STUDENT SOLUTIONS MANUAL

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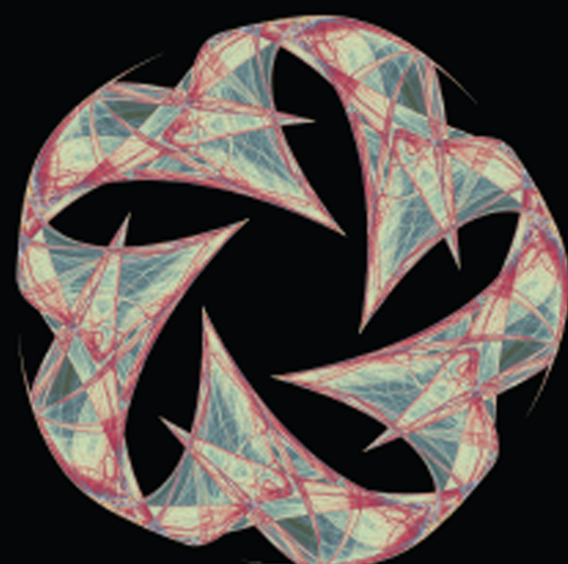
A CHAPMAN & HALL BOOK

Mitchal Dichter

NONLINEAR DYNAMICS AND CHAOS

With Applications to Physics, Biology, Chemistry, and Engineering



Steven H. Strogatz

SECOND EDITION

Student Solutions Manual for Nonlinear Dynamics and Chaos, Second Edition

Mitchal Dichter

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2

Flows on the Line

2.1 A Geometric Way of Thinking

2.1.1

The fixed points of the flow $\dot{x} = \sin(x)$ occur when

$$\dot{x} = 0 \Rightarrow \sin(x) = 0 \Rightarrow x = z\pi \qquad z \in \mathbb{Z}$$

2.1.3

a)

We can find the flow's acceleration $\ddot{x}$ by first deriving an equation containing $\ddot{x}$ by taking the time derivative of the differential equation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}\sin(x) \Rightarrow \ddot{x} = \cos(x)\dot{x}$$

We can obtain $\ddot{x}$ solely as a function of x by plugging in our previous equation for $\dot{x}$.

$$\ddot{x} = \cos(x)\sin(x)$$

b)

We can find what values of x give the acceleration $\ddot{x}$ maximum positive values by using the trigonometric identity

$$\frac{1}{2}\sin(2x) = \sin(x)\cos(x)$$

which can be used to rewrite $\ddot{x}$ as

$$\ddot{x} = \frac{1}{2}\sin(2x)$$

which has maximums when

$$x = \left(z + \frac{1}{4}\right)\pi \qquad z \in \mathbb{Z}$$

2.1.5

a)

A pendulum submerged in honey with the pendulum at the 12 o'clock position corresponding to $x = 0$ is qualitatively similar to $\dot{x} = \sin(x)$. The force near the 12 o'clock position is small, is greatest at the 3 o'clock position, and is again small at the 6 o'clock position.

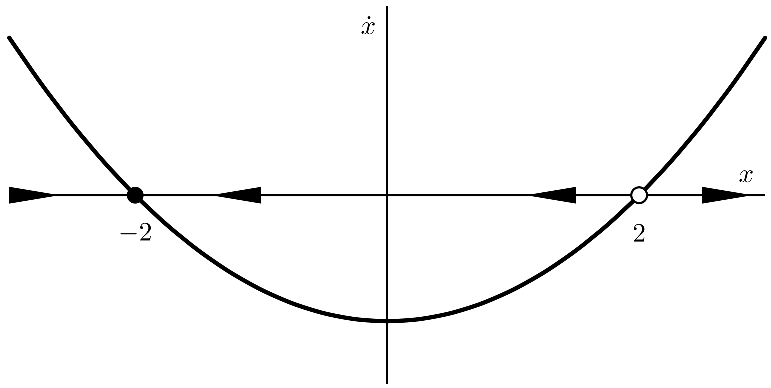
b)

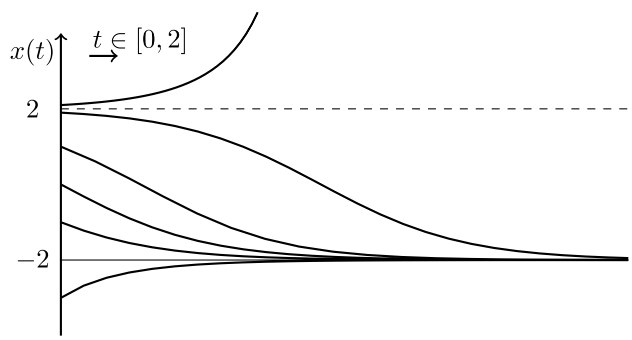
$x = 0$ and $x = \pi$ being unstable and stable fixed points respectively is consistent with our intuitive understanding of gravity.

2.2 Fixed Points and Stability

2.2.1

$$\dot{x} = 4x^2 - 16$$

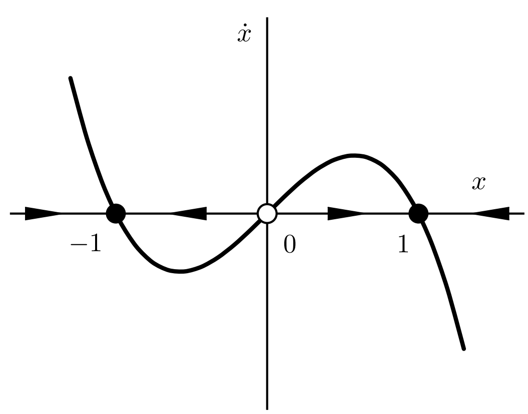


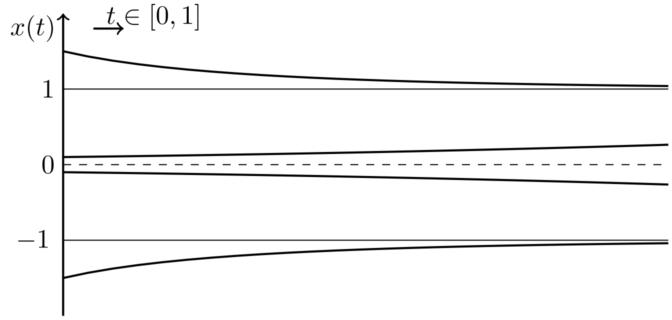


$$x(t) = \frac{2\left(x_0 e^{16t} + x_0 - 2e^{16t} + 2\right)}{-x_0 e^{16t} + x_0 + 2e^{16t} + 2}$$

2.2.3

$$\dot{x} = x - x^3$$

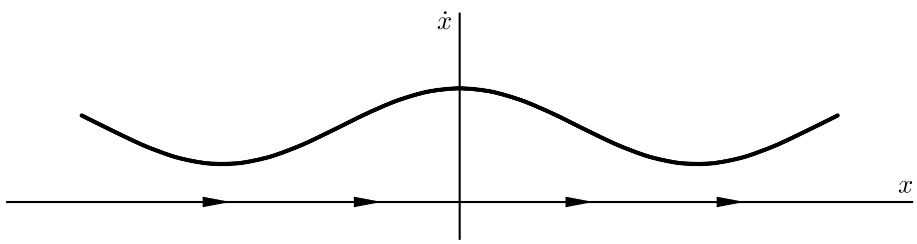




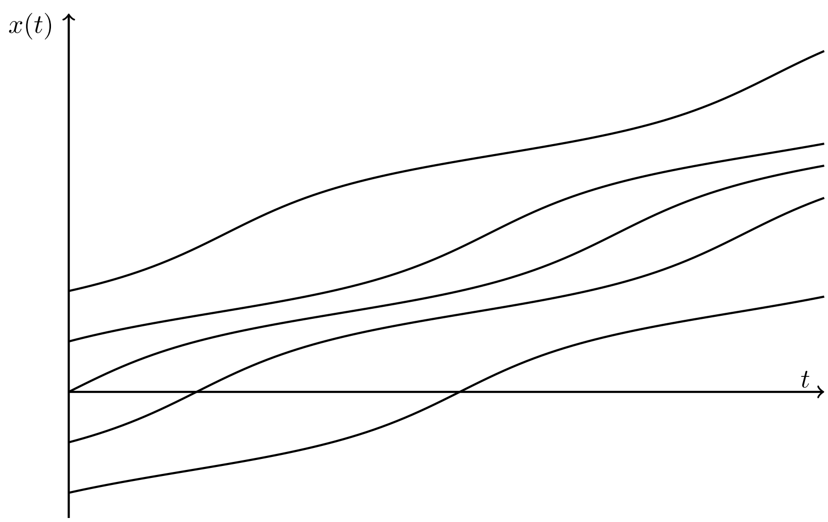
$$x(t) = \frac{\pm e^t}{\sqrt{\frac{1}{x_0^2} + e^{2t} - 1}} \qquad \text{depending on the sign of the initial condition.}$$

2.2.5

$$\dot{x} = 1 + \frac{1}{2}\cos(x)$$



There are no fixed points, but the rate increase for $x(t)$ does vary.

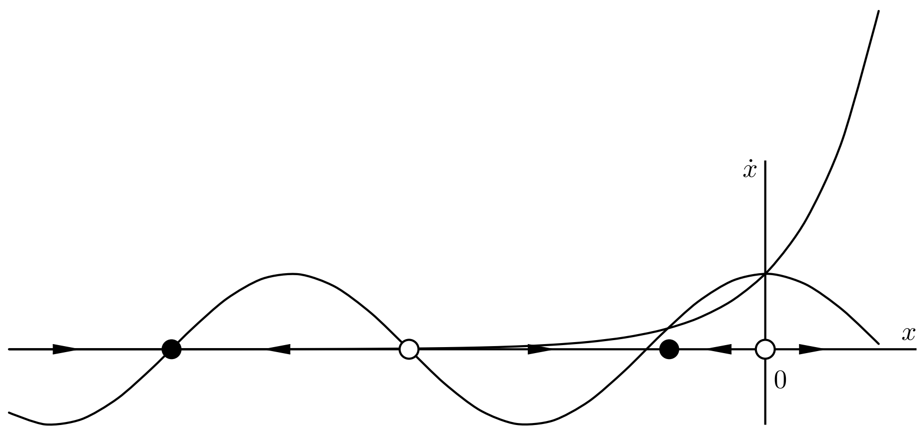


$$x(t) = 2\arctan\left(\sqrt{3}\tan\left(\arctan\left(\frac{\tan\left(\frac{x_0}{2}\right)}{\sqrt{3}}\right) + \frac{\sqrt{3}t}{4}\right)\right)$$

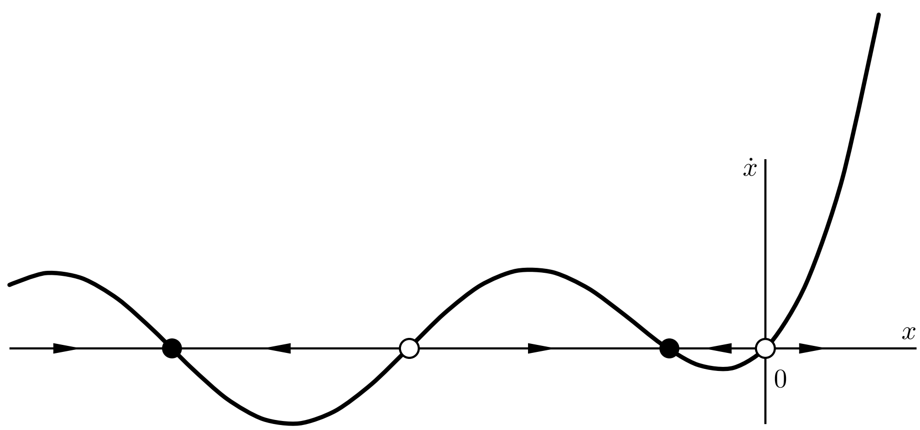
2.2.7

$$\dot{x} = e^x - \cos(x)$$

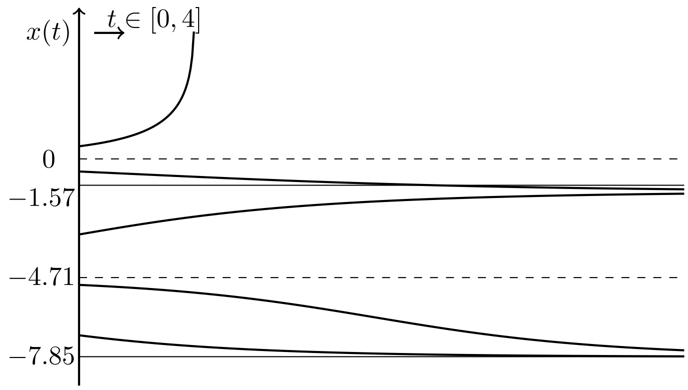
We can't solve for the fixed points analytically, but we can find the fixed points approximately by looking at the intersections of e^x and $\cos(x)$, and determine the stability of the fixed points from which graph is greater than the other nearby.



We could also plot the graph using a computer.



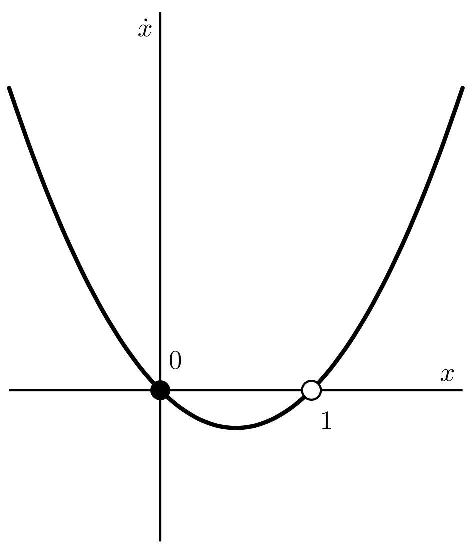
There are fixed points at $x \approx \pi\left(\frac{1}{2} - n\right),\ n \in \mathbb{N}$, and $x = 0$.



Unable to find an analytic solution.

2.2.9

$f(x) = x(x-1)$



2.2.11

RC circuit

$$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC} \Rightarrow \dot{Q} + \frac{1}{RC}Q = \frac{V_0}{R} \qquad Q(0) = 0$$

Multiply by an integrating factor $e^{\frac{t}{RC}}$ to both sides.

$$\dot{Q}e^{\frac{t}{RC}} + \frac{1}{RC}e^{\frac{t}{RC}}Q = \frac{V_0}{R}e^{\frac{t}{RC}} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(Qe^{\frac{t}{RC}}\right) = \frac{V_0}{R}e^{\frac{t}{RC}}$$

Apply an indefinite integral to both sides with respect to t.

$$\int \frac{\mathrm{d}}{\mathrm{d}t}\left(Qe^{\frac{t}{RC}}\right)\mathrm{d}t = \int \frac{V_0}{R} e^{\frac{t}{RC}}\,\mathrm{d}t$$

$$Qe^{\frac{t}{RC}} = V_0 C e^{\frac{t}{RC}} + D \Rightarrow Q(t) = V_0 C + D e^{\frac{-t}{RC}}$$

And using the initial condition.

$$Q(0) = V_0 C + D = 0 \Rightarrow D = -V_0 C \Rightarrow Q(t) = V_0 C\left(1 - e^{\frac{-t}{RC}}\right)$$

2.2.13

Terminal velocity

The velocity $v(t)$ of a skydiver falling follows the equation

$$m\dot{v} = mg - kv^2$$

with m the mass of the skydiver, g the acceleration due to gravity, and $k > 0$ the drag coefficient.

a)

$$v(t) = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}t\right)$$

b)

$$v(t) \xrightarrow{t\to\infty} \sqrt{\frac{mg}{k}}$$

c)

The terminal velocity should occur at a fixed point. The fixed point occurs when

$$\dot{v} = 0 \Rightarrow mg - kv^2 = 0 \Rightarrow v = \sqrt{\frac{mg}{k}}$$

d)

$$v_{avg} = \frac{31400\text{ft} - 2100\text{ft}}{116\text{sec}} \approx 253\frac{\text{ft}}{\text{sec}}$$

e)

$$s(t) = \frac{m}{k}\ln\left(\cosh\left(\sqrt{\frac{gk}{m}}t\right)\right)$$

which can be rewritten using $V = \sqrt{\frac{mg}{k}}$ into

$$s(t) = \frac{V^2}{g}\ln\left(\cosh\left(\frac{g}{V}t\right)\right)$$

Using $s(116\text{sec}) = 29300\text{ft}$ and $g = 32.2\frac{\text{ft}}{\text{sec}^2}$ gives

$$29300\text{ft} = \frac{V^2}{32.2\frac{\text{ft}}{\text{sec}^2}}\ln\left(\cosh\left(\frac{32.2\frac{\text{ft}}{\text{sec}^2}}{V}t\right)\right)$$

which can be solved numerically to give $V \approx 266\frac{\text{ft}}{\text{sec}}$.

2.3 Population Growth

2.3.1

Logistic equation

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) \qquad N(0) = N_0$$

a)

Solve by separating the variables and integrating using partial fractions.

$$rN\left(1-\frac{N}{K}\right) = rN - \frac{rN^2}{K} = \frac{rKN - rN^2}{K} = \frac{r}{K}\left(KN - N^2\right)$$

$$\dot{N} = \frac{\mathrm{d}N}{\mathrm{d}t} = \frac{r}{K}\left(KN - rN^2\right) \Rightarrow \frac{\mathrm{d}N}{KN - N^2} = \frac{r}{K}\mathrm{d}t$$

$$\frac{1}{KN - N^2} = \frac{1}{KN} - \frac{1}{K(N-K)}$$

$$\int \frac{\mathrm{d}N}{KN - N^2} = \int \frac{\mathrm{d}N}{KN} - \int \frac{\mathrm{d}N}{K(N-K)} = \frac{1}{K}\ln(N) - \frac{1}{K}\ln(N-K)$$

$$= \int \frac{r}{K}\mathrm{d}t = \frac{r}{K}t + C$$

$$\frac{1}{K}\ln(N) - \frac{1}{K}\ln(N-K) = \frac{r}{K}t + C$$

$$\ln(N) - \ln(N-K) = rt + KC$$

$$\ln\left(\frac{N}{N-K}\right) = rt + KC$$

$$-\ln\left(\frac{N}{N-K}\right) = \ln\left(\frac{N-K}{N}\right) = \ln\left(1 - \frac{K}{N}\right) = -rt - KC$$

$$1 - \frac{K}{N} = e^{-rt-KC} = e^{-KC}e^{-rt}$$

$$1 - e^{-KC}e^{-rt} = \frac{K}{N}$$

$$N(t) = \frac{K}{1 - e^{-KC}e^{-rt}}$$

$$N(0) = \frac{K}{1 - e^{-KC}} = N_0 \Rightarrow e^{-KC} = 1 - \frac{K}{N_0}$$

$$N(t) = \frac{K}{1 - \left(1 - \frac{K}{N_0}\right)e^{-rt}}$$

b)

Making the change of variables $x = \frac{1}{N}$.

$$N = \frac{1}{x} \Rightarrow \dot{N} = -\frac{\dot{x}}{x^2} = r\frac{1}{x}\left(1 - \frac{1}{xK}\right)$$

$$\dot{x} = \frac{r}{K} - rx$$

$$x(t) = Ce^{-rt} + \frac{1}{K} \Rightarrow \frac{1}{N(t)} = Ce^{-rt} + \frac{1}{K} = \frac{KCe^{-rt} + 1}{K}$$

$$\Rightarrow N(t) = \frac{K}{KCe^{-rt} + 1}$$

$$N(0) = N_0 \Rightarrow N(t) = \frac{K}{1 - \left(1 - \frac{K}{N_0}\right)e^{-rt}}$$

2.3.3

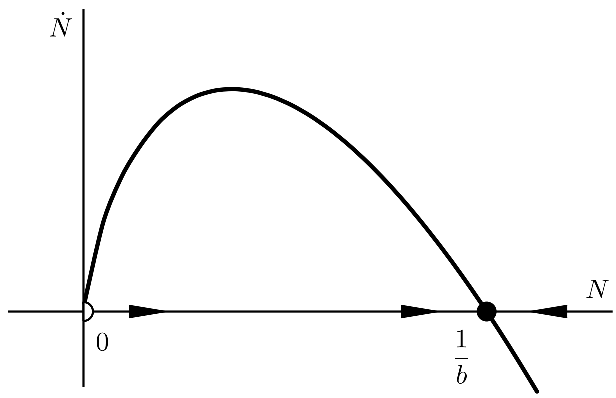
Tumor growth

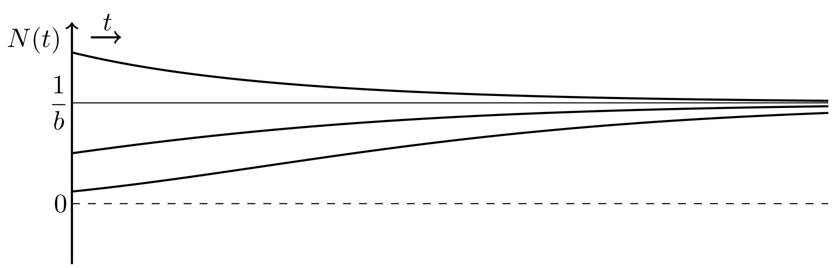
$$\dot{N} = -aN\ln(bN)$$

a)

a can be interpreted as specifying how fast the tumor grows, and $\frac{1}{b}$ specifies the stable size of the tumor.

b)





2.3.5

Dominance of the fittest

$$\dot{X} = aX \qquad \dot{Y} = bY \qquad x(t) = \frac{X(t)}{X(t)+Y(t)} \qquad X_0, Y_0 > 0 \qquad a > b > 0$$

a)

Show as $t \to \infty$, $x(t) \to 1$ by solving for $X(t)$ and $Y(t)$.

$$X(t) = e^{at} \qquad Y(t) = e^{bt} \Rightarrow x(t) = \frac{e^{at}}{e^{at}+e^{bt}} = \frac{1}{1+e^{(b-a)t}}$$

$$a > b > 0 \Rightarrow 0 > b - a \Rightarrow x(t) \to \frac{1}{1+0} = 1 \text{ as } t \to \infty$$

b)

Show as $t \to \infty$, $x(t) \to 1$ by deriving an ODE for $x(t)$.

$$\dot{x} = \frac{\dot{X}Y - X\dot{Y}}{(X+Y)^2} = \frac{aXY - bXY}{(X+Y)^2} = (a-b)\frac{X}{X+Y}\left(1 - \frac{X}{X+Y}\right) = (a-b)x(1-x)$$

This is the logistic equation

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) \qquad r = a - b > 0 \qquad K = 1$$

$$X_0, Y_0 > 0 \Rightarrow 0 < x_0 = \frac{X_0}{X_0 + Y_0} < 1 \Rightarrow x(t) \to 1 = K$$

and $x(t)$ increases monotonically as $t \to \infty$ from our previous analysis of the logistic equation.

2.4 Linear Stability Analysis

2.4.1

$$\begin{aligned}
&\dot{x} = x(1 - x) \\
&\frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = 1 - 2x \\
&\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}(0) = 1 \Rightarrow \text{unstable fixed point} \\
&\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}(1) = -1 \Rightarrow \text{stable fixed point}
\end{aligned}$$

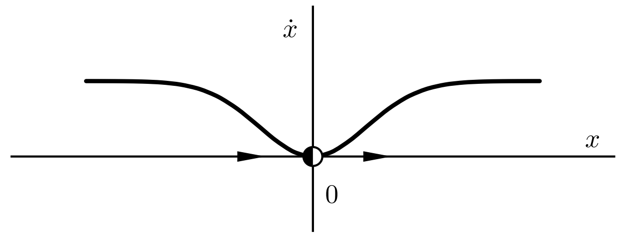
2.4.3

$$\begin{aligned}
&\dot{x} = \tan(x) \\
&\frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = \sec^2(x) \\
&\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}(\pi z) = 1 \Rightarrow \text{unstable fixed point} \quad z \in \mathbb{Z}
\end{aligned}$$

There are also stable fixed-like points at $\pi z + \frac{\pi}{2}$ because $\dot{x}$ is positive on the left and negative on the right. However, these points aren't true fixed points because $\tan(x)$ has infinite discontinuities at these points and hence is not defined there.

2.4.5

$$\begin{aligned}
&\dot{x} = 1 - e^{-x^2} \\
&\frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = 2xe^{-x^2} \\
&\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}(0) = 0 \Rightarrow \text{inconclusive}
\end{aligned}$$



The fixed point 0 is stable from the left and unstable from the right.

2.4.7

$$\dot{x} = ax - x^3 = x(\sqrt{a} - x)(\sqrt{a} + x)$$

$$\frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = a - 3x^2$$

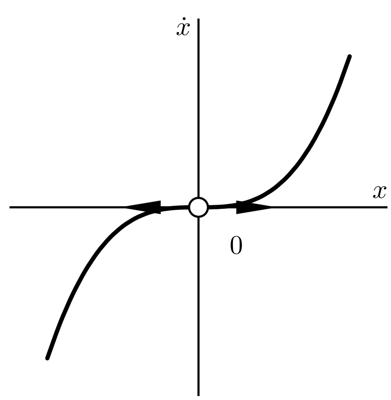
Assuming $a \geq 0$ in order for $\sqrt{a}$ to be a real root.

$$\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}(-\sqrt{a}) = -2a \Rightarrow \begin{cases} \text{unstable} & : a < 0 \\ \text{see 0 root in this case} & : a = 0 \\ \text{stable} & : 0 < a \end{cases}$$

$$\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}(0) = a \Rightarrow \begin{cases} \text{stable} & : a < 0 \\ \text{inconclusive, but graph implies unstable} & : a = 0 \\ \text{unstable} & : 0 < a \end{cases}$$

Assuming $a \geq 0$ in order for $\sqrt{a}$ to be a real root.

$$\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}(\sqrt{a}) = -2a \Rightarrow \begin{cases} \text{unstable} & : a < 0 \\ \text{see 0 root in this case} & : a = 0 \\ \text{stable} & : 0 < a \end{cases}$$



2.4.9

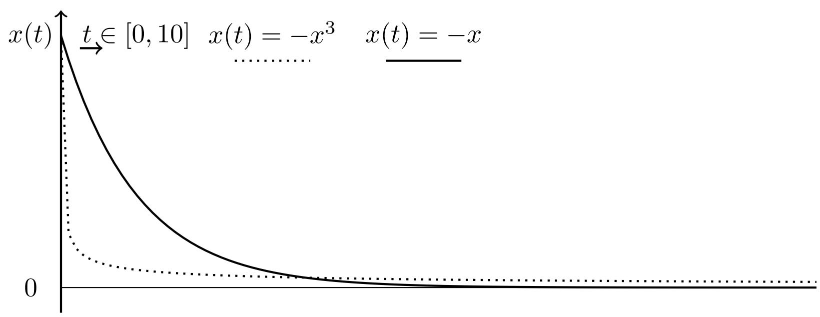
a)

$$x(t) = \frac{\pm 1}{\sqrt{\frac{1}{x_0^2} + 2t}}$$

Depending on the sign of the initial condition

$$\lim_{t\to\infty} x(t) \to \frac{\pm 1}{\sqrt{2t}} \to 0$$

b)



2.5 Existence and Uniqueness

2.5.1

$$\dot{x} = -x^c \qquad x(t) \geq 0 \qquad c \in \mathbb{R}$$

a)

$x = 0$ is the only fixed point $\iff$ $c \neq 0$ and $x(0) \geq 0 \Rightarrow x(t) \to 0$ so $x = 0$ is stable from the right.

b)

$$x(t) = ((c-1)t + x_0^{1-c})^{\frac{1}{1-c}}$$

The particle will reach the origin in finite time if $x_0 = 0$ or if $x_0 > 0$ and $c < 1$ so that a $t > 0$ will make

$$(c-1)t + x_0^{1-c} = 0$$

If $x_0 = 1$ then

$$x(t) = (1 + (c-1)t)^{\frac{1}{1-c}}$$

And with $c < 1$ then

$$t^* = \frac{1}{1-c} > 0 \qquad x(t^*) = 0$$

2.5.3

$$\dot{x} = rx + x^3 \qquad r > 0 \Rightarrow x(t) = \pm\frac{x_0\sqrt{r}e^{rt}}{\sqrt{x_0^2(1-e^{2rt})+r}}$$

Which will blow up when the denominator vanishes at

$$t = \frac{\ln(1+\frac{r}{x_0^2})}{2r}$$

2.5.5

$$\dot{x} = |x|^{\frac{p}{q}} \qquad x(0) = 0 \qquad p, q \text{ are coprime}$$

$x(t) = 0$ is a solution, but we can find another solution through

$$\begin{aligned}
&\int |x|^{\frac{-p}{q}}\,\mathrm{d}x = \int \mathrm{d}t \\
&\frac{1}{1-\frac{p}{q}}|x|^{1-\frac{p}{q}} = t + C \\
&|x| = \left(\left(1-\frac{p}{q}\right)(t+C)\right)^{\frac{1}{1-\frac{p}{q}}} \\
&x(0) = 0 \Rightarrow C = 0 \\
&|x| = \left(\left(1-\frac{p}{q}\right)t\right)^{\frac{1}{1-\frac{p}{q}}}
\end{aligned}$$

a)

$$p < q \Rightarrow \frac{p}{q} < 1 \Rightarrow 0 < 1 - \frac{p}{q}$$

Here there are an infinite number of solutions because we can specify that $x(t) = 0$ for an arbitrary amount of time t_0 and then switch to

$$x(t) = \left(\left(1-\frac{p}{q}\right)(t-t_0)\right)^{\frac{1}{1-\frac{p}{q}}}$$

b)

$$q < p \Rightarrow 1 < \frac{p}{q} \Rightarrow 1 - \frac{p}{q} < 0$$

Here we only have $x(t) = 0$ for all time because the RHS of

$$|x| = \left(\left(1-\frac{p}{q}\right)t\right)^{\frac{1}{1-\frac{p}{q}}}$$

would not always be nonnegative, which is inconsistent with the absolute value on the LHS.

2.6 Impossibility of Oscillations

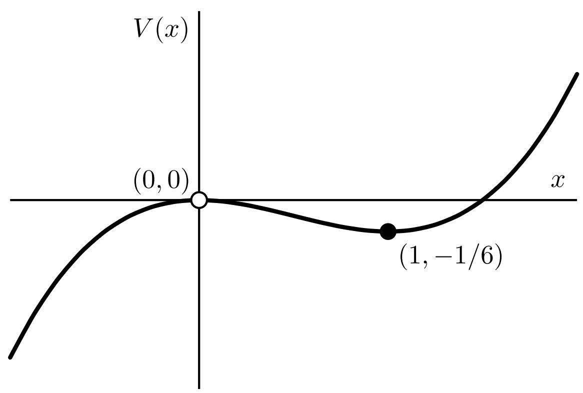
2.6.1

The harmonic oscillator does oscillate along the x-axis, but the position alone does not uniquely describe the state of the system. The system is not uniquely determined unless both the position $x(t)$ and the velocity $\dot{x}(t)$ are specified. The harmonic oscillator is a two-dimensional system so does not contradict the text by the fact that solutions can oscillate.

2.7 Potentials

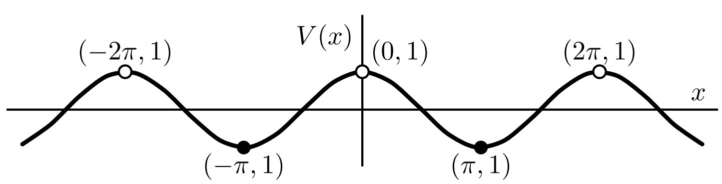
2.7.1

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{3}x^3$$



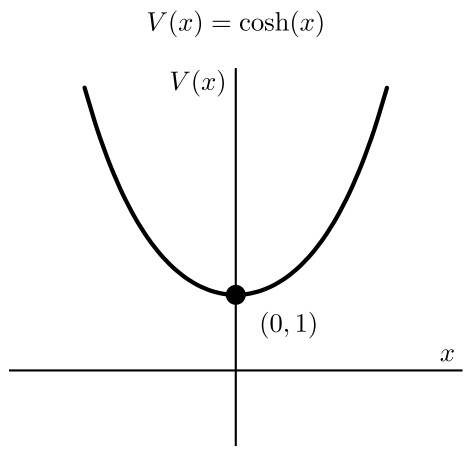
2.7.3

$$V(x) = \cos(x)$$



The pattern of equilibrium points continues in both directions.

2.7.5



2.7.7

Assume for

$$\dot{x} = f(x) = -\frac{\mathrm{d}V}{\mathrm{d}x}$$

there is an oscillating solution $x(t)$ with period $T \neq 0$. Then

$$x(t) = x(t + T) \Rightarrow V(x(t)) = V(x(t + T))$$

but

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leq 0 \Rightarrow x(t)$$

is constant. In other words, the solution $x(t)$ does not oscillate and never moves. Contradiction!

2.8 Solving Equations on the Computer

2.8.1

The slope is constant along horizontal lines because the equation for the slope $\dot{x}$ explicitly depends on x but not on t. Although t varies along the horizontal lines, x remains constant, so the slope remains constant.

2.8.3

a)

$$\dot{x} = -x, x(0) = 1 \Rightarrow x(t) = e^{-t} \qquad x(1) = e^{-1} \approx 0.3678794411$$

b)

$\Delta t = 10^0 \Rightarrow \hat{x}(1) = 0$

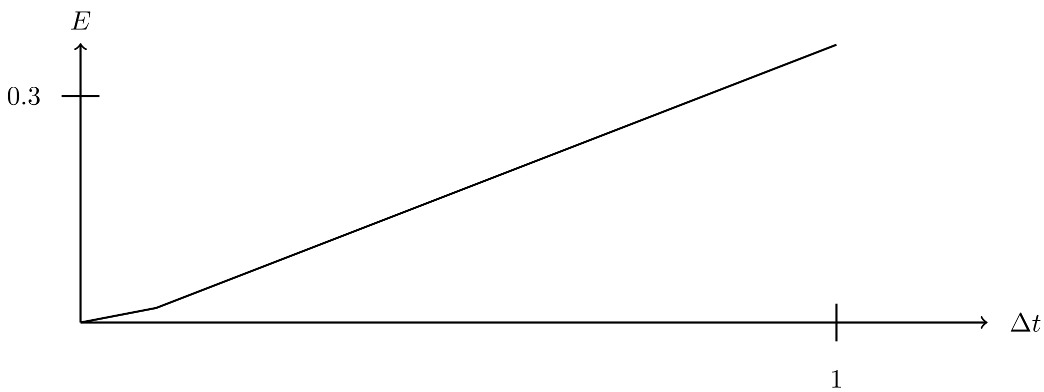
$\Delta t = 10^{-1} \Rightarrow \hat{x}(1) \approx 0.3486784401$

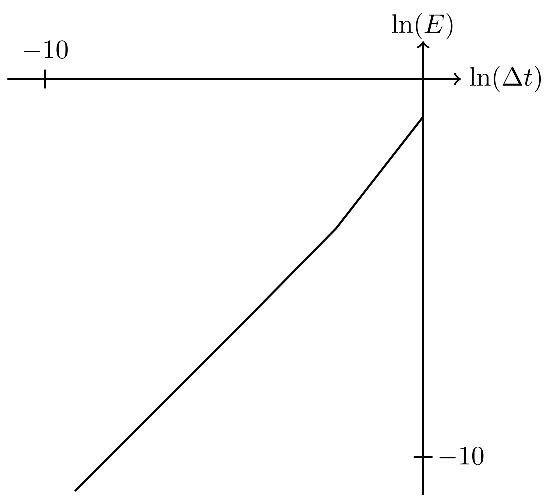
$\Delta t = 10^{-2} \Rightarrow \hat{x}(1) \approx 0.3660323413$

$\Delta t = 10^{-3} \Rightarrow \hat{x}(1) \approx 0.3676954248$

$\Delta t = 10^{-4} \Rightarrow \hat{x}(1) \approx 0.3678610464$

c)





The slope is approximately 1 for small values of Δt, which indicates that the error of the method is proportional to Δt.

$$\frac{\ln(E)}{\ln(\Delta t)} \approx \frac{\ln(C\Delta t)}{\ln(\Delta t)} = \frac{\ln(C) + \ln(\Delta t)}{\ln(\Delta t)} = \frac{\ln(C)}{\ln(\Delta t)} + \frac{\ln(\Delta t)}{\ln(\Delta t)} \approx 0 + 1 = 1 \text{ for small } \Delta t$$

2.8.5

a)

$$\dot{x} = -x, x(0) = 1 \Rightarrow x(t) = e^{-t} \qquad x(1) = e^{-1} \approx 0.3678794411$$

b)

$\Delta t = 10^0 \Rightarrow \hat{x}(1) = 0.375$

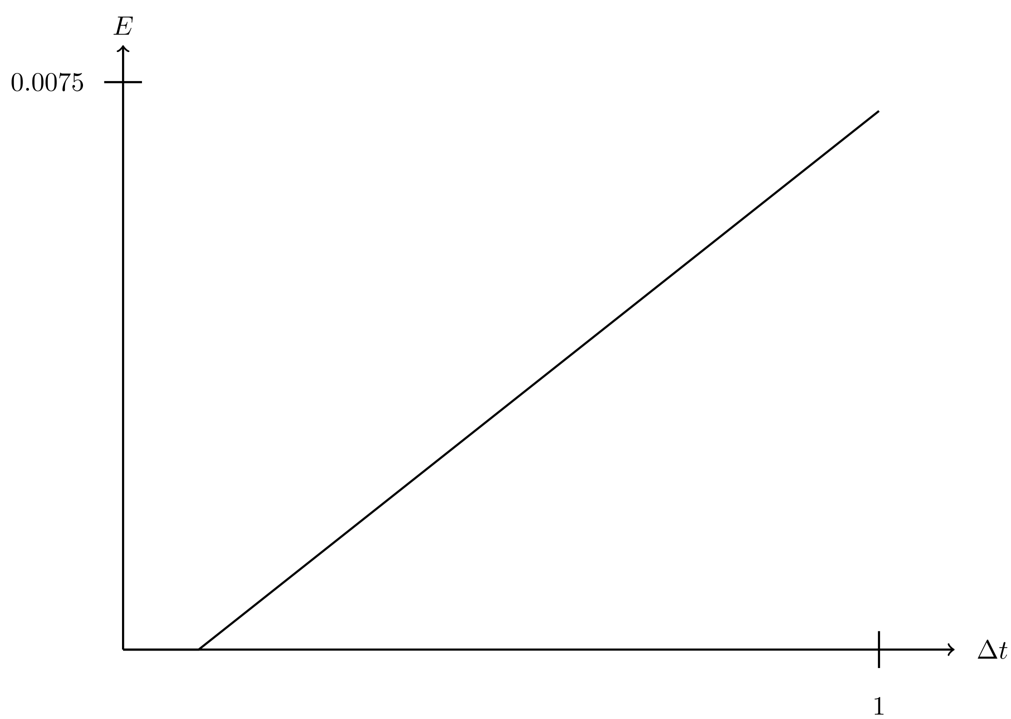
$\Delta t = 10^{-1} \Rightarrow \hat{x}(1) \approx 0.367879774412498$

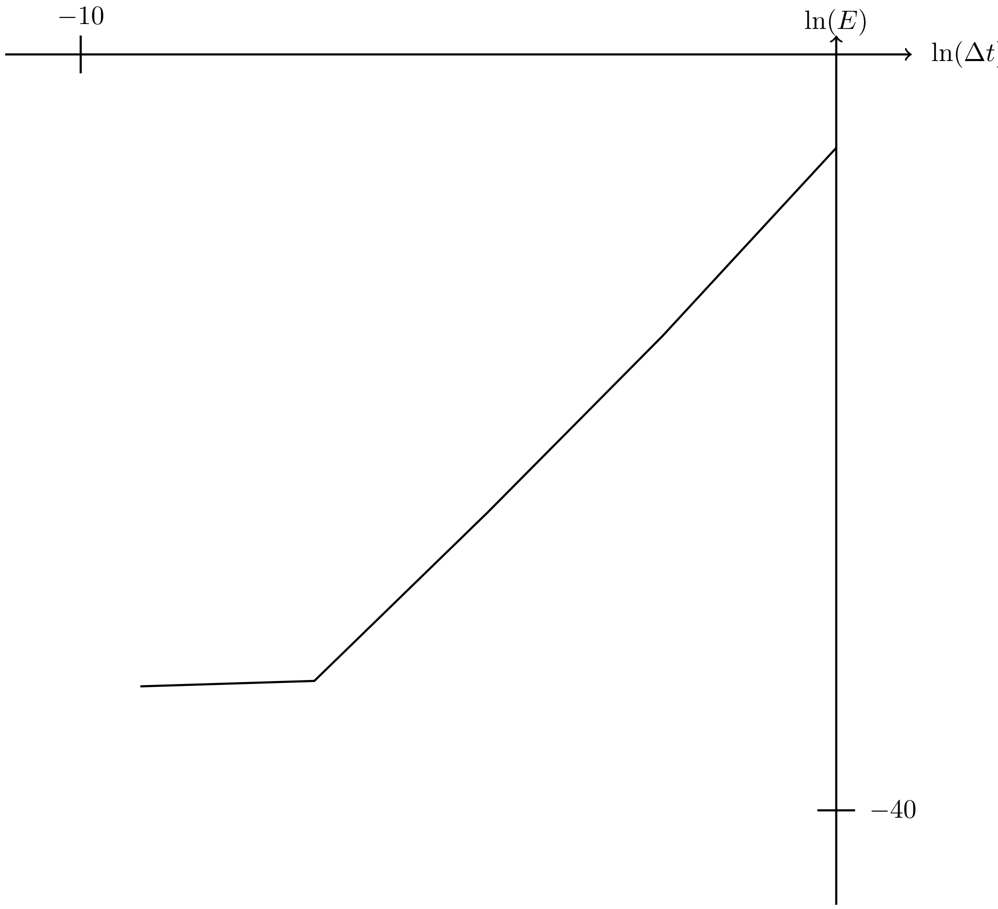
$\Delta t = 10^{-2} \Rightarrow \hat{x}(1) \approx 0.367879441202355$

$\Delta t = 10^{-3} \Rightarrow \hat{x}(1) \approx 0.367879441171446$

$\Delta t = 10^{-4} \Rightarrow \hat{x}(1) \approx 0.367879441171445$

c)





The slope is approximately 4 for small values of Δt, which indicates that the error of the method is proportional to $(\Delta t)^4$.

$$\frac{\ln(E)}{\ln(\Delta t)} \approx \frac{\ln(C(\Delta t)^4)}{\ln(\Delta t)} = \frac{\ln(C) + \ln((\Delta t)^4)}{\ln(\Delta t)} = \frac{\ln(C)}{\ln(\Delta t)} + \frac{4\ln(\Delta t)}{\ln(\Delta t)} \approx 0 + 4 = 4 \text{ for small } \Delta t$$

2.8.7

Euler's method

$$x_{n+1} = x_n + \Delta t f(x_n)$$

a)

$$\begin{aligned} x(t_0 + \Delta t) &= x(t_0) + \Delta t \dot{x}(t_0) + \frac{(\Delta t)^2}{2}\ddot{x}(\xi) \qquad \xi \in (t_0, t_0 + \Delta t) \\ &= x(t_0) + \Delta t f(x(t_0)) + \frac{(\Delta t)^2}{2} f'(x(\xi)) f(x(\xi)) \end{aligned}$$

b)

$$|x(t_1) - x_1| = |x(t_0 + \Delta t) - (x_0 + \Delta t f(x_0))| = \frac{(\Delta t)^2}{2}|f'(x(\xi))f(x(\xi))| \le C(\Delta t)^2$$

$$C = \max\left\{\frac{|f'(x(\xi))f(x(\xi))|}{2} : \xi \in (t_0, t_0 + \Delta t)\right\}$$

2.8.9

Runge-Kutta

$k_1 = \Delta t f(x_n)$

$k_2 = \Delta t f(x_n + \frac{1}{2}k_1)$

$k_3 = \Delta t f(x_n + \frac{1}{2}k_2)$

$k_4 = \Delta t f(x_n + k_3)$

$x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

a)

$$\begin{aligned} x(t_0+\Delta t) &= x(t_0) + \Delta t \tfrac{\mathrm{d}x}{\mathrm{d}t}(t_0) + \tfrac{(\Delta t)^2}{2}\tfrac{\mathrm{d}^2x}{\mathrm{d}t^2}(t_0) + \tfrac{(\Delta t)^3}{3!}\tfrac{\mathrm{d}^3x}{\mathrm{d}t^3}(t_0) + \tfrac{(\Delta t)^4}{4!}\tfrac{\mathrm{d}^4x}{\mathrm{d}t^4}(t_0) + \tfrac{(\Delta t)^5}{5!}\tfrac{\mathrm{d}^5x}{\mathrm{d}t^5}(\xi),\ \xi \in (t_0, t_0+\Delta t) \\ &= x(t_0) + \Delta t f(t_0) + \tfrac{(\Delta t)^2}{2} f'(t_0) + \tfrac{(\Delta t)^3}{3!} f''(t_0) + \tfrac{(\Delta t)^4}{4!} f'''(t_0) + \tfrac{(\Delta t)^5}{5!} f''''(\xi),\ \xi \in (t_0, t_0+\Delta t) \end{aligned}$$

Here we have to translate each derivative into $f(x(t_0))$'s.

$k_1 = \Delta t f(x_0)$

$$\begin{aligned} k_2 &= \Delta t f\big(x_0 + \tfrac{1}{2}k_1\big) \\ &= \Delta t\Big(f(x_0) + \tfrac{1}{2}k_1 f^{(1)}(x_0) + \tfrac{\left(\frac{1}{2}k_1\right)^2}{2!} f^{(2)}(x_0) + \tfrac{\left(\frac{1}{2}k_1\right)^3}{3!} f^{(3)}(x_0) + \tfrac{\left(\frac{1}{2}k_1\right)^4}{4!} f^{(4)}(x_0) + \tfrac{\left(\frac{1}{2}k_1\right)^5}{5!} f^{(5)}(\xi_2)\Big),\ \xi_2 \in (x_0, x_0 + \tfrac{1}{2}k_1) \end{aligned}$$

And then we need to substitute all the k_1's as $f(x_0)$'s.

$$\begin{aligned} k_3 &= \Delta t f\big(x_0 + \tfrac{1}{2}k_2\big) \\ &= \Delta t\Big(f(x_0) + \tfrac{1}{2}k_2 f^{(1)}(x_0) + \tfrac{\left(\frac{1}{2}k_2\right)^2}{2!} f^{(2)}(x_0) + \tfrac{\left(\frac{1}{2}k_2\right)^3}{3!} f^{(3)}(x_0) + \tfrac{\left(\frac{1}{2}k_2\right)^4}{4!} f^{(4)}(x_0) + \tfrac{\left(\frac{1}{2}k_2\right)^5}{5!} f^{(5)}(\xi_3)\Big),\ \xi_3 \in (x_0, x_0 + \tfrac{1}{2}k_2) \end{aligned}$$

And then we need to substitute all the k_2's as $f(x_0)$'s.

$$\begin{aligned} k_4 &= \Delta t f\big(x_0 + k_3\big) \\ &= \Delta t\Big(f(x_0) + k_3 f^{(1)}(x_0) + \tfrac{(k_3)^2}{2!} f^{(2)}(x_0) + \tfrac{(k_3)^3}{3!} f^{(3)}(x_0) + \tfrac{(k_3)^4}{4!} f^{(4)}(x_0) + \tfrac{(k_3)^5}{5!} f^{(5)}(\xi_4)\Big),\ \xi_4 \in (x_0, x_0 + k_3) \end{aligned}$$

And then we need to substitute all the k_3's as $f(x_0)$'s.

b) $|x(t_1) - x_1| = \Big|x(t_0 + \Delta t) - \Big(x_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\Big)\Big|$

And then we need to substitute the $x(t_0 + \Delta t)$, k_1, k_2, k_3, and k_4 as $f(x_0)$'s. This is a herculean task, but at the end we will get a bound on the absolute value of the local truncation error, $C(\Delta t)^5$.

3

Bifurcations

3.1 Saddle-Node Bifurcation

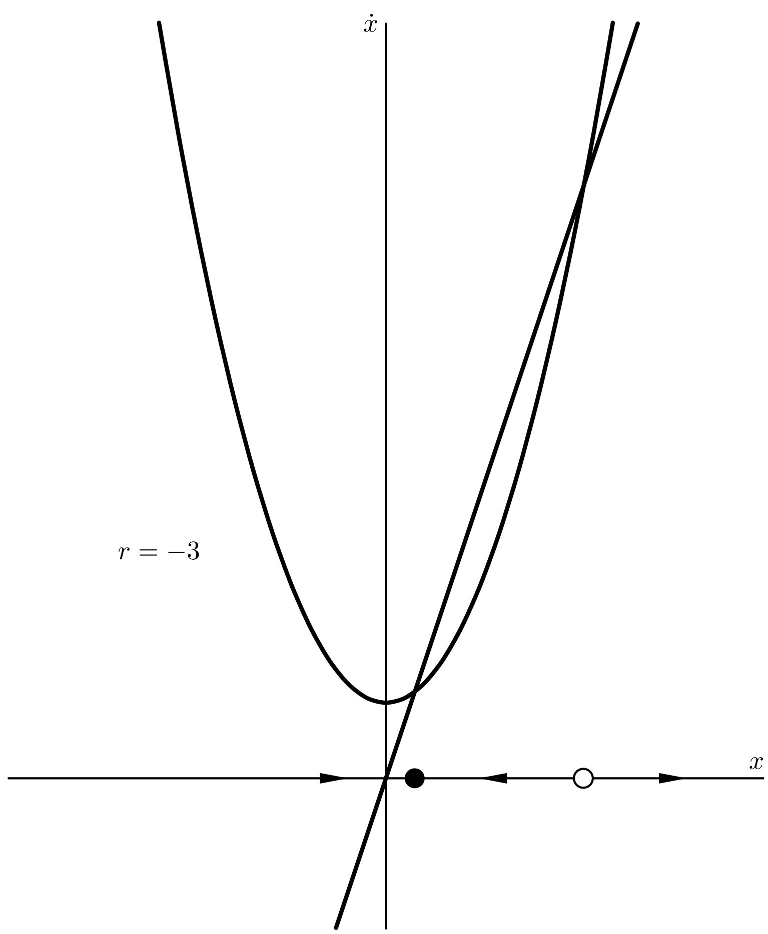
3.1.1

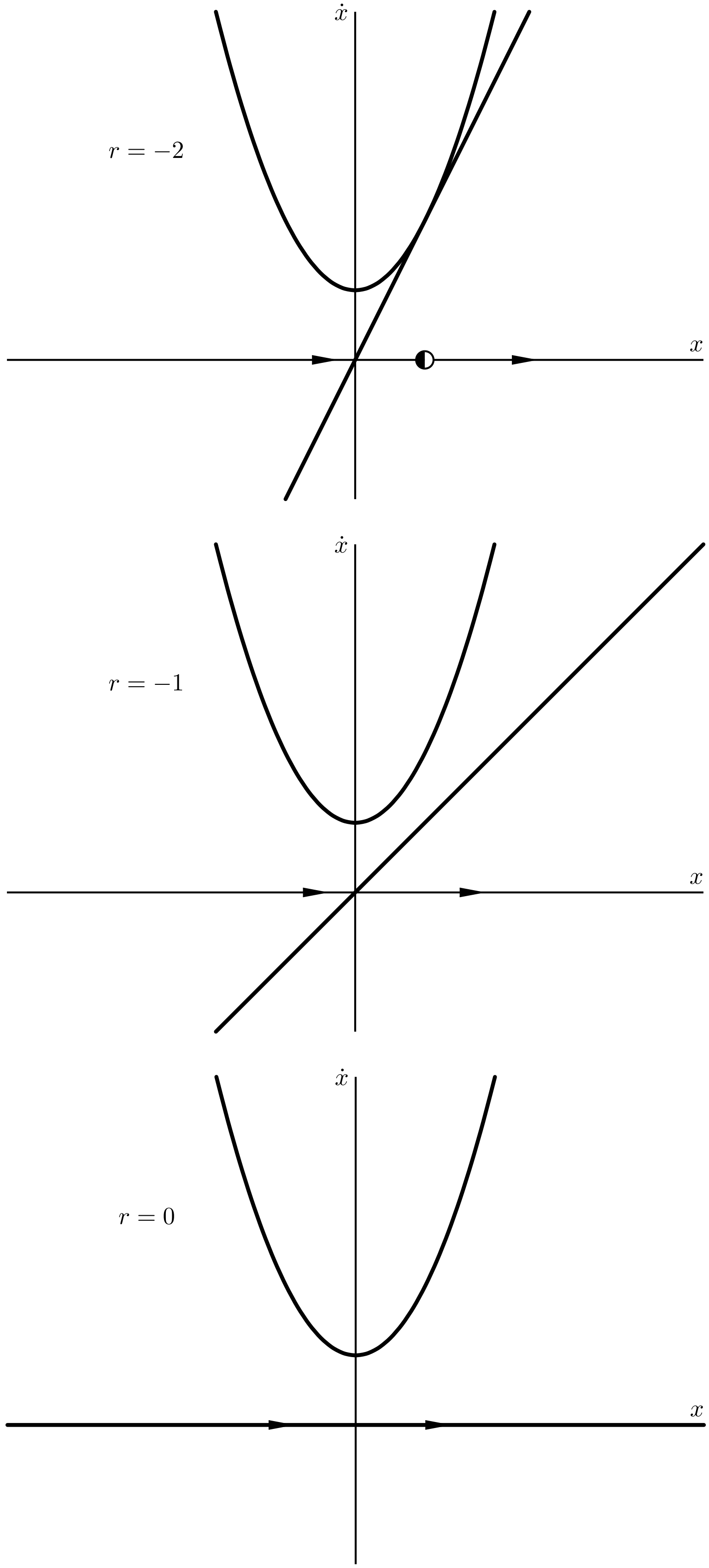
$$\dot{x} = 1 + rx + x^2$$

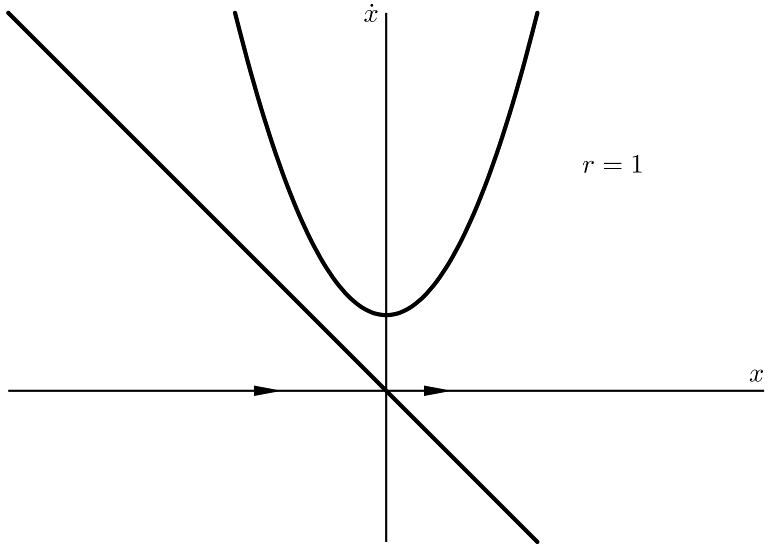
The fixed points are

$$x = \frac{-r \pm \sqrt{r^2 - 4}}{2} \qquad \begin{array}{ll} \text{2 fixed points exist if} & |r| > 2 \\ \text{1 fixed point exists if} & r = \pm 2 \\ \text{0 fixed points exist if} & |r| < 2 \end{array} \quad \Rightarrow \text{Saddle-node bifurcation occurs at } r = \pm 2$$

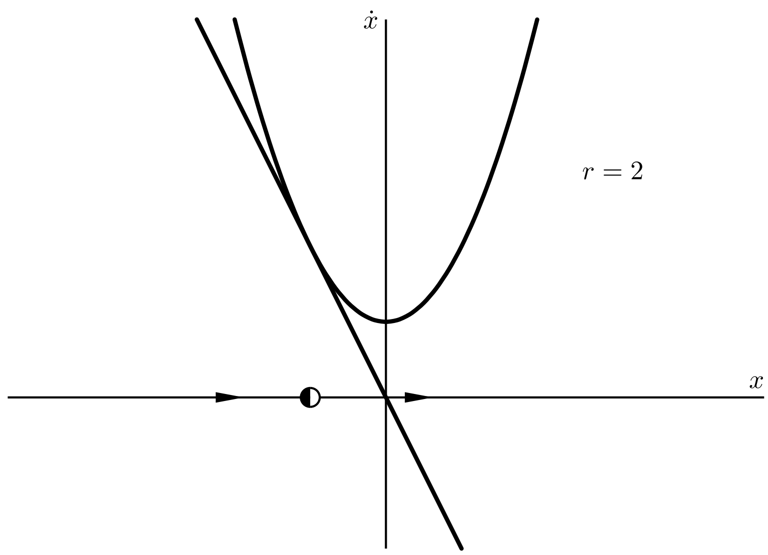
To sketch the vector fields for different values of r, draw $y = 1+x^2$ and $y = -rx$. The intersections represent the fixed points, and the flow is to the right and left if $1 + x^2 > -rx$ and $1 + x^2 < -rx$ respectively.



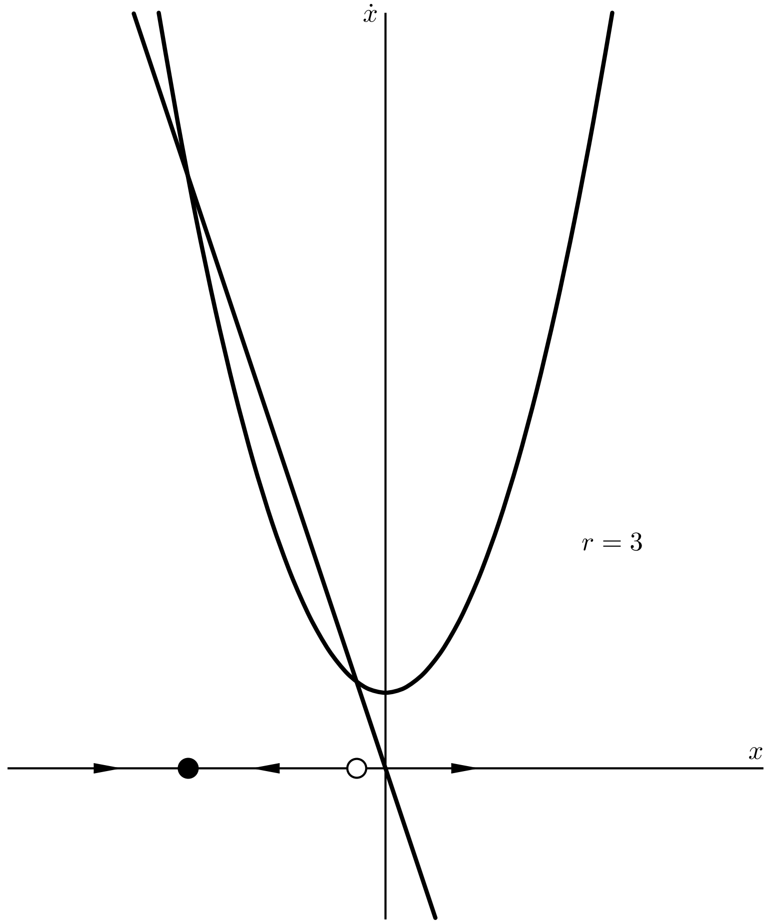

$\dot{x}$
$r = -2$
x
$\dot{x}$
$r = -1$
x
$\dot{x}$
$r = 0$
x



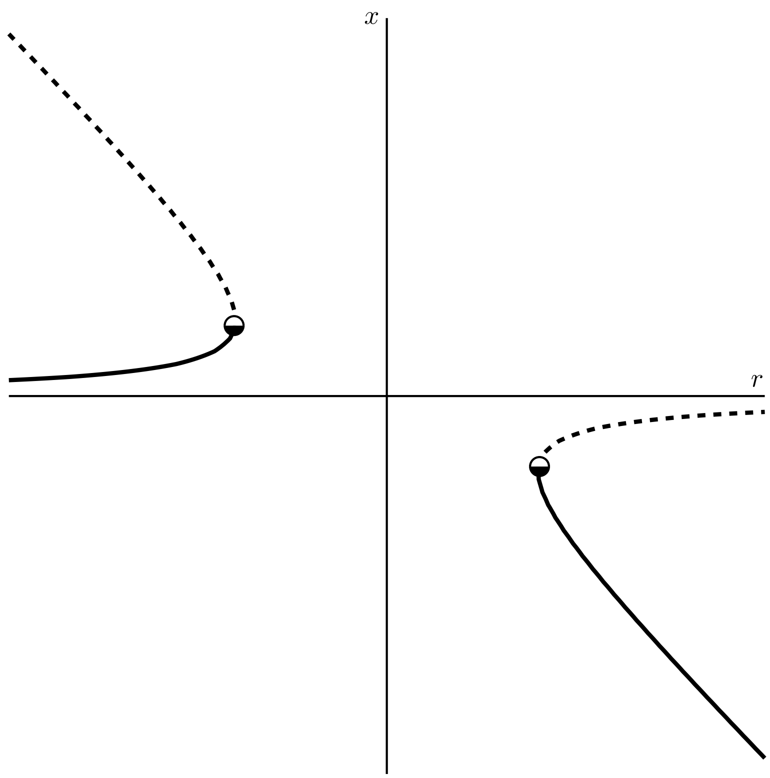
$\dot{x}$
$r = 1$
x



$\dot{x}$
$r = 2$
x


$\dot{x}$
$r = 3$
x

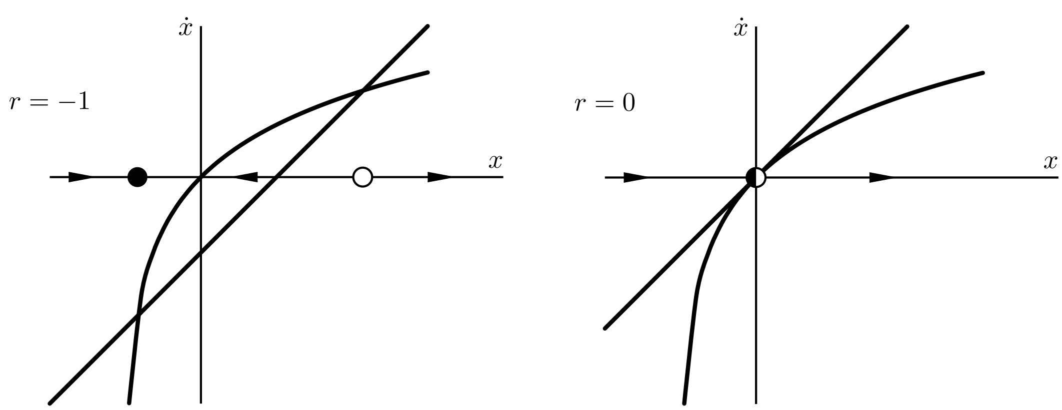
The bifurcation diagram is $x = \dfrac{-r \pm \sqrt{r^2-4}}{2}$ or equivalently $r = -\left(x + \dfrac{1}{x}\right)$, which is easier to plot.

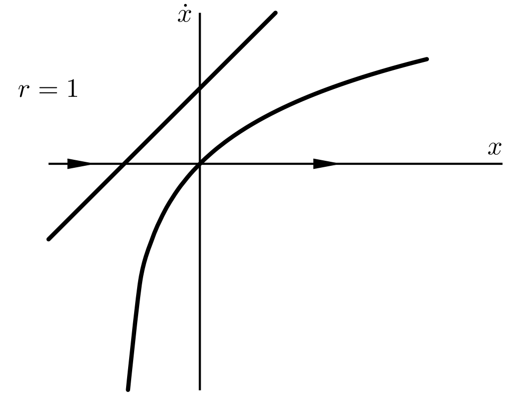


3.1.3

$$\dot{x} = r + x - \ln(1+x)$$

To sketch the vector fields for different values of r, draw $y = r + x$ and $y = \ln(1+x)$. The intersections represent the fixed points, and the flow is to the right and left if $r + x > \ln(1+x)$ and $r + x < \ln(1+x)$ respectively.



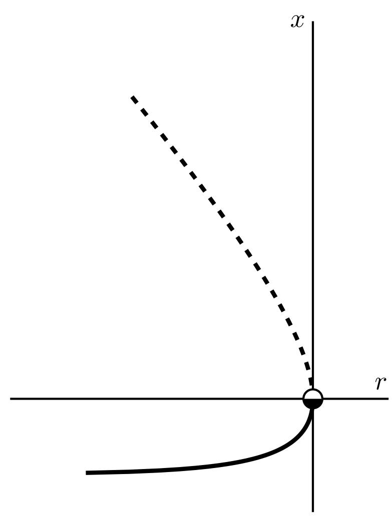


The bifurcation occurs when the curves $y = r + x$ and $y = \ln(1 + x)$ have a tangential intersection. We can find this r value analytically by solving the equations

$$r + x = \ln(1 + x) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\Big(r + x\Big) = 1 = \frac{1}{1 + x} = \frac{\mathrm{d}}{\mathrm{d}x}\Big(\ln(1 + x)\Big)$$

$$1 = \frac{1}{1 + x} \Rightarrow x = 0 \qquad r + x = \ln(1 + x) \longrightarrow r = \ln(1) = 0$$

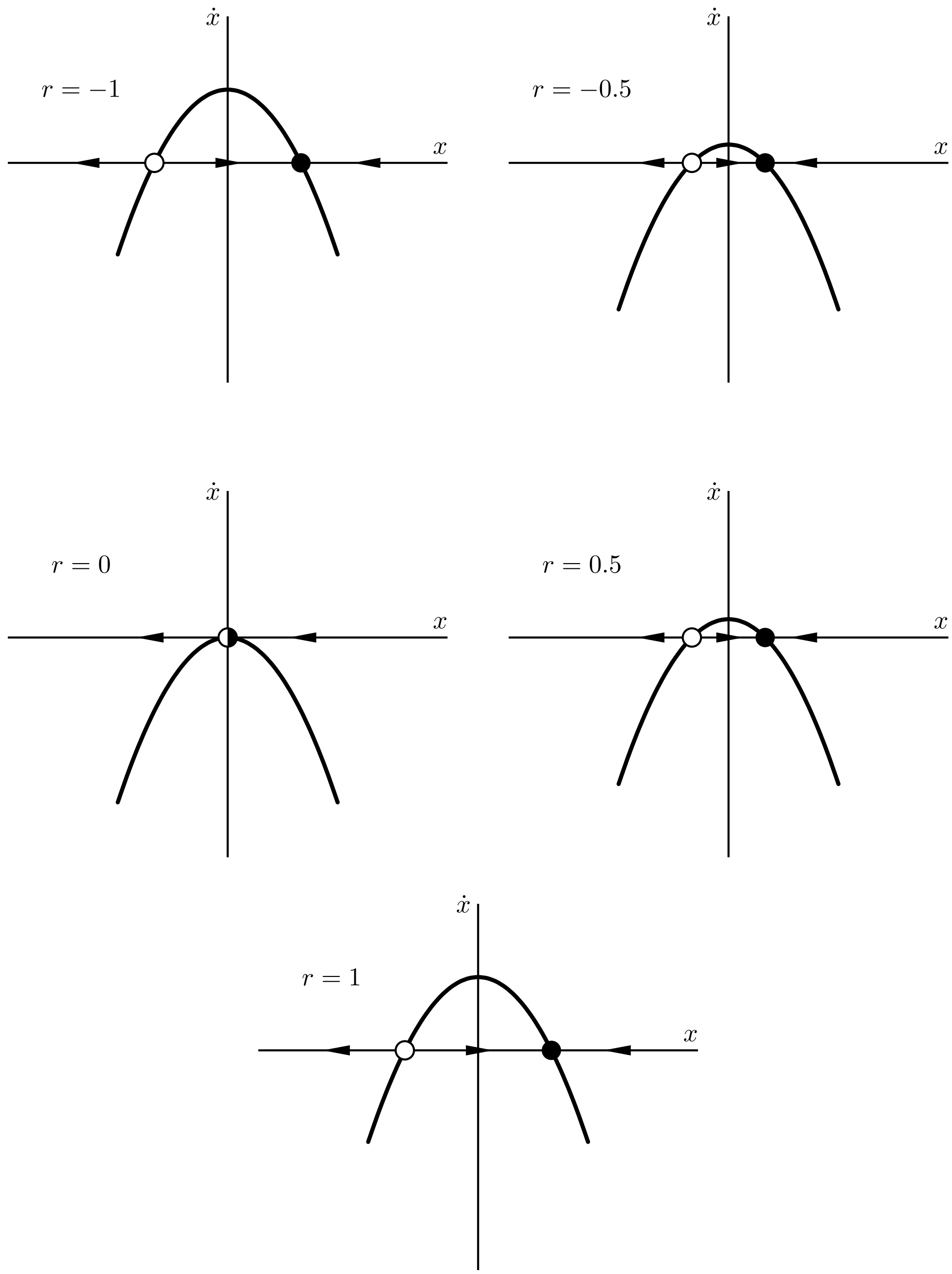
Hence, a saddle-node bifurcation occurs at $r = 0$.

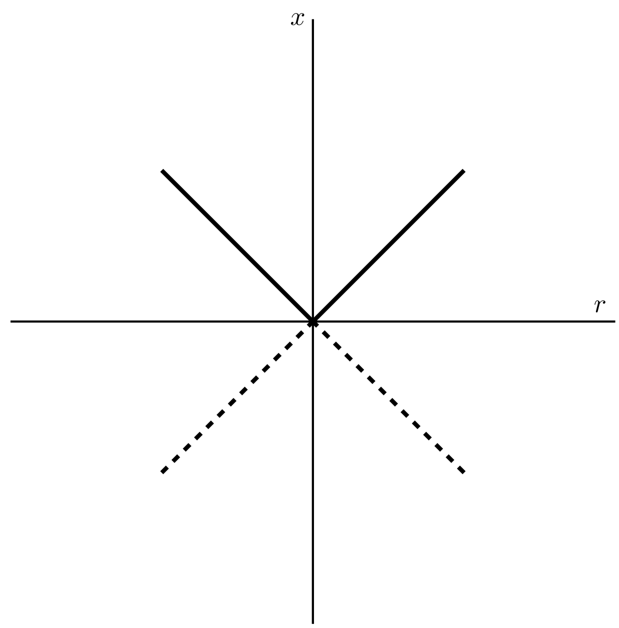


3.1.5

a)

$$\dot{x} = r^2 - x^2$$

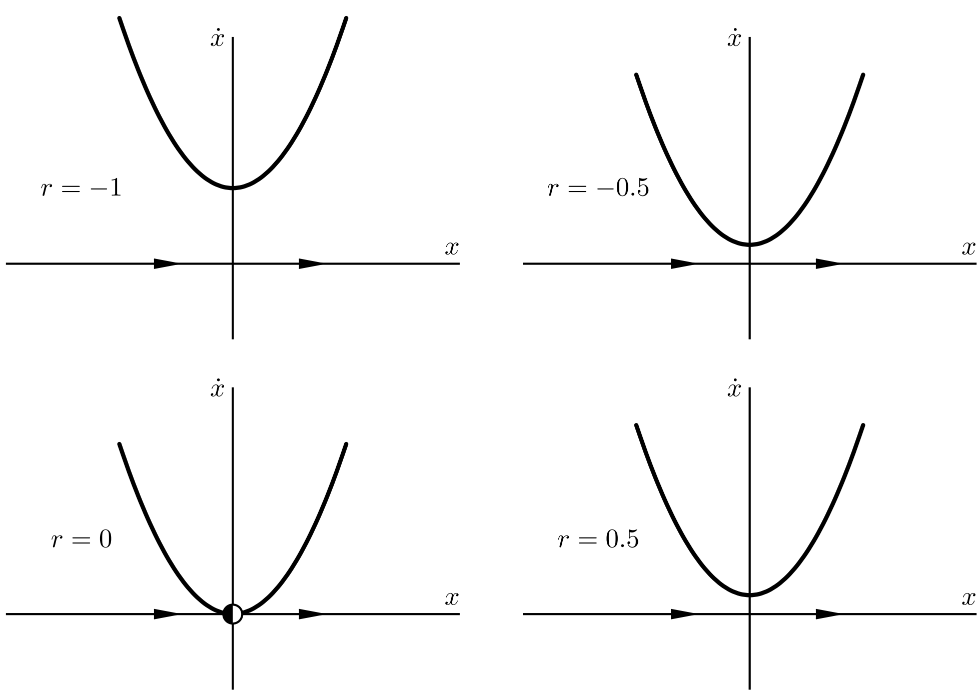


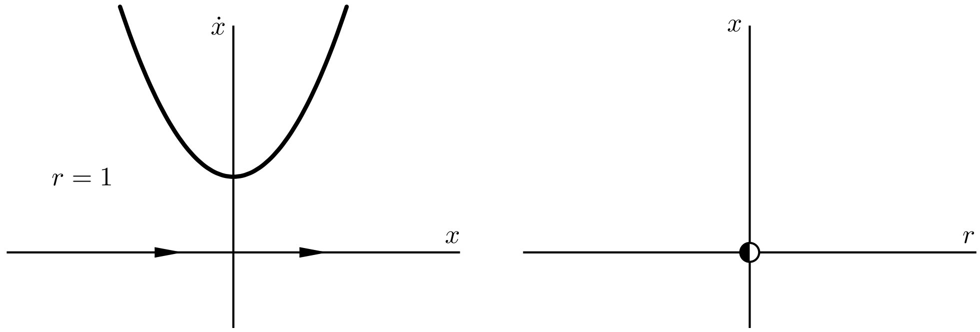


A saddle-node bifurcation doesn't occur. Instead there is more of a transcritical bifurcation, but not quite since both of the fixed-point positions change with r.

b)

$$\dot{x} = r^2 + x^2$$

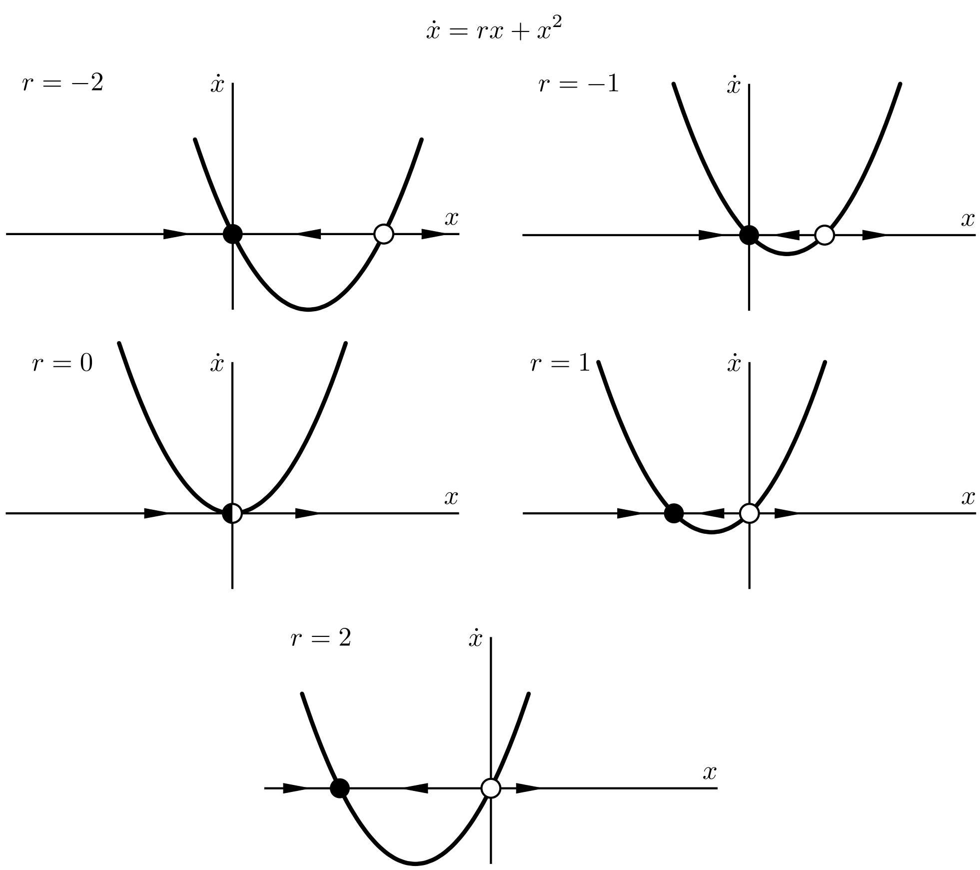




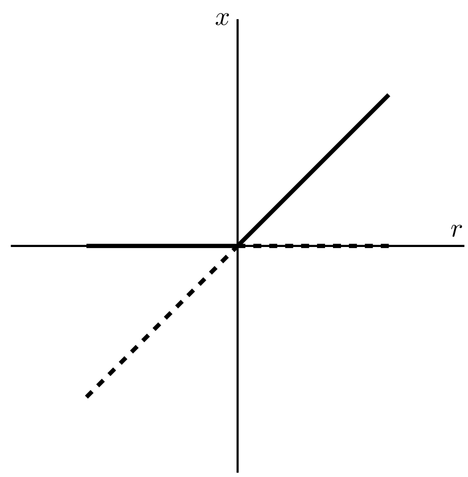
Again, a saddle-node bifurcation doesn't occur. In this case there is almost nothing to plot on the bifurcation diagram.

3.2 Transcritical Bifurcation

3.2.1

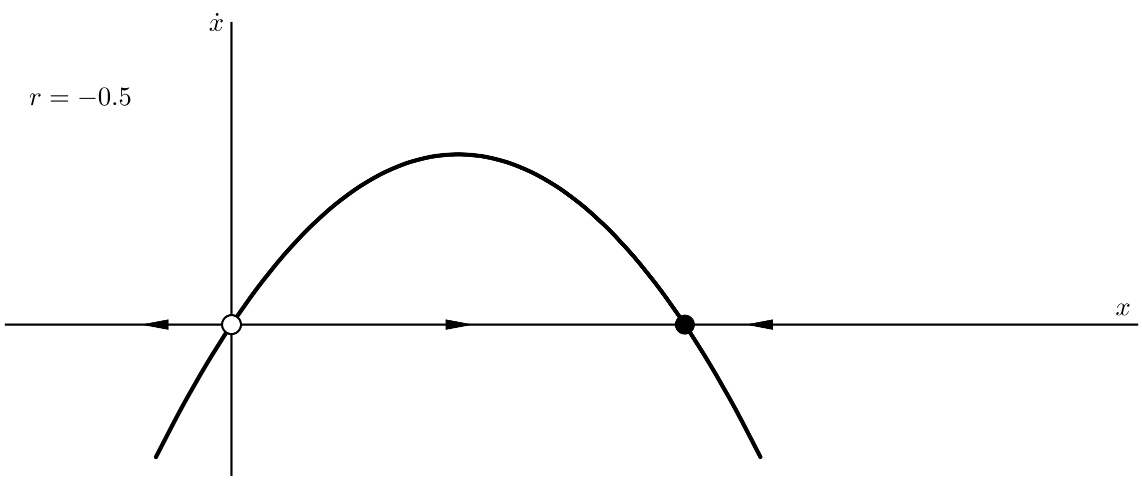


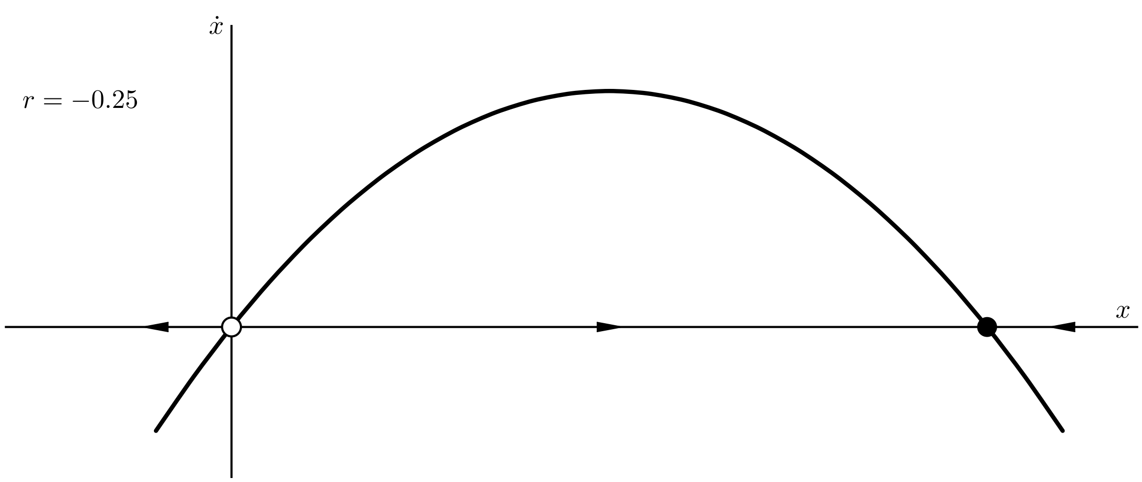
Transcritical bifurcation occurs at $r = 0$.

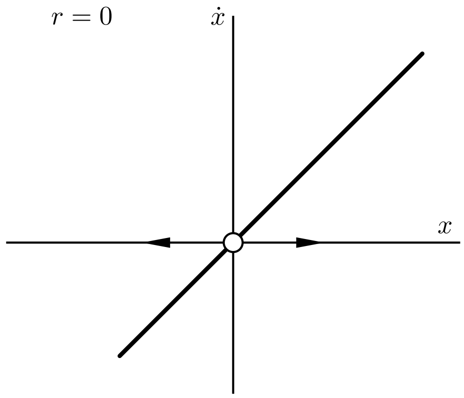


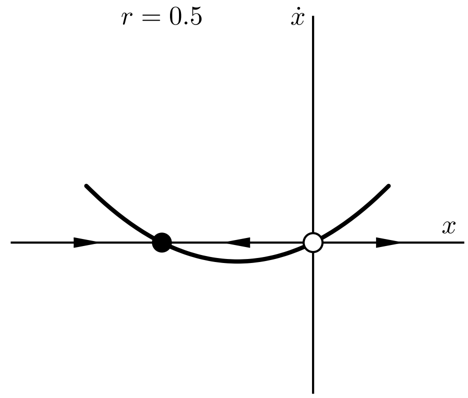
3.2.3

$$\dot{x} = x - rx(1-x)$$

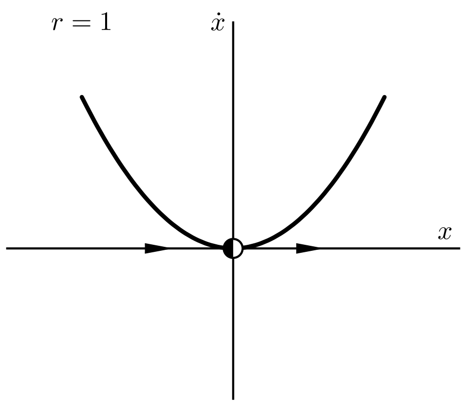


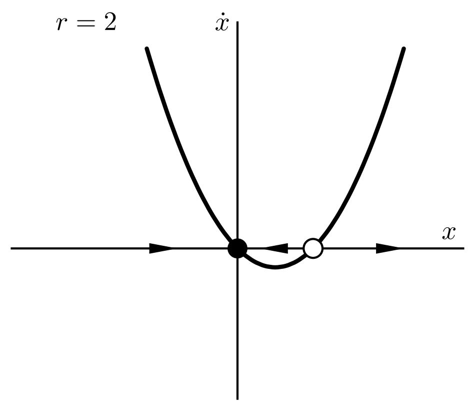


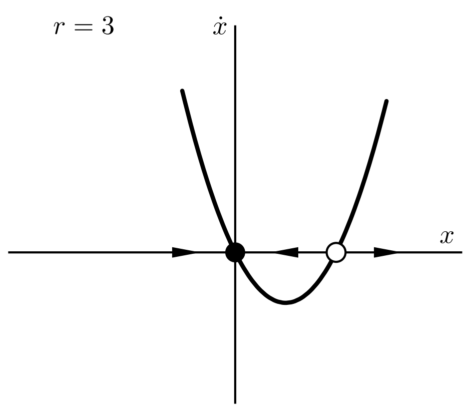




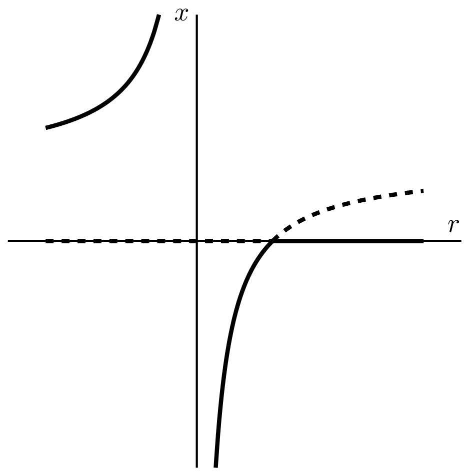
Note: The positive fixed point goes to $x = \infty$ as $r \to 0^-$, disappears when $r = 0$, then reappears at $x = -\infty$ when r becomes slightly positive.







Transcritical bifurcation occurs at $r = 1$.



3.2.5

a)

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X$$

$$X + B \xrightarrow{k_2} C$$

The k_1 reaction has a net gain of one X at rate $k_1[A][X]$, the k_{-1} reaction has a net loss of one X at rate $k_{-1}[X][X] = k_{-1}[X]^2$, and the k_2 reaction has a net loss of one X at rate $k_2[B][X]$, making the rate of concentration change $[\dot{X}] = \dot{x}$ the sum of the rates for positive gains and negative losses.

$$\begin{aligned}\dot{x} &= k_1ax - k_{-1}x^2 - k_2bx \\ &= (k_1a - k_2b)x - k_{-1}x^2 \\ &= c_1 - c_2x_2\end{aligned}$$

So $c_1 = k_1a - k_2b$ and $c_2 = k_{-1}$.

b)

$$\begin{aligned}\frac{\mathrm{d}\dot{x}}{\mathrm{d}x} &= k_1a - k_2b - 2k_{-1}x \\ \left.\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}\right|_{x^*=0} &= k_1a - k_2b < 0 \\ &\Rightarrow k_1a < k_2b\end{aligned}$$

This makes sense chemically because the amount of chemical X will always either increase or decrease to 0. Chemicals A and B are considered inexhaustible, so if the first reaction creates X faster than the second reaction irreversibly destroys $X(k_2b < k_1a)$, then the amount of X will grow unbounded. But if the first reaction creates X slower than the second reaction irreversibly destroys $X(k_1a < k_2b)$, then the amount of X will approach 0.

3.3 Laser Threshold

3.3.1

$$
\begin{aligned}
\dot{n} &= GnN - kn \\
\dot{N} &= -GnN - fN + p
\end{aligned}
$$

a)

If $\dot{N} \approx 0$, then

$$
\begin{aligned}
0 &\approx -GnN - fN + p \\
N &\approx \frac{p}{Gn + f} \\
\dot{n} &= GnN - kn \approx \frac{Gpn}{Gn + f} - kn
\end{aligned}
$$

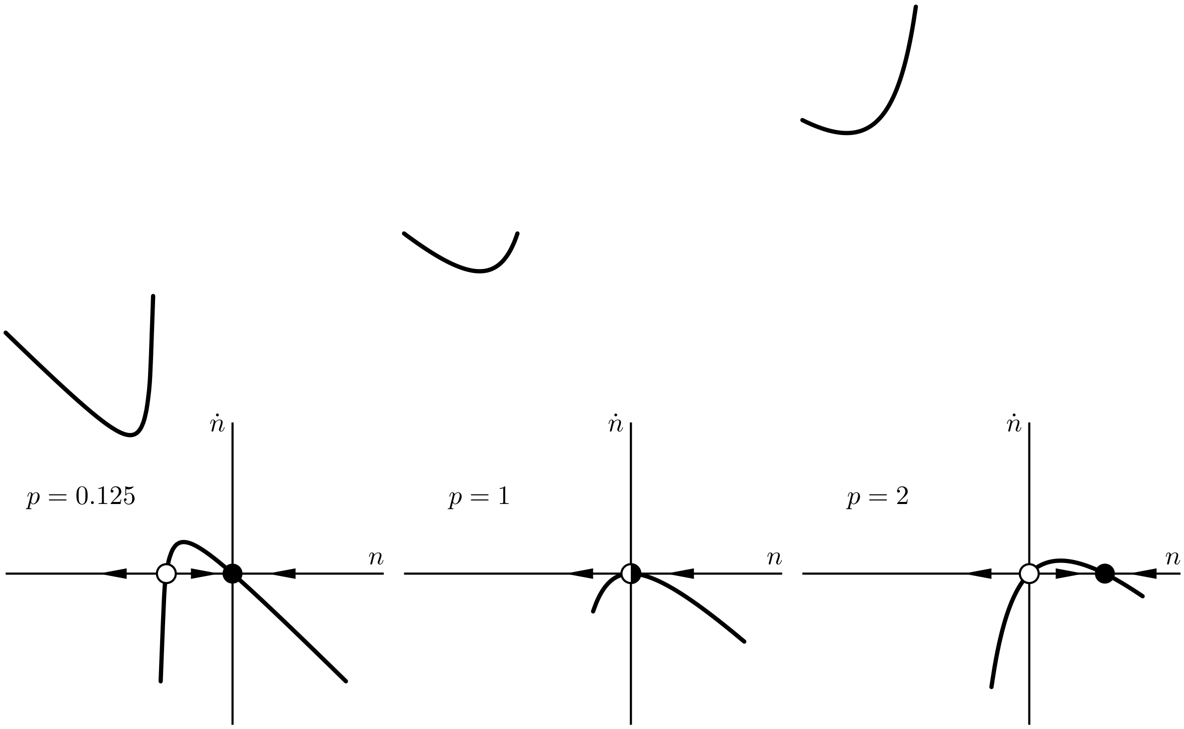
b)

$$
\begin{aligned}
\frac{\mathrm{d}\dot{n}}{\mathrm{d}n} &= \frac{Gpf}{(Gn + f)^2} - k \\
\left.\frac{\mathrm{d}\dot{n}}{\mathrm{d}n}\right|_{n^*=0} &= \frac{Gpf}{f^2} - k = \frac{Gp}{f} - k > 0 \\
&\Rightarrow p > \frac{kf}{Gp} = p_c
\end{aligned}
$$

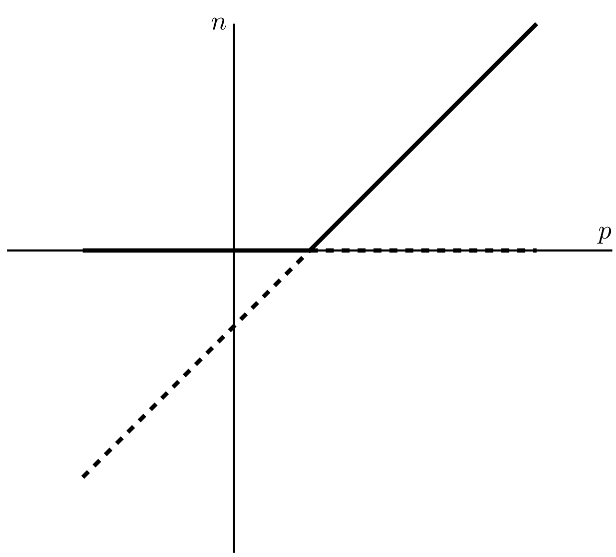
c)

If we set $G = k = f = 1$,

$$
\dot{n} \approx \frac{pn}{n + 1} - n = 0 \Rightarrow n = 0 \text{ or } n = p - 1
$$



A transcritical bifurcation occurs at p_c.



d)

$$\dot{n} = GnN - kn = (GN - k)n$$

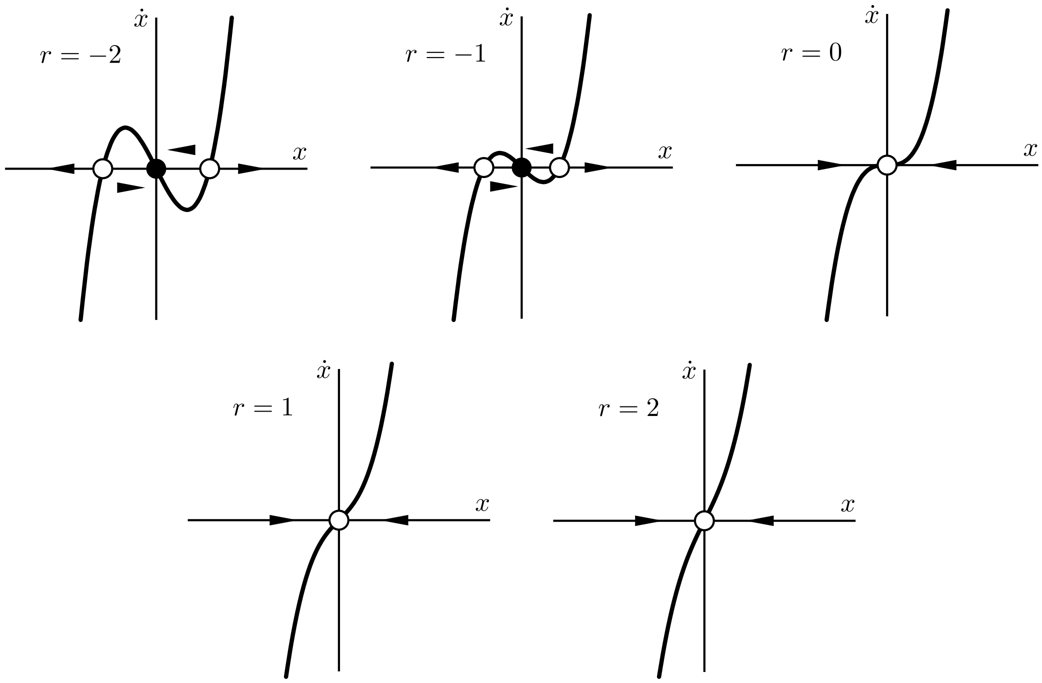
$$\dot{N} = -GnN - fN + p = -(Gn + f)N + p$$

The assumption that N almost isn't changing relative to n is valid if $|GN - k| \ll |Gn + f)|$. Then N will initially change far more rapidly than n, but then stay put once it reaches an equilibrium with the current n value. Now n will still change, but N will reach the equilibrium with the newer n value almost instantly, keeping $\dot{N} \approx 0$.

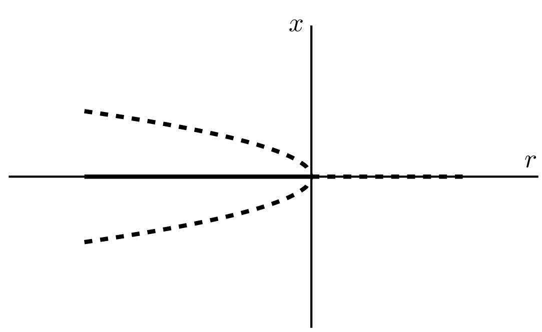
3.4 Pitchfork Bifurcation

3.4.1

$$\dot{x} = rx + 4x^3$$

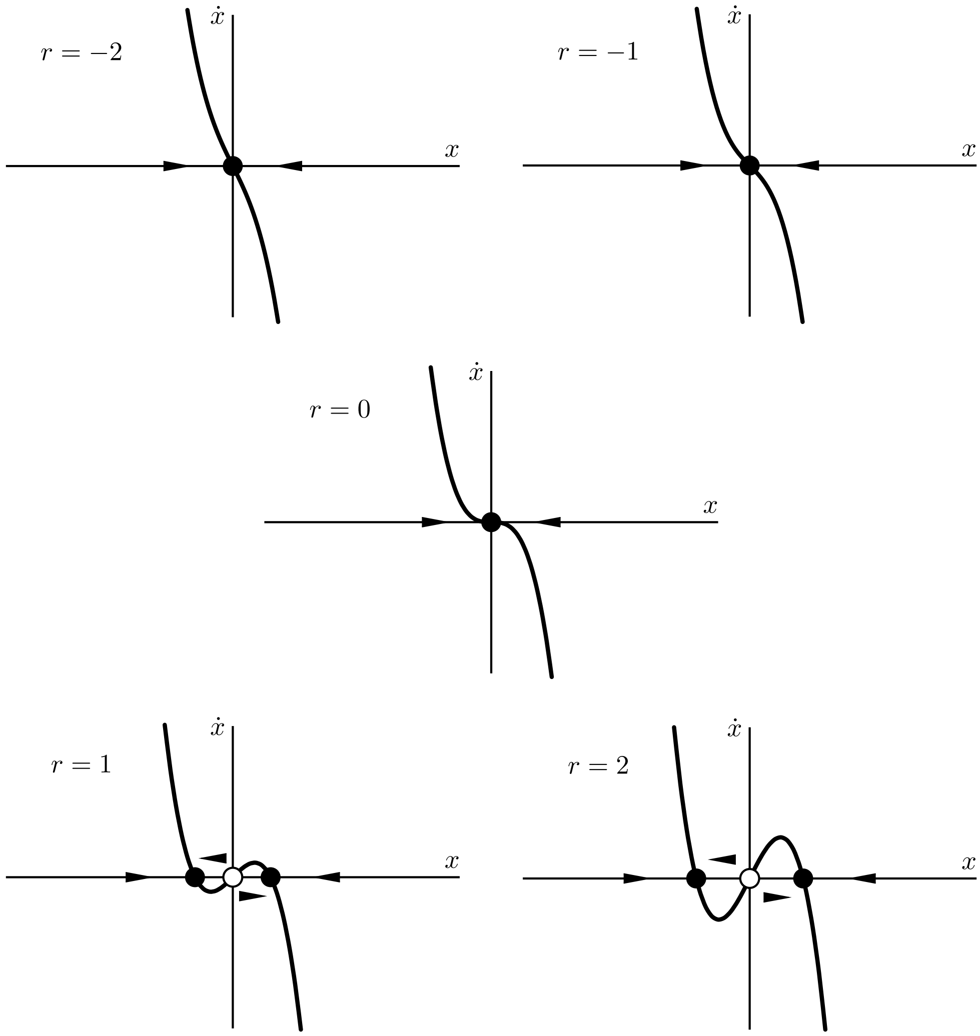


A subcritical pitchfork bifurcation occurs at $r = 0$.

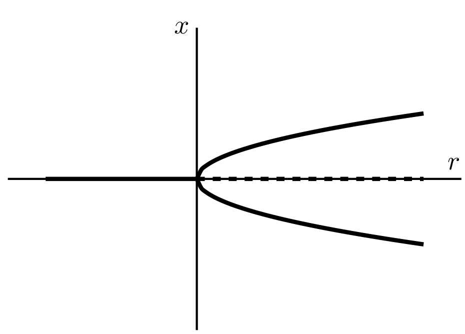


3.4.3

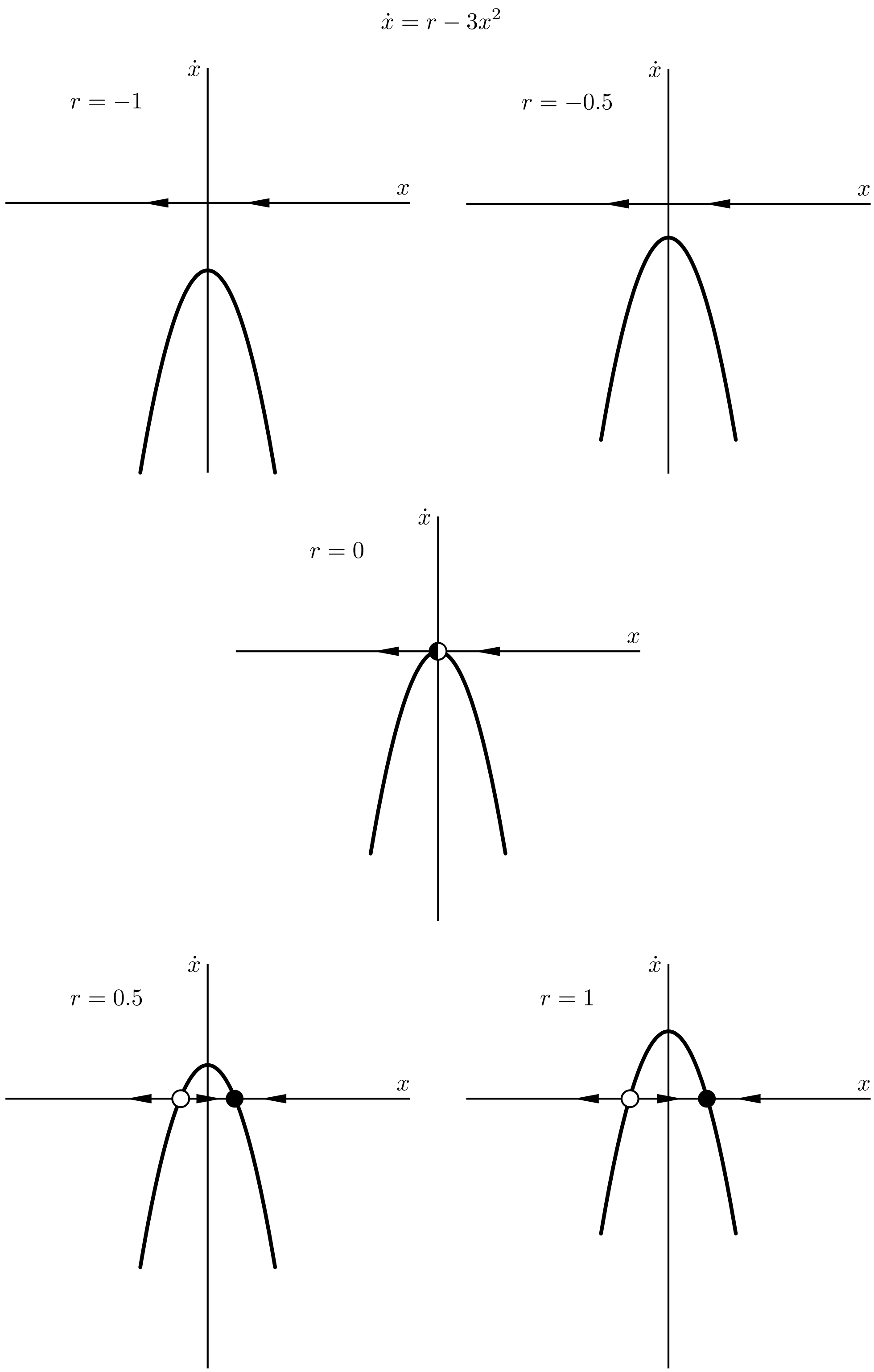
$$\dot{x} = rx - 4x^3$$



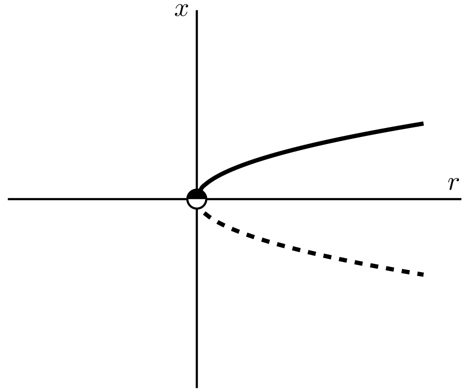
A supercritical pitchfork bifurcation occurs at $r = 0$.



