Comprehensive Tutorial on Probability Concepts

Prepared for Beginners

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Contents

1 Probability

1.1 Definition

Probability is a measure of how likely an event is to occur. It is always a value between 0 and 1.

- 0: The event will not happen.
- 1: The event will certainly happen.
- Values between 0 and 1 show varying levels of likelihood.

1.2 Formula

$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}}$$

1.3 Example

When a fair six-sided die is rolled:

$$P(\text{getting a 4}) = \frac{1}{6}$$

1.4 Types of Probability

1. **Theoretical Probability:** Based on reasoning or logic.
2. **Experimental Probability:** Based on actual experiments.

$$P(E) = \frac{\text{Number of times event occurred}}{\text{Total number of trials}}$$

3. **Subjective Probability:** Based on personal judgment or estimation.

1.5 Basic Terms

- **Sample Space (S):** Set of all possible outcomes.
- **Event (E):** A subset of the sample space.
- **Complementary Event:** $P(\text{not E}) = 1 - P(E)$
- **Mutually Exclusive Events:** Two events that cannot happen together.
- **Independent Events:** One event does not affect the other.

1.6 Rules of Probability

- **Addition Rule (Mutually Exclusive):**

$$P(A \text{ or } B) = P(A) + P(B)$$

- **Addition Rule (Not Mutually Exclusive):**

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

- **Multiplication Rule (Independent):**

$$P(A \text{ and } B) = P(A) \times P(B)$$

- **Complement Rule:**

$$P(\text{not A}) = 1 - P(A)$$

1.7 Example Problem

A coin is tossed twice. Find the probability of getting at least one head.

$$S = \{HH, HT, TH, TT\}$$
$$E = \{HH, HT, TH\}$$
$$P(E) = \frac{3}{4}$$

2 Joint, Marginal, and Conditional Probability

2.1 Joint Probability

It is the probability of two or more events happening together.

$$P(A \text{ and } B) = P(A \cap B)$$

2.2 Marginal Probability

It is the probability of a single event occurring, regardless of the outcomes of other events.

$$P(A) = \sum_B P(A, B)$$

or

$$P(B) = \sum_A P(A, B)$$

2.3 Conditional Probability

It measures the probability of one event occurring given that another event has already occurred.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

where $P(B) \neq 0$.

2.4 Example

In a deck of 52 cards:

- A: drawing a King
- B: drawing a Heart

Then,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{52}}{\frac{13}{52}} = \frac{1}{13}$$

3 Probability Distributions

A probability distribution gives the probability of each possible outcome in a random experiment.

$$\text{For discrete variables: } \sum P(x_i) = 1$$
$$\text{For continuous variables: } \int_{-\infty}^{\infty} f(x)\,dx = 1$$

3.1 Types of Distributions

- **Discrete Probability Distribution:** Outcomes are countable.
- **Continuous Probability Distribution:** Outcomes are within an interval.

4 Discrete Probability Distributions

4.1 Binomial Distribution

It represents the number of successes in a fixed number of independent Bernoulli trials.

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

4.2 Poisson Distribution

It models the number of occurrences in a fixed interval of time or space.

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

5 Continuous Probability Distributions

5.1 Uniform Distribution

All outcomes are equally likely in a continuous interval.

$$f(x) = \frac{1}{b-a}, \quad a \le x \le b$$

5.2 Normal Distribution

It is a bell-shaped curve symmetric around the mean.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where μ is the mean and σ is the standard deviation.

5.3 Exponential Distribution

Represents time between events in a Poisson process.

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

6 Bayesian Probability

Bayesian probability interprets probability as a measure of belief, updated as new evidence appears.

6.1 Bayes' Theorem

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

6.2 Explanation

- $P(A)$: Prior probability (initial belief)
- $P(B|A)$: Likelihood (probability of evidence given the event)
- $P(B)$: Evidence (total probability of observation)
- $P(A|B)$: Posterior probability (updated belief)

6.3 Example

Suppose 1% of a population has a disease. A test has:

- 99% sensitivity: $P(\text{Positive}|\text{Disease}) = 0.99$
- 95% specificity: $P(\text{Negative}|\text{No Disease}) = 0.95$

Find $P(\text{Disease}|\text{Positive})$.

$$P(\text{Positive}) = P(\text{Positive}|\text{Disease})P(\text{Disease}) + P(\text{Positive}|\text{No Disease})P(\text{No Disease})$$

$$= (0.99)(0.01) + (0.05)(0.99) = 0.0099 + 0.0495 = 0.0594$$

$$P(\text{Disease}|\text{Positive}) = \frac{0.0099}{0.0594} \approx 0.166$$

So the probability the person actually has the disease after testing positive is about 16.6%.

6.4 Interpretation

Bayes' theorem updates our belief in an event (having disease) after receiving new evidence (test positive).

7 Conclusion

This tutorial introduced:

- Basic concepts of probability
- Joint, marginal, and conditional probabilities
- Discrete and continuous probability distributions
- Bayesian probability and Bayes' theorem

These concepts are foundational for machine learning, statistics, and data science.