is "$a + bi$"; that is, real-part first and i-part last.

Complex numbers are "binomials" of a sort, and are added, subtracted, and multiplied in a similar way. (Division, which is further down the page, is a bit different.) First, though, you'll probably be asked to demonstrate that you understand the definition of complex numbers.

- **Solve $3 - 4i = x + yi$**

  Finding the answer to this involves nothing more than knowing that two complex numbers can be equal only if their real and imaginary parts are equal. In other words, **$3 = x$ and $-4 = y$.**

To simplify complex-valued expressions, you combine "like" terms and apply the various other methods you learned for working with polynomials.

- **Simplify $(2 + 3i) + (1 - 6i)$.**

  $(2 + 3i) + (1 - 6i) = (2 + 1) + (3i - 6i) = 3 + (-3i) = \mathbf{3 - 3}\boldsymbol{i}$

- **Simplify $(5 - 2i) - (-4 - i)$.**

  $(5 - 2i) - (-4 - i)$

$= (5 - 2i) - \mathbf{1}(-4 - i) = 5 - 2i - 1(-4) - 1(-i)$

$= 5 - 2i + 4 + i = (5 + 4) + (-2i + i)$

$= (9) + (-1i) = \mathbf{9 - i}$

You may find it helpful to insert the "1" in front of the second set of parentheses (highlighted in red above) so you can better keep track of the "minus" being multiplied through the parentheses.

- **Simplify $(2 - i)(3 + 4i)$.**

$(2 - i)(3 + 4i) = (2)(3) + (2)(4i) + (-i)(3) + (-i)(4i)$

$= 6 + 8i - 3i - 4i^2 = 6 + 5i - 4(-1)$

$= 6 + 5i + 4 = \mathbf{10 + 5i}$

Remember that multiplying and adding with complexes works just like multiplying and adding polynomials, except that, while x^2 is just x^2, i^2 is -1. You can use the exact same techniques for simplifying complex-number expressions as you do for polynomial expressions, but you can simplify even further with complexes because i^2 reduces to the number -1.

Adding and multiplying complexes isn't too bad. It's when you work with fractions (that is, with division) that things turn ugly. Most of the reason for this ugliness is actually arbitrary. Remember back in elementary school, when you first learned fractions? Your teacher would get her panties in a wad if you used "improper" fractions. For instance, you couldn't say " $^3/_2$ "; you had to convert it to "$1\,^1/_2$". But now that you're in algebra, nobody cares, and you've probably noticed that "improper" fractions are often more useful than "mixed" numbers. The issue with complex numbers is that your professor will get his boxers in a bunch if you leave imaginaries in the denominator. So how do you handle this?

Suppose you have the following exercise:

- **Simplify $\dfrac{3}{2i}$**

This is pretty "simple", but they want me to get rid of that i underneath, in the denominator. The 2 in the denominator is fine, but the i has got to go. To do this, I will use the fact that $i^2 = -1$. If I multiply the fraction, top and bottom, by i, then the i underneath will vanish in a puff of negativity:

$$\frac{3}{2i} = \frac{3}{2i} \cdot \frac{i}{i} = \frac{3i}{2i^2} = \frac{3i}{2(-1)} = \frac{3i}{-2} = -\frac{3i}{2} = -\frac{3}{2}i$$

So the answer is $\mathbf{-\dfrac{3}{2}i}$

This was simple enough, but what if they give you something more complicated?

- **Simplify $\dfrac{3}{2+i}$**

If I multiply this fraction, top and bottom, by i, I'll get:

$$\frac{3}{2+i} = \frac{3}{2+i} \cdot \frac{i}{i} = \frac{3i}{2i+i^2} = \frac{3i}{2i-1} = \frac{3i}{-1+2i}$$

Since I still have an i underneath, this didn't help much. So how do I handle this simplification? I use something called "conjugates". The conjugate of a complex number $a + bi$ is the same number, but with the opposite sign in the middle: $a - bi$. When you multiply conjugates, you are, in effect, multiplying to create something in the pattern of a difference of squares:

$$\begin{aligned}(a+bi)(a-bi) &= a^2 - abi + abi - (bi)^2 \\ &= a^2 - b^2(i^2) \\ &= a^2 - b^2(-1) \\ &= a^2 + b^2\end{aligned}$$

Note that the i's disappeared, and the final result was a *sum* of squares. This is what the conjugate is for, and here's how it is used:

$$\begin{aligned}\frac{3}{2+i} &= \frac{3}{2+i} \cdot \frac{2-i}{2-i} = \frac{3(2-i)}{(2+i)(2-i)} \\ &= \frac{6-3i}{4-2i+2i-i^2} = \frac{6-3i}{4-(-1)} \\ &= \frac{6-3i}{4+1} = \frac{6-3i}{5} = \frac{6}{5} - \frac{3}{5}i\end{aligned}$$

So the answer is $\mathbf{\frac{6}{5} - \frac{3}{5}i}$

In the last step, note how the fraction was split into two pieces. This is because, technically speaking, a complex number is in two parts, the real part and the i part. They aren't supposed to "share" the denominator. To be sure your answer is completely correct, split the complex-valued fraction into its two separate terms.

You'll probably only use complexes in the context of solving quadratics for their zeroes. There are many other *practical* uses for complexes, for which you can go to the following link:

http://en.wikipedia.org/wiki/Complex_number#Applications