3.4.5

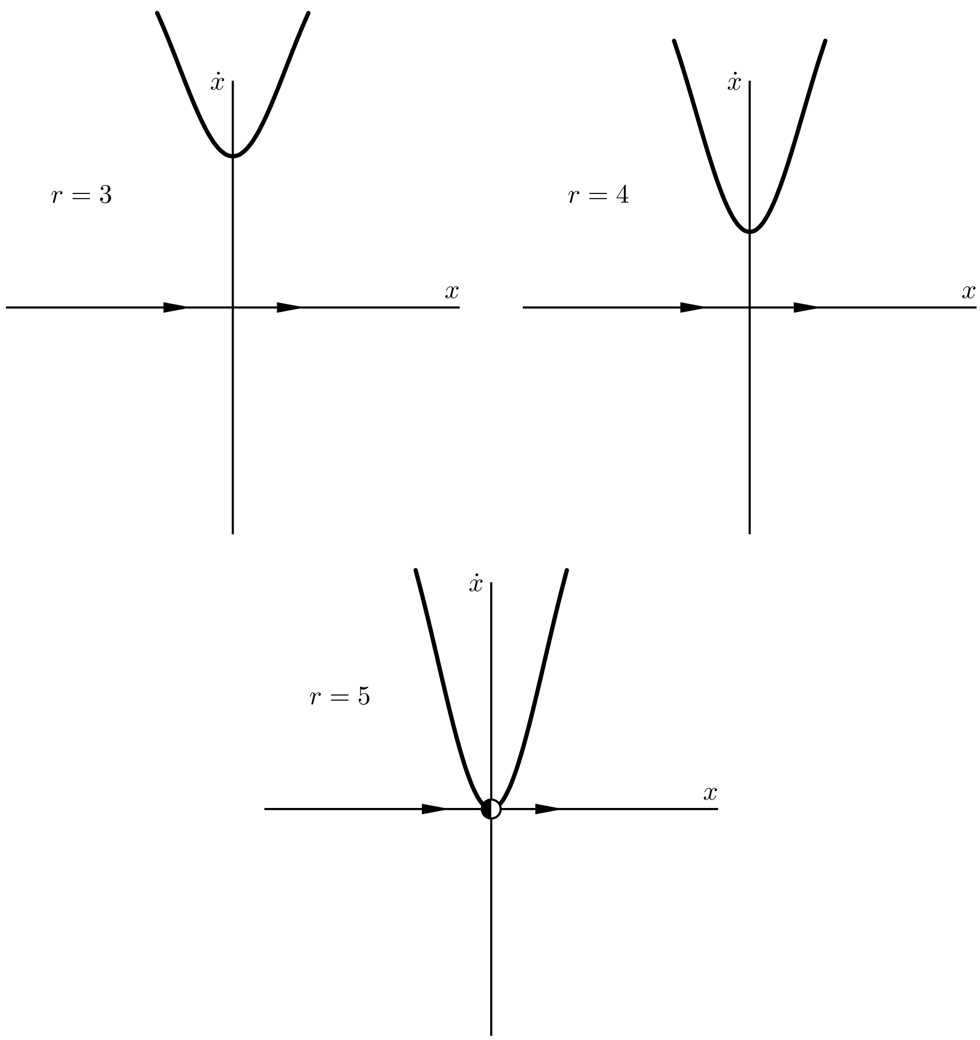


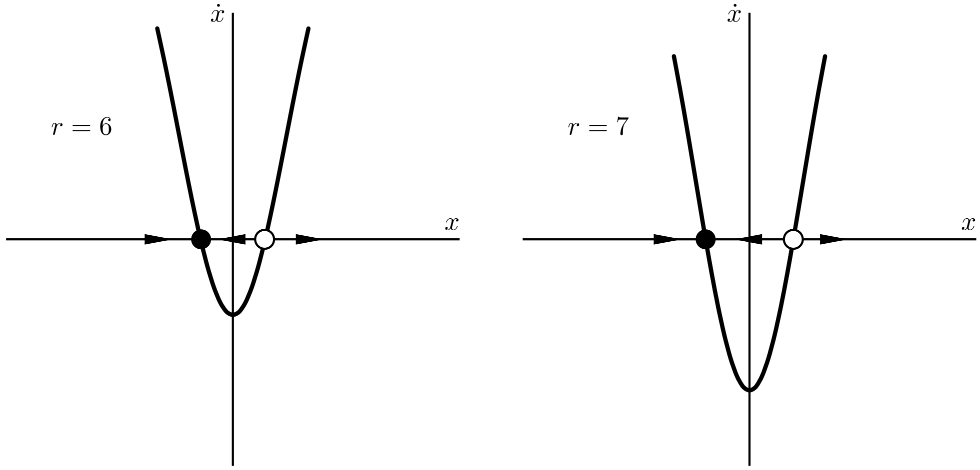
A saddle-node bifurcation occurs at $r = 0$.



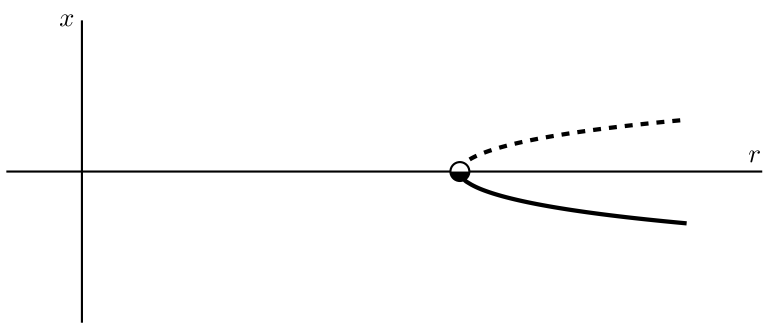
3.4.7

$$\dot{x} = 5 - re^{-x^2}$$



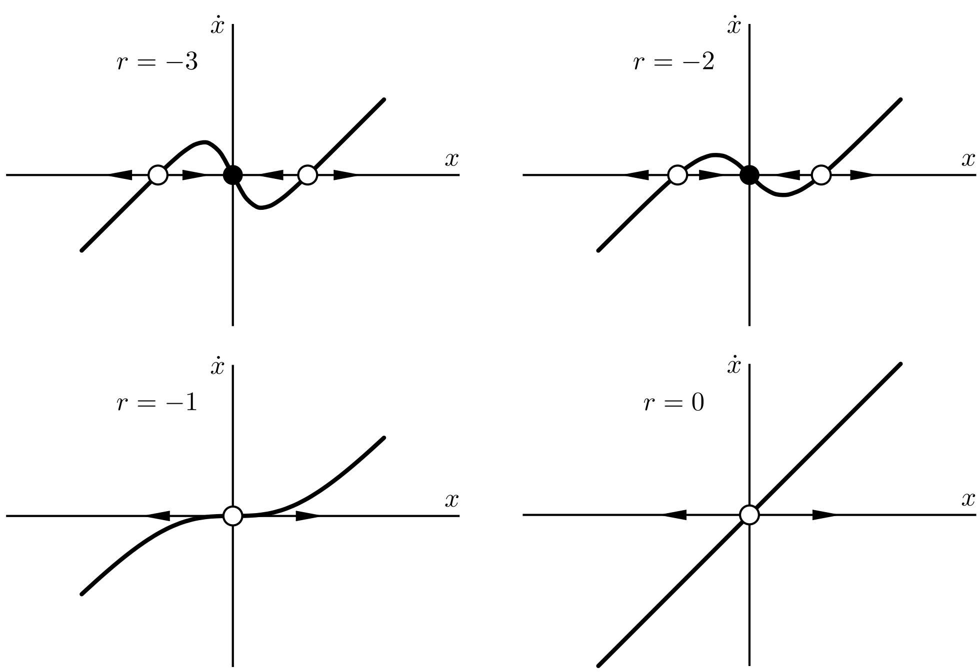


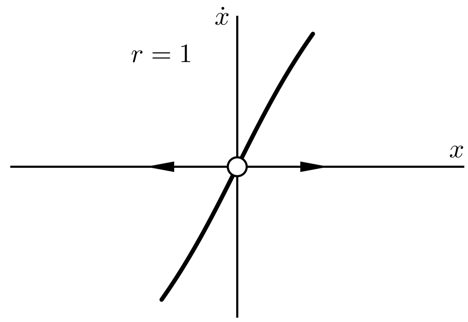
A saddle-node bifurcation occurs at $r = 5$.



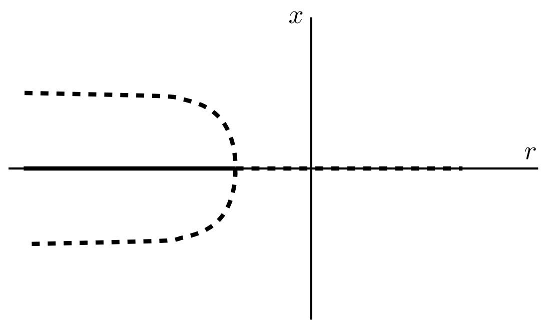
3.4.9

$$\dot{x} = x + \tanh(rx)$$





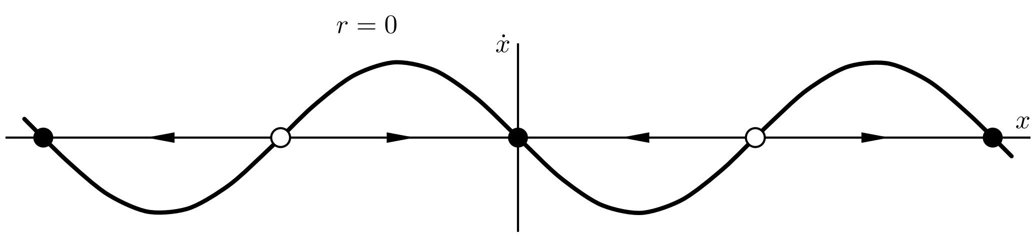
A subcritical pitchfork bifurcation occurs at $r = -1$.



3.4.11

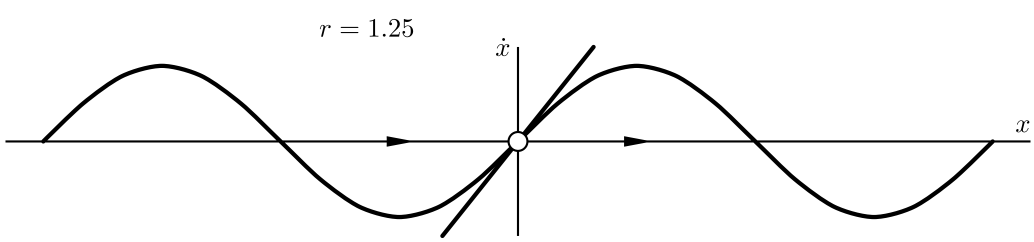
$$\dot{x} = rx - \sin(x)$$

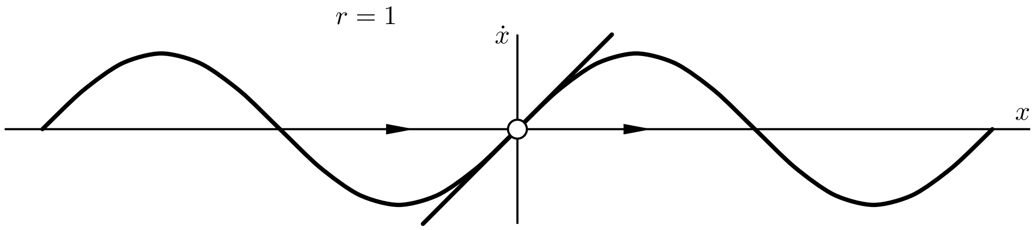
a)



There are an infinite number of fixed points when $r = 0$ at every integer multiple of π. All even multiples of π are stable, and all odd multiples of π are unstable.

b)

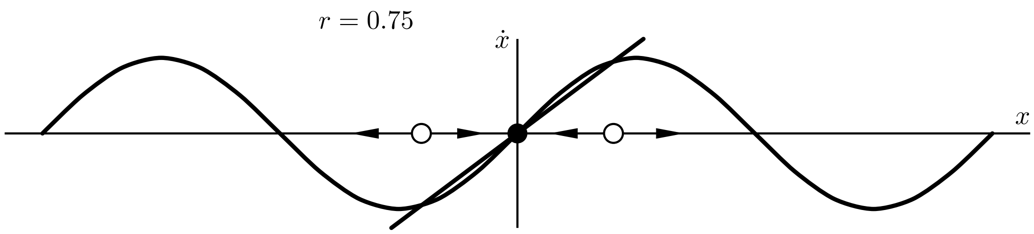




For $r \geq 1$, there is only one intersection between $y = rx$ and $y = \sin(x)$, which is an unstable fixed point at the origin.

c)

There are no bifurcations until $r \leq 1$, and the first bifurcation is a subcritical pitchfork bifurcation at $r = 1$.



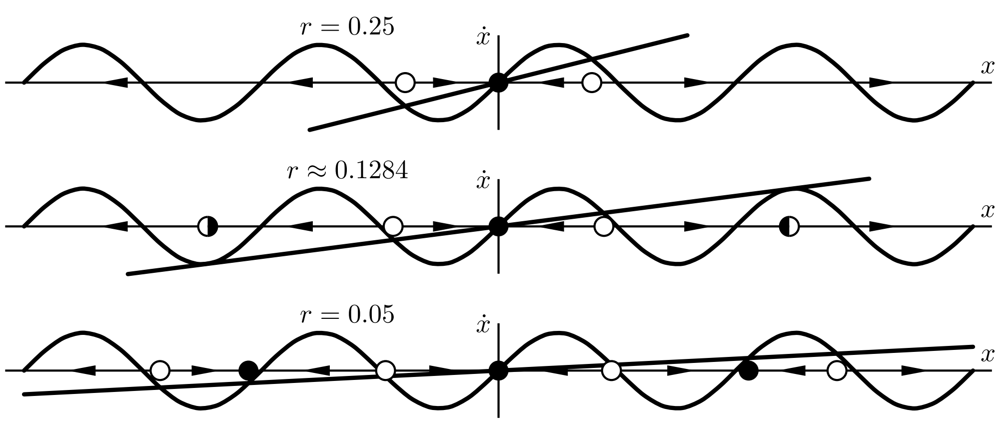
Note: To add more rigor to our bifurcation classification, we can expand in a Taylor series about $x = 0$, which gives

$$\dot{x} = rx - \sin(x) \approx rx - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \approx (r-1)x + \frac{x^3}{3!}$$

This is the normal form of a subcritical pitchfork bifurcation at $r - 1 = 0$.

There are more bifurcations as r continues to decrease toward zero. However, all these subsequent bifurcations are saddle-node bifurcations.

The first of these bifurcations occurs at $r \approx 0.1284$ when the line $y = rx$ becomes tangent to a hilltop of the sine wave.



Saddle-node bifurcations occur repeatedly as the line $y = rx$ becomes tangent to the next hilltop of the sine wave, and an infinite number of hilltops implies an infinite number of saddle-node bifurcations.

d)

To find a formula for the r values of the saddle-node bifurcations, we can use the fact that $y = rx$ and $y = \sin(x)$ tangentially intersect at the bifurcation values.

$$rx = \sin(x) \qquad \frac{\mathrm{d}}{\mathrm{d}x}\Big(rx\Big) = r = \cos(x) = \frac{\mathrm{d}}{\mathrm{d}x}\Big(\sin(x)\Big) \Rightarrow x = \tan(x) \Rightarrow r = \cos(x)$$

Therefore, the saddle-node bifurcation values are $r = \cos(x^*)$ where x^* is a root of the equation $x^* = \tan(x^*)$.

For an approximate formula when $0 < r \ll 1$, the saddle-node bifurcations occur approximately at the hilltops of the sine wave, since $y = rx$ intersects $y = \sin(x)$ tangentially and $y = rx$ is almost flat. Therefore, the bifurcations on the positive x-axis and negative x-axis respectively occur approximately at the local minimums and local maximums of $\sin(x)$. These points are

$$\left\{\frac{\pm(4n+1)\pi}{2} : n \in \mathbb{N}\right\}$$

We have to skip the first local minimum and local maximum on the positive x-axis and negative x-axis respectively because they are involved in the subcritical pitchfork bifurcation.

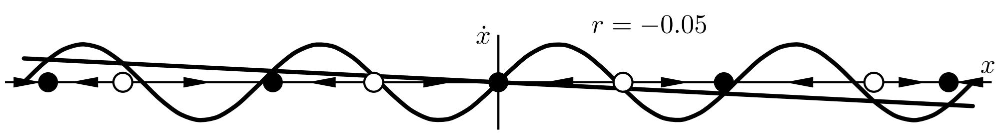
The approximate values of r at which the saddle-node bifurcations occur are the reciprocals of that set.

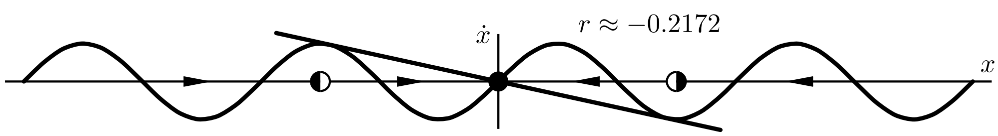
$$r \in \left\{\frac{\pm 2}{(4n+1)\pi} : n \in \mathbb{N}\right\}$$

So x and r in the first and second set respectively implies $rx = \pm 1$ and $\sin(x) = \pm 1$, which will cancel out in $\dot{x} = rx - \sin(x)$.

e)

For $-\infty < r < 0$, we again have saddle-node bifurcations occurring.

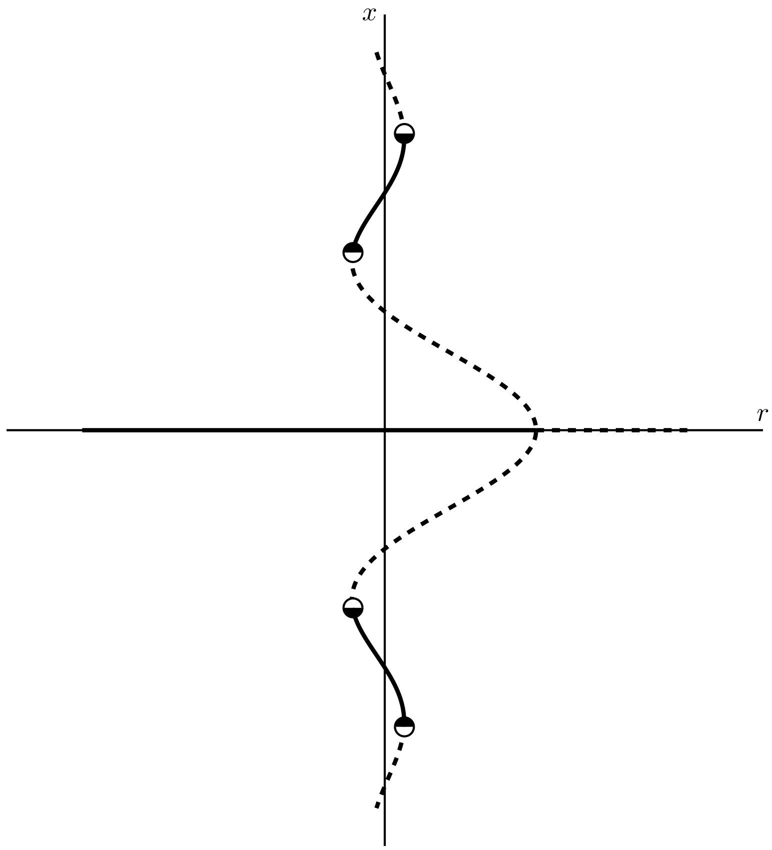




But there is no accompanying pitchfork bifurcation because after $r < -0.2172$ there is only one fixed point left at $x = 0$.

f)

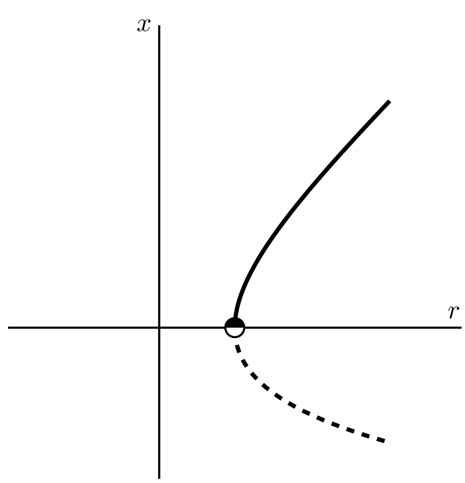
The bifurcation diagram is made of the line $x = 0$ and the curve $r = \dfrac{\sin(x)}{x}$ plotted sideways. Hence, x can take on any value from positive to negative infinity.



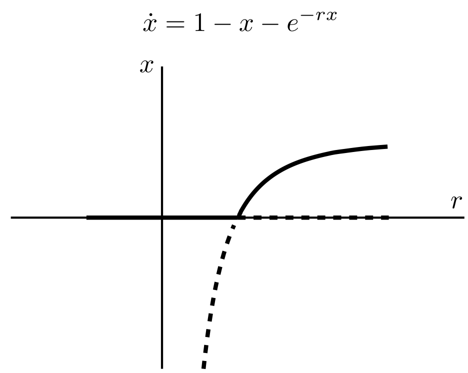
3.4.13

a)

$$\dot{x} = r - x - e^{-x}$$



b)



3.4.15

$$\dot{x} = rx + x^3 - x^5 = -\frac{\mathrm{d}V}{\mathrm{d}x} \Rightarrow V = \frac{-rx^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} + C$$

The critical points occur when

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}x} &= -rx - x^3 + x^5 \\
&= -x(r + x^2 - x^4) = 0 \\
x_1 &= 0 \\
x^2 &= \frac{-1 \pm \sqrt{1^2 - 4(-1)r}}{2(-1)} = \frac{1 \pm \sqrt{1+4r}}{2} \\
x_2 &= \sqrt{\frac{1+\sqrt{1+4r}}{2}} \qquad x_3 = \sqrt{\frac{1-\sqrt{1+4r}}{2}} \\
x_4 &= -\sqrt{\frac{1+\sqrt{1+4r}}{2}} \qquad x_5 = -\sqrt{\frac{1-\sqrt{1+4r}}{2}}
\end{aligned}$$

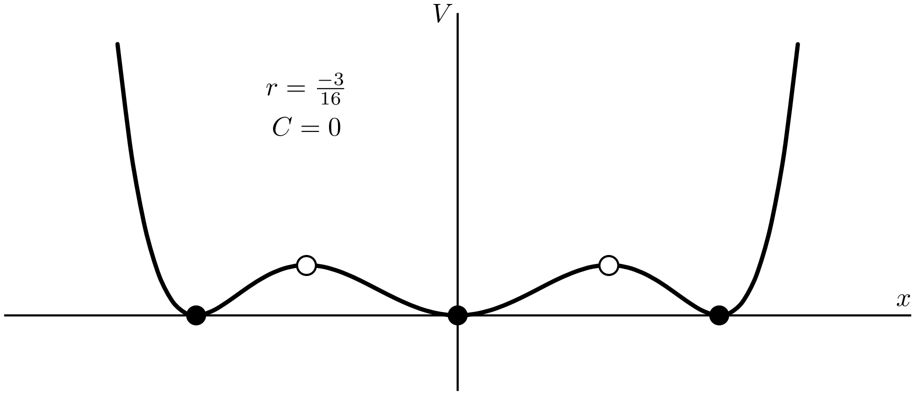
and the local minima occur when

$$\begin{aligned}
&\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = -r - 3x^2 + 5x^4 > 0 \\
&\frac{\mathrm{d}^2V}{\mathrm{d}x^2}(x_1) = -r > 0 \text{ if } r < 0 \\
&\frac{\mathrm{d}^2V}{\mathrm{d}x^2}(x_2) = \frac{\mathrm{d}^2V}{\mathrm{d}x^2}(x_4) = 4r + \sqrt{4r+1} + 1 > 0 \text{ if } \frac{-1}{4} < r \\
&\frac{\mathrm{d}^2V}{\mathrm{d}x^2}(x_3) = \frac{\mathrm{d}^2V}{\mathrm{d}x^2}(x_5) = 4r - \sqrt{4r+1} + 1 > 0 \text{ if } 0 < r
\end{aligned}$$

so the three candidate wells occur at x_1, x_2, and x_4 with $\frac{-1}{4} < r < 0$.

$V(x_0) = C$, so for simplicity we'll set $C = 0$, which leaves

$$V(x_2) = V(x_4) = V(x_0)$$
$$\frac{1}{24}\left(-(4r+1)^{\frac{3}{2}} - 6r - 1\right) = 0 \Rightarrow r = \frac{-3}{16}$$



So $r_c = \frac{-3}{16}$ and the three minima x_1, x_2, and x_4 occur at the value of the integrating constant C.

3.5 Overdamped Bead on a Rotating Hoop

3.5.1

There shouldn't be an equilibrium between $\frac{\pi}{2}$ and π because the rotation would force the bead outward and down, and gravity would force the bead down. The forces wouldn't cancel, so the bead would move.

3.5.3

A Taylor series expansion of $f(\phi) = \sin(\phi)\big(\gamma\cos(\phi) - 1\big)$ centered at $\phi = 0$ gives

$$(\gamma - 1)\phi + \left(\frac{1}{6} - \frac{2\gamma}{3}\right)\phi^3 + O\left(\phi^5\right)$$

3.5.5

a)

Starting with the dimensionless equation

$$\epsilon\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} + \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = f(\phi)$$

We already have $T_{\text{slow}} = \dfrac{t}{\tau} = \dfrac{b}{mg}$ from the text, but we don't know what order of magnitude T_{faster} is in relation to T_{slow}, so we'll guess a new time scale $\tau = \epsilon^k\xi$. The dimensionless equation is now

$$\epsilon^{1-2k}\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} + \epsilon^{-k}\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = f(\phi)$$

We still need to be able to solve the equation, meaning each of the terms must be of the same order or negligible compared to all the other terms.

To have all the terms of the same order, we need $\epsilon^{1-2k} = \epsilon^{-k} = 1$, which is impossible.

To have two terms of the same order and one term negligible, we need

$$\epsilon^{1-2k} = \epsilon^{-k} \gg 1 \Rightarrow k = 1 \quad \text{or} \quad \epsilon^{1-2k} = 1 \gg \epsilon^{-k} \Rightarrow k = \frac{1}{2} \quad \text{or} \quad \epsilon^{-k} = 1 \gg \epsilon^{1-2k} \Rightarrow k = 0$$

Choosing $k = 0$ gives the equation we started with, and $k = \frac{1}{2}$ gives

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} + \frac{1}{\sqrt{\epsilon}}\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = f(\phi)$$

which is bad because $\epsilon \ll 1$ makes the $\frac{1}{\sqrt{\epsilon}}$ term enormous. But the $k = 1$ choice looks good.

If we choose $\tau = \epsilon\xi$ for the new time scale, then we can find T_{fast} in terms of m, g, r, ω, and b by using

$$T_{\text{slow}} = \frac{t}{\tau} = \frac{b}{mg}$$

from the text.

$$T_{\text{fast}} = \frac{t}{\xi} = \frac{t}{\xi}\left(\frac{\tau}{t}\frac{b}{mg}\right) = \frac{\tau}{\xi}\frac{b}{mg} = \epsilon\frac{b}{mg} = \frac{m^2gr}{b^2}\frac{b}{mg} = \frac{mr}{b}$$

b)

Rescaling with the fast time scale

$$\begin{aligned}
\frac{r}{gT_{\text{fast}}^2}\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} &= \frac{-b}{mgT_{\text{fast}}}\frac{\mathrm{d}\phi}{\mathrm{d}\xi} - \sin(\phi) + \frac{r\omega^2}{g}\sin(\phi)\cos(\phi) \\
\frac{b^2}{m^2gr}\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} &= \frac{-b^2}{m^2gr}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} - \sin(\phi) + \frac{r\omega^2}{g}\sin(\phi)\cos(\phi) \\
\frac{b^2}{m^2gr}\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} + \frac{b^2}{m^2gr}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} &= -\sin(\phi) + \frac{r\omega^2}{g}\sin(\phi)\cos(\phi) \\
\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} + \frac{\mathrm{d}\phi}{\mathrm{d}\tau} &= \frac{m^2gr}{b^2}\left(-\sin(\phi) + \frac{r\omega^2}{g}\sin(\phi)\cos(\phi)\right) \\
&= \epsilon f(\phi)
\end{aligned}$$

Now the $\epsilon f(\phi)$ is the negligible term.

c)

$$T_{\text{fast}} = \epsilon\frac{b}{mg} = \epsilon T_{\text{slow}}$$

So if $\epsilon \ll 1$ then $T_{\text{fast}} \ll T_{\text{slow}}$.

3.5.7

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) \qquad N(0) = N_0$$

a)

$$r \sim \frac{1}{\text{time}} \quad K \sim \text{population} \quad N_0 \sim \text{population}$$

b)

$$\begin{aligned}
& \dot{N} = rN\left(1 - \frac{N}{K}\right) \quad N(0) = N_0 \\
x = \frac{N}{K} \qquad & K\dot{x} = rKx(1-x) \quad x(0) = \frac{N_0}{K} = x_0 \\
& \dot{x} = rx(1-x) \quad x(0) = x_0 \\
\tau = rt \qquad & r\frac{\mathrm{d}x}{\mathrm{d}\tau} = rx(1-x) \quad x(0) = x_0 \\
& \frac{\mathrm{d}x}{\mathrm{d}\tau} = x(1-x) \quad x(0) = x_0
\end{aligned}$$

c)

$$\begin{aligned}
& \dot{N} = rN\left(1 - \frac{N}{K}\right) \quad N(0) = N_0 \\
u = \frac{N}{N_0} \qquad & N_0\dot{u} = rN_0u\left(1 - \frac{N_0}{K}u\right) \quad u(0) = \frac{N(0)}{N_0} = 1 \\
& \frac{1}{r}\dot{u} = u\left(1 - \frac{N_0}{K}u\right) \\
\tau = rt \qquad & \frac{\mathrm{d}u}{\mathrm{d}\tau} = u\left(1 - \frac{N_0}{K}u\right) \\
\kappa = rt \qquad & \frac{\mathrm{d}u}{\mathrm{d}\tau} = u\left(1 - \kappa u\right) \quad u(0) = 1
\end{aligned}$$

d)

The x nondimensionalization would be useful for investigating the effect of the initial condition relative to carrying capacity on the solution, whereas the u nondimensionalization would be useful for studying the rate of population growth relative to carrying capacity.

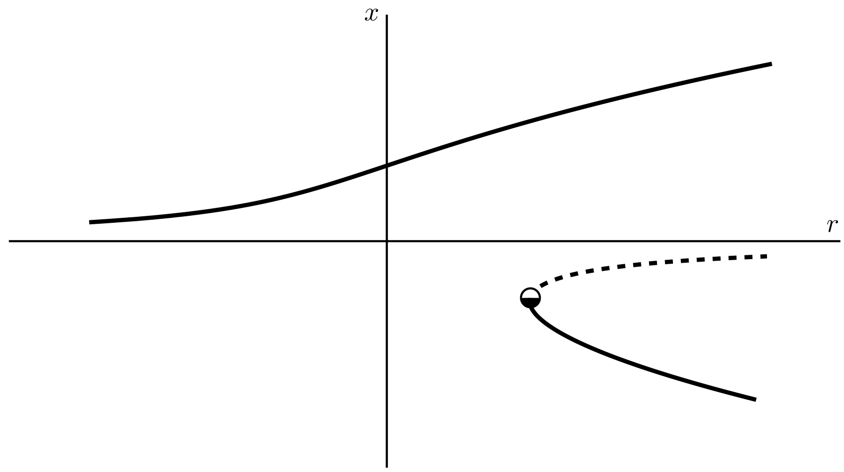
By the way, scaling in such a way that all solutions have the same initial condition is mathematically valid but runs counter to the geometric approach we are using in this book. Our whole approach is based on visualizing different solutions emanating from different initial conditions.

3.6 Imperfect Bifurcations and Catastrophes

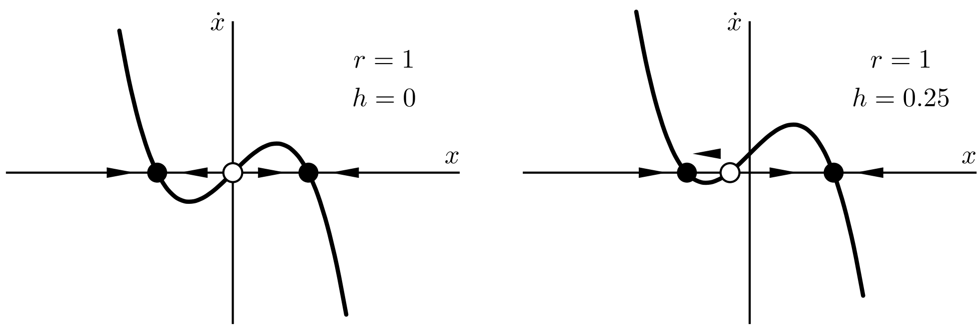
3.6.1

$$\dot{x} = h + rx - x^3$$

This x versus r bifurcation diagram



has a saddle-node bifurcation at a negative x value. Vertically shifting the cubic equation and then varying r will achieve this.

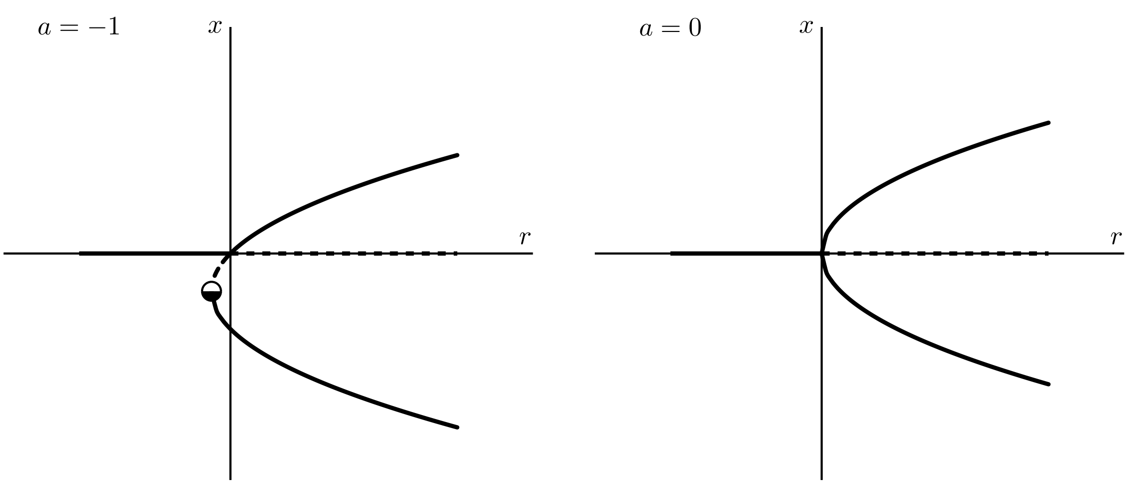


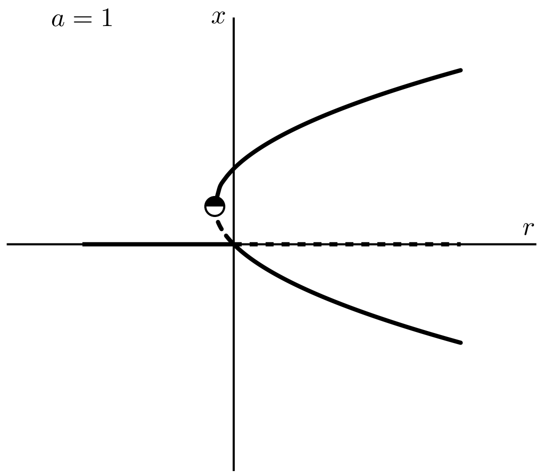
So h is positive.

3.6.3

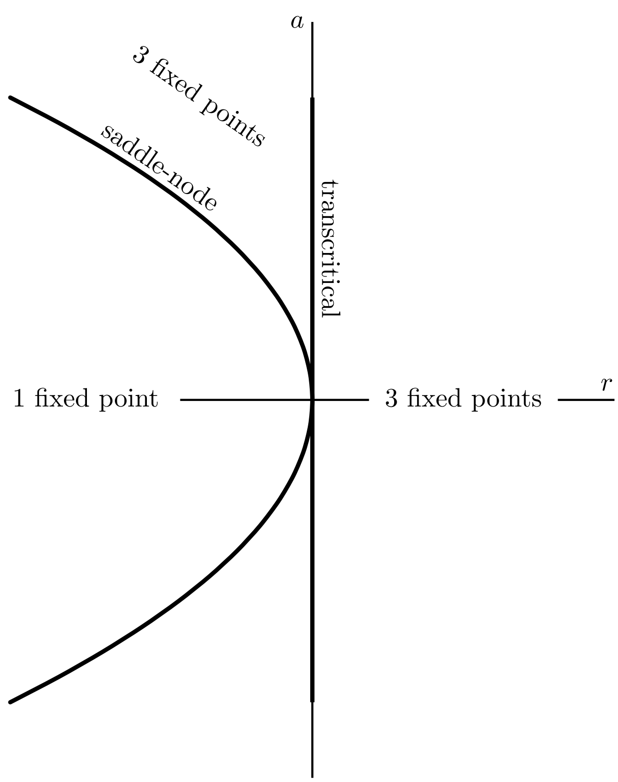
$$\dot{x} = rx + ax^2 - x^3$$

a)





b)



3.6.5

a)

The spring has spring constant k and rest length L_0, but only the force from the spring that is parallel, and not perpendicular, with the wire will affect the bead.

$$
\begin{aligned}
k(L - L_0) &= k\left(\sqrt{x^2 + a^2} - L_0\right)\frac{x}{\sqrt{x^2 + a^2}} \\
&= -kx\left(1 - \frac{L_0}{\sqrt{x^2 + a^2}}\right)
\end{aligned}
$$

The force of gravity will also move the bead along the wire, but only the fraction of the force that is parallel with the wire.

$$mg\sin(\theta)$$

At equilibrium, the force of the spring and the force from gravity have to be equal and opposite, which gives

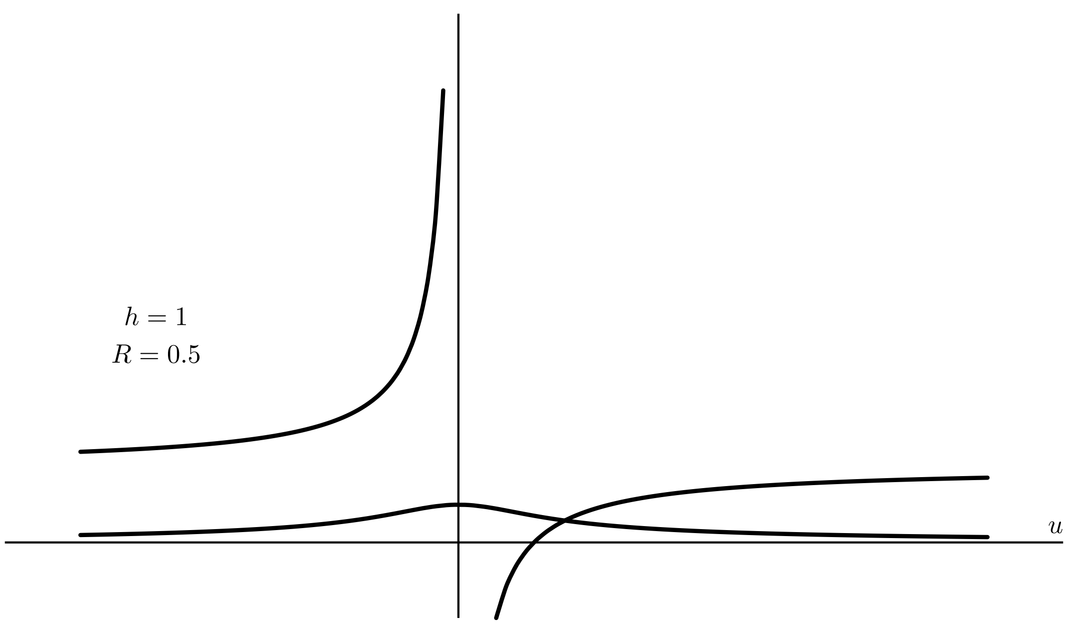
$$mg\sin(\theta) = kx\left(1 - \frac{L_0}{\sqrt{x^2+a^2}}\right)$$

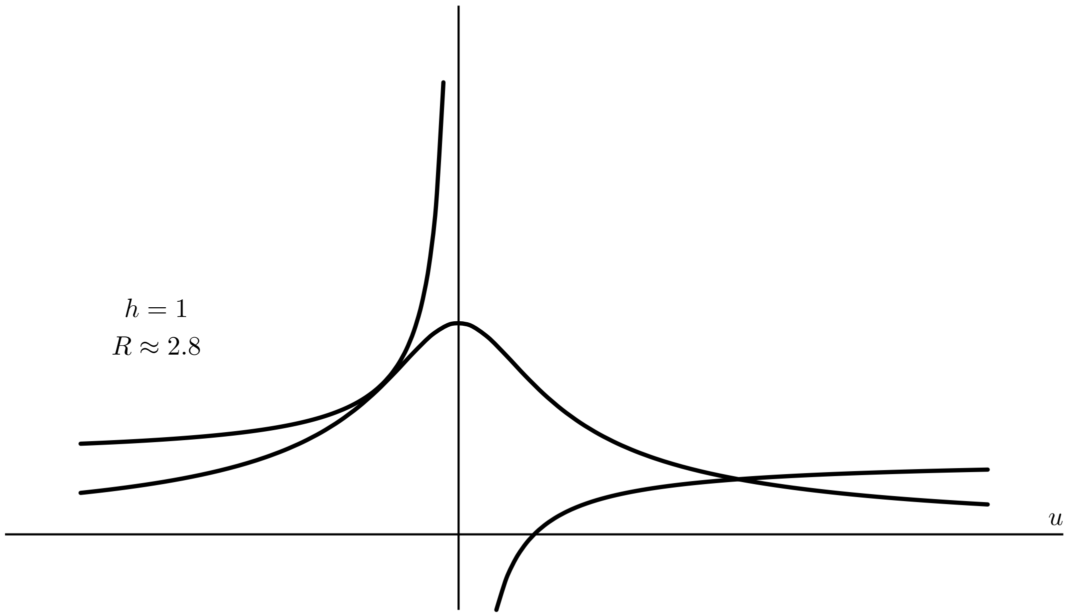
b)

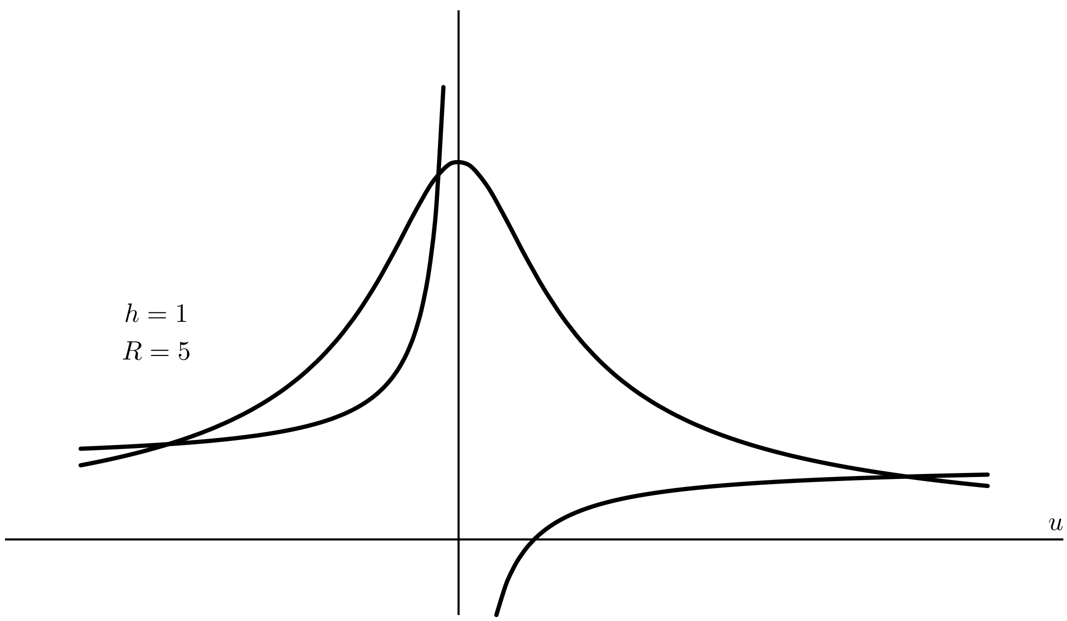
$$\begin{aligned}
mg\sin(\theta) &= kx\left(1 - \frac{L_0}{\sqrt{x^2+a^2}}\right)\\
mg\sin(\theta) &= kx - kx\frac{L_0}{\sqrt{x^2+a^2}}\\
mg\sin(\theta) - kx &= -kx\frac{L_0}{a\sqrt{\frac{x^2}{a^2}+1}}\\
\frac{-mg\sin(\theta)}{kx} + 1 &= \frac{L_0}{a\sqrt{\frac{x^2}{a^2}+1}}\\
1 - \frac{mg\sin(\theta)}{kx} &= \frac{L_0}{a\sqrt{1+\frac{x^2}{a^2}}}\\
u = \frac{x}{a} \qquad R &= \frac{L_0}{a} \qquad h = \frac{mg\sin(\theta)}{ka}\\
1 - \frac{h}{u} &= \frac{R}{\sqrt{1+u^2}}
\end{aligned}$$

c)

Both h and R are strictly positive quantities from the physical constants they're made of. The following graphs show the different types and number of possible intersections. In each graph, h shapes the discontinuous graph and R shapes the bell-shaped graph with u as the independent variable.







For $0 < R < 1$ there is only one intersection since the other curve asymptotes to height 1 from above. As R increases, there is a point when the two curves just touch, making another fixed point in a saddle-node bifurcation. As R increases further there are three fixed points.

d)

$$1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}$$

$$r = R - 1$$

$$1 - \frac{h}{u} = \frac{r+1}{\sqrt{1+u^2}}$$

$$u\sqrt{1+u^2} - h\sqrt{1+u^2} = (r+1)u$$
$$h\sqrt{1+u^2} + (r+1)u - u\sqrt{1+u^2} = 0$$

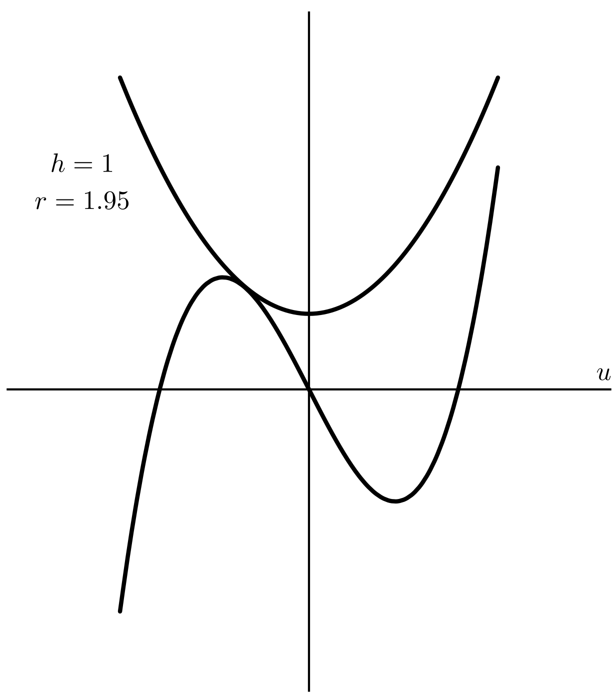
Taylor series expansion

$$h\left(1+\frac{1}{2}u^2+O\left(u^4\right)\right)+(r+1)u-u\left(1+\frac{1}{2}u^2+O\left(u^4\right)\right)=0$$
$$h+\frac{h}{2}u^2+ru+u-u-\frac{1}{2}u^3=O\left(u^4\right)$$
$$h+ru+\frac{h}{2}u^2-\frac{1}{2}u^3=O\left(u^4\right)$$
$$h+ru+\frac{h}{2}u^2-\frac{1}{2}u^3\approx 0$$

e)

The saddle-node bifurcations will occur approximately at the local maximum of the approximate equation when the slopes of these equations are equal at an equal height.

$$\frac{1}{2}u^3-ru=h+\frac{h}{2}u^2$$



$$\frac{1}{2}u^3-ru=h+\frac{h}{2}u^2$$
$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{2}u^3-ru\right)=\frac{3}{2}u^2-r=hu=\frac{\mathrm{d}}{\mathrm{d}u}\left(h+\frac{h}{2}u^2\right)$$
$$\frac{3}{2}u^2-hu=r$$

$$\frac{1}{2}u^3 - ru = \frac{1}{2}u^3 - \left(\frac{3}{2}u^2 - hu\right)u = h + \frac{h}{2}u^2$$
$$\frac{1}{2}u^3 - \frac{3}{2}u^3 + hu^2 = h + \frac{h}{2}u^2$$
$$\frac{h}{2}u^2 - h = u^3$$
$$h\left(\frac{1}{2}u^2 - 1\right) = u^3$$
$$h = \frac{2u^3}{u^2 - 2}$$
$$r = \frac{3}{2}u^2 - hu = \frac{3}{2}u^2 - \frac{2u^3}{u^2-2}u = \frac{u^4 + 3u^2}{2(1-u^2)}$$

f)

Here we use the same procedure as in part (e). The saddle-node bifurcations will occur at the local maximum of the equation when the slopes of these equations are equal at an equal height.

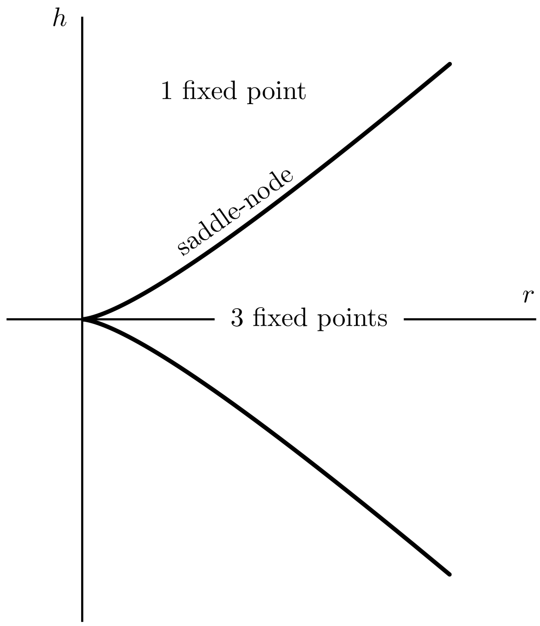
$$1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}u}\left(1 - \frac{h}{u}\right) = \frac{h}{u^2} = \frac{-Ru}{(1+u^2)^{\frac{3}{2}}} = \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{R}{\sqrt{1+u^2}}\right)$$
$$h(1+u^2)^{\frac{3}{2}} = -Ru^3$$
$$R = \frac{-h(1+u^2)^{\frac{3}{2}}}{u^3}$$
$$1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}} = \frac{-h(1+u^2)^{\frac{3}{2}}}{u^3\sqrt{1+u^2}} = \frac{-h(1+u^2)}{u^3}$$
$$u^3 - hu^2 = -h(1+u^2)$$
$$h = -u^3$$
$$R = \frac{-h(1+u^2)^{\frac{3}{2}}}{u^3} = \frac{-(-u^3)(1+u^2)^{\frac{3}{2}}}{u^3} = (1+u^2)^{\frac{3}{2}}$$

The approximate result and the exact result agree in the limit of $u \to 0$.

$$h = -u^3 \approx \frac{2u^3}{0-2} \approx \frac{2u^3}{u^2-2}$$
$$R = (1+u^2)^{\frac{3}{2}} \approx (1+0^2)^{\frac{3}{2}} \approx \frac{0^4 + 3(0)^2}{2(1-0^2)} + 1 \approx \frac{u^4+3u^2}{2(1-u^2)} + 1 = r + 1$$

g)

$$h(u) = -u^3 \qquad R(u) = \left(1+u^2\right)^{\frac{3}{2}} \Rightarrow r(u) = \left(1+u^2\right)^{\frac{3}{2}} - 1$$



h)

$$h(u) = -u^3 \qquad R(u) = \left(1+u^2\right)^{\frac{3}{2}}$$

$$u = \frac{x}{a}$$

$$h = -\left(\frac{x}{a}\right)^3 = \frac{mg\sin(\theta)}{ka} \qquad R = \left(1+\left(\frac{x}{a}\right)^2\right)^{\frac{3}{2}} = \frac{L_0}{a}$$

Solving for x and equating the two equations gives the conditions that the catastrophe will occur and the position x of its occurrence. Since many of the parameters are fixed, we could find the catastrophe angle θ in terms of the physical constants of the experiment.

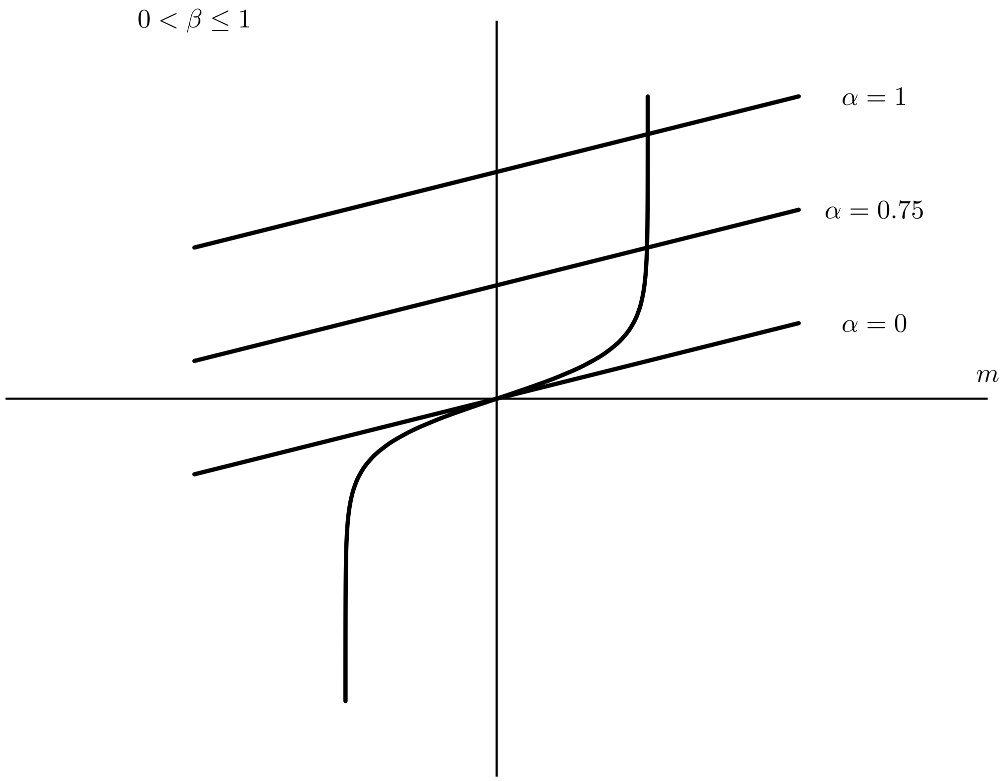
3.6.7

a)

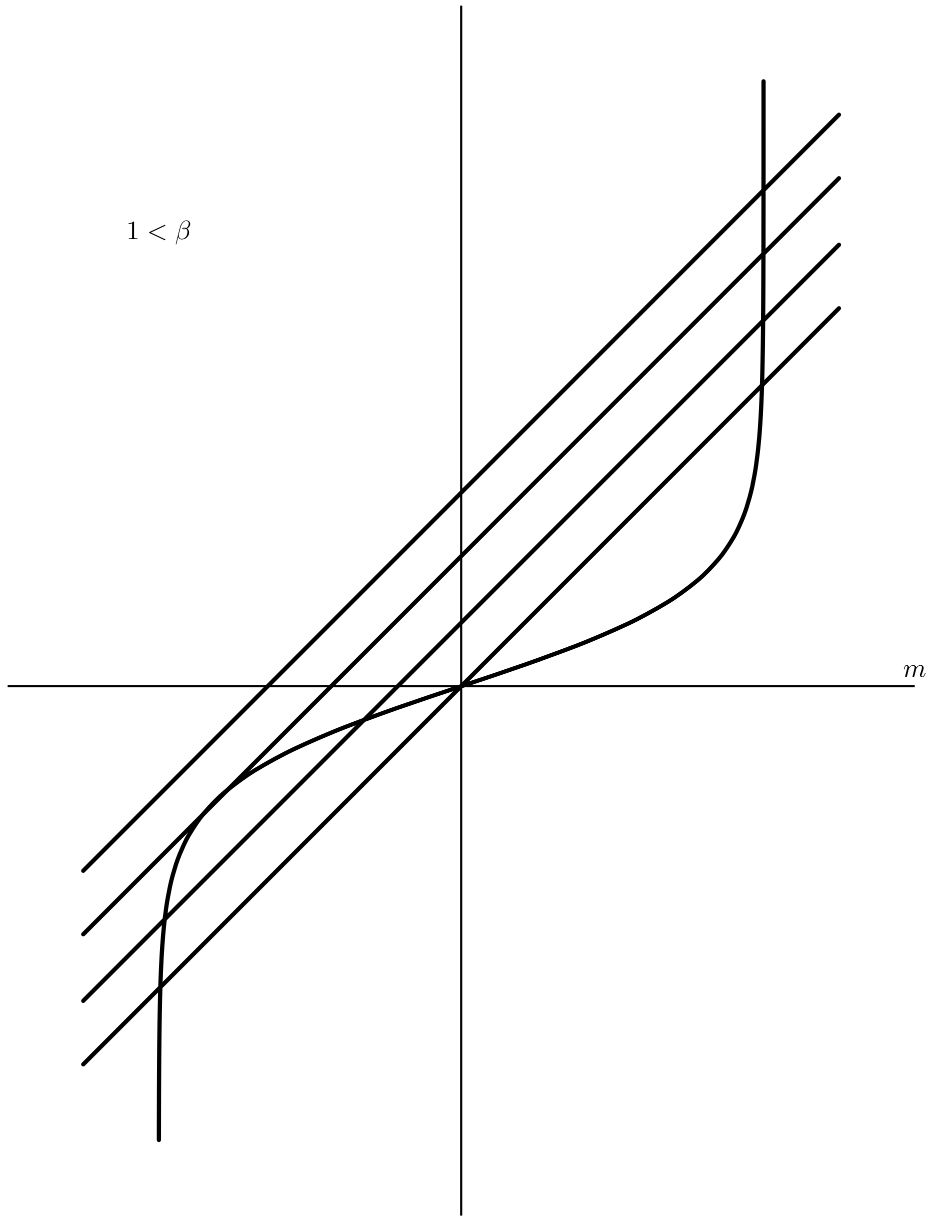
$$h = T\text{arctanh}(m) - Jnm$$

We can nondimensionalize to

$$\alpha + \beta m = \text{arctanh}(m) \qquad \alpha = \frac{h}{T} \quad \beta = \frac{Jn}{T} > 0$$



There is always a unique intersection point with this range of β.



Now there are one, two, or three solutions depending on the values depending on α for this range of β.

b)

If $h = 0$ then $\alpha = 0$ and the phase transition occurs when the slope of the line $\beta = \frac{Jn}{T_c} = 1 \Rightarrow T_c = Jn$.

3.7 Insect Outbreak

3.7.1

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}\tau} &= rx\left(1-\frac{x}{k}\right) - \frac{x^2}{1+x^2} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(rx\left(1-\frac{x}{k}\right) - \frac{x^2}{1+x^2}\right) &= \frac{-2rx}{k} + r - \frac{2x}{(x^2+1)^2} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(rx\left(1-\frac{x}{k}\right) - \frac{x^2}{1+x^2}\right)\Bigg|_{x=0} &= r > 0
\end{aligned}
$$

So $x = 0$ is always unstable.

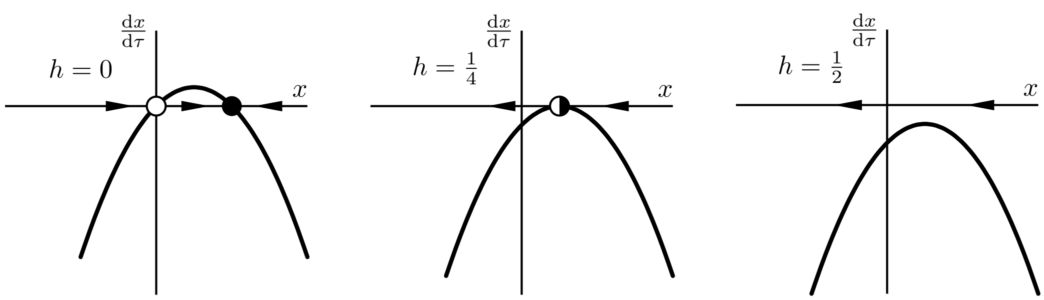
3.7.3

$$
\dot{N} = rN\left(1-\frac{N}{K}\right) - H
$$

a)

$$
\begin{aligned}
\frac{\dot{N}}{rK} &= \frac{N}{K}\left(1-\frac{N}{K}\right) - \frac{H}{rK} \\
x = \frac{N}{K} \quad \tau &= rt \quad h = \frac{H}{rK} \\
\frac{\mathrm{d}x}{\mathrm{d}\tau} &= x(1-x) - h
\end{aligned}
$$

b)



c)

A saddle-node bifurcation occurs at $h_c = \frac{1}{4}$ and $x = \frac{1}{2}$.

d)

For $h < h_c$ the fish are sustainably harvested so that their population never decreases or increases.

For $h_c < h$ the fish are hunted to extinction (assuming that the fish population can't become negative).

3.7.5

a)

$$\dot{g} = \frac{\mathrm{d}g}{\mathrm{d}t} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2}$$

$$\frac{k_4}{k_3}\frac{\mathrm{d}\frac{g}{k_4}}{\mathrm{d}t} = \frac{k_1 s_0}{k_3} - \frac{k_2 k_4}{k_3}\frac{g}{k_4} + \frac{\frac{g^2}{k_4^2}}{1 + \frac{g^2}{k_4^2}}$$

$$x = \frac{g}{k_4} \qquad s = \frac{k_1 s_0}{k_3} \qquad r = \frac{k_2 k_4}{k_3} \qquad \tau = \frac{k_3}{k_4}t$$

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = s - rx + \frac{x^2}{1+x^2}$$

b)

If $s = 0$

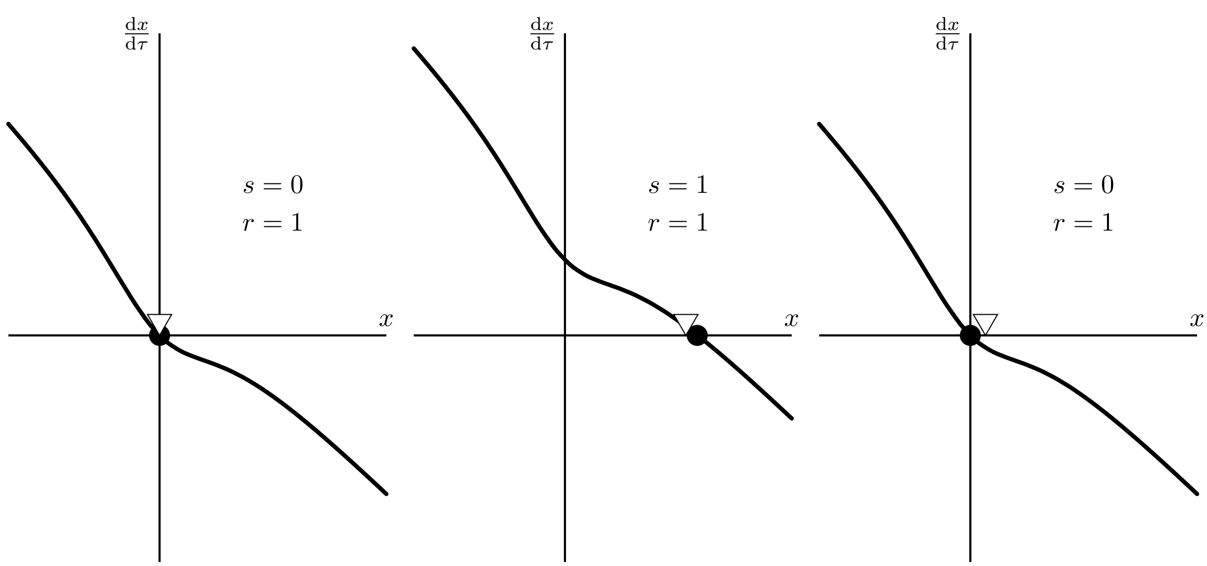
$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = 0 - rx + \frac{x^2}{1+x^2} = 0$$

$$x = 0, \frac{1 \pm \sqrt{1-4r^2}}{2r}$$

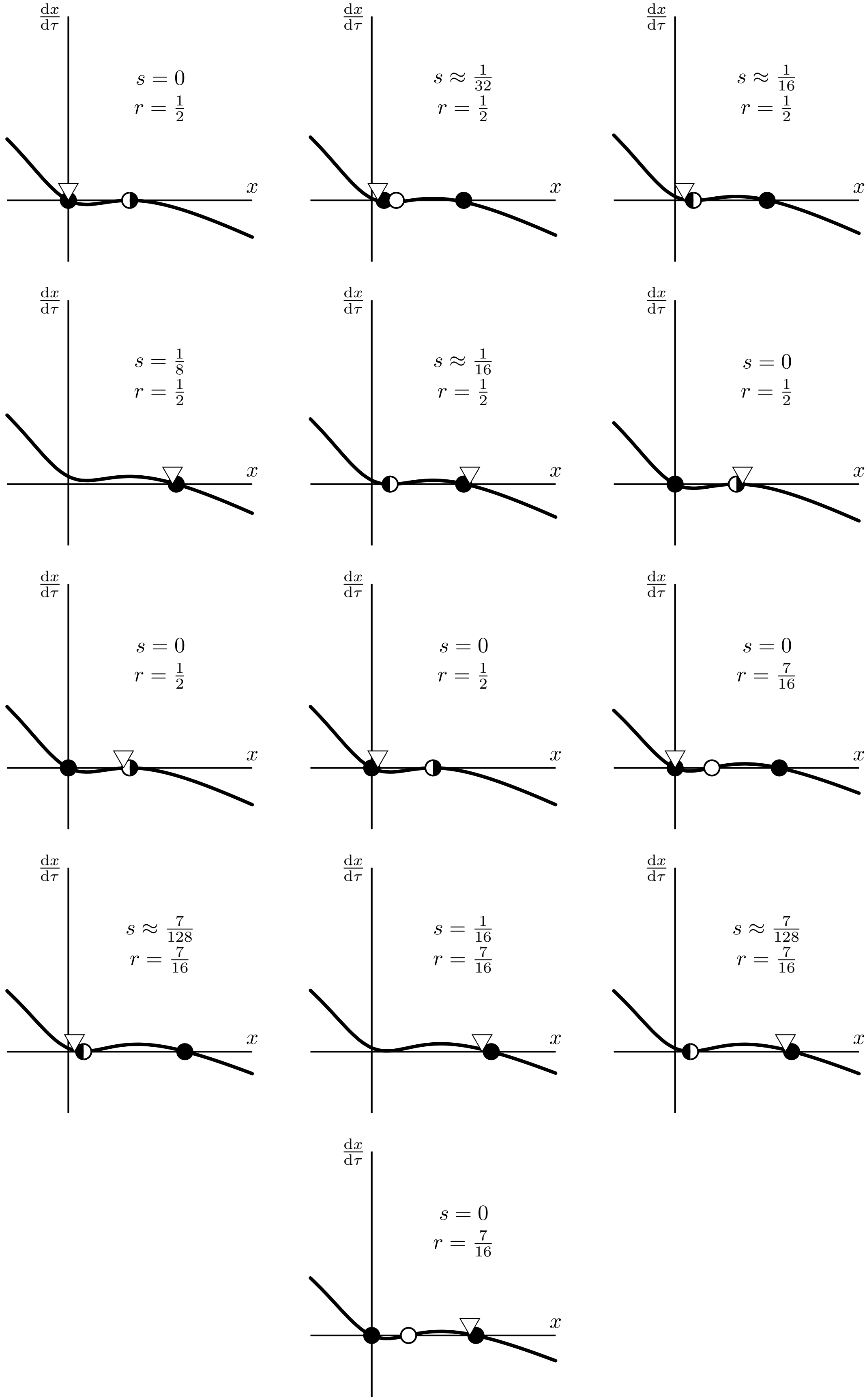
And there are three real fixed points, two of which are positive since the square root is at most one, if $r < r_c = \frac{1}{2}$.

c)

$g(0) = 0$ corresponds to $x(0) = 0$, and increasing the parameter s translates the graph vertically. There is only one fixed point at $x = 0$ for $\frac{1}{2} < r$ since the graph is strictly decreasing, and x will stay close to the fixed point as s is increased.



At $r \leq \frac{1}{2}$ one, two, and three fixed points are possible with two saddle-node bifurcations occurring as s increases.


$\frac{dx}{d\tau}$
$s = 0$
$r = \frac{1}{2}$
x
$\frac{dx}{d\tau}$
$s \approx \frac{1}{32}$
$r = \frac{1}{2}$
x
$\frac{dx}{d\tau}$
$s \approx \frac{1}{16}$
$r = \frac{1}{2}$
x
$\frac{dx}{d\tau}$
$s = \frac{1}{8}$
$r = \frac{1}{2}$
x
$\frac{dx}{d\tau}$
$s \approx \frac{1}{16}$
$r = \frac{1}{2}$
x
$\frac{dx}{d\tau}$
$s = 0$
$r = \frac{1}{2}$
x
$\frac{dx}{d\tau}$
$s = 0$
$r = \frac{1}{2}$
x
$\frac{dx}{d\tau}$
$s = 0$
$r = \frac{1}{2}$
x
$\frac{dx}{d\tau}$
$s = 0$
$r = \frac{7}{16}$
x
$\frac{dx}{d\tau}$
$s \approx \frac{7}{128}$
$r = \frac{7}{16}$
x
$\frac{dx}{d\tau}$
$s = \frac{1}{16}$
$r = \frac{7}{16}$
x
$\frac{dx}{d\tau}$
$s \approx \frac{7}{128}$
$r = \frac{7}{16}$
x
$\frac{dx}{d\tau}$
$s = 0$
$r = \frac{7}{16}$
x

Starting with $s = 0$, increasing s, and then returning to $s = 0$:

If $\frac{1}{2} < r$, the system will go towards the $x = 0$ stable fixed point since that is the only fixed point.

If $r = \frac{1}{2}$, the system can go back to the $x = 0$ fixed point or the stable half of the semistable fixed point depending on how fast s changes, but noise in a real system would most likely bump x to the unstable side of the semistable fixed point and then the system would go towards the $x = 0$ fixed point.

If $r < \frac{1}{2}$, the system will almost surely be near the far right stable fixed point, despite s returning to its original value.

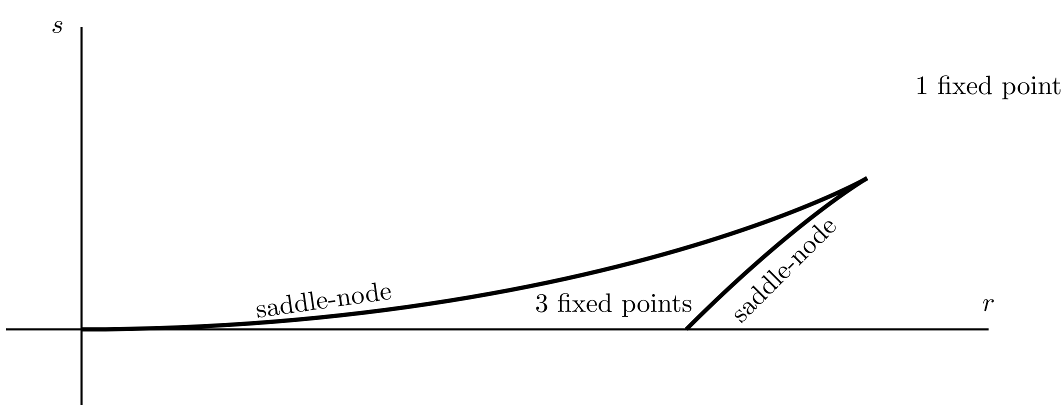
d)

The saddle-node bifurcations will occur when $\frac{\mathrm{d}x}{\mathrm{d}\tau} = 0$ and $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\tau} = 0$, which gives us two equations to solve for r and s in terms of x.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(s - rx + \frac{x^2}{1+x^2}\right) = -r + \frac{2x}{(1+x^2)^2} = 0$$
$$r = \frac{2x}{(1+x^2)^2}$$
$$s - rx + \frac{x^2}{1+x^2} = s - \frac{2x}{(1+x^2)^2}x + \frac{x^2}{1+x^2} = 0$$
$$s = \frac{x^2(1-x^2)}{(x^2+1)^2}$$

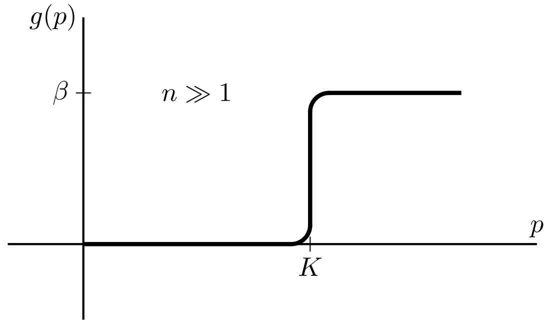
These are saddle-node bifurcations.

e)



3.7.7

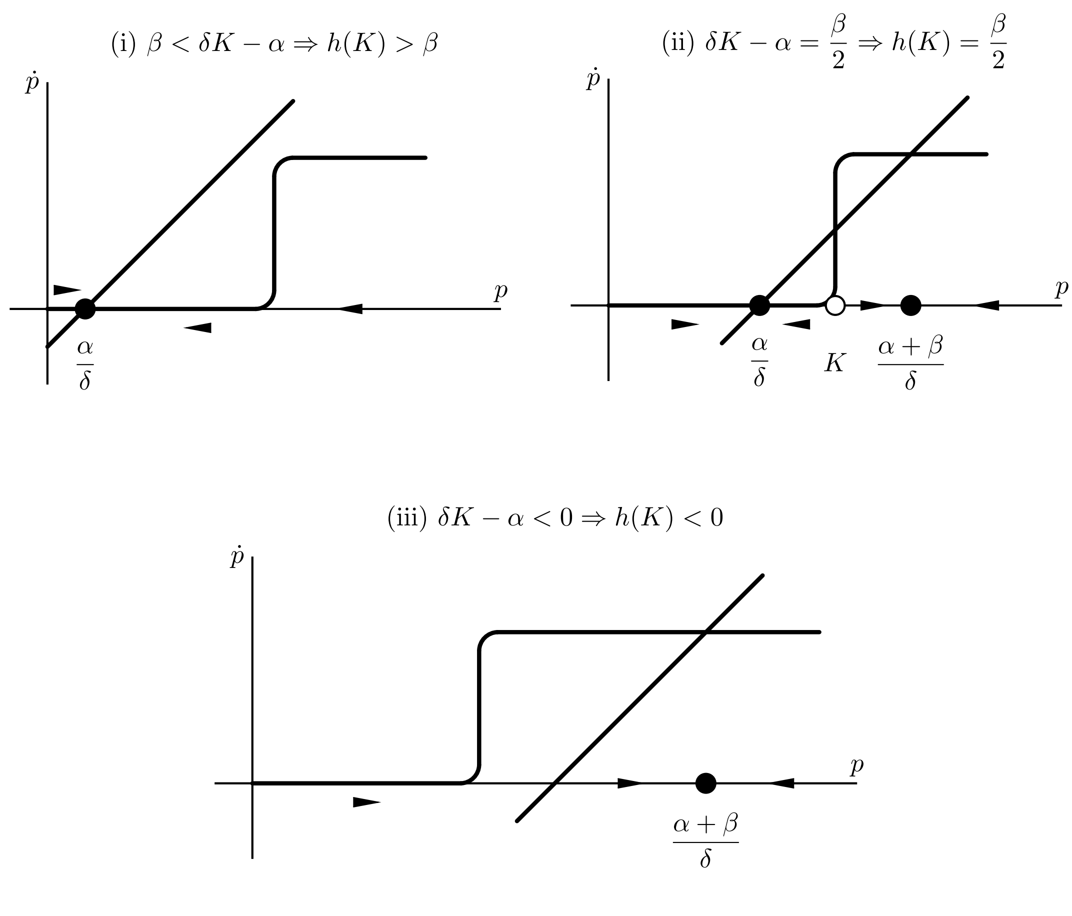
a)



$$\lim_{n\to\infty} \frac{\beta p^n}{K^n + p^n} = \lim_{n\to\infty} \frac{\beta \frac{p^n}{p^n}}{\frac{K^n}{p^n} + \frac{p^n}{p^n}} = \lim_{n\to\infty} \frac{\beta}{\frac{K^n}{p^n} + 1} = \beta H(p - K)$$

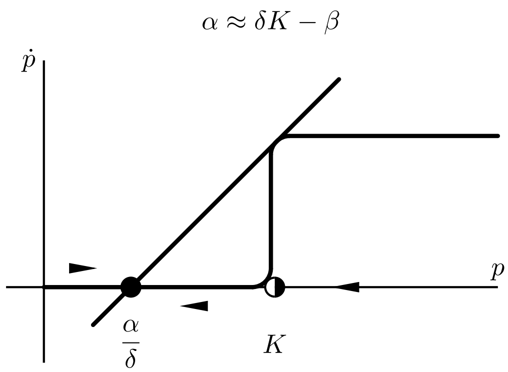
where $H(p)$ is the Heaviside step function.

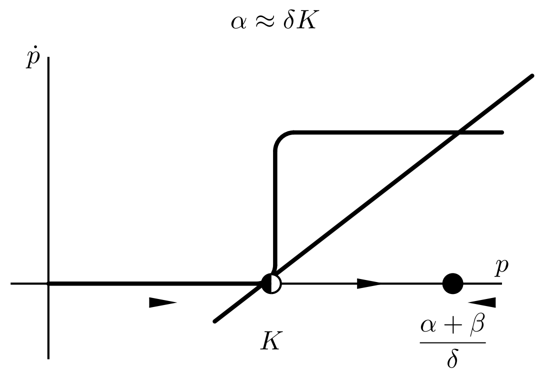
b)



c)

We'll assume $\alpha \geq 0$ and $\beta > 0$ due to being biological measurements, as well as $\delta K > \beta$ from the problem statement. If we start $\alpha = 0$ and increase α, we'll get a saddle-node bifurcation at $p \approx K$ when the line $h(p)$ touches the top corner of $g(p)$. Then we get another saddle-node bifurcation when $h(p)$ touches the bottom corner of $g(p)$.

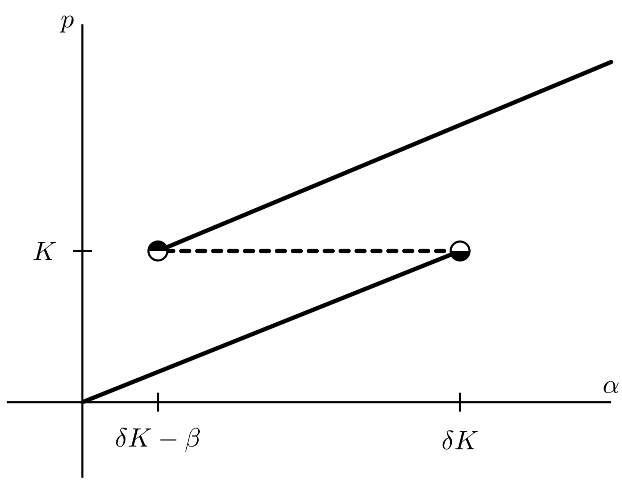




In summary, the fixed points are

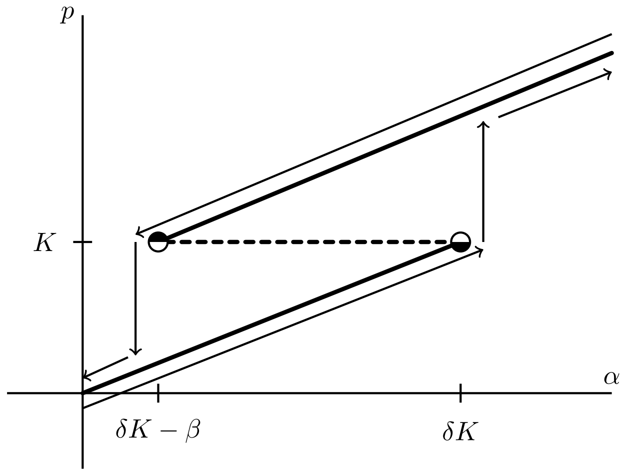
$$\begin{array}{ccrl}
0 < \alpha < \delta K - \beta & \Rightarrow & p \approx \dfrac{\alpha}{\delta} & \text{stable} \\
\alpha \approx \delta K - \beta & \Rightarrow & p \approx \dfrac{\alpha}{\delta}, K & \text{stable, semistable} \\
\delta K - \beta < \alpha < \delta K & \Rightarrow & p \approx \dfrac{\alpha}{\delta}, K, \dfrac{\alpha+\beta}{\delta} & \text{stable, unstable, stable} \\
\alpha \approx \delta K & \Rightarrow & p \approx K, \dfrac{\alpha+\beta}{\delta} & \text{semistable, stable} \\
\delta K < \alpha & \Rightarrow & p \approx \dfrac{\alpha+\beta}{\delta} & \text{stable}
\end{array}$$

and now we can plot the bifurcation diagram.



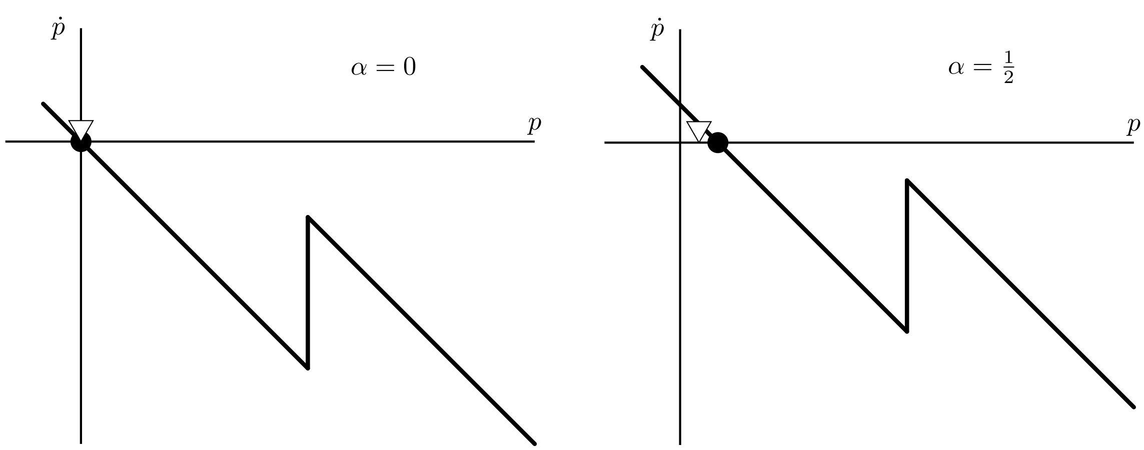
d)

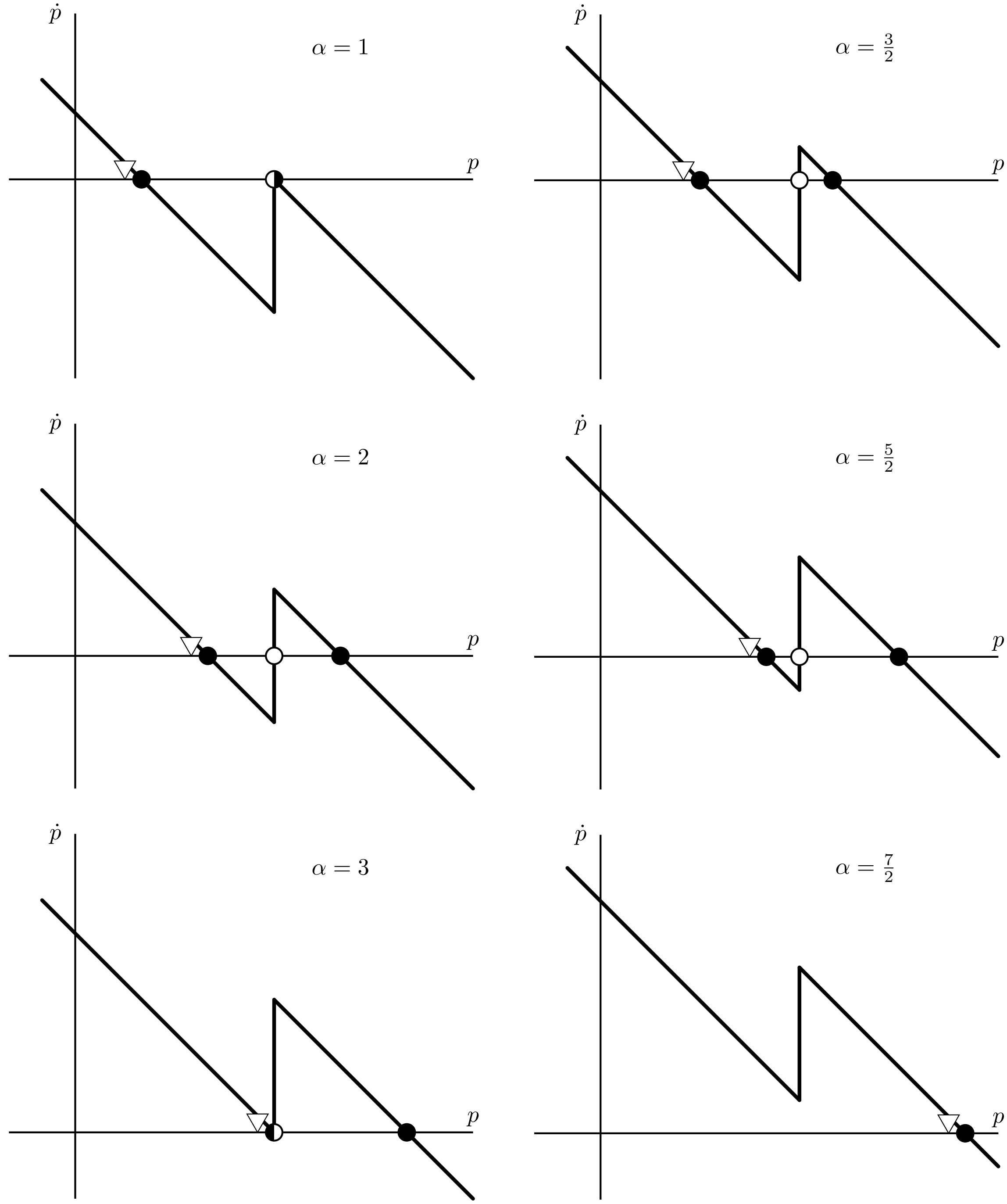
Hysteresis is possible for this system, in the path shown through parameter space below for example.

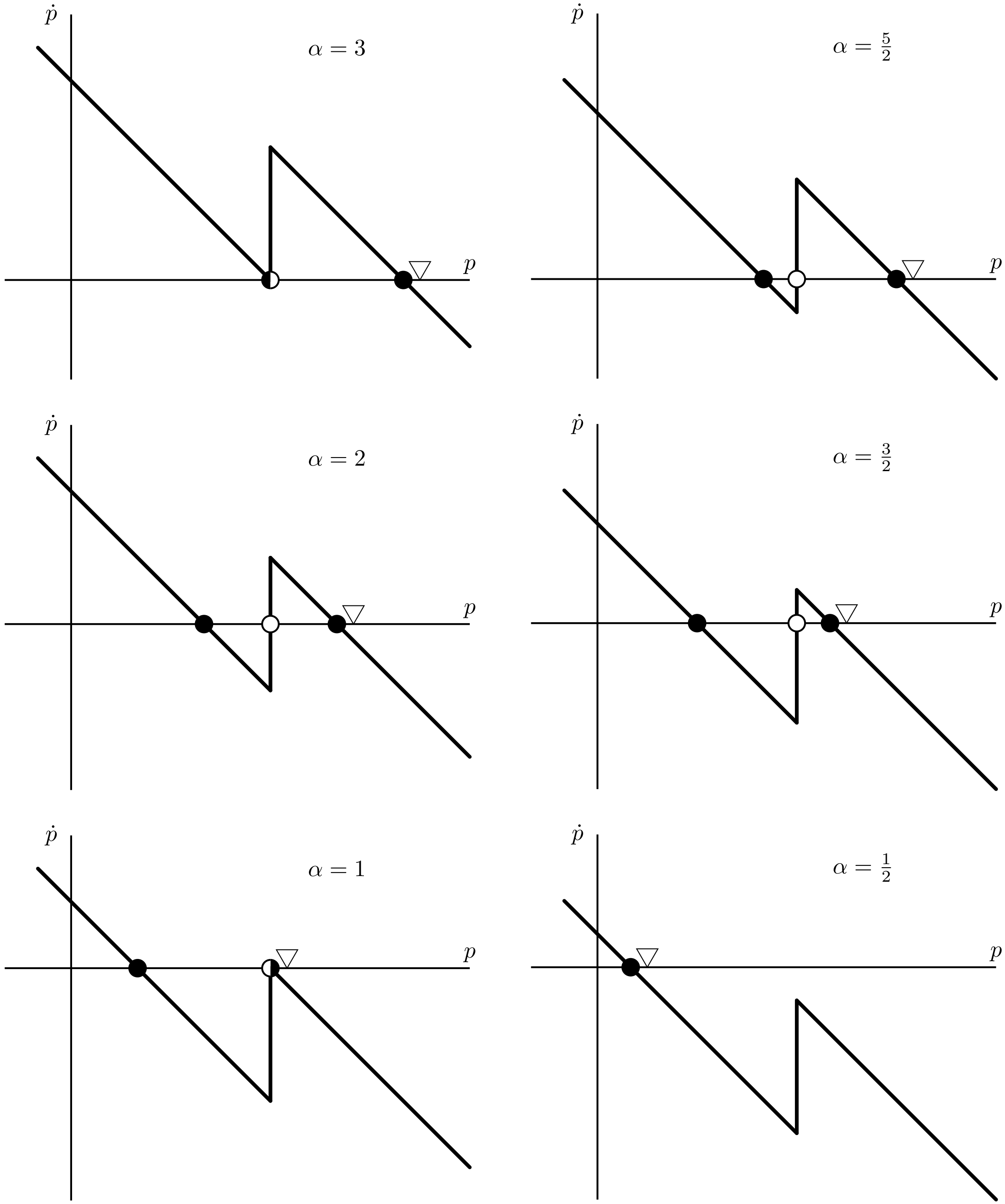


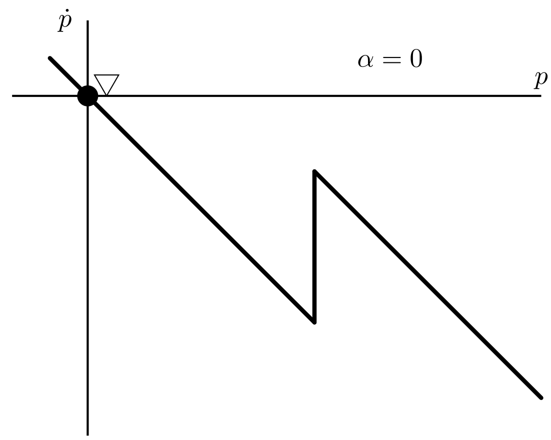
Starting at $\alpha = 0$, $p = 0$, the system stays at the lower fixed point until α increases past δK and the system jumps to the upper fixed point. The system exhibits hysteresis because the system does not jump back to the lower fixed point once α decreases past δK, despite the lower fixed point existing. Instead, we have to decrease α past $\delta K - \beta$ in order for the system to jump back to the lower fixed point.

Here is a similar path viewed on the phase line where we have combined the graph of both functions as $g(p) - h(p)$ with a very large n value, $K = 3$, $\delta = 1$, and $\beta = 2$.




$\dot{p}$
$\alpha = 1$
p
$\dot{p}$
$\alpha = \frac{3}{2}$
p
$\dot{p}$
$\alpha = 2$
p
$\dot{p}$
$\alpha = \frac{5}{2}$
p
$\dot{p}$
$\alpha = 3$
p
$\dot{p}$
$\alpha = \frac{7}{2}$
p


$\dot{p}$
$\alpha = 3$
p
$\dot{p}$
$\alpha = \frac{5}{2}$
p
$\dot{p}$
$\alpha = 2$
p
$\dot{p}$
$\alpha = \frac{3}{2}$
p
$\dot{p}$
$\alpha = 1$
p
$\dot{p}$
$\alpha = \frac{1}{2}$
p



The system exhibits hysteresis when p starts at the left fixed point but moves to the right fixed point once $3 = \delta K < \alpha$. But then α must be decreased to $\alpha < \delta K - \beta = 1$ instead of $\alpha < \delta K = 3$ for p to go back to the original left fixed point.

4

Flows on the Circle

4.1 Examples and Definitions

4.1.1

For what real values of a does the equation $\dot{\theta} = \sin(a\theta)$ give a well-defined vector field on the circle?

The value of a must ensure that $f(\theta) = \sin(a\theta)$ is 2π periodic.

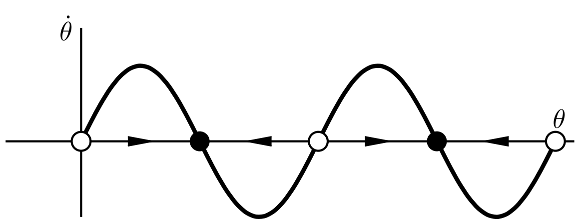
$$f(\theta) = f(\theta + 2\pi)$$

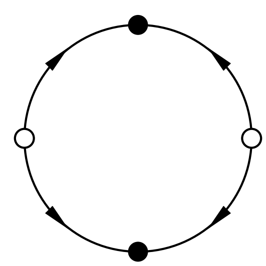
$$\sin(a\theta) = \sin\big(a(\theta + 2\pi)\big) = \sin(a\theta + 2\pi a) \Rightarrow 2\pi a = 2\pi z, z \in \mathbb{Z} \Rightarrow a \in \mathbb{Z}$$

4.1.3

$$\dot{\theta} = \sin(2\theta)$$

$$\dot{\theta} = 0 \Rightarrow \sin(2\theta) = 0 \Rightarrow \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

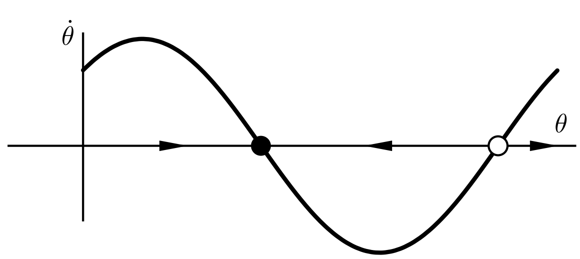


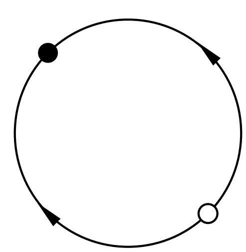


4.1.5

$$\dot{\theta} = \sin(\theta) + \cos(\theta)$$

$$\dot{\theta} = 0 \Rightarrow \sin(\theta) + \cos(\theta) = \sqrt{2}\sin\left(\theta + \frac{pi}{4}\right) = 0 \Rightarrow \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$$



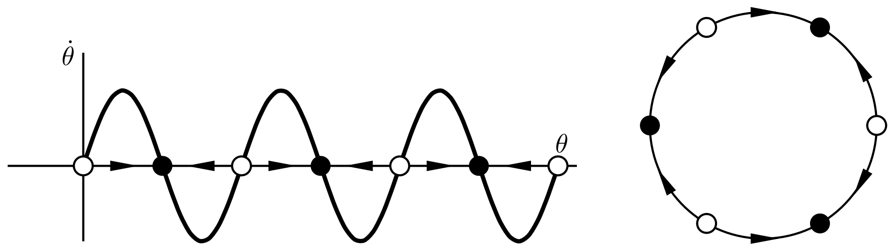


4.1.7

$$\dot{\theta} = \sin(k\theta) \text{ where } k \text{ is a positive integer.}$$

$$\dot{\theta} = 0 \Rightarrow \sin(k\theta) = 0 \Rightarrow \theta = \pi\frac{p}{k}, p = 0, 1, 2, ...2k$$

Example: $k = 3$



4.1.9

Exercise 2.6.2 and 2.7.7 does not carry over to vector fields on the circle because of the different interpretations of points. Put simply, the definition of periodic for vector fields on the line is fundamentally different from vector fields on the circle.

A vector field on the line is an interval with each value in the interval corresponding to a value of x that is interpreted as a unique point. Periodicity is defined as the particle returning to the same location after a time T, which can't occur.

A vector field on a circle with interval $[0, 2\pi)$ interprets $\theta(t) = 1$ and $\theta(t+T) = 1 + 2\pi$ as the same point. Periodicity is defined as the particle returning to the same location modulo 2π, which can occur. The particle isn't really where it started, but unlike vector fields on the line, the two locations are considered identical.

4.2 Uniform Oscillator

4.2.1

Using common sense, we know the two bells will ring again in 12 seconds.

Using the method of Example 4.2.1

$$\left(\frac{1}{T_1} - \frac{1}{T_2}\right)^{-1} = \left(\frac{1}{3} - \frac{1}{4}\right)^{-1} = \left(\frac{1}{12}\right)^{-1} = 12$$

4.2.3

Using the methods of this section

$$\left(\frac{1}{T_1} - \frac{1}{T_2}\right)^{-1} = \left(\frac{1}{1} - \frac{1}{12}\right)^{-1} = \left(\frac{11}{12}\right)^{-1} = \frac{12}{11} \text{ hours}$$

So the hour and minute hands will be aligned again 1 hour, 5 minutes, and 27 seconds, to the nearest second, later.

We also know that the hour and minute hands will be aligned again in 12 hours. During this time, the minute hand will pass the hour hand 11 times. The time between passes is equal, so the minute hand will lap the hour hand every $\frac{1}{11}$ of the perimeter, which is 5 minutes and 27 seconds, to the nearest second.

4.3 Nonuniform Oscillator

4.3.1

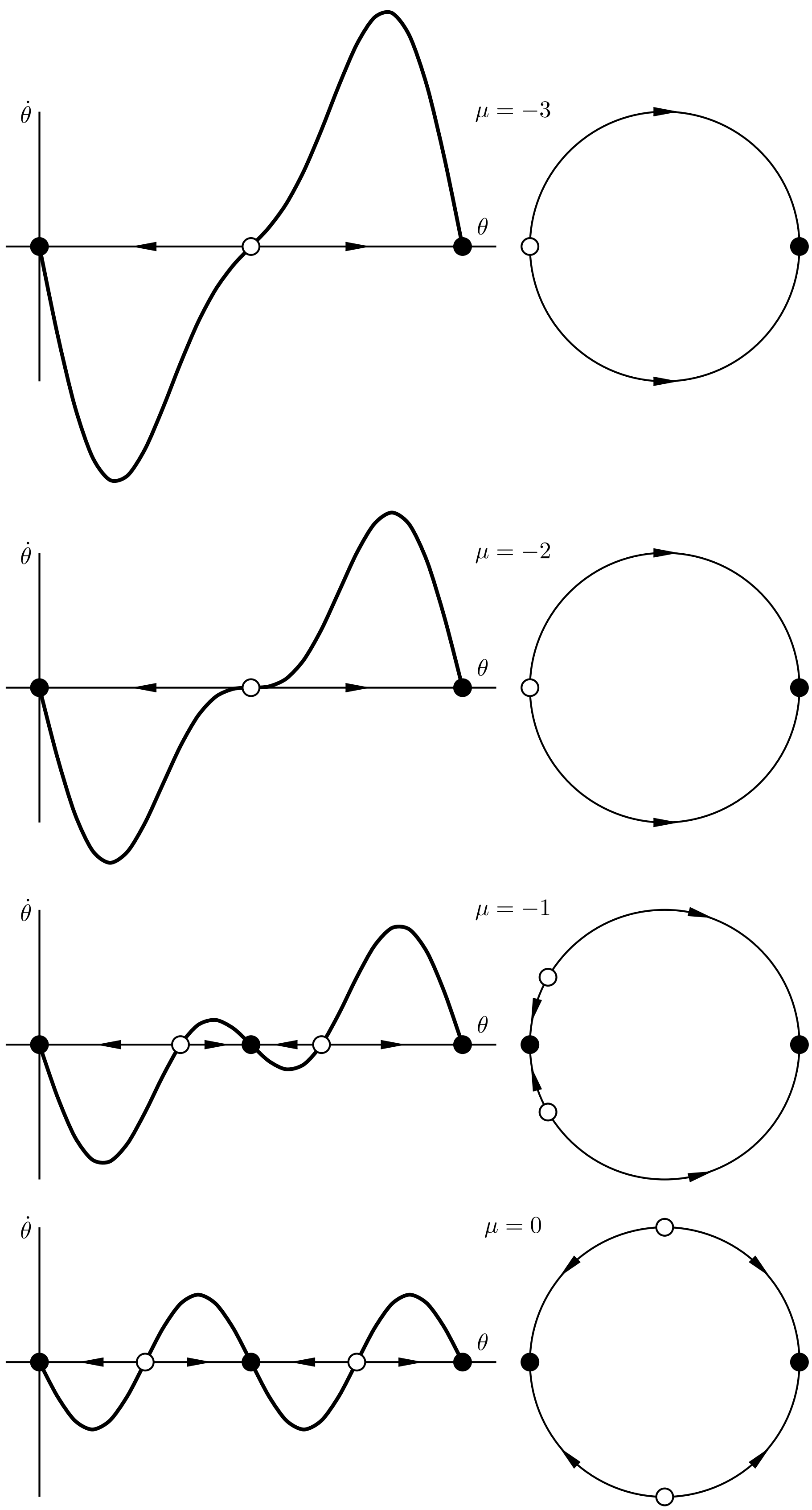
$$\begin{aligned}
T_{\text{bottleneck}} &= \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{r + x^2} \qquad x = \sqrt{r}\tan(\theta) \quad \mathrm{d}x = \sqrt{r}\sec^2(\theta)\mathrm{d}\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{r}\sec^2(\theta)}{r + r\tan^2(\theta)}\mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{r}\sec^2(\theta)}{r(1+\tan^2(\theta))}\mathrm{d}\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sec^2(\theta)}{\sqrt{r}\sec^2(\theta)}\mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\sqrt{r}}\mathrm{d}\theta \\
&= \frac{\pi}{\sqrt{r}}
\end{aligned}$$

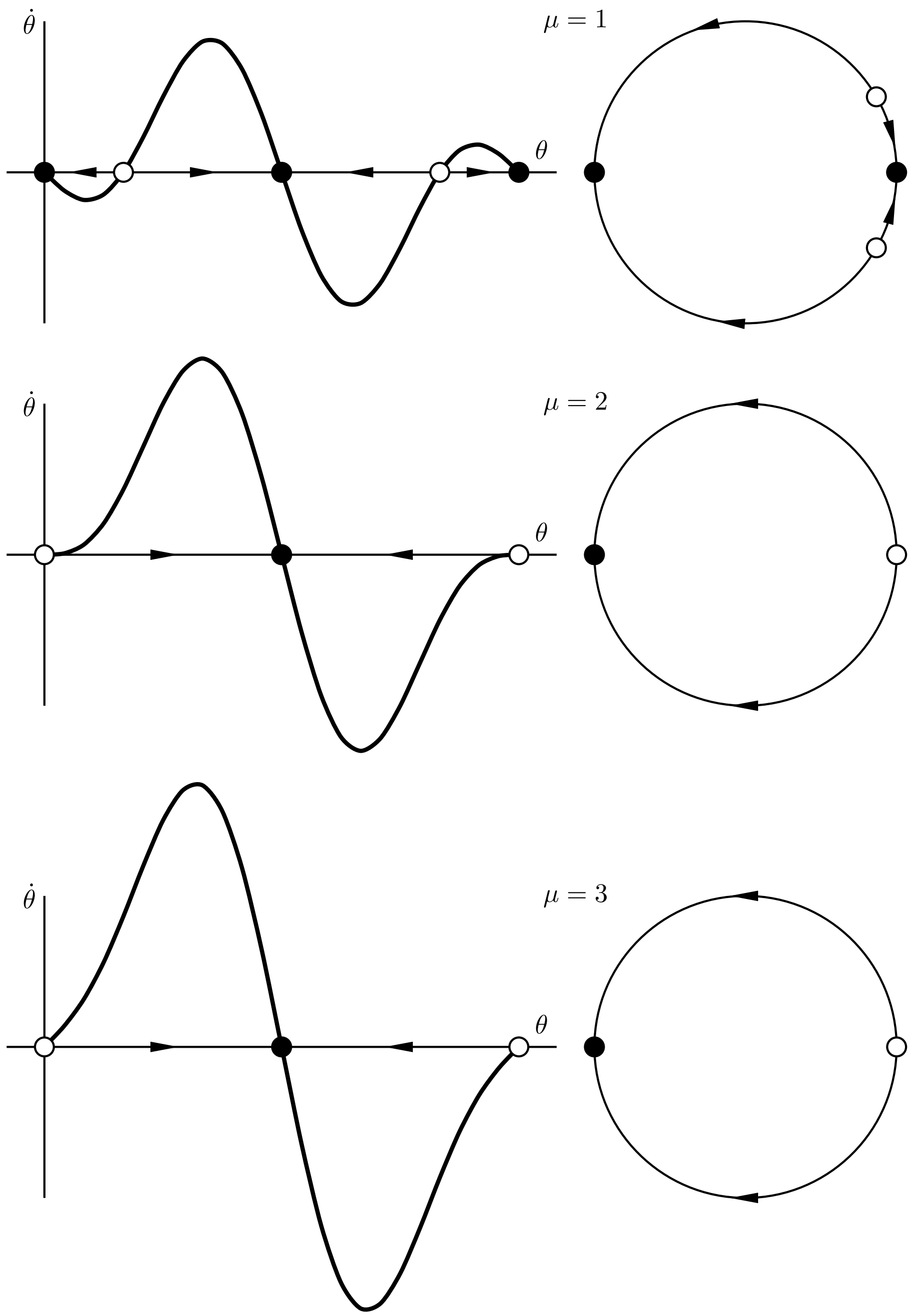
4.3.3

$$\dot{\theta} = \mu\sin(\theta) - \sin(2\theta)$$

$\mu = -2$ subcritical pitchfork bifurcation at $\theta = \pi$

$\mu = 2$ subcritical pitchfork bifurcation at $\theta = 0$


$\dot{\theta}$
$\mu = -3$
θ
$\dot{\theta}$
$\mu = -2$
θ
$\dot{\theta}$
$\mu = -1$
θ
$\dot{\theta}$
$\mu = 0$
θ



4.3.5

$$\dot{\theta} = \mu + \cos(\theta) + \cos(2\theta)$$

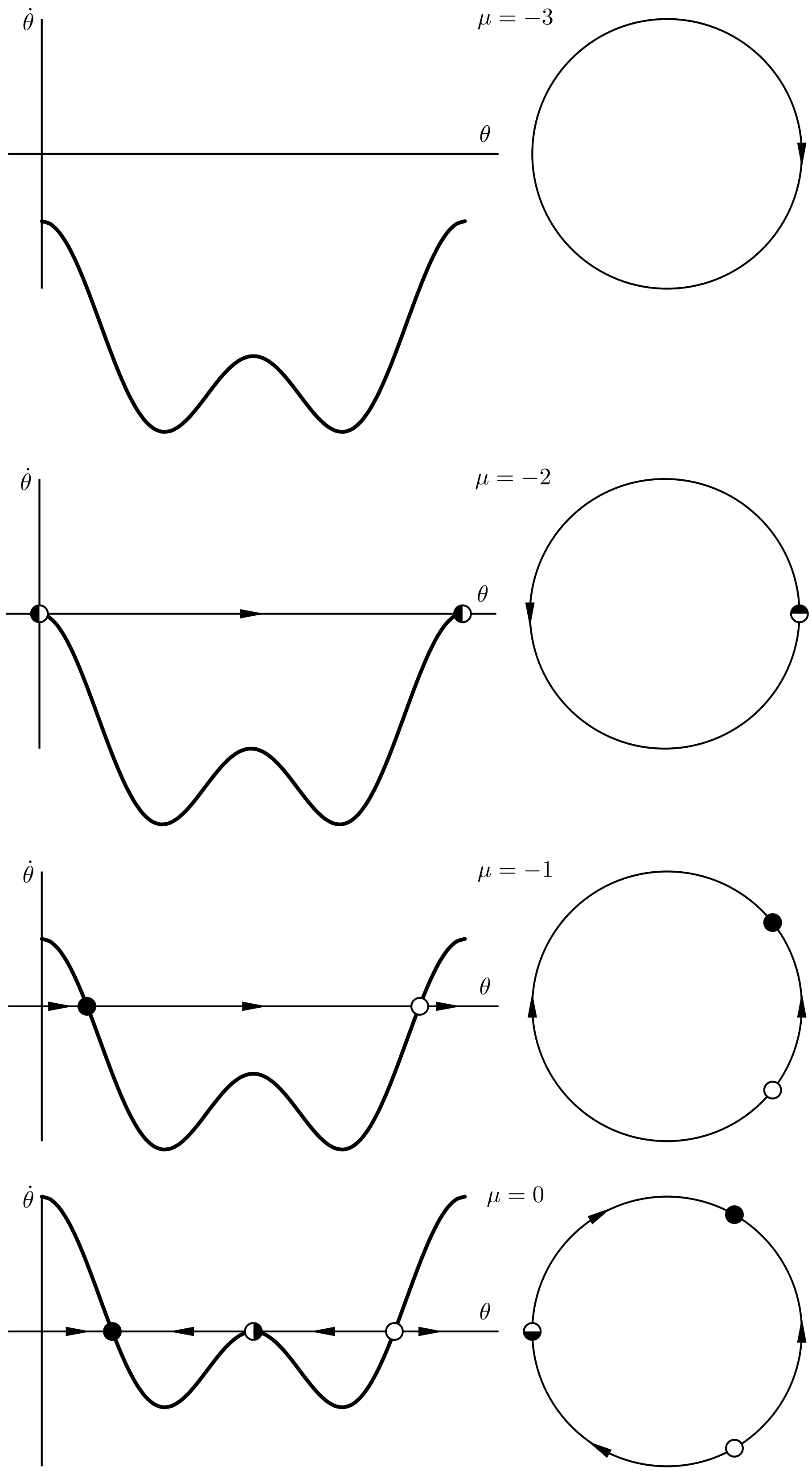
$\mu = -2$ saddle-node bifurcation at $\theta = 0$

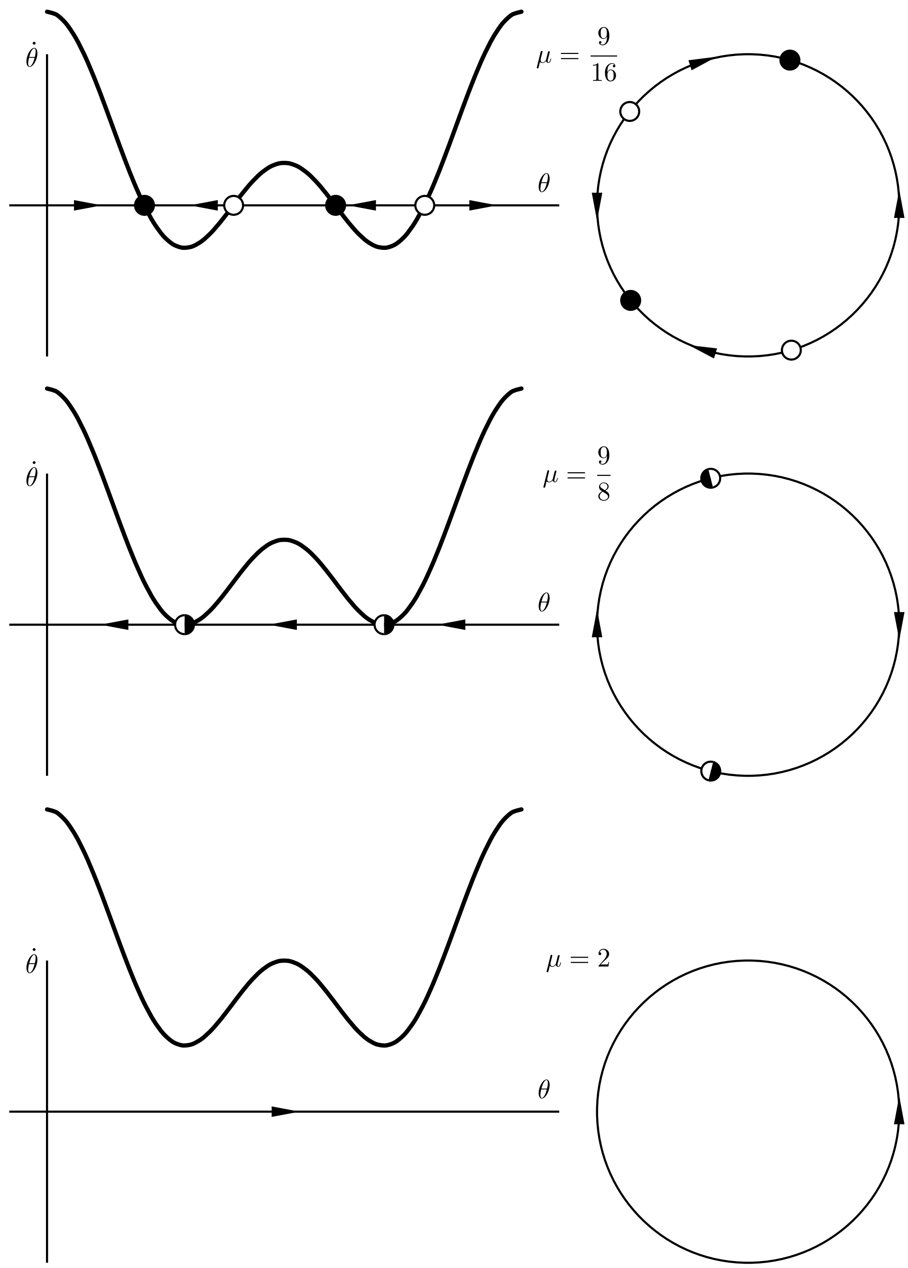
$\mu = 0$ saddle-node bifurcation at $\theta = \pi$

$\mu = \frac{9}{8}$ saddle-node bifurcation at

$\theta = 2\arctan\left(\sqrt{\frac{5}{3}}\right)$

$\theta = 2\pi - 2\arctan\left(\sqrt{\frac{5}{3}}\right)$


$\dot{\theta}$
$\mu = -3$
θ
$\dot{\theta}$
$\mu = -2$
θ
$\dot{\theta}$
$\mu = -1$
θ
$\dot{\theta}$
$\mu = 0$
θ



4.3.7

$$\dot\theta = \frac{\sin(\theta)}{\mu + \sin(\theta)}$$

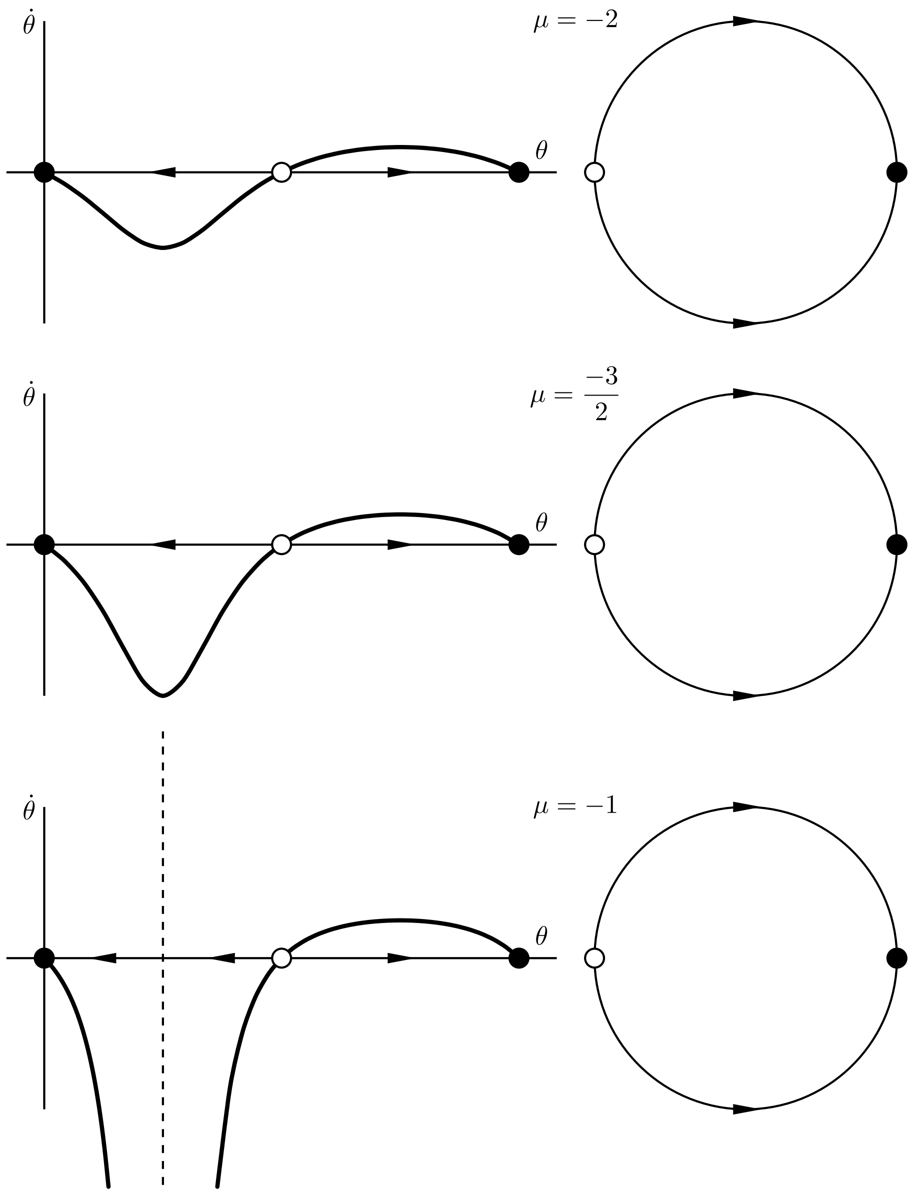
$\mu = -1$ saddle-node bifurcation at $\theta = \frac{\pi}{2}$

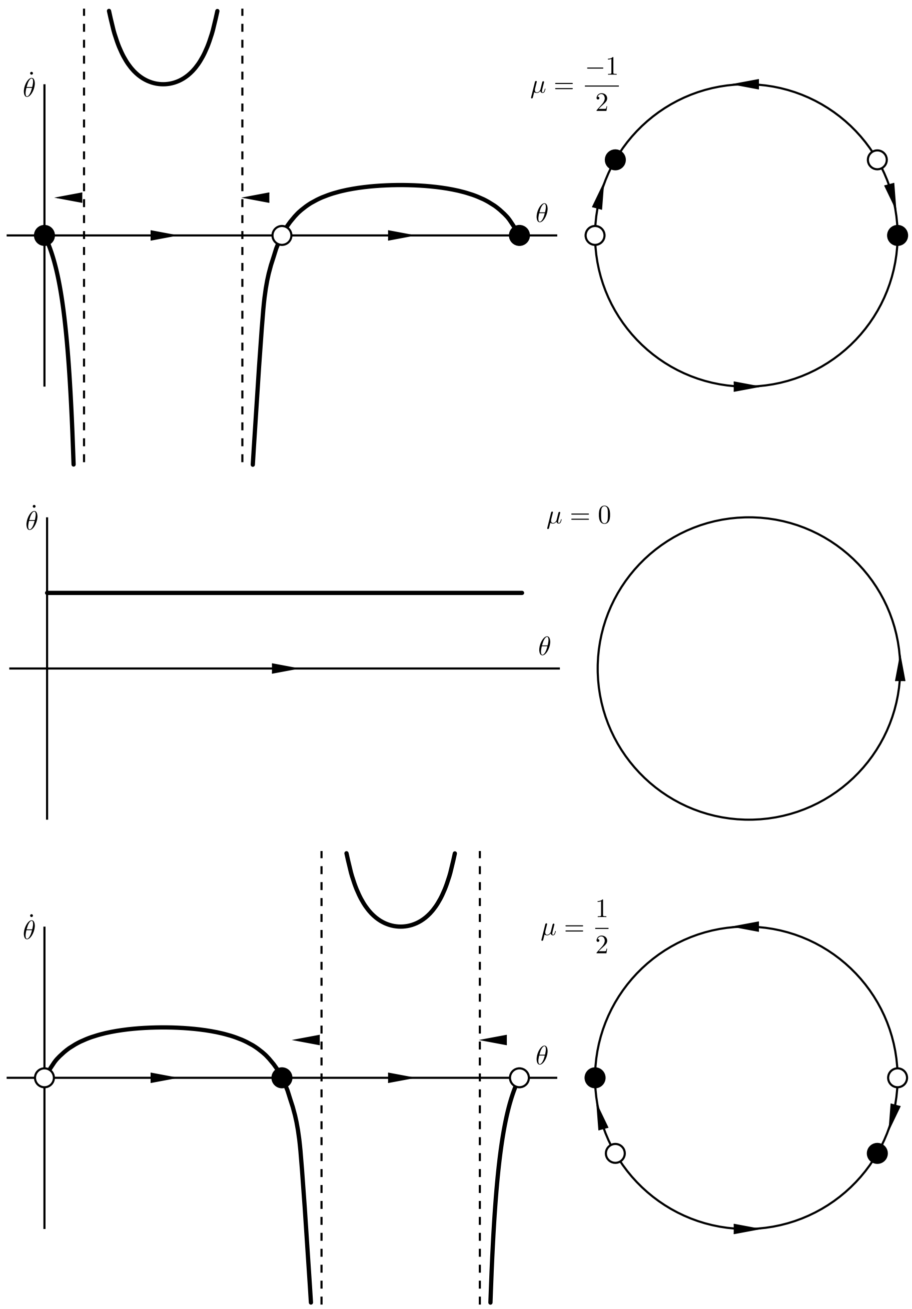
$\mu = 0$ weird

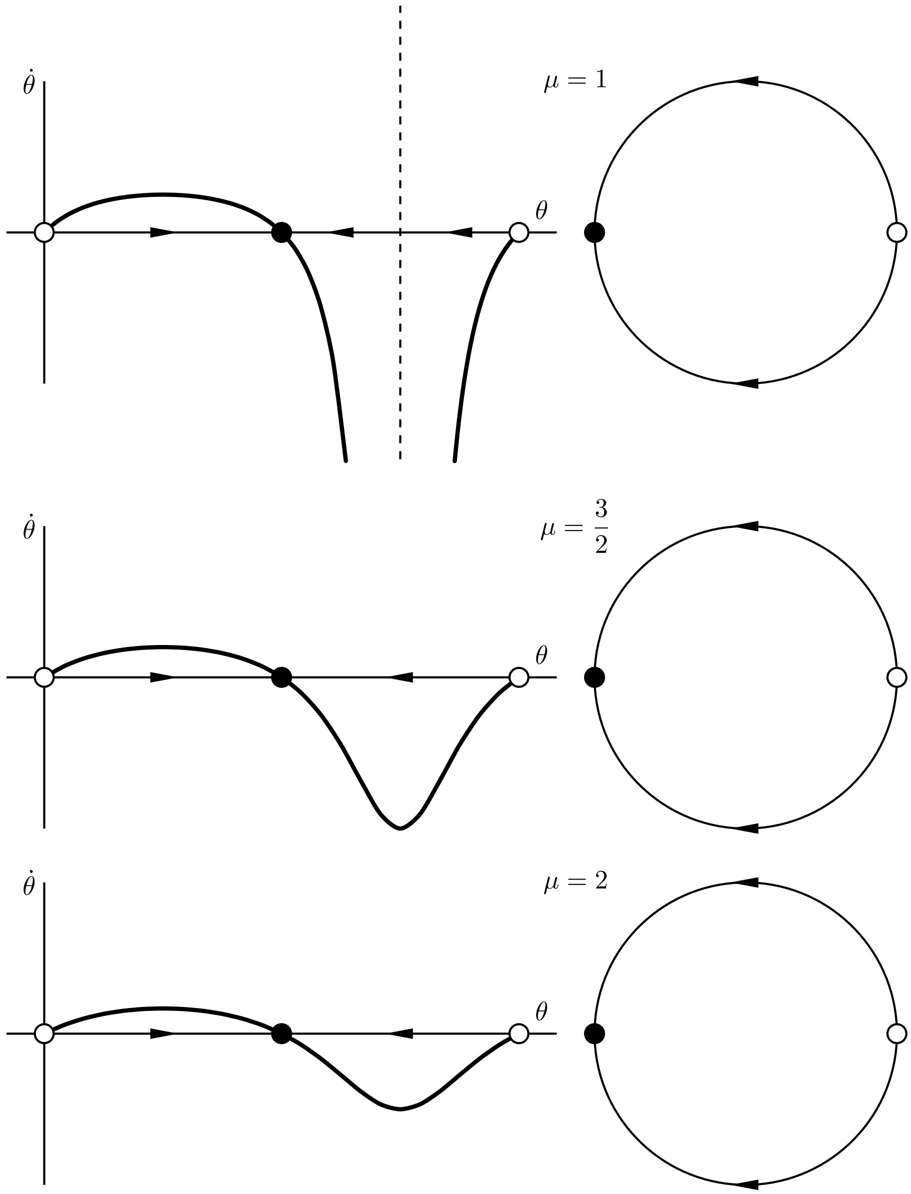
The function hugs closer and closer to the $\dot\theta = 1$ line and the two vertical asymptotes, forming right angles there while simultaneously disappearing.

$\mu = 1$ saddle-node bifurcation at $\theta = \frac{3\pi}{2}$

Note: Some of the fixed points on the circles are not actually fixed points but correspond to infinite discontinuities. However, the discontinuities act like stable or unstable fixed points, since the point of discontinuity is either attracting or repelling respectively.




$\dot{\theta}$
$\mu = \frac{-1}{2}$
θ
$\dot{\theta}$
$\mu = 0$
θ
$\dot{\theta}$
$\mu = \frac{1}{2}$
θ



4.3.9

a)

$$\dot{x} = r + x^2 \qquad x = r^a u \quad t = r^b \tau$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{r^a}{r^b}\frac{\mathrm{d}u}{\mathrm{d}\tau} = r^{a-b}\frac{\mathrm{d}u}{\mathrm{d}\tau} = r + x^2 = r + (r^a u)^2 = r + r^{2a}u^2$$

b)

$$r^{2a}u^2 = ru^2 \Rightarrow a = \frac{1}{2}$$
$$r^{a-b}\frac{\mathrm{d}u}{\mathrm{d}\tau} = r\frac{\mathrm{d}u}{\mathrm{d}\tau} \Rightarrow a - b = \frac{1}{2} - b = 1 \Rightarrow b = \frac{-1}{2}$$

4.4 Overdamped Pendulum

4.4.1

$$
\begin{aligned}
&mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin(\theta) = \Gamma \\
&mL^2\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + b\frac{\mathrm{d}\theta}{\mathrm{d}t} + mgL\sin(\theta) = \Gamma \\
&t = T\tau \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{T}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} \qquad \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \frac{1}{T^2}\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} \\
&\frac{mL^2}{T^2}\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} + \frac{b}{T}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + mgL\sin(\theta) = \Gamma \\
&\frac{L}{gT^2}\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} + \frac{b}{mgLT}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + \sin(\theta) = \frac{\Gamma}{mgL}
\end{aligned}
$$

We want the $\frac{\mathrm{d}\theta}{\mathrm{d}\tau}$ to be $O(1)$ so that all the terms except the $\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2}$ term are of the same order.

$$
\begin{aligned}
&\frac{b}{mgLT} = 1 \Rightarrow T = \frac{b}{mgL} \\
&\frac{m^2gL^3}{b^2}\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} + \frac{\mathrm{d}\theta}{\mathrm{d}\tau} + \sin(\theta) = \frac{\Gamma}{mgL}
\end{aligned}
$$

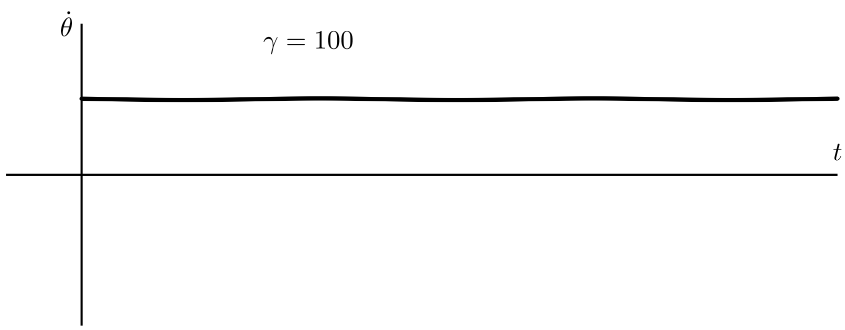
Then we want the $\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2}$ coefficient to be very small so the system relaxes quickly and we can neglect the $\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2}$ term.

$$\frac{m^2gL^3}{b^2} \ll 1 \Rightarrow m^2gL^3 \ll b^2$$

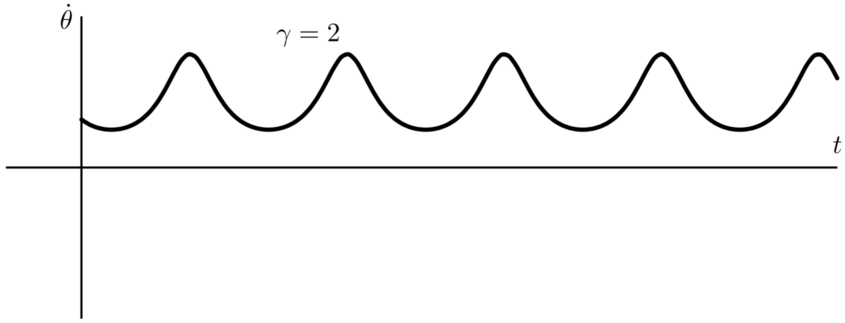
So as long as the above condition is satisfied, the second-order differential equation can be well approximated by the first-order differential equation after an initial transient.

4.4.3

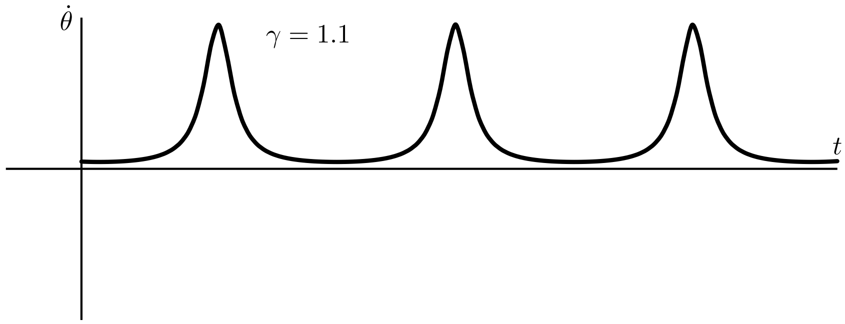
The length of the simulation increases in each graph in order to compare the oscillations.



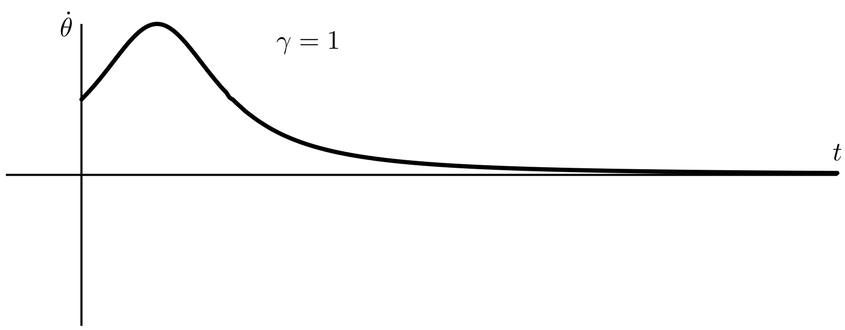
The oscillations are almost indistinguishable due to the vertical scaling for $1 \ll \gamma$ because the derivative is approximately 100 all the time.

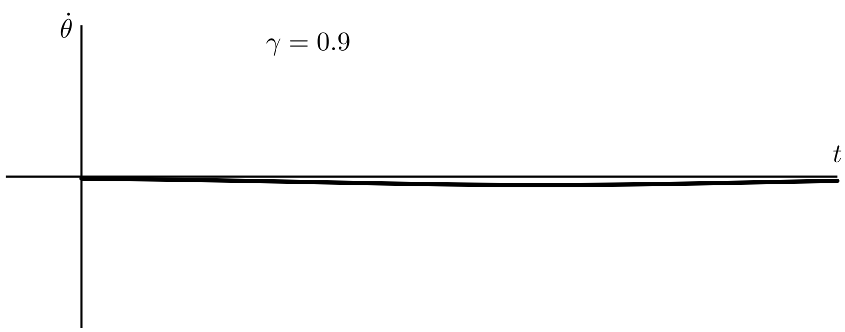


The oscillations look irregular as $\gamma \to 1$ because the pendulum is slowing down significantly when going through the horizontal portion of the swing.

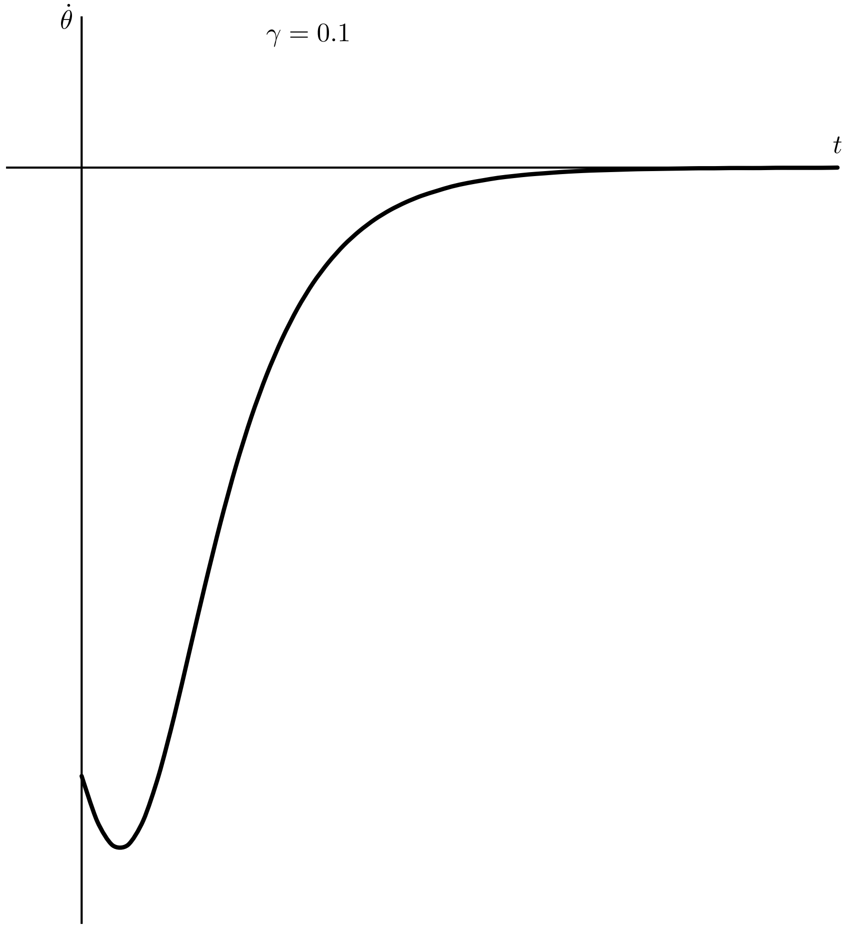


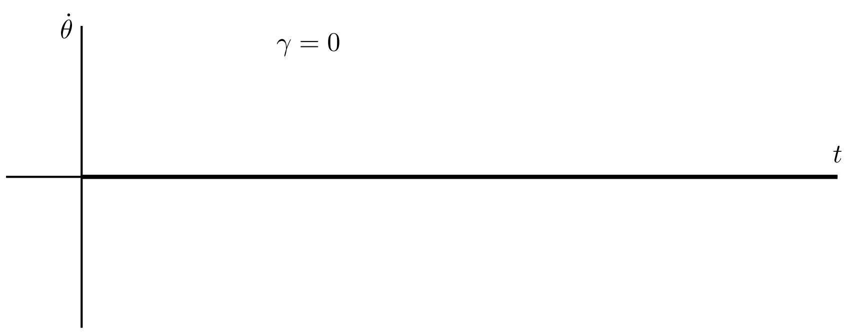
These oscillations continue to sharpen and lengthen as $\gamma \to 1$, and the period will become infinity if $\gamma = 1$ because the pendulum never passes through the horizontal portion of the swing.





Now the torque isn't strong enough to continue the oscillations, so the pendulum comes to rest where the torque balances the force of gravity.





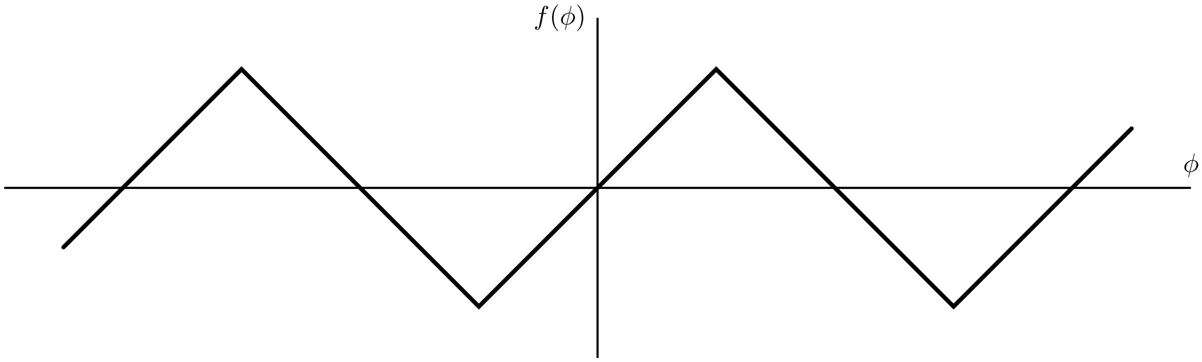
Now there is no torque and the pendulum settles to the bottom of the swing.

4.5 Fireflies

4.5.1

$$\dot{\phi} = \Omega - \omega - Af(\phi) \qquad f(\phi) = \begin{cases} \phi & \frac{-\pi}{2} \le \phi \le \frac{\pi}{2} \\ \pi - \phi & \frac{\pi}{2} \le \phi \le \frac{3\pi}{2} \end{cases}$$

a)



b)

Entrainment occurs when there are still fixed points of the $\dot{\phi}$ equation. The maximum and minimum of $f(\phi)$ are $\pm\frac{\pi}{2}$, so

$$\begin{aligned}
&\dot{\phi} = 0 = \Omega - \omega - Af(\phi) \\
&\Omega - \omega = Af(\phi) \\
&\frac{-\pi}{2}A \leq \frac{\Omega - \omega}{A} \leq \frac{\pi}{2}A \\
&|\Omega - \omega| \leq \frac{\pi}{2}A
\end{aligned}$$

c)

The phase difference is simply the ϕ value of the stable fixed point.

$$\begin{aligned}
&\dot{\phi} = 0 = \Omega - \omega - Af(\phi^*) = \Omega - \omega - A(\pi - \phi^*) \\
&\phi^* = \frac{\omega - \Omega}{A} + \pi \qquad |\Omega - \omega| \leq \frac{\pi}{2}A
\end{aligned}$$

d)

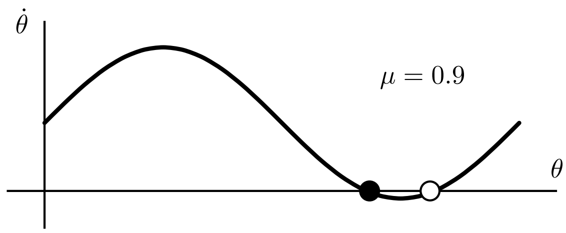
$$\begin{aligned}
T_{\text{drift}} &= \int_0^{2\pi} \frac{\mathrm{d}\phi}{\dot{\phi}} = \int_0^{2\pi} \frac{\mathrm{d}\phi}{\Omega - \omega - Af(\phi)} = \int_{\frac{-\pi}{2}}^{\frac{3\pi}{2}} \frac{\mathrm{d}\phi}{\Omega - \omega - Af(\phi)} \\
&= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\Omega - \omega - Af(\phi)} + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\mathrm{d}\phi}{\Omega - \omega - Af(\phi)} \\
&= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\Omega - \omega - A\phi} + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\mathrm{d}\phi}{\Omega - \omega - A(\pi - \phi)} \\
&= \left[\frac{-1}{A}\ln\left(\Omega - \omega - A\phi\right)\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}} + \left[\frac{1}{A}\ln\left(\Omega - \omega - A(\pi - \phi)\right)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}
\end{aligned}$$

$$= \left[\frac{-1}{A}\ln\left(\Omega-\omega-A\phi\right)\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}} + \left[\frac{1}{A}\ln\left(\Omega-\omega-A\phi\right)\right]_{\frac{\pi}{2}}^{\frac{-\pi}{2}}$$
$$= \left[\frac{1}{A}\ln\left(\Omega-\omega-A\phi\right)\right]_{\frac{\pi}{2}}^{\frac{-\pi}{2}} + \left[\frac{1}{A}\ln\left(\Omega-\omega-A\phi\right)\right]_{\frac{\pi}{2}}^{\frac{-\pi}{2}}$$
$$= \left[\frac{2}{A}\ln\left(\Omega-\omega-A\phi\right)\right]_{\frac{\pi}{2}}^{\frac{-\pi}{2}}$$
$$= \frac{2}{A}\left(\ln\left(\Omega-\omega-A\frac{\pi}{2}\right)-\ln\left(\Omega-\omega-A\frac{-\pi}{2}\right)\right)$$
$$= \frac{2}{A}\left(\ln\left(\Omega-\omega-A\frac{\pi}{2}\right)-\ln\left(\Omega-\omega+A\frac{\pi}{2}\right)\right)$$

4.5.3

$$\dot{\theta} = \mu + \sin(\theta)$$

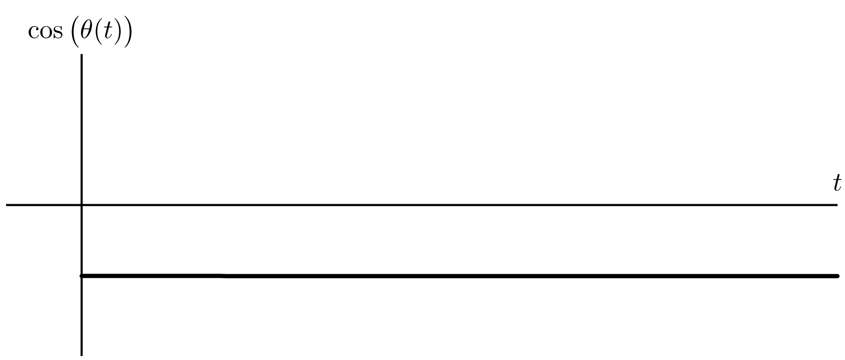
a)

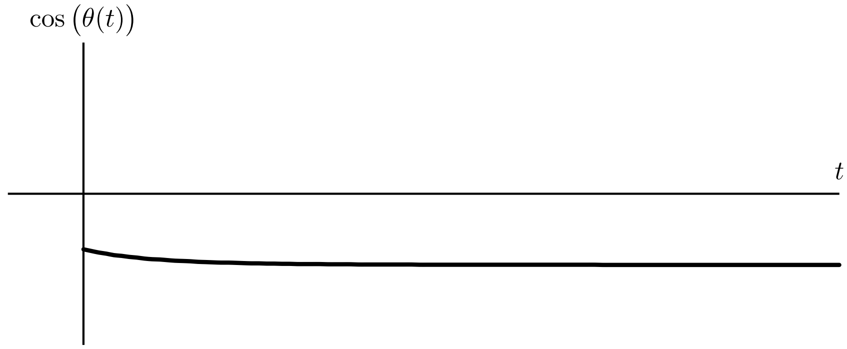


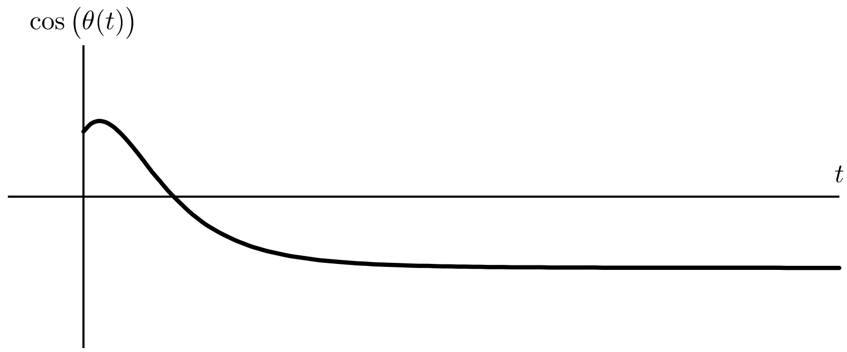
The system is periodic, so the unique globally attracting "rest state" and "threshold" are the stable and unstable fixed points respectively.

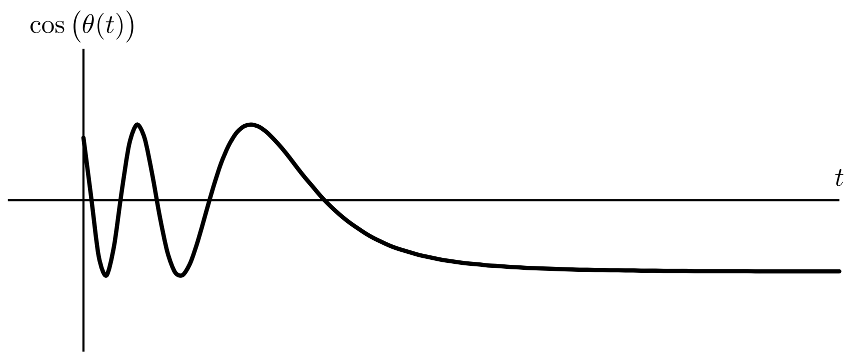
Also, a large enough stimulus will shift the system past the unstable-fixed-point "threshold" and will only return to the stable-fixed-point "rest state" after going through most of the 2π-length periodicity.

b)







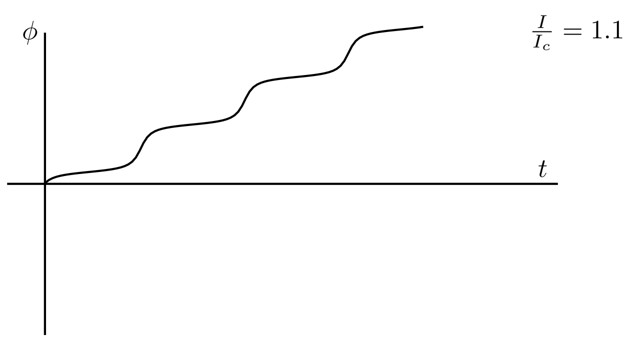


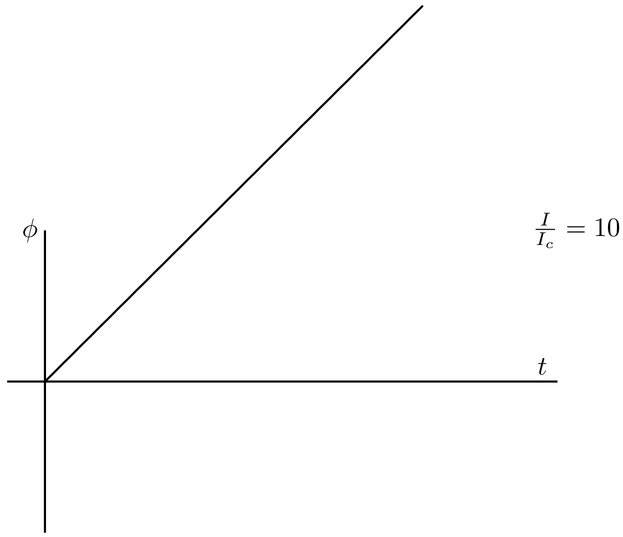
4.6 Superconducting Josephson Junctions

4.6.1

a)

$$\dot{\phi} = \frac{I}{I_c} - \sin(\phi)$$

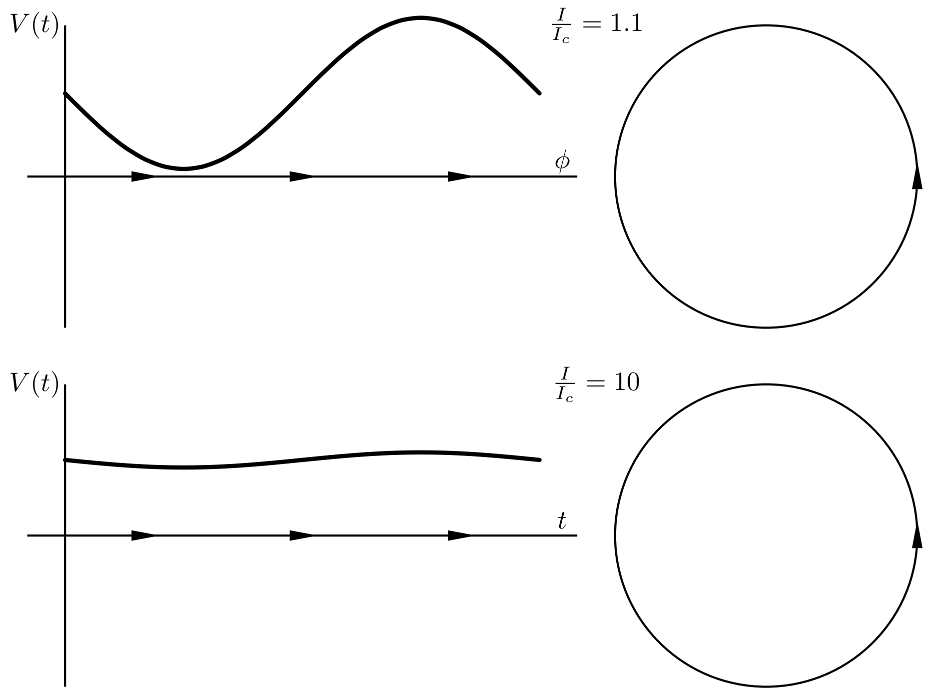




The vertical axis in the second case has been scaled for a reasonable size.

b)

$$V(t) = \frac{\hbar}{2\mathrm{e}}\dot{\phi} = \frac{\hbar}{2\mathrm{e}}\left(\frac{I}{I_c} - \sin(\phi)\right)$$



The vertical axis in the second case has been scaled for a reasonable size.

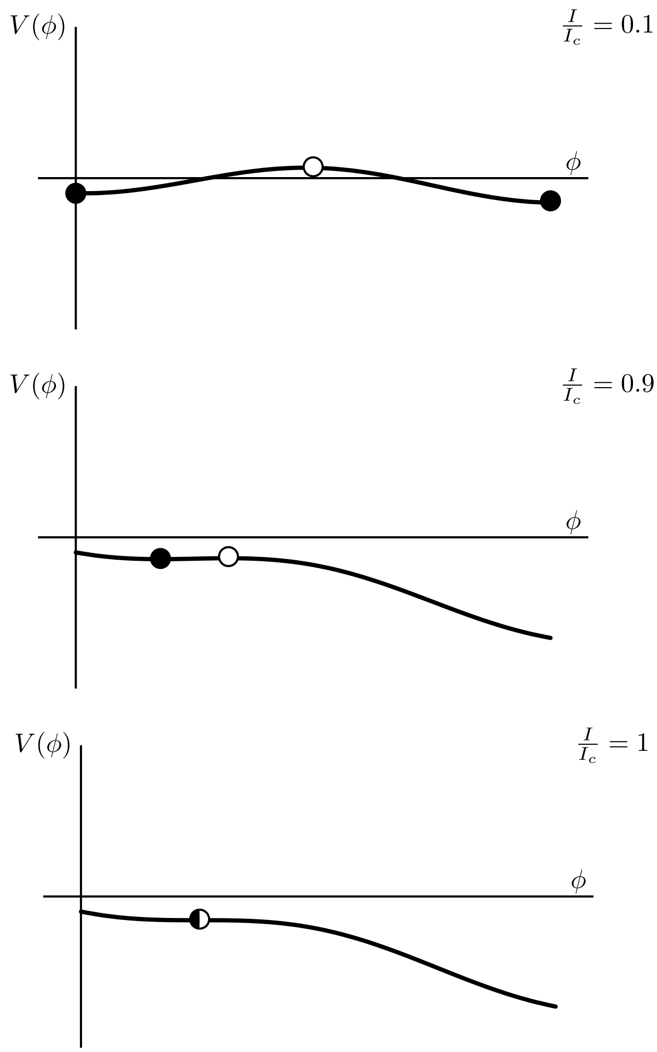
4.6.3

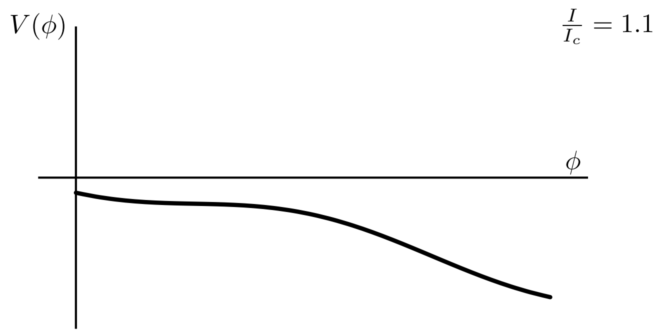
a)

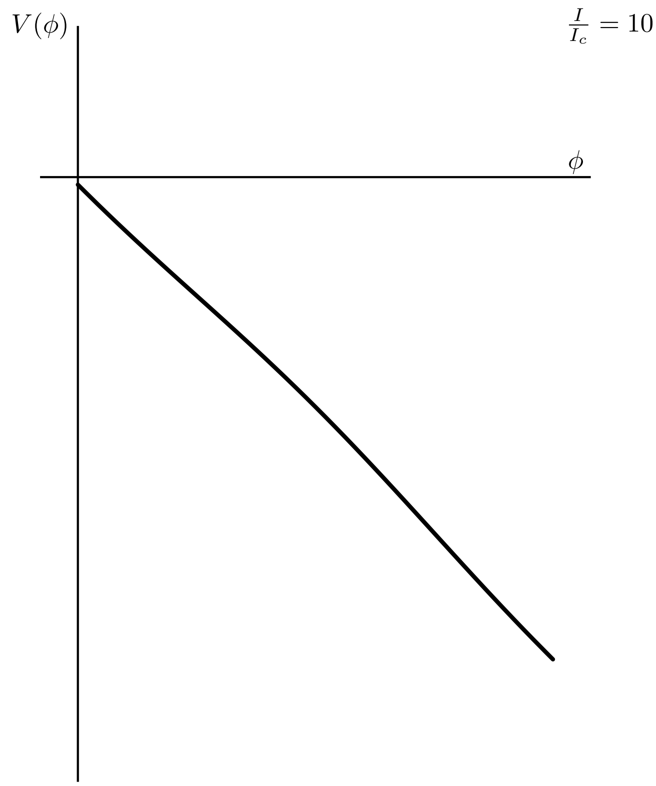
$$
\begin{aligned}
&\dot{\phi} = \frac{I}{I_c} - \sin(\phi) \\
&-\frac{\mathrm{d}V}{\mathrm{d}\phi} = \frac{I}{I_c} - \sin(\phi) \Rightarrow V(\phi) = \frac{-I}{I_c}\phi - \cos(\phi) \\
&V(\phi) \neq V(\phi + 2\pi)
\end{aligned}
$$

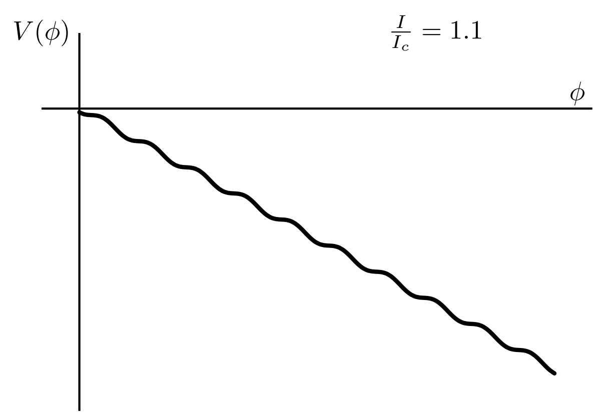
So $V(\phi)$ is *not* a single-valued function on the circle.

b)









c)

Increasing I brings the potential well and hill closer together, which eventually collide and annihilate.

4.6.5

a)

$$I_b = I_a + I_R$$

b)

The current running through the middle is I_a, which has to be the same current running through all sections because they are in series. The current in a section is split between the current through the Josephson junction $I_c \sin(\phi_i)$ and the current running through the resistor because they are in parallel. This gives

$$I_a = I_c \sin(\phi_k) + \frac{V_k}{r}$$

c)

$$\begin{aligned}
I_a &= I_c \sin(\phi_k) + \frac{V_k}{r} \\
V_k &= rI_a - rI_c \sin(\phi_k) \\
V &= \frac{\hbar}{2\mathrm{e}}\dot{\phi} \qquad \dot{\phi} = \frac{I}{I_c} - \sin(\phi) \\
V_k &= rI_c\left(\frac{I_a}{I_c} - \sin(\phi_k)\right) = \frac{\hbar}{2\mathrm{e}}\dot{\phi}_k
\end{aligned}$$

d)

Using Kirchhoff's voltage law, the voltage from the top to the bottom of the circuit is the same going through the Josephson junctions and the resistor.

$$\begin{aligned}
&\sum_{i=1}^{N} V_i = V_R \\
&\frac{\hbar}{2\mathrm{e}}\sum_{i=1}^{N} \dot{\phi}_i = I_R R = (I_b - I_a)\, R \\
&I_b = I_a + \frac{\hbar}{2\mathrm{e}R}\sum_{i=1}^{N} \dot{\phi}_i \\
&\quad = I_c \sin(\phi_k) + \frac{V_k}{r} + \frac{\hbar}{2\mathrm{e}R}\sum_{i=1}^{N} \dot{\phi}_i \\
&\quad = I_c \sin(\phi_k) + \frac{\hbar}{2\mathrm{e}r}\dot{\phi}_k + \frac{\hbar}{2\mathrm{e}R}\sum_{i=1}^{N} \dot{\phi}_i
\end{aligned}$$

e)

$$
\begin{aligned}
I_b &= I_c \sin(\phi_k) + \frac{\hbar}{2\mathrm{e}r}\dot{\phi}_k + \frac{\hbar}{2\mathrm{e}R}\sum_{i=1}^{N}\dot{\phi}_i \\
NI_b &= I_c \sum_{i=1}^{N}\sin(\phi_i) + \frac{\hbar}{2\mathrm{e}r}\sum_{i=1}^{N}\dot{\phi}_i + \frac{N\hbar}{2\mathrm{e}R}\sum_{i=1}^{N}\dot{\phi}_i \\
I_b &= \frac{I_c}{N}\sum_{i=1}^{N}\sin(\phi_i) + \frac{\hbar}{2N\mathrm{e}r}\sum_{i=1}^{N}\dot{\phi}_i + \frac{\hbar}{2\mathrm{e}R}\sum_{i=1}^{N}\dot{\phi}_i \\
&= I_c \sin(\phi_k) + \frac{\hbar}{2\mathrm{e}r}\dot{\phi}_k + \frac{\hbar}{2\mathrm{e}R}\sum_{i=1}^{N}\dot{\phi}_i
\end{aligned}
$$

$$
\begin{aligned}
\frac{I_c}{N}\sum_{i=1}^{N}\sin(\phi_i) + \frac{\hbar}{2N\mathrm{e}r}\sum_{i=1}^{N}\dot{\phi}_i &= I_c\sin(\phi_k) + \frac{\hbar}{2\mathrm{e}r}\dot{\phi}_k \\
\frac{\hbar}{2N\mathrm{e}r}\sum_{i=1}^{N}\dot{\phi}_i &= I_c\sin(\phi_k) + \frac{\hbar}{2\mathrm{e}r}\dot{\phi}_k - \frac{I_c}{N}\sum_{i=1}^{N}\sin(\phi_i) \\
\frac{\hbar}{2\mathrm{e}R}\sum_{i=1}^{N}\dot{\phi}_i &= \frac{Nr}{R}I_c\sin(\phi_k) + \frac{N\hbar}{2\mathrm{e}R}\dot{\phi}_k - \frac{r}{R}I_c\sum_{i=1}^{N}\sin(\phi_i)
\end{aligned}
$$

$$
\begin{aligned}
I_b &= I_c \sin(\phi_k) + \frac{\hbar}{2\mathrm{e}r}\dot{\phi}_k + \frac{\hbar}{2\mathrm{e}R}\sum_{i=1}^{N}\dot{\phi}_i \\
&= I_c \sin(\phi_k) + \frac{\hbar}{2\mathrm{e}r}\dot{\phi}_k + \frac{Nr}{R}I_c\sin(\phi_k) + \frac{N\hbar}{2\mathrm{e}R}\dot{\phi}_k - \frac{r}{R}I_c\sum_{i=1}^{N}\sin(\phi_i) \\
&= \left(1+\frac{Nr}{R}\right)I_c\sin(\phi_k) + \hbar\left(\frac{1}{2\mathrm{e}r}+\frac{N}{2\mathrm{e}R}\right)\dot{\phi}_k - \frac{r}{R}I_c\sum_{j=1}^{2}\sin(\phi_j) \\
\hbar\left(\frac{1}{2\mathrm{e}r}+\frac{N}{2\mathrm{e}R}\right)\dot{\phi}_k &= I_b - \left(1+\frac{Nr}{R}\right)I_c\sin(\phi_k) + \frac{r}{R}I_c\sum_{j=1}^{2}\sin(\phi_j) \\
\frac{\hbar(R+Nr)}{2\mathrm{e}rR}\dot{\phi}_k &= I_b - \frac{R+Nr}{R}I_c\sin(\phi_k) + \frac{r}{R}I_c\sum_{j=1}^{2}\sin(\phi_j) \\
\frac{\hbar(R+Nr)}{2N\mathrm{e}r^2 I_c}\dot{\phi}_k &= \frac{RI_b}{NrI_c} - \frac{R+Nr}{Nr}\sin(\phi_k) + \frac{1}{N}\sum_{j=1}^{2}\sin(\phi_j) \\
\frac{\mathrm{d}\phi_k}{\mathrm{d}\tau} &= \Omega + a\sin(\phi_k) + \frac{1}{N}\sum_{j=1}^{2}\sin(\phi_j) \\
\Omega &= \frac{RI_b}{NrI_c} \qquad a = \frac{-(R+Nr)}{Nr} \qquad \tau = \frac{2N\mathrm{e}r^2 I_c}{\hbar(R+Nr)}t
\end{aligned}
$$

5

Linear Systems

5.1 Definitions and Examples

5.1.1

Harmonic oscillator

$$\dot{x} = v \qquad \dot{v} = -\omega^2 x$$

a)

Show that the orbits are given by $\omega^2 x^2 + v^2 = C$ where C is any nonnegative constant.

Divide the first equation by the second equation.

$$\frac{\dot{x}}{\dot{v}} = \frac{v}{-\omega^2 x} \Rightarrow -\omega^2 x\dot{x} = v\dot{v} \Rightarrow \omega^2 x\dot{x} + v\dot{v} = 0$$

Integrate

$$\frac{\omega^2}{2}x^2 + \frac{1}{2}v^2 = D \Rightarrow \omega^2 x^2 + v^2 = C$$

b)

Show that this is equivalent to conservation of energy.

The harmonic oscillator with mass m and spring constant k stores a constant amount of energy as the sum of kinetic and potential energy.

Potential energy

$$PE = \int -kx\mathrm{d}x = \frac{-k}{2}x^2 + D$$

Kinetic energy

$$KE = \frac{-m}{2}v^2$$

Total energy

$$E = PE + KE = \frac{-k}{2}x^2 + D + \frac{-m}{2}v^2 \Rightarrow \frac{-2}{m}E + \frac{2D}{m} = \frac{k}{m}x^2 + v^2$$

If we set

$$C = \frac{-2}{m}E + \frac{2D}{m} \qquad \omega^2 = \frac{k}{m}$$

then we get the same equation.

5.1.3

$$\dot{x} = -y \qquad \dot{y} = -x$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

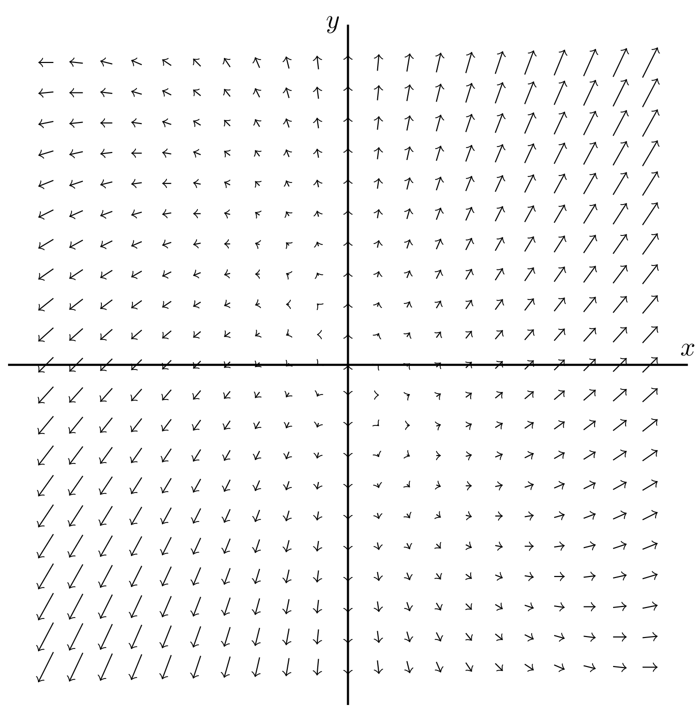
5.1.5

$$\dot{x} = 0 \qquad \dot{y} = x + y$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

5.1.7

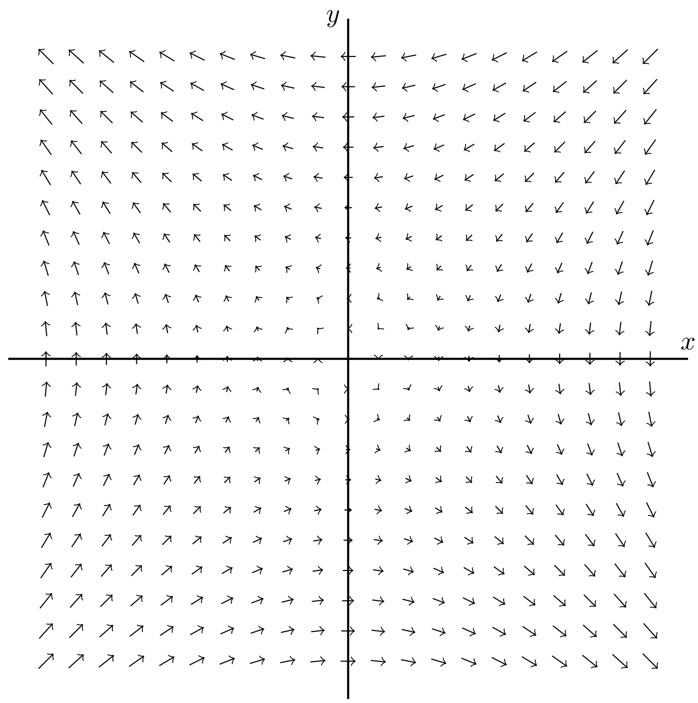
$$\dot{x} = x \qquad \dot{y} = x + y$$



5.1.9

$$\dot{x} = -y \qquad \dot{y} = -x$$

a)



b)

$$x\dot{x} - y\dot{y} = x(-y) - y(-x) = -xy + xy = 0$$
$$\int x\dot{x} - y\dot{y}\ \mathrm{d}t = \int 0\ \mathrm{d}t$$
$$x^2 - y^2 = C$$

c)

From the vector field, it looks like $y = x$ and $y = -x$ are the stable and unstable manifolds respectively. We can check by plugging $y = x$ and $y = -x$ into the system.

$y = x$ gives $\dot{x} = -x$ and $\dot{y} = -y$, which has a stable fixed point at the origin.
$y = -x$ gives $\dot{x} = x$ and $\dot{y} = y$, which has an unstable fixed point at the origin.

d)

$$u = x + y \qquad v = x - y$$

$$\dot{u} = \dot{x} + \dot{y} = -y - x = -u$$
$$\dot{u} = -u \Rightarrow u(t) = Ce^{-t}$$
$$u(0) = u_0 \Rightarrow C = u_0$$
$$u(t) = u_0 e^{-t}$$

$$\dot{v} = \dot{x} - \dot{y} = -y + x = v$$
$$\dot{v} = v \Rightarrow v(t) = Ce^{t}$$
$$v(0) = v_0 \Rightarrow C = v_0$$
$$v(t) = v_0 e^{t}$$

e)

The stable and unstable manifold occurs when $u = 0$ and $v = 0$ respectively.

f)

$$x = \frac{u+v}{2} = \frac{u_0 e^{-t} + v_0 e^{t}}{2} = \frac{(x_0 + y_0)e^{-t} + (x_0 - y_0)e^{t}}{2}$$
$$y = \frac{u-v}{2} = \frac{u_0 e^{-t} - v_0 e^{t}}{2} = \frac{(x_0 + y_0)e^{-t} - (x_0 - y_0)e^{t}}{2}$$

5.1.11

a)

$$\dot{x} = y \qquad \dot{y} = -4x$$
$$\ddot{x} = \dot{y} = -4x$$
$$x(t) = x_0 \cos(2t) + \frac{y_0}{2}\sin(2t)$$
$$y(t) = -2x_0 \sin(2t) + y_0 \cos(2t)$$
$$\delta < |x_0| \Rightarrow \big|\big|\big(x(t), y(t)\big)\big|\big| < 2|x_0| = \epsilon$$

Liapunov stable

b)

$$\begin{aligned}
&\dot{x} = 2y \qquad \dot{y} = x \\
&\ddot{x} = 2\dot{y} = 2x \\
&x = x_0 \cosh\left(\sqrt{2}t\right) + \sqrt{2}y_0 \sinh\left(\sqrt{2}t\right) \\
&y = \frac{x_0}{\sqrt{2}} \sinh\left(\sqrt{2}t\right) + y_0 \cosh\left(\sqrt{2}t\right)
\end{aligned}$$

Trajectories go away from the origin

None of the above

c)

$$\begin{aligned}
&\dot{x} = 0 \qquad \dot{y} = x \\
&x = x_0 \qquad y = x_0 t + y_0
\end{aligned}$$

Trajectories go away from the origin

None of the above

d)

$$\begin{aligned}
&\dot{x} = 0 \qquad \dot{y} = -y \\
&x = x_0 \qquad y = y_0 e^{-t} \\
&\delta = \left|\left|\left(x_0, y_0\right)\right|\right| \geq \left|\left|\left(x(t), y(t)\right)\right|\right| \geq \left|\left|\left(x_0, 0\right)\right|\right| \text{ for } t \geq 0
\end{aligned}$$

Liapunov stable

e)

$$\begin{aligned}
&\dot{x} = -x \qquad \dot{y} = -5y \\
&x = x_0 e^{-t} \qquad y = y_0 e^{-5t} \\
&\delta = \left|\left|\left(x_0, y_0\right)\right|\right| \geq \left|\left|\left(x(t), y(t)\right)\right|\right| \text{ for } t \geq 0
\end{aligned}$$

Asymptotically stable

f)

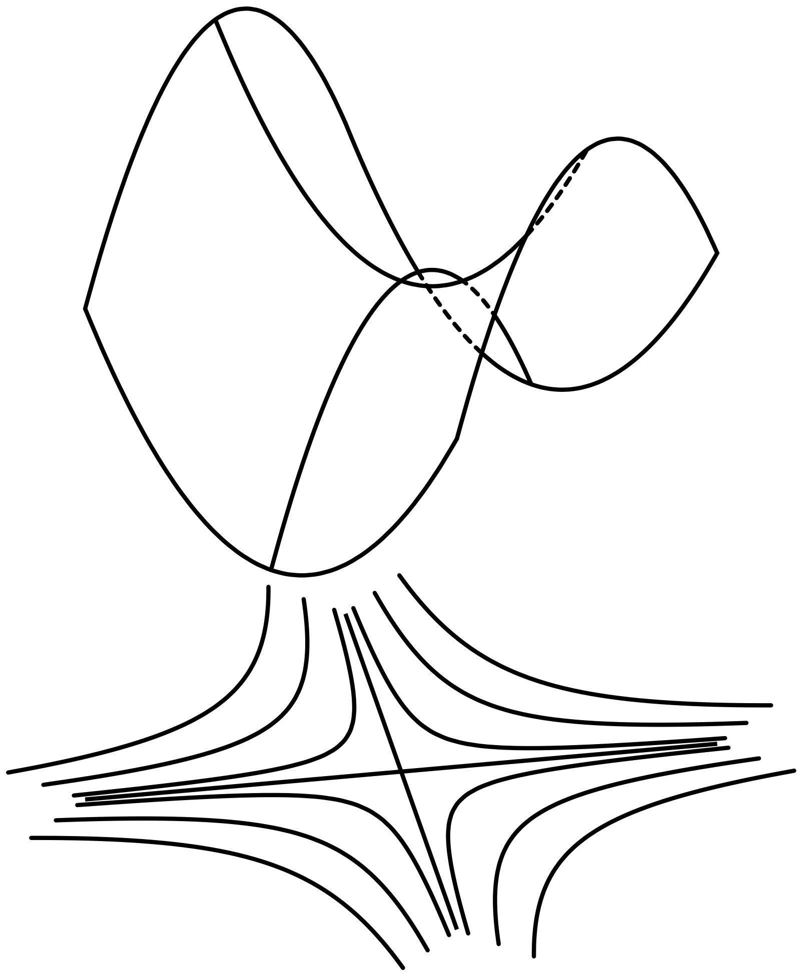
$$\begin{aligned}
&\dot{x} = x \qquad \dot{y} = y \\
&x = x_0 e^{t} \qquad y = y_0 e^{t}
\end{aligned}$$

Trajectories go away from the origin

None of the above

5.1.13

Going back to multivariable calculus, a saddle point on a graph is neither a local minimum nor a local maximum, but the first partial derivatives of x and y both equal zero at the saddle point. The typical example



looks just like a horse saddle, and the level sets of graph make this pattern. The phase plane of a saddle point for an ODE looks the same, so we call these fixed points *saddle points*.

5.2 Classification of Linear Systems

5.2.1

$$\dot{x} = 4x - y \qquad \dot{y} = 2x + y$$

a)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda)+2 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3) = 0$$

$$\lambda_1 = 2 \qquad \lambda_2 = 3$$

$$\begin{pmatrix} 4-\lambda_1 & -1 \\ 2 & 1-\lambda_1 \end{pmatrix} \vec{v}_1 = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \vec{v}_1 = \vec{0} \quad \Rightarrow \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 4-\lambda_2 & -1 \\ 2 & 1-\lambda_2 \end{pmatrix} \vec{v}_2 = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \vec{v}_2 = \vec{0} \quad \Rightarrow \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

b)

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

c)

Unstable node

d)

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad c_1 = 1 \text{ and } c_2 = 2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{3t} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

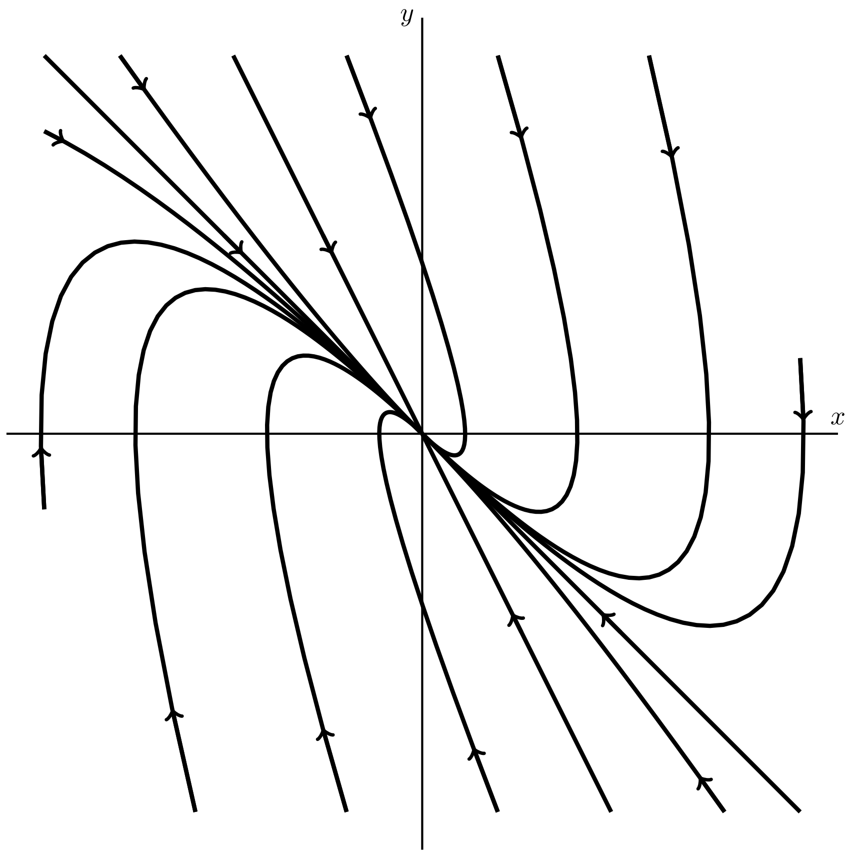
5.2.3

$$\dot{x} = y \qquad \dot{y} = -2x - 3y$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\lambda_1 = -2 \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \lambda_2 = -1 \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Stable node



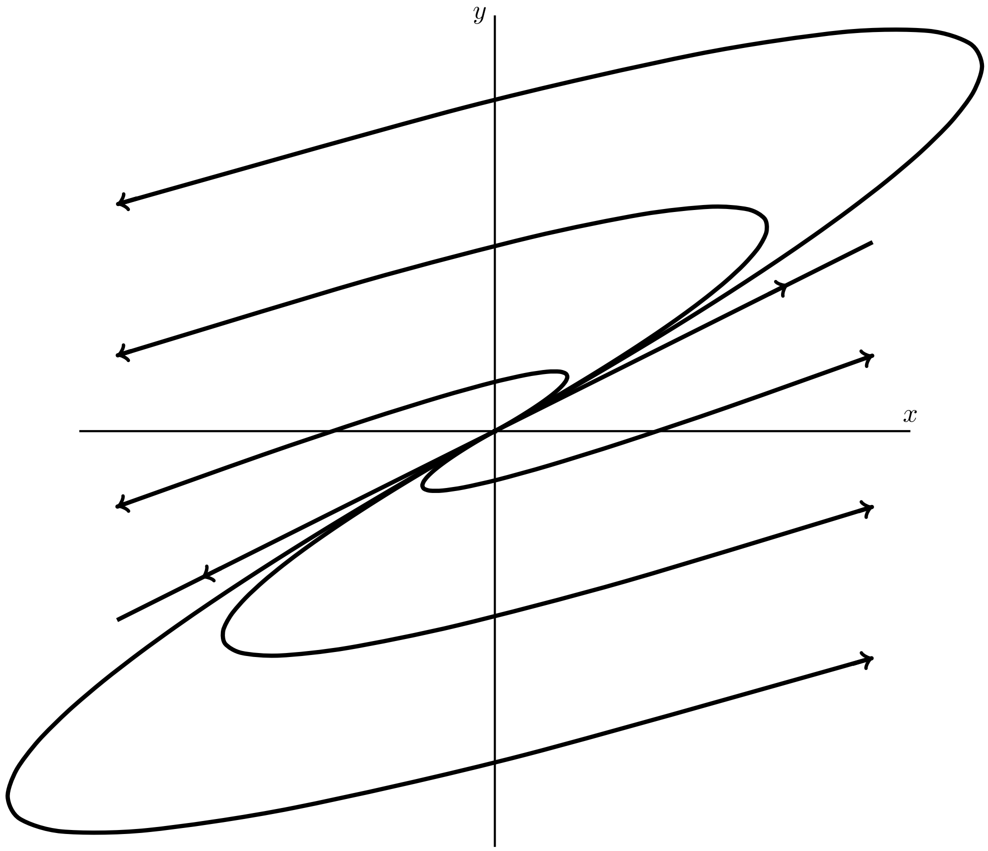
5.2.5

$$\dot{x} = 3x - 4y \qquad \dot{y} = x - y$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\lambda_1 = 1 \quad \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Unstable degenerate node



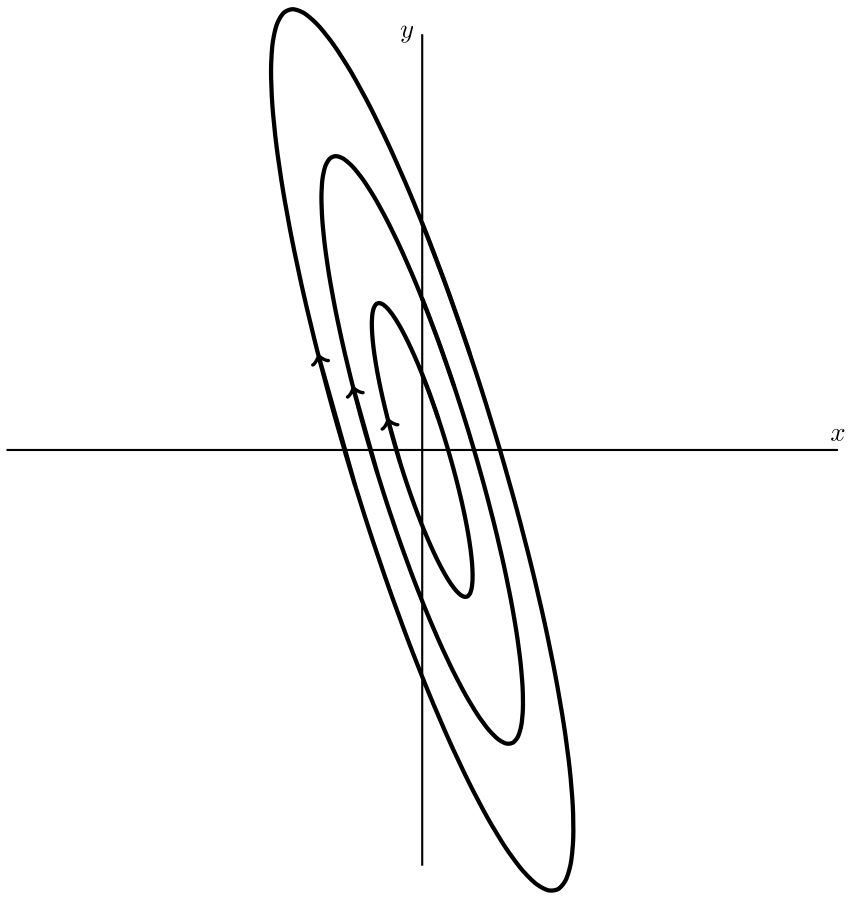
5.2.7

$$\dot{x} = 5x + 2y \qquad \dot{y} = -17x - 5y$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ -17 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\lambda_1 = 3i \quad \vec{v}_1 = \begin{pmatrix} 5+3i \\ -17 \end{pmatrix} \qquad \lambda_2 = -3i \quad \vec{v}_2 = \begin{pmatrix} 5-3i \\ -17 \end{pmatrix}$$

Center



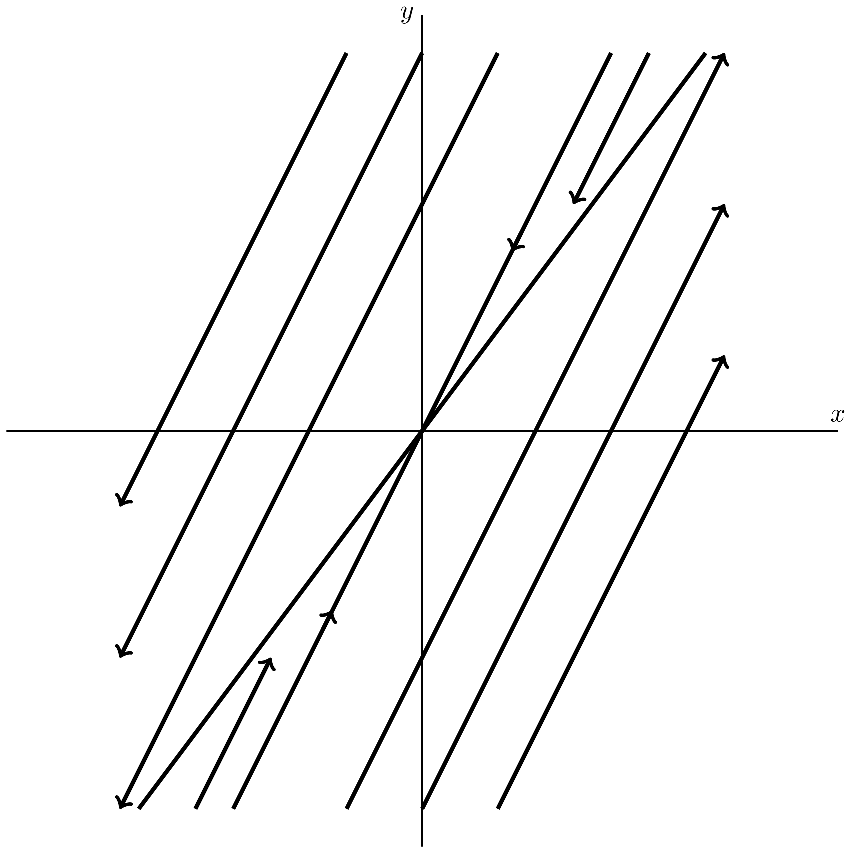
5.2.9

$$\dot{x} = 4x - 3y \qquad \dot{y} = 8x - 6y$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\lambda_1 = -2 \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \lambda_2 = 0 \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Nonisolated fixed point



5.2.11

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{vmatrix} a-\lambda & b \\ 0 & a-\lambda \end{vmatrix} = (a-\lambda)^2 = 0 \quad \Rightarrow \quad \lambda = a$$

$$\begin{pmatrix} a-\lambda & b \\ 0 & a-\lambda \end{pmatrix} \vec{v} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \vec{v} = \vec{0} \quad \Rightarrow \quad \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\dot{y} = ay \Rightarrow y = c_2 e^{at}$$

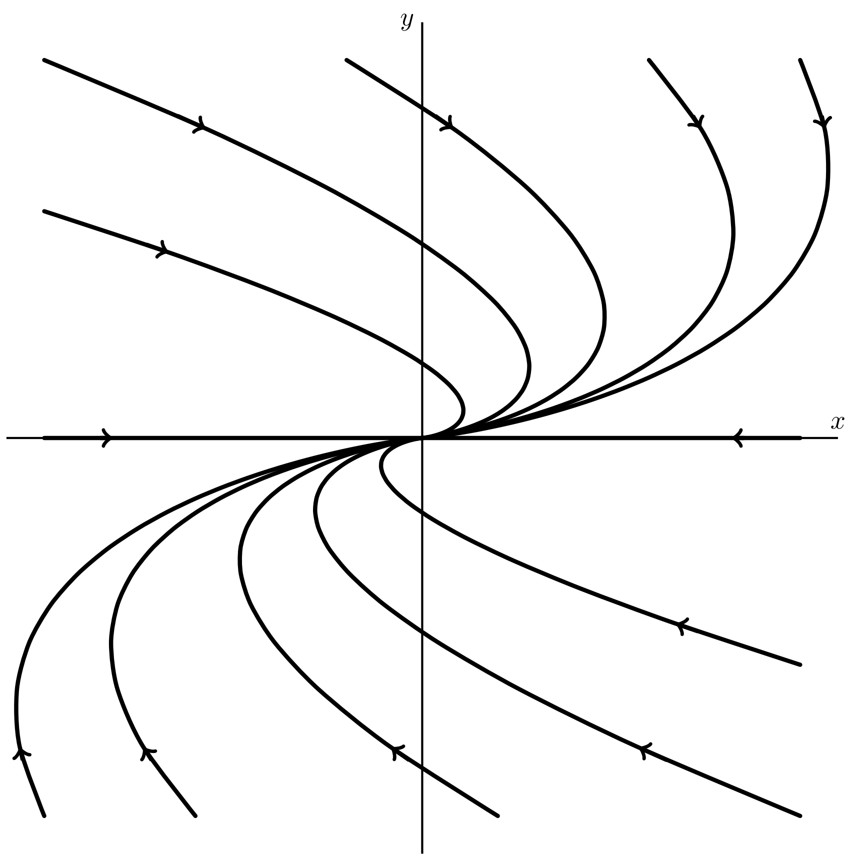
$$\dot{x} = ax + by = ax + c_2 b e^{at}$$

$$\dot{x} - ax = c_2 b e^{at}$$

$$
\begin{aligned}
&e^{-at}\dot{x} - ae^{-at}x = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-at}x\right) = c_2 b \\
&e^{-at}x = \int c_2 b \ \mathrm{d}t = c_1 + c_2 bt \\
&x = c_1 e^{at} + c_2 bte^{at}
\end{aligned}
$$

$$
\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{at}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{at}\left(t\begin{pmatrix} b \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)
$$

Stable degenerate node



5.2.13

$$
m\ddot{x} + b\dot{x} + kx = 0 \qquad b > 0
$$

a)

$$
\dot{x} = y \qquad \dot{y} = \ddot{x} = \frac{-k}{m}x - \frac{b}{m}\dot{x} = \frac{-k}{m}x - \frac{b}{m}y
$$

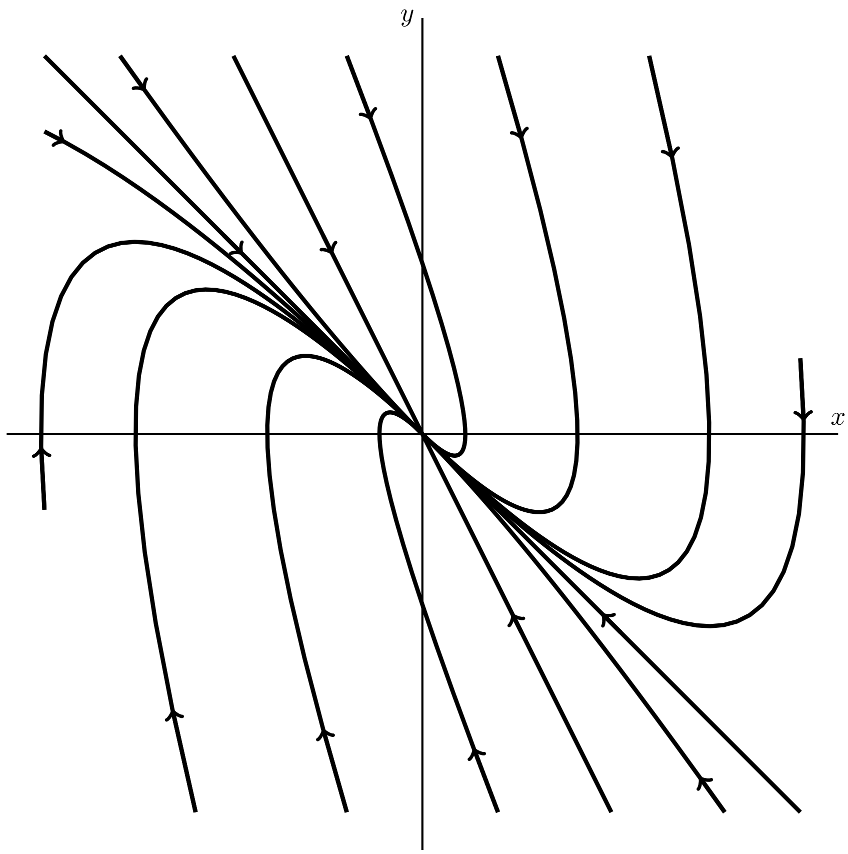
$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-b}{m} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}
$$

b)

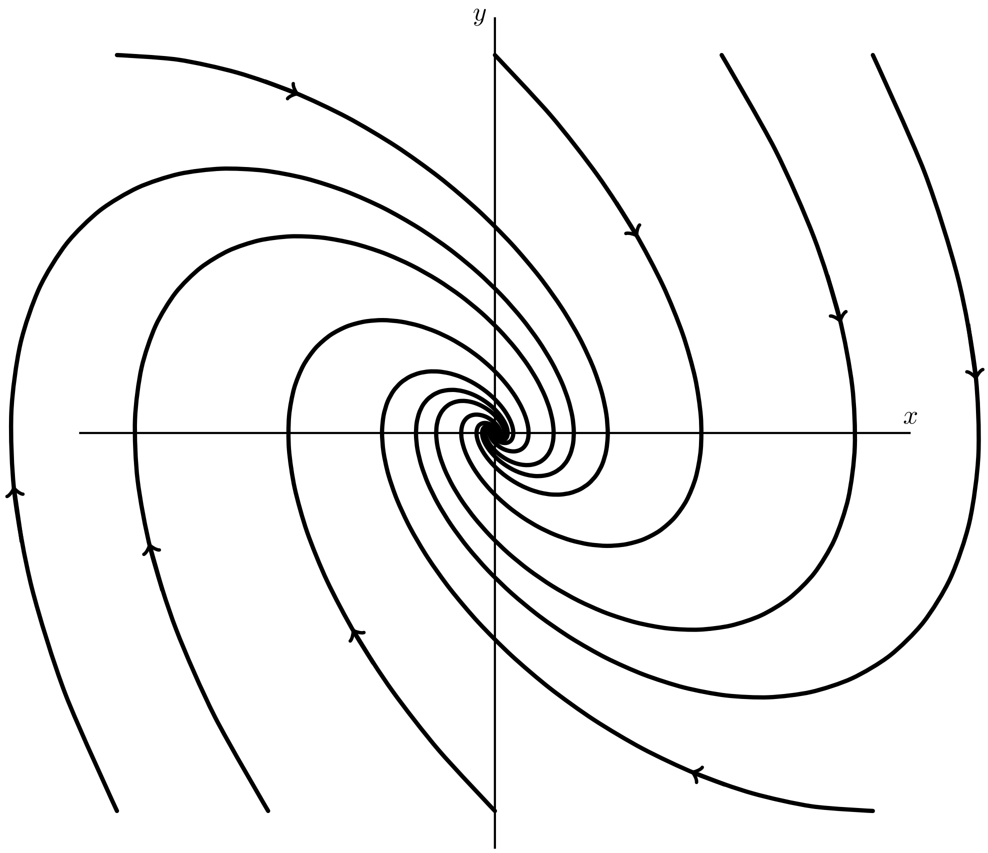
$$\begin{vmatrix} 0-\lambda & 1 \\ \frac{-k}{m} & \frac{-b}{m}-\lambda \end{vmatrix} = -\lambda\left(\frac{-b}{m}-\lambda\right)+\frac{k}{m}=\frac{1}{m}(m\lambda^2+b\lambda+k)=0$$

$$\lambda=\frac{-b\pm\sqrt{b^2-4mk}}{2m}$$

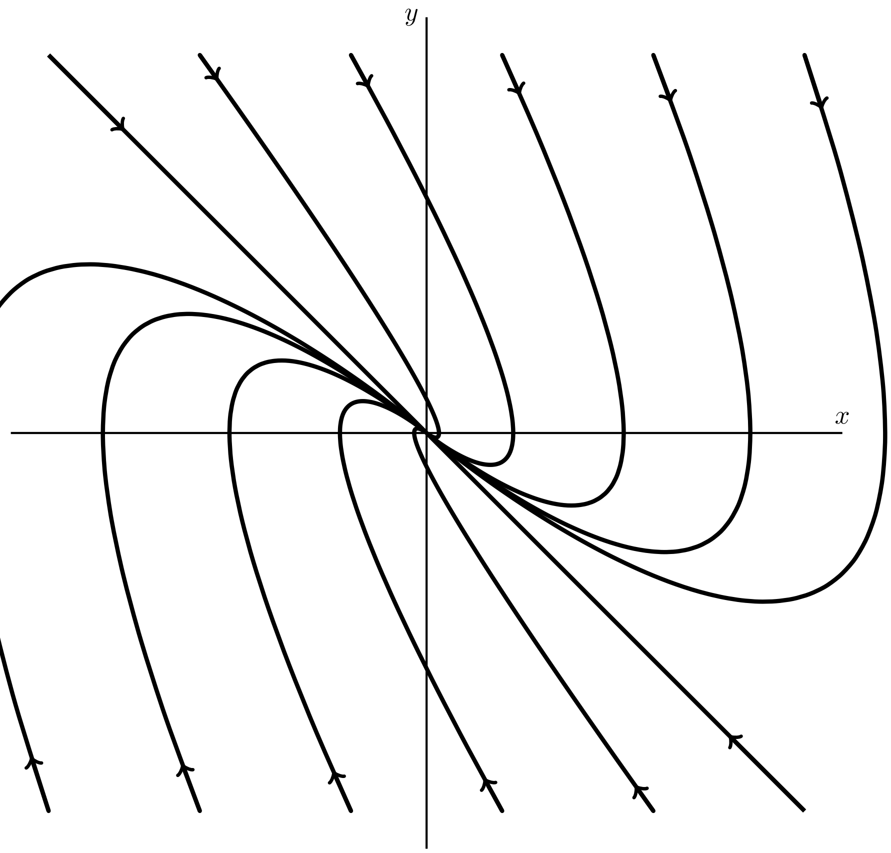
If the inside of the square root is positive, then the origin is stable because the square root can't exceed $-b$, and both roots will be nonnegative. If the inside of the square root is negative, then the $-b$ term still determines the stability. $b>0$ ensures the origin is asymptotically stable.



$m=\frac{1}{2}$, $b=\frac{3}{2}$, $k=1\Rightarrow b^2-4mk>0\Rightarrow$ Stable node



$m = 1,\, b = 1,\, k = 1 \Rightarrow b^2 - 4mk < 0 \Rightarrow$ stable spiral



$m = 1,\ b = 2,\ k = 1 \Rightarrow b^2 - 4mk = 0 \Rightarrow$ stable degenerate node

c)

Stable node $\longleftrightarrow$ overdamped

Stable degenerate node $\longleftrightarrow$ critically damped

Stable spiral $\longleftrightarrow$ underdamped

5.3 Love Affairs

5.3.1

See the "Answers to Selected Exercises" section in the textbook.

5.3.3

$$\dot{R} = aJ \qquad \dot{J} = bR$$

$$\begin{pmatrix} \dot{R} \\ \dot{J} \end{pmatrix} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix}$$

$$\begin{vmatrix} 0-\lambda & a \\ b & 0-\lambda \end{vmatrix} = \lambda^2 - ab = 0$$

$ab < 0 \Rightarrow$ center

$0 < ab \Rightarrow$ saddle point

5.3.5

$$\dot{R} = aR + bJ \qquad \dot{J} = bR + aJ$$

$$\begin{pmatrix} \dot{R} \\ \dot{J} \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix}$$

$$\begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 - b^2 = 0 \quad \Rightarrow \quad \lambda = a \pm b$$

$a < b \Rightarrow$ saddle point

$a < b < 0 \Rightarrow$ stable node

$0 < b < a \Rightarrow$ unstable node

6

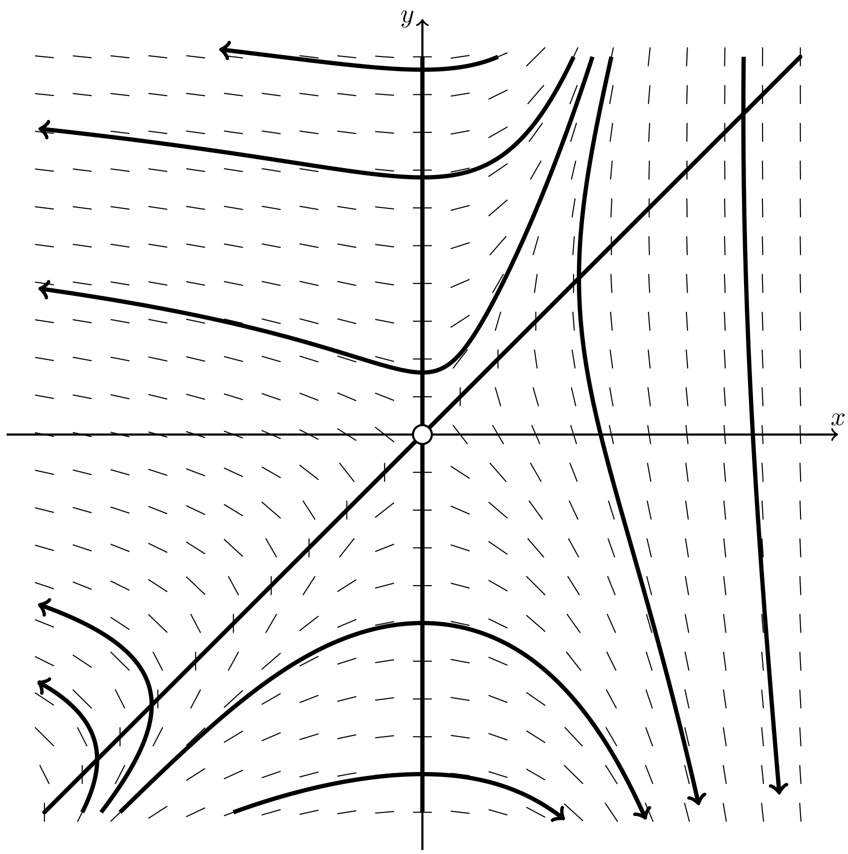
Phase Plane

6.1 Phase Portraits

6.1.1

$$\dot{x} = x - y \qquad \dot{y} = 1 - e^x$$

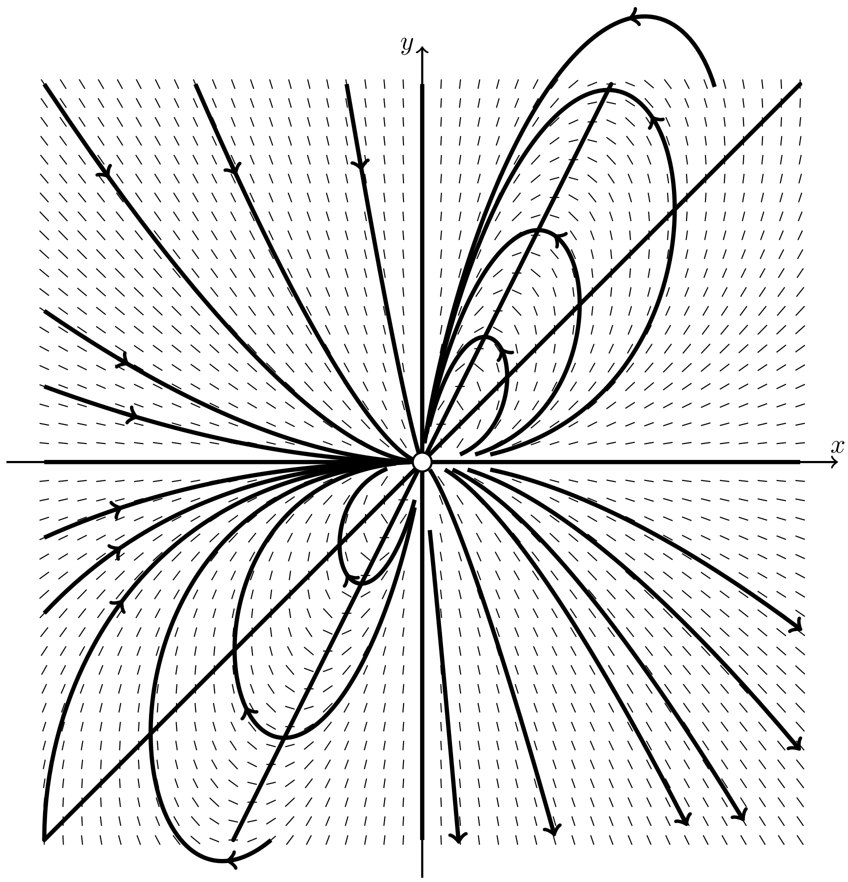
$(x, y) = (0, 0)$ is the fixed point.



6.1.3

$$\dot{x} = x(x - y) \qquad \dot{y} = y(2x - y)$$

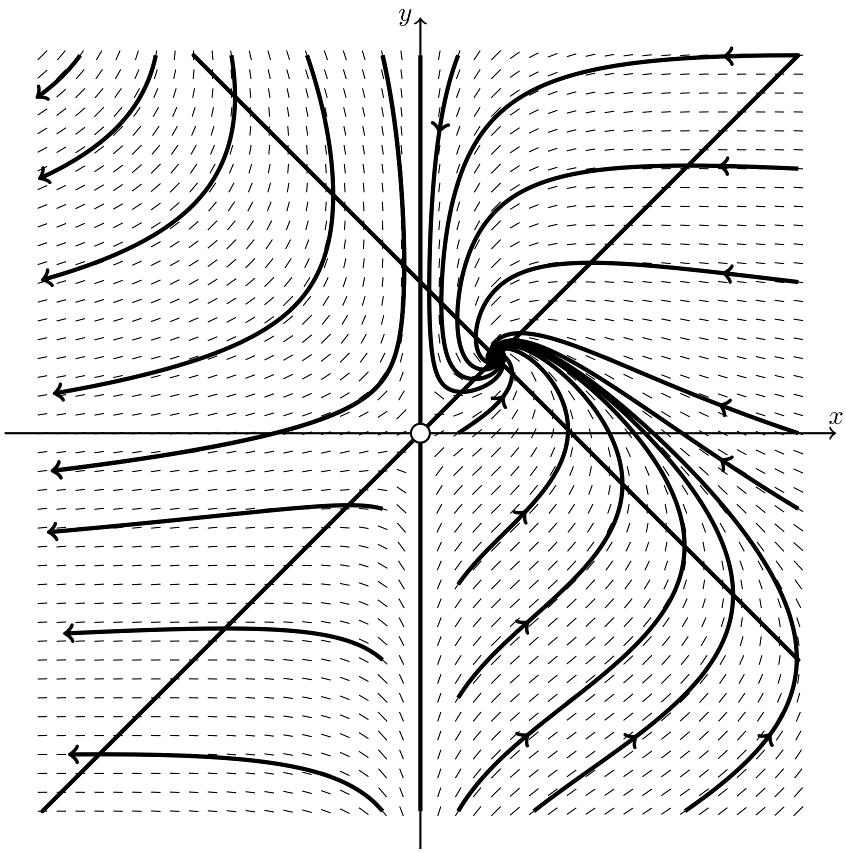
$(x, y) = (0, 0)$ is the fixed point.



6.1.5

$$\dot{x} = x(2 - x - y) \qquad \dot{y} = x - y$$

$(x, y) = (0, 0)$ and $(1, 1)$ are the fixed points.



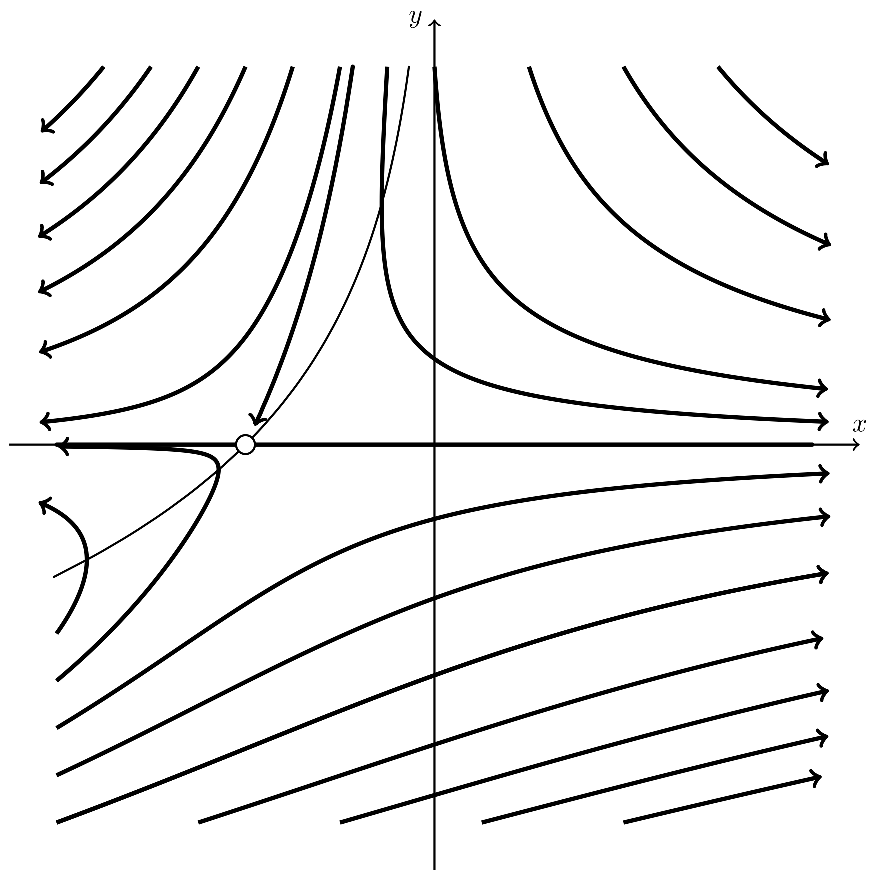
6.1.7

$$\dot{x} = x + e^{-y} \qquad \dot{y} = -y$$

$$y = c_2 e^{-t} \qquad x = c_1 e^t - \frac{e^{t-c_2}}{c_2} + \frac{e^{t-c_2 e^{-t}}}{c_2}$$

If we pick $c_1 = 2$ and $c_1 = \frac{e^{-2}-1}{2}$, then the growing exponentials destructively interfere with each other, leaving the stable manifold.

$$y = 2e^{-t} \qquad x = \frac{e^t}{2}\left(e^{-2e^{-t}} - 1\right)$$

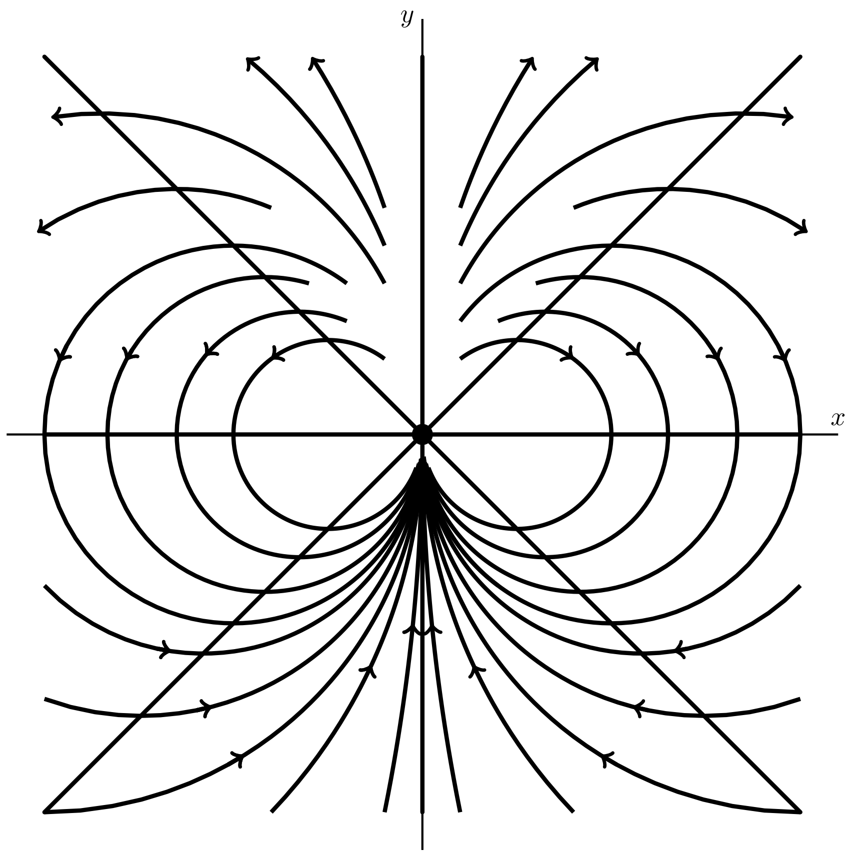


6.1.9

Dipole fixed point

$$\dot{x} = 2xy \qquad \dot{y} = y^2 - x^2$$

$(x, y) = (0, 0)$ is the fixed point.

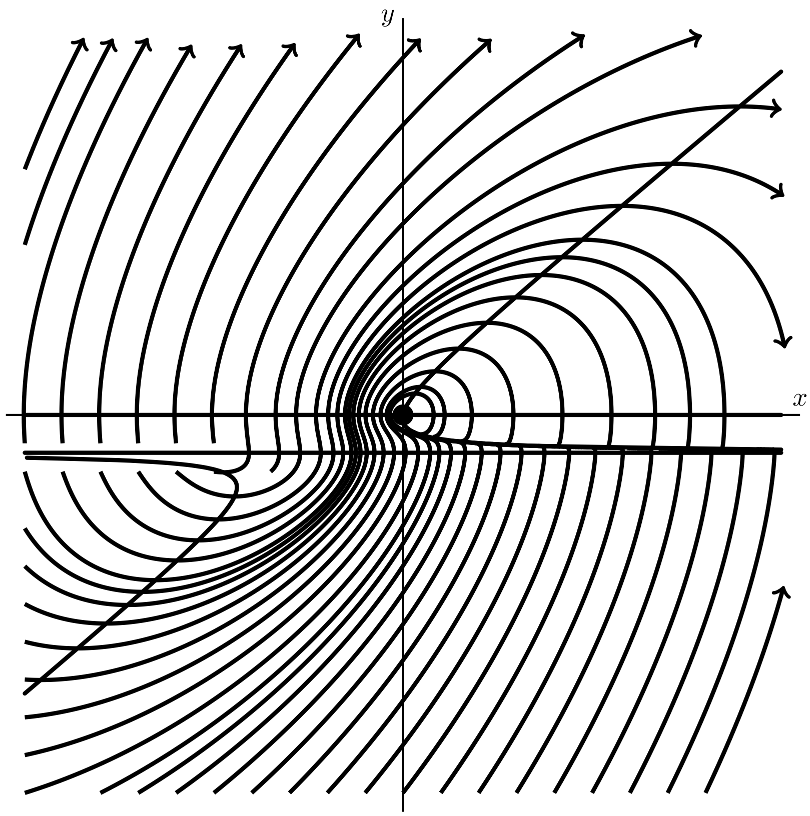


6.1.11

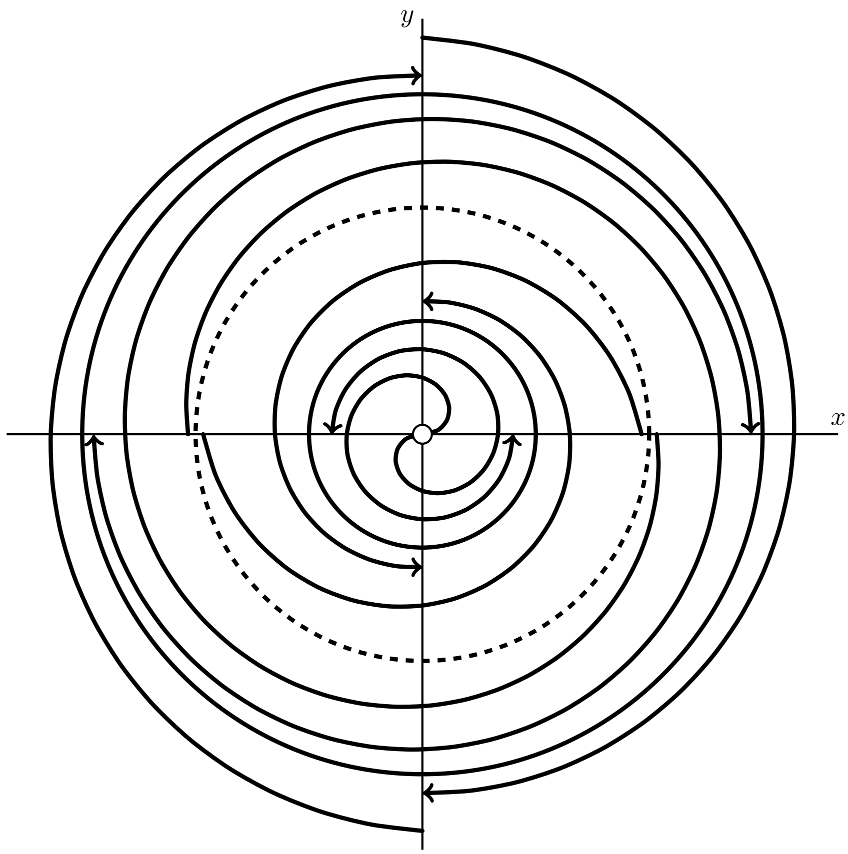
Parrot

$$\dot{x} = y + y^2 \qquad \dot{y} = -x + \frac{1}{5}y - xy + \frac{6}{5}y^2$$

$(x, y) = (0, 0)$ is the fixed point.



6.1.13



6.2 Existence, Uniqueness, and Topological Consequences

6.2.1

Trajectories in a phase portrait never intersect, despite trajectories that approach a stable fixed point. Trajectories never actually reach the stable fixed point because they decay like an exponential function, never vanishing but always decreasing.

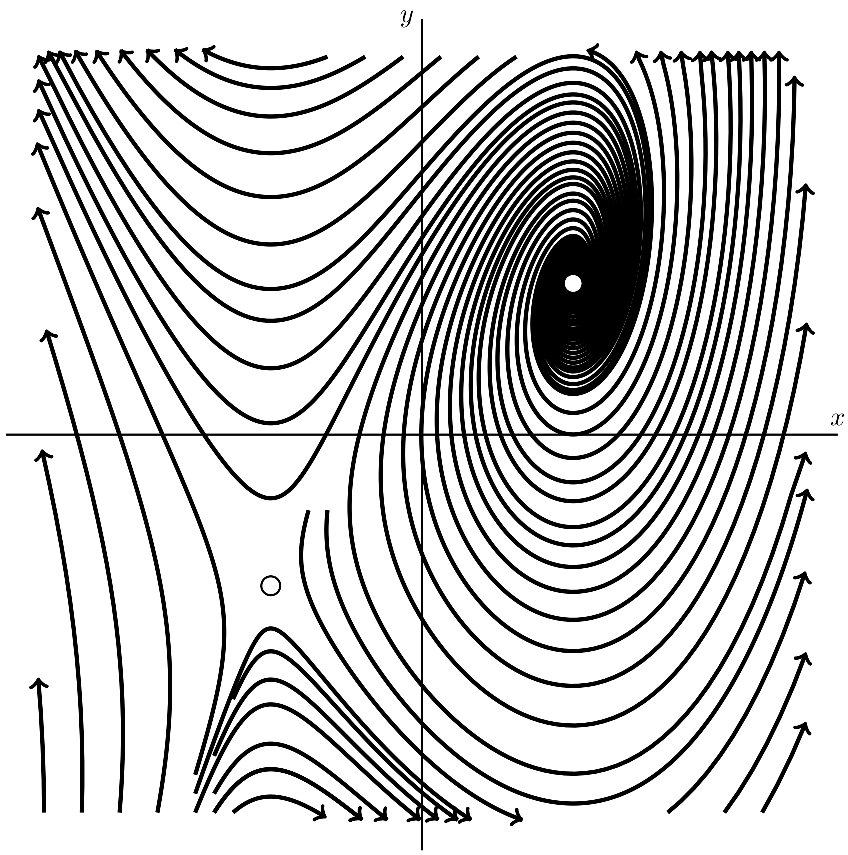
However, less well-behaved ODEs such as having nonunique solutions can reach a fixed point in finite time.

6.3 Fixed Points and Linearization

6.3.1

$$\dot{x} = x - y \qquad \dot{y} = x^2 - 4$$

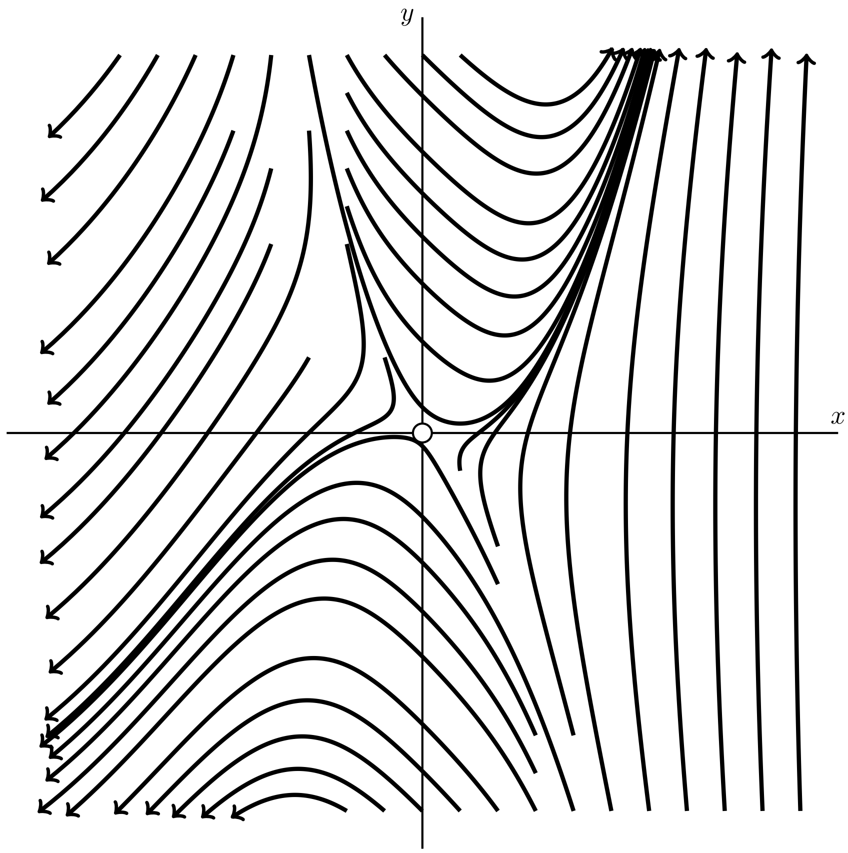
$(x, y) = (-2, -2)$ and $(2, 2)$ are the fixed points.



6.3.3

$$\dot{x} = 1 + y - e^{-x} \qquad \dot{y} = x^3 - y$$

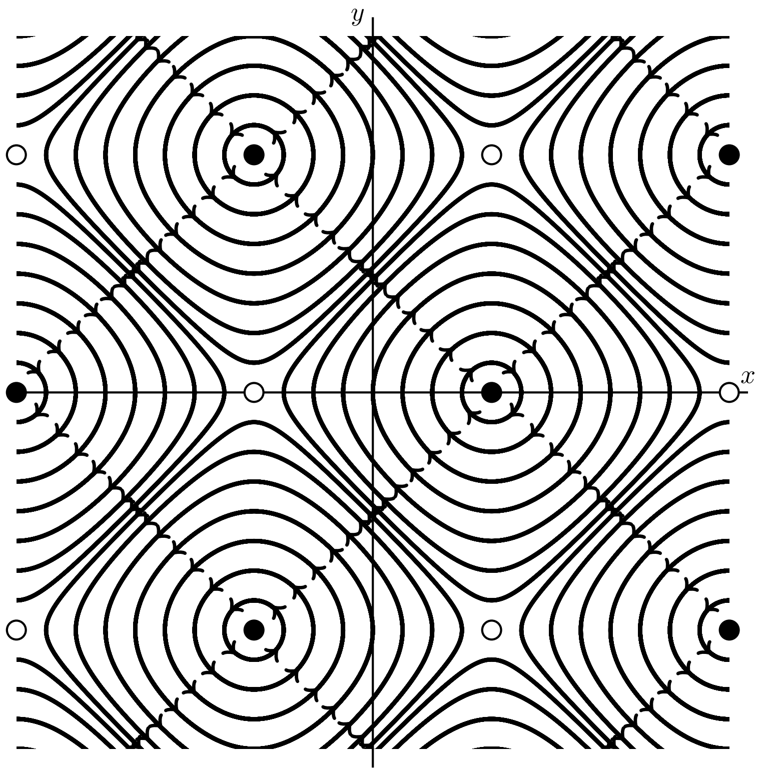
$(x, y) = (0, 0)$ is the fixed point.



6.3.5

$$\dot{x} = \sin(y) \qquad \dot{y} = \cos(x)$$

$(x, y) = \left(\left(n_1 - \frac{1}{2}\right)\pi, n_2\pi\right)$ are the fixed points.



6.3.7

See graphs for Exercises 6.3.1, 6.3.3, and 6.3.5.

6.3.9

$$\dot{x} = y^3 - 4x \qquad \dot{y} = y^3 - y - 3x$$

a)

$(x, y) = (-2, -2)$, $(0, 0)$, and $(2, 2)$ are the fixed points.

$$A = \begin{pmatrix} -4 & 3y^2 \\ -3 & 3y^2 - 1 \end{pmatrix}$$

$$A_{(-2,-2)} = \begin{pmatrix} -4 & 12 \\ -3 & 11 \end{pmatrix} \qquad \Delta = -8 \Rightarrow \text{saddle point}$$

$$A_{(0,0)} = \begin{pmatrix} -4 & 0 \\ -3 & -1 \end{pmatrix} \qquad \Delta = 4 \quad \tau = -5 \Rightarrow \text{unstable spiral}$$

$$A_{(2,2)} = \begin{pmatrix} -4 & 12 \\ -3 & 11 \end{pmatrix} \qquad \Delta = -8 \Rightarrow \text{saddle point}$$

b)

$$x = y \Rightarrow \dot{x} = x^3 - 4x = y^3 - 4y = \dot{y}$$

So as long as the initial condition satisfies $x(0) = y(0)$, then $x(t) = y(t)$ since the two variables and equations are indistinguishable.

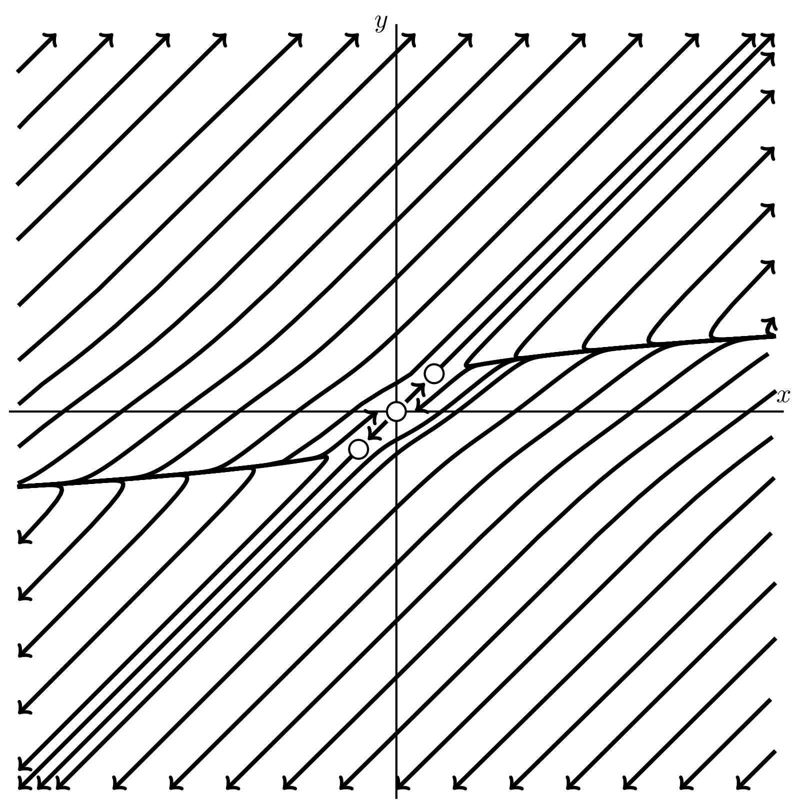
c)

$$u = x - y$$
$$\dot{u} = \dot{x} - \dot{y} = y^3 - 4x - (y^3 - y - 3x) = -x + y = -u$$
$$u(t) = ce^{-t}$$
$$|x(t) - y(t)| = |u(t)| = \left|ce^{-t}\right|$$
$$\lim_{t\to\infty} |x(t) - y(t)| = \lim_{t\to\infty} \left|ce^{-t}\right| \to 0$$

d)

See part (e).

e)



6.3.11

$$\dot{r} = -r \qquad \dot{\theta} = \frac{1}{\ln(r)}$$

a)

$$\begin{aligned}
r(t) &= r_0 e^{-t} \\
\dot{\theta} &= \frac{1}{\ln(r)} = \frac{1}{\ln(r_0 e^{-t})} = \frac{1}{\ln(r_0) + \ln(e^{-t})} = \frac{1}{\ln(r_0) - t} \\
\theta(t) &= -\ln|\ln(r_0) - t| + \ln(\ln(r_0)) + \theta_0 \\
&= \ln\left(\frac{\ln(r_0)}{|\ln(r_0) - t|}\right) + \theta_0
\end{aligned}$$

b)

$$\begin{aligned}
&\lim_{t\to\infty} r(t) = \lim_{t\to\infty} r_0 e^{-t} \to 0 \Rightarrow \text{stable} \\
&\lim_{t\to\infty} |\theta(t)| = \lim_{t\to\infty} \left|\ln\left(\frac{\ln(r_0)}{|\ln(r_0) - t|}\right) + \theta_0\right| \to \infty \Rightarrow \text{spiral}
\end{aligned}$$

c)

$$\begin{aligned}
r &= \sqrt{x^2 + y^2} \qquad \theta = \arctan\left(\frac{y}{x}\right) \\
\dot{r} &= \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{x^2 + y^2} = \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}} = -r = -\sqrt{x^2 + y^2} \\
\dot{\theta} &= \frac{\mathrm{d}}{\mathrm{d}t}\arctan\left(\frac{y}{x}\right) = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2} = \frac{1}{\ln(r)} = \frac{1}{\ln\left(\sqrt{x^2 + y^2}\right)} = \frac{2}{\ln(x^2 + y^2)}
\end{aligned}$$

$$\begin{aligned}
&\frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}} = -\sqrt{x^2 + y^2} \qquad x\dot{x} + y\dot{y} = -(x^2 + y^2) \\
&\frac{x\dot{y} - y\dot{x}}{x^2 + y^2} = \frac{2}{\ln(x^2 + y^2)} \qquad x\dot{y} - y\dot{x} = \frac{2(x^2 + y^2)}{\ln(x^2 + y^2)} \\
&x(x\dot{x} + y\dot{y}) - y(x\dot{y} - y\dot{x}) = (x^2 + y^2)\dot{x} = -x(x^2 + y^2) - y\frac{2(x^2 + y^2)}{\ln(x^2 + y^2)} \\
&\dot{x} = -x - \frac{2y}{\ln(x^2 + y^2)} \\
&y(x\dot{x} + y\dot{y}) + x(x\dot{y} - y\dot{x}) = (x^2 + y^2)\dot{y} = -y(x^2 + y^2) + x\frac{2(x^2 + y^2)}{\ln(x^2 + y^2)} \\
&\dot{y} = -y + \frac{2x}{\ln(x^2 + y^2)}
\end{aligned}$$

d)

$$A = \begin{pmatrix} \dfrac{4xy}{(x^2+y^2)\ln^2(x^2+y^2)} - 1 & \dfrac{4y^2}{(x^2+y^2)\ln^2(a^2+y^2)} - \dfrac{2}{\ln(a^2+y^2)} \\ \dfrac{2}{\ln(x^2+y^2)} - \dfrac{4x^2}{(x^2+y^2)\ln^2(x^2+y^2)} & \dfrac{-4xy}{(x^2+y^2)\ln^2(x^2+y^2)} - 1 \end{pmatrix}$$

$$A_{(0,0)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \dot{x} = -x \quad \dot{y} = -y \Rightarrow \text{stable star}$$

6.3.13

$$\dot{x} = -y - x^3 \qquad \dot{y} = x$$

$$r^2 = x^2 + y^2 \qquad 2r\dot{r} = 2x\dot{x} + 2y\dot{y}$$

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{x(-y-x^3) + yx}{\sqrt{x^2+y^2}} = \frac{-x^4}{\sqrt{x^2+y^2}} < 0 \text{ for } (x,y) \neq (0,0)$$

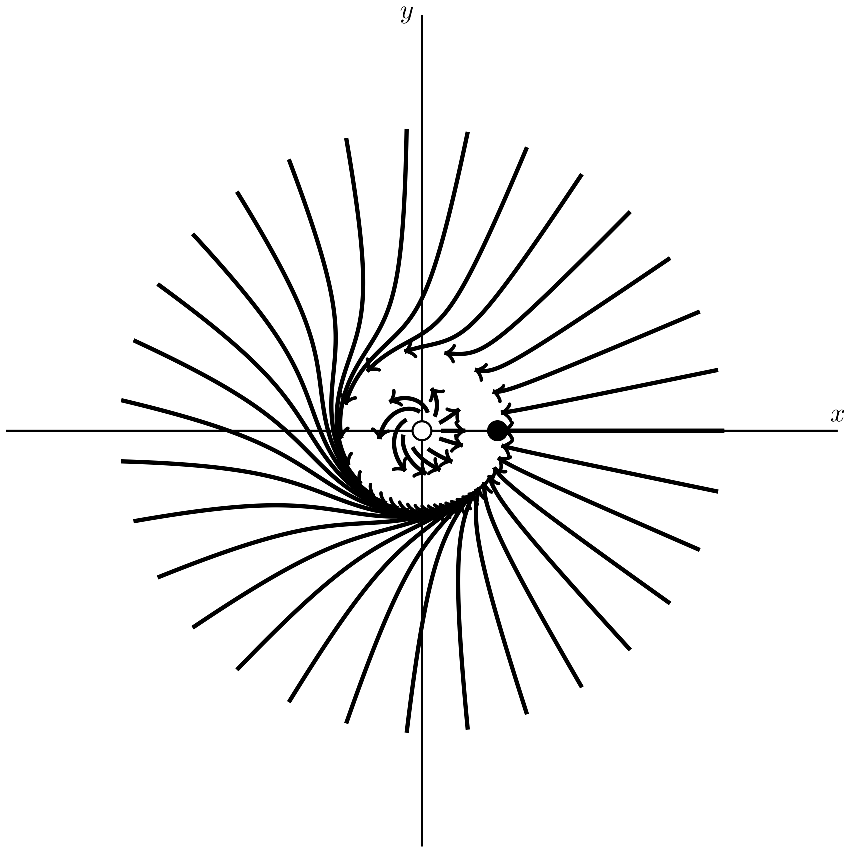
$$A = \begin{pmatrix} -3x^2 & -1 \\ 1 & 0 \end{pmatrix}$$

$$A_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \Delta > 0 \quad \tau = 0 \Rightarrow \text{center}$$

6.3.15

$$\dot{r} = r(1 - r^2) \qquad \dot{\theta} = 1 - \cos(\theta)$$

$(r, \theta) = (0, 0)$ and $(1, 0)$ are the fixed points.

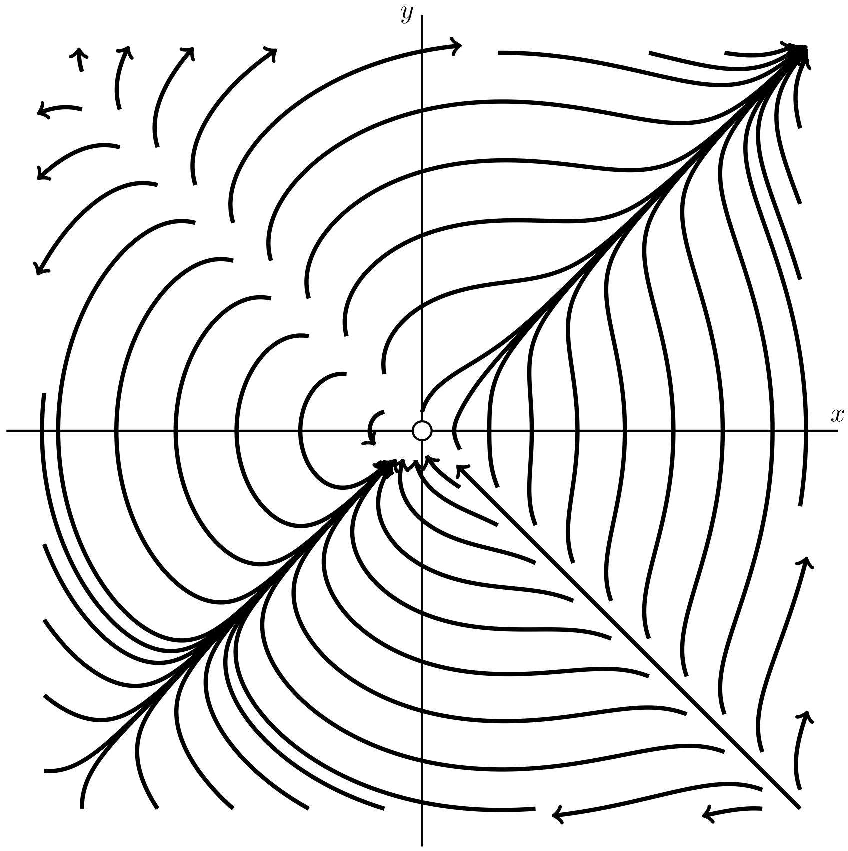


(1,0) is attracting, but all trajectories except those starting on the positive x-axis have to traverse near the unit circle before reaching the fixed point.

6.3.17

$$\dot{x} = xy - x^2y + y^3 \qquad \dot{y} = y^2 + x^3 - xy^2$$

$(x, y) = (0, 0)$ is the fixed point.

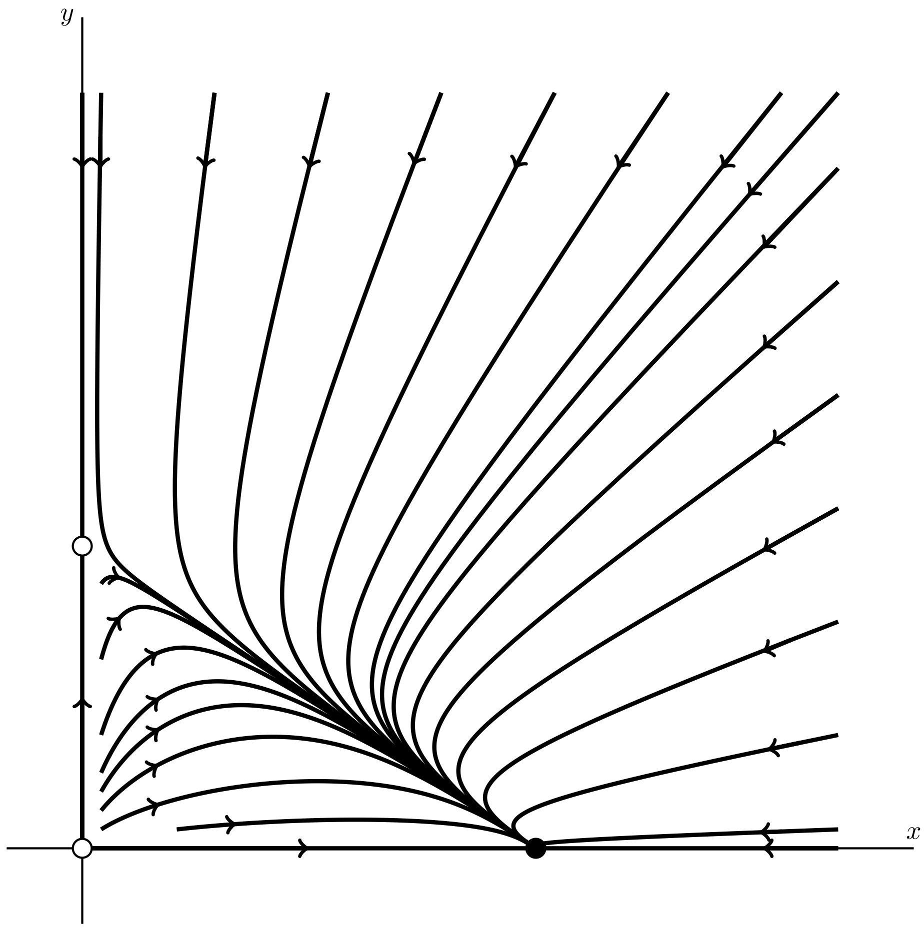


6.4 Rabbits versus Sheep

6.4.1

$$\dot{x} = x(3 - x - y) \qquad \dot{y} = y(2 - x - y)$$

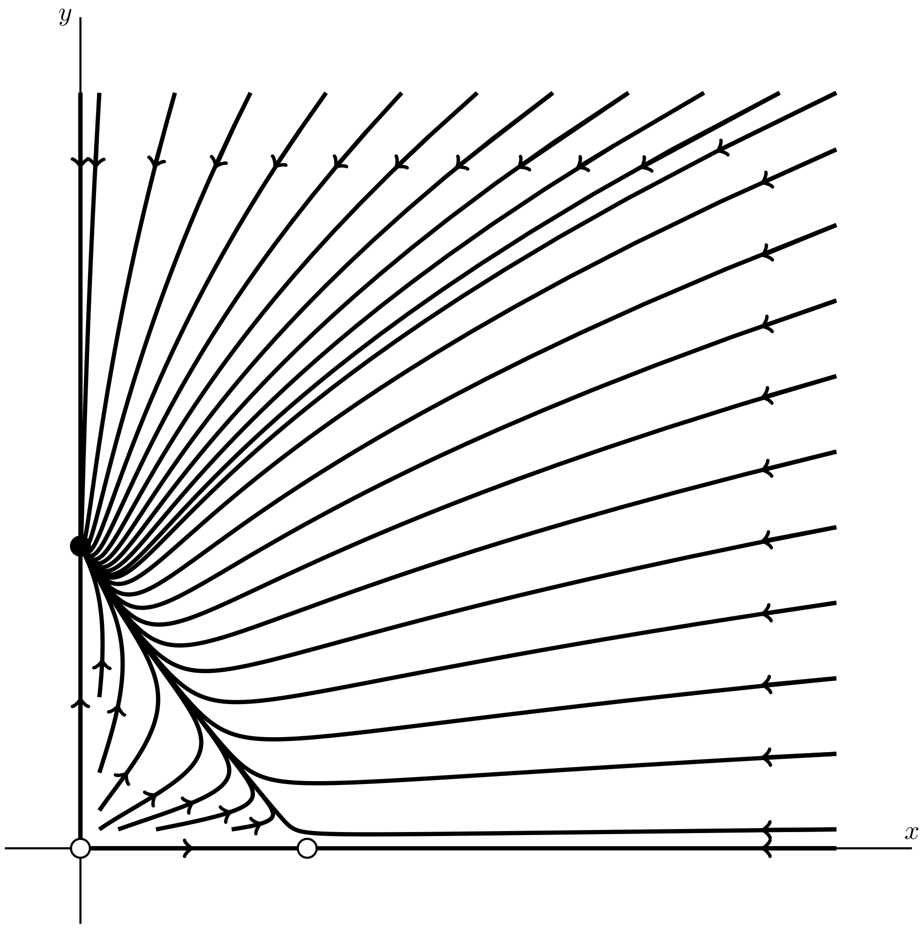
$(x, y) = (0, 0)$, $(0, 2)$, and $(3, 0)$ are the fixed points.



6.4.3

$$\dot{x} = x(3 - 2x - 2y) \qquad \dot{y} = y(2 - x - y)$$

$(x, y) = (0, 0)$, $(0, 2)$, and $(\frac{3}{2}, 0)$ are the fixed points.



6.4.5

$$\dot{N}_1 = r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - b_1 N_1 N_2 \qquad \dot{N}_2 = r_2 N_2 - b_2 N_1 N_2$$

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - b_1 N_1 N_2 \qquad \frac{\mathrm{d}N_2}{\mathrm{d}t} = r_2 N_2 - b_2 N_1 N_2$$

$$t = u\tau \qquad N_1 = vx \qquad N_2 = wy$$

$$\frac{v}{u}\frac{\mathrm{d}x}{\mathrm{d}\tau} = r_1 vx \left(1 - \frac{vx}{K_1}\right) - b_1 vwxy \qquad \frac{w}{u}\frac{\mathrm{d}y}{\mathrm{d}\tau} = r_2 wy - b_2 vwxy$$

$$\frac{1}{r_1 u}\frac{\mathrm{d}x}{\mathrm{d}\tau} = x\left(1 - \frac{vx}{K_1}\right) - \frac{b_1 w}{r_1} xy \qquad \frac{1}{b_2 uv}\frac{\mathrm{d}y}{\mathrm{d}\tau} = \frac{r_2}{b_2 v} y - xy$$

$$\frac{1}{r_1 u} = 1 \Rightarrow u = \frac{1}{r_1} \quad \frac{b_1 w}{r_1} = 1 \Rightarrow w = \frac{r_1}{b_1} \quad \frac{1}{b_2 uv} = \frac{r_1}{b_2 v} = 1 \Rightarrow v = \frac{r_1}{b_2}$$

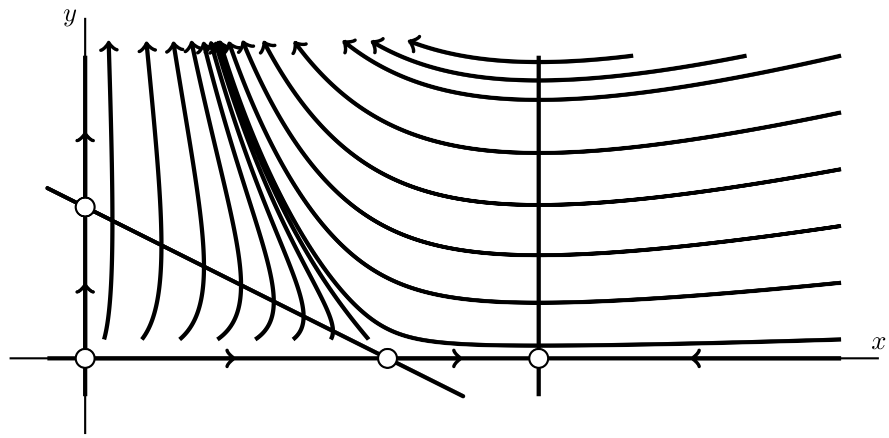
$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = x\left(1 - \frac{r_1 x}{b_2 K_1}\right) - xy \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau} = \frac{r_2}{r_1} y - xy$$

$$\rho = \frac{r_2}{r_1} \quad \kappa = \frac{r_1}{b_2 K_1}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = x(1 - \kappa x - y) \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau} = y(\rho - x)$$

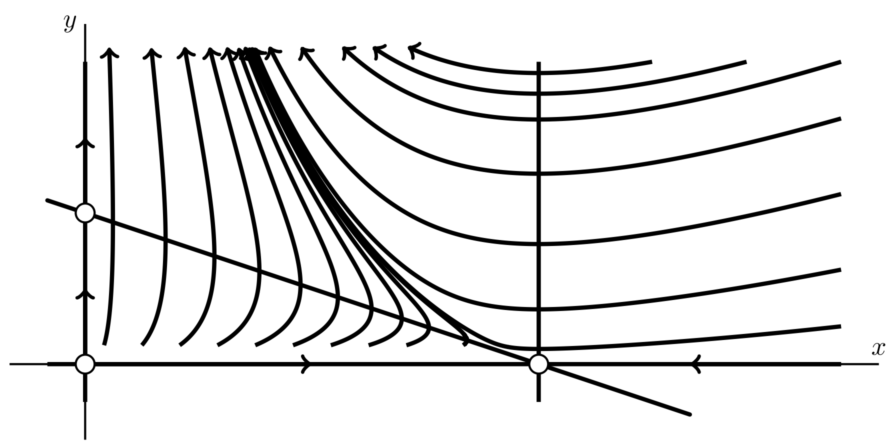
$\rho = 3 \quad \kappa = \frac{1}{2}$

$(x, y) = (0, 0)$, $(0, 1)$, $(2, 0)$, and $(3, 0)$ are the fixed points.



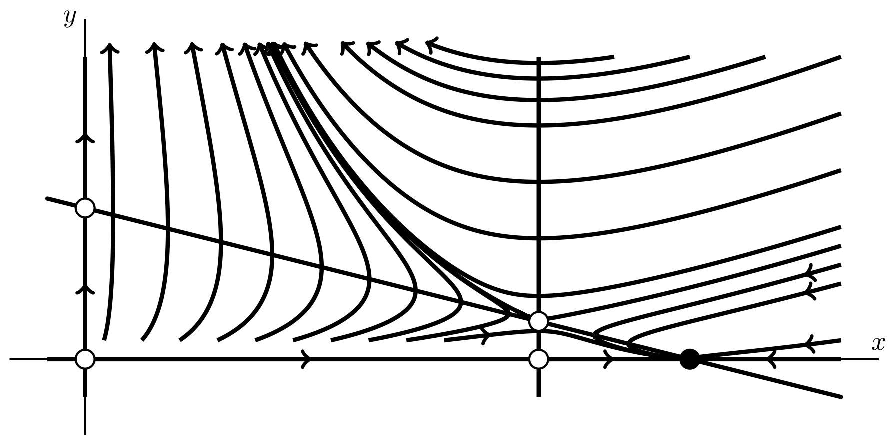
$\rho = 3 \quad \kappa = \frac{1}{3}$

$(x, y) = (0, 0)$, $(0, 1)$, and $(3, 0)$ are the fixed points.



$\rho = 3 \quad \kappa = \frac{1}{4}$

$(x, y) = (0, 0)$, $(0, 1)$, $(3, 0)$, $\left(3, \frac{1}{4}\right)$, and $(4, 0)$ are the fixed points.



The top and bottom graphs have a different number of fixed points, with the middle graph being the borderline case. Except by starting with an initial condition of zero population, the top two graphs grow infinitely in y, and x decays to zero. Depending on the initial condition, the bottom graph can grow infinitely in y, and x decays to zero, but it can also go towards a stable fixed point where x becomes finite and y decays to zero.

6.4.7

$$\dot{n_1} = G_1(N_0 - \alpha_1 n_1 - \alpha_2 n_2)n_1 - K_1 n_1 \qquad \dot{n_2} = G_2(N_0 - \alpha_1 n_1 - \alpha_2 n_2)n_2 - K_2 n_2$$

a)

$$A = \begin{pmatrix} G_1(N_0 - 2\alpha_1 n_1 - \alpha_2 n_2) - K_1 & -G_1\alpha_2 n_2 \\ -G_2\alpha_1 n_1 & G_2(N_0 - \alpha_1 n_1 - 2\alpha_2 n_2) - K_2 \end{pmatrix}$$

$$A_{(0,0)} = \begin{pmatrix} G_1 N_0 - K_1 & 0 \\ 0 & G_2 N_0 - K_2 \end{pmatrix}$$

$$\lambda_1 = G_1 N_0 - K_1 \qquad \lambda_2 = G_2 N_0 - K_2$$

So the origin is unstable if $G_1 N_0 - K_1 > 0$ or $G_2 N_0 - K_2 > 0$.

b)

The other two fixed points that exist are $\left(0, \frac{G_2 N_0 - K_2}{\alpha_2 G_2}\right)$ and $\left(\frac{G_1 N_0 - K_1}{\alpha_1 G_1}, 0\right)$.

$$A_{\left(0, \frac{G_2 N_0 - K_2}{\alpha_2 G_2}\right)} = \begin{pmatrix} \frac{G_1 K_2}{G_2} - K_1 & -G_1 N_0 + \frac{G_1 K_2}{G_2} \\ 0 & -G_2 N_0 + K_2 \end{pmatrix}$$

$$A_{\left(\frac{G_1 N_0 - K_1}{\alpha_1 G_1}, 0\right)} = \begin{pmatrix} -G_1 N_0 + K_1 & 0 \\ -G_2 N_0 + \frac{G_2 K_1}{G_1} & \frac{G_2 K_1}{G_1} - K_2 \end{pmatrix}$$

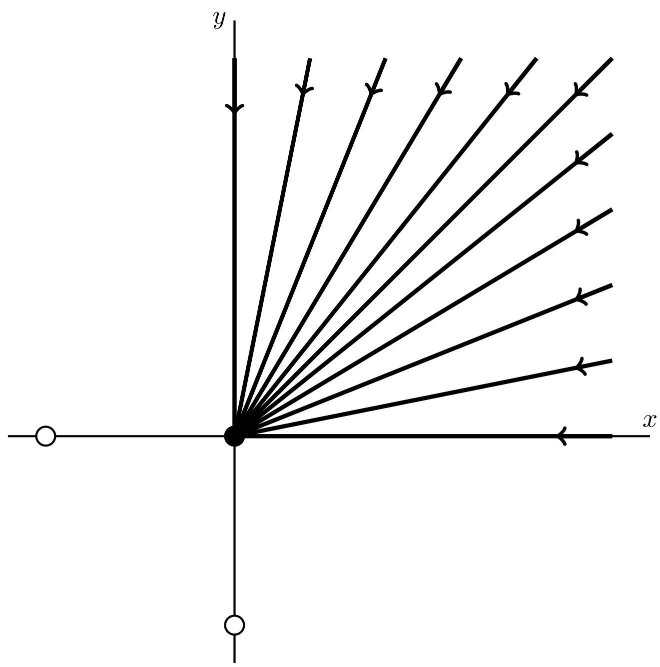
Since these are triangular matrices, the eigenvalues are on the diagonals. Notice that two of the eigenvalues, $\frac{G_1K_2}{G_2} - K_1$ and $\frac{G_2K_1}{G_1} - K_2$, will have opposite signs or both be zero, meaning at most one of the fixed points can be stable.

The other pair of eigenvalues doesn't have that relationship and can be positive, negative, or zero depending on how big N_0 is.

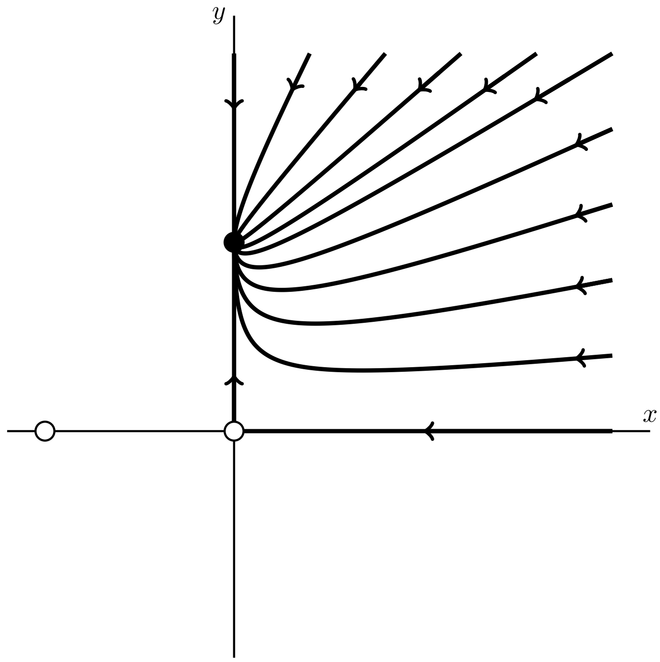
If you look closely, you'll see that the eigenvalues of a fixed point and its coordinate are somewhat simply related. This relationship makes for some interesting graphs.

c)

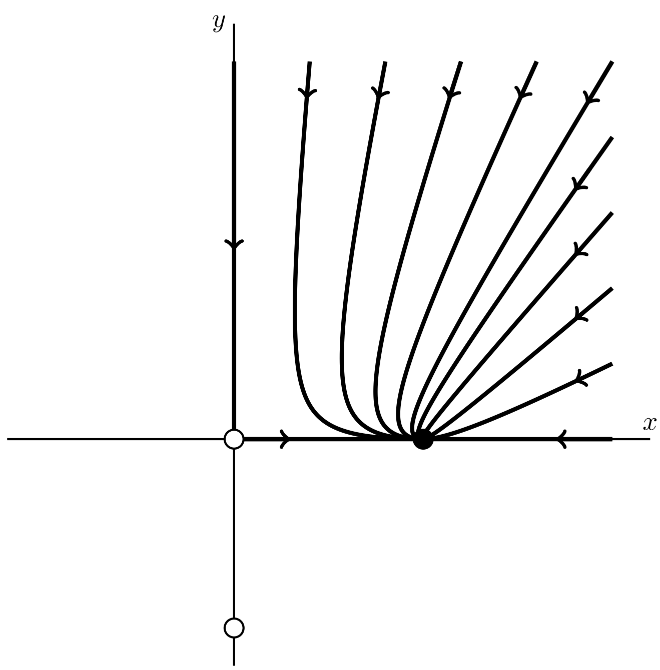
$N_0 = 5 \quad G_1 = 1 \quad G_2 = 1 \quad \alpha_1 = 1 \quad \alpha_2 = 1 \quad K_1 = 7.5 \quad K_2 = 7.5$



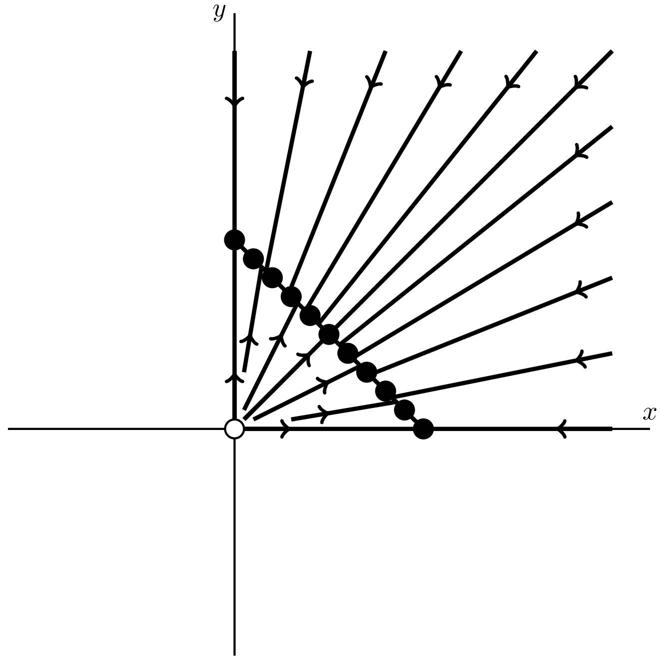
$N_0 = 5 \quad G_1 = 1 \quad G_2 = 1 \quad \alpha_1 = 1 \quad \alpha_2 = 1 \quad K_1 = 7.5 \quad K_2 = 2.5$



$N_0 = 5 \quad G_1 = 1 \quad G_2 = 1 \quad a_1 = 1 \quad a_2 = 1 \quad K_1 = 2.5 \quad K_2 = 7.5$



$N_0 = 5 \quad G_1 = 1 \quad G_2 = 1 \quad \alpha_1 = 1 \quad \alpha_2 = 1 \quad K_1 = 2.5 \quad K_2 = 2.5$



There are four qualitatively different phase portraits. n_1 and n_2 decay to the origin, n_1 axis, n_2 axis, or an infinite number of fixed points. No other phase portraits are possible because the nullclines are the axes and parallel lines.

6.4.9

$$\dot{I} = I - \alpha C \qquad \dot{C} = \beta(I - C - G) \qquad I, C, G \geq 0, \quad 1 < \alpha < \infty, \quad 1 \leq \beta < \infty$$

a)

$(x, y) = \left(\frac{\alpha G}{\alpha-1}, \frac{G}{\alpha-1}\right)$ is the fixed point.

$$A = \begin{pmatrix} 1 & -\alpha \\ \beta & -\beta \end{pmatrix}$$

$$A_{\left(\frac{\alpha G}{\alpha-1}, \frac{G}{\alpha-1}\right)} = \begin{pmatrix} 1 & -\alpha \\ \beta & -\beta \end{pmatrix}$$

$$\lambda_{1,2} = \frac{-(\beta-1) \pm \sqrt{(\beta-1)^2 - 4(\alpha-1)\beta}}{2}$$

As long as $\beta > 1$, the real part is negative, making the fixed point stable. The sign of the expression inside the square root determines the type of stable behavior.

$\alpha > \frac{(\beta-1)^2}{4\beta} + 1 \Rightarrow$ stable spiral

$\alpha = \frac{(\beta-1)^2}{4\beta} + 1 \Rightarrow$ stable star

$\alpha < \frac{(\beta-1)^2}{4\beta} + 1 \Rightarrow$ stable node

And the expression is purely imaginary when

$\beta = 1 \Rightarrow$ Center

b)

$G = G_0 + kI \Rightarrow (x, y) = \left(\frac{\alpha G_0}{\alpha(1-k)-1}, \frac{G_0}{\alpha(1-k)-1}\right)$ is the fixed point, and if $k < 1 - \frac{1}{\alpha} = k_c$, then it's in the positive quadrant.

$$A = \begin{pmatrix} 1 & -\alpha \\ \beta(1-k) & -\beta \end{pmatrix}$$

$$A_{\left(\frac{\alpha G_0}{\alpha(1-k)-1}, \frac{G_0}{\alpha(1-k)-1}\right)} = \begin{pmatrix} 1 & -\alpha \\ \beta(1-k) & -\beta \end{pmatrix}$$

$$\lambda_1 = \frac{-(\beta-1) + \sqrt{(\beta-1)^2 - 4\big(\alpha(1-k)-1\big)\beta}}{2}$$

$$\lambda_2 = \frac{-(\beta-1) - \sqrt{(\beta-1)^2 - 4\big(\alpha(1-k)-1\big)\beta}}{2}$$

If $k > k_c$ and $\beta > 1$, then $\lambda_2 < 0 < \lambda_1$.

$$v_1 = \begin{pmatrix} \frac{\beta+\lambda_1}{\beta(1-k)} \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} \frac{\beta+\lambda_2}{\beta(1-k)} \\ 1 \end{pmatrix}$$

The attracting eigendirection is v_1, which has positive slope and is present in the positive quadrant. The economy will follow this line, with I and C always increasing.

c)

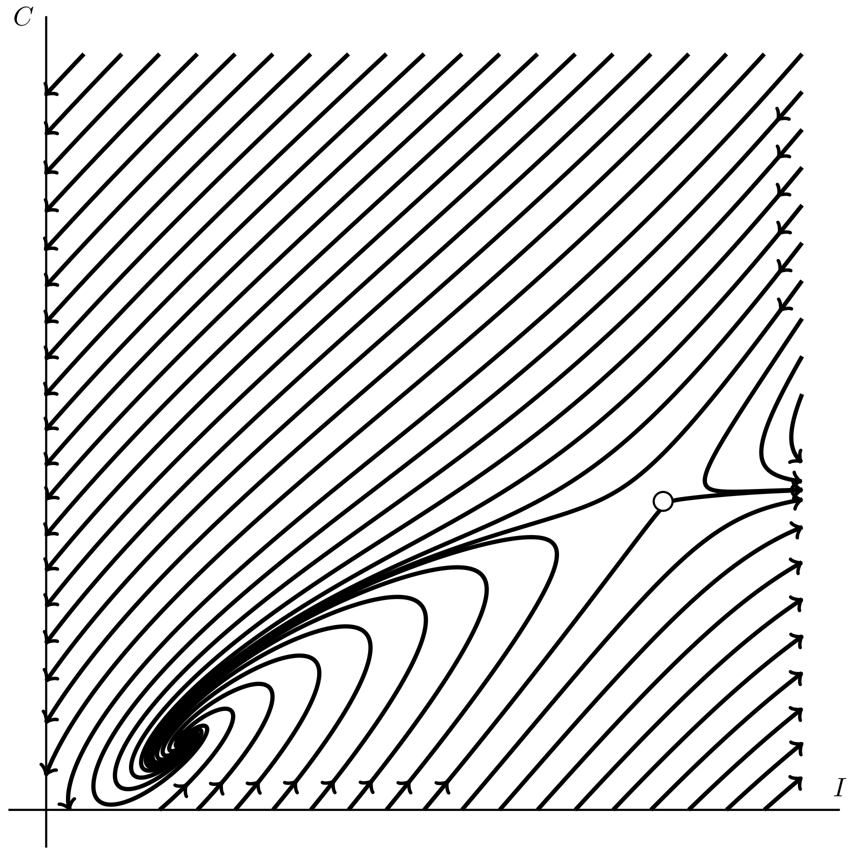
$G = G_0 + kI^2 \Rightarrow (x, y) = \left(\frac{\alpha-1\pm\sqrt{(\alpha-1)^2-4\alpha^2 G_0 k}}{2\alpha k}, \frac{\alpha-1\pm\sqrt{(\alpha-1)^2-4\alpha^2 G_0 k}}{2\alpha^2 k}\right)$ are the fixed points.

If $G_0 < \frac{(\alpha-1)^2}{4\alpha^2 k}$ then there are two fixed points in the positive quadrant.

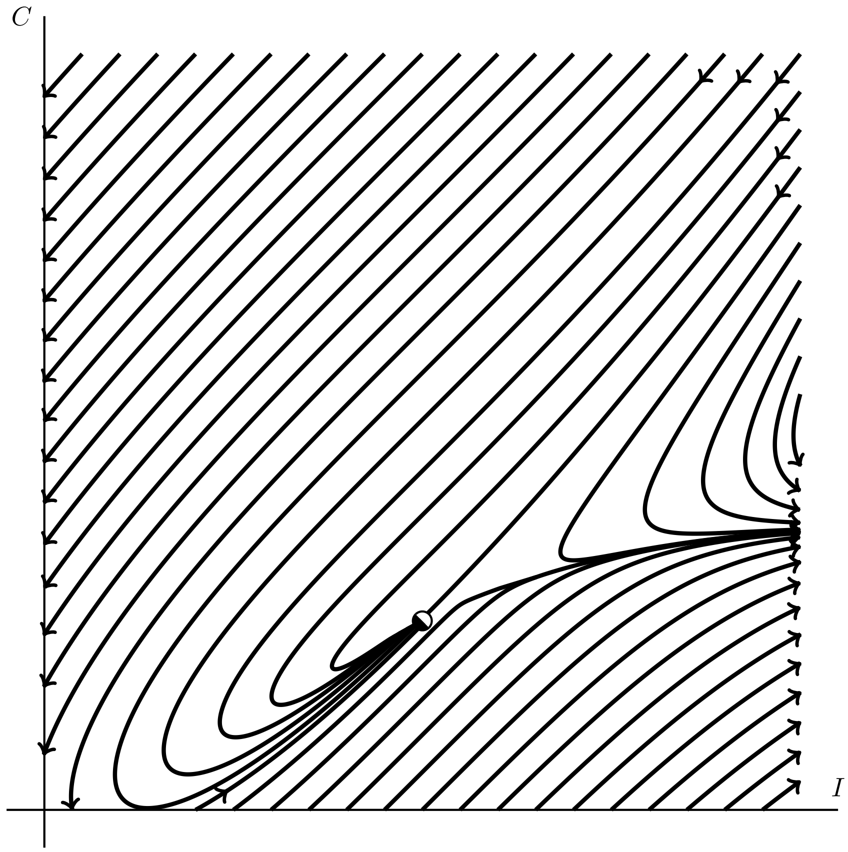
If $G_0 = \frac{(\alpha-1)^2}{4\alpha^2 k}$ then there is only one fixed point in the positive quadrant.

If $G_0 > \frac{(\alpha-1)^2}{4\alpha^2 k}$ then there are no real fixed points.

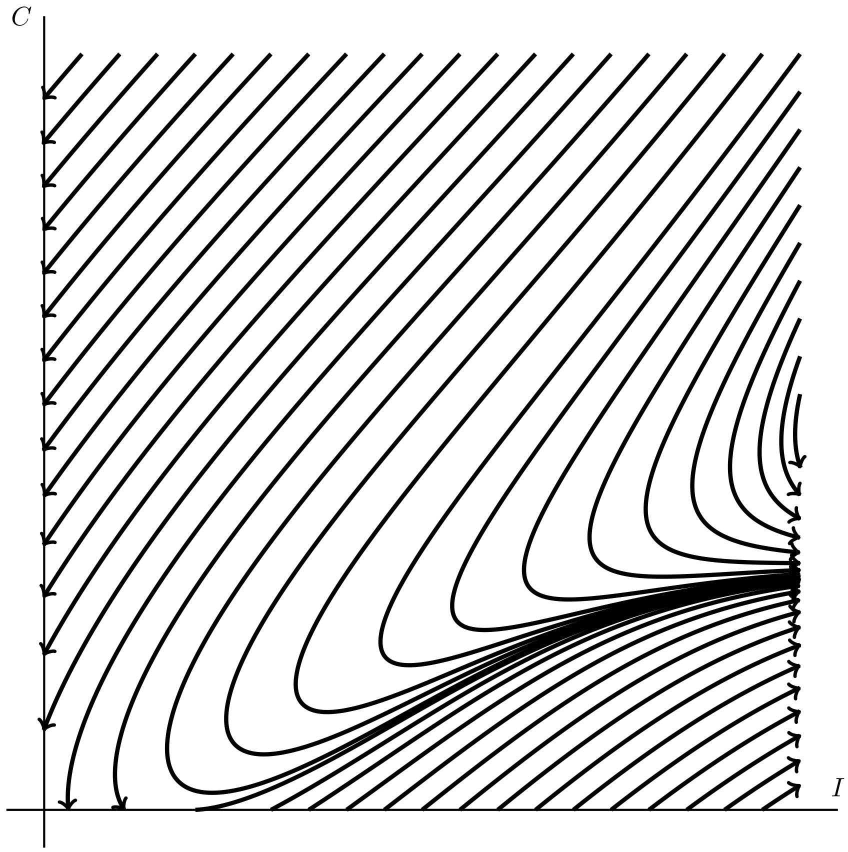
$\alpha = 2 \quad \beta = 2 \quad k = 0.05 \quad G_0 = 0.75$



$\alpha = 2 \quad \beta = 2 \quad k = 0.01 \quad G_0 = 1.25$



$\alpha = 2 \quad \beta = 2 \quad k = 0.01 \quad G_0 = 1.75$



In all cases, the trajectories can result in the national income I going negative, but the amount of consumer spending C is still positive. Also, it looks like the government spending lots of money is a good thing. With too little spending the economy can get stuck at a fixed point, but a lot of government spending with the right initial condition will result in the national income growing continually, with consumer spending not increasing much. This is the unstable manifold of the saddle point.

6.4.11

$$\dot{x} = rxz \qquad \dot{y} = ryz \qquad \dot{z} = -rxz - ryz$$

a)

$$x + y + z = 1 \quad \text{(The total sum of leftists, centrists, and rightists should add up to 1.)}$$
$$\dot{x} + \dot{y} + \dot{z} = rxz + ryz - rxz - ryz = 0$$

So the sum never changes, also known as invariant.

b)

$(x, y, z) = (x, y, 0)$ and $(0, 0, z)$ are the fixed points.

$$A = \begin{pmatrix} rz & 0 & rx \\ 0 & rz & ry \\ -rz & -rz & -rx - ry \end{pmatrix}$$

$$A_{(x,y,0)} = \begin{pmatrix} 0 & 0 & rx \\ 0 & 0 & ry \\ 0 & 0 & -rx - ry \end{pmatrix}$$

$$\lambda_1 = 0 \quad \lambda_2 = 0 \quad \lambda_3 = -rx - ry$$

$$A_{(0,0,z)} = \begin{pmatrix} rz & 0 & 0 \\ 0 & rz & 0 \\ -rz & -rz & 0 \end{pmatrix} \qquad \lambda_1 \qquad = 0 \quad \lambda_2 = 0 \quad \lambda_3 = rz$$

c)

For $r > 0$, a purely centrist government $z = 1$ is unstable. If there are any leftists or rightists, then the centrists are converted to each side with the limiting number of leftists and rightists determined by the initial values.

For $r < 0$, a purely centrist government $z = 1$ is stable. If there are no centrists $z = 0$, then the leftists and rightists will stay at their initial value because neither side can ever convert opposite members.

6.5 Conservative Systems

6.5.1

$$\ddot{x} = x^3 - x$$

a)

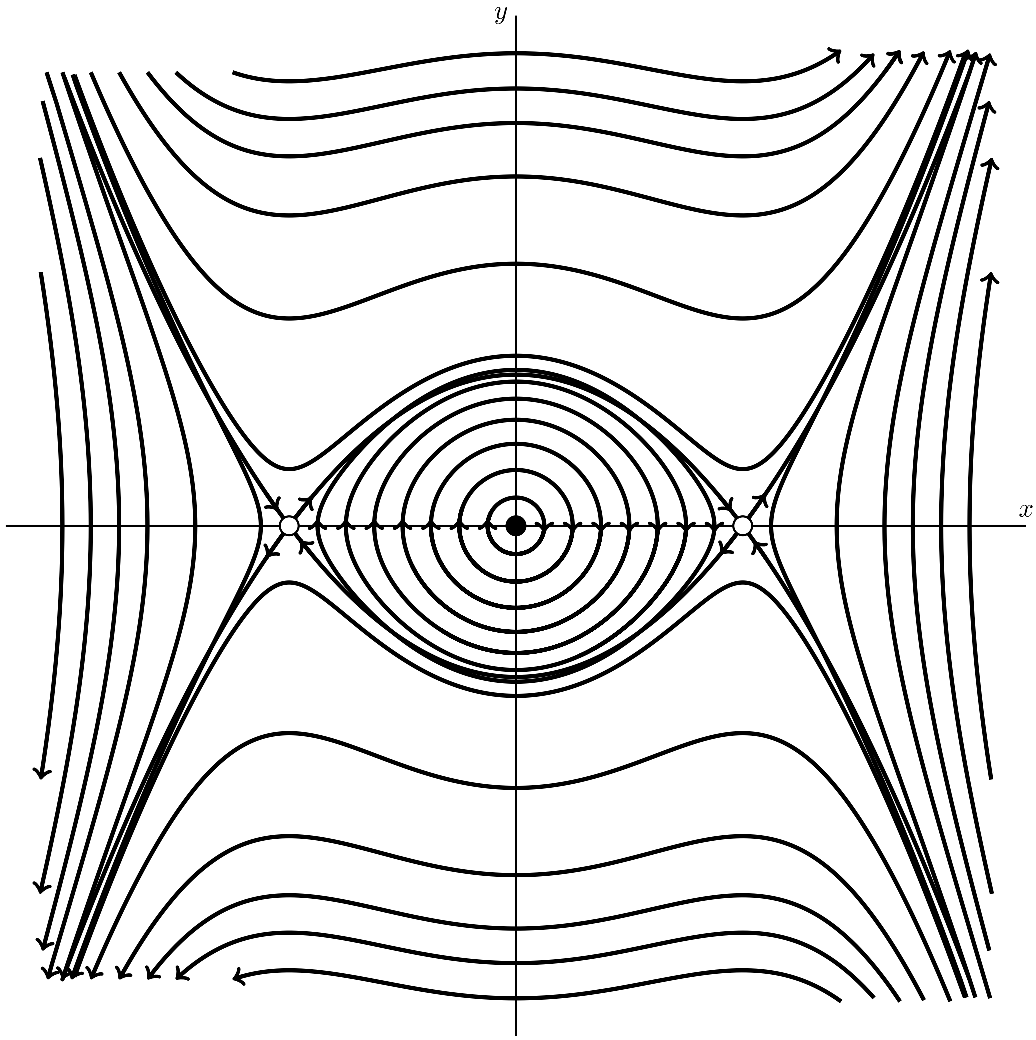
$$\dot{x} = y \qquad \dot{y} = x^3 - x$$

$(x, y) = (-1, 0)$, $(0, 0)$, and $(1, 0)$ are the fixed points.

b)

$$E = \frac{1}{2}\dot{x}^2 + \int -(x^3 - x)\, \mathrm{d}x = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4$$

c)



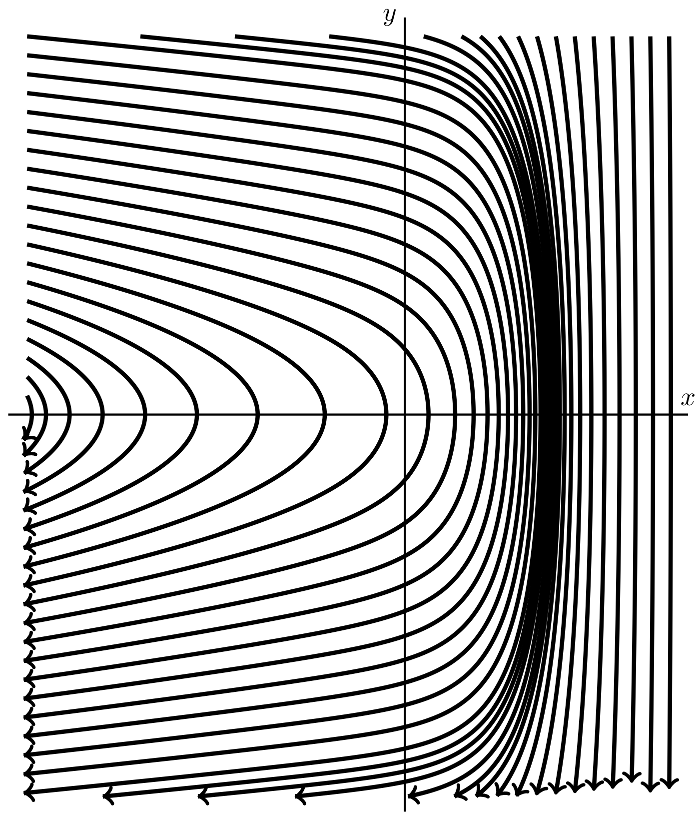
6.5.3

$$\ddot{x} = a - e^x$$

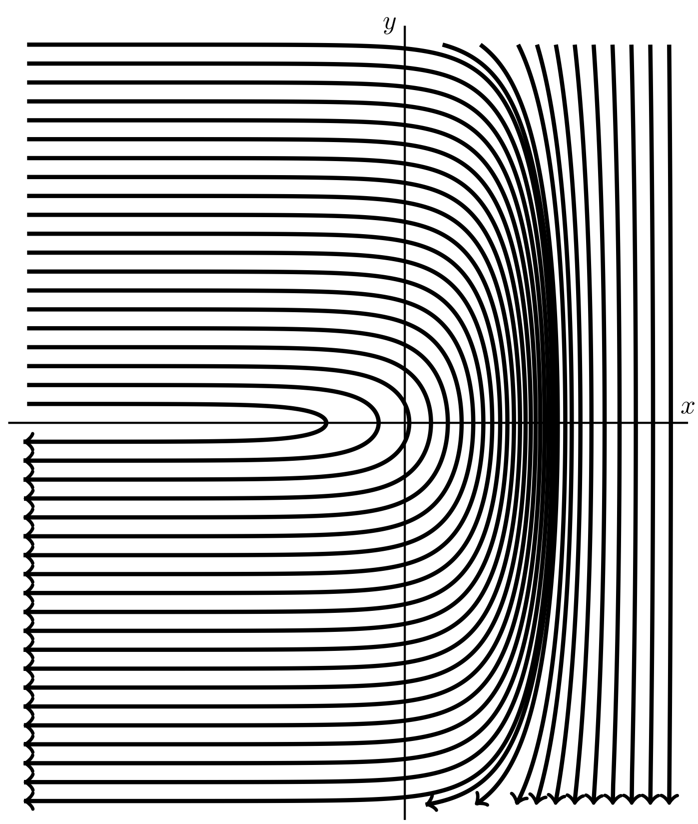
$$E = \frac{1}{2}\dot{x}^2 + \int -(a - e^x)\,\mathrm{d}x = \frac{1}{2}\dot{x}^2 - ax + e^x$$

$$\dot{x} = y \qquad \dot{y} = a - e^x$$

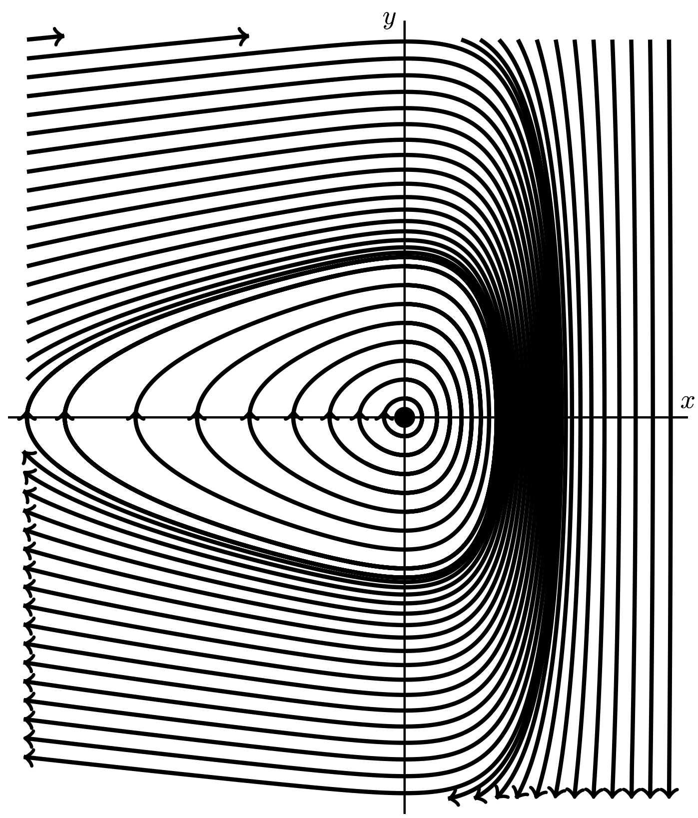
$a < 0$



$a = 0$



$a > 0$



6.5.5

$$\ddot{x} = (x - a)(x^2 - a)$$

$(x, \dot{x}) = (a, 0)$ and $(\pm\sqrt{a}, 0)$ are the fixed points.

$$\dot{x} = y \qquad \dot{y} = (x - a)(x^2 - a)$$

$$A = \begin{pmatrix} 0 & 1 \\ -2ax - a + 3x^2 & 0 \end{pmatrix}$$

$$A_{(a,0)} = \begin{pmatrix} 0 & 1 \\ a^2 - a & 0 \end{pmatrix} \qquad \lambda_{1,2} = \pm\sqrt{a^2 - a}$$

$$A_{(\sqrt{a},0)} = \begin{pmatrix} 0 & 1 \\ -2a^{\frac{3}{2}} + 2a & 0 \end{pmatrix} \qquad \lambda_{1,2} = \pm\sqrt{-2a^{\frac{3}{2}} + 2a}$$

$$A_{(-\sqrt{a},0)} = \begin{pmatrix} 0 & 1 \\ 2a^{\frac{3}{2}} + 2a & 0 \end{pmatrix} \qquad \lambda_{1,2} = \pm\sqrt{2a^{\frac{3}{2}} + 2a}$$

The eigenvalues tell us that the fixed points, if they are real, are saddle points. The only ones to be unsure about are when $a = -1, 0, 1$, which causes some eigenvalues to be zero.

We can determine what happens at these troublesome values of a by finding a conserved quantity E and classifying the critical points.

$$E = \frac{1}{2}\dot{x}^2 - \int (x-a)(x^2-a)\ \mathrm{d}x = \frac{1}{2}y^2 - \frac{1}{4}x^4 + \frac{a}{3}x^3 - \frac{a}{2}x^2 + a^2x$$

$$\begin{vmatrix} E_{xx} & E_{xy} \\ E_{yx} & E_{yy} \end{vmatrix} = \begin{vmatrix} -3x^2 + 2ax - a & y - x^3 + ax^2 - ax + a^2 \\ y - x^3 + ax^2 - ax + a^2 & 1 \end{vmatrix}$$

$a = -1$

$$\begin{vmatrix} E_{xx} & E_{xy} \\ E_{yx} & E_{yy} \end{vmatrix}_{(1,0)} = \begin{vmatrix} -4 & 0 \\ 0 & 1 \end{vmatrix} = -4 < 0 \Rightarrow \text{saddle point}$$

$a = 0$

$$\begin{vmatrix} E_{xx} & E_{xy} \\ E_{yx} & E_{yy} \end{vmatrix}_{(0,0)} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 < 0 \Rightarrow \text{inconclusive}$$

$a = 1$

$$\begin{vmatrix} E_{xx} & E_{xy} \\ E_{yx} & E_{yy} \end{vmatrix}_{(1,0)} = \begin{vmatrix} -2 & 0 \\ 0 & 1 \end{vmatrix} = -2 < 0 \Rightarrow \text{saddle point}$$

$$\begin{vmatrix} E_{xx} & E_{xy} \\ E_{yx} & E_{yy} \end{vmatrix}_{(-1,0)} = \begin{vmatrix} -6 & 4 \\ 4 & 1 \end{vmatrix} = -22 < 0 \Rightarrow \text{saddle point}$$

So everything was a saddle point, except possibly $(x, y) = (0, 0)$ when $a = 0$, but we can figure it out by looking at the conserved quantity.

$$E = \frac{1}{2}y^2 - \frac{1}{4}x^4$$

which is most definitely a saddle point since the x-axis cross section is concave down and the y-axis cross section is concave up.

6.5.7

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \alpha + \epsilon u^2 \qquad u = \frac{1}{r}$$

a)

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = v$$
$$\frac{\mathrm{d}v}{\mathrm{d}\theta} = \alpha + \epsilon u^2 - u$$

b)

$(u,v) = \left(\frac{1+\sqrt{1-4\alpha\epsilon}}{2\epsilon}, 0\right)$ and $\left(\frac{1-\sqrt{1-4\alpha\epsilon}}{2\epsilon}, 0\right)$ are the fixed points.

c)

$$A = \begin{pmatrix} 0 & 1 \\ 2\epsilon u - 1 & 0 \end{pmatrix}$$

$$A_{\left(\frac{1+\sqrt{1-4\alpha\epsilon}}{2\epsilon},0\right)} = \begin{pmatrix} 0 & 1 \\ \sqrt{1-4\alpha\epsilon} & 0 \end{pmatrix}$$

$$\Delta = -\sqrt{1-4\alpha\epsilon} \quad \tau = 0 \Rightarrow \text{Saddle point}$$

$$A_{\left(\frac{1-\sqrt{1-4\alpha\epsilon}}{2\epsilon},0\right)} = \begin{pmatrix} 0 & 1 \\ -\sqrt{1-4\alpha\epsilon} & 0 \end{pmatrix}$$

$$\Delta = \sqrt{1-4\alpha\epsilon} \quad \tau = 0 \Rightarrow \text{linear center}$$

This is also a nonlinear center by Theorem 6.5.1. The only difficult condition to check is that the fixed point is a local minimum of a conserved quantity.

$$E = \frac{1}{2}\dot{x}^2 + \int -(\alpha + \epsilon x^2 - x)\ \mathrm{d}x = \frac{1}{2}\dot{x}^2 - \alpha x - \frac{\epsilon}{3}x^3 + \frac{1}{2}x^2$$

$$\nabla E = \langle \alpha + \epsilon x^2 - x, \dot{x} \rangle$$

$$\nabla E\left(\frac{1-\sqrt{1-4\alpha\epsilon}}{2\epsilon}, 0\right) = (0,0)$$

$$\begin{vmatrix} E_{xx} & E_{x\dot{x}} \\ E_{\dot{x}x} & E_{\dot{x}\dot{x}} \end{vmatrix} = \begin{vmatrix} -2\epsilon & \dot{x} - \alpha - \epsilon x^2 + x \\ \dot{x} - \alpha - \epsilon x^2 + x & 1 \end{vmatrix}$$

$$\nabla E = \langle \alpha + \epsilon x^2 - x, \dot{x} \rangle$$

$$\nabla E\left(\frac{1-\sqrt{1-4\alpha\epsilon}}{2\epsilon}, 0\right) = (0,0)$$

$$\begin{vmatrix} E_{xx} & E_{x\dot{x}} \\ E_{\dot{x}x} & E_{\dot{x}\dot{x}} \end{vmatrix}_{\left(\frac{1-\sqrt{1-4\alpha\epsilon}}{2\epsilon},0\right)} = -2\epsilon < 0$$

$$\Rightarrow \left(\frac{1-\sqrt{1-4\alpha\epsilon}}{2\epsilon}, 0\right) \text{ is a local minimum}$$

d)

If the fixed point corresponds to a circular orbit, then the radius is constant.

$$\frac{1}{r} = u = \frac{1 - \sqrt{1 - 4\alpha\epsilon}}{2\epsilon} \Rightarrow r = \frac{2\epsilon}{1 - \sqrt{1 - 4\alpha\epsilon}}$$

which is a constant.

6.5.9

If $H(x, p)$ is a conserved quantity, then H is constant in time.

$$\frac{\mathrm{d}}{\mathrm{d}t} H(x, p) = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial p}\dot{p} = \dot{p}\dot{x} - \dot{x}\dot{p} = 0$$

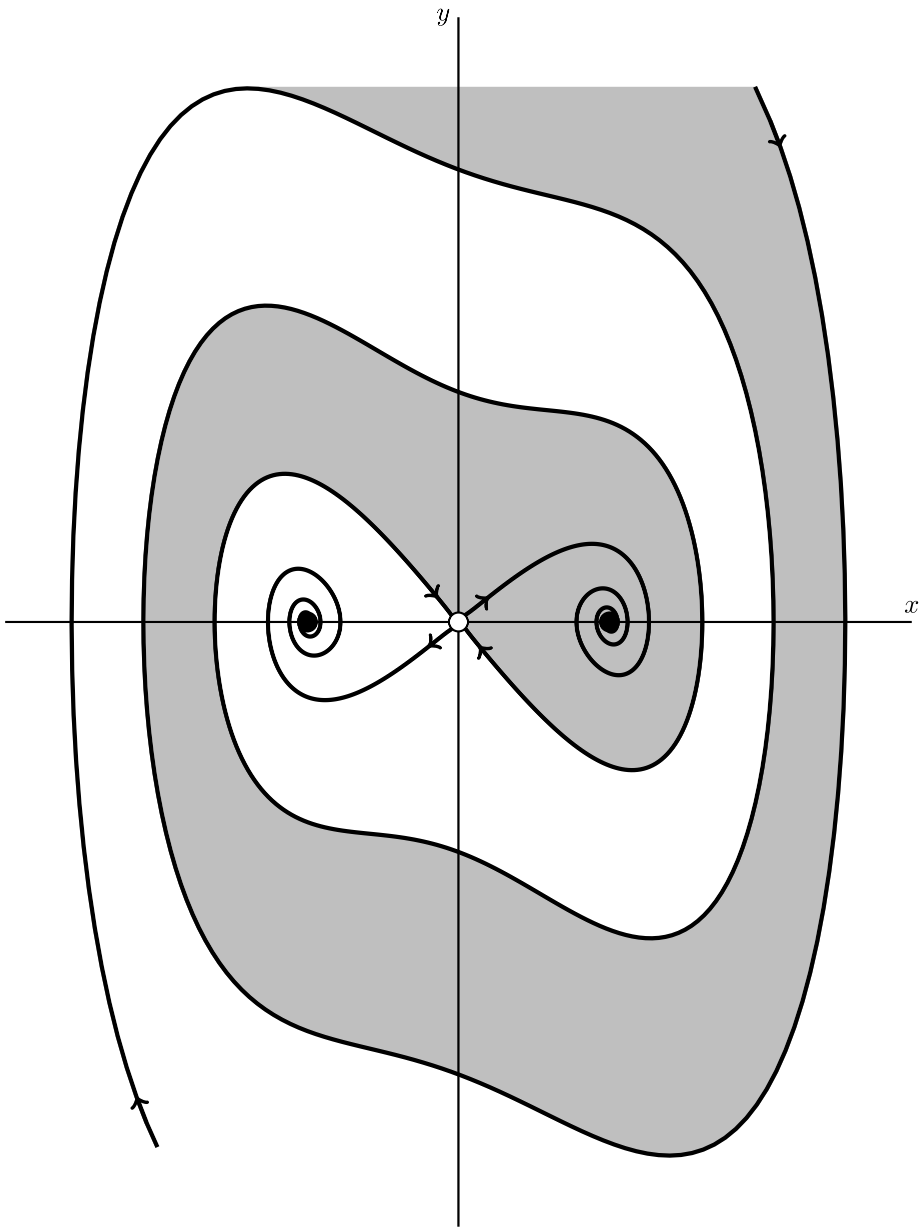
So $\dot{H} = 0$ and H is a conserved quantity.

6.5.11

$$\dot{x} = y \qquad \dot{y} = -by + x - x^3 \qquad 0 < b \ll 1$$

Adding the small amount of damping to the double-well oscillator of Example 6.5.2 changes the nonlinear centers to stable spirals and destroys the homoclinic orbits of the saddle point.

Using a bit of intuition, we know all the trajectories will go to either the left or right stable spiral, with the exception of the stable manifold of the saddle point. The two trajectories leading to the saddle point form the separatrices for the basins of attraction of each spiral.



6.5.13

$$\ddot{x} + x + \epsilon x^3 = 0$$

a)

By checking that the isolated fixed point $(x, \dot{x}) = (0, 0)$ is the minimum of a conserved quantity

$$E = \frac{1}{2}\dot{x}^2 - \int (-x - \epsilon x^3)\ \mathrm{d}x = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{\epsilon}{4}x^4$$

$$\begin{vmatrix} E_{xx} & E_{x\dot{x}} \\ E_{\dot{x}x} & E_{\dot{x}\dot{x}} \end{vmatrix}_{(0,0)} = \begin{vmatrix} 1 + 3\epsilon x^2 & \dot{x} + x + \epsilon x^3 \\ \dot{x} + x + \epsilon x^3 & 1 \end{vmatrix}_{(0,0)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 > 0$$

So $E_{xx}(0,0) = 1 > 0$ means the fixed point is a local minimum and therefore a nonlinear center of the system.

b)

The origin will still be a nonlinear center for any value of ϵ because the origin is always a local minimum of E and therefore will be surrounded by closed trajectories. However, we don't know how far away those closed trajectories exist for any value of ϵ, or even if the value of ϵ matters at all.

To find out, we can look at the level sets of E for arbitrary ϵ.

$$\begin{aligned}
E &= \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{\epsilon}{4}x^4 \\
4E\epsilon &= 2\epsilon\dot{x}^2 + 2\epsilon x^2 + \epsilon^2 x^4 \\
&= 2\epsilon\dot{x}^2 + (\epsilon^2 x^4 + 2\epsilon x^2 + 1) - 1 \\
&= 2\epsilon\dot{x}^2 + (\epsilon x^2 + 1)^2 - 1 \\
4E\epsilon + 1 &= 2\epsilon\dot{x}^2 + (\epsilon x^2 + 1)^2
\end{aligned}$$

This equation looks a lot like, but not quite due to the x exponent, the equations for the conic sections (hyperbola, parabola, ellipse, circle). Even more interesting is that by varying ϵ we can get all the different conic-like sections as level sets.

$\epsilon > 0 \Rightarrow$ ellipse-like trajectories

$\epsilon = 0 \Rightarrow (x, \dot{x}) = (0, 0)$

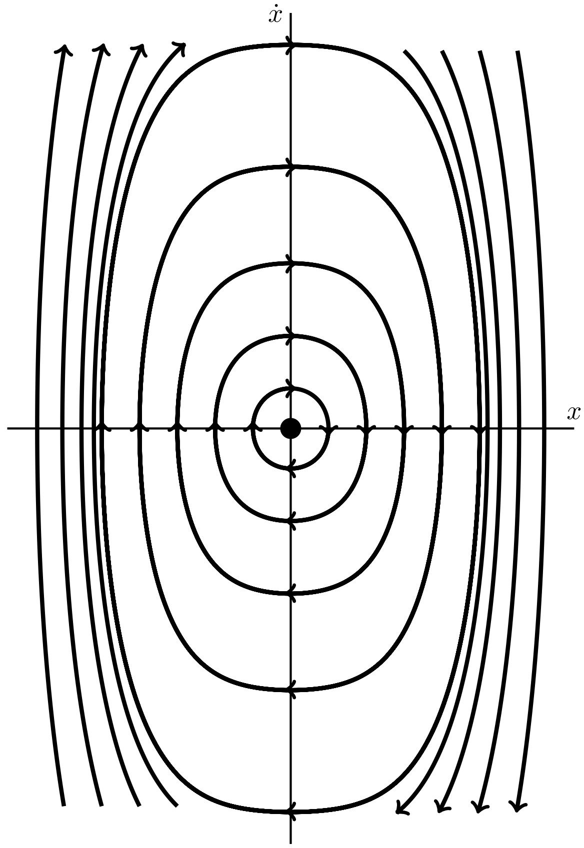
$\frac{-1}{4E} < \epsilon < 0 \Rightarrow$ hyperbola-like trajectories

$\epsilon = \frac{-1}{4E} \Rightarrow (x, \dot{x}) = \left(\frac{\pm 1}{\sqrt{-\epsilon}}, 0\right) = \left(\pm 2\sqrt{E}, 0\right)$

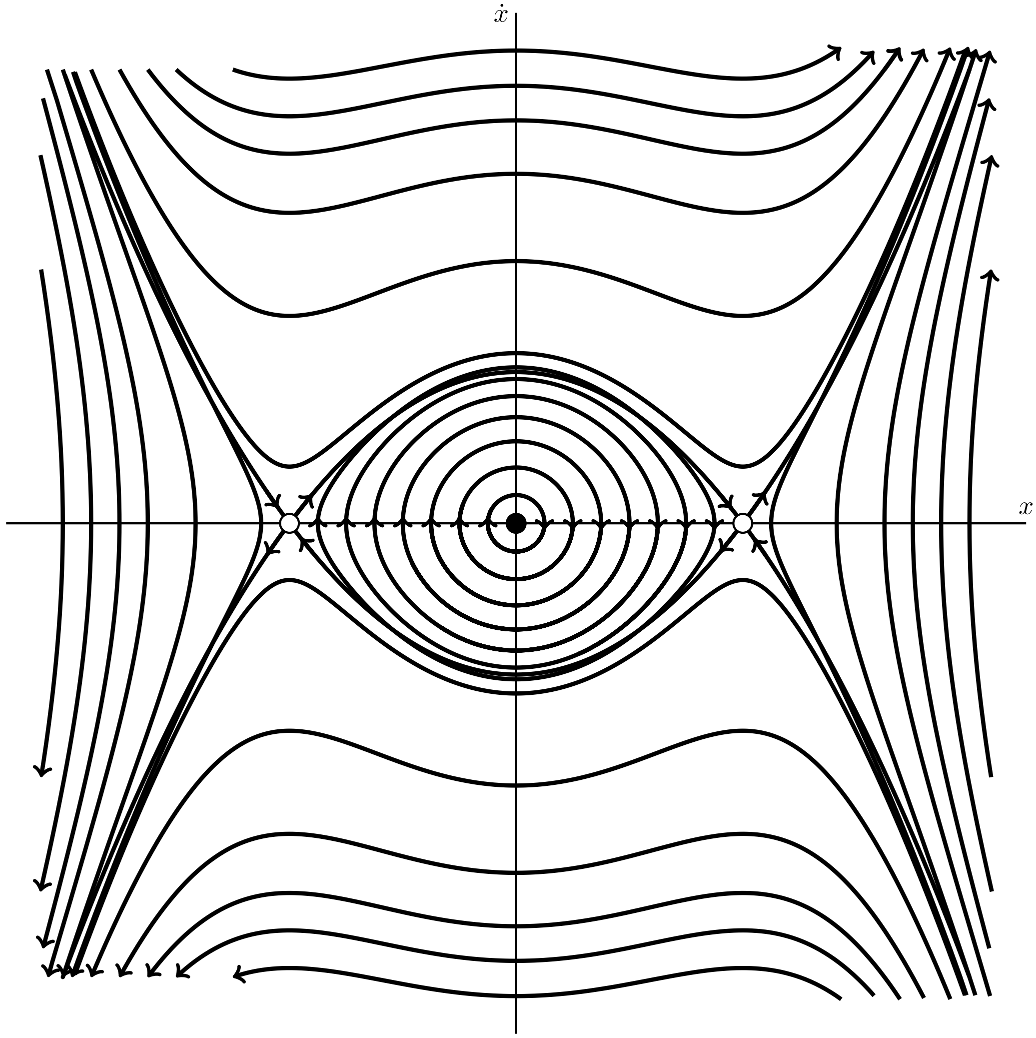
$\epsilon < \frac{-1}{4E} \Rightarrow$ hyperbola-like trajectories

Next we can look at the graphs to see what these conic-like sections actually look like.

Here $\epsilon > 0$ and all trajectories loop around the origin no matter how far away we are.



$\epsilon < 0$



By picking different initial conditions we specify different values of E and can transition from $\dfrac{-1}{4E} < \epsilon < 0$ to $\epsilon = \dfrac{-1}{4E}$ to $\epsilon < \dfrac{-1}{4E}$.

We can see that the origin is still a nonlinear center, but the trajectories are no longer closed after going past the heteroclinic trajectories.

6.5.15

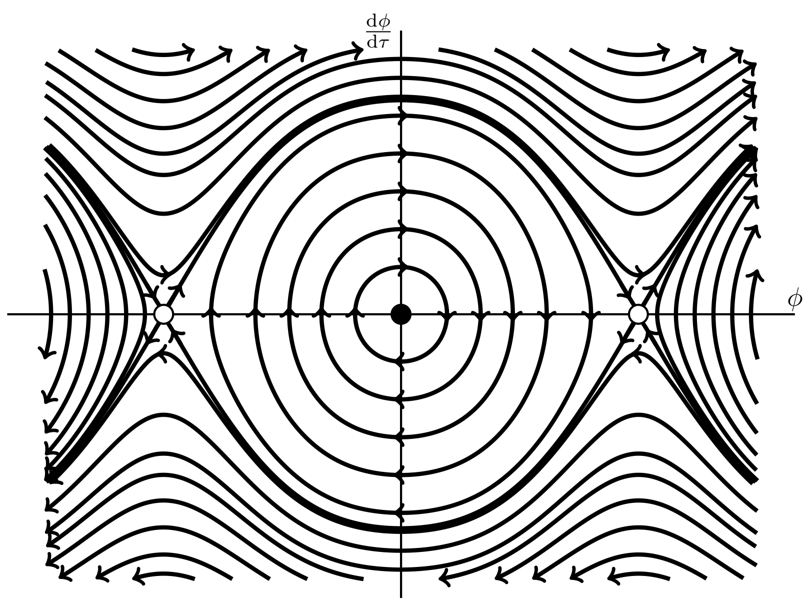
a)

$$
\begin{aligned}
mr\ddot{\phi} &= -mg\sin(\phi) + mr\omega^2\sin(\phi)\cos(\phi) \\
\frac{1}{\omega^2}\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} &= \sin(\phi)\left(\cos(\phi) - \frac{g}{r\omega^2}\right) \\
\tau = \omega t \qquad & \gamma = \frac{r\omega^2}{g} \\
\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} &= \sin(\phi)\left(\cos(\phi) - \frac{1}{\gamma}\right)
\end{aligned}
$$

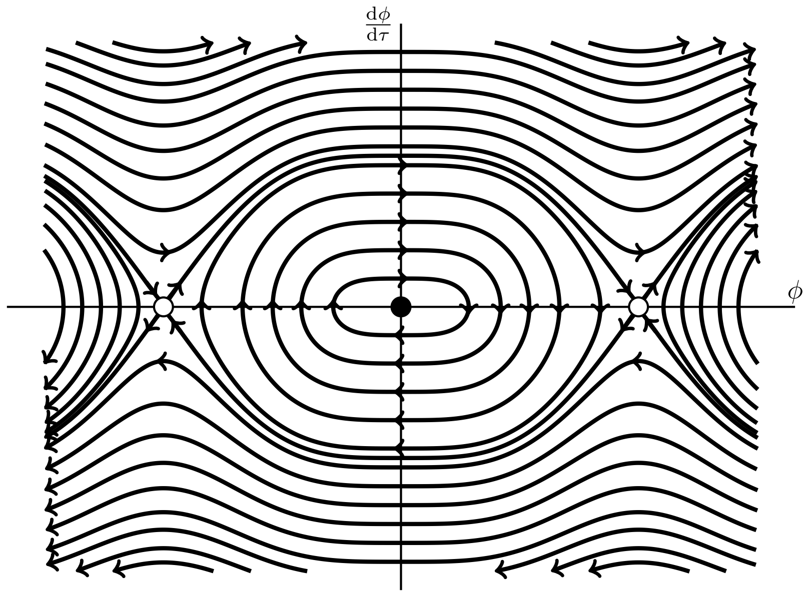
b)

$\left(\phi, \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right) = (z\pi, 0)$ with $z \in \mathbb{Z}$ and $\left(\pm\arccos\left(\frac{1}{\gamma}\right), 0\right)$ if $\frac{1}{\gamma} < 1$ are the fixed points.

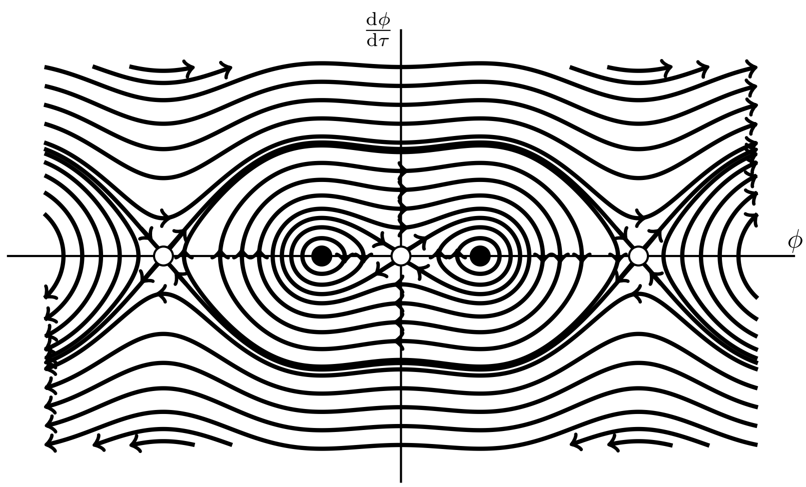
$\frac{1}{\gamma} > 1$



$\frac{1}{\gamma} = 1$



$0 < \frac{1}{\gamma} < 1$



c)

The phase portraits imply that the bead either oscillates in position as in one of the closed trajectories, or the bead keeps making loops around the entire hoop as in the upper and lower periodic trajectories.

6.5.17

$$
\begin{aligned}
mr\ddot{\phi} &= -mg\sin(\phi) + mr\omega^2 \sin(\phi)\cos(\phi) \\
\ddot{\phi} &= \omega^2 \sin(\phi) \left(\cos(\phi) - \frac{g}{r\omega^2}\right) \\
E &= \frac{1}{2}\dot{\phi}^2 - \int \omega^2 \sin(\phi) \left(\cos(\phi) - \frac{g}{r\omega^2}\right) \mathrm{d}\phi \\
&= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega^2 \cos(\phi) \left(\cos(\phi) - \frac{2g}{r\omega^2}\right) \\
&= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega^2 \cos^2(\phi) - \frac{g}{r}\cos(\phi)
\end{aligned}
$$

We define the bottom of the hoop as zero potential energy. Then the potential energy of the bead is mass times gravity times the vertical distance that the bead has traveled.

$$\mathrm{PE} = mgr(1 - \cos(\phi))$$

There are two kinetic energies to calculate. One is from the velocity that the bead is moving in the plane of the hoop. We can calculate that using the rate of change of angle $\dot{\phi}$ and the radius r.

$$\mathrm{KE}_{\text{translational}} = \frac{1}{2}mv^2 = \frac{1}{2}m(r\dot{\phi})^2$$

The other is from the bead's rotation around the vertical rotation axis of the hoop, which depends on the hoop's rotation rate and the bead's horizontal distance from the rotation axis.

$$\mathrm{KE}_{\text{rotational}} = \frac{1}{2}mv^2 = \frac{1}{2}m(r\sin(\phi)\omega)^2$$

Then we add these together to get the total kinetic energy.

$$\mathrm{KE} = \frac{1}{2}mv^2 = \frac{1}{2}m(r\dot{\phi})^2 + \frac{1}{2}m(r\sin(\phi)\omega)^2$$

which leads us to the bead's total energy.

$$E_{\text{bead}} = \mathrm{PE} + \mathrm{KE} = mgr(1 - \cos(\phi)) + \frac{1}{2}m(r\dot{\phi})^2 + \frac{1}{2}m(r\sin(\phi)\omega)^2$$

If we check whether $\dot{E}_{\text{bead}} = 0$, we find that

$$
\begin{aligned}
\dot{E}_{\text{bead}} &= mr\dot{\phi}\Big(g\sin(\phi) + r\ddot{\phi} + r\omega^2 \sin(\phi)\cos(\phi)\Big) \\
&= mr\dot{\phi}\left(g\sin(\phi) + r\omega^2 \sin(\phi)\left(\cos(\phi) - \frac{g}{r\omega^2}\right) + r\omega^2 \sin(\phi)\cos(\phi)\right) \\
&= 2mr^2\omega^2 \sin(\phi)\cos(\phi)\dot{\phi} \neq 0
\end{aligned}
$$

So the bead's total energy is not constant. This actually makes sense because the hoop transfers energy to the bead when it moves away from the bottom. This hoop has a constant rotation rate ω, so whatever is rotating the hoop has to put in more energy to maintain the constant rotation rate when the bead gets farther away from the rotation axis and take away energy when the bead gets closer to the rotation axis.

As for what the real conserved quantity is actually conserving,

$$\begin{aligned}
E &= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega^2\cos^2(\phi) - \frac{g}{r}\cos(\phi) \\
Emr^2 &= \frac{1}{2}m(r\dot{\phi})^2 + \frac{1}{2}mr^2\omega^2\cos^2(\phi) - mgr\cos(\phi) \\
&= \mathrm{KE}_{\text{translational}} + \frac{1}{2}mr^2\omega^2\big(1-\sin^2(\phi)\big) + \mathrm{PE} - mgr \\
Emr^2 - \frac{1}{2}mr^2\omega^2 + mgr &= \mathrm{KE}_{\text{translational}} - \frac{1}{2}m\big(r\sin(\phi)\omega\big)^2 + \mathrm{PE} \\
Emr^2 - \frac{1}{2}mr^2\omega^2 + mgr &= \mathrm{KE}_{\text{translational}} - \mathrm{KE}_{\text{rotational}} + \mathrm{PE}
\end{aligned}$$

If we define a new conserved quantity by lumping all the constants together on the other side (keeping in mind there is a new zero for potential energy), we get

$$E = \mathrm{PE} + \mathrm{KE}_{\text{translational}} - \mathrm{KE}_{\text{rotational}}$$

This equation makes sense. The bead has an initial amount of energy in the rotating frame, and the hoop-rotating energy source transfers energy to and from the bead's total energy in the rotating frame. A conserved quantity of energy is what the bead starts with in the rotating frame minus any energy the hoop-rotating energy source transfers to the bead.

The conserved quantity is H, the "Hamiltonian" for the system. When the potential energy V is a function only of the coordinates for a stationary coordinate system, H is the total energy. But for problems with moving constraints (as here), H is not the total energy. The force moving the hoop is doing work on it, and that work (= energy) needs to be included in the energy accounting. Another way to handle the problem is to go into a rotating frame with the spinning hoop; then you get a fictitious centrifugal force and its associated potential energy. This lets you pretend the system has stationary coordinates, and in that frame the conserved quantity represents the total "energy" of the system.

6.5.19

$$\dot{R} = aR - bRF \qquad \dot{F} = -cF + dRF \qquad a, b, c, d > 0$$

a)

a represents the growth rate of the rabbits.

b represents the probability for a rabbit to be eaten when encountering a fox, assuming rabbits can only die by being eaten.

c represents the death rate for foxes due to illness, accidents, etc.

d represents the rate at which rabbits can be converted into foxes when eaten.

This model is unrealistic in that the rabbits have no carrying capacity in the absence of foxes, the foxes would all die if there were no rabbits (meaning there is nothing else for the foxes to eat), the foxes might mate seasonally, both species could have a threshold population above and below which the species would grow and die off respectively, etc.

b)

$$\frac{\mathrm{d}R}{\mathrm{d}t} = aR - bRF \qquad \frac{\mathrm{d}F}{\mathrm{d}t} = -cF + dRF$$
$$\frac{1}{a}\frac{\mathrm{d}R}{\mathrm{d}t} = R\left(1 - \frac{b}{a}F\right) \qquad \frac{\mathrm{d}F}{\mathrm{d}t} = cF\left(\frac{d}{c}R - 1\right)$$
$$x = \frac{d}{c}R \quad y = \frac{b}{a}F \quad \tau = at$$
$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = x(1-y) \qquad a\frac{\mathrm{d}y}{\mathrm{d}\tau} = cy(x-1)$$
$$\mu = \frac{c}{a}$$
$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = x(1-y) \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau} = \mu y(x-1)$$

c)

Believe it or not, we can use the spirit of solving exact equations and separable equations from an introductory ordinary differential equations course to find the conserved quantity.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}E(x,y) =& E_x x' + E_y y' \\
=& E_x x(1-y) + E_y \mu y(x-1) = 0 \\
& \frac{1-y}{y}\mathrm{d}y + \mu\frac{x-1}{x}\mathrm{d}x = 0 \\
& \frac{1-y}{y}\mathrm{d}y = \mu\frac{1-x}{x}\mathrm{d}x \\
& \int \frac{1}{y} - 1 \ \mathrm{d}y = \mu \int \frac{1}{x} - 1 \ \mathrm{d}x \\
& \ln|y| - y = \mu(\ln|x| - x) + C \\
& \mu x - y - \mu\ln|x| - \ln|y| = C
\end{aligned}$$

This expression equals a constant whose value is determined by a point (x_0, y_0) on the curve, so the LHS must be the conserved quantity. (We can also drop the absolute value signs since we only consider nonnegative populations of rabbits and foxes.)

$$E(x,y) = \mu x - y - \mu\ln(x) - \ln(y)$$

d)

$(x, y) = (0, 0)$ and $(1, 1)$ are the fixed points.

We can show that the fixed point $(1,1)$ is a nonlinear center since we know the conserved quantity along with the theorem in the section. In fact, everything in the first quadrant is a closed orbit around $(1,1)$ except for trajectories starting on the axes, which go to infinity on that axis. This can be verified by showing that $(1,1)$ is a global minimum and each level set of E is bounded in the first quadrant. $\big(x - \ln(x)$ and $y - \ln(y)$ grow unbounded near their axes and far away from their axes.$\big)$

6.6 Reversible Systems

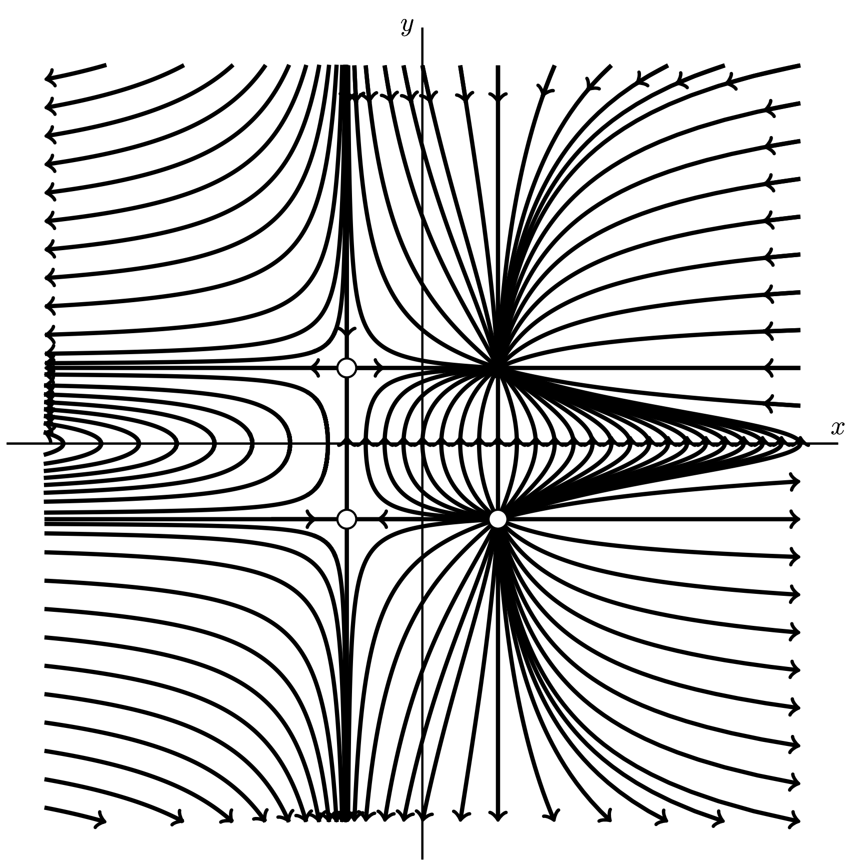
6.6.1

We can prove the system is reversible by verifying that $f(x,y)$ is odd in y and $g(x,y)$ is even in y.

$$\dot{x} = y(1-x^2) = f(x,y) \qquad f(x,-y) = -y(1-x^2) = -f(x,y)$$
$$\dot{y} = 1 - y^2 = g(x,y) \qquad g(x,-y) = 1 - (-y)^2 = 1 - y^2 = g(x,y)$$

So the system is reversible.

$(x,y) = (1,1)$, $(1,-1)$, $(-1,1)$, and $(-1,-1)$ are the fixed points.



6.6.3

a)

$$\dot{x} = \sin(y) \qquad \dot{y} = \sin(x)$$

We can prove the system is reversible by verifying that $f(x, y)$ is odd in y and $g(x, y)$ is even in y.

$$f(x,y) = \sin(y) \qquad f(x,-y) = \sin(-y) = -\sin(y) = -f(x,y)$$
$$g(x,y) = \sin(x) \qquad g(x,-y) = \sin(x) = g(x,y)$$

So the system is reversible.

b)

$(x, y) = (z_1\pi, z_2\pi)$ are the fixed points. $\quad z_1, z_2 \in \mathbb{Z}$

$$A = \begin{pmatrix} 0 & \cos(y) \\ \cos(x) & 0 \end{pmatrix}$$

The eigenvalues are $\lambda_{1,2} = \pm 1$, which is a saddle point if z_1 and z_2 have the same parity.

The eigenvalues are $\lambda_{1,2} = \pm i$, which is a center if z_1 and z_2 have the opposite parity.

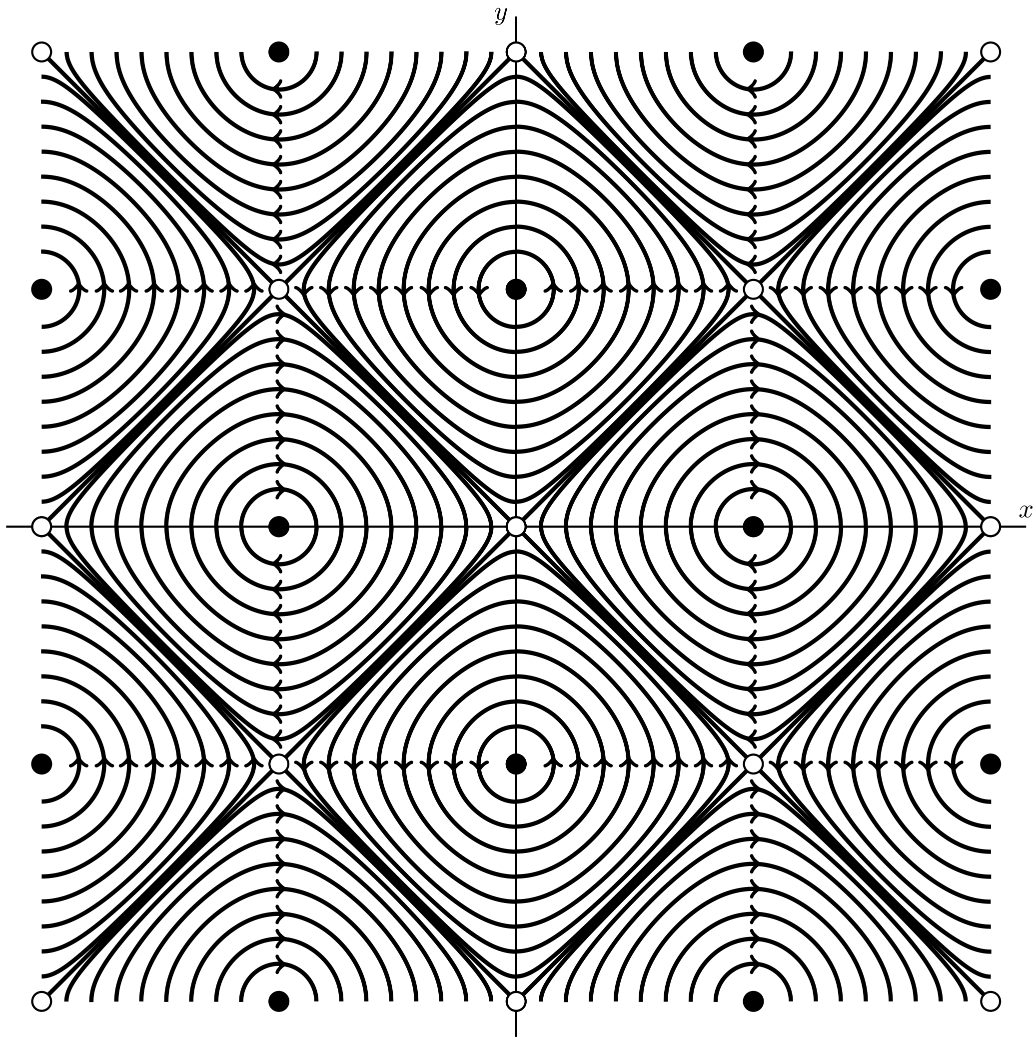
c)

If the lines $y = \pm x$ are invariant, then the derivative will be zero.

$$\begin{aligned} y = x \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(y - x) &= \dot{y} - \dot{x} = \sin(x) - \sin(y) \\ &= \sin(x) - \sin(x) = 0 \\ y = -x \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(y + x) &= \dot{y} + \dot{x} = \sin(x) + \sin(y) = \sin(x) + \sin(-x) \\ &= \sin(x) - \sin(x) = 0 \end{aligned}$$

Hence the trajectories that start on $y = \pm x$ stay on that line forever.

d)



6.6.5

$$\ddot{x} + f(\dot{x}) + g(x) = 0 \qquad f \text{ is even and } f, g \text{ are smooth.}$$

a)

$$\begin{aligned}
\ddot{x}(t) + f\big(\dot{x}(t)\big) + g\big(x(t)\big) &= \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(t) + f\left(\frac{\mathrm{d}x}{\mathrm{d}t}(t)\right) + g\big(x(t)\big) = 0 \\
t \to -t \Rightarrow & \frac{1}{(-1)^2}\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(-t) + f\left(\frac{1}{-1}\frac{\mathrm{d}x}{\mathrm{d}t}(-t)\right) + g\big(x(-t)\big) \\
& \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(-t) + f\left(-\frac{\mathrm{d}x}{\mathrm{d}t}(-t)\right) + g\big(x(-t)\big) \\
& \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(-t) + f\left(\frac{\mathrm{d}x}{\mathrm{d}t}(-t)\right) + g\big(x(-t)\big) = 0
\end{aligned}$$

So $x(-t)$ is also a solution if $x(t)$ is a solution.

b)

The system transforms into

$$\dot{x} = y \qquad \dot{y} = -f(y) - g(x)$$

Since the system is reversible, the trajectories for $y < 0$ are reflections of trajectories with $y > 0$ with the arrows reversed.

Now supposing we have a trajectory with $y > 0$ leading into a stable node or spiral, we know that the trajectory has to always travel towards the fixed point sufficiently close to the fixed point. However, the twin trajectory with $y < 0$ will necessarily travel away from the fixed point because the arrows are reversed. This is a contradiction because the trajectory has to travel toward the fixed point whether or not y is positive or negative.

So a reversible system will never contain stable nodes or spirals.

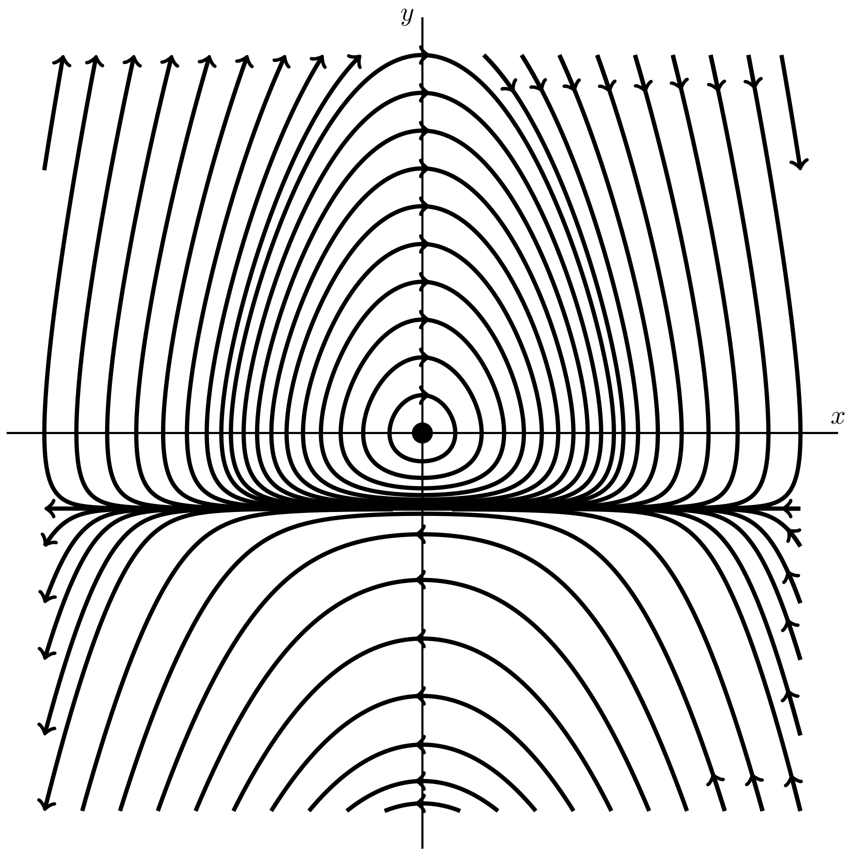
6.6.7

$$\ddot{x} + x\dot{x} + x = 0$$

This system's symmetry is across the $\dot{x}$-axis. As a consequence, we'll check that $f(x, y)$ and $g(x, y)$ are even and odd in x respectively.

$$\dot{x} = y = f(x, y) \qquad f(-x, y) = y = f(x, y)$$

$$\dot{y} = -xy - x = g(x, y) \qquad g(-x, y) = xy + x = -g(x, y)$$



6.6.9

$$\frac{\mathrm{d}\phi_k}{\mathrm{d}\tau} = \Omega + a\sin(\phi_k) + \frac{1}{N}\sum_{j=1}^{N}\sin(\phi_j) \quad \text{for } k = 1, 2$$

a)

$$\theta_k = \phi_k - \frac{\pi}{2}$$

$$\frac{\mathrm{d}\phi_k}{\mathrm{d}\tau} = \Omega + a\sin\left(\theta_k + \frac{\pi}{2}\right) + \frac{1}{N}\sum_{j=1}^{N}\sin\left(\theta_j + \frac{\pi}{2}\right)$$

$$\frac{\mathrm{d}\theta_k}{\mathrm{d}\tau} = \Omega + a\cos(\theta_k) + \frac{1}{N}\sum_{j=1}^{N}\cos(\theta_j)$$

Applying the transformation

$$\theta_k \to -\theta_k \quad \tau \to -\tau$$

$$\frac{-1}{-1}\frac{\mathrm{d}\theta_k}{\mathrm{d}\tau} = \Omega + a\cos(-\theta_k) + \frac{1}{N}\sum_{j=1}^{N}\cos(-\theta_j)$$

$$\frac{\mathrm{d}\theta_k}{\mathrm{d}\tau} = \Omega + a\cos(\theta_k) + \frac{1}{N}\sum_{j=1}^{N}\cos(\theta_j)$$

results in the same equations, so the system is reversible.

b)

$$\frac{\mathrm{d}\theta_k}{\mathrm{d}\tau} = 0 = \Omega + a\cos(\theta_k) + \frac{1}{N}\sum_{j=1}^{N}\cos(\theta_j)$$

$$\Omega + a\cos(\theta_k) + \frac{1}{2}\big(\cos(\theta_1) + \cos(\theta_2)\big) = 0$$

$$2a\cos(\theta_k) + \cos(\theta_1) + \cos(\theta_2) = -2\Omega$$

$$(2a+1)\cos(\theta_1) + \cos(\theta_2) = -2\Omega$$

$$\cos(\theta_1) + (2a+1)\cos(\theta_2) = -2\Omega$$

$$(2a+1)^2\cos(\theta_1) + (2a+1)\cos(\theta_2) = -2(2a+1)\Omega$$

$$-\cos(\theta_1) - (2a+1)\cos(\theta_2) = 2\Omega$$

$$\big((2a+1)^2 - 1\big)\cos(\theta_1) = 2\Omega - 2(2a+1)\Omega$$

$$(4a^2 + 4a + 1 - 1)\cos(\theta_1) = (2 - 4a - 2)\Omega$$

$$4a(a+1)\cos(\theta_1) = -4a\Omega$$

$$\cos(\theta_1) = \frac{-\Omega}{a+1}$$

which has two solutions if $\left|\dfrac{\Omega}{a+1}\right| < 1$,

one solution if $\left|\dfrac{\Omega}{a+1}\right| = 1$,

and zero solutions if $\left|\dfrac{\Omega}{a+1}\right| > 1$.

$$(2a+1)\cos(\theta_1) + \cos(\theta_2) = -2\Omega$$

$$\cos(\theta_1) + (2a+1)\cos(\theta_2) = -2\Omega$$

$$-\cos(\theta_2) - (2a+1)\cos(\theta_1) = 2\Omega$$

$$(2a+1)^2\cos(\theta_2) + (2a+1)\cos(\theta_1) = -2(2a+1)\Omega$$

$$\vdots$$

$$\cos(\theta_2) = \frac{-\Omega}{a+1}$$

which has the same conditions on the number of solutions.

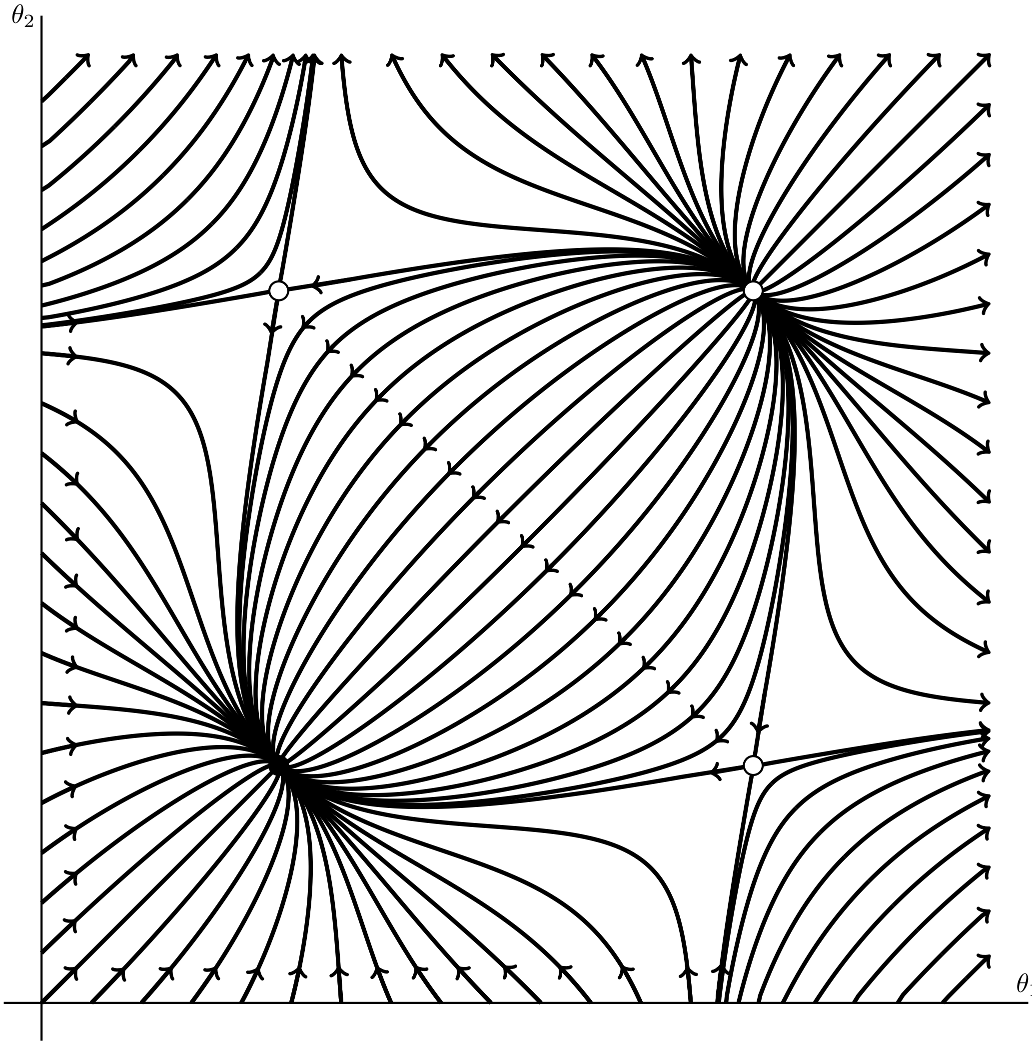
There are four (θ_1, θ_2) fixed points if $\left|\dfrac{\Omega}{a+1}\right| < 1$.

There is one (θ_1, θ_2) fixed point if $\left|\dfrac{\Omega}{a+1}\right| = 1.$

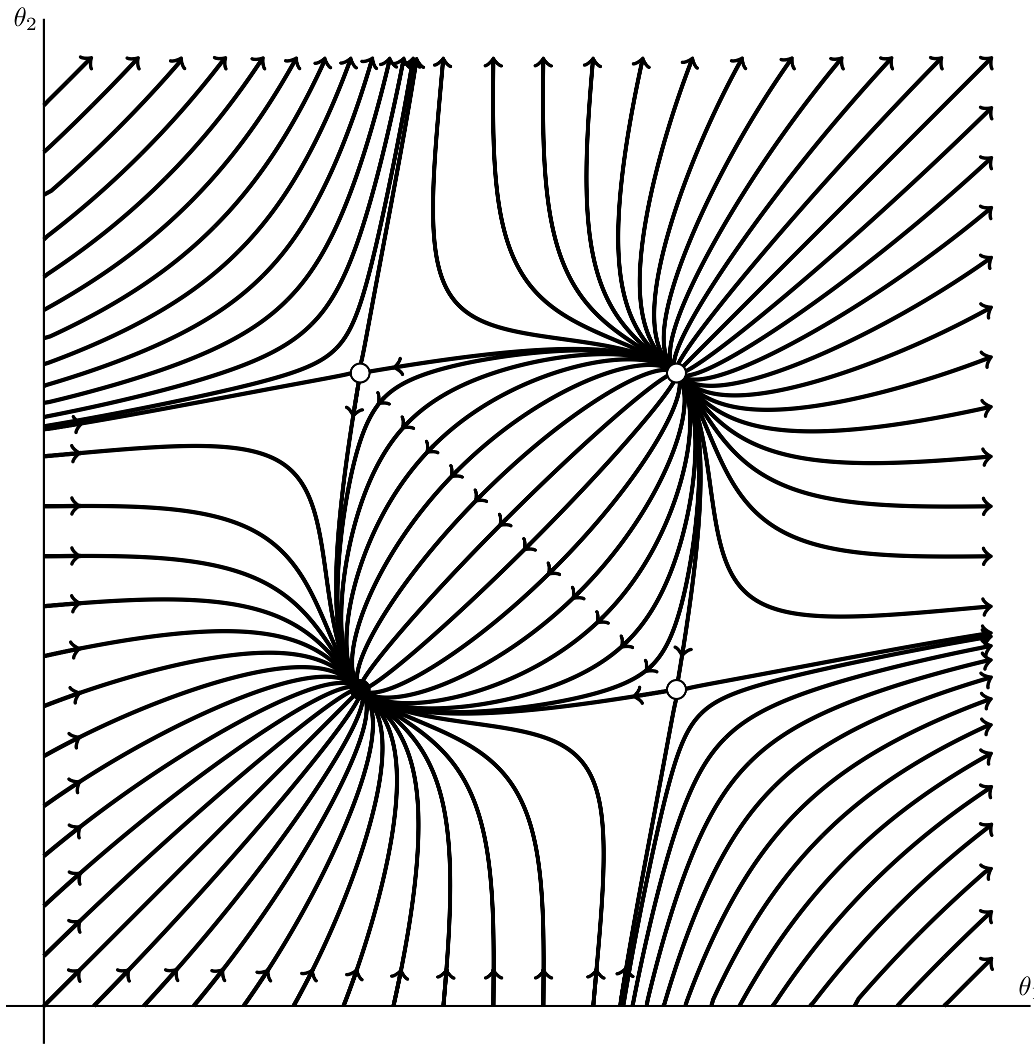
There are zero (θ_1, θ_2) fixed points if $\left|\dfrac{\Omega}{a+1}\right| > 1.$

c)

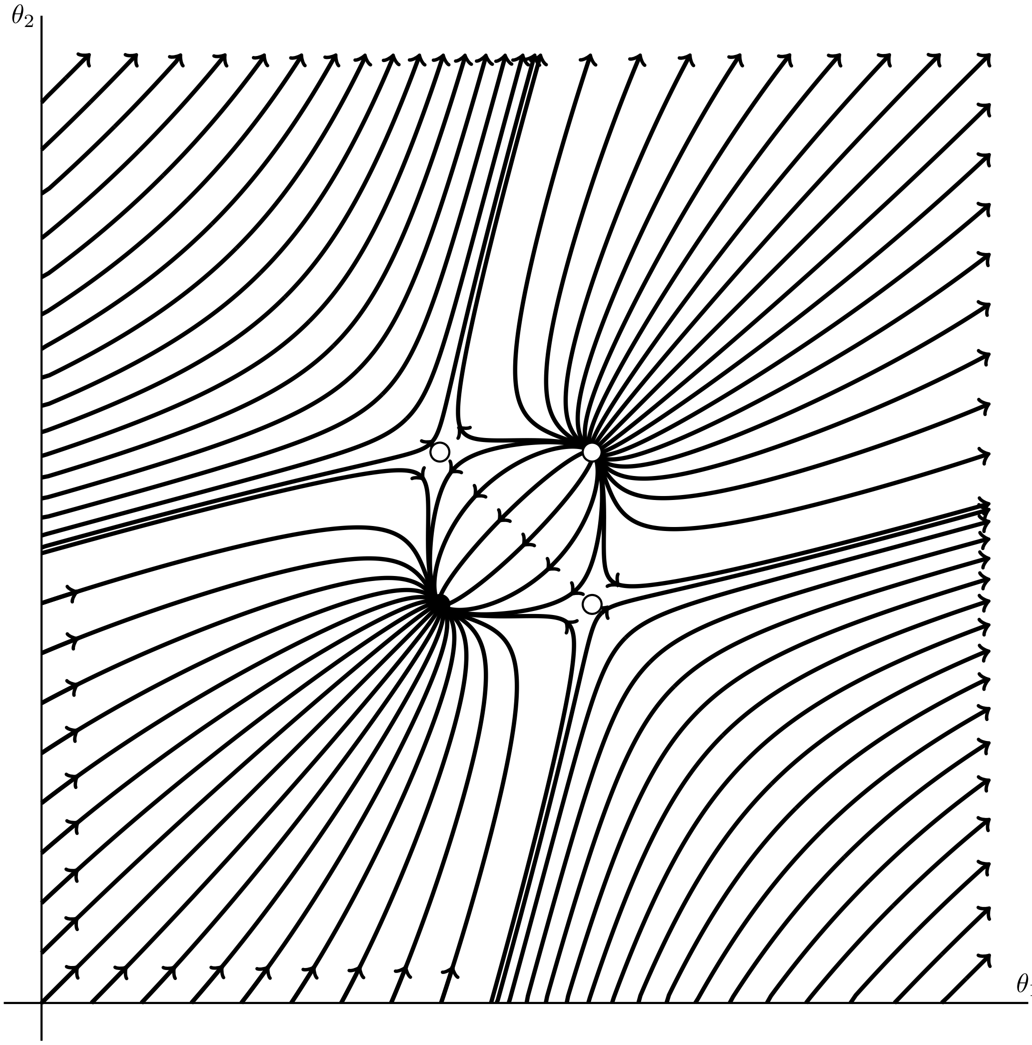
$\Omega = 0$



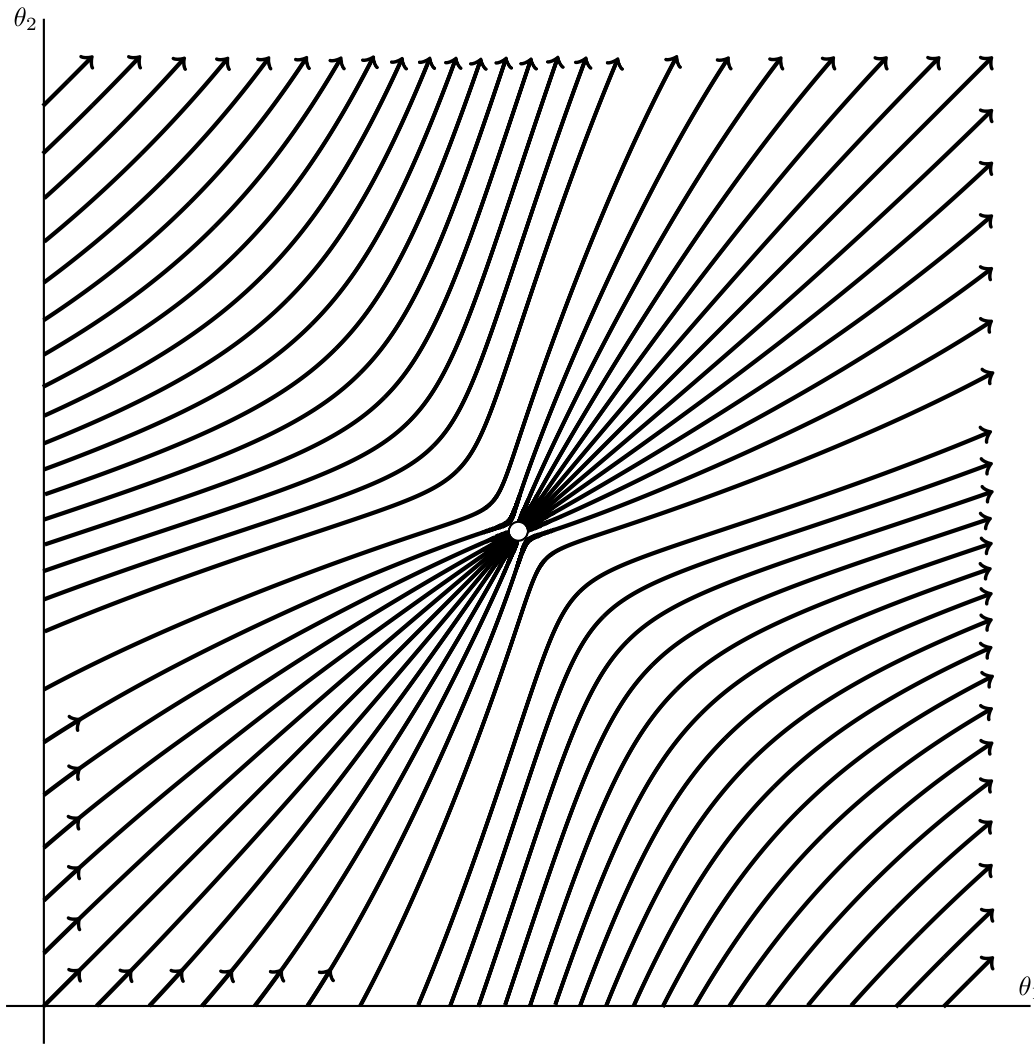
$\Omega = 1$



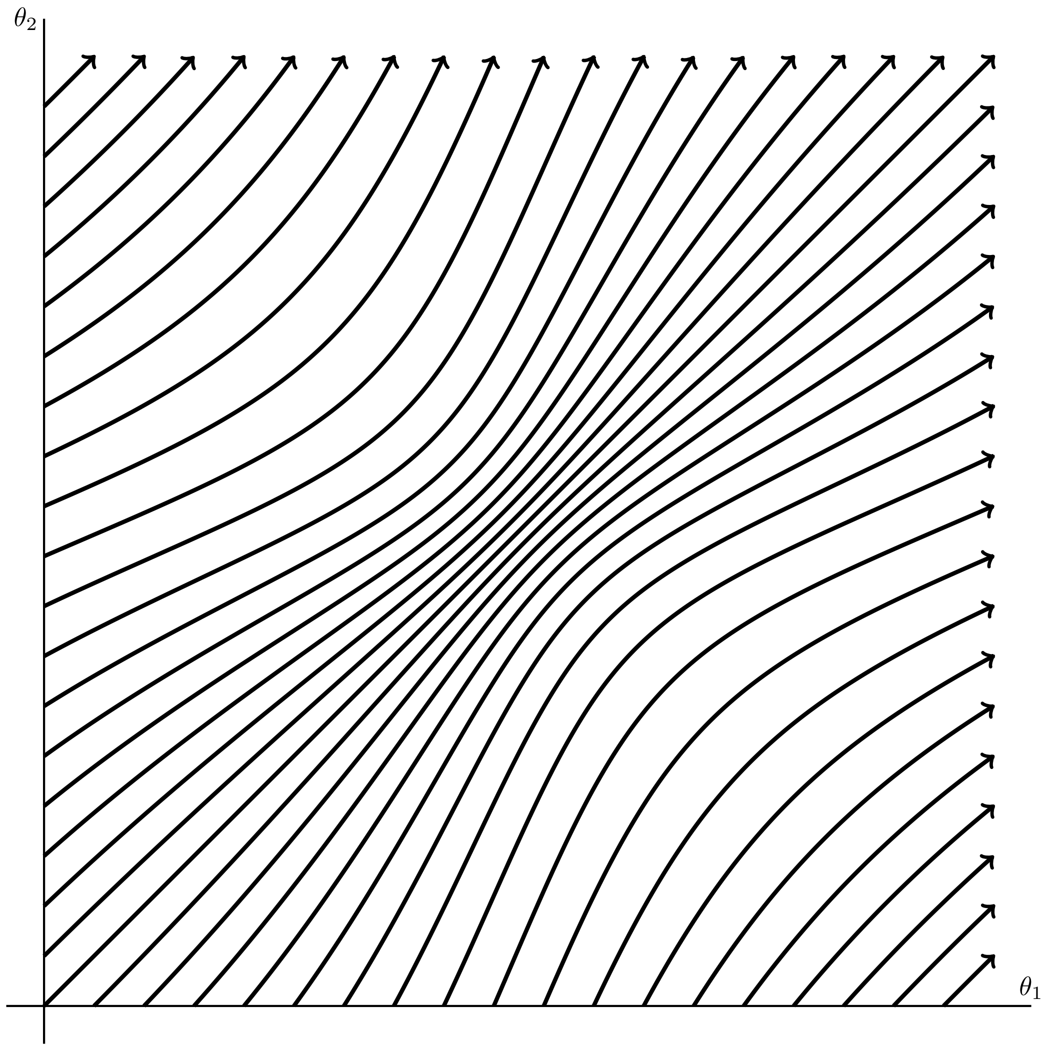
$\Omega = 1.75$



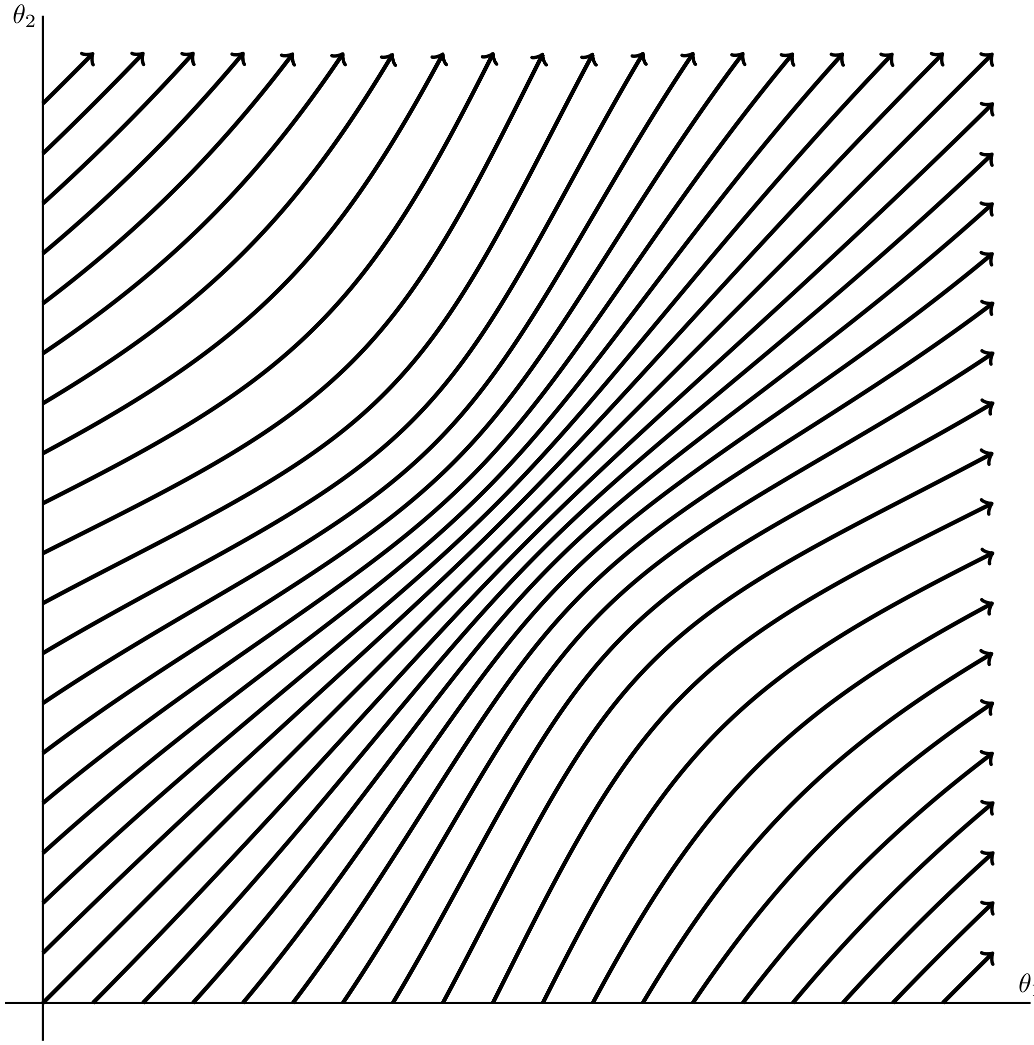
$\Omega = 2$



$\Omega = 2.5$



$\Omega = 3$



6.6.11

$$\dot{\theta} = \cot(\phi)\cos(\theta) \qquad \dot{\phi} = \big(\cos^2(\phi) + A\sin^2(\phi)\big)\sin(\theta)$$

a)

$t \to -t \quad \theta \to -\theta$

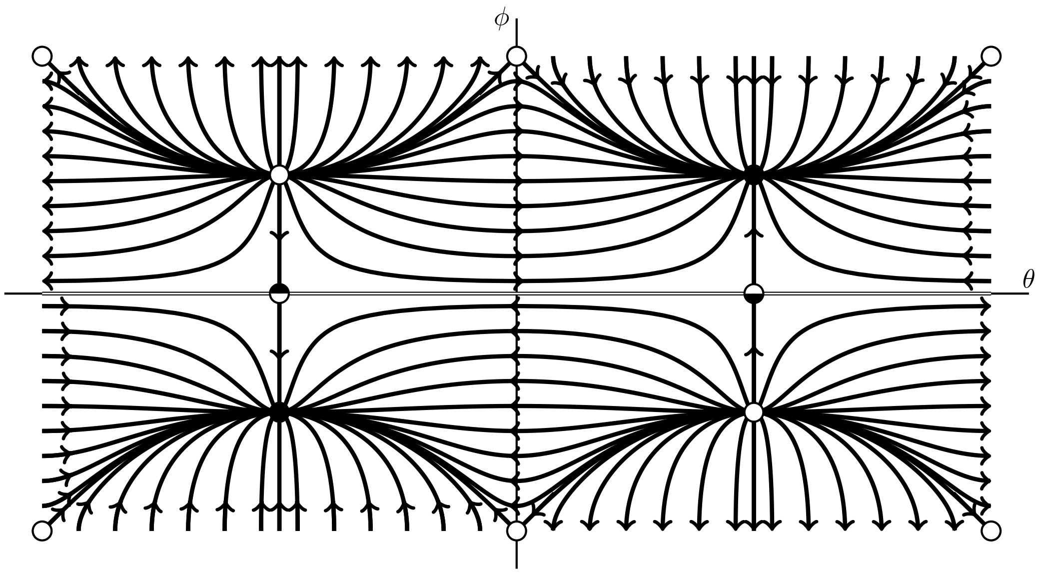
$$\begin{aligned} \frac{-1}{-1}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \cot(\phi)\cos(-\theta) & \frac{1}{-1}\frac{\mathrm{d}\phi}{\mathrm{d}t} &= \big(\cos^2(\phi) + A\sin^2(\phi)\big)\sin(-\theta) \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \cot(\phi)\cos(-\theta) & -\frac{\mathrm{d}\phi}{\mathrm{d}t} &= \big(\cos^2(\phi) + A\sin^2(\phi)\big)\big(-\sin(\theta)\big) \\ & & \frac{\mathrm{d}\phi}{\mathrm{d}t} &= \big(\cos^2(\phi) + A\sin^2(\phi)\big)\sin(\theta) \end{aligned}$$

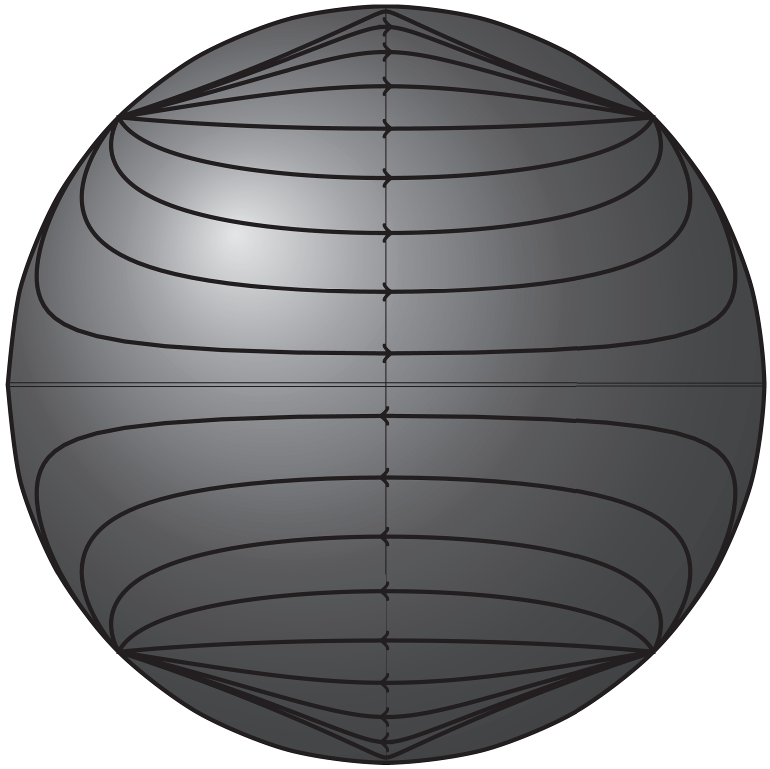
$t \to -t \quad \phi \to -\phi$

$$\begin{aligned} \frac{1}{-1}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \cot(-\phi)\cos(\theta) & \frac{-1}{-1}\frac{\mathrm{d}\phi}{\mathrm{d}t} &= \big(\cos^2(-\phi) + A\sin^2(-\phi)\big)\sin(\theta) \\ -\frac{\mathrm{d}\theta}{\mathrm{d}t} &= -\cot(\phi)\cos(\theta) & \frac{\mathrm{d}\phi}{\mathrm{d}t} &= \big(\cos^2(\phi) + A\big(-\sin(\phi)\big)^2\big)\sin(\theta) \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \cot(\phi)\cos(\theta) & \frac{\mathrm{d}\phi}{\mathrm{d}t} &= \big(\cos^2(\phi) + A\sin^2(\phi)\big)\sin(\theta) \end{aligned}$$

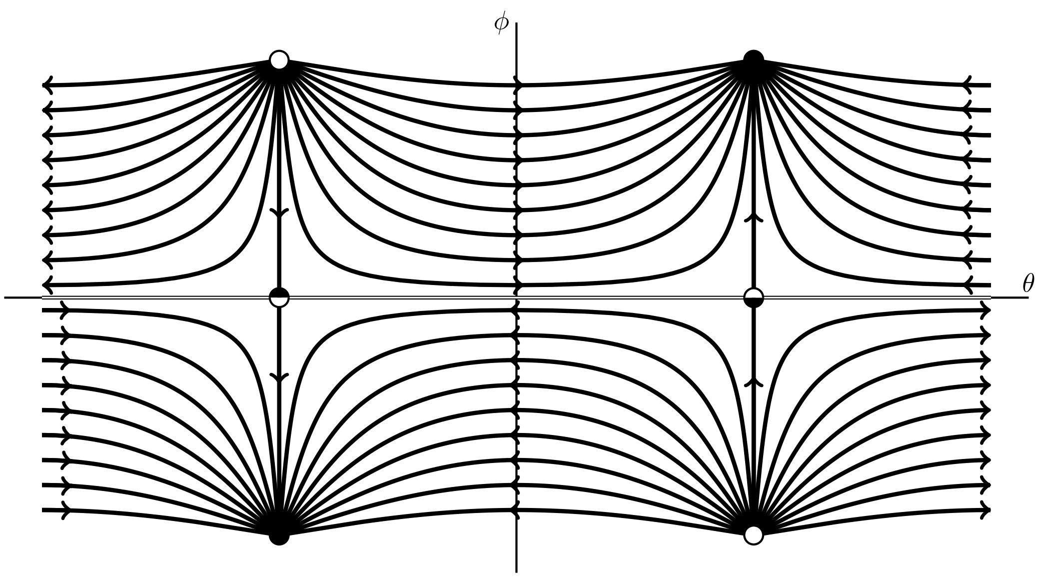
b)

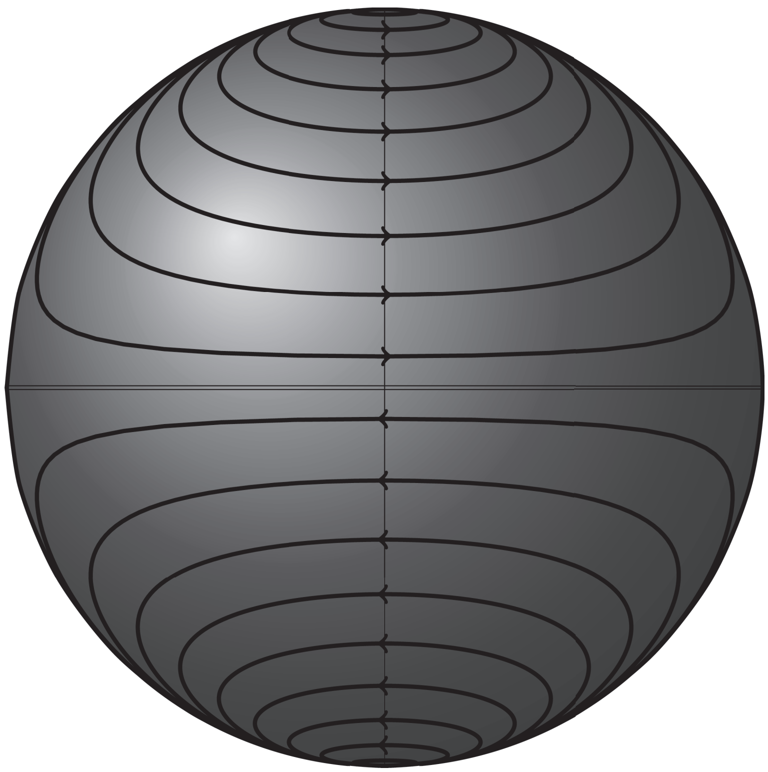
$A = -1$



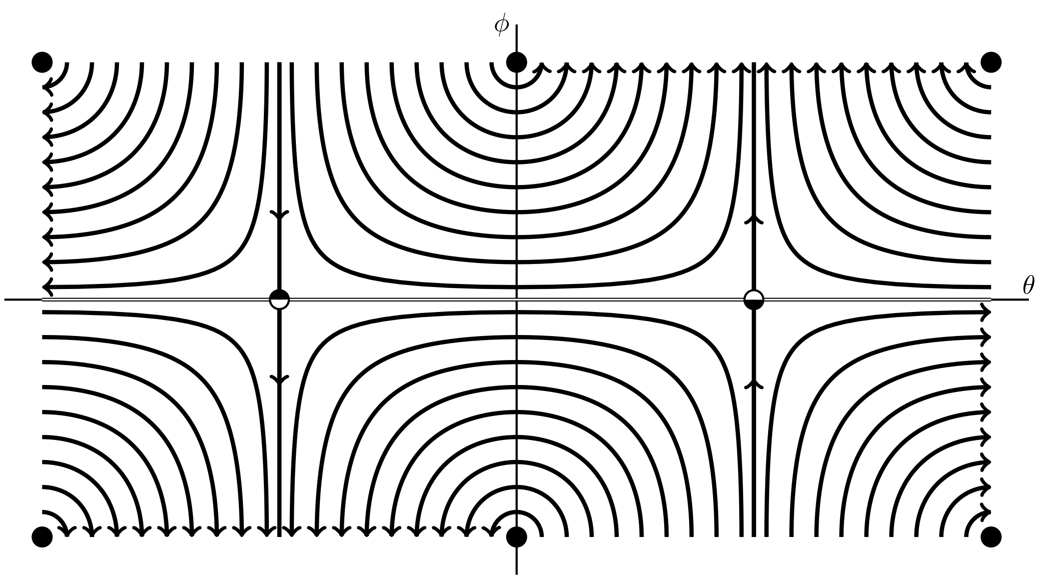


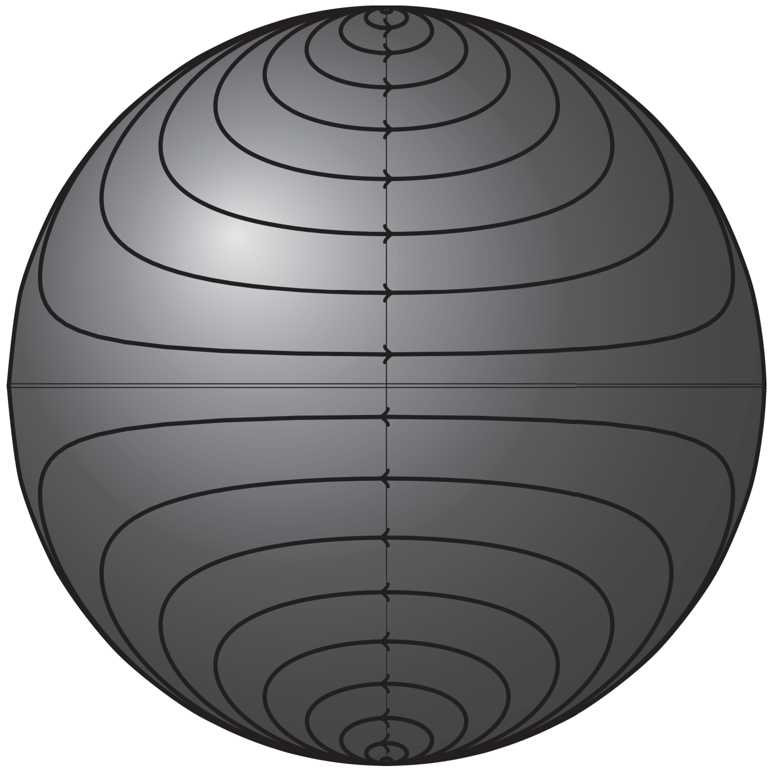
$A = 0$





$A = 1$





c)

From the phase portraits, it seems that almost all trajectories approach one of two stable fixed points when $A=-1$ or $A=0$. This means that the object settles down to a fixed orientation in the shear flow. In contrast, when $A=1$, the trajectories trace out neutrally stable periodic orbits starting from almost all initial conditions. This means the object tumbles periodically in the shear flow without ever converging to a fixed orientation.

6.7 Pendulum

6.7.1

$$\ddot{\theta}+b\dot{\theta}+\sin\theta=0 \qquad b>0$$

$$\dot{x}=y \quad \dot{y}=-\sin(x)-by$$

$(x,y)=(z\pi,0)$ are the fixed points, $z\in\mathbb{Z}$.

$$A=\begin{pmatrix}0 & 1\\ -\cos(x) & -b\end{pmatrix}$$

$$A_{(2z\pi,0)} = \begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix}$$

$\Delta = 1, \tau = -b < 0 \Rightarrow$ stable fixed point

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2-4}}{2}$$

$0 < b < 2 \Rightarrow$ stable spiral

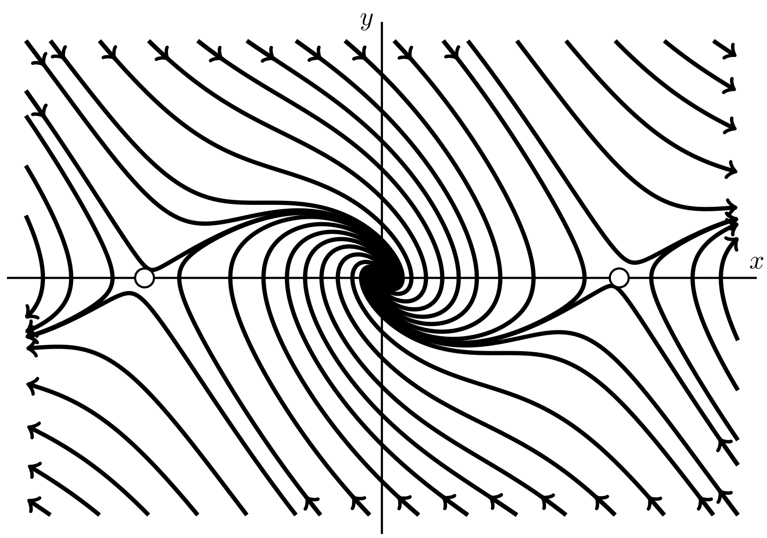
$b = 2 \Rightarrow$ stable degenerate node

$2 < b \Rightarrow$ stable node

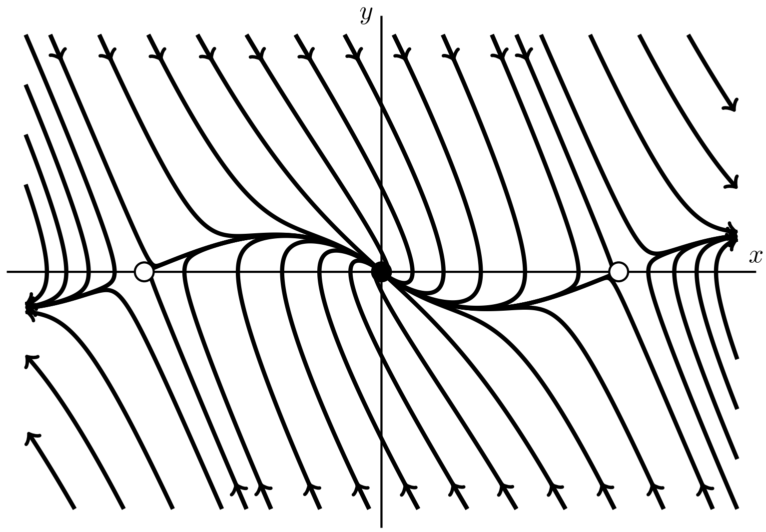
$$A_{\left((2z+1)\pi,0\right)} = \begin{pmatrix} 0 & 1 \\ 1 & -b \end{pmatrix}$$

$\Delta = -1, \tau = -b < 0 \Rightarrow$ saddle point

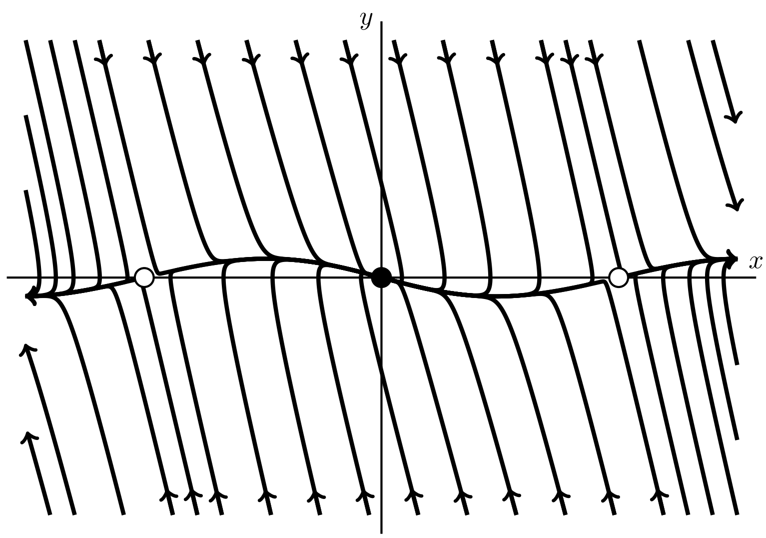
$b = 1$



$b = 2$



$b = 4$



6.7.3

$$\ddot{\theta} + \big(1 + a\cos(\theta)\big)\dot{\theta} + \sin\theta = 0 \qquad a \geq 0$$

$$\dot{x} = y \quad \dot{y} = -\sin(x) - \big(1 + a\cos(x)\big)y$$

$(x, y) = (z\pi, 0)$ are the fixed points, $z \in \mathbb{Z}$.

$$A = \begin{pmatrix} 0 & 1 \\ -\cos(x) + ay\sin(x) & -1 - a\cos(x) \end{pmatrix}$$

$$A_{(2z\pi,0)} = \begin{pmatrix} 0 & 1 \\ -1 & -1-a \end{pmatrix}$$

$$\Delta = 1, \tau = -1 - a < 0 \Rightarrow \text{stable fixed point}$$

$$A_{\left((2z+1)\pi,0\right)} = \begin{pmatrix} 0 & 1 \\ 1 & a-1 \end{pmatrix}$$

$$\Delta = -1 \Rightarrow \text{saddle point}$$

So as expected, the low point of the swing is stable and the very top of the swing is unstable for this damped pendulum.

6.7.5

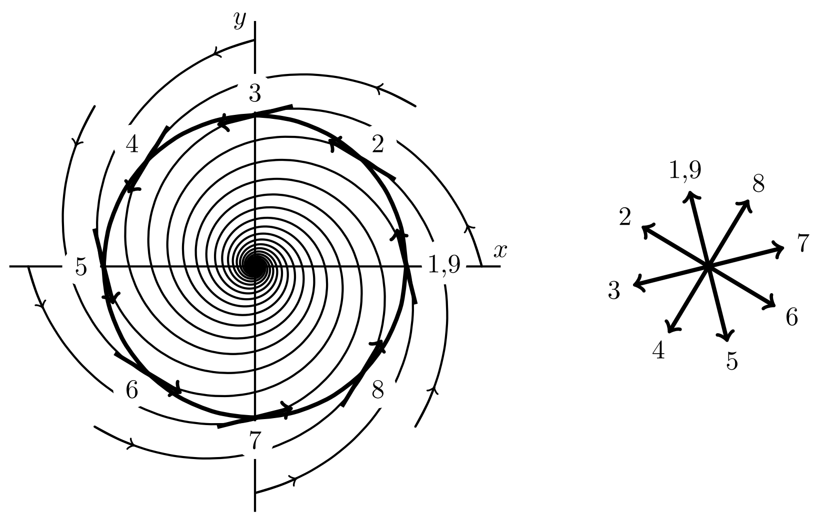
α	$2\pi\left(1+\frac{1}{16}\alpha^2\right)$	Quadrature
$\frac{0}{18}\pi$	2π	2π
$\frac{1}{18}\pi$	6.29514	6.29517
$\frac{2}{18}\pi$	6.3310	6.3313
$\frac{3}{18}\pi$	6.3908	6.3925
$\frac{4}{18}\pi$	6.4746	6.4801
$\frac{5}{18}\pi$	6.5822	6.5960
$\frac{6}{18}\pi$	6.7138	6.7430
$\frac{7}{18}\pi$	6.8693	6.9250
$\frac{8}{18}\pi$	7.0488	7.1471
$\frac{9}{18}\pi$	7.2521	7.4163
$\frac{10}{18}\pi$	7.4794	7.7423
$\frac{11}{18}\pi$	7.7306	8.1389
$\frac{12}{18}\pi$	8.0058	8.6261
$\frac{13}{18}\pi$	8.3048	9.2352
$\frac{14}{18}\pi$	8.6278	10.0182
$\frac{15}{18}\pi$	8.9747	11.0723
$\frac{16}{18}\pi$	9.3455	12.6135
$\frac{17}{18}\pi$	9.7403	15.3270
$\frac{18}{18}\pi$	10.1590	82.0530

6.8 Index Theory

6.8.1

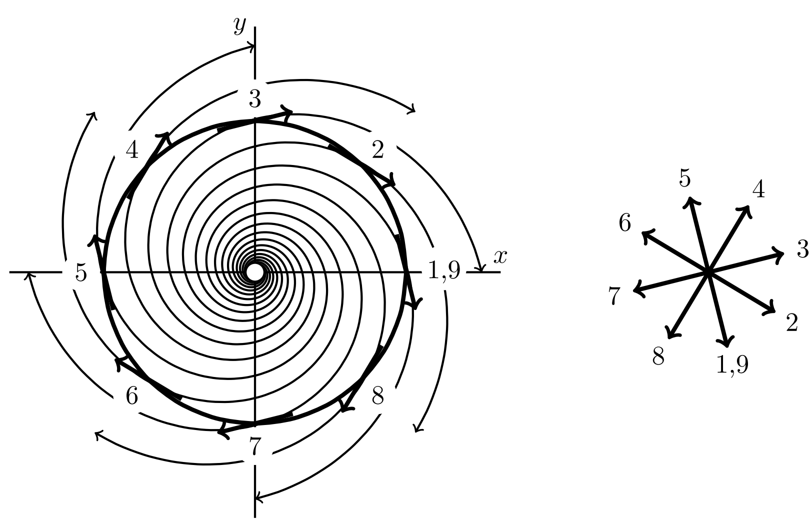
a)

Stable spiral



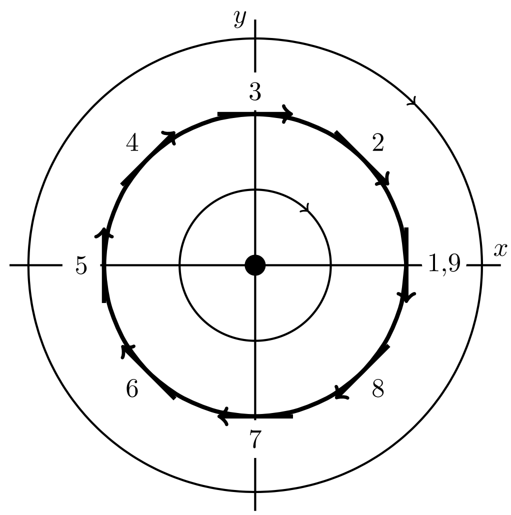
b)

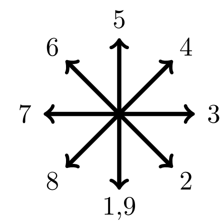
Unstable spiral



c)

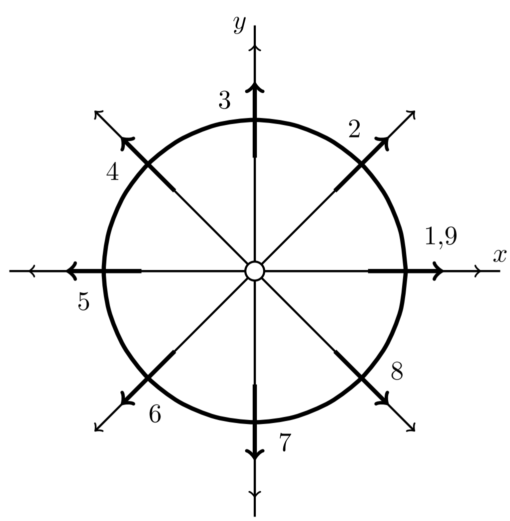
Center

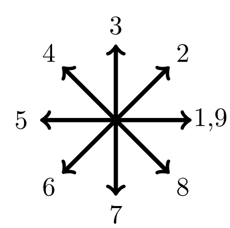




d)

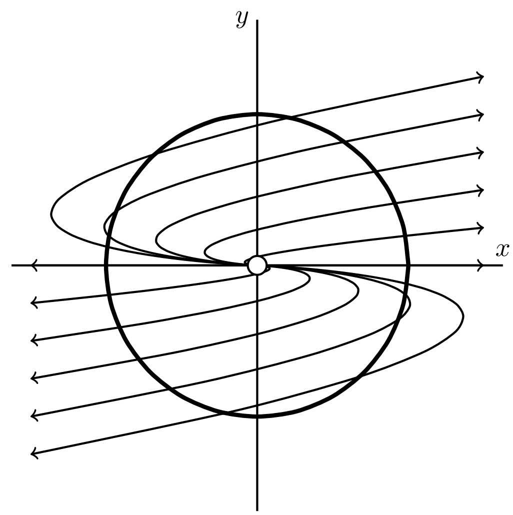
Star



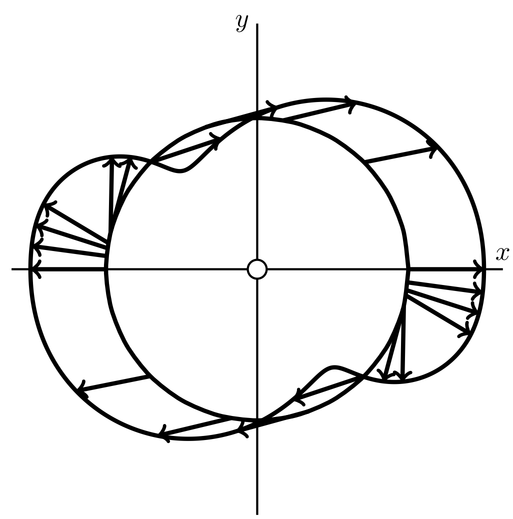


e)

Degenerate node



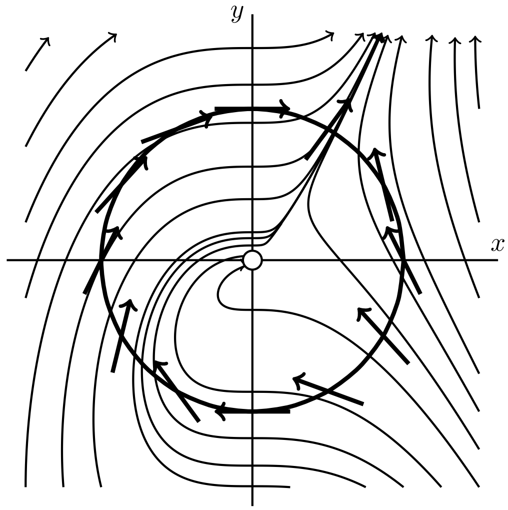
The vector field on this circle is pretty finicky near the point when the trajectory turns around and starts going the opposite direction. Due to this, we'll be graphing the path of the vector heads instead of the individual vectors themselves since they really bunch around $\theta = \pi$ and the side directly opposite.



And the degenerate node has an index of $+1$ as expected.

6.8.3

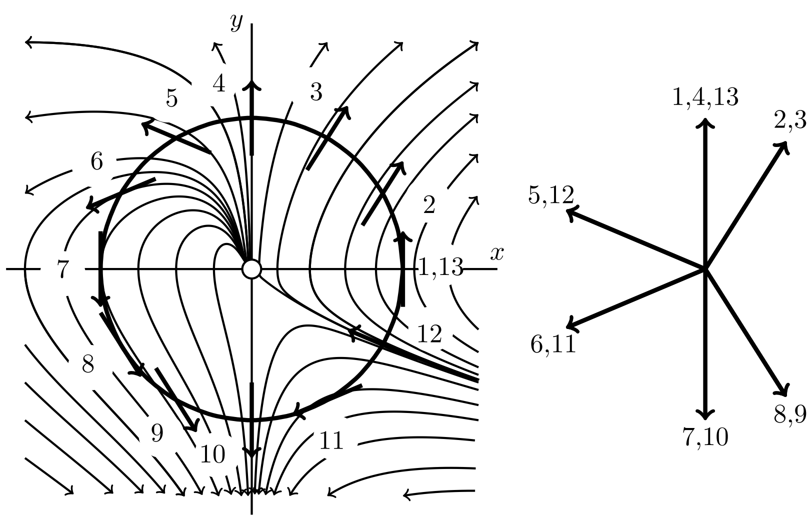
$$\dot{x} = y - x \qquad \dot{y} = x^2$$



Despite looking complicated, this curve has an index of 0 because the arrows never point down so can't make a full rotation.

6.8.5

$$\dot{x} = xy \qquad \dot{y} = x + y$$



The arrows almost make a full counterclockwise rotation but then turn around, so the index is 0.

6.8.7

$$\dot{x} = x(4 - y - x^2) \qquad \dot{y} = y(x - 1)$$

$(x, y) = (0, 0)$, $(2, 0)$, $(1, 3)$, and $(-2, 0)$ are the fixed points.

$$A = \begin{pmatrix} 4 - y - 3x^2 & -x \\ y & x - 1 \end{pmatrix}$$

$$A_{(0,0)} = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$$

$\lambda_1 = 4 \quad \lambda_2 = -1 \Rightarrow$ saddle point

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$A_{(2,0)} = \begin{pmatrix} -8 & -2 \\ 0 & 1 \end{pmatrix}$$

$\lambda_1 = -8 \quad \lambda_2 = 1 \Rightarrow$ saddle point

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} -2 \\ 9 \end{pmatrix}$$

$$A_{(1,3)} = \begin{pmatrix} -2 & -1 \\ 3 & 0 \end{pmatrix}$$

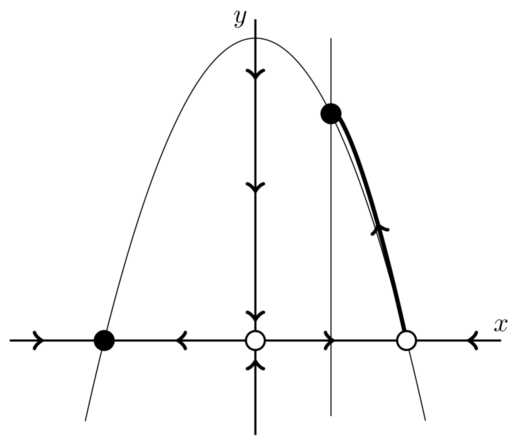
$\lambda_{1,2} = -1 \pm \sqrt{2}i \Rightarrow$ stable spiral

$$A_{(-2,0)} = \begin{pmatrix} -8 & 2 \\ 0 & -3 \end{pmatrix}$$

$\lambda_1 = -8 \quad \lambda_2 = -3 \Rightarrow$ stable node

Also notice from the equations that $x(0) = 0 \Rightarrow x(t) = 0$ and $y(0) = 0 \Rightarrow y(t) = 0$. This means the x-axis and y-axis are invariant. This means a closed orbit can't cross the axes.

A closed orbit must enclose a fixed point, and the only candidate is the stable spiral. However, this is also impossible since the unstable manifold of the saddle point goes into the stable spiral. We didn't actually prove this but use numerical results as evidence.



An intuitive argument is that the upper branch of the unstable manifold of the saddle point (2,0) must approach the spiral. It has to go somewhere. If it does not go to the stable spiral, where else could it go?

i) Both the x and y axes are invariant, so the unstable manifold can't get out of the first quadrant. Said another way, both of the axes contain trajectories and trajectories cannot cross by the uniqueness theorem for solutions of ODEs.

ii) The unstable manifold can't approach the saddle point at (0,0) since the only trajectories that do lie on the y-axis, which is the stable manifold of the saddle point at (0,0).

iii) Conceivably the upper branch of the unstable manifold could escape to infinity. However, the flow in the first quadrant pushes the unstable manifold upward and to the left until it crosses the vertical line $x = 1$. Then the unstable manifold moves left because $\dot{x} < 0$ when x lies to the right of the nullcline given the parabola $4 - y - x^2 = 0$ and moves upward because $\dot{y} > 0$ when $x > 1$, which is the case initially since the unstable manifold starts at (2,0). Once the manifold crosses the line $x = 1$ it has to move downward. And after that, there's no way for the trajectory to escape out to infinity.

Clearly this is only a plausibility argument, not a proof. To really prove that the unstable manifold cannot escape to infinity, one could construct a trapping region, a technique discussed in Section 7.3.

6.8.9

False. A counter example is

$$\dot{r} = r(r-1)(r-3)$$
$$\dot{\theta} = r^2 - 4$$

There is only one fixed point at the origin. The inner closed orbit is attracting and $\dot{\theta} < 0$, so rotation is in the clockwise direction. The outer closed orbit is repelling with $\dot{\theta} > 0$, so rotation is in the counterclockwise direction.

6.8.11

a)

$k = 1$

$$\dot{z} = z = x + iy = re^{i\theta} \qquad r = \sqrt{x^2 + y^2} \quad \tan(\theta) = \frac{y}{x}$$
$$\dot{z} = \bar{z} = x - iy = re^{-i\theta}$$

$k = 2$

$$\begin{aligned}\dot{z} = z^2 &= (x + iy)^2 = x^2 - y^2 + 2ixy \\ &= \left(re^{i\theta}\right)^2 = r^2 e^{2i\theta} \\ \dot{z} = (\bar{z})^2 &= (x - iy)^2 = x^2 - y^2 - 2ixy \\ &= \left(re^{-i\theta}\right)^2 = r^2 e^{-2i\theta}\end{aligned}$$

$k = 3$

$$\begin{aligned}\dot{z} = z^3 &= (x + iy)^3 = x^3 - 3xy^2 + i(3x^2 y - y^3) \\ &= \left(re^{i\theta}\right)^3 = r^3 e^{3i\theta} \\ \dot{z} = (\bar{z})^3 &= (x - iy)^3 = x^3 + i(y^3 - 3x^2 y) - 3xy^2 \\ &= \left(re^{-i\theta}\right)^3 = r^3 e^{-3i\theta}\end{aligned}$$

bc)

It's not so easy to see that the origin is the only stable fixed point in the Cartesian coordinates, but in the polar coordinates it's clear that the only way to satisfy $\dot{z} = 0$ is $r = 0$ in both cases, which is the origin.

The index, also not easy to see from the Cartesian coordinates, is $\pm k$ in the z^k and $(\bar{z})^k$ cases respectively. The arrows will do k rotations on a loop enclosing the origin when θ goes through a 2π rotation due to the k in $e^{\pm ki\theta}$, with $+ki\theta$ causing counterclockwise and $-ki\theta$ causing clockwise rotations.

6.8.13

a)

$$\phi = \arctan\left(\frac{\dot{y}}{\dot{x}}\right) = \arctan\left(\frac{g(x,y)}{f(x,y)}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi = \frac{\mathrm{d}}{\mathrm{d}t}\arctan\left(\frac{g(x,y)}{f(x,y)}\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\frac{g(x,y)}{f(x,y)}}{1+\left(\frac{g(x,y)}{f(x,y)}\right)^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{g}{f}\right)\frac{1}{1+\frac{g^2}{f^2}}$$

$$\mathrm{d}\phi = \frac{f\mathrm{d}g - g\mathrm{d}f}{f^2}\frac{1}{1+\frac{g^2}{f^2}} = \frac{f\mathrm{d}g - g\mathrm{d}f}{f^2+g^2}$$

b)

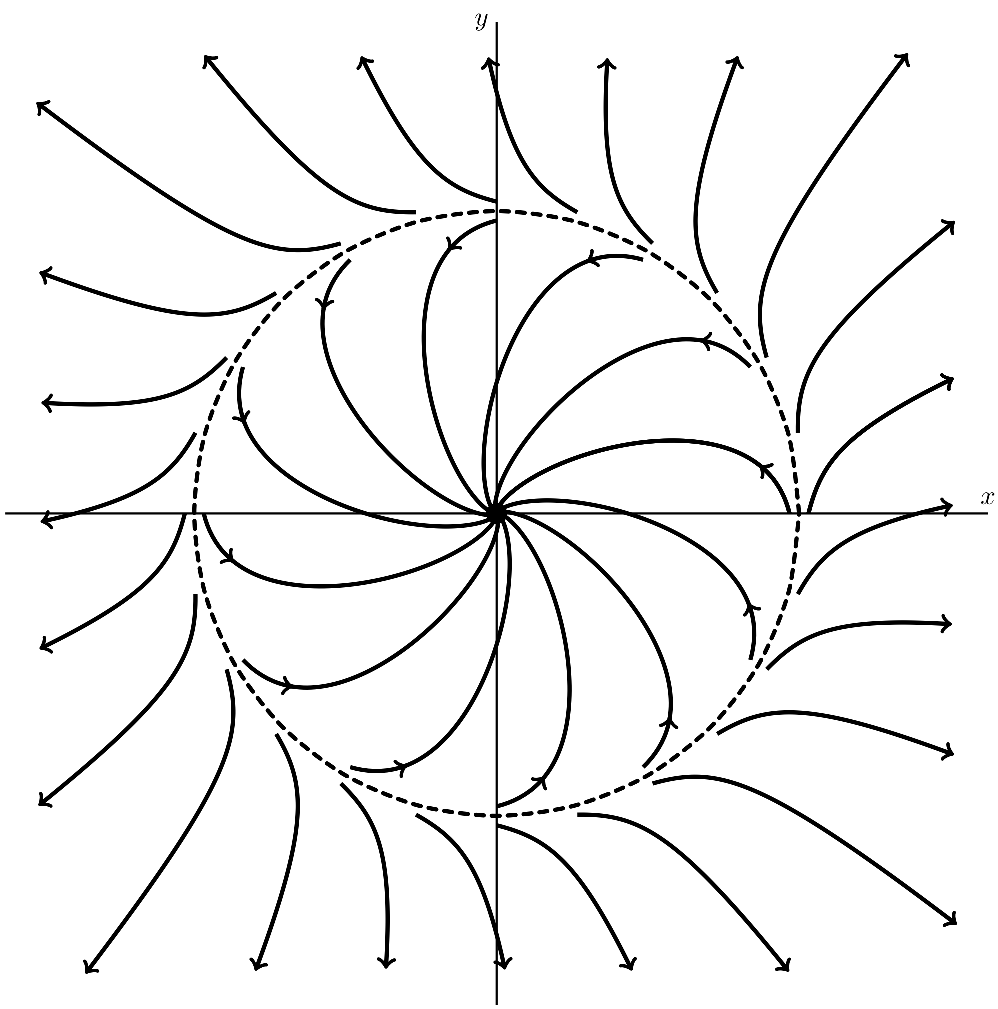
$$I_C = \frac{1}{2\pi}\oint_C \mathrm{d}\phi = \frac{1}{2\pi}\oint_C \frac{f\mathrm{d}g - g\mathrm{d}f}{f^2+g^2}$$

7
Limit Cycles

7.1 Examples

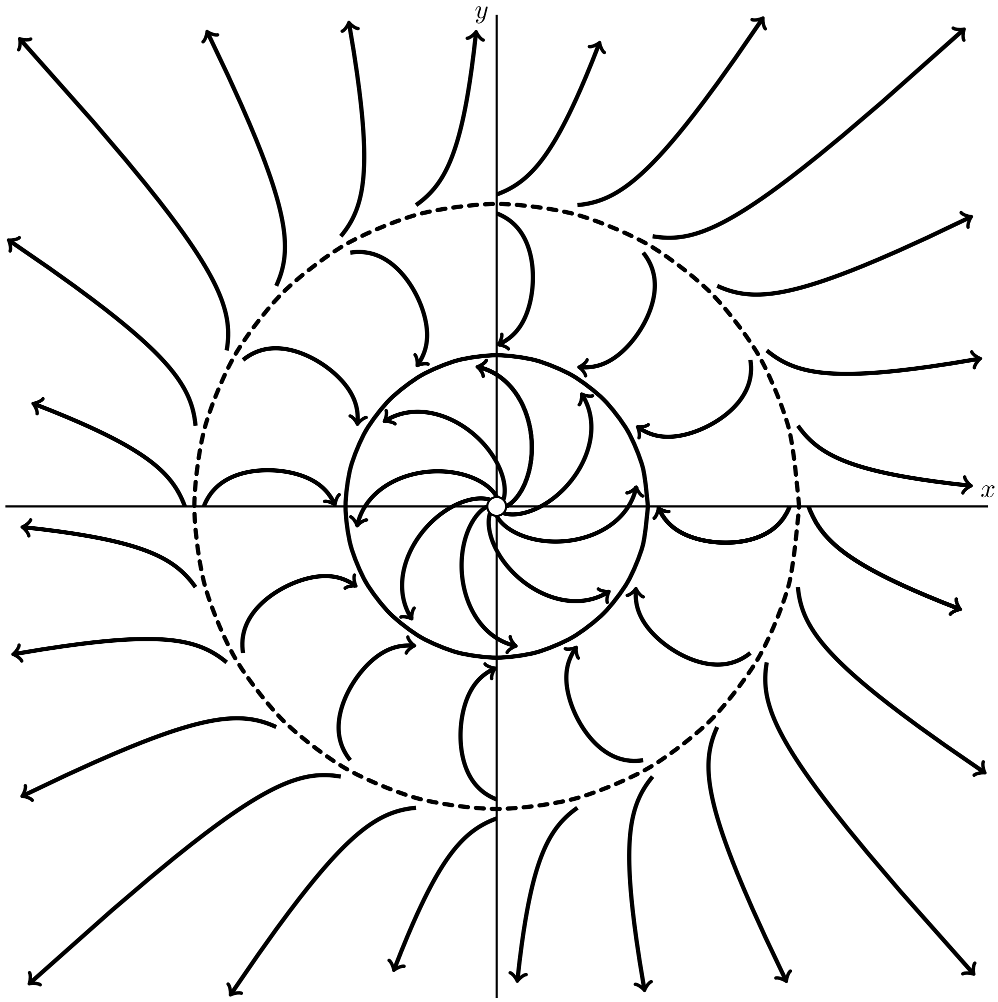
7.1.1

$$\dot{r} = r^3 - 4r \qquad \dot{\theta} = 1$$



7.1.3

$$\dot{r} = r(1-r^2)(4-r^2) \qquad \dot{\theta} = 2 - r^2$$



7.1.5

$$\dot{r} = r(1-r^2) \qquad \dot{\theta} = 1$$

$$\begin{aligned}
\dot{x} &= \frac{\mathrm{d}}{\mathrm{d}t}\big(r\cos(\theta)\big) \\
&= \dot{r}\cos(\theta) - r\sin(\theta)\dot{\theta} \\
&= r(1-r^2)\cos(\theta) - r\sin(\theta) \\
&= (1-r^2)r\cos(\theta) - r\sin(\theta) \\
&= (1-x^2-y^2)x - y \\
&= x - y - x(x^2+y^2)
\end{aligned}$$

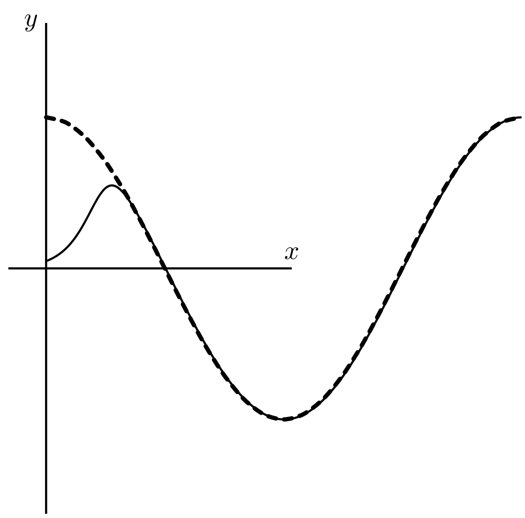
$$
\begin{aligned}
\dot{y} &= \frac{\mathrm{d}}{\mathrm{d}t}\big(r\sin(\theta)\big) \\
&= \dot{r}\sin(\theta) + r\cos(\theta)\dot{\theta} \\
&= r(1-r^2)\sin(\theta) + r\cos(\theta) \\
&= (1-r^2)r\sin(\theta) + r\cos(\theta) \\
&= (1-x^2-y^2)y + x \\
&= x + y - y(x^2+y^2)
\end{aligned}
$$

7.1.7

$$\dot{r} = r(4-r^2) \qquad \dot{\theta} = 1$$

The initial condition of $(r,\theta) = (0.1, 0)$ in Cartesian coordinates is $(x,y) = (0.1, 0)$. The system will approach the $r = 2$ limit cycle, which would make $x(t) = 2\cos(t)$ the limiting waveform for any initial condition except the origin.

We can see this in the following graph.

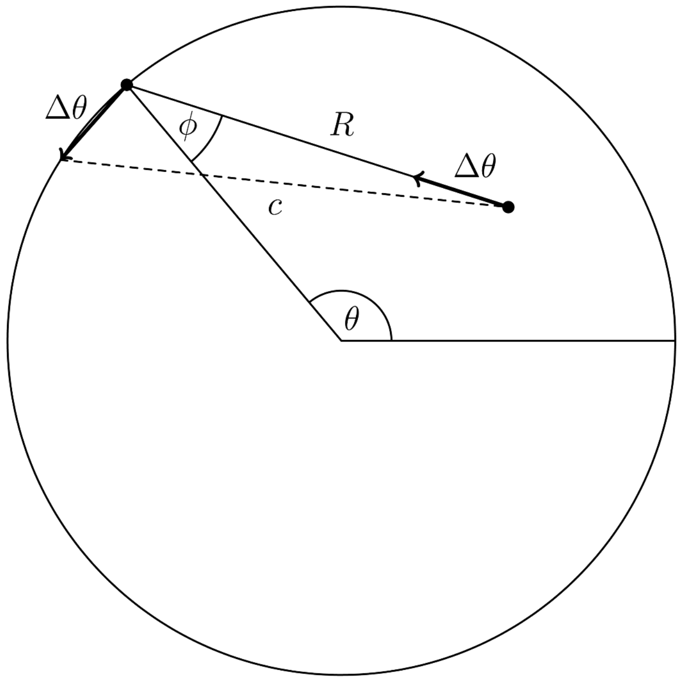


7.1.9

a)

Since the dog and the duck are moving at the same speed, they will both move the same amount in the time it takes θ to change by $\Delta\theta$. Thus the problem can be done using $\Delta\theta$ instead of Δt in the derivation.

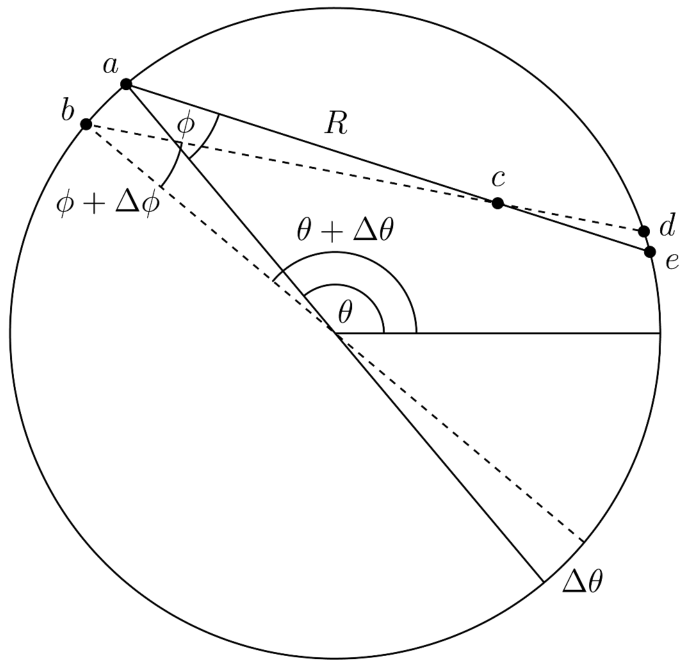
During a small amount of time Δt, the length of the segment R decreases from the dog swimming a length of $\Delta\theta$ and increases from the duck moving along the perimeter of the circle also by a length of $\Delta\theta$. We'll eventually be taking a limit, so we can make our lives easier with some approximations.



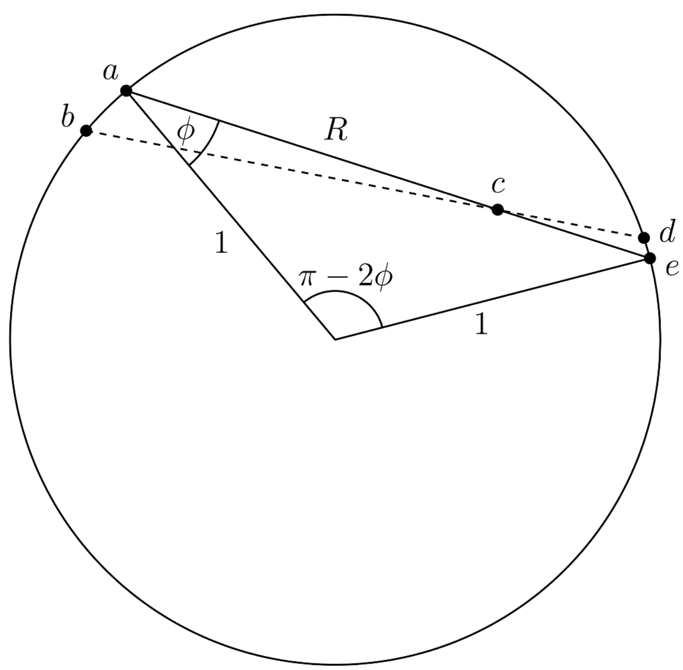
The line segment along the perimeter of the circle is approximately the length of the arc that the line segment spans, and the angle bordering ϕ is approximately a right angle. Using these approximations, the law of cosines, and a trick for taking a limit, we can arrive at a differential equation for R.

$$\begin{aligned}
\Delta R &\approx -\Delta\theta + c - R \\
&= -\Delta\theta + \sqrt{(\Delta\theta)^2 + R^2 - 2\Delta\theta R\cos\left(\phi + \frac{\pi}{2}\right)} - R \\
&= -\Delta\theta + \frac{(\Delta\theta)^2 + R^2 - 2\Delta\theta R\cos\left(\phi + \frac{\pi}{2}\right) - R^2}{\sqrt{(\Delta\theta)^2 + R^2 - 2\Delta\theta R\cos\left(\phi + \frac{\pi}{2}\right)} + R} \\
&= -\Delta\theta + \frac{(\Delta\theta)^2 + 2\Delta\theta R\sin(\phi)}{\sqrt{(\Delta\theta)^2 + R^2 + 2\Delta\theta R\sin(\phi)} + R} \\
\frac{\Delta R}{\Delta\theta} &= -1 + \frac{\Delta\theta + 2R\sin(\phi)}{\sqrt{(\Delta\theta)^2 + R^2 + 2\Delta\theta R\sin(\phi)} + R} \\
\lim_{\Delta\theta\to 0}\frac{\Delta R}{\Delta\theta} &= -1 + \frac{\Delta\theta + 2R\sin(\phi)}{\sqrt{(\Delta\theta)^2 + R^2 + 2\Delta\theta R\sin(\phi)} + R} \\
R' &= -1 + \frac{2R\sin(\phi)}{\sqrt{R^2} + R} = -1 + \sin(\phi)
\end{aligned}$$

For the ϕ' equation we can use some basic geometry.



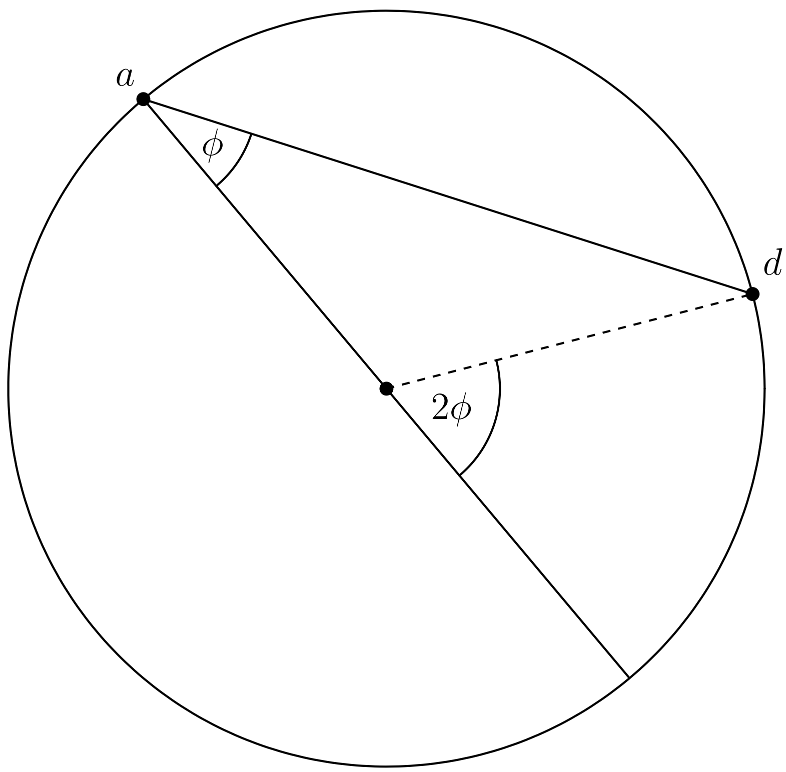
Next we have to remember that $\triangle acb$ is similar to $\triangle dce$ (angles subtended by the same arc are equal), meaning the ratio between $\overline{ab}$ and $\overline{de}$ is equal to the ratio of $\overline{ac}$ and $\overline{ce}$.



Approximating $\overline{ab}$ as $\widehat{ab} = \Delta\theta$ and $\overline{de}$ as $\widehat{de}$

$$\overline{ab} : \overline{de} = \overline{ac} : \overline{cd} \Rightarrow \widehat{de} \approx \frac{2\cos(\phi) - R}{R}\Delta\theta$$

Next we have to remember that the arc made by the inscribed angle ϕ is half of the arc of the central angle (inscribed angle theorem).



Now putting it all together, the change in the central angle $2\Delta\phi$ is from an increase of arc $\widehat{de}$ and a decrease of arc $\Delta\theta$.

$$
\begin{aligned}
2\Delta\phi = \widehat{de} - \Delta\theta &\approx \frac{2\cos(\phi) - R}{R}\Delta\theta - \Delta\theta = \frac{2\cos(\phi) - 2R}{R}\Delta\theta \\
R\frac{\Delta\phi}{\Delta\theta} &\approx \cos(\phi) - R \\
\lim_{\Delta\theta\to 0} R\frac{\Delta\phi}{\Delta\theta} &= R\phi' = \cos(\phi) - R
\end{aligned}
$$

As for whether or not the dog catches the duck, the answer is no. The proof is difficult enough that it is the subject of a published paper. Please take a look at *Applications of Center Manifolds* by Keith Promislow if you're interested.

b)

Repeating the derivation now with the speed of the dog as $k\Delta\theta$

$$
\begin{aligned}
R' &= -k + \sin(\phi) \\
R\phi' &= \cos(\phi) - R
\end{aligned}
$$

c)

Assuming there is a limit cycle

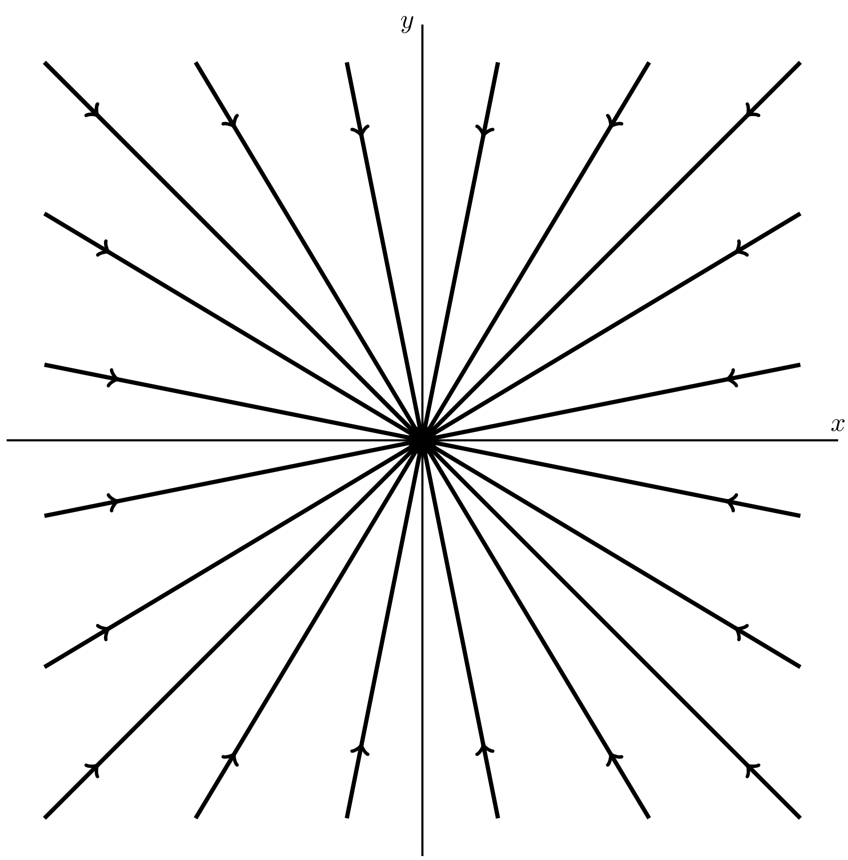
$$R' = -k + \sin(\phi) = 0 \Rightarrow \phi = \arcsin(k)$$
$$R\phi' = \cos(\phi) - R = 0 \Rightarrow R = \cos(\phi) = \sqrt{1-k^2}$$

So for $k = \frac{1}{2}$ the dog will endlessly chase the duck while asymptotically approaching a circle of radius $\frac{\sqrt{3}}{2}$ as a limit cycle.

7.2 Ruling Out Closed Orbits

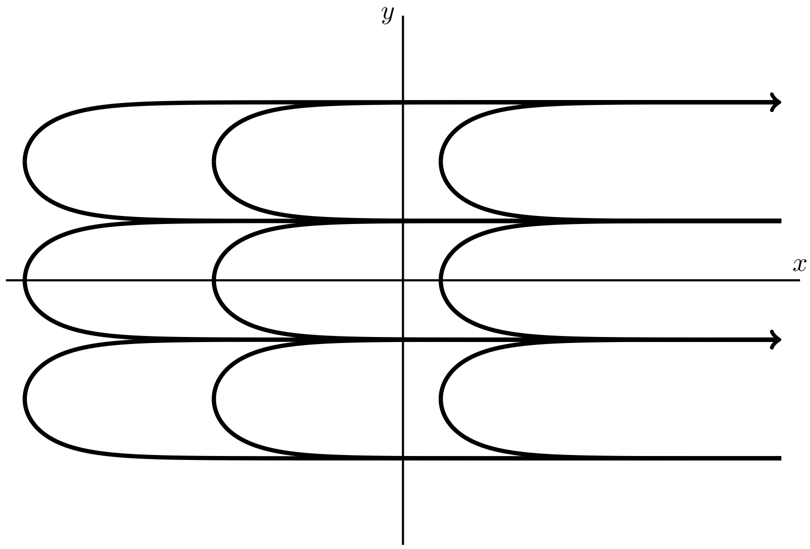
7.2.1

$$V = x^2 + y^2$$



7.2.3

$$V = e^x \sin(y)$$



7.2.5

Let $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$ be a smooth vector field defined on the phase plane.

a)

If this is a gradient system, then

$$-\nabla V = \left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}\right) = (f(x,y), g(x,y))$$

$$-\frac{\partial V}{\partial x} = f \text{ and } -\frac{\partial V}{\partial y} = g \Rightarrow \frac{\partial f}{\partial y} = -\frac{\partial^2 V}{\partial y \partial x} = -\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial g}{\partial x}$$

b)

Yes. If f and g are smooth on $\mathbb{R}^2$ and $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, the system is a gradient system.

$f : \mathbb{R}^2 \to \mathbb{R}$ being smooth and the fundamental theorem of calculus together imply there exists $V : \mathbb{R}^2 \to \mathbb{R}$ such that $V(x, y)$ is continuous and $V(x, y) = \int f(x, y)\mathrm{d}x$.

Leibniz integral rule $\Rightarrow \frac{\partial V}{\partial y} = \frac{\partial}{\partial y} \int f(x, y)\mathrm{d}x = \int \frac{\partial f}{\partial y}(x, y)\mathrm{d}x = \int \frac{\partial g}{\partial x}(x, y)\mathrm{d}x = g$

Hence there exists $V(x, y)$ such that $\frac{\partial V}{\partial x} = f$ and $\frac{\partial V}{\partial y} = g$.

7.2.7

$$\dot{x} = y + 2xy \qquad \dot{y} = x + x^2 - y^2$$

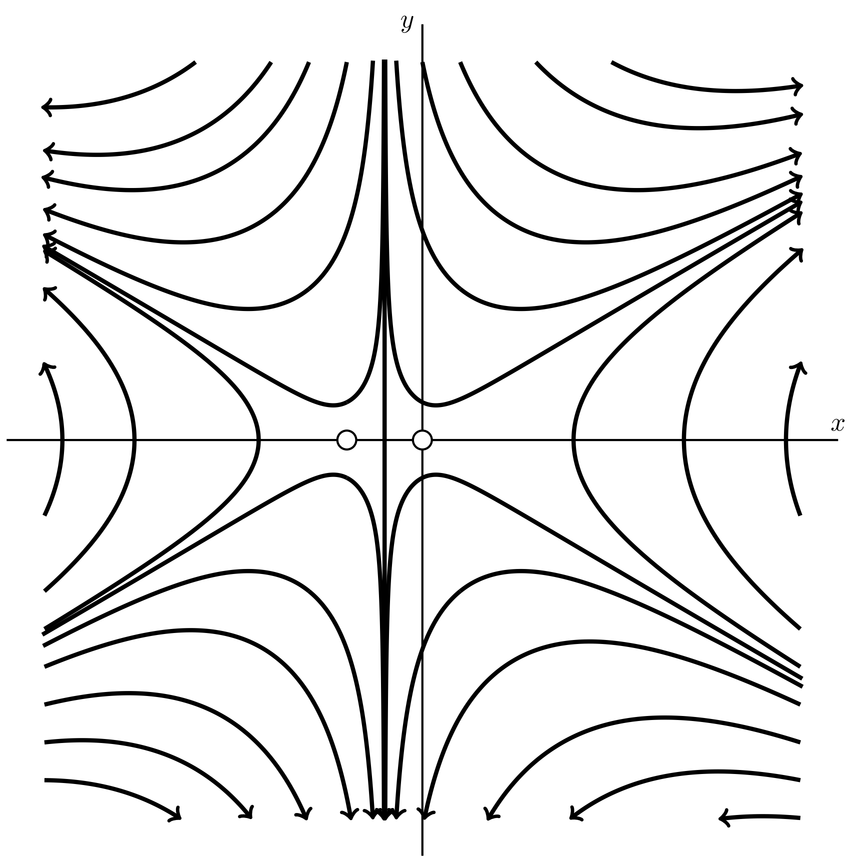
a)

$$\frac{\partial}{\partial x}\big(y + 2xy\big) = 0 + 2y = 2y = 0 + 0 + 2y = \frac{\partial}{\partial y}\big(x + x^2 - y^2\big)$$

b)

$$
\begin{aligned}
-\frac{\partial V}{\partial x} &= \dot{x} = y + 2xy \\
&\Rightarrow -V = \int y + 2xy \ \mathrm{d}x = xy + x^2 y + g(y) \\
-\frac{\partial V}{\partial y} &= \dot{y} = x + x^2 - y^2 \\
&= \frac{\partial}{\partial y}\big(xy + x^2 y + g(y)\big) = x + x^2 + g'(y) \\
&\Rightarrow g'(y) = -y^2 \Rightarrow g(y) = -\frac{1}{3}y^3 + C \quad \text{(set to 0 for convenience)} \\
V(x,y) &= -xy - x^2 y + \frac{1}{3}y^3
\end{aligned}
$$

c)



7.2.9

Recall that a gradient system will have

$$\langle \dot{x}, \dot{y} \rangle = \langle f(x,y), g(x,y) \rangle = \langle -V_x, -V_y \rangle \Rightarrow f_y = -V_{xy} = -V_{yx} = g_x$$

a)

$$\dot{x} = y + x^2 y \qquad \dot{y} = -x + 2xy$$

$$f_y = \frac{\partial}{\partial y}(y + x^2 y) = 1 + x^2 \neq -1 + 2y = \frac{\partial}{\partial x}(-x + 2xy) = g_x$$

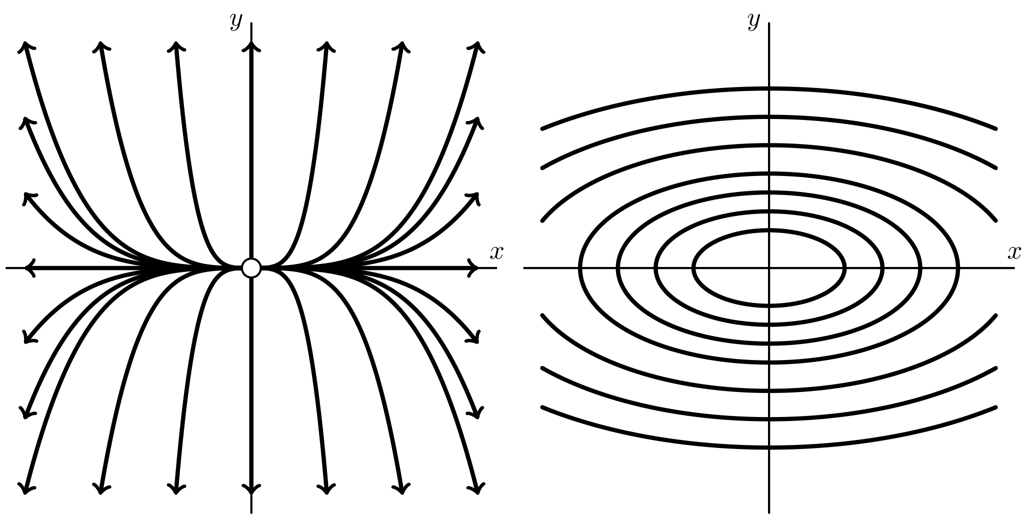
So this is not a gradient system.

b)

$$\dot{x} = 2x \qquad \dot{y} = 8y$$

$$f_y = \frac{\partial}{\partial y}(2x) = 0 = \frac{\partial}{\partial x}(8y) = g_x$$

So this is a gradient system with $V(x, y) = -x^2 - 4y^2$.

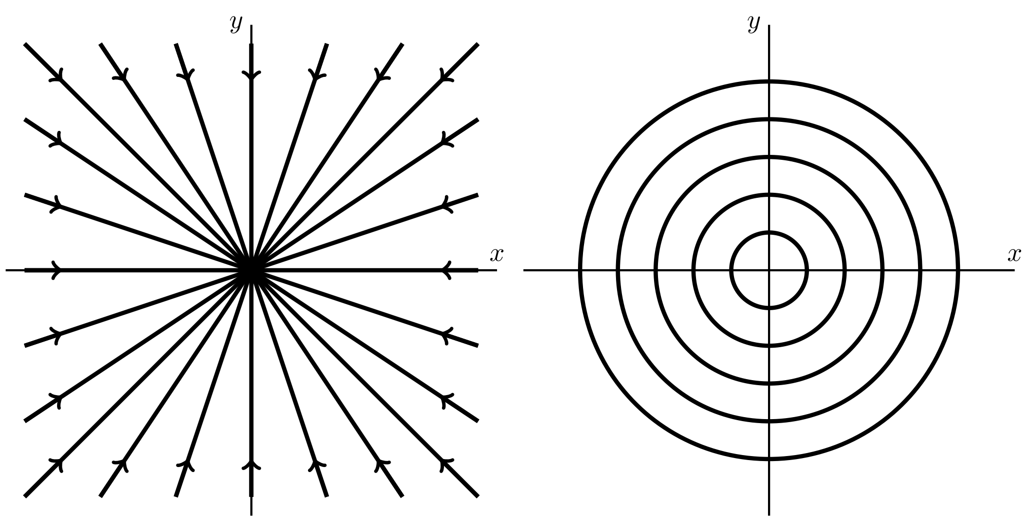


c)

$$\dot{x} = -2xe^{x^2+y^2} \qquad \dot{y} = -2ye^{x^2+y^2}$$

$$f_y = \frac{\partial}{\partial y}\Big(-2xe^{x^2+y^2}\Big) = -4xye^{x^2+y^2} = \frac{\partial}{\partial x}\Big(-2ye^{x^2+y^2}\Big) = g_x$$

So this is a gradient system with $V(x, y) = e^{x^2+y^2}$.



7.2.11

$$V = ax^2 + 2bxy + cy^2$$

The shape is paraboloid-like, but the middle term can make the origin into a saddle point for the surface. This problem is exactly like a multivariable calculus problem to find an absolute minimum, so we will be using the second derivative test.

$$V_{xx}V_{yy} - \left(V_{xy}\right)^2\Big|_{(0,0)} = (2a)(2c) - (2b)^2\Big|_{(0,0)} = 4\left(ac - b^2\right)$$

The second derivative test states that the origin is a local extrema if $ac - b^2 > 0$ and a local minimum if $V_x(0,0) = V_y(0,0) = 0$ for the origin to be a critical point and for $V_{xx}(0,0) = 2a > 0$. (We could have used V_{yy} instead of V_{xx} but we simply had to use one of them.)

Hence, $V(x,y)$ is positive definite if and only if $a > 0$ and $ac - b^2 > 0$.

7.2.13

$$\dot{N}_1 = r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - b_1 N_1 N_2 \qquad \dot{N}_2 = r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - b_2 N_1 N_2$$

Using Dulac's criterion with weighting function $g(N_1, N_2) = \dfrac{1}{N_1 N_2}$

$$\begin{aligned}
&\nabla \cdot \left(g(\dot{N}_1, \dot{N}_2)\langle N_1, N_2\rangle\right) \\
&\quad = \nabla \cdot \left\langle r_1 \left(\frac{1}{N_2} - \frac{1}{K_1}\frac{N_1}{N_2}\right) - b_1, r_2 \left(\frac{1}{N_1} - \frac{1}{K_2}\frac{N_2}{N_1}\right) - b_2 \right\rangle \\
&\quad = -\frac{r_1}{K_1}\frac{1}{N_2} - \frac{r_2}{K_2}\frac{1}{N_1} < 0
\end{aligned}$$

The last step holds since all the parameters and variables are strictly positive. Hence, by Dulac's criterion there are no periodic orbits contained entirely in the first quadrant.

7.2.15

$$\dot{x} = x(2 - x - y) \qquad \dot{y} = y(4x - x^2 - 3)$$

a)

(x, y) =(0,0), (1,1), (2,0), and (3,-1) are the fixed points.

$$A = \begin{pmatrix} 2 - 2x - y & -x \\ y(4 - 2x) & 4x - x^2 - 3 \end{pmatrix}$$

$$A_{(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$

$$\lambda_1 = 2 \quad \lambda_2 = -3 \Rightarrow \text{saddle point}$$

$$A_{(1,1)} = \begin{pmatrix} -1 & -1 \\ 2 & 0 \end{pmatrix}$$

$$\lambda_{1,2} = \frac{-1 \pm i\sqrt{7}}{2} \Rightarrow \text{stable spiral}$$

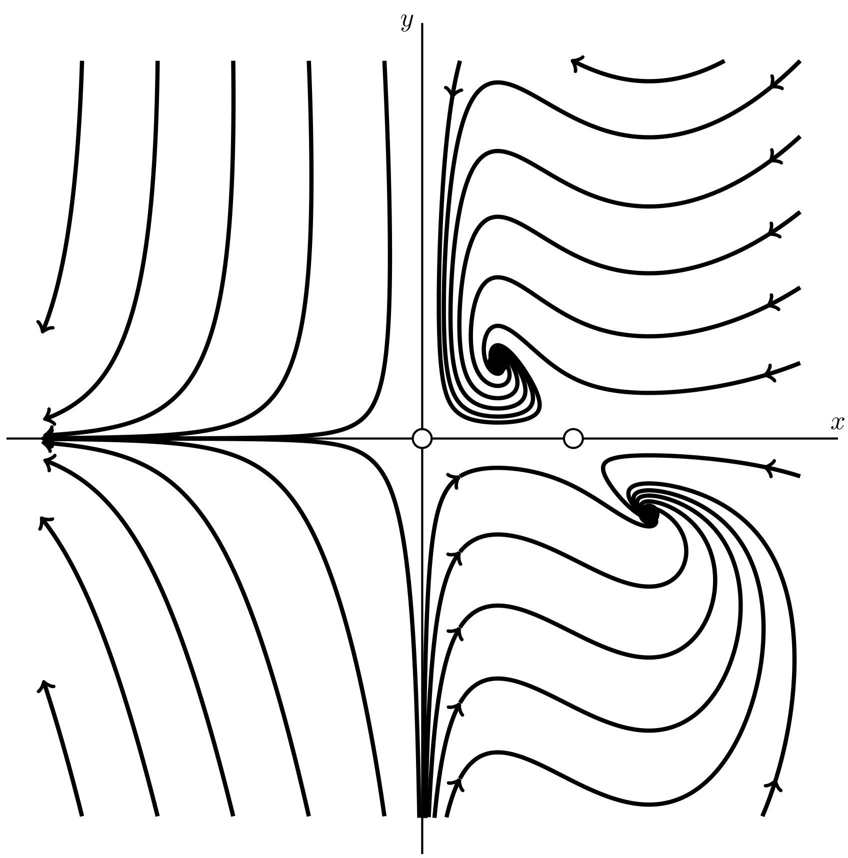
$$A_{(2,0)} = \begin{pmatrix} -2 & -2 \\ 0 & 1 \end{pmatrix}$$

$$\lambda_1 = -2 \quad \lambda_2 = 1 \Rightarrow \text{saddle point}$$

$$A_{(3,-1)} = \begin{pmatrix} -3 & -3 \\ 2 & 0 \end{pmatrix}$$

$$\lambda_{1,2} = \frac{-3 \pm i\sqrt{15}}{2} \Rightarrow \text{stable spiral}$$

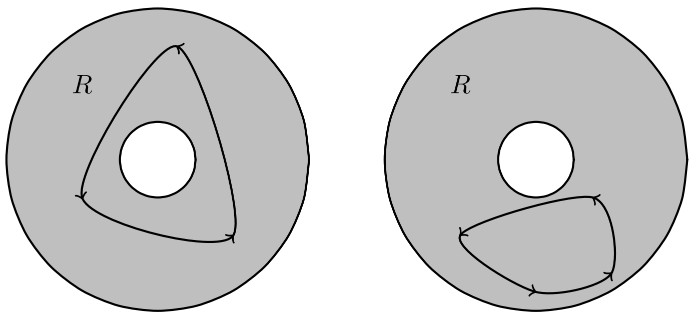
b)



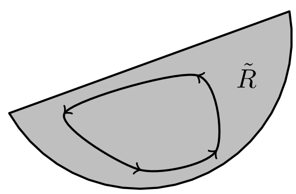
7.2.17

We'll prove this by contradiction by assuming that there are two or more closed orbits in R. Next we assume all the conditions of Dulac's criterion, except that R is not simply connected and has exactly one hole in it.

There are two possible properties for the closed orbits. Either a closed orbit does not encircle the hole or the closed orbit does encircle the hole.



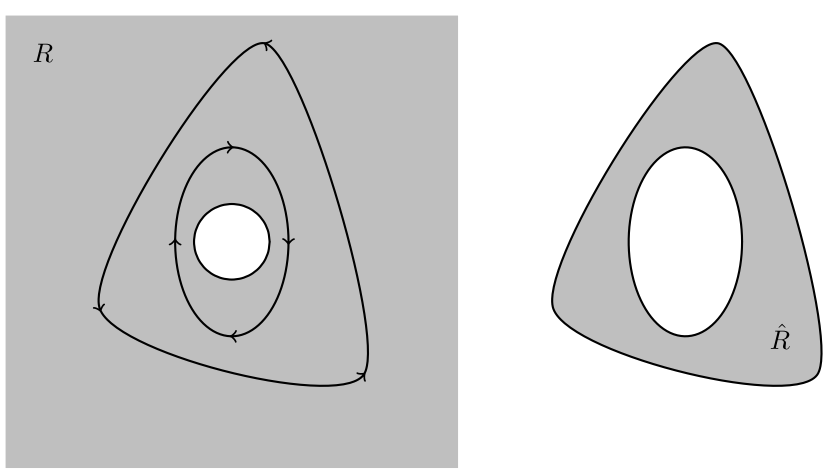
We can actually rule out the former case immediately by trimming the R into a region $\tilde{R}$ that is simply connected.



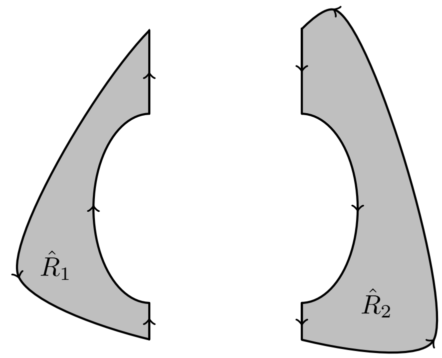
Then applying Dulac's criterion rules out the existence of the closed orbit.

As for when the closed orbit encircles the hole, zero or one closed orbit are possible inside R, but two or more leads to a contradiction.

If there are two closed orbits encircling the hole, then one closed orbit must encircle the inner closed orbit, and both closed orbits are the boundary to a region $\hat{R}$ they enclose.



We can then split the region $\hat{R}$ into two simply connected regions and apply Green's theorem to each.



$$\begin{aligned} 0 \neq \int_{\hat{R}} \nabla \cdot (g\dot{\mathbf{x}}) \, \mathrm{d}A &= \int_{\hat{R}_1} \nabla \cdot (g\dot{\mathbf{x}}) \, \mathrm{d}A + \int_{\hat{R}_2} \nabla \cdot (g\dot{\mathbf{x}}) \, \mathrm{d}A \\ &= \oint_{\partial \hat{R}_1} g\dot{\mathbf{x}} \cdot \mathbf{n} \, \mathrm{d}\ell + \oint_{\partial \hat{R}_2} g\dot{\mathbf{x}} \cdot \mathbf{n} \, \mathrm{d}\ell \\ &= \oint_{\partial \hat{R}} g\dot{\mathbf{x}} \cdot \mathbf{n} \, \mathrm{d}\ell = 0 \end{aligned}$$

(The orientation of the closed orbits does not matter since the the normal vector is perpendicular regardless if the closed orbit is traversed backwards or forwards.)

And here is our contradiction of $0 \neq 0$. All the steps are correct, which leaves the assumption at the very beginning that this region exists at all, meaning there can't be two closed orbits encircling the hole in R. More than two closed orbits are also ruled out by the same reasoning.

Hence the conditions of Dulac's criterion with R as a region topologically equivalent to an annulus instead of a simply connected region implies that there is at most one closed orbit in the region R.

7.2.19

$$\dot{R} = -R + A_S + kSe^{-S} \qquad \dot{S} = -S + A_R + kRe^{-R} \qquad k, A_S, A_R > 0$$

a)

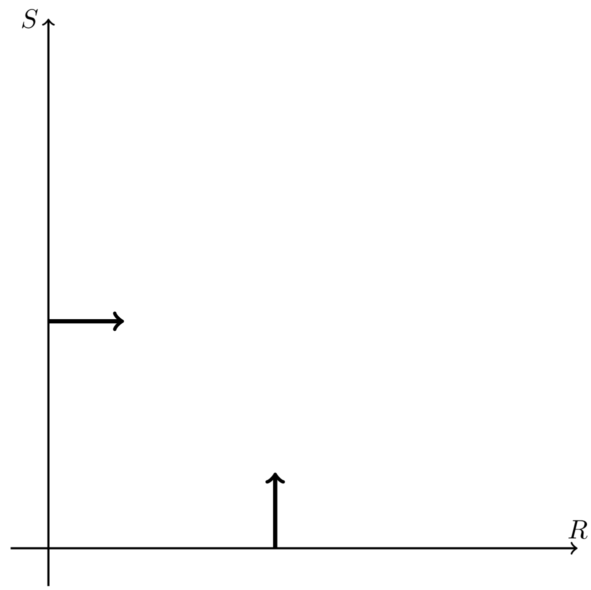
The $-R$ and $-S$ terms signify that the more love Rhett and Scarlett feel for the other, the faster their love for the other decreases.

The A_S and A_R are the baseline amounts of love one feels for the other. For instance, if Scarlett's love for Rhett is fixed at 0, then Rhett will still love Scarlett an amount A_S, and similarly for Scarlett when the roles are reversed.

The kSe^{-S} and kRe^{-R} are 0 for $R, S = 0$ and approximately 0 when R, S are large, with the maximum somewhere in between. The function having a maximum implies too little love. Too little love for each partner has effectively no effect, and there is an optimal amount of love for each partner that will cause the greatest increase in each partner's love.

b)

Making normal vectors pointing inward on the boundary of the first quadrant



and then dot product with the vector field on the boundary

$$\begin{aligned}
\langle 1,0\rangle \cdot \langle \dot{R}, \dot{S}\rangle\Big|_{R=0} &= \langle 1,0\rangle \cdot \langle -R + A_S + kSe^{-S}, -S + A_R + kRe^{-R}\rangle\Big|_{R=0} \\
&= \langle 1,0\rangle \cdot \langle A_S + kSe^{-S}, -S + A_R\rangle \\
&= A_S + kSe^{-S} > 0 \\
\langle 0,1\rangle \cdot \langle \dot{R}, \dot{S}\rangle\Big|_{S=0} &= \langle 0,1\rangle \cdot \langle -R + A_S + kSe^{-S}, -S + A_R + kRe^{-R}\rangle\Big|_{S=0} \\
&= \langle 0,1\rangle \cdot \langle -R + A_S, A_R + kRe^{-R}\rangle \\
&= A_R + kRe^{-R} > 0
\end{aligned}$$

meaning the vector field always points inward on the positive axes. Thus the trajectory can never escape the first quadrant.

c)

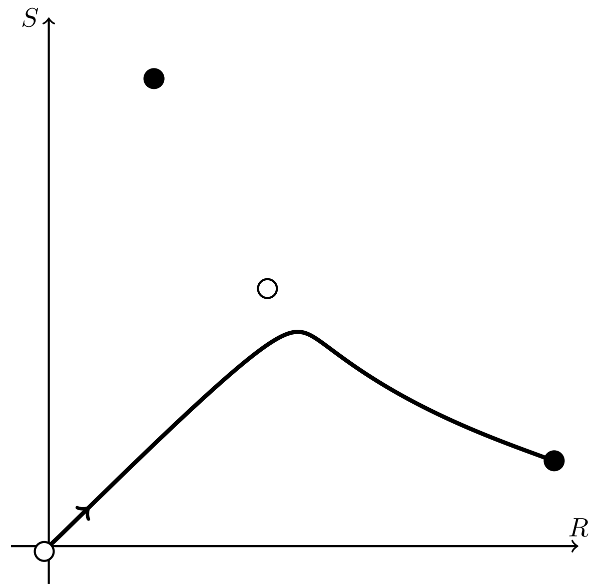
Picking $g(R, S) = 1$

$$\begin{aligned}
\nabla \cdot \langle \dot{R}, \dot{S}\rangle &= \nabla \cdot \langle -R + A_S + kSe^{-S}, -S + A_R + kRe^{-R}\rangle \\
&= (-1) + (-1) = -2
\end{aligned}$$

then by Dulac's criterion there are no periodic solutions, and not just in the first quadrant.

d)

$A_S = 1.2 \quad A_R = 1 \quad k = 15 \qquad R(0) = S(0) = 0$



7.3 Poincaré-Bendixson Theorem

7.3.1

$$\dot{x} = x - y - x(x^2 + 5y^2) \qquad \dot{y} = x + y - y(x^2 + y^2)$$

a)

Classify the fixed point at the origin.

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$(1-\lambda)^2 + 1 = 0 \Rightarrow \lambda^2 - 2\lambda + 2 = (\lambda - (1+i))\,(\lambda - (1-i)) = 0$$

$$\lambda_1 = 1 + i \quad \lambda_2 = 1 - i \Rightarrow \text{ unstable spiral}$$

b)

Rewrite the system in polar coordinates using

$$r\dot{r} = x\dot{x} + y\dot{y} \qquad \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$$

$$x\dot{x} = x^2 - xy - x^2(x^2 + 5y^2) \qquad y\dot{y} = xy + y^2 - y^2(x^2 + y^2)$$

$$\begin{aligned}
x\dot{x} + y\dot{y} &= x^2 - xy - x^2(x^2+5y^2) + xy + y^2 - y^2(x^2+y^2) \\
&= x^2 + y^2 - x^2(x^2+5y^2) - y^2(x^2+y^2) \\
&= x^2 + y^2 - x^4 - y^4 - 6x^2y^2 \\
&= r^2 - r^4\cos^4(\theta) - r^4\sin^4(\theta) - 6r^4\cos^2(\theta)\sin^2(\theta) \\
&= r^2 - \left(r^4\cos^4(\theta) + r^4\cos^2(\theta)\sin^2(\theta)\right) - \left(r^4\sin^4(\theta) + r^4\cos^2(\theta)\sin^2(\theta)\right) - 4r^4\cos^2(\theta)\sin^2(\theta) \\
&= r^2 - r^4\cos^2(\theta)\left(\cos^2(\theta)+\sin^2(\theta)\right) - r^4\sin^2(\theta)\left(\sin^2(\theta)+\cos^2(\theta)\right) - 4r^4\cos^2(\theta)\sin^2(\theta) \\
&= r^2 - r^4\cos^2(\theta) - r^4\sin^2(\theta) - 4r^4\cos^2(\theta)\sin^2(\theta) \\
&= r^2 - r^4\left(\cos^2(\theta)+\sin^2(\theta)\right) - 4r^4\cos^2(\theta)\sin^2(\theta) \\
&= r^2 - r^4 - 4r^4\cos^2(\theta)\sin^2(\theta) \\
&= r^2 - r^4 - r^4\left(2\cos(\theta)\sin(\theta)\right)^2 \\
&= r^2 - r^4 - r^4\sin^2(2\theta)
\end{aligned}$$

$$r\dot{r} = x\dot{x} + y\dot{y} = r^2 - r^4 - r^4\sin^2(2\theta) \Rightarrow \dot{r} = r - r^3 - r^3\sin^2(2\theta) = r\left(1 - r^2 - r^2\sin^2(2\theta)\right)$$

$$\begin{aligned}
x\dot{y} &= x\left(x + y - y(x^2+y^2)\right) = x^2 + xy - xy(x^2+y^2) \\
y\dot{x} &= y\left(x - y - x(x^2+5y^2)\right) = xy - y^2 - xy(x^2+5y^2)
\end{aligned}$$

$$\begin{aligned}
x\dot{y} - y\dot{x} &= x^2 + y^2 - xy(x^2+y^2) - \left(xy - y^2 - xy(x^2+5y^2)\right) \\
&= x^2 + y^2 + xy(-x^2 - y^2 + x^2 + 5y^2) \\
&= x^2 + y^2 + 4xy^3 \\
&= r^2 + 4r^4\cos(\theta)\sin^3(\theta)
\end{aligned}$$

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = 1 + 4r^2\cos(\theta)\sin^3(\theta)$$

c)

Determine the circle of maximum radius, r_1, centered on the origin such that all trajectories have a radially *outward* component on it.

This is equivalent to $0 \leq \dot{r}$ for all θ.

$$\begin{aligned}
0 \leq r\left(1 - r^2 - r^2\sin^2(2\theta)\right) &\Rightarrow 0 \leq 1 - r^2 - r^2\sin^2(2\theta) \\
&\Rightarrow 0 \leq r \leq \sqrt{\frac{1}{1+\sin^2(2\theta)}} \\
&\Rightarrow 0 \leq r \leq \frac{1}{\sqrt{2}} = r_1
\end{aligned}$$

d)

Determine the circle of minimum radius, r_2, centered on the origin such that all trajectories have a radially *inward* component on it.

$$r\left(1 - r^2 - r^2\sin^2(2\theta)\right) \leq 0 \Rightarrow 1 - r^2 - r^2\sin^2(2\theta) \leq 0 \qquad \Rightarrow \sqrt{\frac{1}{1+\sin^2(2\theta)}} \leq r \Rightarrow r_2 = 1 \leq r$$

e)

We know that there is at least one limit cycle in the region $\frac{1}{\sqrt{2}} \leq r \leq 1$ by the Poincaré-Bendixson theorem.

The trapping region is a closed and bounded subset of the plane.

The vector field is continuously differentiable, easily seen in the Cartesian representation, on an open set containing the trapping region.

The trapping region does not contain any fixed points.

$$\dot{r} = 0 \Rightarrow r\left(1 - r^2 - r^2\sin^2(2\theta)\right) = 0 \Rightarrow r = 0 \text{ or } r^2 = \frac{1}{1+\sin^2(2\theta)}$$

$$\dot{\theta} = 0 \Rightarrow 1 + 4r^2\cos(\theta)\sin^3(\theta) = 0 \Rightarrow r^2 = \frac{-1}{4\cos(\theta)\sin^3(\theta)}$$

$$\begin{aligned}
\frac{1}{1+\sin^2(2\theta)} = \frac{-1}{4\cos(\theta)\sin^3(\theta)} &\Rightarrow 4\cos(\theta)\sin^3(\theta) = -1 - \sin^2(2\theta) \\
&\Rightarrow \sin^2(2\theta)\left(4\cos(\theta)\sin(\theta) + 1\right) = -1 \\
&\Rightarrow \sin^2(2\theta)\left(2\sin(2\theta) + 1\right) = -1 \\
&\Rightarrow 2\sin^3(2\theta) + \sin^2(2\theta) + 1 = 0 \\
&\Rightarrow \left(\sin(2\theta) + 1\right)\left(2\sin^2(2\theta) - \sin(2\theta) + 1\right) = 0
\end{aligned}$$

The only real solution is $\sin(2\theta) = -1 \Rightarrow \theta = \frac{3\pi}{2} \Rightarrow x = 0 \Rightarrow y = 0$.

7.3.3

$$\dot{x} = x - y - x^3 \qquad \dot{y} = x + y - y^3$$

The only fixed point of the system is the origin, and linearization predicts the origin is an unstable spiral.

We'll make a trapping region by converting to polar coordinates.

$$\begin{aligned}
r^2 &= x^2 + y^2 \\
2r\dot{r} &= 2x\dot{x} + 2y\dot{y} \\
\dot{r} &= \frac{x(x - y - x^3) + y(x + y - y^3)}{r} \\
&= \frac{x^2 + y^2 - x^4 - y^4}{r} \\
&= \frac{r^2 - r^4\cos^4(\theta) - r^4\sin^4(\theta)}{r}
\end{aligned}$$

$$= r - r^3 \cos^4(\theta) - r^3 \sin^4(\theta)$$
$$= r - r^3\big(\cos^4(\theta) + \sin^4(\theta)\big)$$

We can bound the θ term with $\frac{1}{2} \leq \cos^4(\theta) + \sin^4(\theta) \leq 1$, which gives

$$r - r^3 \leq \dot{r} \leq r - \frac{r^3}{2}$$

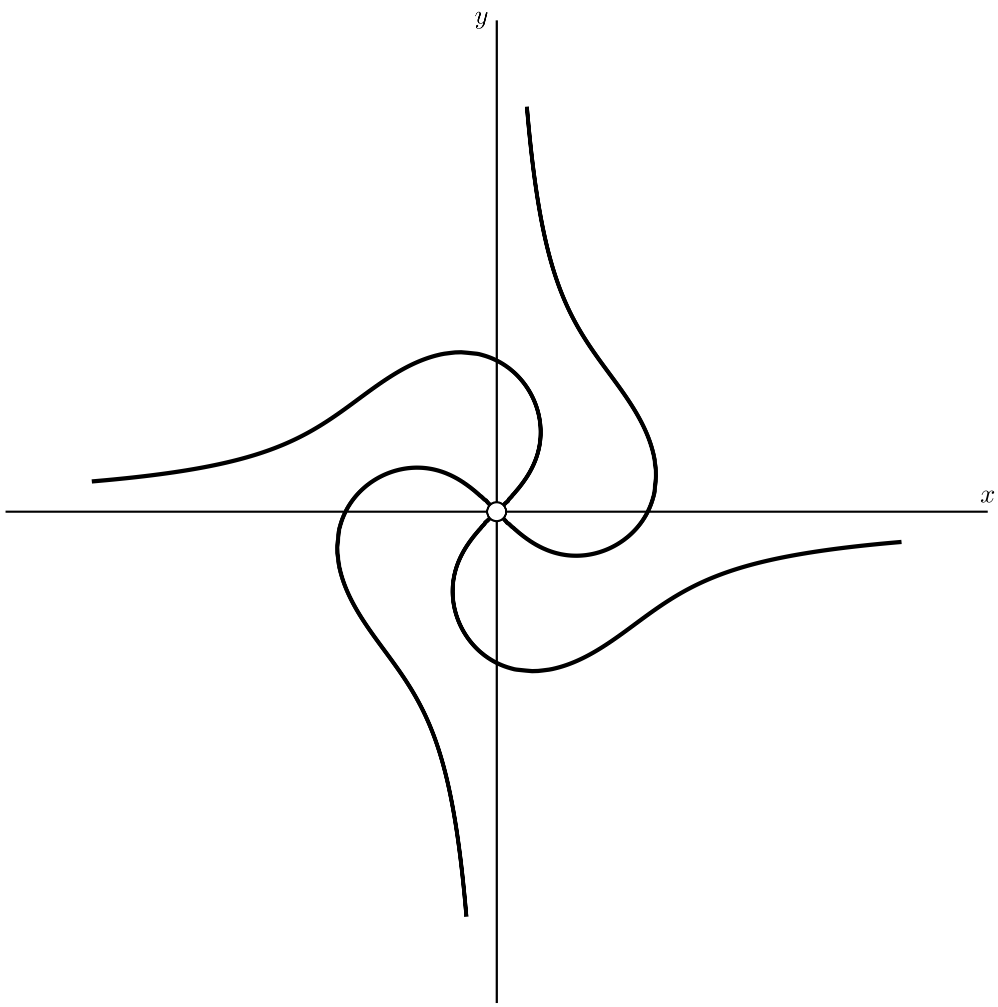
So a suitable trapping region is bounded by a circle of radius less than 1, which has a positive radial component, and a circle of radius greater than $\sqrt{2}$, which has a negative radial component.

Then by the Poincaré-Bendixson theorem, there is at least one periodic solution in the trapping region.

7.3.5

$$\dot{x} = -x - y + x(x^2 + 2y^2) \qquad \dot{y} = x - y + y(x^2 + 2y^2)$$

This system has a fixed point at $(x, y) = (0, 0)$, which is best seen using a graph of the nullclines.



$$A = \begin{pmatrix} 3x^2 + 2y^2 - 1 & 4xy - 1 \\ 2xy + 1 & x^2 + 6y^2 - 1 \end{pmatrix}$$

$$A_{(0,0)} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$\lambda_{1,2} = -1 \pm i \Rightarrow \text{stable spiral}$$

The Poincaré-Bendixson theorem can still be applied here, despite the fixed point being unstable. In this case the trapping region will be more of a repulsion region, as the vector field will point outward on the boundaries. Thus there will be at least one unstable limit cycle in the region. (You can see this by transforming $t \to -t$, which runs time backwards for the system. Then we're showing existence for the usual stable limit cycle.)

Next we'll transform the system into polar coordinates to find the boundaries of the trapping region.

$$\begin{aligned}
r^2 &= x^2 + y^2 \\
2r\dot{r} &= 2x\dot{x} + 2y\dot{y} \\
\dot{r} &= \frac{x\dot{x} + y\dot{y}}{r} \\
&= \frac{x\big(-x - y + x(x^2 + 2y^2)\big) + y\big(x - y + y(x^2 + 2y^2)\big)}{r} \\
&= \frac{x^4 + 3x^2y^2 - x^2 + 2y^4 - y^2}{r} \\
&= \frac{(x^2 + y^2)(x^2 + 2y^2 - 1)}{r} \\
&= \frac{r^2(r^2 + y^2 - 1)}{r} \\
&= r\big(r^2 + r^2\sin^2(\theta) - 1\big)
\end{aligned}$$

From here we can upper and lower bound the θ term to bound $\dot{r}$.

$$r(r^2 - 1) \leq \dot{r} \leq r(2r^2 - 1)$$

So a suitable trapping region is bounded by a circle of radius less than $\frac{1}{\sqrt{2}}$, which has a negative radial component, and a circle of radius greater than 1, which has a positive radial component.

Then by the Poincaré-Bendixson theorem, there is at least one periodic solution in the trapping region.

7.3.7

$$\dot{x} = y + ax(1-2b-r^2) \qquad \dot{y} = -x + ay(1-r^2) \qquad 0 < a \leq 1 \quad 0 \leq b < \frac{1}{2}$$

a)

$$\begin{aligned}
\dot{r} &= \frac{x\dot{x} + y\dot{y}}{r} \\
&= \frac{x\big(y + ax(1-2b-r^2)\big) + y\big(-x + ay(1-r^2)\big)}{r} \\
&= \frac{-2abx^2 - ar^2x^2 - ar^2y^2 + ax^2 + ay^2}{r} \\
&= \frac{a\big(-2bx^2 - r^4 + r^2\big)}{r} \\
&= \frac{a\big(-2br^2\cos^2(\theta) - r^4 + r^2\big)}{r} \\
&= ar\big(1 - r^2 - 2b\cos^2(\theta)\big)
\end{aligned}$$

$$\begin{aligned}
\theta &= \tan\left(\frac{y}{x}\right) \\
\dot{\theta} &= \frac{x\dot{y} - \dot{x}y}{x^2 + y^2} \\
&= \frac{x\big(-x + ay(1-r^2)\big) - \big(y + ax(1-2b-r^2)\big)y}{r^2} \\
&= \frac{2abxy - x^2 - y^2}{r^2} \\
&= \frac{2abr^2\sin(\theta)\cos(\theta) - r^2}{r^2} \\
&= 2ab\sin(2\theta) - 1
\end{aligned}$$

b)

A fixed point can occur only if $\dot{r} = \dot{\theta} = 0$ or $r = \dot{r} = 0$ is true. The ranges of a and b and the $\dot{\theta}$ equation make the latter impossible, leaving the origin as the only fixed point.

We can make an annular trapping region using $ar(1-r^2) \leq \dot{r} \leq ar(1-2b-r^2)$, for which $1 < r \Rightarrow \dot{r} < 0$, and $\sqrt{1-2b} < r \Rightarrow \dot{r} > 0$ will have all vectors pointing inward on the boundary. Therefore, by the Poincaré-Bendixson theorem, there is at least one limit cycle in the trapping region.

We also know that all limit cycles have the same period because $\dot{\theta} = 2ab\sin(2\theta) - 1$ is independent of r, meaning the rotation rate is the same at every θ value no matter which limit cycle is being traversed.

c)

Setting $b = 0$ simplifies the equations to

$$\dot{r} = ar\big(1 - r^2\big) \quad \dot{\theta} = -1$$

for which $r < 1 \Rightarrow \dot{r} > 0$ and $1 < r \Rightarrow \dot{r} > 0$, making $r = 1$ the only limit cycle.

7.3.9

$$\dot{r} = r(1-r^2) + \mu r\cos(\theta) \quad \dot{\theta} = 1$$

a)

$$\begin{aligned}
&\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\theta} = r(1-r^2) + \mu r\cos(\theta) \\
&r(\theta) = 1 + \mu r_1(\theta) + O\big(\mu^2\big) \\
&\frac{\mathrm{d}r}{\mathrm{d}\theta} = \mu r_1'(\theta) \\
&\dot{r} = r(1-r^2) + \mu r\cos(\theta) \\
&\mu r_1'(\theta) = \Big(1 + \mu r_1(\theta)\Big)\Big(1 - \big(1 + \mu r_1(\theta)\big)^2\Big) + \mu\Big(1 + \mu r_1(\theta)\Big)\cos(\theta) \\
&r_1' = -2r_1 + \cos(\theta) \\
&r_1 = \frac{1}{5}\sin(t) + \frac{2}{5}\cos(\theta) + Ce^{-2t}
\end{aligned}$$

The exponential term eventually dies off, leaving only the periodic trajectory.

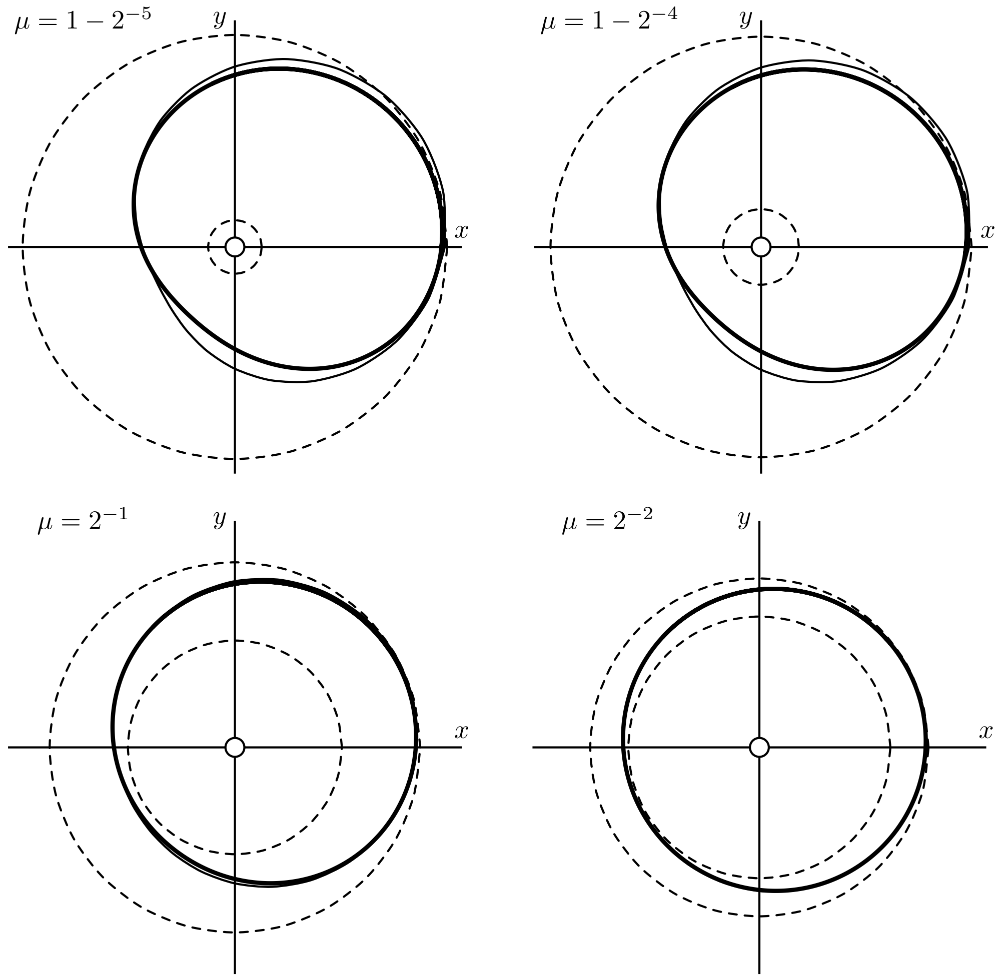
$$r_1 = \frac{1}{5}\sin(\theta) + \frac{2}{5}\cos(\theta) \qquad r(\theta) = 1 + \mu\left(\frac{1}{5}\sin(\theta) + \frac{2}{5}\cos(\theta)\right) + O\big(\mu^2\big)$$

b)

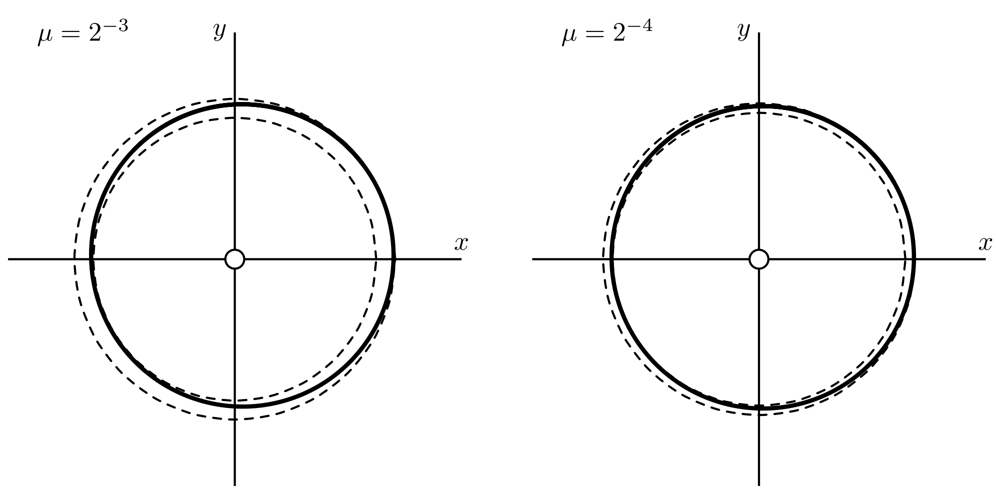
$$\begin{aligned}
&\frac{\mathrm{d}r}{\mathrm{d}\theta} = \mu\left(\frac{1}{5}\cos(\theta) - \frac{2}{5}\sin(\theta)\right) = 0 \\
&\frac{1}{5}\cos(\theta) - \frac{2}{5}\sin(\theta) = 0 \Rightarrow \theta = \arctan\left(\frac{1}{2}\right)\ ,\ \arctan\left(\frac{1}{2}\right) + \pi \\
&r\left(\arctan\left(\frac{1}{2}\right)\right) = 1 + \mu\left(\frac{1}{5}\frac{1}{\sqrt{5}} + \frac{2}{5}\frac{2}{\sqrt{5}}\right) + O\big(\mu^2\big) = 1 + \frac{\mu}{\sqrt{5}} + O\big(\mu^2\big) \\
&r\left(\arctan\left(\frac{1}{2}\right) + \pi\right) = 1 + \mu\left(\frac{1}{5}\frac{-1}{\sqrt{5}} + \frac{2}{5}\frac{-2}{\sqrt{5}}\right) + O\big(\mu^2\big) = 1 - \frac{\mu}{\sqrt{5}} + O\big(\mu^2\big) \\
&\sqrt{1-\mu} < 1 - \frac{\mu}{\sqrt{5}} < r < 1 + \frac{\mu}{\sqrt{5}} < \sqrt{1+\mu} \qquad \text{for } \mu \ll 1
\end{aligned}$$

c)

The dashed circles are the error bounds $r = \sqrt{1-\mu}$ and $r = \sqrt{1+\mu}$, the thick solid line is the numerical solution, and the thin solid line is the series approximation solution.



The numerical solution and series approximation solution become indistinguishable on the graphs, but the upper and lower bounds continue to restrict the closed orbit to a more and more circular path as μ decreases.



As for how the maximum error in r depends on μ, some values have been computed below.

μ	Error	μ	Error
$1/8$	0.0016312	$1/32$	9.8704e-05
$2/8$	0.0068165	$2/32$	3.9907e-04
$3/8$	0.016030	$3/32$	9.0765e-04
$4/8$	0.029790	$4/32$	0.0016312
$5/8$	0.048636	$5/32$	0.0025766
$6/8$	0.073110	$6/32$	0.0037510
$7/8$	0.10371	$7/32$	0.0051618

Using only the right side as the data to a quadratic fit gives

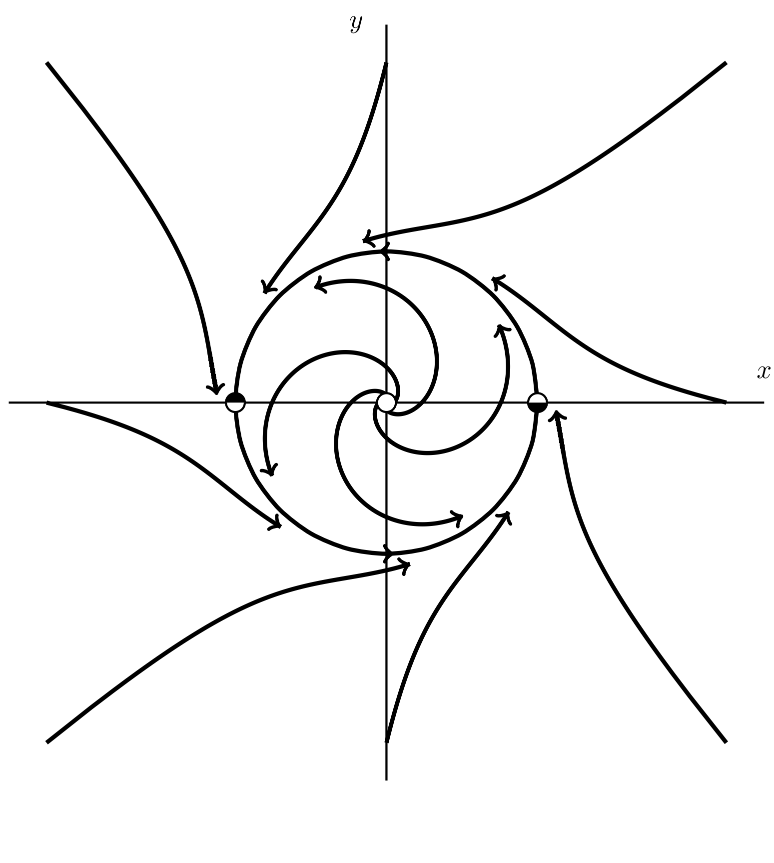
$$(1.1368\text{e-}01)\mu^2 + (-1.4908\text{e-}03)\mu + (4.1271\text{e-}05)$$

The μ^2 coefficient is dominant in the polynomial, giving us a little reassurance that the error really is $O\big(\mu^2\big)$ for $\mu \ll 1$.

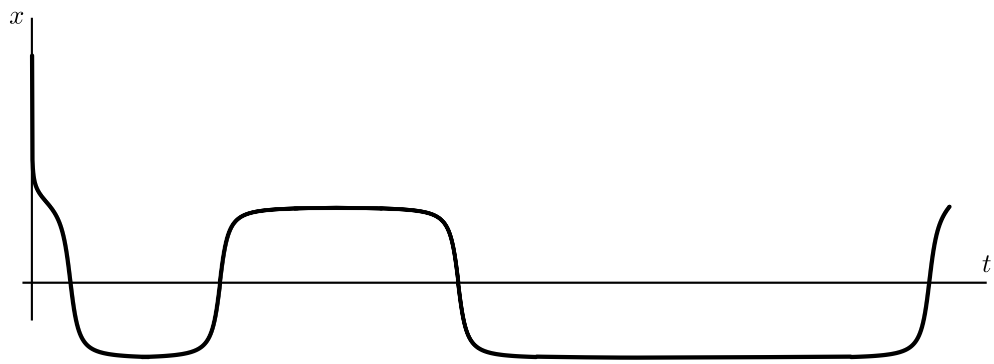
7.3.11

$$\dot{r} = r(1-r^2)\big[r^2\sin^2(\theta) + (r^2\cos^2(\theta)-1)^2\big] \qquad \dot{\theta} = r^2\sin^2(\theta) + (r^2\cos^2(\theta)-1)^2$$

a)



b)



The above graph has the qualitative behavior of the system, but the real graph has greatly exaggerated pauses.

The trajectory here starts at $(r, \theta) = (3, 0)$, which is $(x, y) = (3, 0)$, and quickly asymptotes to the circle $r = 1$. The trajectory then follows the circumference for all time, but what shows in the x versus t graph is that the trajectory is spending more and more time near the fixed points $(x, y) = (\pm 1, 0)$ with each traversal because $r \to 1$ as $t \to \infty$, which makes $\dot{\theta} \approx 0$ near the fixed points.

7.4 Liénard Systems

7.4.1

$$\ddot{x} + \mu(x^2 - 1) + \tanh(x) = 0$$

Applying Liénard's theorem

(1)

$$f(x) = \mu(x^2 - 1) \quad g(x) = \tanh(x)$$

which are both continuously differentiable for all x.

(2)

$$g(-x) = \tanh(-x) = -\tanh(x) = -g(x)$$

$g(x)$ is odd.

(3)

$$g(x) > 0 \quad \text{for } x > 0$$

(4)

$$f(-x) = \mu\big((-x)^2 - 1\big) = \mu(x^2 - 1) = f(x)$$

$f(x)$ is even.

(5)

$$\begin{aligned} F(x) &= \int_0^x f(u)\mathrm{d}u = \int_0^x \mu(u^2-1)\mathrm{d}u = \mu\left(\frac{x^3}{3} - x\right) = \frac{1}{3}\mu(x^3 - 3x) \\ &= \frac{1}{3}\mu x(x+\sqrt{3})(x-\sqrt{3}) \end{aligned}$$

$F(x)$ has exactly one positive root at $x = \sqrt{3}$.
$F(x)$ is negative for $0 < x < \sqrt{3}$.
$F(x)$ is positive and nondecreasing for $\sqrt{3} < x$.
and $F(x) \to \infty$ as $x \to \infty$.

All the properties of Liénard's theorem are satisfied; therefore the system has a unique stable limit cycle surrounding the origin.

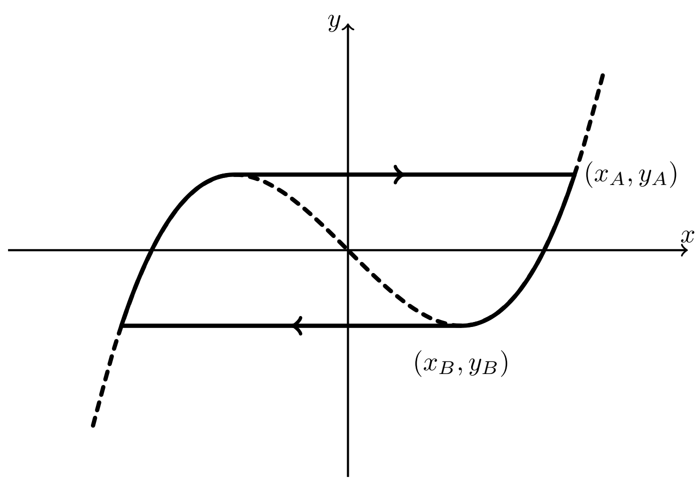
7.5 Relaxation Oscillations

7.5.1

For the van der Pol oscillator with $\mu >> 1$, show that the positive branch of the cubic nullcline begins at $x_A = 2$ and ends at $x_B = 1$.

The cubic nullcline is from

$$\dot{x} = \mu[y - F(x)] \qquad F(x) = \frac{1}{3}x^3 - x$$



x_B occurs at the local minimum
$F'(x) = x^2 - 1 = 0 \Rightarrow x = \pm 1 \Rightarrow x_B = 1$

x_A occurs at the intersection of the tangent line of the local maximum with $F(x)$

$F(-1) = y_A = \frac{2}{3} = F(x) \Rightarrow x = -1, 2 \Rightarrow x_A = 2$

7.5.3

$$\ddot{x} + k(x^2 - 4)\dot{x} + x = 1$$

We can do a substitution to transform the equation into the form of Example 7.5.1.

$$x \to z + 1 \qquad \ddot{x} + k(x^2 - 4)\dot{x} + x = 1 \to \ddot{z} + k\big((z+1)^2 - 4\big)\dot{z} + (z+1) = 1$$

$$\begin{aligned}
&0 = \ddot{z} + k\big((z+1)^2 - 4\big)\dot{z} + z = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{z} + k\left(\frac{1}{3}(z+1)^3 - 4z\right)\right) \\
&F(z) = \frac{1}{3}(z+1)^3 - 4z \\
&w = \dot{z} + kF(z) \quad \dot{w} = \ddot{z} + k\big((z+1)^2 - 4\big)\dot{z} = -z \\
&y = \frac{w}{k} \quad \dot{z} = k\big[y - F(z)\big] \quad \dot{y} = -\frac{z}{k}
\end{aligned}$$

Now that we transformed the equation into something similar to Example 7.5.1, we can perform the integral over one branch, then multiply by 2 for the approximate period.

$$\begin{aligned}
&T \approx 2\int_{t_A}^{t_B} \mathrm{d}t \\
&-\frac{z}{k} = \frac{\mathrm{d}y}{\mathrm{d}t} \approx F'(z)\frac{\mathrm{d}z}{\mathrm{d}t} = \big((z+1)^2 - 4\big)\frac{\mathrm{d}z}{\mathrm{d}t} \\
&\mathrm{d}t \approx \frac{-k\big((z+1)^2 - 4\big)}{z}\mathrm{d}z
\end{aligned}$$

Next we have to find the turning point of the limit cycle for the integral bounds.

$$\begin{aligned}
&F'(z) = (z+1)^2 - 4 = z^2 + 2z - 3 = (z+3)(z-1) \\
&z_A = 1 \\
&z_B \neq -3 \text{ and } F(-3) = \frac{28}{3} = F(z_B) \Rightarrow z_B = 3
\end{aligned}$$

And now we can compute the integral for the period.

$$T \approx 2\int_{t_A}^{t_B} \mathrm{d}t = -2k\int_3^1 \frac{(z+1)^2 - 4}{z}\mathrm{d}z = 2k\ln\big(8 - \ln(27)\big)$$

7.5.5

$$\ddot{x} + \mu\big(|x| - 1\big)\dot{x} + x = 0$$

Define

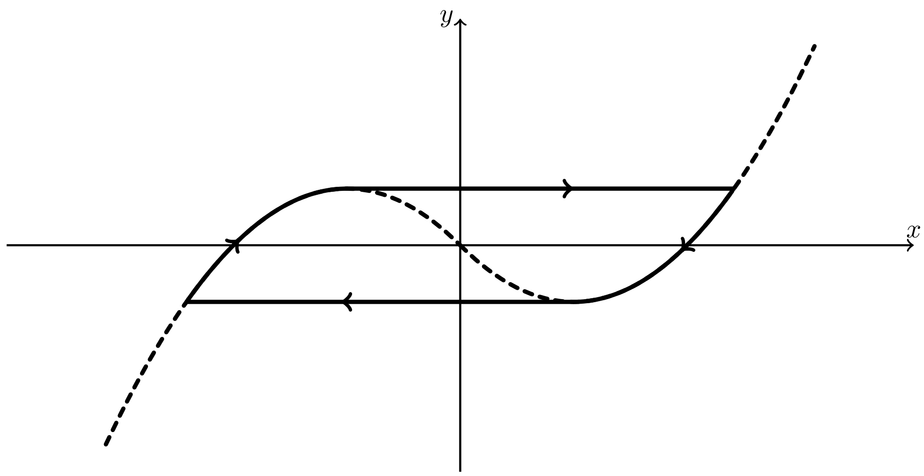
$$\dot{x} = \mu\big(y - F(x)\big) \qquad \dot{y} = -\frac{x}{\mu}$$

$$F(x) = \begin{cases} -\frac{x^2}{2} - x & x \le 0 \\ \frac{x^2}{2} - x & 0 < x \end{cases} \quad \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x} F(x) = \begin{cases} x - 1 & x \le 0 \\ -x - 1 & 0 < x \end{cases} \quad = |x| - 1$$

Verifying these two systems are the same,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\dot{x} = \ddot{x} &= \frac{\mathrm{d}}{\mathrm{d}t}\mu\big(y - F(x)\big) = \mu\big(\dot{y} - \mu F'(x)\dot{x}\big) = \mu\dot{y} - \mu\big(|x| - 1\big)\dot{x} \\ &= \mu\left(-\frac{x}{\mu}\right) - \mu\big(|x| - 1\big)\dot{x} = -x - \mu\big(|x| - 1\big)\dot{x} \\ \ddot{x} &= -x - \mu\big(|x| - 1\big)\dot{x} \Rightarrow \ddot{x} + \mu\big(|x| - 1\big)\dot{x} + x = 0 \end{aligned}$$

The period is approximately twice the traversal time of the right branch.



$$\begin{aligned} &T \approx 2\int_{t_A}^{t_B} \mathrm{d}t \\ &\dot{y} = \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{x}{\mu} \Rightarrow \mathrm{d}t = -\mu\frac{\mathrm{d}y}{x} \\ &0 \le x \quad y = F(x) = -\frac{x^2}{2} - x \Rightarrow x = 1 \pm \sqrt{1 + 2y} \end{aligned}$$

And here we need to pick the correct side of the parabola to integrate.

$$T \approx -2\mu \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{\mathrm{d}y}{x} = -2\mu \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{\mathrm{d}y}{1 + \sqrt{1 + 2y}} = 2\mu\Big(\sqrt{2} - \ln\big(1 + \sqrt{2}\big)\Big)$$

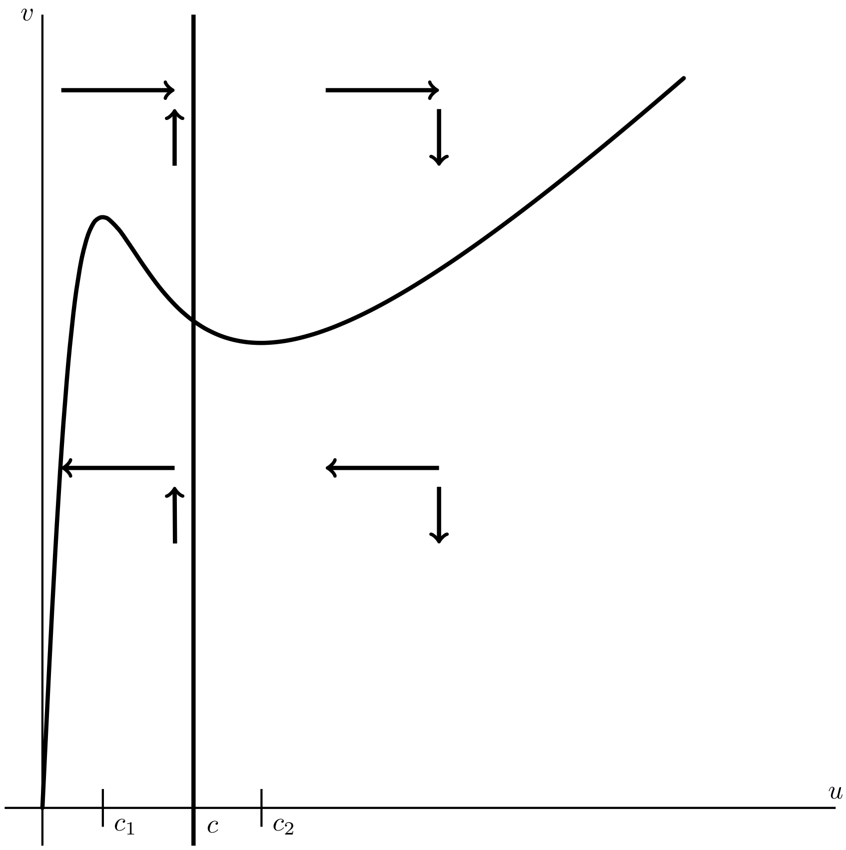
7.5.7

$$\dot{u} = b(v - u)(\alpha + u^2) - u \qquad \dot{v} = c - u \qquad b \gg 1 \quad \alpha \ll 1 \quad 8\alpha b < 1$$

ab)

The nullclines are

$$u = c \qquad v = u\left(\frac{1}{b(\alpha + u^2)} + 1\right)$$



The vector field is similar to Example 7.5.1 as shown in the graph. Therefore there should be a stable limit cycle as long as the vertical nullcline lies between the local maximum and local minimum, but we also have to check that the intersection is an unstable fixed point.

Assuming $8\alpha b \ll 1$, the local maximum and minimum occur when

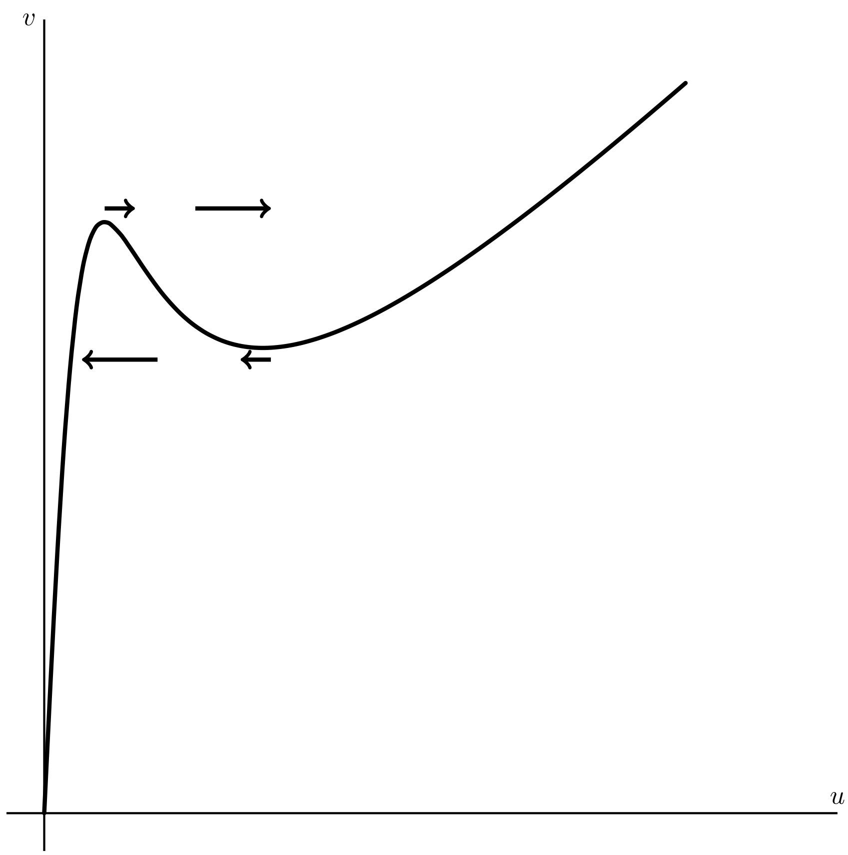
$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}u}v = \frac{\mathrm{d}}{\mathrm{d}u}\left(u\left(\frac{1}{b(\alpha+u^2)}+1\right)\right) = \frac{(\alpha+u^2)-2u^2}{b(\alpha+u^2)^2}+1=0\\
&(\alpha+u^2)-2u^2+b(\alpha+u^2)^2=0\\
&bu^4+(2\alpha b-1)u^2+\alpha^2 b+\alpha=0\\
&u^2=\frac{-(2\alpha b-1)\pm\sqrt{(2\alpha b-1)^2-4(b)(\alpha^2 b+\alpha)}}{2b}\\
&\quad=\frac{1-2\alpha b\pm\sqrt{1-8\alpha b}}{2b}=\frac{1-2\alpha b\pm\left(1-\frac{8\alpha b}{2}+O\big((\alpha b)^2\big)\right)}{2b}\\
&\quad\approx\frac{1-2\alpha b\pm(1-4\alpha b)}{2b}=2\alpha,\ \frac{1-3\alpha b}{b}
\end{aligned}
$$

We can determine stability by linearizing.

$$A = \begin{pmatrix} -\alpha b - 3bu^2 + 2buv - 1 & b(\alpha + u^2) \\ -1 & 0 \end{pmatrix}$$

$$\Delta = b(\alpha + c^2) > 0 \quad \tau = -\alpha b - 3bu^2 + 2buv - 1$$

The stability of the fixed point is determined by the sign on τ, which is the upper-left element. $\frac{\partial \dot{u}}{\partial u} < 0$ implies the point is stable, and $\frac{\partial \dot{u}}{\partial u} > 0$ implies it is unstable. We can actually see from the graph that the unstable point occurs only between the local maximum and local minimum.

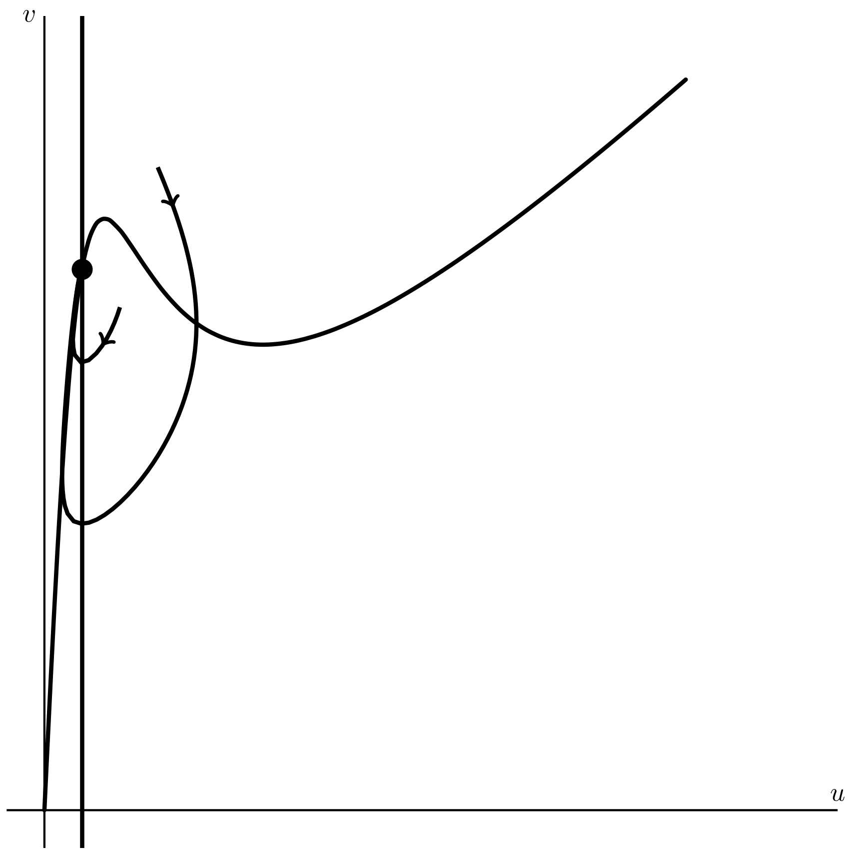


The $\dot{u}$ has a positive increase to the rightward direction, either starting positive and becoming more positive above the nullcline, or starting negative and increasing towards zero below the nullcline.

Therefore $\frac{\partial \dot{u}}{\partial u} > 0$ when c lies on the middle branch. Thus the fixed point is unstable and there is a stable limit cycle when

$$2\alpha \approx c_1 < c < c_2 \approx \frac{1 - 3\alpha b}{b}$$

c)



The fixed point is stable when $c < c_1$ and all trajectories eventually approach the fixed point, but the route can be very long depending on the initial condition. Consequently a bit of noise in the system can have varying amounts of effect depending on which way the noise nudges the system.

7.6 Weakly Nonlinear Oscillators

7.6.1

$$
\begin{aligned}
x(t,\epsilon) &= \frac{\sin(t\sqrt{1-\epsilon^2})}{e^{\epsilon t}\sqrt{1-\epsilon^2}} \\
&= x(t,0) + \epsilon \frac{\partial x}{\partial \epsilon}\bigg|_{(t,0)} + \frac{\epsilon^2}{2}\frac{\partial^2 x}{\partial \epsilon^2}\bigg|_{(t,\xi)} \qquad 0 < \xi < \epsilon \\
x(t,0) &= \frac{\sin(t\sqrt{1-\epsilon^2})}{e^{\epsilon t}\sqrt{1-\epsilon^2}}\bigg|_{(t,0)} = \sin(t) \\
\frac{\partial x}{\partial \epsilon} &= -\frac{te^{-\epsilon t}\sin(\sqrt{1-\epsilon^2}t)}{\sqrt{1-\epsilon^2}} + \frac{\epsilon e^{-\epsilon t}\sin(\sqrt{1-\epsilon^2}t)}{(1-\epsilon^2)^{3/2}} - \frac{\epsilon t e^{-\epsilon t}\cos(\sqrt{1-\epsilon^2}t)}{1-\epsilon^2} \\
\frac{\partial x}{\partial \epsilon}\bigg|_{(t,0)} &= -t\sin(t) \\
x(t,\epsilon) &= \sin(t) - \epsilon t \sin(t) + O\big(\epsilon^2\big)
\end{aligned}
$$

7.6.3

$$\ddot{x} + x = \epsilon \qquad x(0) = 1 \quad \dot{x}(0) = 0$$

a)

First we solve the homogeneous equation.

$$\ddot{x} + x = 0$$
$$x(t) = c_1 \sin(t) + c_2 \cos(t)$$

Next we can use the method of undetermined coefficients to deduce the nonhomogeneous solution.

$$\ddot{x} + x = \epsilon$$
$$x = At^2 + Bt + C \quad \dot{x} = 2At + B \quad \ddot{x} = 2A$$
$$2A + At^2 + Bt + C = \epsilon \Rightarrow x = \epsilon$$

Now applying the initial conditions

$$x = \epsilon + c_1 \sin(t) + c_2 \cos(t)$$
$$x(0) = \epsilon + c_2 = 1 \Rightarrow c_2 = 1 - \epsilon$$
$$\dot{x}(0) = c_1 = 0$$
$$x = \epsilon + (1 - \epsilon) \cos(t)$$

b)

$$x(t, \epsilon) = x_0(t, \epsilon) + \epsilon x_1(t, \epsilon) + \epsilon^2 x_2(t, \epsilon) + O(\epsilon^3)$$
$$\ddot{x} + x = \epsilon \to (\ddot{x}_0 + \epsilon \ddot{x}_1 + \epsilon^2 \ddot{x}_2) + (x_0 + \epsilon x_1 + \epsilon^2 x_2) = \epsilon$$
$$(\ddot{x}_0 + x_0) + \epsilon(\ddot{x}_1 + x_1 - 1) + \epsilon^2(\ddot{x}_2 + x_2) = 0$$

Now we can solve for each of the ϵ order terms separately by using the method of undetermined coefficients and applying the initial conditions

$$\ddot{x}_0 + x_0 = 0 \Rightarrow x_0 = \cos(t)$$
$$\ddot{x}_2 + x_2 = 0 \Rightarrow x_2 = \cos(t)$$
$$\ddot{x}_1 + x_1 - 1 = 0 \Rightarrow x_1 = 1$$

Now we can plug these all back into the starting equation

$$x(t, \epsilon) = x_0(t, \epsilon) + \epsilon x_1(t, \epsilon) + \epsilon^2 x_2(t, \epsilon) + O(\epsilon^3)$$
$$\approx \cos(t) + \epsilon + \epsilon^2 \cos(t) = \epsilon + (1 + \epsilon^2) \cos(t)$$

c)

The perturbation does not contain any secular terms because there is a single time scale in this problem. The governing equation describes an undamped harmonic oscillator with a constant drive. The most general formula for the solution is $x(t) = A\cos(\omega t + \phi) + d$, with appropriately chosen constants to satisfy the initial conditions and forcing term. The solution can also be interpreted by a shift.

$$\begin{aligned} & \ddot{x} + x = \epsilon \qquad x(0) = 1 \quad \dot{x}(0) = 0 \\ z = x - \epsilon \rightarrow \quad & \ddot{z} + z = 0 \qquad z(0) = 1 - \epsilon \quad \dot{z}(0) = 0 \end{aligned}$$

which now describes an undamped harmonic oscillator with a zero drive, but in either case the only time scale is the natural period of the harmonic oscillator.

7.6.5

$h(x, \dot{x}) = x\dot{x}^2 \qquad x(0) = a \quad \dot{x}(0) = 0$

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}T} &= \langle h\sin(\theta)\rangle = \frac{1}{2\pi}\int_0^{2\pi} h\big(r\cos(\theta), -r\sin(\theta)\big)\sin(\theta)\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} r^3\cos(\theta)\sin^3(\theta)\mathrm{d}\theta = 0 \Rightarrow r(T) = r_0 \\ r\frac{\mathrm{d}\phi}{\mathrm{d}T} &= \langle h\cos(\theta)\rangle = \frac{1}{2\pi}\int_0^{2\pi} h\big(r\cos(\theta), -r\sin(\theta)\big)\cos(\theta)\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} r^3\cos^2(\theta)\sin^2(\theta)\mathrm{d}\theta = \frac{r^3}{8} \\ \frac{\mathrm{d}\phi}{\mathrm{d}T} &= \frac{r^2}{8} = \frac{r_0^2}{8} \Rightarrow \phi(T) = \frac{r_0^2}{8}T + \phi_0 \end{aligned}$$

$$r = r_0 + O(\epsilon) \quad \omega = 1 + \epsilon\phi' = 1 + \frac{r_0^2}{8}\epsilon + O\big(\epsilon^2\big)$$

The amplitude of the closed orbit can be anything and the closed orbit is approximately circular.

$$\begin{aligned} r(0) &= \sqrt{x(0)^2 + \dot{x}(0)^2} = a \qquad r(T) = a \\ \phi(0) &= \arctan\left(\frac{\dot{x}(0)}{x(0)}\right) - \tau = 0 - 0 = 0 \qquad \phi(T) = \frac{a^2}{8}T \end{aligned}$$

$$\begin{aligned} x_0 &= r(T)\cos\big(\phi(T) + \tau\big) = a\cos\left(\frac{a^2}{8}T + \tau\right) \\ x(t, \epsilon) &= a\cos\left(\frac{a^2}{8}T + \tau\right) + O(\epsilon) = a\cos\left(\left(\frac{a^2}{8}\epsilon + 1\right)t\right) + O(\epsilon) \end{aligned}$$

7.6.7

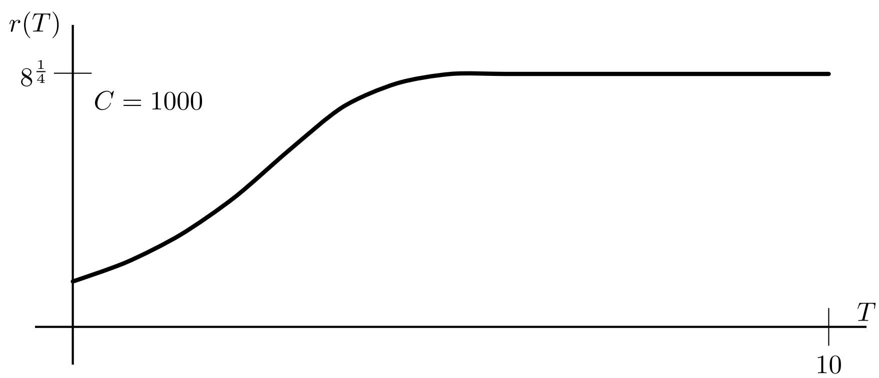
$h(x,\dot{x}) = (x^4-1)\dot{x} \qquad x(0)=a \quad \dot{x}(0)=0$

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}T} &= \langle h\sin(\theta)\rangle = \frac{1}{2\pi}\int_0^{2\pi} h\big(r\cos(\theta), -r\sin(\theta)\big)\sin(\theta)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} -r(r^4\cos^4(\theta)-1)\sin^2(\theta)\mathrm{d}\theta = \frac{1}{16}r(8-r^4) \\
&\Rightarrow r(T) = \frac{8^{\frac14}e^{\frac{T}{2}}}{(C+e^{2T})^{\frac14}} \\
r\frac{\mathrm{d}\phi}{\mathrm{d}T} &= \langle h\cos(\theta)\rangle = \frac{1}{2\pi}\int_0^{2\pi} h\big(r\cos(\theta), -r\sin(\theta)\big)\cos(\theta)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} -r(r^4\cos^4(\theta)-1)\sin(\theta)\cos(\theta)\mathrm{d}\theta = 0 \\
\frac{\mathrm{d}\phi}{\mathrm{d}T} &= 0 \Rightarrow \phi(T) = \phi_0
\end{aligned}$$

$$\frac{\mathrm{d}r}{\mathrm{d}T} = \frac{1}{16}r(8-r^4) = 0 \Rightarrow r = 0, 8^{\frac14}$$
$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{16}r(8-r^4)\Big|_{r=0} = \frac12 \qquad \frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{16}r(8-r^4)\Big|_{r=8^{\frac14}} = -2$$

The origin is an unstable fixed point, and there is a stable limit cycle at $r = 8^{\frac14}$.

$$r(T) = \frac{8^{\frac14}e^{\frac{T}{2}}}{(C+e^{2T})^{\frac14}} + O(\epsilon) \quad \omega = 1+\epsilon\phi' = 1 + O\big(\epsilon^2\big)$$



Next, we solve the initial value problem.

$$r(0) = \sqrt{x(0)^2+\dot{x}(0)^2} = a \qquad r(T) = \frac{8^{\frac14}e^{\frac{T}{2}}}{\left(\frac{8}{a^4}-1+e^{2T}\right)^{\frac14}}$$

$$\phi(0) = \arctan\left(\frac{\dot{x}(0)}{x(0)}\right) - \tau = 0 - 0 = 0 \qquad \phi(T) = 0$$

$$x_0 = r(T)\cos\big(\phi(T)+\tau\big) = \frac{8^{\frac14}e^{\frac{T}{2}}}{\left(\frac{8}{a^4}-1+e^{2T}\right)^{\frac14}}\cos(\tau)$$

$$x(t,\epsilon) = \frac{8^{\frac14}e^{\frac{T}{2}}}{\left(\frac{8}{a^4}-1+e^{2T}\right)^{\frac14}}\cos(\tau) + O(\epsilon) = \frac{8^{\frac14}e^{\frac{\epsilon t}{2}}}{\left(\frac{8}{a^4}-1+e^{2\epsilon t}\right)^{\frac14}}\cos(t) + O(\epsilon)$$

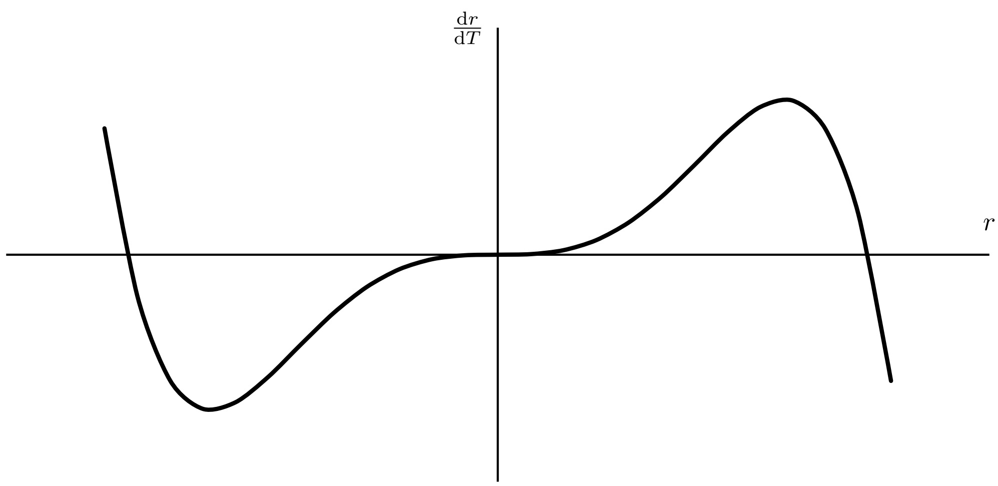
7.6.9

$$h(x,\dot{x}) = (x^2-1)\dot{x}^3 \qquad x(0)=a \quad \dot{x}(0)=0$$

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}T} &= \langle h\sin(\theta)\rangle = \frac{1}{2\pi}\int_0^{2\pi} h\big(r\cos(\theta), -r\sin(\theta)\big)\sin(\theta)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} -r^3(r^2\cos^2(\theta)-1)\sin^4(\theta)\mathrm{d}\theta = \frac{1}{16}r^3(6-r^2) \\
&\Rightarrow 2\ln(r) - \frac{6}{r^2} - \ln(6-r^2) = \frac{9}{2}T + C \\
r\frac{\mathrm{d}\phi}{\mathrm{d}T} &= \langle h\cos(\theta)\rangle = \frac{1}{2\pi}\int_0^{2\pi} h\big(r\cos(\theta), -r\sin(\theta)\big)\cos(\theta)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} -r^3(r^2\cos^2(\theta)-1)\sin^3(\theta)\cos(\theta)\mathrm{d}\theta = 0 \\
\frac{\mathrm{d}\phi}{\mathrm{d}T} &= 0 \Rightarrow \phi(T) = \phi_0
\end{aligned}$$

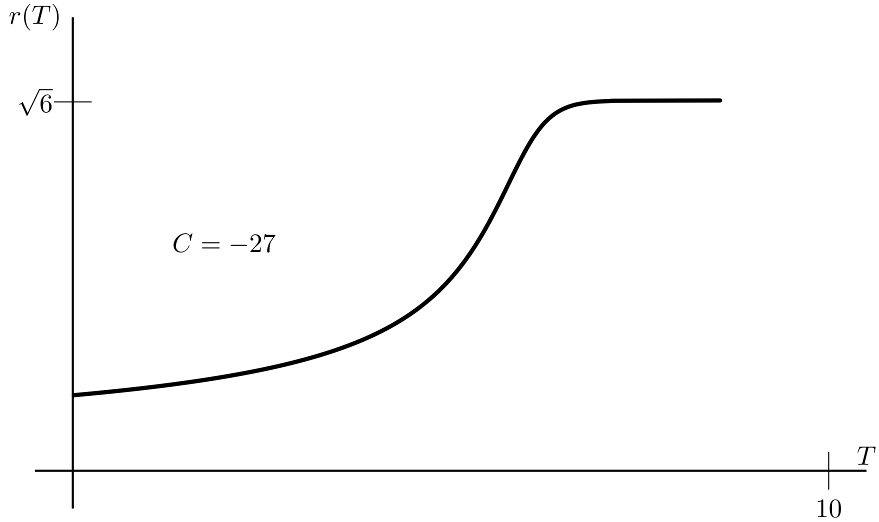
$$\frac{\mathrm{d}r}{\mathrm{d}T} = \frac{1}{16}r^3(6-r^2) = 0 \Rightarrow r = 0, \sqrt{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{16}r^3(6-r^2)\Big|_{r=0} = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{16}r^3(6-r^2)\Big|_{r=\sqrt{6}} = -\frac{9}{2}$$



The origin is an unstable fixed point from the right, and there is a stable limit cycle at $r=\sqrt{6}$.

$$2\ln(r) - \frac{6}{r^2} - \ln(6-r^2) = \frac{9}{2}T + C \quad \omega = 1 + \epsilon\phi' = 1 + O\big(\epsilon^2\big)$$



We can't solve for the general solution with an initial condition, but we can solve for the limit cycle.

$$x_0 = r(T)\cos\big(\phi(T)+\tau\big) = \sqrt{6}\cos(\tau)$$
$$x(t,\epsilon) = \sqrt{6}\cos(\tau) + O(\epsilon) = \sqrt{6}\cos(t) + O(\epsilon)$$

7.6.11

We start off by taking the expansion out one more term.

$$\begin{aligned}
x(t,\epsilon) &= x_0(\tau,T) + \epsilon x_1(\tau,T) + \epsilon^2 x_2(\tau,T) + O\big(\epsilon^3\big)\\
\dot{x} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(x_0(\tau,T) + \epsilon x_1(\tau,T) + \epsilon^2 x_2(\tau,T) + O\big(\epsilon^3\big)\Big)\\
&= \partial_\tau x_0 + \epsilon\partial_T x_0 + \epsilon\Big(\partial_\tau x_1 + \epsilon\partial_T x_1\Big) + \epsilon^2\Big(\partial_\tau x_2 + \epsilon\partial_T x_2\Big) + O\big(\epsilon^3\big)\\
&= \partial_\tau x_0 + \epsilon\Big(\partial_T x_0 + \partial_\tau x_1\Big) + \epsilon^2\Big(\partial_T x_1 + \partial_\tau x_2\Big) + O\big(\epsilon^3\big)\\
\ddot{x} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\partial_\tau x_0 + \epsilon\Big(\partial_T x_0 + \partial_\tau x_1\Big) + \epsilon^2\Big(\partial_T x_1 + \partial_\tau x_2\Big) + O\big(\epsilon^3\big)\Big)\\
&= \partial_{\tau\tau} x_0 + \epsilon\partial_{\tau T} x_0 + \epsilon\Big(\partial_{T\tau} x_0 + \epsilon\partial_{TT} x_0 + \partial_{\tau\tau} x_1 + \epsilon\partial_{\tau T} x_1\Big)\\
&\quad + \epsilon^2\Big(\partial_{T\tau} x_1 + \epsilon\partial_{TT} x_1 + \partial_{\tau\tau} x_2 + \epsilon\partial_{\tau T} x_2\Big) + O\big(\epsilon^3\big)\\
&= \partial_{\tau\tau} x_0 + \epsilon\Big(\partial_{\tau T} x_0 + \partial_{T\tau} x_0 + \partial_{\tau\tau} x_1\Big)\\
&\quad + \epsilon^2\Big(\partial_{TT} x_0 + \partial_{\tau T} x_1 + \partial_{T\tau} x_1 + \partial_{\tau\tau} x_2\Big) + O\big(\epsilon^3\big)\\
&= \partial_{\tau\tau} x_0 + \epsilon\Big(2\partial_{T\tau} x_0 + \partial_{\tau\tau} x_1\Big) + \epsilon^2\Big(\partial_{TT} x_0 + 2\partial_{T\tau} x_1 + \partial_{\tau\tau} x_2\Big) + O\big(\epsilon^3\big)
\end{aligned}$$

Then we plug these into the van der Pol oscillator and collecting terms.

$$
\begin{aligned}
&\ddot{x} + x + \epsilon\left(x^2 - 1\right)\dot{x} = 0\\
&O\left(1\right) : \partial_{\tau\tau}x_0 + x_0 = 0\\
&O\left(\epsilon\right) : \partial_{\tau\tau}x_1 + x_1 = -2\partial_{T\tau}x_0 - \left(x_0^2 - 1\right)\partial_\tau x_0\\
&O\left(\epsilon^2\right) : \partial_{\tau\tau}x_2 + x_2 = -\partial_{TT}x_0 - 2\partial_{T\tau}x_1 - \left(x_0^2 - 1\right)\left(\partial_T x_0 + \partial_\tau x_1\right) - 2x_0x_1\partial_T x_0
\end{aligned}
$$

The $O\left(\epsilon\right)$ is Equation (39) from the text.

$$\partial_{\tau\tau}x_1 + x_1 = \left[-2r' + r - \frac{1}{4}r^3\right]\sin(\tau + \phi) + [-2r\phi']\cos(\tau + \phi) - \frac{1}{4}r^3 \sin\left(3(\tau + \phi)\right)$$

The first two terms on the RHS are constrained to be zero as in Example 7.6.2, so the differential equation simplifies to

$$\partial_{\tau\tau}x_1 + x_1 = -\frac{1}{4}r^3 \sin\left(3(\tau + \phi)\right)$$

with general solution

$$x_1 = \frac{1}{32}r^3(T)\sin\left(3(\tau + \phi)\right) + A(T)\cos(\tau) + B(T)\sin(\tau)$$

and applying the initial conditions gives

$$x_1 = \frac{1}{4}\sin(3\tau) + A(T)\cos(\tau) + B(T)\sin(\tau)$$

Unfortunately we don't know much about $A(T)$ and $B(T)$ right away, but looking at the $O\left(\epsilon^2\right)$ term for the $x(t, \epsilon)$ expansion we can see that the $\cos(\tau)$ and $\sin(\tau)$ attached to $A(T)$ and $B(T)$ will create at least one secular term.

The work required to solve explicitly for $A(T)$ and $B(T)$ is not worth the trouble, but we can derive differential equations for $A(T)$ and $B(T)$, just as in the $O\left(\epsilon\right)$ terms, in order to cancel out all the secular terms in the $O\left(\epsilon^2\right)$ terms.

So all in all the leftover bit of $x_1 = \frac{1}{4}\sin(3\tau)$ resulting from the higher harmonic term $-\frac{1}{4}r^3 \sin\left(3(\tau + \phi)\right)$ has hardly any effect at all. However, for the practically minded we could use these results to make a more accurate prediction of the error of our approximate solution.

7.6.13

$$\ddot{x} + x + \epsilon x^3 = 0 \qquad 0 < \epsilon \ll 1 \quad x(0) = a \quad \dot{x}(0) = 0$$

a)

Since there's no $\dot{x}$ term, it's quite straightforward to find the conserved energy equation

$$E = \frac{1}{2}\dot{x}^2 + \int \left(x + \epsilon x^3\right)\mathrm{d}x = \frac{1}{2}\dot{x}^2 + \frac{x^2}{2} + \epsilon\frac{x^4}{4}$$

which we'll use later.

Looking at our initial conditions $x(0) = a$ and $\dot{x}(0) = 0$, we see that the $t = 0$ corresponds to the turning-around point for the oscillation. The amplitude of the oscillation is therefore a.

An obvious relationship for the period of the limit cycle is

$$T = \int_0^T \mathrm{d}t \Rightarrow \frac{T}{4} = \int_0^{\frac{T}{4}} \mathrm{d}t \Rightarrow T = 4\int_0^{\frac{T}{4}} \mathrm{d}t$$

but we can change this into an integral in terms of x.

$$T = 4\int_0^{\frac{T}{4}} \mathrm{d}t = 4\int_a^0 \frac{\mathrm{d}x}{\dot{x}}$$

We can solve for $\dot{x}$ in the conservation of energy equation for this particular limit cycle and substitute it into the above integral.

$$E = \frac{1}{2}\dot{x}^2(0) + \frac{x^2(0)}{2} + \epsilon\frac{x^4(0)}{4} \Rightarrow E = \frac{a^2}{2} + \epsilon\frac{a^4}{4}$$
$$\dot{x} = \pm\sqrt{a^2 + \epsilon\frac{a^4}{2} - x^2 - \epsilon\frac{x^4}{2}}$$

Since x is decreasing from a to 0 on this quarter of the cycle, $\dot{x} < 0$ and we pick the negative square root.

$$T = 4\int_a^0 \frac{\mathrm{d}x}{\dot{x}} = 4\int_a^0 \frac{\mathrm{d}x}{-\sqrt{a^2 + \epsilon\frac{a^4}{2} - x^2 - \epsilon\frac{x^4}{2}}} = 4\int_0^a \frac{\mathrm{d}x}{\sqrt{a^2 + \epsilon\frac{a^4}{2} - x^2 - \epsilon\frac{x^4}{2}}}$$

We can simplify this a bit with a substitution of $x = ay$, then factor.

$$\begin{aligned}
T &= 4\int_0^a \frac{\mathrm{d}x}{\sqrt{a^2 + \epsilon\frac{a^4}{2} - x^2 - \epsilon\frac{x^4}{2}}} = 4\int_0^1 \frac{a\mathrm{d}y}{\sqrt{a^2 + \epsilon\frac{a^4}{2} - a^2y^2 - \epsilon\frac{a^4y^4}{2}}} \\
&= 4\int_0^1 \frac{\mathrm{d}y}{\sqrt{1 + \epsilon\frac{a^2}{2} - y^2 - \epsilon\frac{a^2y^4}{2}}} = \frac{4}{\sqrt{1 + \epsilon\frac{a^2}{2}}}\int_0^1 \frac{\mathrm{d}y}{\sqrt{\left(1 - y^2\right)\left(1 - k^2y^2\right)}} \\
T(\epsilon) &= \frac{4}{\sqrt{1 + \epsilon\frac{a^2}{2}}}K(k) \qquad k^2 = \frac{-\epsilon\frac{a^2}{2}}{1 + \epsilon\frac{a^2}{2}}
\end{aligned}$$

where $K(k)$ is the complete elliptic integral of the first kind.

b)

The power series is a tad easier to do if we go back a few steps and expand from there.

$$
\begin{aligned}
T(\epsilon) &= 4\int_0^1 \frac{\mathrm{d}y}{\sqrt{1+\epsilon\frac{a^2}{2}-y^2-\epsilon\frac{a^2y^4}{2}}} = 4\int_0^1 \frac{\mathrm{d}y}{\sqrt{1-y^2+\epsilon\left(\frac{a^2}{2}-\frac{a^2y^4}{2}\right)}} \\
&= 4\int_0^1 \frac{1}{\sqrt{1-y^2}} - \epsilon\frac{\left(\frac{a^2}{2}-\frac{a^2y^4}{2}\right)}{2\left(1-y^2\right)^{\frac{3}{2}}} + \epsilon^2\frac{3\left(\frac{a^2}{2}-\frac{a^2y^4}{2}\right)^2}{8\left(1-y^2\right)^{\frac{5}{2}}} + O\left(\epsilon^3\right)\,\mathrm{d}y \\
&= \int_0^1 \frac{4}{\sqrt{1-y^2}} - \epsilon\frac{a^2\left(1-y^4\right)}{\left(1-y^2\right)^{\frac{3}{2}}} + \epsilon^2\frac{3a^4\left(1-y^4\right)^2}{8\left(1-y^2\right)^{\frac{5}{2}}} + O\left(\epsilon^3\right)\,\mathrm{d}y \\
&= 2\pi - \frac{3a^2\pi}{4}\epsilon + \frac{57a^4\pi}{128}\epsilon^2 + O\left(\epsilon^3\right) = 2\pi\left(1-\frac{3a^2}{8}\epsilon+\frac{57a^4}{256}\epsilon^2+O\left(\epsilon^3\right)\right)
\end{aligned}
$$

To check against Equation 7.6.57, we have to convert angular frequency ω to period T.

$$
\begin{aligned}
\omega &= 1+\frac{3}{8}\epsilon a^2+O\left(\epsilon^2\right) \\
T &= \frac{2\pi}{\omega} \approx \frac{2\pi}{1+\frac{3}{8}\epsilon a^2} = 2\pi\left(1-\frac{3a^2}{8}\epsilon+\left(\frac{3}{8}\epsilon a^2\right)^2+\cdots\right) \\
&= 2\pi\left(1-\frac{3a^2}{8}\epsilon+O\left(\epsilon^2\right)\right)
\end{aligned}
$$

So the two methods have agreement in at least the first two terms.

7.6.15

a)

The pendulum equation can be changed into

$$
\begin{aligned}
&\ddot{x}+\sin(x)=0 \qquad x(0)=a \quad \dot{x}(0)=0 \\
&\sin(x) = x-\frac{1}{6}x^3+O(x^5) \\
&\ddot{x}+\sin(x)=0 \rightarrow \ddot{x}+x-\frac{1}{6}x^3 = O(x^5)\approx 0 \\
&\ddot{x}+x+\epsilon h(x,\dot{x})=0 \qquad \epsilon=\frac{1}{6} \quad h(x,\dot{x})=-x^3
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}T} &= \langle h\sin(\theta)\rangle = \frac{1}{2\pi}\int_0^{2\pi} h\big(r\cos(\theta),-r\sin(\theta)\big)\sin(\theta)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} -r^3\cos^4(\theta)\mathrm{d}\theta = 0 \Rightarrow r(T)=r_0 \\
r\frac{\mathrm{d}\phi}{\mathrm{d}T} &= \langle h\cos(\theta)\rangle = \frac{1}{2\pi}\int_0^{2\pi} h\big(r\cos(\theta),\sin(\theta)\big)\cos(\theta)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} r^3\sin^3(\theta)\cos(\theta)\mathrm{d}\theta = \frac{3}{8}r^3 \\
\frac{\mathrm{d}\phi}{\mathrm{d}T} &= \frac{3}{8}r^2 \Rightarrow \phi(T)=\frac{3}{8}r_0^2+\phi_0
\end{aligned}
$$

$$r(0) = \sqrt{x(0)^2 + \dot{x}(0)^2} = a \qquad r(T) = a$$
$$\omega = 1 + \epsilon\phi' = 1 + \epsilon\frac{3}{8}r_0^2 + O\big(\epsilon^2\big) \approx 1 - \frac{1}{16}a^2$$

b)

The exact period from Exercise 6.7.4 is

$$T = 2\pi\left(1 + \frac{1}{16}a^2 + O\big(a^4\big)\right)$$

We can compare the exact result to our approximation by first converting the angular frequency ω to period T, and then converting the expression for T into an infinite geometric series since $\alpha \ll 1$.

$$\begin{aligned} T &= \frac{2\pi}{\omega} \approx \frac{2\pi}{1 - \frac{1}{16}a^2} = 2\pi\left(1 + \frac{1}{16}a^2 + \left(\frac{1}{16}a^2\right)^2 + \left(\frac{1}{16}a^2\right)^4 + \cdots\right) \\ &= 2\pi\left(1 + \frac{1}{16}a^2 + O\big(a^4\big)\right) \end{aligned}$$

And the two derivations do agree!

7.6.17

$$\ddot{x} + \big(1 + \epsilon\gamma + \epsilon\cos(2t)\big)\sin(x) = 0 \qquad 0 < \epsilon \ll 1$$

a)

$$\ddot{x} + \big(1 + \epsilon\gamma + \epsilon\cos(2t)\big)x \approx 0$$

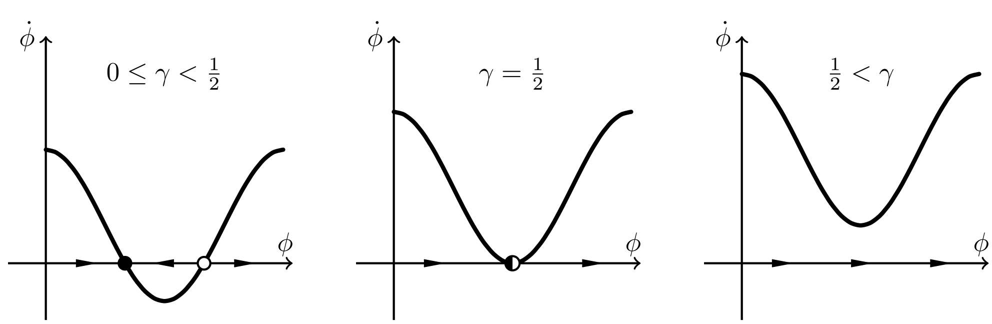
$$\begin{aligned} &h(x,\dot{x}) = \big(\gamma + \cos(2t)\big)x \Rightarrow h(\theta) = r\big(\gamma + \cos\big(2(\theta - \phi)\big)\big)\cos(\theta) \\ &r' = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\sin(\theta)\mathrm{d}\theta = \frac{1}{4}r\sin(2\phi) \\ &r\phi' = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\cos(\theta)\mathrm{d}\theta = \frac{1}{2}r\left(\gamma + \frac{1}{2}\cos(2\phi)\right) \Rightarrow \phi' = \frac{1}{2}\left(\gamma + \frac{1}{2}\cos(2\phi)\right) \end{aligned}$$

b)

Due to the r present in the r' equation, $r' \ll \phi'$ is a good approximation for $r(0) \approx 0$. Thus ϕ should reach equilibrium quickly enough relative to r that we can use the ϕ stable fixed point, assuming it exists, in the r' equation and solve.

$$\phi' = \frac{1}{2}\left(\gamma + \frac{1}{2}\cos(2\phi)\right) = 0 \Rightarrow \cos(2\phi) = -2\gamma \qquad |\gamma| \le \frac{1}{2}$$

The graphs below illustrate the phase line for γ positive, and γ negative is similar.



We won't say much about $|\gamma| = \frac{1}{2}$ because convergence to the fixed point is slower, so our assumptions may no longer hold.

We can now approximate the solution to the r' equation when $r \approx 0$.

$$r' = \frac{1}{4}r\sin(2\phi) \approx \frac{1}{4}r\sin\left(\arccos(-2\gamma)\right) = \frac{1}{4}r\sqrt{1-4\gamma^2} \qquad |\gamma| < \frac{1}{2}$$

This equation has an unstable fixed point at $r = 0$ and therefore $\gamma < \frac{1}{2} = \gamma_c$ is the critical value.

c)

Now solving for r is relatively straightforward.

$$r' = \frac{\mathrm{d}r}{\mathrm{d}T} = \frac{1}{4}r\sqrt{1-4\gamma^2} \Rightarrow r(T) = r_0 e^{kT} \qquad k = \frac{1}{4}\sqrt{1-4\gamma^2} \quad |\gamma| < \frac{1}{2}$$

d)

For $\gamma_c < |\gamma|$ we can see that $\phi' > 0$, but we can still make a solvable differential equation.

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}\phi} &= \frac{\mathrm{d}r}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}\phi} = \frac{r'}{\phi'} \\
&= \frac{\frac{1}{4}r\sin(2\phi)}{\frac{1}{2}\left(\gamma + \frac{1}{2}\cos(2\phi)\right)} = \frac{r\sin(2\phi)}{2\gamma + \cos(2\phi)} \\
\int \frac{\mathrm{d}r}{r} &= \int \frac{\sin(2\phi)}{2\gamma + \cos(2\phi)}\mathrm{d}\phi \\
\ln(r) &= -\frac{1}{2}\ln\left(2\gamma + \cos(2\phi)\right) + C \\
r(\phi) &= \frac{C}{\sqrt{2\gamma + \cos(2\phi)}}
\end{aligned}$$

$r(\phi)$ is bounded, so trajectories that start near the origin stay near the origin and are closed orbits.

e)

As for the physical interpretation, let's start with a pendulum. A frictionless pendulum will continue its oscillation with the same period forever; but imagine that we have the power to raise or lower the mass at the end of the rod whenever we want, which effectively changes the length of the rod. If we move the mass up (shorten the rod) at the bottom of the swing, and move the mass farther away (lengthen the rod) at the top of the swing, then the pendulum will swing higher each time.

The rationale is from conservation of angular momentum. Decreasing the rod length at the bottom of the swing, which is when the mass is rotating the fastest, makes the mass swing faster in order to maintain the same rotational inertia. Therefore we can put energy into the system this way. Also, increasing the rod length at the top of the swing does not add or remove energy from the system because the mass has zero velocity at the top of the swing, and all the energy is stored as potential energy.

So repeating in this fashion, which is twice the frequency of the pendulum since one-half of the pendulum swing contains one full up-down-up cycle, will increase the oscillation amplitude more and more.

This reasoning also works for a regular swing with flexible chains because the leg-pumping motion causes the chain to bend at the swinger's handholds, effectively raising the swinger's center of mass and changing the length of the chain. (Consequently, pumping your legs on a swing with rigid rods instead of chains won't work because the swinger can't bend the rods.) The differential equation for this problem doesn't have an instantaneous up-down motion, but instead has a continuously changing rod length.

$$\ddot{\theta} + \frac{g}{L}\sin(\theta) = 0 \quad \longleftrightarrow \quad \ddot{x} + \big(1 + \epsilon\gamma + \epsilon\cos(2t)\big)\sin(x) = 0$$

and the value of $|\gamma|$ affects when the length of the rod is changed. Only some values of $|\gamma|$ will change the length of the rod at the correct values of ϕ to impart more energy into the system.

7.6.19

$$\ddot{x} + x + \epsilon x^3 = 0 \qquad 0 < \epsilon \ll 1 \quad x(0) = a \quad \dot{x}(0) = 0$$

a)

$$\tau = \omega t \Rightarrow \ddot{x} = \frac{\mathrm{d}^2}{\mathrm{d}t}x\big(\tau(t)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}x\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\omega\frac{\mathrm{d}x}{\mathrm{d}\tau}\right) = \omega^2\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2}$$

$$\ddot{x} + x + \epsilon x^3 = 0 \to \omega^2\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} + x(\tau) + \epsilon\big(x(\tau)\big)^3 = \omega^2 x'' + x + \epsilon x = 0$$

b)

$$x(\tau, \epsilon) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + O\big(\epsilon^3\big)$$
$$\omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + O\big(\epsilon^3\big)$$

$$\begin{aligned}
&\omega^2 x'' + x + \epsilon x^3 = 0 \\
&\Big(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + O\big(\epsilon^3\big)\Big)^2\Big(x_0'' + \epsilon x_1'' + \epsilon^2 x_2'' + O\big(\epsilon^3\big)\Big) \\
&\qquad + \Big(x_0 + \epsilon x_1 + \epsilon^2 x_2 + O\big(\epsilon^3\big)\Big) + \epsilon\Big(x_0 + \epsilon x_1 + \epsilon^2 x_2 + O\big(\epsilon^3\big)\Big)^3 = 0 \\
&\Big(1 + 2\epsilon\omega_1 + 2\epsilon^2\omega_2 + \epsilon^2\omega_1^2 + O\big(\epsilon^3\big)\Big)\Big(x_0'' + \epsilon x_1'' + \epsilon^2 x_2'' + O\big(\epsilon^3\big)\Big) \\
&\qquad + \Big(x_0 + \epsilon x_1 + \epsilon^2 x_2 + O\big(\epsilon^3\big)\Big) + \epsilon\Big(x_0 + \epsilon x_1 + \epsilon^2 x_2 + O\big(\epsilon^3\big)\Big)^3 = 0
\end{aligned}$$

There's no need to expand everything out. We only need the $O\big(1\big)$ and $O\big(\epsilon\big)$ terms, which we can pick out.

$$O\big(1\big) : x_0'' + x_0 = 0$$
$$O\big(\epsilon\big) : x_1'' + x_1 = -2\omega_1 x_0'' - x_0^3$$

c)

The x_0 term has to match the initial conditions of the expand function $x(t, \epsilon)$ since the series is supposed to hold for all ϵ values, and all the other terms in the series consequently have zero as their initial conditions.

$$x(0) = a \quad \dot{x}(0) = 0 \Rightarrow x_0(0) = a \quad x_0'(0) = 0 \qquad x_k(0) = 0 \quad x_k'(0) = 0 \text{ for all } k > 0$$

d)

$$x_0'' + x_0 = 0 \quad x_0(0) = a \quad x_0'(0) = 0 \Rightarrow x_0 = a\cos(\tau)$$

e)

$$\begin{aligned}
x_1'' + x_1 &= -2\omega_1 x_0'' - x_0^3 \\
&= 2a\omega_1\cos(\tau) - a^3\cos^3(\tau) \\
\Big(\text{Triple angle formula}: \ &\cos(3x) = 4\cos^3(x) - 3\cos(x)\Big) \\
&= 2a\omega_1\cos(\tau) - \frac{1}{4}a^3\big(\cos(3\tau) + 3\cos(\tau)\big) \\
&= \left(2a\omega_1 - \frac{3}{4}a^3\right)\cos(\tau) - \frac{1}{4}a^3\cos(3\tau) \\
\text{No secular terms} &\Rightarrow 2a\omega_1 - \frac{3}{4}a^3 = 0 \Rightarrow \omega_1 = \frac{3}{3}a^2
\end{aligned}$$

f)

$$\begin{aligned}
x_1'' + x_1 &= -\frac{1}{4}a^3\cos(3\tau) \qquad x_1(0) = 0 \quad x_1'(0) = 0 \\
x_1 &= \frac{1}{32}\big(a^3\cos(3\tau) - a^3\cos(\tau)\big)
\end{aligned}$$

7.6.21

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0 \qquad x(0) = a \quad \dot{x}(0) = 0$$

$$x(\tau,\epsilon) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + O\big(\epsilon^3\big) \qquad \omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + O\big(\epsilon^3\big)$$

$$\begin{aligned}
&\omega^2 x'' + \epsilon(x^2 - 1)\omega x' + x = 0 \\
&O\big(1\big): x_0'' + x_0 = 0 \qquad x_0(0) = a \quad x_0'(0) = 0 \quad x_0 = a\cos(\tau) \\
&O\big(\epsilon\big): x_1'' + x_1 = -2\omega_1 x_0'' + \big(1 - x_0^2\big)x_0' \qquad x_1(0) = 0 \quad x_1'(0) = 0 \\
&\qquad\qquad\quad = 2\omega_1 a\cos(\tau) - \big(1 - a^2\cos^2(\tau)\big)a\sin(\tau) \\
&\qquad\qquad\quad = 2\omega_1 a\cos(\tau) - a\left(1 - \frac{1}{4}a^2\right)\sin(\tau) + \frac{1}{4}a^3\sin(3\tau) \\
&\text{No secular terms} \Rightarrow \omega_1 = 0 \quad a = 2 \\
&x_0 = 2\cos(\tau) \qquad x_1 = \frac{1}{4}\big(3\sin(\tau) - \sin(3\tau)\big) \\
&O\big(\epsilon^2\big): x_2'' + x_2 = -\big(\omega_1^2 + 2\omega_2\big)x_0'' - 2\omega_1 x_1'' + \big(1 - x_0^2\big)\big(x_1' + \omega_1 x_0'\big) - 2x_0 x_1 x_0' \\
&\qquad\qquad\quad = -2\omega_2 x_0'' + \big(1 - x_0^2\big)x_1' - 2x_0 x_1 x_0'
\end{aligned}$$

$$
\begin{aligned}
&= 4\omega_2 \cos(\tau) + \frac{3}{4}\big(1 - 4\cos^2(\tau)\big)\big(\cos(\tau) - \cos(3\tau)\big) \\
&\quad + 2\cos(\tau)\big(3\sin(\tau) - \sin(3\tau)\big)\sin(\tau) \\
&= \left(4\omega_2 + \frac{1}{4}\right)\cos(\tau) + \cdots
\end{aligned}
$$

No secular terms $\Rightarrow \omega_2 = -\dfrac{1}{16}$

$$\omega = 1 - \frac{1}{16}\epsilon^2 + O\big(\epsilon^3\big)$$

7.6.23

$$\ddot{x} - \epsilon x\dot{x} + x = 0 \qquad x(0) = a \quad \dot{x}(0) = 0$$

$$x(\tau, \epsilon) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + O\big(\epsilon^3\big) \qquad \omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + O\big(\epsilon^3\big)$$

$$
\begin{aligned}
&\omega^2 x'' + \epsilon\omega x x' + x = 0 \\
&O\big(1\big) : x_0'' + x_0 = 0 \qquad x_0(0) = a \quad x_0'(0) = 0 \quad x_0 = a\cos(\tau) \\
&O\big(\epsilon\big) : x_1'' + x_1 = -2\omega_1 x_0 + x_0 x_0' \qquad x_1(0) = 0 \quad x_1'(0) = 0 \\
&\qquad\qquad\qquad = -2\omega_1 a\cos(\tau) - a^2\cos(\tau)\sin(\tau) \\
&x_1 = \frac{1}{6}\big(-2a^2\sin(\tau) + a^2\sin(2\tau) - 6a\omega_1\tau\sin(\tau)\big) \\
&\text{No secular terms} \Rightarrow \omega_1 = 0 \\
&x_1 = \frac{1}{6}a^2\big(-2\sin(\tau) + \sin(2\tau)\big) \\
&O\big(\epsilon^2\big) : x_2'' + x_2 = x_1 x_0' + x_0 x_1' + \omega_1 x_0 x_0' - 2\omega_2 x_0'' - 2\omega_1 x_1'' - w_1^2 x_0'' \\
&\qquad\qquad\qquad = x_1 x_0' + x_0 x_1' - 2\omega_2 x_0'' \\
&\qquad\qquad\qquad = -\frac{1}{6}a^3\big(-2\sin(\tau) + \sin(2\tau)\big)\sin(\tau) \\
&\qquad\qquad\qquad\quad + \frac{1}{6}a^3\cos(\tau)\big(-2\cos(\tau) - 2\cos(2\tau)\big) + 2\omega_2 a\cos(\tau) \\
&\qquad\qquad\qquad = \frac{1}{12}\big((-3a^3 + 24a\omega_2)\cos(\tau) - 4a^3\cos(2\tau) - a^3\cos(3\tau)\big) \\
&\text{No secular terms} \Rightarrow -3a^3 + 24a\omega_2 = 0 \Rightarrow \omega_2 = -\frac{1}{8}a^2
\end{aligned}
$$

$$\omega(a) = 1 - \frac{1}{8}\epsilon^2 a^2 + O\big(\epsilon^3\big)$$

7.6.25

$$\ddot{x} + x + \epsilon h(x, \dot{x}, t) = 0$$

$$x(t) = r(t)\cos\big(t + \phi(t)\big) \qquad \dot{x}(t) = -r(t)\sin\big(t + \phi(t)\big)$$

a)

We can derive a differential equation for $r(t)$ and $\phi(t)$ by comparing the derivative of the $x(t)$ definition to the $\dot{x}(t)$ definition.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}x(t) &= \frac{\mathrm{d}}{\mathrm{d}t}r(t)\cos\big(t + \phi(t)\big) = \dot{r}\cos(t+\phi) - r(1+\dot{\phi})\sin(t+\phi) \\
\dot{x}(t) &= -r\sin(t+\phi) \\
-r\sin(t+\phi) &= \dot{r}\cos(t+\phi) - r(1+\dot{\phi})\sin(t+\phi) \\
0 &= \dot{r}\cos(t+\phi) - r\dot{\phi}\sin(t+\phi)
\end{aligned}$$

We can derive another differential equation for $r(t)$ and $\phi(t)$ by substituting the derivative of the $\dot{x}(t)$ definition into the main differential equation.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\dot{x}(t) = \ddot{x} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(-r(t)\sin\big(t+\phi(t)\big)\Big) &= -\dot{r}\sin(t+\phi) - r(1+\dot{\phi})\cos(t+\phi) \\
0 = \ddot{x} + x + \epsilon h(x, \dot{x}, t) & \\
= -\dot{r}\sin(t+\phi) - r(1+\dot{\phi})\cos(t+\phi) + r\cos(t+\phi) + \epsilon h & \\
= -\dot{r}\sin(t+\phi) - r\dot{\phi}\cos(t+\phi) + \epsilon h &
\end{aligned}$$

Now we have two linear equations of $\dot{r}(t)$ and $\dot{\phi}(t)$ and can solve for each explicitly.

$$\begin{aligned}
\dot{r}\cos(t+\phi) - r\dot{\phi}\sin(t+\phi) &= 0 \\
\dot{r}\sin(t+\phi) + r\dot{\phi}\cos(t+\phi) &= \epsilon h
\end{aligned}$$

$$\begin{aligned}
&\dot{r}\cos^2(t+\phi) - r\dot{\phi}\sin(t+\phi)\cos(t+\phi) = 0 \\
&\dot{r}\sin^2(t+\phi) + r\dot{\phi}\cos(t+\phi)\sin(t+\phi) = \epsilon h\sin(t+\phi) \\
&\dot{r}\cos^2(t+\phi) + \dot{r}\sin^2(t+\phi) = \dot{r} = \epsilon h\sin(t+\phi)
\end{aligned}$$

$$\begin{aligned}
-\dot{r}\cos(t+\phi)\sin(t+\phi) + r\dot{\phi}\sin^2(t+\phi) &= 0 \\
\dot{r}\sin(t+\phi)\cos(t+\phi) + r\dot{\phi}\cos^2(t+\phi) &= \epsilon h \\
r\dot{\phi}\sin^2(t+\phi) + r\dot{\phi}\cos^2(t+\phi) = r\dot{\phi} &= \epsilon h\cos(t+\phi)
\end{aligned}$$

b)

$$\langle r\rangle(t) = \bar{r}(t) = \frac{1}{2\pi}\int_{t-\pi}^{t+\pi} r(\tau)\ \mathrm{d}\tau$$

We can use Leibniz's integral rule here, assuming that $\dot{r}(t)$ is continuous for $\tau \in [t-\pi, t+\pi]$.

$$\begin{aligned}
\frac{\mathrm{d}\langle r\rangle}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2\pi}\int_{t-\pi}^{t+\pi} r(\tau)\ \mathrm{d}\tau\right)\\
&= \frac{1}{2\pi}\left(r(t+\pi)\left(\frac{\mathrm{d}}{\mathrm{d}t}(t+\pi)\right) - r(t-\pi)\left(\frac{\mathrm{d}}{\mathrm{d}t}(t-\pi)\right) + \int_{t-\pi}^{t+\pi}\frac{\partial}{\partial t}r(\tau)\ \mathrm{d}\tau\right)\\
&= \frac{1}{2\pi}\left(r(t+\pi)(1) - r(t-\pi)(1) + \int_{t-\pi}^{t+\pi} 0\ \mathrm{d}\tau\right)\\
&= \frac{1}{2\pi}\left(r(t+\pi) - r(t-\pi)\right) = \frac{1}{2\pi}\int_{t-\pi}^{t+\pi}\frac{\mathrm{d}}{\mathrm{d}\tau}r(\tau)\ \mathrm{d}\tau = \left\langle\frac{\mathrm{d}r}{\mathrm{d}t}\right\rangle
\end{aligned}$$

c)

$$\begin{aligned}
\frac{\mathrm{d}\langle r\rangle}{\mathrm{d}t} &= \left\langle\frac{\mathrm{d}r}{\mathrm{d}t}\right\rangle = \big\langle\dot{r}(t)\big\rangle = \big\langle\epsilon h(x,\dot{x},t)\sin(t+\phi)\big\rangle\\
&= \epsilon\Big\langle h\big(r(t)\cos(t+\phi), -r(t)\sin(t+\phi), t\big)\sin(t+\phi)\Big\rangle
\end{aligned}$$

d)

We can intuitively see that these equations

$$r = \bar{r} + O\big(\epsilon\big) \qquad \phi = \bar{\phi} + O\big(\epsilon\big)$$

are true because the variable will be the average value over one oscillation plus the rate of change over that cycle.

$$\begin{aligned}
r &= \bar{r} + \langle\dot{r}\rangle = \bar{r} + \epsilon\big\langle h\sin(t+\phi)\big\rangle = \bar{r} + O\big(\epsilon\big)\\
\phi &= \bar{\phi} + \langle\dot{\phi}\rangle = \bar{\phi} + \frac{\epsilon}{\bar{r}}\big\langle h\cos(t+\phi)\big\rangle = \bar{\phi} + O\big(\epsilon\big)
\end{aligned}$$

The other two equations follow easily from here.

$$\begin{aligned}
\frac{\mathrm{d}\bar{r}}{\mathrm{d}t} &= \langle\dot{r}\rangle = \epsilon\Big\langle h\big(r(t)\cos(t+\phi), -r(t)\sin(t+\phi), t\big)\sin(t+\phi)\Big\rangle\\
&= \epsilon\Big\langle h\big(\bar{r}\cos(t+\bar{\phi}), -\bar{r}\sin(t+\bar{\phi}), t\big)\sin(t+\bar{\phi}) + O\big(\epsilon\big)\Big\rangle\\
&= \epsilon\Big\langle h\big(\bar{r}\cos(t+\bar{\phi}), -\bar{r}\sin(t+\bar{\phi}), t\big)\sin(t+\bar{\phi})\Big\rangle + O\big(\epsilon^2\big)
\end{aligned}$$

$$\begin{aligned}
\bar{r}\frac{\mathrm{d}\bar{\phi}}{\mathrm{d}t} &= \langle r\dot{\phi}\rangle = \epsilon\Big\langle h\big(r(t)\cos(t+\phi), -r(t)\sin(t+\phi), t\big)\cos(t+\phi)\Big\rangle\\
&= \epsilon\Big\langle h\big(\bar{r}\cos(t+\bar{\phi}), -\bar{r}\sin(t+\bar{\phi}), t\big)\cos(t+\bar{\phi}) + O\big(\epsilon\big)\Big\rangle\\
&= \epsilon\Big\langle h\big(\bar{r}\cos(t+\bar{\phi}), -\bar{r}\sin(t+\bar{\phi}), t\big)\cos(t+\bar{\phi})\Big\rangle + O\big(\epsilon^2\big)
\end{aligned}$$

8

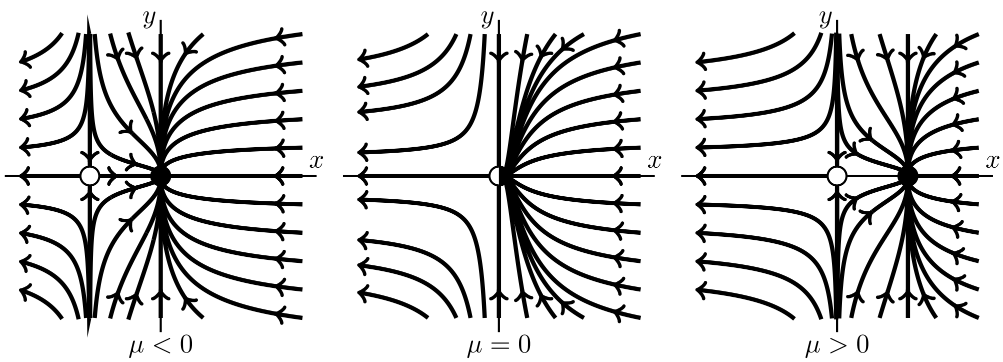
Bifurcations Revisited

8.1 Saddle-Node, Transcritical, and Pitchfork Bifurcations

8.1.1

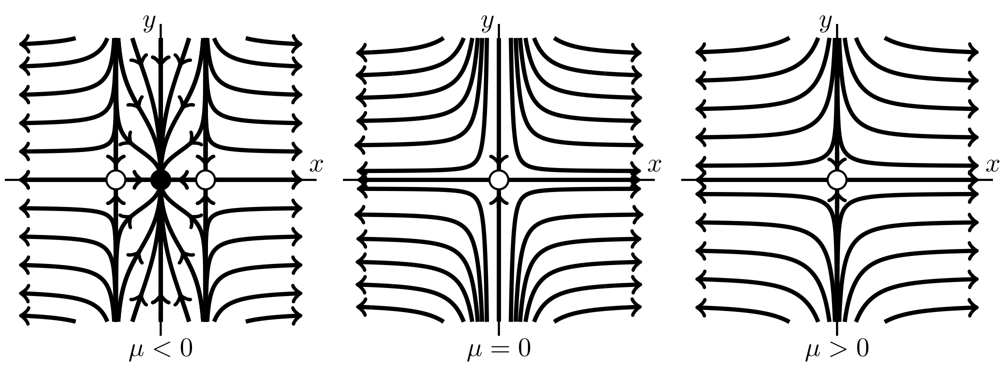
a)

$$\dot{x} = \mu x - x^2 \quad \dot{y} = -y$$



b)

$$\dot{x} = \mu x + x^3 \quad \dot{y} = -y$$



8.1.3

$$\dot{x} = \mu x - x^2 \qquad \dot{y} = -y$$

$(x, y) = (0, 0)$ and $(\mu, 0)$ are the fixed points.

$$A = \begin{pmatrix} \mu - 2x & 0 \\ 0 & -1 \end{pmatrix}$$

$$A_{(0,0)} = \begin{pmatrix} \mu & 0 \\ 0 & -1 \end{pmatrix}$$

$$\lambda_1 = \mu \quad \lambda_2 = -1 \Rightarrow \text{stable for } \mu < 0$$

$$\lambda_1 = \mu \to 0 \text{ as } \mu \to 0^-$$

$$A_{(\mu,0)} = \begin{pmatrix} -\mu & 0 \\ 0 & -1 \end{pmatrix}$$

$$\lambda_1 = -\mu \quad \lambda_2 = -1 \Rightarrow \text{stable for } \mu > 0$$

$$\lambda_1 = -\mu \to 0 \text{ as } \mu \to 0^+$$

One fixed point is always at the origin and the other fixed point moves along the x-axis. The stable fixed point moves along the negative half of the x-axis while the origin is an unstable fixed point. The two fixed points combine at the origin with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -1$, then separate into a stable fixed point at the origin and an unstable fixed point that moves along the positive half of the x-axis.

The fixed points exchange stability, but for the stable fixed point we have $\lambda_1 = \mu \to 0,$ as $\mu \to 0$ regardless of which side of 0 μ is on.

8.1.5

True. The nullclines always intersect tangentially at a zero-eigenvalue bifurcation in two dimensions.

For a system with (x^*, y^*) as a fixed point

$$\dot{x} = f(x, y) \qquad \dot{y} = g(x, y)$$

$$f(x^*, y^*) = 0 \qquad g(x^*, y^*) = 0$$

$$A_{(x^*,y^*)} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} \nabla f(x^*, y^*) \\ \nabla g(x^*, y^*) \end{pmatrix}$$

With a zero eigenvalue, the two rows of the A matrix evaluated at the fixed point must be multiples of each other, which are the gradient of $f(x, y)$ and the gradient of $g(x, y)$. The gradient vectors are orthogonal to the level curves, which includes the nullclines. Therefore the tangent lines of the nullclines at the fixed point are parallel because their normal vectors are parallel.

8.1.7

$$\dot{x} = y - ax \qquad \dot{y} = -by + \frac{x}{1+x}$$

First find the fixed points, which are the intersections of the nullclines.

$$y = ax \qquad y = \frac{x}{b(1+x)} \Rightarrow (x, y) = (0, 0) \text{ and } \left(\frac{1}{ab} - 1, \frac{1}{b} - a\right)$$

Next find the linearization at the fixed points.

$$A = \begin{pmatrix} -a & 1 \\ \frac{1}{(1+x)^2} & -b \end{pmatrix}$$

$(x, y) = (0, 0) \Rightarrow \Delta = ab - 1 \quad \tau = -(a + b)$
$(x, y) = \left(\frac{1}{ab} - 1, \frac{1}{b} - a\right) \Rightarrow \Delta = ab - (ab)^2 \quad \tau = -(a + b)$

Looking more closely at this system, we see that the fixed point goes to infinity when either a or b is zero. Also notice that there is only one fixed point when $ab = 1$, and there are two fixed points when $ab \neq 1$, $a \neq 0$, and $b \neq 0$. There are several cases for the stability of the two fixed points.

$ab > 1$ and $a + b > 0 \Rightarrow$ The origin is a stable node and the other fixed point is a saddle point.
$0 < ab < 1$ and $a + b > 0 \Rightarrow$ The origin is a saddle point and the other fixed point is a stable node.
$ab > 1$ and $a + b < 0 \Rightarrow$ The origin is an unstable node and the other fixed point is a saddle point.
$0 < ab < 1$ and $a + b < 0 \Rightarrow$ The origin is a saddle point and the other fixed point is an unstable node.
$ab < 0 \Rightarrow$ The origin and the other fixed point are saddle points. This last case completes all possible cases.

From this we can conclude that transcritical bifurcations occur along the boundary $a = \dfrac{1}{b}$ in parameter space.

8.1.9

$$\ddot{x} + b\dot{x} - kx + x^3 = 0$$

First we transform this into a 2-dimensional system.

$$\dot{x} = y \qquad \dot{y} = kx - by - x^3$$

Next we find the fixed points, which are the intersections of the nullclines.

$$y = 0 \qquad y = \frac{kx - x^3}{b} \Rightarrow (x, y) = (0, 0) \text{ and } (\pm\sqrt{k}, 0)$$

Now we find the linearization at the fixed points.

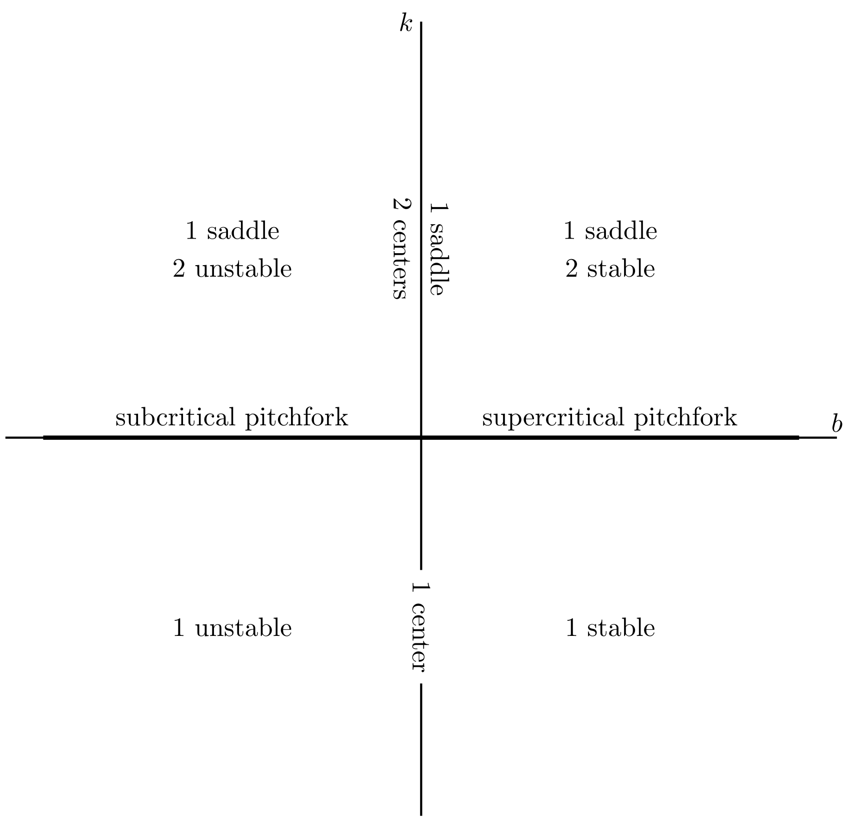
$$A = \begin{pmatrix} 0 & 1 \\ k - 3x^2 & -b \end{pmatrix}$$

$$A_{(0,0)} = \begin{pmatrix} 0 & 1 \\ k & -b \end{pmatrix} \qquad \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 + 4k}}{2}$$

$$A_{(\pm\sqrt{k},0)} = \begin{pmatrix} 0 & 1 \\ -2k & -b \end{pmatrix} \qquad \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 8k}}{2}$$

The second pair of fixed points comes into existence and splits off from the origin when $k > 0$, and the stability of all three fixed points is determined by the sign of b. If $b < 0$ and k become positive, then there is a subcritical pitchfork bifurcation. If $b > 0$ and k become positive, there is a supercritical pitchfork bifurcation. When $b = 0$ there is a bifurcation when k becomes positive too. The origin changes from a center to a saddle point, and the two new fixed points are centers.

Note that the new fixed points can be spirals or nodes depending on k and b, but we only care about the stability for the stability diagram.



8.1.11

$$\dot{u} = a(1-u) - uv^2 \qquad \dot{v} = uv^2 - (a+k)v \qquad a, k > 0$$

The fixed points of the system are $(u, v) = (1, 0)$, which always exists, and

$$(u, v) = \left(\frac{a \pm \sqrt{a^2 - 4(a+k)^2}}{2a}, \frac{a \mp \sqrt{a^2 - 4(a+k)^2}}{2(a+k)} \right)$$

if the inside of the square root is positive.

We definitely have a saddle-node bifurcation when the inside of the square root is negative with zero fixed-points, the inside of the square root is zero with one fixed point, and the inside of the square root is positive with two fixed points.

The inside of the square root is zero when

$$a^2 - 4(a+k)^2 = 0 \Rightarrow (a+k)^2 = \frac{a}{4} \Rightarrow k = -a \pm \frac{\sqrt{a}}{2}$$

and thus saddle-node bifurcations occur on this curve in $a - k$ parameter space.

8.1.13

$$\dot{n} = GnN - kn \qquad \dot{N} = -GnN - fN + p \qquad G, k, f > 0$$

a)

$$\begin{aligned}
&\frac{\mathrm{d}n}{\mathrm{d}t} = GnN - kn \quad \frac{\mathrm{d}N}{\mathrm{d}t} = -GnN - fN + p \\
&\frac{G}{k^2}\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{G^2nN}{k^2} - \frac{Gn}{k} \quad \frac{G}{k^2}\frac{\mathrm{d}N}{\mathrm{d}t} = -\frac{G^2nN}{k^2} - \frac{fGN}{k^2} + \frac{pG}{k^2} \\
&\tau = kt \quad x = \frac{Gn}{k} \quad y = \frac{GN}{k} \quad a = \frac{f}{k} \quad b = \frac{pG}{k^2} \\
&\frac{\mathrm{d}x}{\mathrm{d}\tau} = x(y-1) \quad \frac{\mathrm{d}y}{\mathrm{d}\tau} = -xy - ay + b
\end{aligned}$$

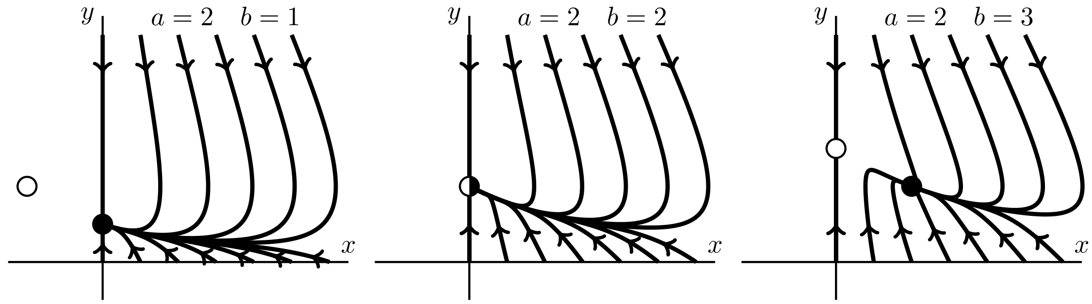
b) $(x, y) = \left(0, \frac{b}{a}\right)$ and $(b - a, 1)$ are the fixed points.

$$A = \begin{pmatrix} y-1 & x \\ -y & -x-a \end{pmatrix}$$

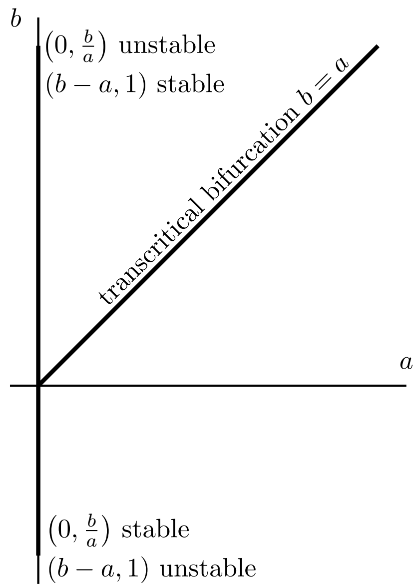
$$A_{\left(0, \frac{b}{a}\right)} = \begin{pmatrix} \frac{b}{a} - 1 & 0 \\ -\frac{b}{a} & -a \end{pmatrix} \quad \Delta = a - b \quad \tau = \frac{b}{a} - 1 - a$$

$$A_{(b-a,1)} = \begin{pmatrix} 0 & b-a \\ -1 & -b \end{pmatrix} \quad \Delta = b - a \quad \tau = -b$$

c)



d)



The picture only includes positive values of a due to the sign restrictions of the nondimensionalized variables.

8.1.15

The power of true believers

$$\dot{n}_A = (p + n_A)n_{AB} - n_A n_B \qquad \dot{n}_B = n_B n_{AB} - (p + n_A)n_B \qquad n_{AB} = 1 - (p + n_A) - n_B$$

a)

The first equation is the rate of change of the A-believers, which is the sum of gaining A-believers and losing A-believers. Gain comes from AB-believers converting at a rate proportional to the fraction of A-believers, and loss comes from A-believers converting to AB-believers at a rate proportional to the fraction of B-believers.

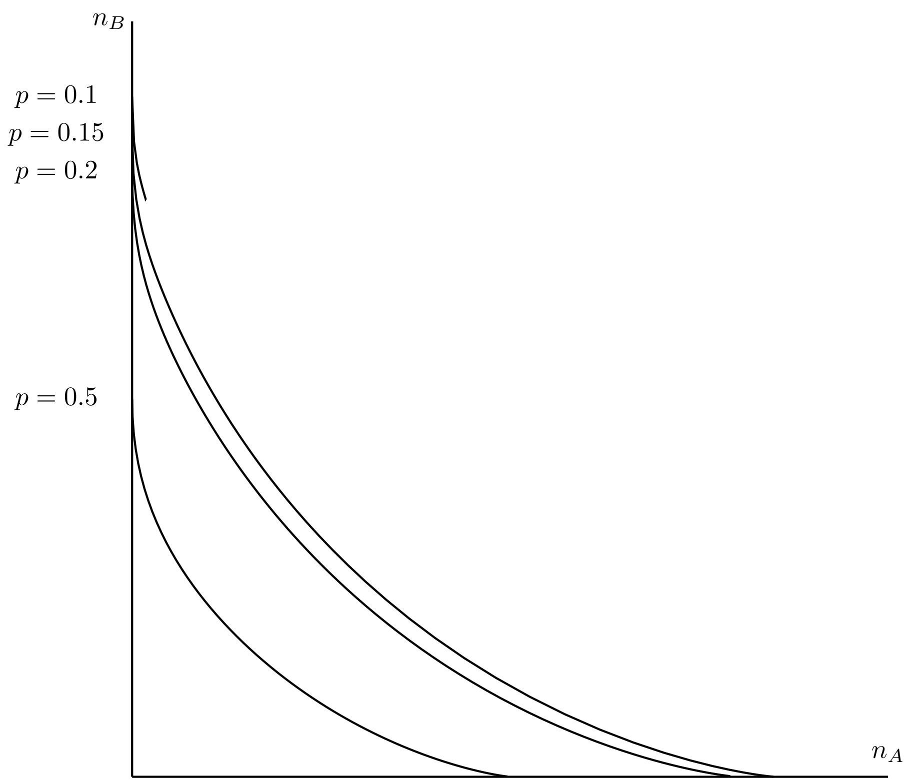
The second equation is the rate of change of the B-believers, which is the sum of gaining B-believers and losing B-believers. Gain comes from AB-believers converting at a rate proportional to the fraction of

B-believers, and loss comes from B-believers converting to AB-believers at a rate proportional to the fraction of A-believers.

While there is a fixed fraction of unconvertable A-believers, p, there is no analogous group for the B-believers. So the total fraction of A-believers is $(p + n_A)$, but the total number of B-believers is n_B.

An equation for $\dot{n}_{AB}$ is unnecessary because n_{AB} can be found by conservation of believers.

b)



An initial condition of $p = 0.15$ successfully converts everyone to A-believers, but an initial condition of $p = 0.1$ is unsuccessful. Then there should be a $0.1 < p_c < 0.15$ that is a critical point for the change in behavior.

c)

$$\dot{n}_B = 0 \Rightarrow n_B = 0 \text{ or } n_B = -2n_A - 2p + 1$$

$$\dot{n}_A = 0 \text{ and } n_B = 0 \Rightarrow n_A = 1 - p$$

$$\dot{n}_A = 0 \text{ and } n_B = -2n_A - 2p + 1 \Rightarrow n_A = \frac{-4p + 1 \pm \sqrt{4p^2 - 8p + 1}}{6}$$

Then the fixed points in terms of p are

$$(1 - p, 0) \qquad \left(\frac{-4p + 1 \pm \sqrt{4p^2 - 8p + 1}}{6}, \frac{4p - 1 \mp \sqrt{4p^2 - 8p + 1}}{3} - 2p + 1 \right)$$

The first fixed point is always there, but the other two fixed points will collide when

$$4p^2 - 8p + 1 = 0 \Rightarrow p = 1 \pm \frac{\sqrt{3}}{2}$$

out of which we have $p_c = 1 - \frac{\sqrt{3}}{2}$ to be in the [0,1] interval.

The last two fixed points are complex if $p_c < p$, and therefore to check if the remaining fixed point $(1-p, 0)$ is globally attracting we only need to look at its linearization.

$$A = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}n_A}\dot{n}_A & \frac{\mathrm{d}}{\mathrm{d}n_A}\dot{n}_B \\ \frac{\mathrm{d}}{\mathrm{d}n_B}\dot{n}_A & \frac{\mathrm{d}}{\mathrm{d}n_B}\dot{n}_B \end{pmatrix} = \begin{pmatrix} -2n_A - 2n_B - 2p + 1 & -2n_A - 2p \\ -2n_B & -2n_A - 2n_B - p + 1 \end{pmatrix}$$

$$A_{(1-p,0)} = \begin{matrix} \frac{\mathrm{d}}{\mathrm{d}n_A}\dot{n}_A & \frac{\mathrm{d}}{\mathrm{d}n_A}\dot{n}_B \\ \frac{\mathrm{d}}{\mathrm{d}n_B}\dot{n}_A & \frac{\mathrm{d}}{\mathrm{d}n_B}\dot{n}_B \end{matrix} = \begin{pmatrix} -1 & -1 \\ 0 & p-1 \end{pmatrix} \qquad \lambda_1 = -1 \quad \lambda_2 = p - 1$$

The eigenvalues are always negative. Therefore $(1-p, 0)$ is a stable node and when $p_c < p$ all trajectories will converge to it.

d)

A saddle-node bifurcation occurs at $p = p_c$ as two of the fixed points approach, collide, and then disappear.

8.2 Hopf Bifurcations

8.2.1

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$$

First convert the ODE into a system.

$$\dot{x} = y \qquad \dot{y} = a - \mu(x^2 - 1)y - x$$

Now we solve for fixed points, which gives $(x, y) = (a, 0)$, and then compute the Jacobian.

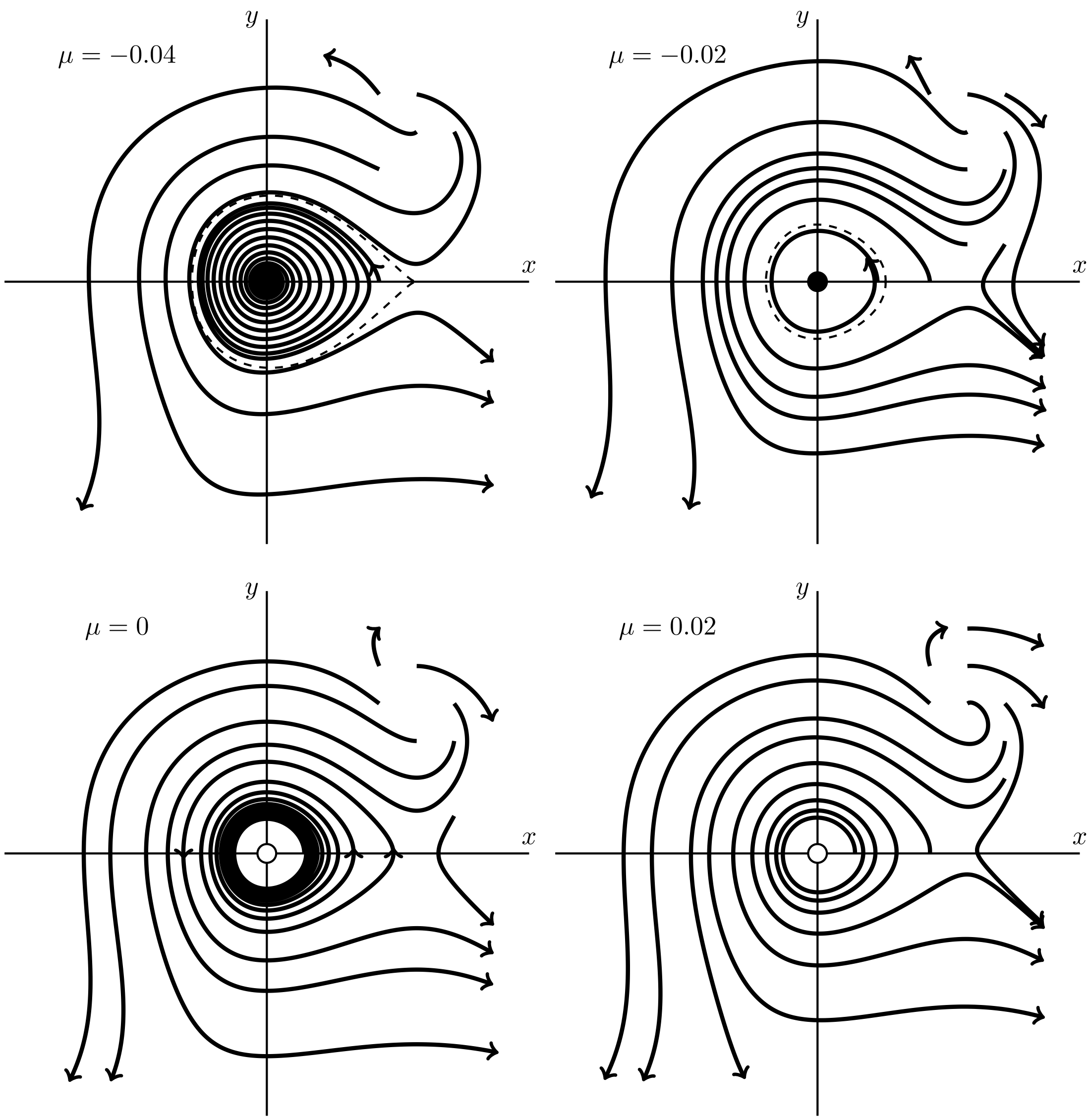
$$A = \begin{pmatrix} 0 & 1 \\ -2\mu xy - 1 & -\mu(x^2 - 1) \end{pmatrix}$$

$$A_{(a,0)} = \begin{pmatrix} 0 & 1 \\ -1 & -\mu(a^2 - 1) \end{pmatrix} \qquad \Delta = 1 \quad \tau = -\mu(a^2 - 1)$$

Hopf bifurcations occur when the trace changes from negative to zero to positive, and vice versa. The trace changes sign when crossing the lines $\mu = 0$ and $a = \pm 1$ in parameter space.

8.2.3

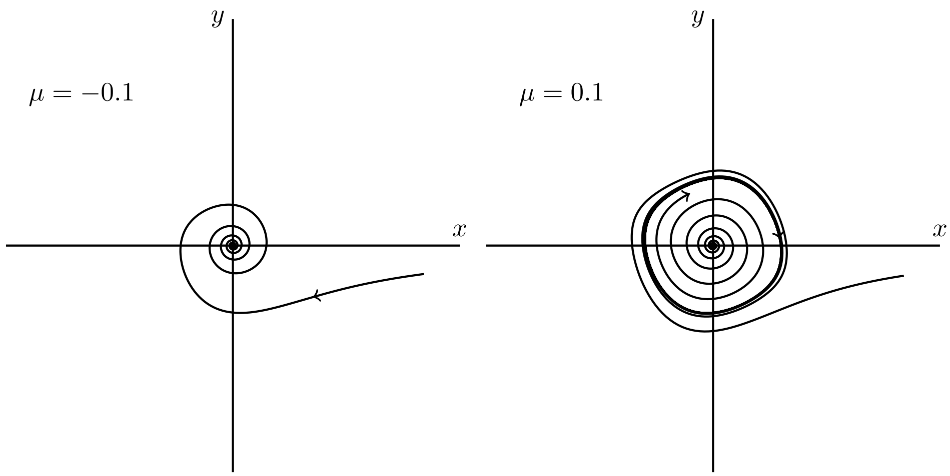
$$\dot{x} = -y + \mu x + xy^2 \qquad \dot{y} = x + \mu y - x^2$$



There's an unstable limit cycle for μ negative, and the limit cycle constricts towards the origin as μ increases to zero. (Only a portion of the spiral trajectory is shown for $\mu = -0.02$ because the individual spiral trajectories are very densely packed.) The limit cycle disappears when $\mu = 0$ with a very slight unstable spiral now at the origin. The unstable spiral's radial growth increases as μ increases past zero. This is a subcritical Hopf bifurcation.

8.2.5

$$\dot{x} = y + \mu x \qquad \dot{y} = -x + \mu y - x^2 y$$

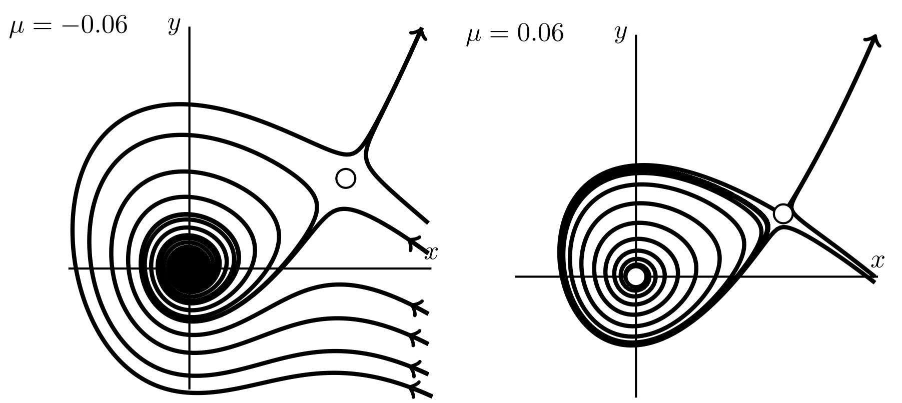


Supercritical Hopf bifurcation

8.2.7

$$\dot{x} = \mu x + y - x^2 \qquad \dot{y} = -x + \mu y + 2x^2$$

These graphs have bounds of $(x, y) \in [-0.4, 0.8] \times [-0.4, 0.8]$.



This is a subcritical Hopf bifurcation. For $\mu < 0$ the origin is stable and there is an unstable limit cycle, but for $\mu > 0$ the origin is unstable and there is no limit cycle.

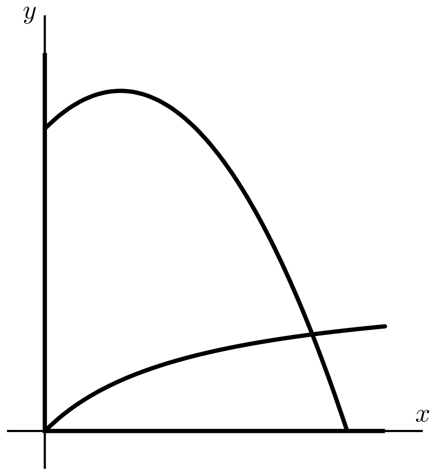
8.2.9

$$\dot{x} = x\left(b - x - \frac{y}{1+x}\right) \qquad \dot{y} = y\left(\frac{x}{1+x} - ay\right) \qquad x, y \geq 0 \quad a, b > 0$$

a)

The x and y nullclines are

$$x = 0 \quad y = (1+x)(b-x) \qquad y = 0 \quad y = \frac{x}{a(1+x)}$$



The fixed points are $(x, y) = (0, 0)$, $(b, 0)$, and the intersection of the curves, which is difficult to find because we need to solve a cubic equation.

Next, we check stability.

$$A = \begin{pmatrix} b - \frac{y}{(x+1)^2} - 2x & \frac{-x}{1+x} \\ \frac{y}{(x+1)^2} & \frac{x}{1+x} - 2ay \end{pmatrix}$$

$$A_{(0,0)} = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \quad b > 0 \Rightarrow \text{unstable}$$

$$A_{(b,0)} = \begin{pmatrix} -b & \frac{-b}{1+b} \\ 0 & \frac{b}{1+b} \end{pmatrix} \quad \lambda_1 = -b < 0 < \frac{b}{1+b} = \lambda_2 \Rightarrow \text{unstable}$$

As for the third fixed point, there can't be any fixed point creating bifurcations since there is only one intersection. There could be a Hopf bifurcation though.

b)

We're pretty certain from the nullcline graph above that there is always a fixed point in the positive quadrant. Being careful though, we can note that the x nullcline always has a positive y intercept and the y nullcline always starts at the origin. Also, the x nullcline has an x intercept at b, and the y nullcline has a horizontal asymptote of $\frac{1}{a}$. Therefore the x and y nullcline will always have an intersection at a strictly positive coordinate.

We should also check that the strictly positive fixed point is unique because of the y nullcline intersecting on different sides of the x nullcline maximum. The easiest way to do this is to set the x and y nullclines equal and set up the cubic equation we would need to solve to find all the roots.

$$(1 + x)(b - x) = y = \frac{x}{a(1 + x)}$$

$$ax^3 + (2a - ab)x^2 + (1 + a - 2ab)x - ab = (x - x_1)(x - x_2)(x - x_3) = 0$$

From this we know that $x_1x_2x_3 = ab > 0$, so we have to have exactly one strictly positive intersection, or exactly three strictly positive intersections. We can't have three intersections graphically, which means the strictly positive fixed point is unique.

c)

A Hopf bifurcation can only occur if the trace of the linearized system is zero. So we'll plug in the strictly positive fixed point (x^*, y^*) and find conditions on a and b.

$$\begin{aligned}
\tau &= b - \frac{y}{(x+1)^2} - 2x + \frac{x}{1+x} - 2ay \\
\tau|_{(x^*,y^*)} &= b - \frac{y^*}{(x^*+1)^2} - 2x^* + \frac{x^*}{1+x^*} - 2ay^* \\
y^* &= (1+x^*)(b-x^*) \qquad y^* = \frac{x^*}{a(1+x^*)} \\
\tau|_{(x^*,y^*)} &= b - \frac{(1+x^*)(b-x^*)}{(x^*+1)^2} - 2x^* + \frac{x^*}{1+x^*} - \frac{2x^*}{1+x^*} \\
&= b - \frac{b-x^*}{1+x^*} - 2x^* + \frac{x^*}{1+x^*} - \frac{2x^*}{1+x^*} \\
&= b - 2x^* - \frac{b}{1+x^*} = 0 \Rightarrow x^* = \frac{b-2}{2}
\end{aligned}$$

$$\begin{aligned}
(1+x^*)(b-x^*) &= \frac{x^*}{a(1+x^*)} \\
a = \frac{x^*}{(1+x^*)^2(b-x^*)} &= \frac{\frac{b-2}{2}}{\left(1+\frac{b-2}{2}\right)^2\left(b-\frac{b-2}{2}\right)} \\
= \frac{\frac{b-2}{2}}{\left(\frac{b}{2}\right)^2\frac{b+2}{2}} &= \frac{4(b-2)}{b^2(b+2)} = a_c
\end{aligned}$$

We should also check that the determinant is positive here to show there is a Hopf bifurcation.

$$A_{(x^*,y^*)} = \begin{pmatrix} b - \frac{y^*}{(x^*+1)^2} - 2x^* & \frac{-x^*}{1+x^*} \\ \frac{y^*}{(x^*+1)^2} & \frac{x^*}{1+x^*} - 2ay^* \end{pmatrix}$$

$$y^* = (1+x^*)(b-x^*) \qquad y^* = \frac{x^*}{a(1+x^*)} \qquad x^* = \frac{b-2}{2} \Rightarrow b = 2x^*+2$$

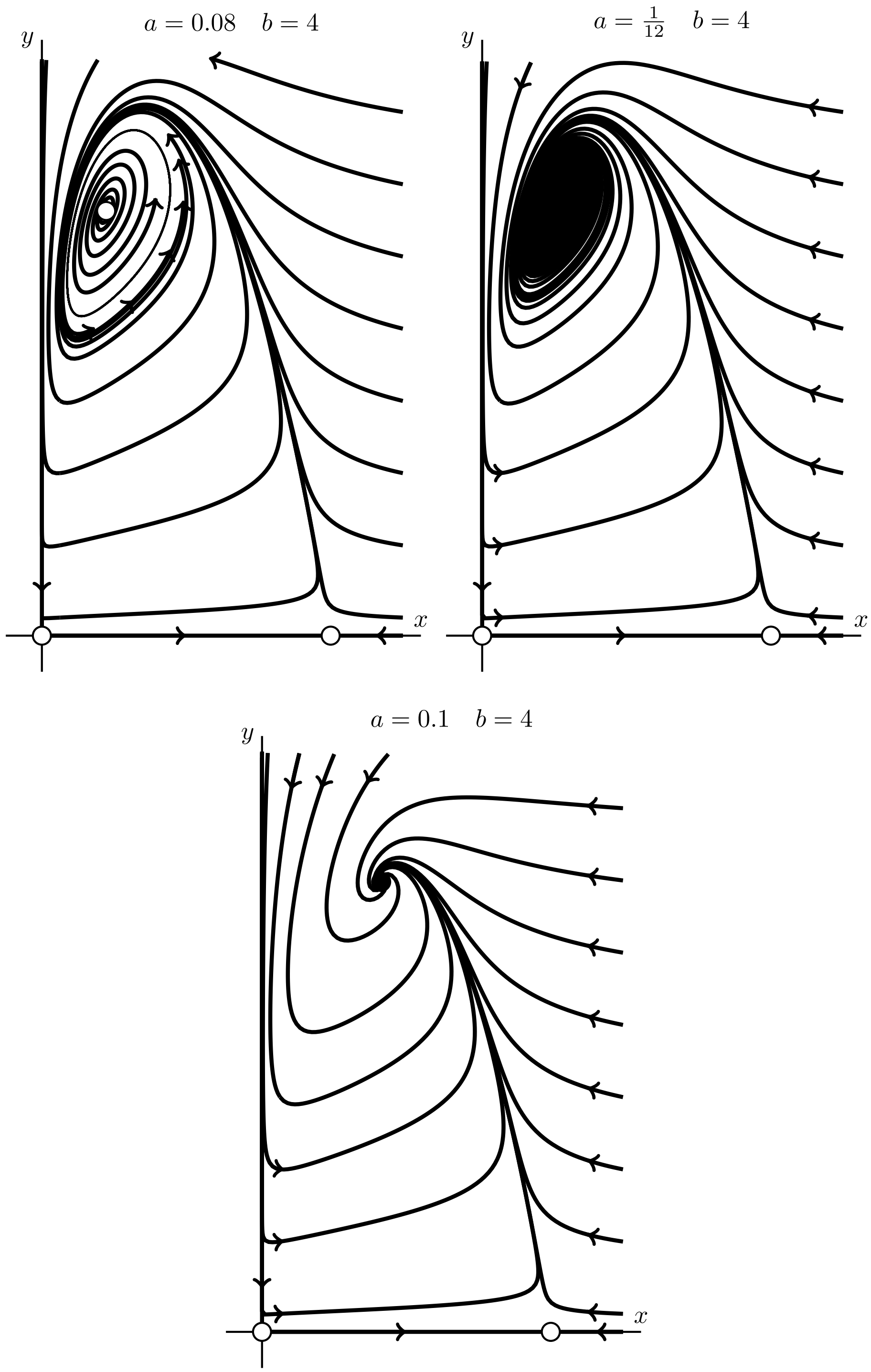
$$\begin{aligned}
A_{(x^*,y^*)} &= \begin{pmatrix} 2x^*+2 - \frac{b-x^*}{1+x^*} - 2x^* & \frac{-x^*}{1+x^*} \\ \frac{b-x^*}{1+x^*} & \frac{x^*}{1+x^*} - 2\frac{x^*}{1+x^*} \end{pmatrix} \\
&= \begin{pmatrix} 2 - \frac{b-x^*}{1+x^*} & \frac{-x^*}{1+x^*} \\ \frac{b-x^*}{1+x^*} & \frac{-x^*}{1+x^*} \end{pmatrix} = \begin{pmatrix} 2 - \frac{x^*+2}{1+x^*} & \frac{-x^*}{1+x^*} \\ \frac{x^*+2}{1+x^*} & \frac{-x^*}{1+x^*} \end{pmatrix} \\
&= \begin{pmatrix} \frac{2(1+x^*)-(x^*+2)}{1+x^*} & \frac{-x^*}{1+x^*} \\ \frac{x^*+2}{1+x^*} & \frac{-x^*}{1+x^*} \end{pmatrix} = \begin{pmatrix} \frac{x^*}{1+x^*} & \frac{-x^*}{1+x^*} \\ \frac{x^*+2}{1+x^*} & \frac{-x^*}{1+x^*} \end{pmatrix}
\end{aligned}$$

$$\Delta = \frac{x^*}{1+x^*}\frac{-x^*}{1+x^*} - \frac{-x^*}{1+x^*}\frac{x^*+2}{1+x^*} = \frac{2x^*}{(1+x^*)^2} > 0$$

The last inequality follows from x^* being strictly positive.

Hence we have a whole curve of Hopf bifurcations in (a, b) parameter space.

d)



8.2.11

$$\ddot{x} + \mu\dot{x} + x - x^3 = 0$$

a)

$$\dot{x} = y \qquad \dot{y} = -\mu y - x + x^3$$

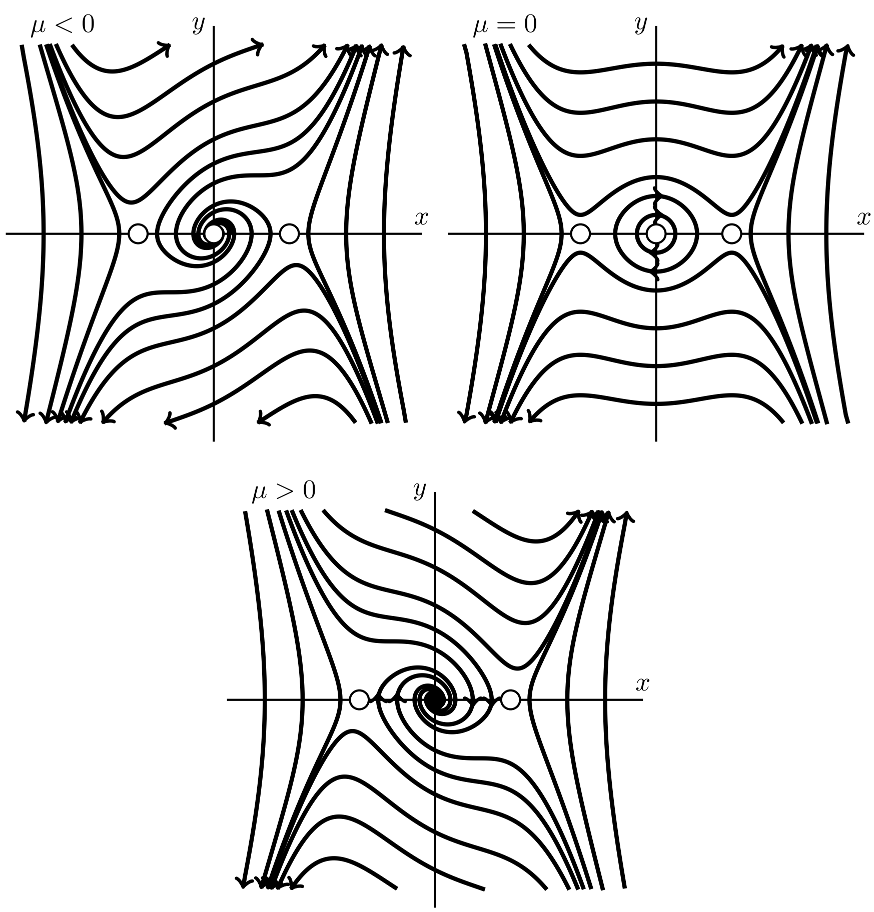
$$A = \begin{pmatrix} 0 & 1 \\ -1 + 3x^2 & -\mu \end{pmatrix}$$

$$A_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & -\mu \end{pmatrix} \quad \Delta = 1 \quad \tau = -\mu$$

A negative μ close to zero implies an unstable spiral.

A positive μ close to zero implies a stable spiral.

b)



The origin becomes a center when $\mu = 0$, which is a degenerate case of a Hopf bifurcation because there are an infinite number of closed orbits instead of a single closed orbit.

8.2.13

$$\dot{x} = y + \mu x \qquad \dot{y} = -x + \mu y - x^2 y \qquad \mu = 0$$

$$\omega = -1 \quad f(x,y) = 0 \quad g(x,y) = -x^2 y$$

$$\begin{array}{lllllll} f_x = 0 & f_y = 0 & & & g_x = -2xy & g_y = -x^2 & \\ f_{xx} = 0 & f_{xy} = 0 & f_{yy} = 0 & & g_{xx} = -2y & g_{xy} = -2x & g_{yy} = 0 \\ f_{xxx} = 0 & f_{xyy} = 0 & & & g_{xxy} = -2 & g_{yyy} = 0 & \end{array}$$

$$\begin{aligned} 16a &= f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} \\ &\quad + \frac{1}{\omega}\Big[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\Big] \\ 16a &= -2 \Rightarrow a = \frac{-1}{8} < 0 \end{aligned}$$

Hence this is a supercritical Hopf bifurcation, and numerical simulations agree.

8.2.15

$$\dot{x} = \mu x + y - x^2 \qquad \dot{y} = -x + \mu y + 2x^2 \qquad \mu = 0$$

$$\omega = -1 \qquad f(x,y) = -x^2 \quad g(x,y) = 2x^2$$

$$\begin{array}{lllllll} f_x = -2x & f_y = 0 & & & g_x = 4x & g_y = 0 & \\ f_{xx} = -2 & f_{xy} = 0 & f_{yy} = 0 & & g_{xx} = 4 & g_{xy} = 0 & g_{yy} = 0 \\ f_{xxx} = 0 & f_{xyy} = 0 & & & g_{xxy} = 0 & g_{yyy} = 0 & \end{array}$$

$$\begin{aligned} 16a &= f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} \\ &\quad + \frac{1}{\omega}\Big[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\Big] \\ 16a &= -8 \Rightarrow a = \frac{-1}{2} < 0 \end{aligned}$$

Hence this is a supercritical Hopf bifurcation, and numerical simulations agree.

8.2.17

$$\begin{aligned} \dot{x}_1 &= -x_1 + F(I - bx_2 - gy_1) \qquad \dot{y}_1 = \frac{-y_1 + x_1}{T} \\ \dot{x}_2 &= -x_2 + F(I - bx_1 - gy_2) \qquad \dot{y}_2 = \frac{-y_2 + x_2}{T} \\ F(x) &= \frac{1}{1 + e^{-x}} \end{aligned}$$

a)

Assuming that $x_1^* = y_1^* = x_2^* = y_2^* = u$, substituting into the equations gives

$$\dot{x}_1 = -u + F(I - bu - gu) \qquad \dot{y}_1 = \frac{-u+u}{T} = 0$$
$$\dot{x}_2 = -u + F(I - bu - gu) \qquad \dot{y}_2 = \frac{-u+u}{T} = 0$$

The $\dot{y}_1$ and $\dot{y}_2$ are satisfied, and what's left are the $\dot{x}_1$ and $\dot{x}_2$ equations.

$$0 = -u + F(I - bu - gu)$$
$$u = \frac{1}{1 + e^{-(I-bu-gu)}}$$

The LHS and RHS are zero and positive respectively when $u = 0$. Also, the LHS and RHS are strictly increasing and decreasing respectively, with the LHS greater than the RHS maximum at $u = 0$ within a positive finite u. Therefore the curves intersect and this symmetric solution is unique.

b)

$$A = \begin{pmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial y_1} & \frac{\partial \dot{x}_1}{\partial x_2} & \frac{\partial \dot{x}_1}{\partial y_2} \\ \frac{\partial \dot{y}_1}{\partial x_1} & \frac{\partial \dot{y}_1}{\partial y_1} & \frac{\partial \dot{y}_1}{\partial x_2} & \frac{\partial \dot{y}_1}{\partial y_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial y_1} & \frac{\partial \dot{x}_2}{\partial x_2} & \frac{\partial \dot{x}_2}{\partial y_2} \\ \frac{\partial \dot{y}_2}{\partial x_1} & \frac{\partial \dot{y}_2}{\partial y_1} & \frac{\partial \dot{y}_2}{\partial x_2} & \frac{\partial \dot{y}_2}{\partial y_2} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & \frac{-ge^{-I+bx_2+gy_1}}{(1+e^{-I+bx_2+gy_1})^2} & \frac{-be^{-I+gx_2+gy_1}}{(1+e^{-I+bx_2+gy_1})^2} & 0 \\ \frac{1}{T} & -\frac{1}{T} & 0 & 0 \\ \frac{-be^{-I+bx_1+gy_2}}{(1+e^{-I+bx_1+gy_2})^2} & 0 & -1 & \frac{-ge^{-I+bx_1+gy_2}}{(1+e^{-I+bx_1+gy_2})^2} \\ 0 & 0 & \frac{1}{T} & -\frac{1}{T} \end{pmatrix}$$

$$\begin{aligned} \frac{e^{-I+bu+gu}}{(1+e^{-I+bu+gu})^2} &= \frac{1+e^{-I+bu+gu}}{(1+e^{-I+bu+gu})^2} - \frac{1}{(1+e^{-I+bu+gu})^2} \\ &= \frac{1}{1+e^{-I+bu+gu}} - \frac{1}{(1+e^{-I+bu+gu})^2} \\ &= F(I-bu-gu) - F^2(I-bu-gu) \\ &= u - u^2 \end{aligned}$$

$$A_{(u,u,u,u)} = \begin{pmatrix} -1 & -g(u-u^2) & -b(u-u^2) & 0 \\ \frac{1}{T} & -\frac{1}{T} & 0 & 0 \\ -b(u-u^2) & 0 & -1 & -g(u-u^2) \\ 0 & 0 & \frac{1}{T} & -\frac{1}{T} \end{pmatrix}$$

Now to prove the determinant law

$$\begin{vmatrix} a_{1,1} & a_{1,2} & b & 0 \\ a_{2,1} & a_{2,2} & 0 & 0 \\ b & 0 & a_{1,1} & a_{1,2} \\ 0 & 0 & a_{2,1} & a_{2,2} \end{vmatrix}$$

(This is in no way a general proof but suffices for our purposes.) Expanding along the last column

$$-a_{1,2} \begin{vmatrix} a_{1,1} & a_{1,2} & b \\ a_{2,1} & a_{2,2} & 0 \\ 0 & 0 & a_{2,1} \end{vmatrix} + a_{2,2} \begin{vmatrix} a_{1,1} & a_{1,2} & b \\ a_{2,1} & a_{2,2} & 0 \\ b & 0 & a_{1,1} \end{vmatrix}$$

And then the bottom row

$$-a_{1,2}a_{2,1} \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} + a_{2,2}b \begin{vmatrix} a_{1,2} & b \\ a_{2,2} & 0 \end{vmatrix} + a_{1,1}a_{2,2} \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}$$

$$-a_{1,2}a_{2,1} \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} - (a_{2,2}b)^2 + a_{1,1}a_{2,2} \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}$$

$$(a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} - (a_{2,2}b)^2$$

$$(a_{1,1}a_{2,2} - a_{1,2}a_{2,1})^2 - (a_{2,2}b)^2$$

$$\big((a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) + a_{2,2}b\big)\big((a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) - a_{2,2}b\big)$$

$$\big((a_{1,1} + b)a_{2,2} - a_{1,2}a_{2,1}\big)\big((a_{1,1} - b)a_{2,2} - a_{1,2}a_{2,1}\big)$$

$$\begin{vmatrix} a_{1,1} + b & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} \begin{vmatrix} a_{1,1} - b & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}$$

Therefore

$$\begin{vmatrix} A & B \\ B & A \end{vmatrix} = |A + B||A - B|$$

We can use this to compute the eigenvalues of the 4×4 matrix.

$$\begin{vmatrix} -1 - \lambda & -g(u - u^2) & -b(u - u^2) & 0 \\ \frac{1}{T} & -\frac{1}{T} - \lambda & 0 & 0 \\ -b(u - u^2) & 0 & -1 - \lambda & -g(u - u^2) \\ 0 & 0 & \frac{1}{T} & -\frac{1}{T} - \lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} -1 - b(u - u^2) - \lambda & -g(u - u^2) \\ \frac{1}{T} & -\frac{1}{T} - \lambda \end{vmatrix} \begin{vmatrix} -1 + b(u - u^2) - \lambda & -g(u - u^2) \\ \frac{1}{T} & -\frac{1}{T} - \lambda \end{vmatrix} = 0$$

c)

Looking at the $A+B$ matrix, we can use that the determinant of a matrix is the product of the eigenvalues and the trace of a matrix is the sum of the eigenvalues.

$$\begin{vmatrix} -1-b(u-u^2) & -g(u-u^2) \\ \frac{1}{T} & -\frac{1}{T} \end{vmatrix}$$

$$\begin{aligned}\Delta &= \big(-1-b(u-u^2)\big)\left(-\frac{1}{T}\right)-\big(-g(u-u^2)\big)\frac{1}{T} \\ &= \frac{1+(b+g)(u-u^2)}{T} = \lambda_1\lambda_2\end{aligned}$$

Going back to the definition of u

$$u = F(I-bu-gu) = \frac{1}{1+e^{-(I-bu-gu)}}$$

We also know that $0<u<1$ because $0<F|_{u=0}<1$ and F is strictly decreasing. Therefore the intersection occurs for $0<u<1$. Hence $u-u^2>0$ and

$$0<\frac{1+(b+g)(u-u^2)}{T} = \lambda_1\lambda_2$$

From this we know that λ_1 and λ_2 are both positive or both negative.

Then from the trace

$$\begin{aligned}\tau &= -1-b(u-u^2)-\frac{1}{T} \\ 0 &> -1-b(u-u^2)-\frac{1}{T} = \lambda_1+\lambda_2\end{aligned}$$

Therefore λ_1 and λ_2 are both negative.

d)

However, for the $A-B$ matrix

$$\Delta = \big(-1+b(u-u^2)\big)\left(-\frac{1}{T}\right)-\big(-g(u-u^2)\big)\frac{1}{T} = \frac{1+(g-b)(u-u^2)}{T}$$

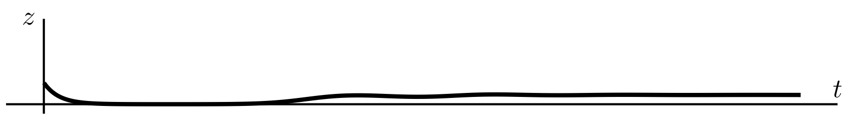
$$\tau = -1+b(u-u^2)-\frac{1}{T}$$

The determinant is positive or negative if $g>b$ or $b>g$ respectively, and the trace can be positive or negative depending on the relative sizes of b and T.

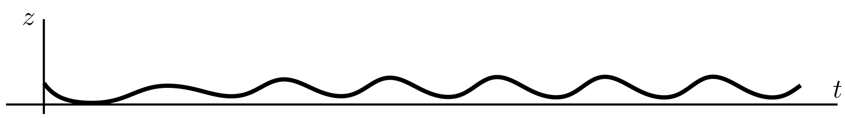
From these facts about the eigenvalues, we can cause either a pitchfork bifurcation or a Hopf bifurcation at (u,u,u,u) by varying g and T respectively. If we start with a negative trace, then by changing g, the system will undergo a pitchfork bifurcation at (u,u,u,u) when the determinant switches sign. If we start with a positive determinant, changing T will cause the system to undergo a Hopf bifurcation at (u,u,u,u) when the trace switches sign.

e)

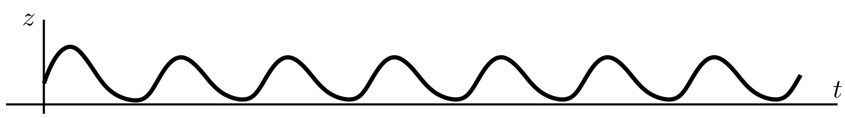
For small T, the variables settle to the stable fixed point (u,u,u,u).



Increasing T causes the stable limit cycle to appear.



And increasing T further enlarges the size of the limit cycle.



We can't plot the limit cycle since it's 4-dimensional, but we can plot the individual variables in time. We used z for the horizontal label because all four of the variables x_1, y_2, x_3, and y_4 have indistinguishable graphs. So not only does the system tend toward a symmetric fixed point, the variables even follow the limit cycle symmetrically.

8.3 Oscillating Chemical Reactions

8.3.1

$$\dot{x} = 1 - (b+1)x + ax^2y \qquad \dot{y} = bx - ax^2y \qquad a, b > 0 \quad x, y \geq 0$$

a)

$\big(x,y\big) = \big(1, \frac{b}{a}\big)$ is the fixed point.

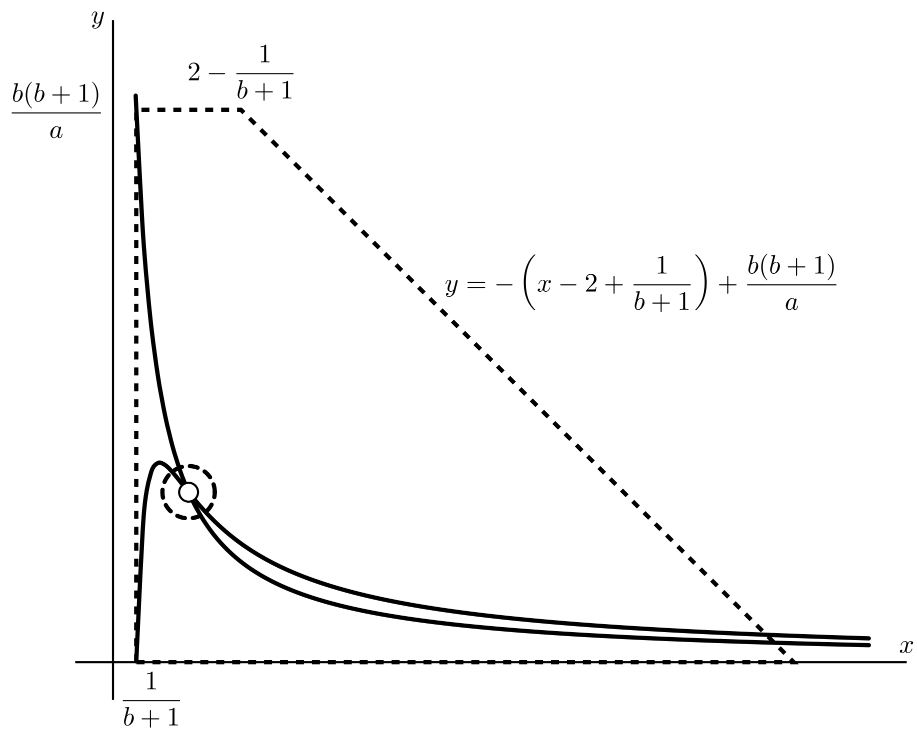
$$A = \begin{pmatrix} -(b+1) + 2axy & ax^2 \\ b - 2axy & -ax^2 \end{pmatrix}$$

$$A_{\left(1,\frac{b}{a}\right)} = \begin{pmatrix} b-1 & a \\ -b & -a \end{pmatrix} \qquad \begin{matrix} \Delta = a \\ \tau = b - (1+a) \end{matrix}$$

The determinant is positive since a is positive, and the trace can vary greatly depending on b and a. Therefore all we really know is that the fixed point can't be a saddle point.

b)

Finding a trapping region implies that we're looking for a limit cycle and $b > 1 + a$ for an unstable fixed point. A trapezoid with a circle around the fixed point will suffice.



First, we should make sure that the fixed point is always inside the trapezoid. This is actually very easy with the choice of trapezoid.

$$x : \frac{1}{b+1} < 1 < 2 - \frac{1}{b+1} \qquad y : 0 < \frac{b}{a} < \frac{b(b+1)}{a}$$

Next, to confirm that the vector field points inward on the trapezoid edges, we'll have to check that the vector field points inward on all the edges (starting with the vertical edge and going clockwise).

$$\langle 1, 0\rangle \cdot \langle \dot{x}, \dot{y}\rangle\Big|_{x=\frac{1}{b+1}} = 1 - (b+1)\frac{1}{b+1} + a\left(\frac{1}{b+1}\right)^2 y = \frac{ay}{(b+1)^2} > 0$$

$$\langle 0, -1\rangle \cdot \langle \dot{x}, \dot{y}\rangle\Big|_{y=\frac{b(b+1)}{a}} = bx - ax^2\frac{b(b+1)}{a} = bx\big(x(b+1) - 1\big) > 0$$

$$\langle -1, -1\rangle \cdot \langle \dot{x}, \dot{y}\rangle\Big|_{x>2-\frac{1}{b+1}} = 1 - (b+1)x + ax^2y - (bx - ax^2y) = -1 + x > 0$$

$$\langle 0, 1\rangle \cdot \langle \dot{x}, \dot{y}\rangle\Big|_{y=0} = bx - ax^2(0) = bx > 0$$

Lastly, since the linearization predicts an unstable fixed point, we can always draw a sufficiently small circle enclosing the fixed point with the vector field pointing outward. Therefore we have our trapping region.

c)

The Hopf bifurcation has to occur at $b = 1 + a = b_c$ where the sign of the trace changes and there are no other fixed points.

d)

The limit cycle exists for $b > b_c$ since we were able to construct a trapping region and apply the Poincaré-Bendixson theorem. (However, we should keep in mind that there could be more than one limit cycle inside the trapping region, but a stable limit cycle will appear after the Hopf bifurcation.)

e)

$b = 1 + a + \mu \quad 0 < \mu \ll 1$

$$A_{\left(1,\frac{b}{a}\right)} = \begin{pmatrix} b-1 & a \\ -b & -a \end{pmatrix} = \begin{pmatrix} a+\mu & a \\ -(1+a+\mu) & -a \end{pmatrix}$$

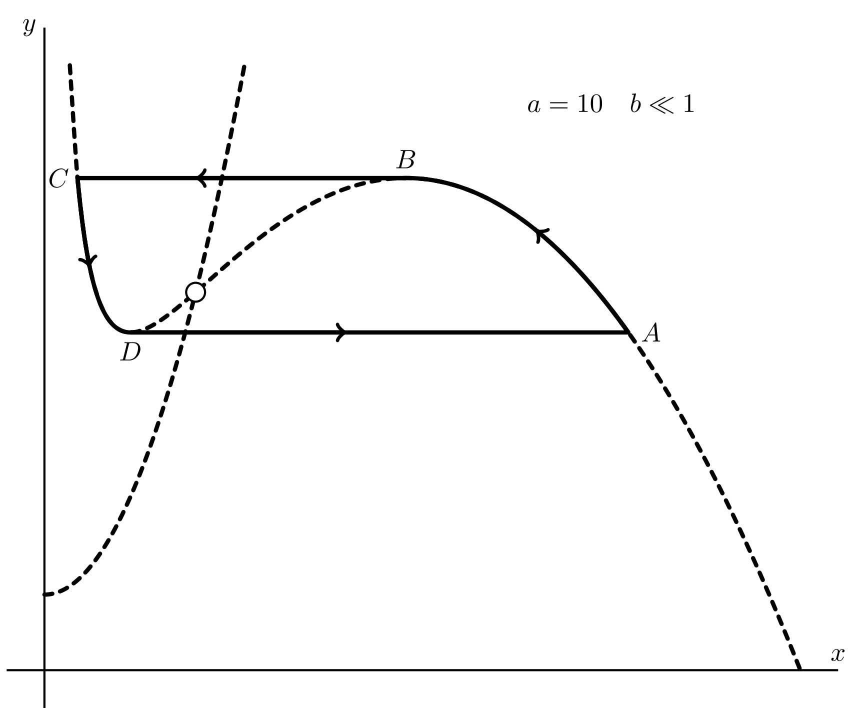
$$\lambda_{1,2} = \frac{\mu - \sqrt{\mu^2 - 4a}}{2} \approx i\sqrt{a} \Rightarrow \omega = \sqrt{a}$$

Therefore the period of the limit cycle is approximately $\dfrac{2\pi}{\sqrt{a}}$ for $0 < \mu \ll 1$.

8.3.3

$$\dot{x} = a - x - \frac{4xy}{1+x^2} \qquad \dot{y} = bx\left(1 - \frac{y}{1+x^2}\right) \qquad a, b, x, y > 0 \quad b \ll 1$$

The solution for the problem done in the text is applicable here as well. The only portion that needs to be redone is estimating the period of the limit cycle since $b \ll b_c$ now.



The entire period of the relaxation oscillation is the time spent on all four branches, but the time spent on the BC and DA branches is negligible compared to the time spent on the AB and CD branches, since $\dot{y} \ll \dot{x}$, so we can neglect the fast branches.

$$T = \int_{t_A}^{t_B} \mathrm{d}t + \int_{t_B}^{t_C} \mathrm{d}t + \int_{t_C}^{t_D} \mathrm{d}t + \int_{t_D}^{t_A} \mathrm{d}t \approx \int_{t_A}^{t_B} \mathrm{d}t + \int_{t_C}^{t_D} \mathrm{d}t$$

Next we change the integrals from $\mathrm{d}t$ to $\mathrm{d}x$.

$$\int_{t_A}^{t_B} \mathrm{d}t = \int_{x_A}^{x_B} \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}t}{\mathrm{d}y}\mathrm{d}x = \int_{x_A}^{x_B} \frac{\mathrm{d}y}{\mathrm{d}x}\frac{1}{\dot{y}}\,\mathrm{d}x = \int_{x_A}^{x_B} \frac{\mathrm{d}y}{\mathrm{d}x}\frac{1}{bx\left(1-\frac{y}{1+x^2}\right)}\,\mathrm{d}x$$

From here we can make a substitution from the $\dot{x}$ nullcline equation, making the integrand a function of x only.

$$y = \frac{(a-x)(1+x^2)}{4x} \Rightarrow \frac{y}{1+x^2} = \frac{a-x}{4x} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ax^2-a-2x^3}{4x^2}$$

$$\int_{t_A}^{t_B} \mathrm{d}t = \int_{x_A}^{x_B} \frac{\mathrm{d}y}{\mathrm{d}x}\frac{1}{bx\left(1-\frac{y}{1+x^2}\right)}\,\mathrm{d}x \approx \int_{x_A}^{x_B} \frac{ax^2-a-2x^3}{4x^2}\frac{1}{bx\left(1-\frac{a-x}{4x}\right)}\,\mathrm{d}x$$

We had to go this very roundabout way because we couldn't use $\dot{x}$ for the approximation, our first of which is in the previous step. We know that the trajectory is close to the x nullcline, so we approximated the (x, y) coordinates as exactly on the x nullcline. The problem is by definition $\dot{x} = 0$ on the x nullcline, which would make an undefined integral.

$$\int_{t_A}^{t_B} \mathrm{d}t = \int_{y_A}^{y_B} \frac{1}{\dot{x}}\,\mathrm{d}x = \int_{x_A}^{x_B} \frac{1}{a-x-\frac{4xy}{1+x^2}}\mathrm{d}x = \int_{x_A}^{x_B} \frac{1}{0}\mathrm{d}x$$

Now back to integrating.

$$\int_{t_A}^{t_B} \mathrm{d}t \approx \int_{x_A}^{x_B} \frac{ax^2-a-2x^3}{4bx^3\left(1-\frac{a-x}{4x}\right)}\,\mathrm{d}x = \int_{x_A}^{x_B} \frac{ax^2-a-2x^3}{4bx^3-bx^2(a-x)}\,\mathrm{d}x$$
$$= \left.\frac{(3a^2-125)x\ln(5x-a)-5a(2x^2+5)+125x\ln(x)}{25abx}\right|_{x_A}^{x_B}$$

Next we can use a bit of basic calculus to find the turnaround points x_A, x_B, x_C, and x_D from the local minimum and maximum of the x nullcline. Unfortunately, this requires solving a cubic equation, and plugging the results back in gets rather messy. We don't show the details here, but the approximate period would be

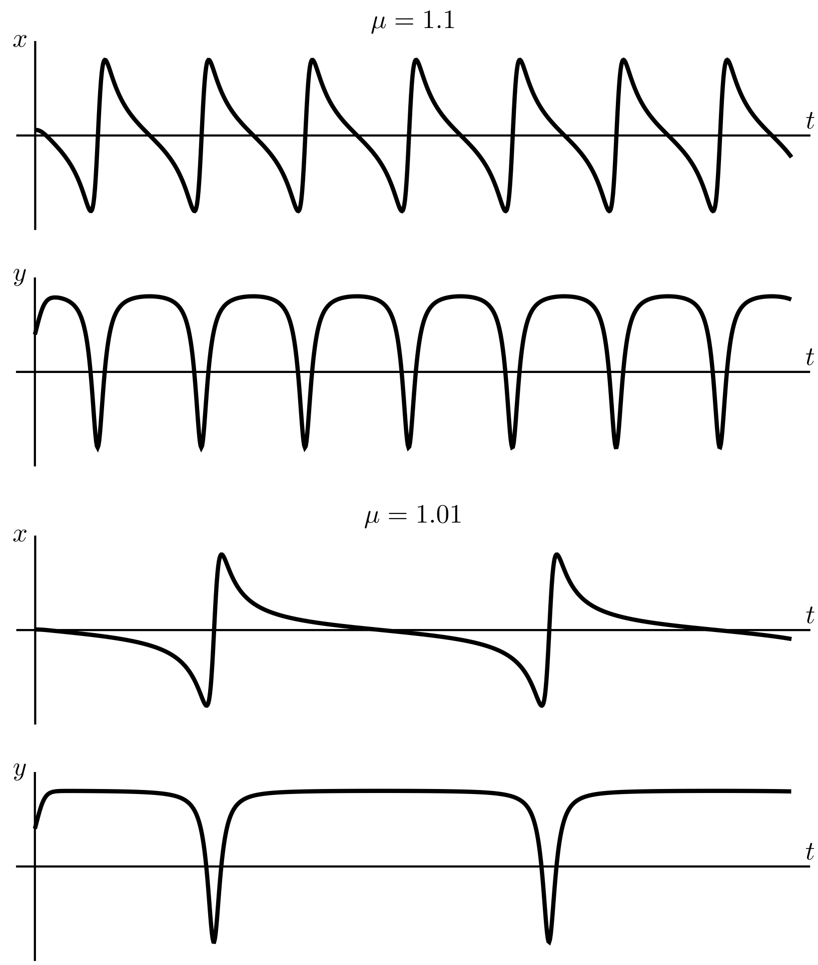
$$T \approx \int_{t_A}^{t_B} \mathrm{d}t + \int_{t_C}^{t_D} \mathrm{d}t$$
$$= \left.\frac{(3a^2-125)x\ln(5x-a)-5a(2x^2+5)+125x\ln(x)}{25abx}\right|_{x_A}^{x_B}$$
$$+ \left.\frac{(3a^2-125)x\ln(5x-a)-5a(2x^2+5)+125x\ln(x)}{25abx}\right|_{x_C}^{x_D}$$

8.4 Global Bifurcations of Cycles

8.4.1

$$\dot{r} = r(1 - r^2) \qquad \dot{\theta} = \mu - \sin(\theta)$$

$x, y \in [-1.25, 1.25] \quad t \in [0, 100]$

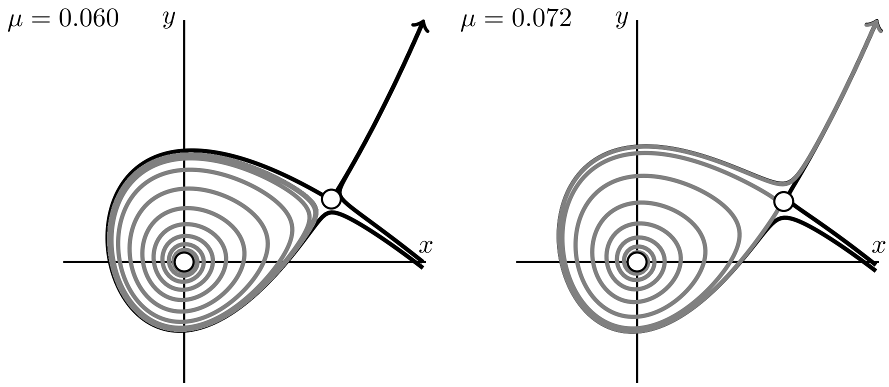


8.4.3

$$\dot{x} = \mu x + y - x^2 \qquad \dot{y} = -x + \mu y + 2x^2$$

Homoclinic bifurcation at $\mu \approx 0.066$.

These graphs have bounds of $(x, y) \in [-0.4, 0.8] \times [-0.4, 0.8]$.



In each case, the gray trajectory starts near the origin. The left graph traps the trajectory in a stable limit cycle, but the the stable limit cycle eventually makes a homoclinic orbit with the saddle point and then disappears in a homoclinic bifurcation. The right graph is after the homoclinic bifurcation.

8.4.5

$$\ddot{x} + x + \epsilon\big(bx^3 + k\dot{x} - ax - F\cos(t)\big) = 0 \qquad 0 < \epsilon << 1 \quad b, k, F > 0$$

$$\begin{aligned} h(x, \dot{x}) &= bx^3 + k\dot{x} - ax - F\cos(t) = bx^3 + k\dot{x} - ax - F\cos(\theta - \phi)h\big(r\cos(\theta), -r\sin(\theta)\big) \\ &= br^3\cos^3(\theta) - kr\sin(\theta) - ar\cos(\theta) - F\cos(\theta - \phi) \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}T} &= \langle h\sin(\theta)\rangle = \frac{1}{2\pi}\int_0^{2\pi} h\big(r\cos(\theta), -r\sin(\theta)\big)\sin(\theta)\mathrm{d}\theta \\ &= \frac{-\big(kr + F\sin(\phi)\big)}{2} \\ r\frac{\mathrm{d}\phi}{\mathrm{d}T} &= \langle h\cos(\theta)\rangle = \frac{1}{2\pi}\int_0^{2\pi} h\big(r\cos(\theta), -r\sin(\theta)\big)\cos(\theta)\mathrm{d}\theta \\ &= \frac{-\big(4ar - 3br^3 + 4F\cos(\phi)\big)}{8} \\ \frac{\mathrm{d}\phi}{\mathrm{d}T} &= \frac{-\big(4ar - 3br^3 + 4F\cos(\phi)\big)}{8r} \end{aligned}$$

8.4.7

$$\mathbf{x}' = (r', r\phi') \qquad g(r\phi) \equiv 1$$

The reason for the $(r', r\phi')$ instead of (r', ϕ') is due to using the polar coordinate unit vectors. $\mathbf{x}' = (r', r\phi') = r'\hat{r} + r\phi'\hat{\phi}$ and the polar coordinate unit vectors, unlike Cartesian coordinates, are dependent on the angle of the point.

$$r' = \frac{-\big(kr + F\sin(\phi)\big)}{2} \qquad \phi' = \frac{-\big(4ar - 3br^3 + 4F\cos(\phi)\big)}{8r}$$

$$\begin{aligned}
\nabla \cdot \mathbf{x}' &= \frac{1}{r}\frac{\partial}{\partial r}(rr') + \frac{1}{r}\frac{\partial}{\partial \phi}(r\phi') \\
&= -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{kr^2 + Fr\sin(\phi)}{2}\right) - \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{4ar - 3br^3 + 4F\cos(\phi)}{8}\right) \\
&= -\frac{1}{r}\left(\frac{2kr + F\sin(\phi)}{2}\right) - \frac{1}{r}\left(\frac{-4F\sin(\phi)}{8}\right) = -k
\end{aligned}$$

Thus, the value of $\nabla \cdot (g\mathbf{x}') < 0$ since $k > 0$ and there are no closed orbits for the averaged system by Dulac's criterion.

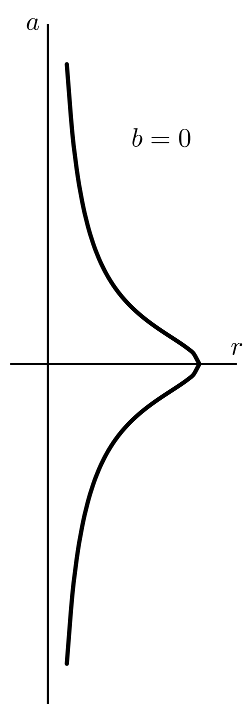
8.4.9

$$r' = \frac{-\big(kr + F\sin(\phi)\big)}{2} \qquad \phi' = \frac{-\big(4ar - 3br^3 + 4F\cos(\phi)\big)}{8r}$$

a)

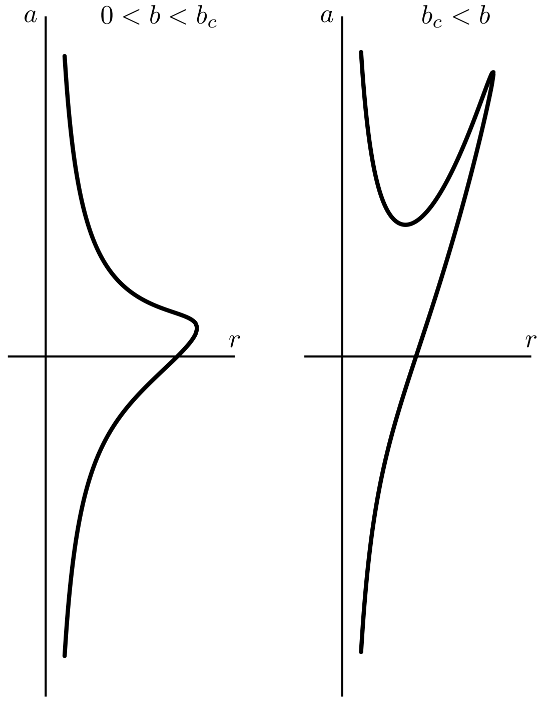
$$\begin{aligned}
&r' = \frac{-\big(kr + F\sin(\phi)\big)}{2} = 0 \Rightarrow -\frac{kr}{F} = \sin\phi \\
&\sin^2(\phi) + \cos^2(\phi) = 1 \Rightarrow \cos^2(\phi) = 1 - \frac{k^2r^2}{F^2} \\
&\phi' = \frac{-\big(4ar - 3br^3 + 4F\cos(\phi)\big)}{8r} = 0 \Rightarrow 4ar - 3br^3 + 4F\cos(\phi) = 0 \\
&3br^3 - 4ar = 4F\cos(\phi) \\
&(3br^3 - 4ar)^2 = 16F^2\cos^2(\phi) = 16F^2\left(1 - \frac{k^2r^2}{F^2}\right) = 16F^2 - 16k^2r^2 \\
&16k^2r^2 + (3br^3 - 4ar)^2 = 16F^2 \\
&r^2\left[k^2 + \left(\frac{3}{4}br^2 - a\right)^2\right] = F^2
\end{aligned}$$

b)



c)

Increasing b slightly positive, and then a bit more gives the following two graphs.



The tip of the $b = 0$ graph continually moves up until the range is eventually overlapped. Now there are three different r values in a range of a values where the graph doubles back.

We can solve for a in terms of r somewhat easily and then find when the solutions overlap.

$$a = \frac{3}{4}br^2 \pm \sqrt{\frac{F^2}{r^2} - k^2}$$

The graph will double back when there is zero slope with respect to r.

$$\frac{\mathrm{d}a}{\mathrm{d}r} = \frac{3}{2}br \pm \frac{F^2}{r^2\sqrt{(F-kr)(F+kr)}} = 0$$

We throw out the plus from the $\pm$ sign because we need the terms to cancel.

$$\frac{\mathrm{d}a}{\mathrm{d}r} = \frac{3}{2}br - \frac{F^2}{r^2\sqrt{(F-kr)(F+kr)}} = 0$$

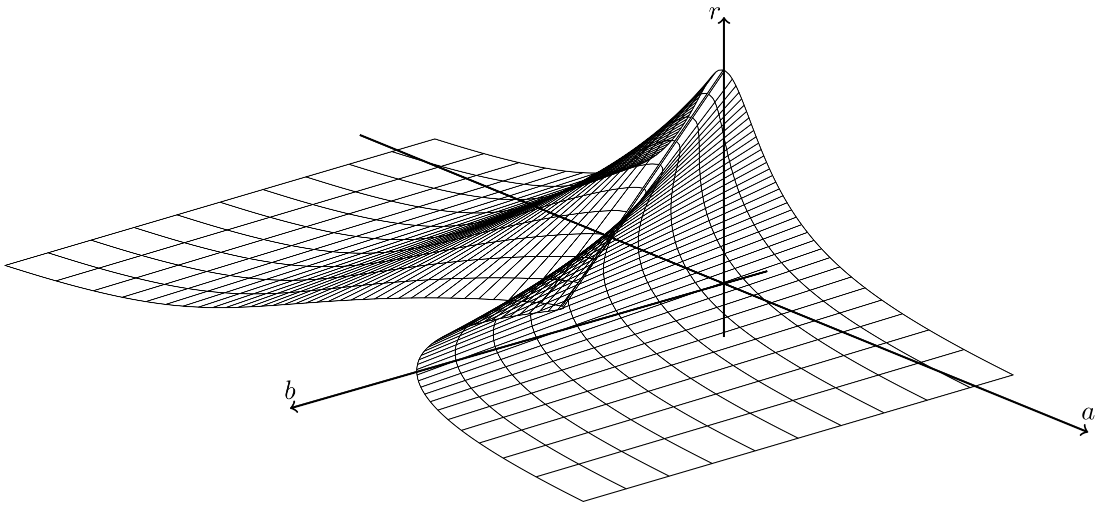
We also know that the second derivative will equal zero at $b = b_c$ when the graph is cubic-shaped and about to develop a local minima.

$$\frac{\mathrm{d}^2a}{\mathrm{d}r^2} = \frac{3}{2}b - \frac{2F^4 - 3F^2k^2r^2}{r^3\big((F-kr)(F+kr)\big)^{\frac{3}{2}}} = 0$$

We can find the conditions under which there is a real value of r as a solution to these two equations. The result is

$$b \geq \frac{32\sqrt{3}k^3}{27F^2} = b_c$$

d)



8.4.11

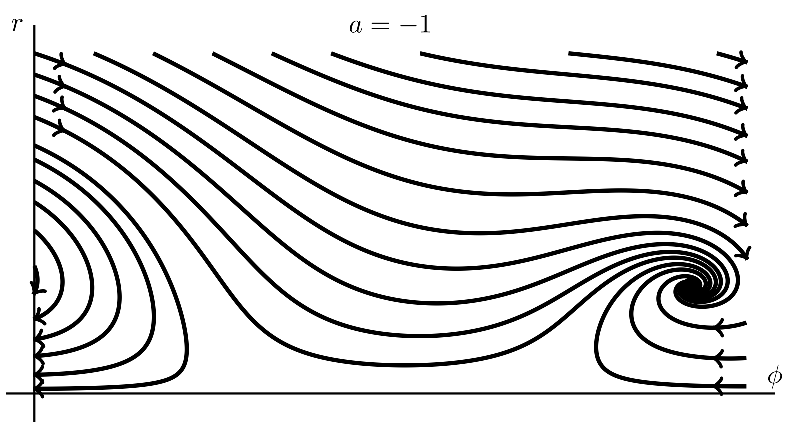
$k = 1 \quad b = \frac{4}{3} \quad F = 2$

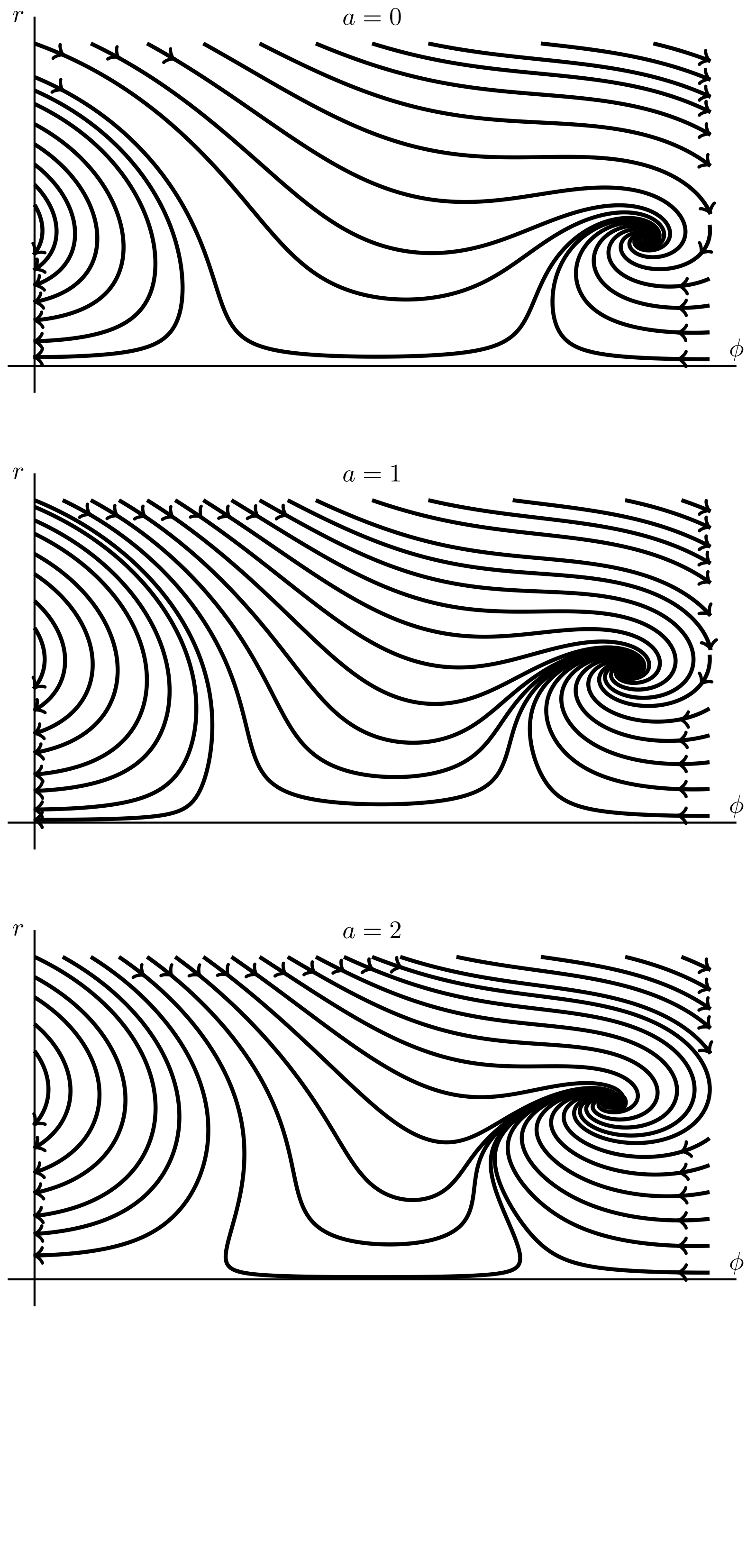
a)

$$\ddot{x} + x + \epsilon\big(bx^3 + k\dot{x} - ax - F\cos(t)\big) = 0 \qquad 0 < \epsilon << 1 \quad b, k, F > 0$$

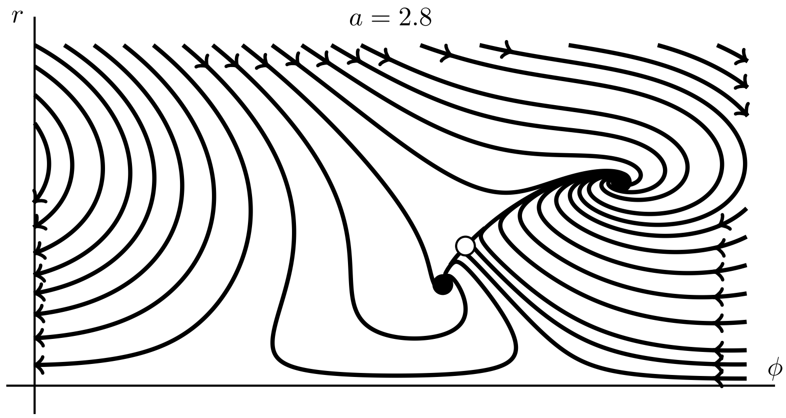
Averaged system

$$r' = \frac{-\big(kr + F\sin(\phi)\big)}{2} \qquad \phi' = \frac{-\big(4ar - 3br^3 + 4F\cos(\phi)\big)}{8r}$$




$a = 0$
r
ϕ
$a = 1$
r
ϕ
$a = 2$
r
ϕ

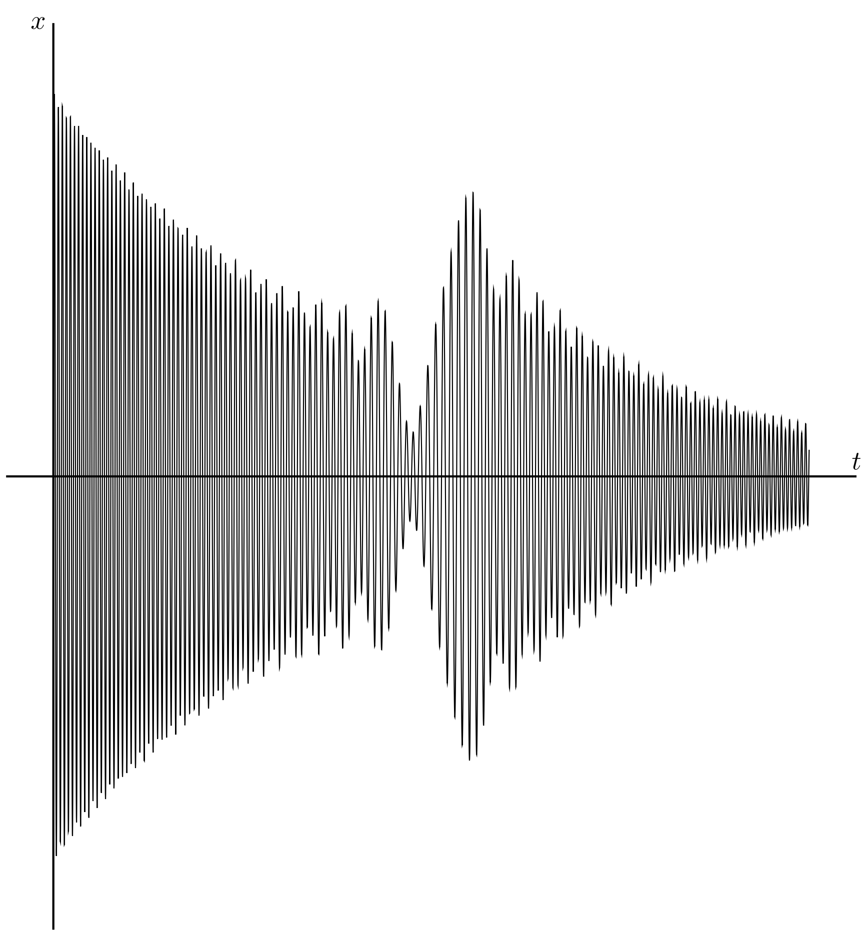
b)



c)

$k = 1, b = \frac{4}{3}, F = 2, \epsilon = 0.01$

The simulation is run from $t = 0$ to $t = 1200$. a starts at -1, linearly increases to 5 at $t = 300$, then linearly decreases back to -1 at $t = 600$.



8.5 Hysteresis in the Driven Pendulum and Josephson Junction

8.5.1

$$f(I) = \frac{1}{\ln(I - I_c)}$$

Taking the first few derivatives

$$\frac{\mathrm{d}}{\mathrm{d}I}\left(\frac{1}{\ln(I - I_c)}\right) = \frac{-1}{(I - I_c)\big(\ln(I - I_c)\big)^2}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}I^2}\left(\frac{1}{\ln(I - I_c)}\right) = \frac{2}{(I - I_c)^2\big(\ln(I - I_c)\big)^3} + \frac{1}{(I - I_c)^2\big(\ln(I - I_c)\big)^2}$$

$$\frac{\mathrm{d}^3}{\mathrm{d}I^2}\left(\frac{1}{\ln(I - I_c)}\right) = \frac{-6}{(I - I_c)^3\big(\ln(I - I_c)\big)^4} + \frac{-6}{(I - I_c)^3\big(\ln(I - I_c)\big)^3} + \frac{-2}{(I - I_c)^3\big(\ln(I - I_c)\big)^2}$$

we can rewrite them all in terms of $f(I) = f$.

$$f^{(1)} = \frac{-f^2}{I - I_c}$$

$$f^{(2)} = \frac{2f^3}{(I - I_c)^2} + \frac{f^2}{(I - I_c)^2} = \frac{2f^3 + f^2}{(I - I_c)^2}$$

$$f^{(3)} = \frac{-6f^4}{(I - I_c)^3} + \frac{-6f^3}{(I - I_c)^3} + \frac{-2f^2}{(I - I_c)^3} = \frac{-6f^4 - 6f^3 - 2f^2}{(I - I_c)^3}$$

Maybe we can rewrite all the derivatives in terms of f.

$$f^{(n)} = \frac{\sum_{p=2}^{n+1} a_p f^p}{(I - I_c)^n}$$

$$\begin{aligned}
f^{(n+1)} &= \frac{\sum_{p=2}^{n+1} a_p p f^{p-1}}{(I - I_c)^n} f^{(1)} - n\frac{\sum_{p=2}^{n+1} a_p f^p}{(I - I_c)^{n+1}} \\
&= \frac{\sum_{p=2}^{n+1} a_p p f^{p-1}}{(I - I_c)^n}\left(\frac{-f^2}{I - I_c}\right) - n\frac{\sum_{p=2}^{n+1} a_p f^p}{(I - I_c)^{n+1}} \\
&= \frac{-\sum_{p=2}^{n+1} a_p p f^{p+1}}{(I - I_c)^{n+1}} - n\frac{\sum_{p=2}^{n+1} a_p f^p}{(I - I_c)^{n+1}} \\
&= \frac{-\sum_{p=2}^{n+1} a_p p f^{p+1} - n\sum_{p=2}^{n+1} a_p f^p}{(I - I_c)^{n+1}} = \frac{\sum_{p=2}^{n+2} \tilde{a}_p f^p}{(I - I_c)^{n+1}}
\end{aligned}$$

The coefficients are a bit confusing to figure out, but we have a pattern. Now we just need to make sure that there is at least one term in the numerator for all derivatives in case everything cancels to zero somehow. Looking a little closer at the recurrence relation, the coefficient of the highest power of f in the numerator is

$$f^{(n)} = \frac{\sum_{p=2}^{n+1} a_p f^p}{(I - I_c)^n} \qquad a_{n+1} = (-1)^n n!$$

So the highest power of f in $f^{(n)}$ has an infinite limit at I_c. (In fact, all terms have an infinite limit as long as the coefficient doesn't cancel out.) Therefore $f(I)$ has infinite derivatives of all orders at I_c.

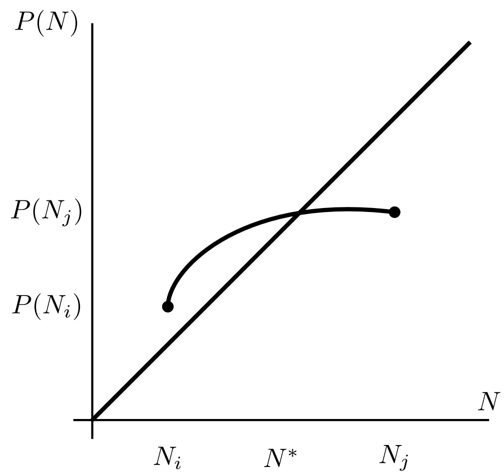
8.5.3

$$\dot{N} = rN\left(1 - \frac{N}{K(t)}\right)$$

a)

$K(t)$ is positive and smooth, and T-periodic in t implies $K(t)$ has a maximum and minimum $K_{\max}$ and $K_{\min}$ on the cylindrical phase space. Notice that $N > K_{\max} \Rightarrow \dot{N} < 0$ and $N < K_{\min} \Rightarrow \dot{N} > 0$ from the differential equation.

Next, we'll define a Poincaré map $P\big(N(t)\big) = N(t+T)$ as successive iterates of trajectories around the cylinder. Just as in the text, we know $P(N)$ is continuous and monotonic, and there must be a fixed point of the Poincaré map $P(N^*) = N^*$ corresponding to a limit cycle of the differential equation.



We also know that $K_{\min} < N^* < K_{\max}$ since $N = K_{\min}$ and $N = K_{\max}$ form a trapping region for the stable limit orbit.

b)

We don't know the stable limit cycle is unique from part (a) since there could be multiple crossings. We'll need to prove it.

First, we do a change of variables.

$$x = \frac{1}{N} \Rightarrow -\frac{\dot{x}}{x^2} = \dot{N} = rN\left(1 - \frac{N}{K(t)}\right) = \frac{r}{x}\left(1 - \frac{1}{xK(t)}\right)$$

$$\Rightarrow \dot{x} + rx = \frac{r}{K(t)}$$

This is a first-order linear ODE, which can be solved with an integrating factor.

$$\begin{aligned}
\dot{x} + rx &= \frac{r}{K(t)} \\
e^{rt}\dot{x} + re^{rt}x &= \frac{re^{rt}}{K(t)} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{rt}x\right) &= \frac{re^{rt}}{K(t)} \\
e^{rt}x &= \int \frac{re^{rt}}{K(t)}\ \mathrm{d}t + C \\
x(t) &= e^{-rt}\left(\int \frac{re^{rt}}{K(t)}\ \mathrm{d}t + C\right)
\end{aligned}$$

We know the integral has a unique solution because $K(t)$ being positive and smooth is well-behaved, and we know that all solutions converge to this unique solution.

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} e^{-rt}\left(\int \frac{re^{rt}}{K(t)}\ \mathrm{d}t + C\right) \longrightarrow e^{-rt}\int \frac{re^{rt}}{K(t)}\ \mathrm{d}t$$

Therefore the stable limit cycle is unique.

8.5.5

$$\ddot{\theta} + \alpha\dot{\theta}|\dot{\theta}| + \sin(\theta) = F \qquad \alpha, F > 0$$

a)

$$\dot{\theta} = \nu \qquad \dot{\nu} = F - \alpha\nu|\nu| - \sin(\theta)$$

The fixed points are $\big(\theta,\nu\big) = \big(\arcsin(F), 0\big)$ and $\big(\pi - \arcsin(F), 0\big) \quad F < 1$

$$A = \begin{pmatrix} 0 & 1 \\ -\cos(\theta) & -2\alpha|\nu| \end{pmatrix}$$

$$A_{\left(\arcsin(F),0\right)} = \begin{pmatrix} 0 & 1 \\ -\sqrt{1-F^2} & 0 \end{pmatrix} \qquad \begin{matrix}\lambda_{1,2} = \pm i(1-F^2)^{\frac{1}{4}} \\ \text{center}\end{matrix}$$

$$A_{\left(\pi-\arcsin(F),0\right)} = \begin{pmatrix} 0 & 1 \\ \sqrt{1-F^2} & 0 \end{pmatrix} \qquad \begin{matrix}\lambda_{1,2} = \pm(1-F^2)^{\frac{1}{4}} \\ \text{saddle point}\end{matrix}$$

Now we aren't sure if the predicted center is really a center. A quick numerical simulation will show it's actually a stable spiral. Now we look for a suitable Liapunov function. A little bit of experimentation with an "energy" relative to the fixed point gives

$$V(\theta,\nu) = \frac{1}{2}\nu^2 - F\theta - \cos(\theta) + F\arcsin(F) + \sqrt{1-F^2}$$

$$\dot{V} = \nu\dot{\nu} - F\dot{\theta} + \sin(\theta)\dot{\theta} = \nu\big(F - \alpha\nu|\nu| - \sin(\theta)\big) - F\nu + \sin(\theta)\nu = -\alpha\nu^2|\nu|$$

Our V is not strictly Liapunov because $\dot{V}(\theta, 0) = 0$, meaning $\dot{V}$ is not negative on the line $\nu = 0$. However, this suffices to prove that the fixed point is a stable spiral. That's because the trajectories always move inward to lower and lower contours of V, except at instants when they cross the line $\nu = 0$. At each such instant they move tangentially to the V-contour, but an instant later they resume moving inward. Also, the V contours approach ellipses as we approach the fixed point, so the geometry of this situation is clear.

Thus the Liapunov "energy" V is almost always decreasing (but never increasing) towards the fixed point after each intersection with $\nu = 0$, and the "energy" of the fixed point is zero. Therefore the fixed point is a stable spiral.

b)

We'll construct a trapping region to prove the existence of the stable limit cycle.

$$\begin{aligned}
&\dot{\nu} = F - \alpha\nu|\nu| - \sin(\theta) \geq 0 \\
&F - \sin(\theta) \geq F - 1 \geq \alpha\nu|\nu| \\
&\nu = \sqrt{\frac{F-1}{\alpha}} \Rightarrow \dot{\nu} \geq 0 \\
&\dot{\nu} = F - \alpha\nu|\nu| - \sin(\theta) \leq 0 \\
&F - \sin(\theta) \leq F + 1 \leq \alpha\nu|\nu| \\
&\nu = \sqrt{\frac{F+1}{\alpha}} \Rightarrow \dot{\nu} \leq 0
\end{aligned}$$

There are no fixed points in this region. Therefore there must be at least one stable limit cycle within

$$0 \leq \theta < 2\pi \qquad \sqrt{\frac{F-1}{\alpha}} \leq \nu \leq \sqrt{\frac{F+1}{\alpha}}$$

Next, we can use the total energy of the system after each cycle to prove there is a unique stable limit cycle within the bounds. Even though the system is losing energy to friction and gaining energy from the externally applied torque, the system will repeat the same velocity at the same angle each cycle. Therefore the total energy at that position will always be the same.

The total energy for a pendulum is given by

$$E = \frac{1}{2}\nu^2 + \big(1 - \cos(\theta)\big)$$

Now we can repeat the reasoning in Section 8.5, "Uniqueness of the Limit Cycle," with some small changes. E is different, and also $y \Rightarrow \nu$ and $\phi \to \theta$. Starting with Equation (7),

$$\frac{\mathrm{d}E}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}\Big(\frac{1}{2}\nu^2 + \big(1 - \cos(\theta)\big)\Big) = \nu\frac{\mathrm{d}\nu}{\mathrm{d}\theta} + \sin(\theta)$$

$$\frac{\mathrm{d}\nu}{\mathrm{d}\theta} = \frac{\dot{\nu}}{\dot{\theta}} = \frac{F - \alpha\nu|\nu| - \sin(\theta)}{\nu}$$

Hence

$$\frac{\mathrm{d}E}{\mathrm{d}\theta} = \nu\frac{F - \alpha\nu|\nu| - \sin(\theta)}{\nu} + \sin(\theta) = F - \alpha\nu|\nu|$$

Going back to the change in energy integral

$$0 = \int_0^{2\pi} \frac{\mathrm{d}E}{\mathrm{d}\theta}\mathrm{d}\theta = \int_0^{2\pi} \big(F - \alpha\nu|\nu|\big)\mathrm{d}\theta \Rightarrow \int_0^{2\pi} \nu(\theta)\big|\nu(\theta)\big|\mathrm{d}\theta = \frac{2\pi F}{\alpha}$$

To prove uniqueness, suppose there were two distinct limit cycles. Following the notation and reasoning in Section 8.5 of the text, denote these by $\nu_U(\theta)$ and $\nu_L(\theta)$. Hence $\nu_U(\theta) > \nu_L(\theta)$ implies

$$\int_0^{2\pi} \nu_U(\theta)\big|\nu_U(\theta)\big|\mathrm{d}\theta > \int_0^{2\pi} \nu_L(\theta)\big|\nu_L(\theta)\big|\mathrm{d}\theta$$

which is a contradiction. Therefore the limit cycle is unique.

c)

$$\begin{aligned}
u(\theta) &= \frac{1}{2}\nu^2\big(t(\theta)\big) \\
\frac{\mathrm{d}u}{\mathrm{d}\theta} &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\nu^2\big(t(\theta)\big) = \nu\frac{\mathrm{d}}{\mathrm{d}\theta}\nu\big(t(\theta)\big) = \nu\frac{\mathrm{d}\nu}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\theta} = \nu\dot{\nu}\frac{1}{\nu} = \dot{\nu} = \ddot{\theta}
\end{aligned}$$

d)

$$\begin{aligned}
&\nu > 0 \longrightarrow \dot{\theta} > 0 \\
&\ddot{\theta} + \alpha\dot{\theta}|\dot{\theta}| + \sin(\theta) = F \longrightarrow \ddot{\theta} + \alpha\dot{\theta}^2 + \sin(\theta) = F \\
&\frac{\mathrm{d}u}{\mathrm{d}\theta} = \ddot{\theta} \quad u = \frac{1}{2}\nu^2 \\
&\ddot{\theta} + \alpha\dot{\theta}^2 + \sin(\theta) = F \longrightarrow \frac{\mathrm{d}u}{\mathrm{d}\theta} + 2\alpha u + \sin(\theta) = F
\end{aligned}$$

This is an integrating factor problem with solution

$$u(\theta) = \frac{F}{2\alpha} + \frac{1}{1+4\alpha^2}\cos(\theta) - \frac{2\alpha}{1+4\alpha^2}\sin(\theta) + Ce^{-2\alpha\theta}$$

And the limit cycle is when the transient decays to zero, leaving

$$u(\theta) = \frac{F}{2\alpha} + \frac{1}{1+4\alpha^2}\cos(\theta) - \frac{2\alpha}{1+4\alpha^2}\sin(\theta)$$

e)

We can find F at the homoclinic bifurcation by constraining $u(\theta)$ to go through one of the fixed points, which has to be the saddle point because homoclinic bifurcations occur with saddles.

The saddle point is

$$(\theta, \nu) = \big(\pi - \arcsin(F), 0\big) \longrightarrow (\theta, u) = \big(\pi - \arcsin(F), 0\big)$$

$$\begin{aligned}
u(\theta) &= \frac{F}{2\alpha} + \frac{1}{1+4\alpha^2}\cos(\theta) - \frac{2\alpha}{1+4\alpha^2}\sin(\theta) \\
0 &= \frac{F}{2\alpha} + \frac{1}{1+4\alpha^2}\left(-\sqrt{1-F^2}\right) - \frac{2\alpha}{1+4\alpha^2}F \\
\frac{2\alpha}{1+4\alpha^2}F - \frac{F}{2\alpha} &= \frac{-1}{1+4\alpha^2}\sqrt{1-F^2} \\
4\alpha^2 F - F(1+4\alpha^2) &= -2\alpha\sqrt{1-F^2} \\
-F &= -2\alpha\sqrt{1-F^2} \\
F^2 &= 4\alpha^2(1-F^2) = 4\alpha^2 - 4\alpha^2 F^2 \\
F^2(1-4\alpha^2) &= 4\alpha^2 \\
F = F_c(\alpha) &= \frac{2\alpha}{\sqrt{1+4\alpha^2}}
\end{aligned}$$

And that's where the homoclinic bifurcation occurs.

8.6 Coupled Oscillators and Quasiperiodicity

8.6.1

$$\dot{\theta}_1 = \omega_1 + \sin(\theta_1)\cos(\theta_2) \qquad \dot{\theta}_2 = \omega_2 + \sin(\theta_2)\cos(\theta_1) \qquad \omega_1, \omega_2 \geq 0$$

a)

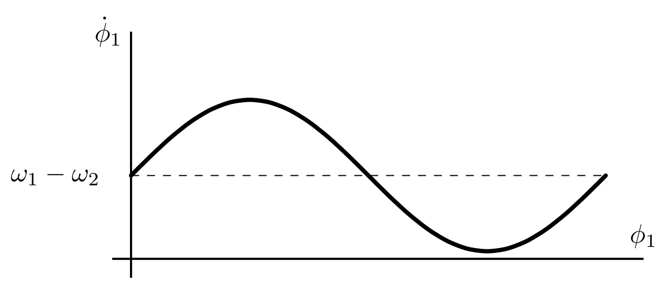
Let $\phi_1 = \theta_1 - \theta_2$, which corresponds to the phase difference of the two oscillators.

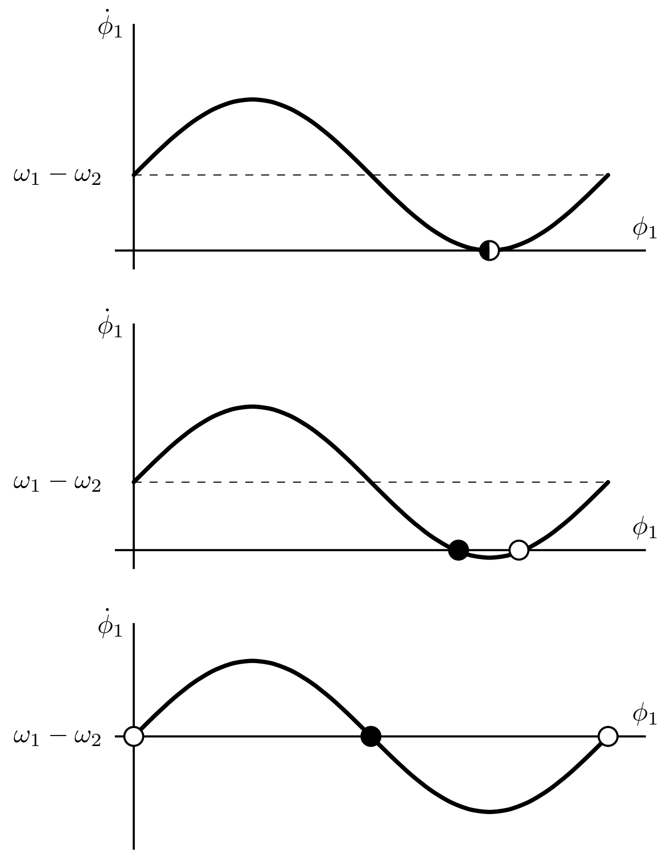
$$\begin{aligned}
\dot{\phi}_1 = \dot{\theta}_1 - \dot{\theta}_2 &= \omega_1 + \sin(\theta_1)\cos(\theta_2) - \omega_2 - \sin(\theta_2)\cos(\theta_1) \\
&= \omega_1 - \omega_2 + \sin(\theta_1 - \theta_2) = \omega_1 - \omega_2 + \sin(\phi_1)
\end{aligned}$$

The fixed points correspond to phase-locked solutions of the original system; i.e., periodic trajectories on the torus.

$$\begin{aligned}
\dot{\phi}_1 &= \omega_1 - \omega_2 + \sin(\phi_1) = 0 \\
\phi_1 &= 2\pi - \arcsin(\omega_1 - \omega_2) \quad \pi + \arcsin(\omega_1 - \omega_2) \qquad |\omega_1 - \omega_2| \leq 1
\end{aligned}$$

A graphical approach for $\omega_1 - \omega_2 \geq 0$ (and similarly for $\omega_1 - \omega_2 \leq 0$) shows





Next, we'll look at $\phi_2 = \theta_1 + \theta_2$, the fixed points of which correspond to periodic trajectories on the torus running in opposite directions, in contrast to the fixed points of ϕ_1. ($\dot{\theta}_1 + \dot{\theta}_2 = 0 \Rightarrow \dot{\theta}_1 = -\dot{\theta}_2$)

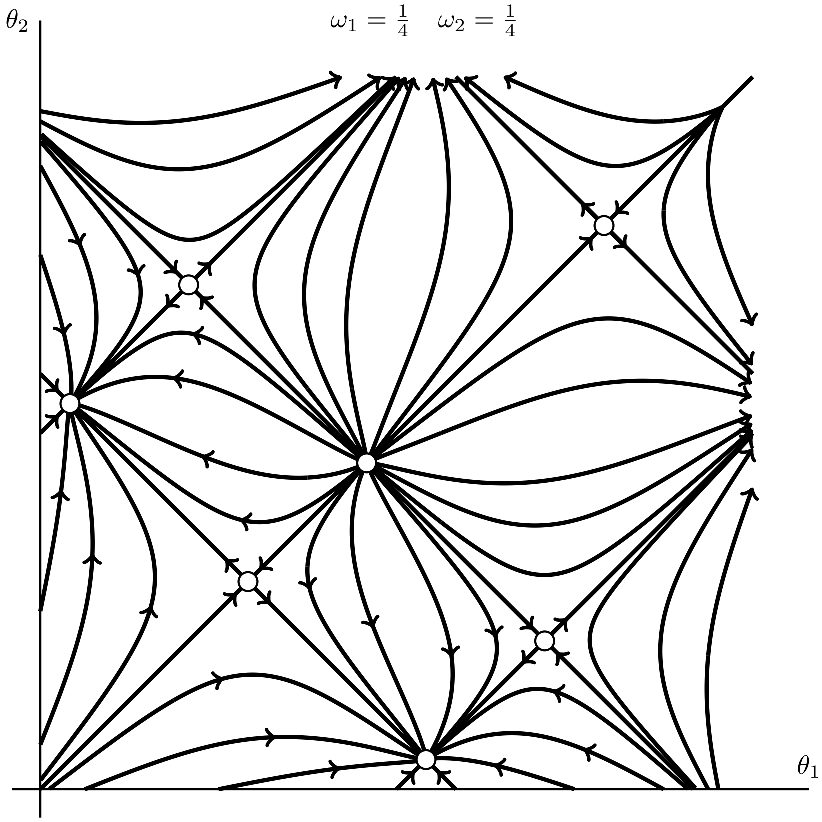
$$\dot{\phi}_2 = \dot{\theta}_1 + \dot{\theta}_2 = \omega_1 + \sin(\theta_1)\cos(\theta_2) + \omega_2 + \sin(\theta_2)\cos(\theta_1)$$
$$= \omega_1 + \omega_2 + \sin(\theta_1 + \theta_2) = \omega_1 + \omega_2 + \sin(\phi_2)$$

The fixed points are

$$\dot{\phi}_2 = \omega_1 + \omega_2 + \sin(\phi_2) = 0$$
$$\phi_2 = 2\pi - \arcsin(\omega_1 + \omega_2) \quad \pi + \arcsin(\omega_1 + \omega_2) \qquad \omega_1 + \omega_2 \leq 1$$

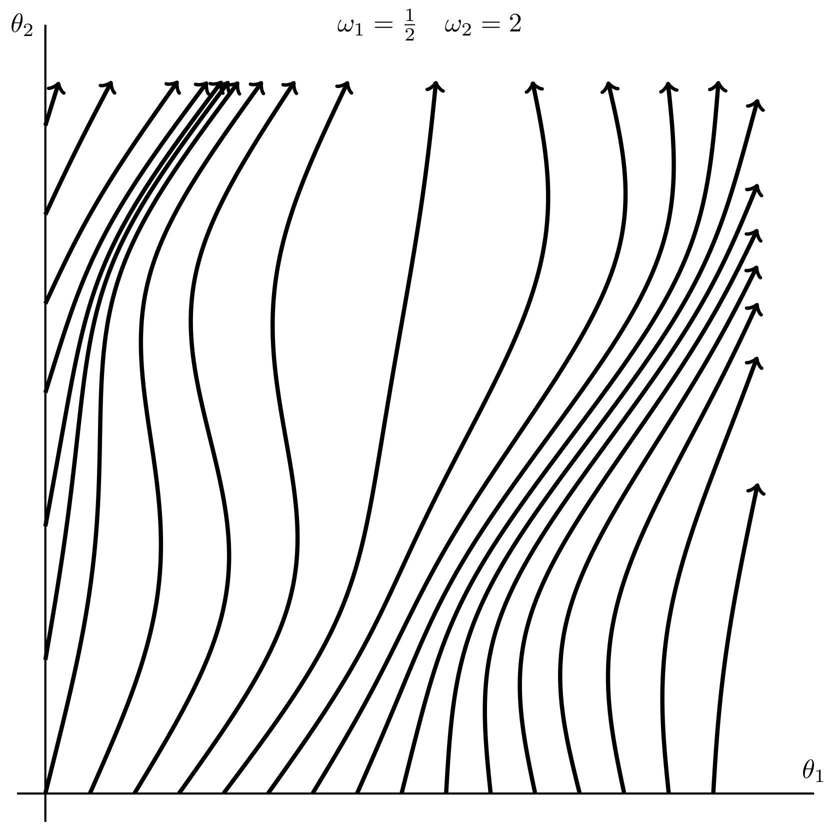
A graphical approach for $\omega_1 - \omega_2 \geq 0$ is the same as previously, except this time there is no $\omega_1 + \omega_2 \leq 0$ case to consider because $\omega_1, \omega_2 \geq 0$.

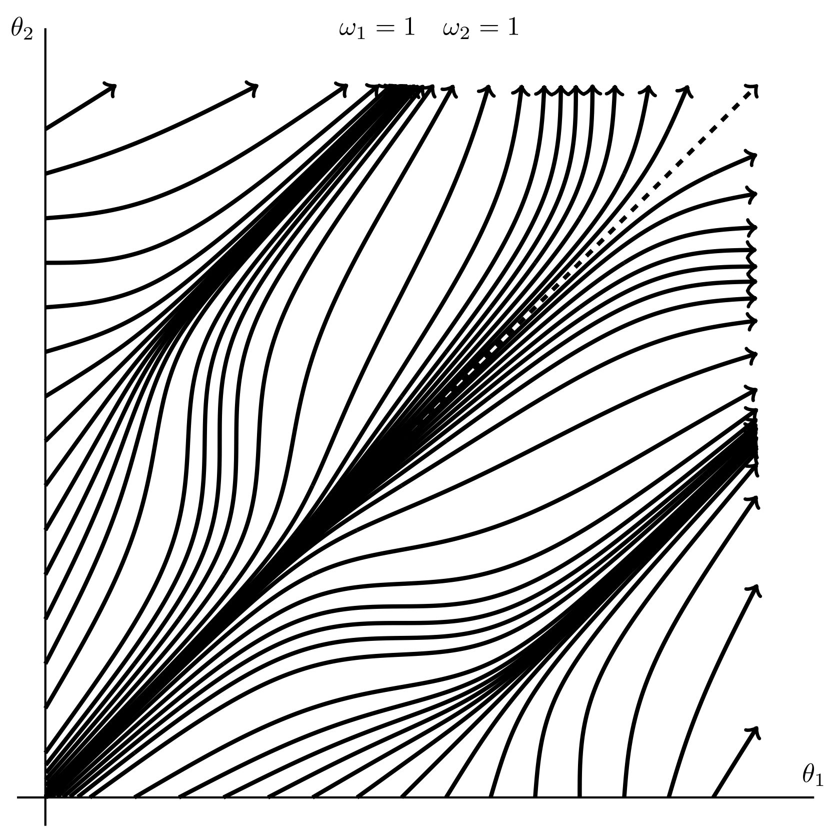
Now we have to go through all the possible cases for ω_1 and ω_2 and the existence of fixed points in the (ϕ_1, ϕ_2) system and see what happens in the (θ_1, θ_2) system. (Actually we only show graphs from about half the regions due to symmetry.)



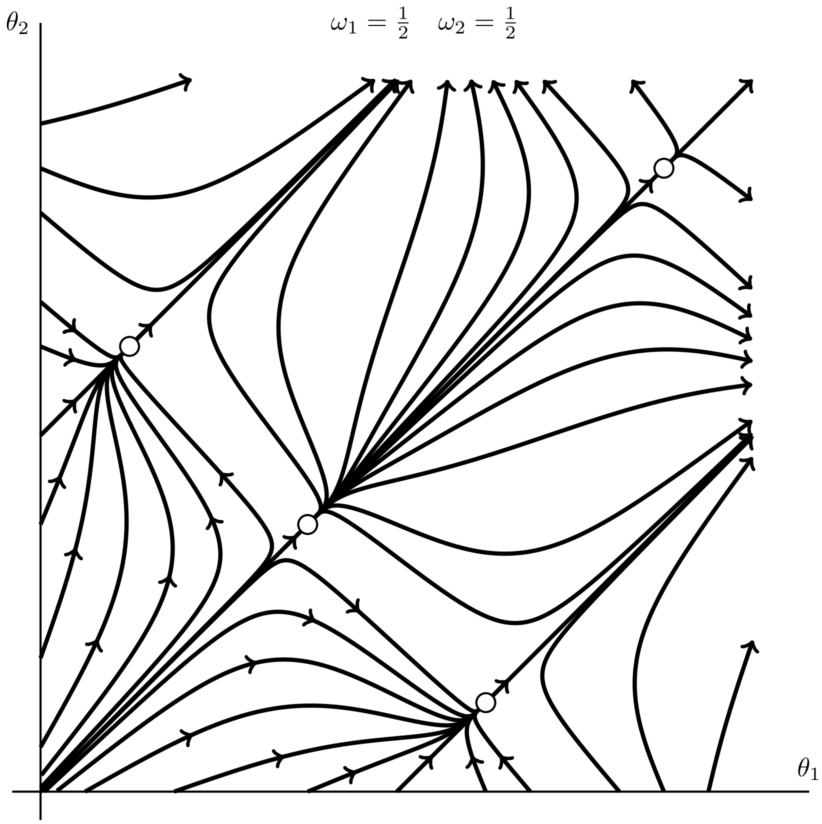
Fixed points

$$(\theta_1, \theta_2) = \left(\frac{\pi}{12}, \frac{13\pi}{12}\right), \left(\frac{5\pi}{12}, \frac{17\pi}{12}\right), \left(\frac{7\pi}{12}, \frac{7\pi}{12}\right), \left(\frac{11\pi}{12}, \frac{11\pi}{12}\right), \left(\frac{13\pi}{12}, \frac{\pi}{12}\right), \left(\frac{17\pi}{12}, \frac{5\pi}{12}\right), \left(\frac{19\pi}{12}, \frac{19\pi}{12}\right)$$


θ_2
$\omega_1 = \frac{1}{2} \quad \omega_2 = 2$
θ_1


θ_2
$\omega_1 = 1 \quad \omega_2 = 1$
θ_1

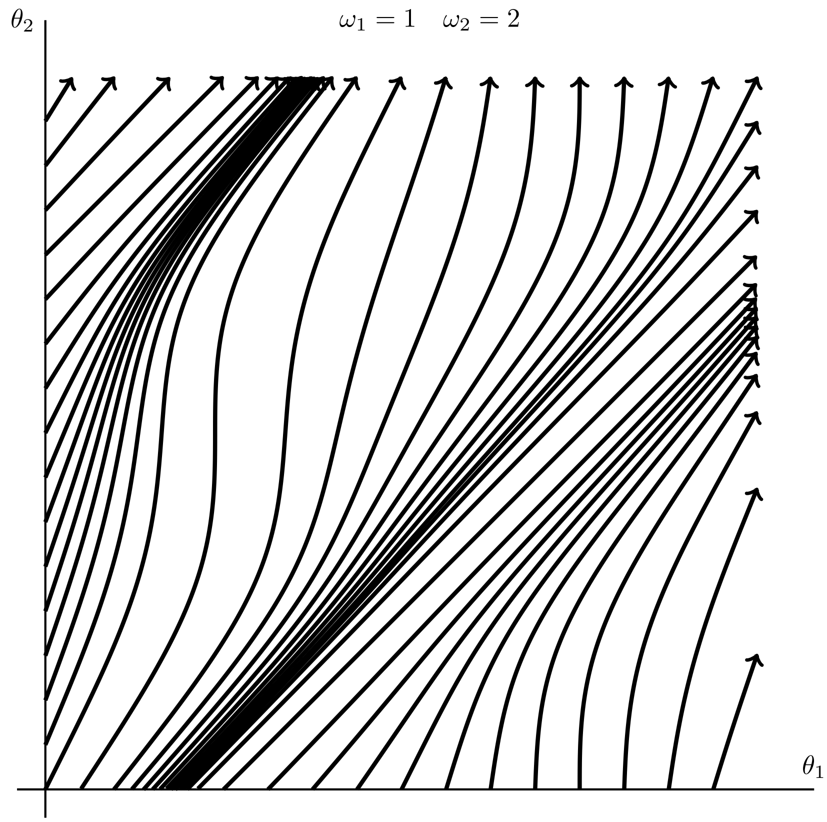
Stable and unstable limit cycles are present.



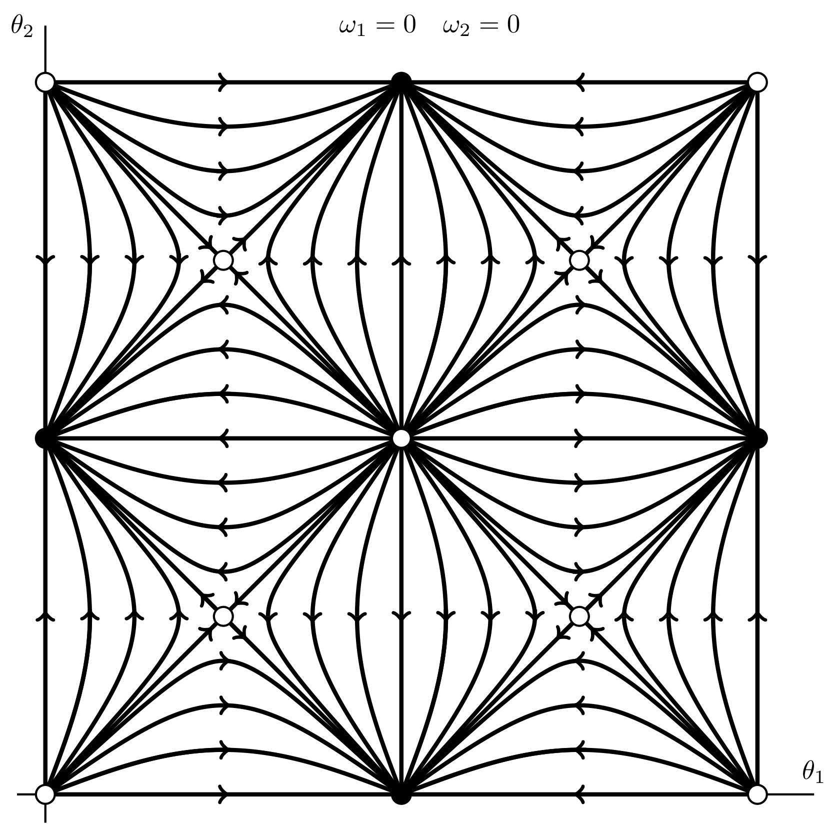
Fixed points

$$(\theta_1, \theta_2) = \left(\frac{\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{3\pi}{4}\right), \left(\frac{5\pi}{4}, \frac{\pi}{4}\right), \left(\frac{7\pi}{4}, \frac{7\pi}{4}\right)$$

In this phase portrait, the fixed points are about to disappear in an infinite-period bifurcation if we view the phase portrait as on a torus. In their places will be limit cycles.



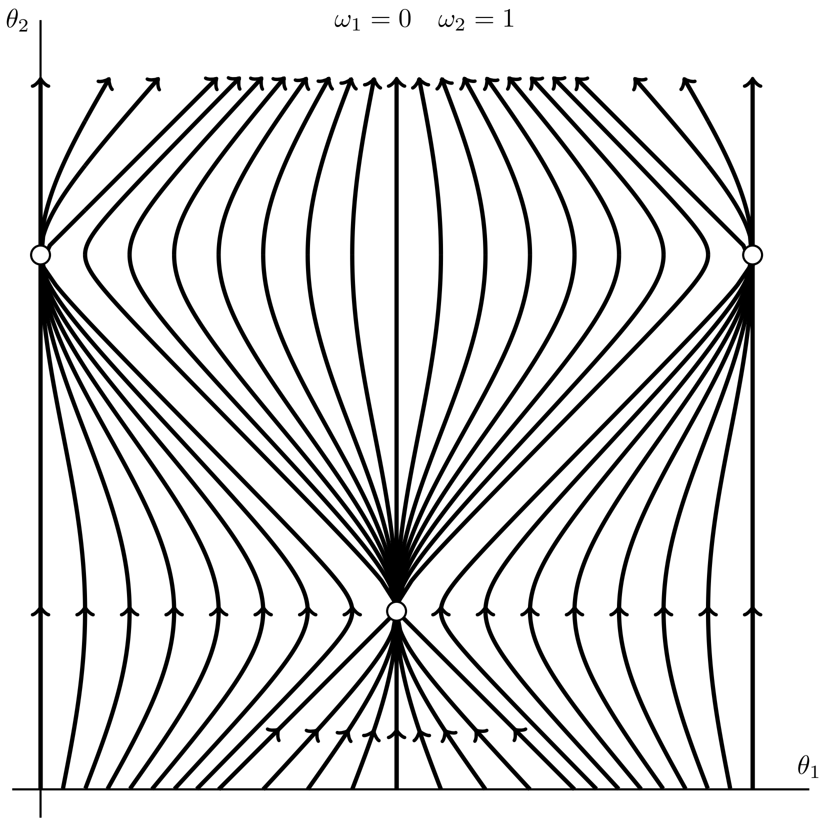
θ_2
$\omega_1 = 1 \quad \omega_2 = 2$
θ_1



θ_2
$\omega_1 = 0 \quad \omega_2 = 0$
θ_1

Fixed points

$$(\theta_1, \theta_2) = \left(\frac{0\pi}{2}, \frac{0\pi}{2}\right), \left(\frac{0\pi}{2}, \frac{2\pi}{2}\right), \left(\frac{0\pi}{2}, \frac{4\pi}{2}\right), \left(\frac{1\pi}{2}, \frac{1\pi}{2}\right), \left(\frac{1\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{2\pi}{2}, \frac{0\pi}{2}\right),$$
$$\left(\frac{2\pi}{2}, \frac{2\pi}{2}\right), \left(\frac{2\pi}{2}, \frac{4\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{1\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{4\pi}{2}, \frac{0\pi}{2}\right), \left(\frac{4\pi}{2}, \frac{2\pi}{2}\right), \left(\frac{4\pi}{2}, \frac{4\pi}{2}\right)$$



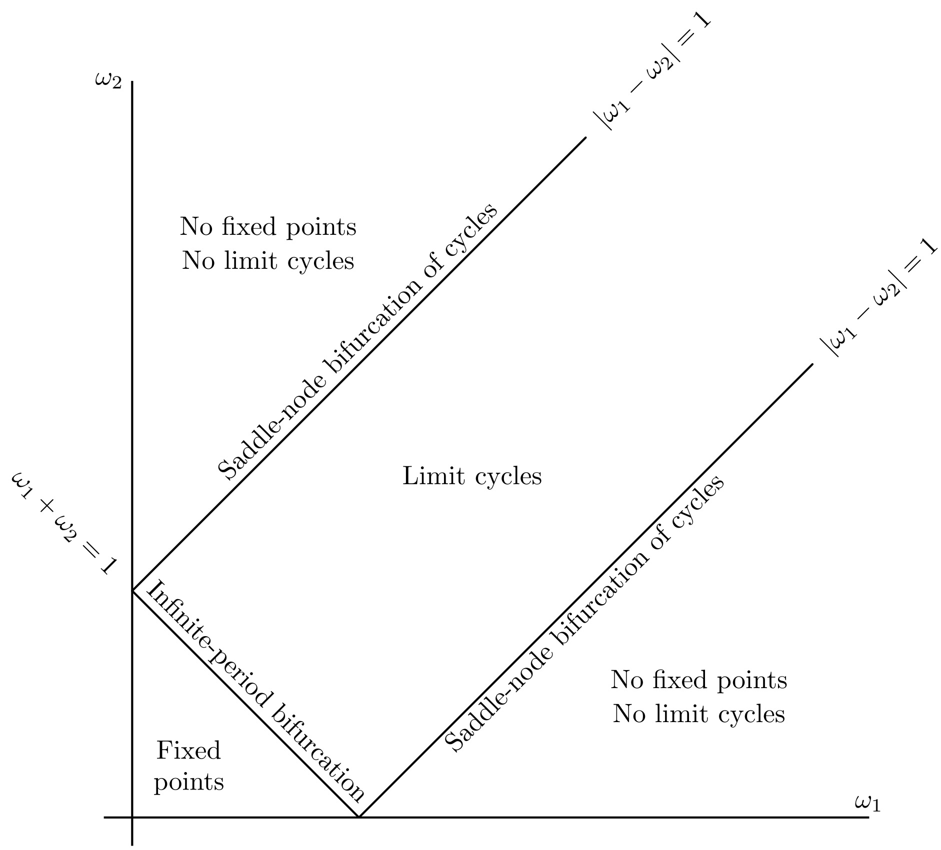
Fixed points

$$(\theta_1, \theta_2) = \left(\frac{0\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{2\pi}{2}, \frac{\pi}{2}\right), \left(\frac{4\pi}{2}, \frac{3\pi}{2}\right)$$

b)

We have saddle-node bifurcation of cycles occurring along the $|\omega_1 + \omega_2| = 1$ lines and infinite-period bifurcations occurring along the $\omega_1 - \omega_2 = 1$ line.

c)



The oscillator death region is the lower-left triangle. Once we cross the infinite-period bifurcation $\omega_1+\omega_2 = 1$ line, the limit cycles are broken by fixed points and the oscillations stop.

8.6.3

This problem is actually quite common and there are multiple proofs. Here is one that doesn't require much prerequisite knowledge.

Without loss of generality, assume that $q = (0, 0)$, as we can translate to any point and $\omega = \dfrac{\omega_1}{\omega} < 1$ since exchanging ω_1 with ω_2 is an equivalent problem. Also, we only need to prove that the irrational orbit is dense in $\{(\theta_1, \theta_2) : \theta_2 = 0\}$ (the bottom of the phase space) since a dense set at a cross section of the torus will also be dense everywhere when translated by the irrational flow. So now we have a somewhat simpler problem to work on.

We'll be using Dirichlet's approximation theorem, which implies that for every irrational number ω there exist integers a and b such that

$$\left|\omega - \frac{a}{b}\right| < \frac{1}{b^2}$$

The orbits that start at $(\theta_1, \theta_2) = (0, 0)$ with slope $\frac{a}{b}$ intersect the bottom of the phase space at

$$\left\{0, \frac{1}{b}, \frac{2}{b}, \dots \frac{b-1}{b}\right\}$$

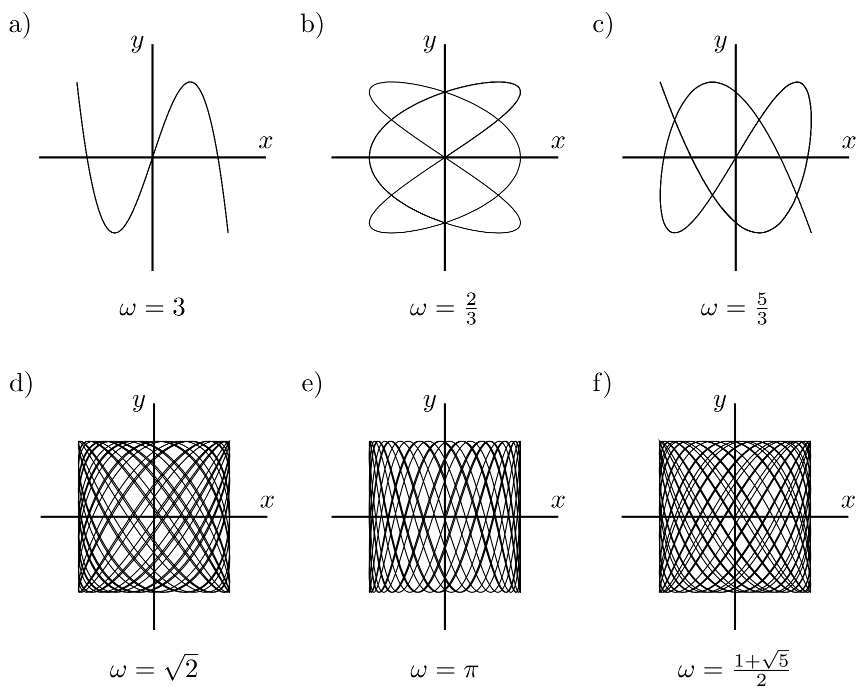
So the rational orbit passes within distance $\frac{1}{2b}$ of every point at the bottom of the phase space.

Now applying the implication of Dirichlet's approximation theorem

$$n\left|\omega - \frac{a}{b}\right| < \frac{n}{b^2} < \frac{1}{b} \qquad n = 0, 1, 2, \dots (b-1)$$

As long as $0 \le n \le b-1$, the nth intersection of the irrational orbit is within $\frac{1}{b}$ distance of $\frac{n}{b}$ and therefore the first $b-1$ iterations of the irrational orbit are within distance $\frac{1}{b}$ of every point at the bottom of the phase space. Hence the irrational orbit can be made arbitrarily close to any point at the bottom of the phase space, and consequently any point on the entire torus, by picking a large enough value for b.

8.6.5



8.6.7

$$m\ddot{r} = \frac{h^2}{mr^3} - k \qquad \dot{\theta} = \frac{h}{mr^2} \qquad m, k, h > 0$$

a)

$$
\begin{aligned}
&r(t) = r_0 \Rightarrow \ddot{r} = 0 \\
&m\ddot{r} = \frac{h^2}{mr^3} - k \longrightarrow 0 = \frac{h^2}{mr_0^3} - k \Rightarrow r_0 = \left(\frac{h^2}{mk}\right)^{\frac{1}{3}} \\
&\dot{\theta} = \frac{h}{mr_0^2} = \frac{h}{m}\left(\frac{mk}{h^2}\right)^{\frac{2}{3}} = \left(\frac{k^2}{mh}\right)^{\frac{1}{3}} = \omega_\theta
\end{aligned}
$$

b)

First, we'll transform the $\ddot{r}$ equation to a system and then linearize about $(r, \dot{r}) = (r_0, 0)$.

$$
x = r \quad y = \dot{r} \qquad \dot{x} = y \quad \dot{y} = \frac{h^2}{m^2x^3} - \frac{k}{m}
$$

$$
A = \begin{pmatrix} 0 & 1 \\ \frac{-3h^2}{m^2x^4} & 0 \end{pmatrix} \qquad A_{(r_0,0)} = \begin{pmatrix} 0 & 1 \\ \frac{-3h^2}{m^2r_0^4} & 0 \end{pmatrix}
$$

Then the frequency of small radial oscillations is

$$
\omega_r = \sqrt{\frac{3h^2}{m^2r_0^4}} = \sqrt{3}\frac{h}{mr_0^2} = \sqrt{3}\omega_\theta
$$

c)

The winding number

$$
\frac{\omega_r}{\omega_\theta} = \frac{\omega_\theta}{\omega_\theta} = \sqrt{3}
$$

is irrational and hence the small radial oscillations are quasiperiodic.

d)

The linearization from part (c) predicts a center, but we didn't actually check that it's a center. It really is a center and we can find a conserved quantity; or, if we look closely, we see that the r equation is a nonlinear spring with no damping and no $r = r_0$ rest position. The solution to $r(t)$ is periodic for any initial conditions.

The periodicity of $r(t)$ implies that $\dot{\theta}$ is periodic. The $\dot{\theta}$ equation is also strictly positive. Both these facts together imply that $\theta(t)$ has a constant finite period, which is determined by the $r(t)$ initial conditions. The period can vary for different $r(t)$ initial conditions, but the period is constant for each choice of $r(t)$ initial conditions. More precisely,

$$
\begin{aligned}
&\big(r(0), \dot{r}(0)\big) = (a, b) \quad \theta(T_{a,b}) - \theta(0) = 2\pi \Rightarrow \theta(t + T_{a,b}) - \theta\big(t\big) = 2\pi \\
&\big(r(0), \dot{r}(0)\big) = (\alpha, \beta) \quad \theta(T_{\alpha,\beta}) - \theta(0) = 2\pi \Rightarrow \theta(t + T_{\alpha,\beta}) - \theta\big(t\big) = 2\pi
\end{aligned}
$$

However, if $(a, b) \neq (\alpha, \beta)$, then $T_{a,b}$ may or may not equal $T_{\alpha,\beta}$ because they correspond to different initial conditions.

Now consider the geometric argument. We just proved that ω_r and consequently ω_θ are constant for any amplitude of radial oscillation. Then the winding number $\frac{\omega_r}{\omega_\theta}$ is also constant. The winding number is either a rational or irrational number that corresponds to periodic and quasiperiodic orbits on a torus with dimensions corresponding to the $r(t)$ and $\theta(t)$ periods respectively. There is no chaos.

e)

Two masses are connected by a string of fixed length. The first mass plays the role of the particle; it moves on a frictionless, horizontal "air table." It is connected to the second mass by a string that passes through a hole in the center of the table. This second mass hangs below the table, bobbing up and down and supplying the constant force of its weight. This mechanical system obeys the equations given in the text, after some rescaling.

8.6.9

$$\begin{aligned}\dot{\theta}_1 &= \omega + H(\theta_2 - \theta_1)\\ \dot{\theta}_2 &= \omega + H(\theta_1 - \theta_2)\end{aligned}$$

$$\begin{aligned}\dot{\theta}_1 &= \omega + H(\theta_2 - \theta_1) + H(\theta_3 - \theta_1)\\ \dot{\theta}_2 &= \omega + H(\theta_1 - \theta_2) + H(\theta_3 - \theta_2)\\ \dot{\theta}_3 &= \omega + H(\theta_1 - \theta_3) + H(\theta_2 - \theta_3)\end{aligned}$$

a)

$$\dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2 = H(\theta_2 - \theta_1) - H(\theta_1 - \theta_2) = H(-\phi) - H(\phi)$$

$$\begin{aligned}\dot{\phi} &= \dot{\theta}_1 - \dot{\theta}_2 = H(\theta_2 - \theta_1) + H(\theta_3 - \theta_1) - H(\theta_1 - \theta_2) - H(\theta_3 - \theta_2)\\ &= H(-\phi) + H(-\phi - \psi) - H(\phi) - H(-\psi)\\ \dot{\psi} &= \dot{\theta}_2 - \dot{\theta}_3 = H(\theta_1 - \theta_2) + H(\theta_3 - \theta_2) - H(\theta_1 - \theta_3) - H(\theta_2 - \theta_3)\\ &= H(\phi) + H(-\psi) - H(\phi + \psi) - H(\psi)\end{aligned}$$

b)

$H(x) = a\sin(x)$

For the two-frog system,

$$\begin{aligned}&\dot{\phi} = H(-\phi) - H(\phi) = a\sin(-\phi) - a\sin(\phi) = -2a\sin(\phi) = 0 \Rightarrow \phi = 0, \pi\\ &\frac{\mathrm{d}\dot{\phi}}{\mathrm{d}\phi} = -2a\cos(\phi) \qquad \left.\frac{\mathrm{d}\dot{\phi}}{\mathrm{d}\phi}\right|_{\phi=0} = -2a \qquad \left.\frac{\mathrm{d}\dot{\phi}}{\mathrm{d}\phi}\right|_{\phi=\pi} = 2a\end{aligned}$$

If $a < 0$ then the $\phi = \pi$ antiphase synchronization state will be stable.

For the three-frog system,

$$\begin{aligned}
\dot{\phi} &= H(-\phi) + H(-\phi-\psi) - H(\phi) - H(-\psi) \\
&= a\sin(-\phi) + a\sin(-\phi-\psi) - a\sin(\phi) - a\sin(-\psi) \\
&= -2a\sin(\phi) + a\sin(\psi) - a\sin(\phi+\psi) \\
\dot{\psi} &= H(\phi) + H(-\psi) - H(\phi+\psi) - H(\psi) \\
&= a\sin(\phi) + a\sin(-\psi) - a\sin(\phi+\psi) - a\sin(\psi) \\
&= -2a\sin(\psi) + a\sin(\phi) - a\sin(\phi+\psi) \\
\dot{\phi}(0,\pi) &= \dot{\psi}(0,\pi) = 0 \\
\dot{\phi}\left(\frac{2\pi}{3},\frac{2\pi}{3}\right) &= \dot{\psi}\left(\frac{2\pi}{3},\frac{2\pi}{3}\right) = 0
\end{aligned}$$

The two experimentally stable states are fixed points according to the model.

Next we linearize

$$A = \begin{pmatrix} -2a\cos(\phi) - a\cos(\phi+\psi) & a\cos(\psi) - a\cos(\phi+\psi) \\ a\cos(\phi) - a\cos(\phi+\psi) & -2a\cos(\psi) - a\cos(\phi+\psi) \end{pmatrix}$$

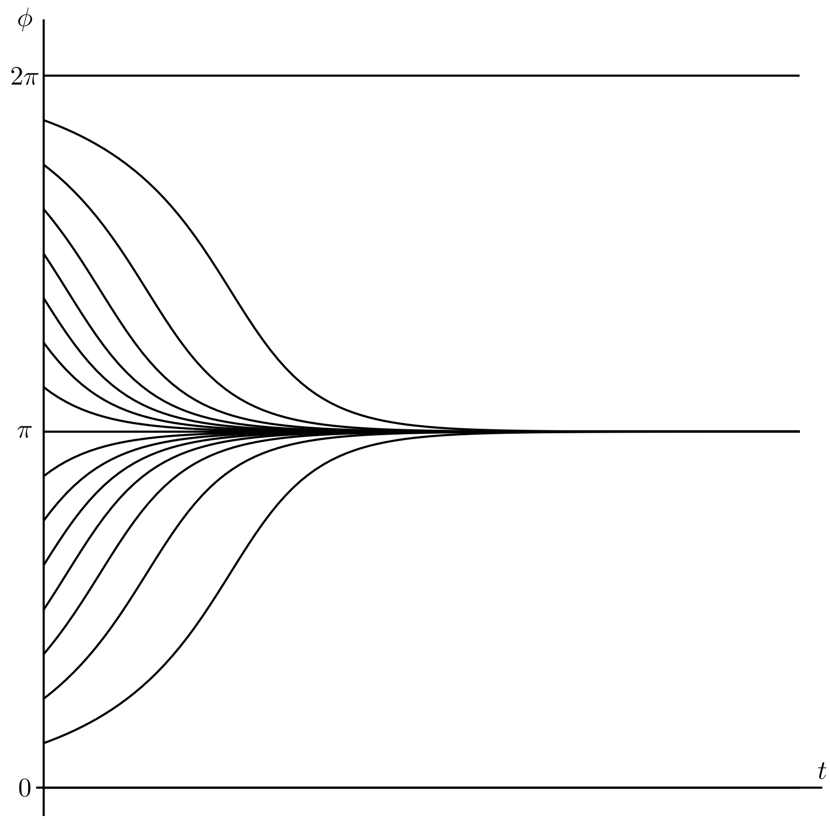
$$A_{(0,\pi)} = \begin{pmatrix} -a & 0 \\ 2a & 3a \end{pmatrix} \qquad \begin{matrix} \Delta = -3a^2 \quad \tau = 2a \\ \Delta < 0 \Rightarrow \text{saddle point} \end{matrix}$$

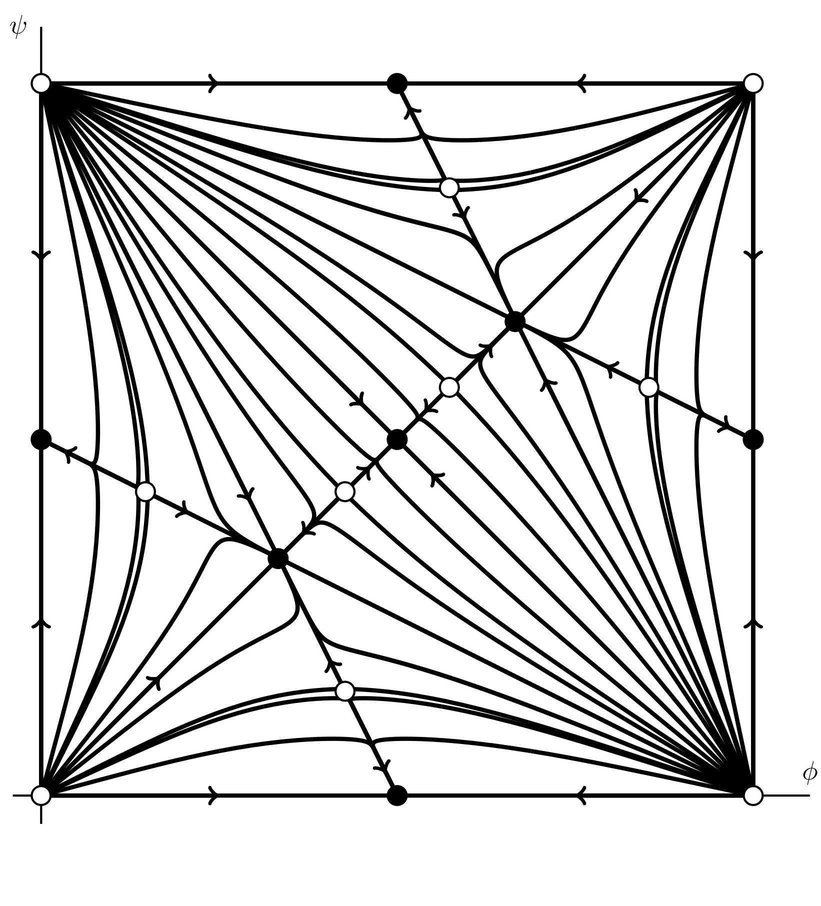
$$A_{\left(\frac{2\pi}{3},\frac{2\pi}{3}\right)} = \begin{pmatrix} \frac{3a}{2} & 0 \\ 0 & \frac{3a}{2} \end{pmatrix} \qquad \begin{matrix} \Delta = \frac{9a^2}{4} \quad \tau = 3a \\ a < 0 \Rightarrow \tau < 0 \quad \text{stable} \end{matrix}$$

So the experimental fixed points exist, but the pair of frogs in unison and the third out of phase are unstable.

c)

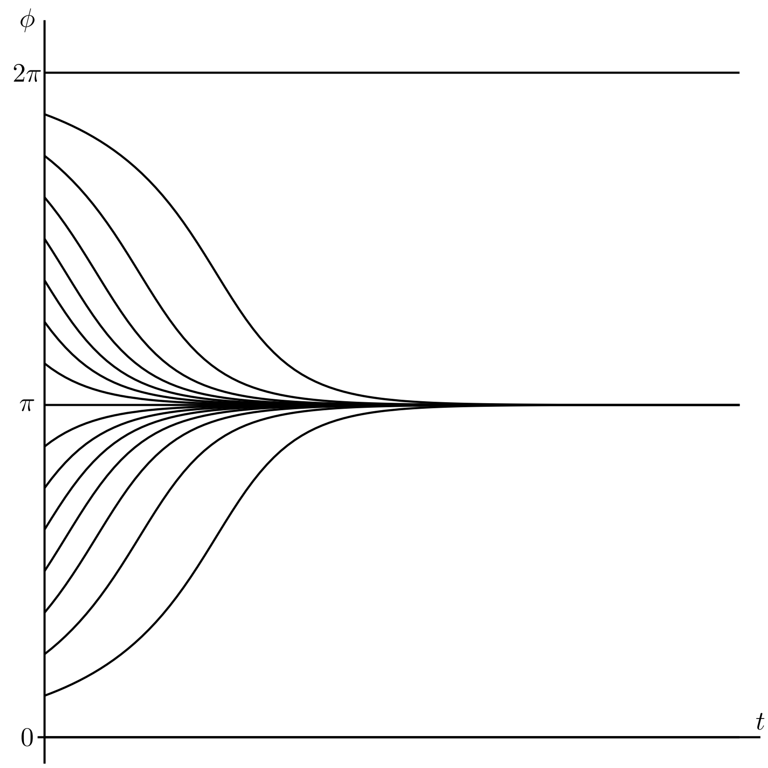
$H(x) = a\sin(x) + b\sin(2x) \qquad a = -5 \quad b = 1$

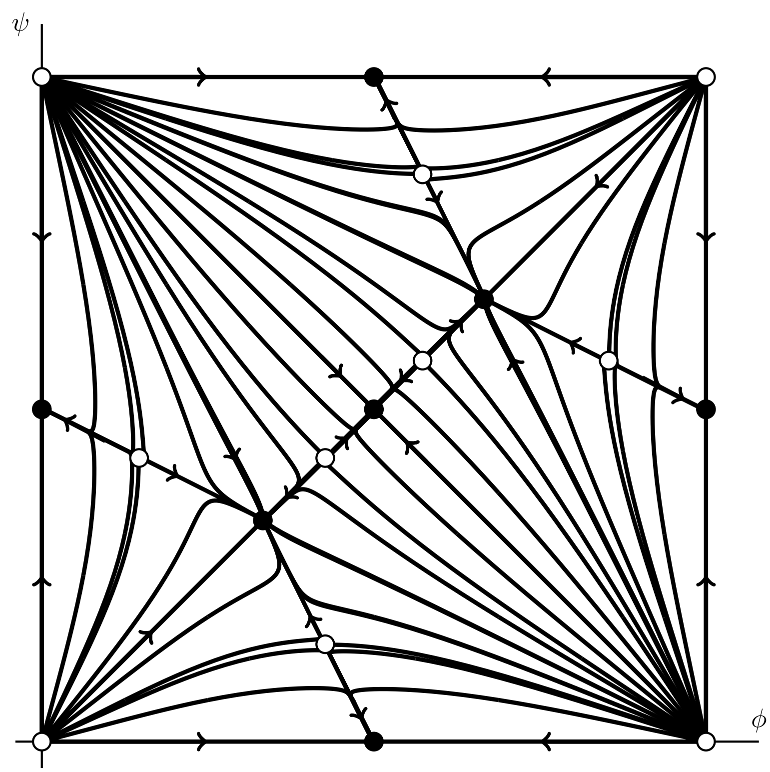




d)

$H(x) = a\sin(x) + b\sin(2x) + c\cos(x) \qquad a = -5 \quad b = 1 \quad c = 0.1$





So the fixed points and their stability remained even after adding a small even component to H.

8.7 Poincaré Maps

8.7.1

$$\int_{r_0}^{r_1} \frac{\mathrm{d}r}{r(1-r^2)} = 2\pi$$

$$\frac{1}{r(1-r^2)} = \frac{1}{r(1+r)(1-r)} = \frac{A}{r} + \frac{B}{1+r} + \frac{C}{1-r}$$

$$\begin{aligned}
1 &= A(1+r)(1-r) + Br(1-r) + Cr(1+r) \\
&= A(1-r^2) + B(-r^2+r) + C(r^2+r) \\
&= (-A-B+C)r^2 + (B+C)r + A \\
&\Rightarrow A = 1 \quad B = \frac{-1}{2} \quad C = \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\int_{r_0}^{r_1} \frac{\mathrm{d}r}{r(1-r^2)} &= \int_{r_0}^{r_1} \left(\frac{1}{r} - \frac{1}{2(1+r)} + \frac{1}{2(1-r)}\right) \mathrm{d}r = \int_{r_0}^{r_1} \left(\frac{1}{r} - \frac{1}{2(r+1)} - \frac{1}{2(r-1)}\right) \mathrm{d}r \\
&= \left[\ln|r| - \frac{1}{2}\ln|r+1| - \frac{1}{2}\ln|r-1|\right]_{r_0}^{r_1} \\
&= \left(\ln|r_1| - \frac{1}{2}\ln|r_1+1| - \frac{1}{2}\ln|r_1-1|\right) - \left(\ln|r_0| - \frac{1}{2}\ln|r_0+1| - \frac{1}{2}\ln|r_0-1|\right) \\
&= \ln\left|\frac{r_1}{\sqrt{r_1^2-1}}\right| - \ln\left|\frac{r_0}{\sqrt{r_0^2-1}}\right| = \ln\left|\frac{r_1\sqrt{r_0^2-1}}{r_0\sqrt{r_1^2-1}}\right| = \frac{1}{2}\ln\left|\frac{r_1^2(r_0^2-1)}{r_0^2(r_1^2-1)}\right| = 2\pi
\end{aligned}$$

We can drop the absolute value because $r = 1$ is a limit cycle, so $(r_1 - 1)$ and $(r_0 - 1)$ have to have the same sign.

$$\begin{aligned}
&\frac{r_1^2(r_0^2-1)}{r_0^2(r_1^2-1)} = e^{4\pi} \\
&r_1^2(r_0^2-1) = r_0^2(r_1^2-1)e^{4\pi} \\
&r_1^2(r_0^2-1) - r_0^2(r_1^2-1)e^{4\pi} = 0 \\
&r_1^2(r_0^2 - 1 - r_0^2 e^{4\pi}) + r_0^2 e^{4\pi} = 0 \\
&r_1^2 = \frac{r_0^2 e^{4\pi}}{-r_0^2 + 1 + r_0^2 e^{4\pi}} = \frac{1}{-e^{-4\pi} + r_0^{-2} e^{-4\pi} + 1} = \frac{1}{1 + e^{-4\pi}(r_0^{-2} - 1)} \\
&r_1 = \frac{1}{\sqrt{1 + e^{-4\pi}(r_0^{-2} - 1)}} \\
&r_{n+1} = P(r_n) = \frac{1}{\sqrt{1 + e^{-4\pi}(r_n^{-2} - 1)}} \\
&\frac{\mathrm{d}}{\mathrm{d}r} P(r) = \frac{e^{-4\pi} r^{-3}}{\sqrt{1 + e^{-4\pi}(r^{-2} - 1)}} \\
&\frac{\mathrm{d}}{\mathrm{d}r} P(1) = e^{-4\pi}
\end{aligned}$$

8.7.3

$$\dot{x} + x = F(t) \qquad F(t) = \begin{cases} +A & 0 < t < \frac{T}{2} \\ -A & \frac{T}{2} < t < T \end{cases}$$

a)

$$\begin{aligned}
&\dot{x} + x = F(t) \\
&e^t\dot{x} + e^t x = \frac{\mathrm{d}}{\mathrm{d}t}\big(e^t x\big) = e^t F(t) \\
&\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\big(e^t x\big)\ \mathrm{d}t = \int_0^T e^t F(t)\ \mathrm{d}t \\
&\Big[e^t x\Big]_0^T = \int_0^{\frac{T}{2}} e^t(+A)\ \mathrm{d}t + \int_{\frac{T}{2}}^T e^t(-A)\ \mathrm{d}t \\
&e^T x(T) - x(0) = A\big(e^{\frac{T}{2}} - 1\big) - A\big(e^T - e^{\frac{T}{2}}\big) \\
&x(T) = e^{-T}x(0) + A\big(e^{-\frac{T}{2}} - e^{-T}\big) - A\big(1 - e^{-\frac{T}{2}}\big) \\
&\qquad = e^{-T}x_0 - A\big(1 - e^{-\frac{T}{2}}\big)^2
\end{aligned}$$

b)

If we assume there is a T-periodic solution and solve

$$\begin{aligned}
&x(T) = x(0) = x_0 = e^{-T}x_0 - A\big(1 - e^{-\frac{T}{2}}\big)^2 \\
&x_0 - e^{-T}x_0 = x_0\big(1 - e^{-T}\big) = -A\big(1 - e^{-\frac{T}{2}}\big)^2 \\
&x_0 = -A\frac{\big(1 - e^{-\frac{T}{2}}\big)^2}{1 - e^{-T}} = -A\frac{\big(1 - e^{-\frac{T}{2}}\big)^2}{\big(1 + e^{-\frac{T}{2}}\big)\big(1 - e^{-\frac{T}{2}}\big)} \\
&\quad = -A\frac{1 - e^{-\frac{T}{2}}}{1 + e^{-\frac{T}{2}}} = -A\tanh\left(\frac{T}{4}\right)
\end{aligned}$$

then we arrive at the result.

c)

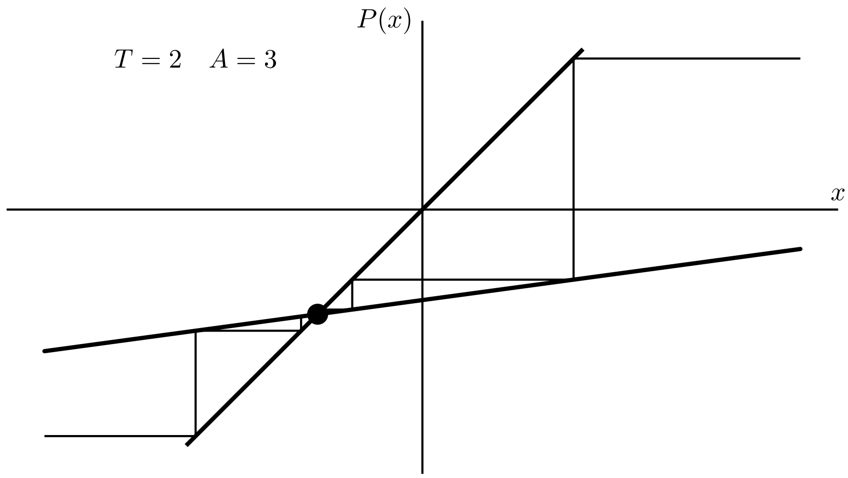
$$\begin{aligned}
&\lim_{T\to 0} x(T) = \lim_{T\to 0} e^{-T}x_0 - A\big(1 - e^{-\frac{T}{2}}\big)^2 = x_0 - A\big(1-1\big)^2 = x_0 \\
&\lim_{T\to\infty} x(T) = \lim_{T\to\infty} e^{-T}x_0 - A\big(1 - e^{-\frac{T}{2}}\big)^2 = (0)x_0 - A\big(1-0\big)^2 = A
\end{aligned}$$

These results are quite plausible. As the period goes to zero, $F(t)$ has almost no time to do anything and $x(t)$ has no time to move anywhere. If $x(0) = x_0$ then $x(T) \approx x_0$ and $T \to \infty$. Then $x(0) = x(T) = x_0$ and the solution doesn't go anywhere.

As the period becomes infinite, $F(t)$ is essentially not periodic. The equation and solution become

$$\dot{x} + x = F(t) = A \qquad x(t) = (x_0 - A)e^{-t} + A$$

d)



The fixed point is stable, and not just for these choices of parameters. $P(x)$ is a line.

$$P(x) = e^{-T}x - A\left(1 - e^{-\frac{T}{2}}\right)^2 = mx + b$$

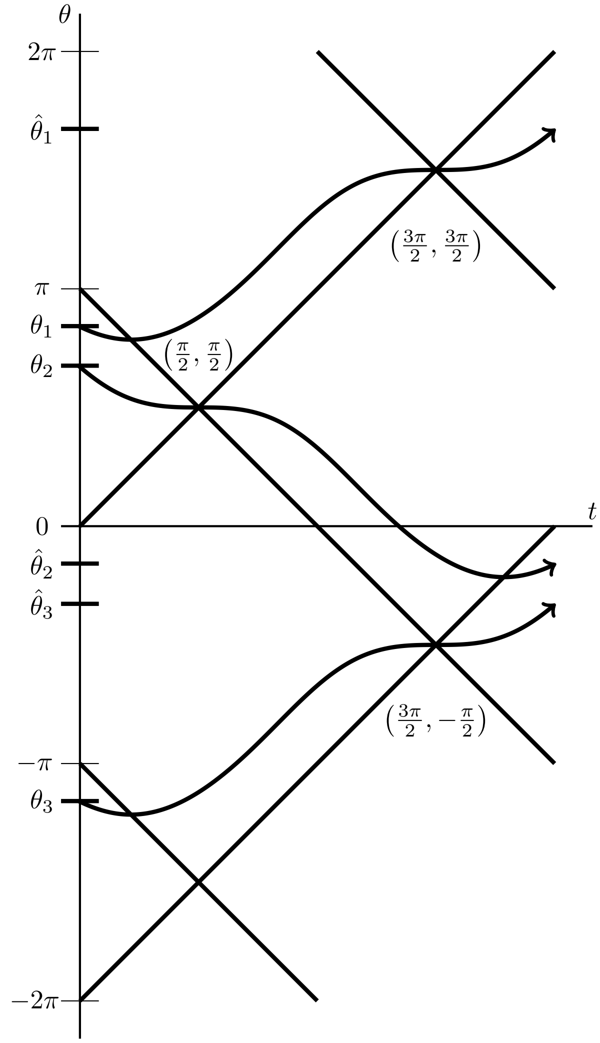
and the slope m is always between 0 and 1 since $T > 0$ and $m = e^{-T}$, guaranteeing the existence of a unique globally stable fixed point.

8.7.5

$$\dot{\theta} + \sin(\theta) = \sin(t)$$

$$\dot{t} = 1 \qquad \dot{\theta} = \sin(t) - \sin(\theta)$$

The trajectories here can be wrapped between $-\pi$ and π, but the modulo operation is removed to more easily follow the paths of the trajectories.



The above trajectories were computed numerically, but the important point analytically is when these trajectories are increasing and decreasing. From the intervals of increase and decrease, we can determine where the trajectories start and end relative to the nullclines.

The Poincaré map $P(\theta)$, while we don't know it explicitly, maps the interval $[\theta_2, \theta_3]$ into $[\hat{\theta}_2, \hat{\theta}_3]$ continuously. Therefore there has to be at least one $\theta^* \in [\theta_2, \theta_3]$ such that $P(\theta^*) = \theta^*$, which represents a periodic orbit.

The same argument can be applied to $[\theta_1, \theta_2]$, meaning there has to be at least one $\theta^* \in [\theta_1, \theta_2]$ such that $P(\theta^*) = \theta^*$, which represents a periodic orbit.

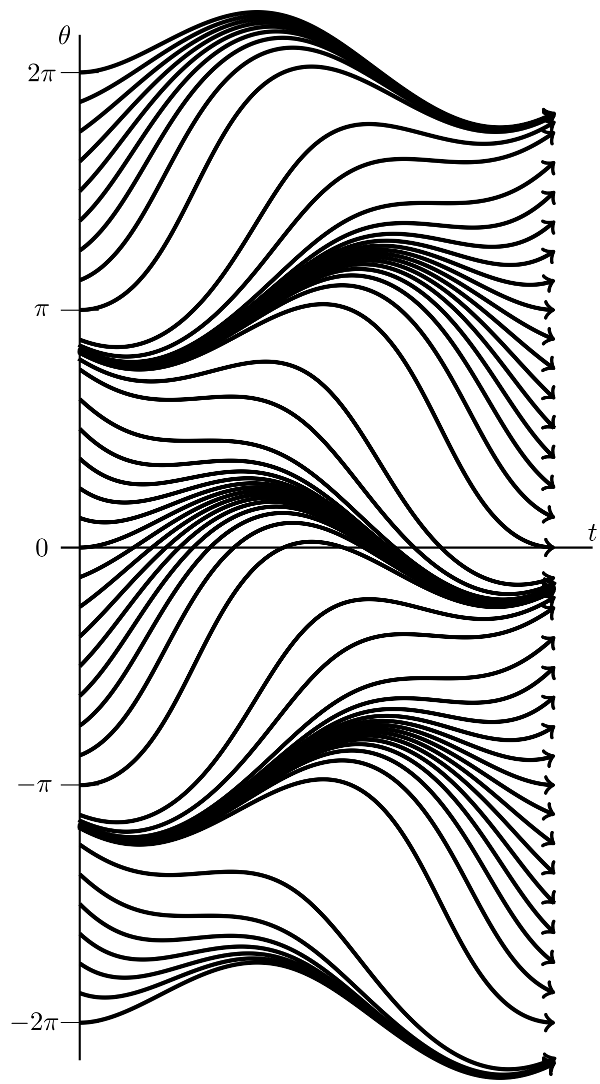
As for stability, there can't be just a single stable limit cycle in the interval $[\theta_1, \theta_2]$ because the interval expanded after one mapping. However, there could be unstable limit cycles surrounding a stable limit cycle, or maybe even an interval of unstable periodic solutions since we haven't explicitly ruled out that possibility. We do know for sure that there is at least one unstable limit cycle in the interval.

The $[\theta_2, \theta_3]$ interval on the other hand contracts after one mapping. Therefore we can't have just a single unstable periodic orbit in $[\theta_2, \theta_3]$. There has to be at least one stable periodic orbit in order to account for the contracting interval.

8.7.7

$$\dot{\theta} + \sin(\theta) = \sin(t)$$

The trajectories here can be wrapped between $-\pi$ and π, but the modulo operation is removed to more easily follow the paths of the trajectories.



And the graph clearly indicates the existence of one stable and one unstable periodic solution.

8.7.9

$$\dot{r} = r - r^2 \qquad \dot{\theta} = 1$$

a)

We can solve for $r(t)$ by separating variables and then integrating using partial fraction decomposition and $\theta(t)$ by direct integration.

$$r(t) = \frac{e^{2\pi} r_0}{(e^{2\pi} - 1)r_0 + 1} \qquad \theta(t) = t + \theta_0$$

The positive x-axis as the surface of section corresponds to $\theta = 0$ and r positive.

$$r_{n+1} = P(r_n) = \frac{e^{2\pi} r_n}{(e^{2\pi} - 1)r_n + 1}$$

b)

Fixed point

$$P(r) = \frac{e^{2\pi} r}{(e^{2\pi} - 1)r + 1} = r \Rightarrow r = 1$$

Stability

$$P'(r) = \frac{e^{2\pi}}{\left((e^{2\pi} - 1)r + 1\right)^2} \qquad P'(1) = e^{-2\pi} < 1 \Rightarrow \text{stable}$$

c)

We need to compute the linearized Poincaré map at the fixed point, but we already did that in part (b). Therefore $e^{-2\pi}$ is the characteristic multiplier for the periodic orbit.

8.7.11

From Example 8.7.4, we know that the linearization of the in-phase fixed point of Equation 8.7.1

$$\dot{\phi}_i = \Omega + a\sin(\phi_i) + \frac{1}{N}\sum_{j=1}^{N} \sin(\phi_j)$$

predicts a center.

Going back to Section 6.6, a higher-order system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is reversible if it is invariant under the transformation

$$t \to -t \qquad \mathbf{x} \to \mathbf{R}(\mathbf{x}) \quad \text{where} \quad \mathbf{R}^2(\mathbf{x}) = \mathbf{x}$$

The simplest choice, $\mathbf{R}(\mathbf{x}) = -\mathbf{x}$, doesn't work, but $\mathbf{R}(\mathbf{x}) = \pi - \mathbf{x}$ does.

$\mathbf{R}^2(\mathbf{x}) = \mathbf{R}(\pi - \mathbf{x}) = \pi - (\pi - \mathbf{x}) = \mathbf{x}$

$$\frac{\mathrm{d}\phi_i}{\mathrm{d}t} \longrightarrow \frac{\mathrm{d}(\pi - \phi_i)}{-\mathrm{d}t} = \frac{-\mathrm{d}\phi_i}{-\mathrm{d}t} = \frac{\mathrm{d}\phi_i}{\mathrm{d}t}$$

$\sin(\pi - \phi_i) = \sin(\phi_i)$

$$\Omega + a\sin(\pi - \phi_i) + \frac{1}{N}\sum_{j=1}^{N} \sin(\pi - \phi_j) = \Omega + a\sin(\phi_i) + \frac{1}{N}\sum_{j=1}^{N} \sin(\phi_j)$$

Then by Theorem 6.6.1, the linear center is a nonlinear center; i.e., the in-phase state is not attracting.

9

Lorenz Equations

9.1 A Chaotic Waterwheel

9.1.1

a)

The moment of inertia of a ring is the mass times radius squared.

$$I_{\text{water}} = Mr^2$$

b)

The rate of change of water is the rate at which water flows in minus the rate at which water flows out.

$$\dot{M} = Q_{\text{total}} - KM$$

c)

First, we make a differential equation for I_{water}

$$\dot{I}_{\text{water}} = \dot{M}r^2 = Q_{\text{total}}r^2 - KMr^2 = Q_{\text{total}}r^2 - KI$$

which has solution

$$I_{\text{water}}(t) = \left(I_{\text{water}}(0) - \frac{Qr^2}{K}\right)e^{-Kt} + \frac{Qr^2}{K} \qquad \lim_{t\to\infty} I_{\text{water}}(t) = \frac{Qr^2}{K}$$

9.1.3

$$\dot{a}_1 = \omega b_1 - Ka_1 \qquad \dot{b}_1 = -\omega a_1 + q_1 - Kb_1 \qquad \dot{\omega} = -\frac{\nu}{I}\omega + \frac{\pi g r_w}{I}a_1$$
$$\dot{x} = \sigma(y - x) \qquad \dot{y} = r_L x - xz - y \qquad \dot{z} = xy - bz$$

We distinguish between the r in the waterwheel equations and Lorenz equations with a w and L subscript respectively.

$$\omega = \alpha x \qquad a_1 = \beta y \qquad b_1 = \gamma z + \phi \qquad t = \xi\tau$$

$$\dot{\omega} = -\frac{\nu}{I}\omega + \frac{\pi g r_w}{I}a_1 \to \alpha\frac{1}{\xi}\frac{\mathrm{d}x}{\mathrm{d}\tau} = -\frac{\nu}{I}\alpha x + \frac{\pi g r_w}{I}\beta y$$
$$\dot{a}_1 = \omega b_1 - Ka_1 \to \beta\frac{1}{\xi}\frac{\mathrm{d}y}{\mathrm{d}\tau} = \alpha x(\gamma z + \phi) - K\beta y$$
$$\dot{b}_1 = -\omega a_1 + q_1 - Kb_1 \to \gamma\frac{1}{\xi}\frac{\mathrm{d}z}{\mathrm{d}\tau} = -\alpha x\beta y + q_1 - K(\gamma z + \phi)$$

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}\tau} &= -\frac{\nu\xi}{I}x + \frac{\pi g r_w \beta\xi}{\alpha I}y = \sigma(y-x)\\
\frac{\mathrm{d}y}{\mathrm{d}\tau} &= \frac{\alpha\xi\gamma}{\beta}xz + \frac{\alpha\phi\xi}{\beta}x - \xi K y = r_L x - xz - y\\
\frac{\mathrm{d}z}{\mathrm{d}\tau} &= -\frac{\alpha\beta\xi}{\gamma}xy + \frac{\xi}{\gamma}(q_1 - K\phi) - \xi K z = xy - bz
\end{aligned}$$

$$\begin{aligned}
\sigma &= \frac{\nu\xi}{I} = \frac{\pi g r_w \beta\xi}{\alpha I}\\
r_L &= \frac{\alpha\phi\xi}{\beta} \qquad -1 = \frac{\alpha\xi\gamma}{\beta} \qquad 1 = \xi K\\
1 &= -\frac{\alpha\beta\xi}{\gamma} \qquad 0 = \frac{\xi}{\gamma}(q_1 - K\phi) \qquad b = \xi K\\
\Rightarrow \xi &= \frac{1}{K} \qquad \phi = \frac{q_1}{K}\\
\sigma &= \frac{\nu}{IK} = \frac{\pi g r_w \beta}{\alpha I K}\\
r_L &= \frac{\alpha q_1}{\beta K^2} \qquad -1 = \frac{\alpha\gamma}{\beta K}\\
1 &= -\frac{\alpha\beta}{\gamma K} \qquad b = 1\\
\Rightarrow \alpha &= \frac{\pi g r_w \beta}{\nu} \qquad \frac{\alpha}{\beta} = \frac{\pi g r_w}{\nu}\\
r_L &= \frac{\pi g r_w q_1}{\nu K^2} \qquad -1 = \frac{\pi g r_w \gamma}{\nu K}\\
1 &= -\frac{\pi g r_w \beta^2}{\gamma\nu K}\\
\Rightarrow \gamma &= -\frac{\nu K}{\pi g r_w} \qquad \frac{1}{\gamma} = -\frac{\pi g r_w}{\nu K}\\
1 &= \left(\frac{\pi g r_w \beta}{\nu K}\right)^2\\
\Rightarrow \beta &= \pm\frac{\nu K}{\pi g r_w} \qquad \alpha = \pm K \qquad \text{Choose } +\\
\beta &= \frac{\nu K}{\pi g r_w} \qquad \alpha = K
\end{aligned}$$

$$\begin{aligned}
&\omega = \frac{\pi g r_w K}{\nu}x \qquad a_1 = \frac{\nu K}{\pi g r_w}y \qquad b_1 = -\frac{\nu K}{\pi g r_w}z + \frac{q_1}{K} \qquad t = \frac{1}{K}\tau\\
&\sigma = \frac{\nu}{IK} \qquad r_L = \frac{\pi g r_w q_1}{\nu K^2} \qquad b = 1
\end{aligned}$$

9.1.5

We start off the problem with the same equations, but now $Q(\theta)$ has $p_n \sin(n\theta)$ terms in the Fourier series. Previously we used only the $q_n \cos(n\theta)$ because we assumed $Q(\theta)$ was an even function.

$$\frac{\partial m}{\partial t} = Q - Km - \omega \frac{\partial m}{\partial \theta}$$
$$m(\theta, t) = \sum_{n=0}^{\infty} \big[a_n(t)\sin(n\theta) + b_n(t)\cos(n\theta)\big]$$
$$Q(\theta) = \sum_{n=0}^{\infty} \big[p_n \sin(n\theta) + q_n \cos(n\theta)\big]$$

If we go back through the derivation

$$\frac{\partial}{\partial t}\left[\sum_{n=0}^{\infty} \big[a_n(t)\sin(n\theta) + b_n(t)\cos(n\theta)\big]\right] = \sum_{n=0}^{\infty} \big[p_n \sin(n\theta) + q_n \cos(n\theta)\big]$$
$$- K\sum_{n=0}^{\infty} \big[a_n(t)\sin(n\theta) + b_n(t)\cos(n\theta)\big] - \omega \frac{\partial}{\partial \theta}\sum_{n=0}^{\infty} \big[a_n(t)\sin(n\theta) + b_n(t)\cos(n\theta)\big]$$
$$\sum_{n=0}^{\infty} \big[\dot{a}_n(t)\sin(n\theta) + \dot{b}_n(t)\cos(n\theta)\big] = \sum_{n=0}^{\infty} \big[p_n \sin(n\theta) + q_n \cos(n\theta)\big]$$
$$- K\sum_{n=0}^{\infty} \big[a_n(t)\sin(n\theta) + b_n(t)\cos(n\theta)\big] - \omega \sum_{n=0}^{\infty} n\big[a_n(t)\cos(n\theta) - b_n(t)\sin(n\theta)\big]$$

The $\dot{\omega}$ equation stays the same because $Q(\theta)$ wasn't involved in the derivation.

$$\dot{\omega} = \frac{-\nu\omega + \pi g r a_1}{I}$$

Equating terms gives

$$\dot{a}_n = n\omega b_n - K a_n + p_n$$
$$\dot{b}_n = -n\omega a_n - K b_n + q_n$$

And all equations for $n = 1$ are

$$\dot{a}_1 = \omega b_1 - K a_1 + p_1$$
$$\dot{b}_1 = -\omega a_1 - K b_1 + q_1$$
$$\dot{\omega} = \frac{-\nu\omega + \pi g r a_1}{I}$$

We have a new term p_1 in the $\dot{a}_1$ equation.

Now solving for the fixed points,

$$0 = \omega b_1 - K a_1 + p_1$$
$$0 = -\omega a_1 - K b_1 + q_1$$
$$0 = \frac{-\nu\omega + \pi g r a_1}{I} \Rightarrow 0 = -\nu\omega + \pi g r a_1$$

Next we eliminate b_1 from the first two equations, solve for a_1, then substitute into the third equation.

$$\begin{aligned}
0 &= K\omega b_1 - K^2 a_1 + Kp_1 \\
0 &= -\omega^2 a_1 - K\omega b_1 + \omega q_1 \\
0 &= (K\omega b_1 - K^2 a_1 + Kp_1) + (-\omega^2 a_1 - K\omega b_1 + \omega q_1) \\
&= -K^2 a_1 + Kp_1 - \omega^2 a_1 + \omega q_1) \\
a_1 &= \frac{Kp_1 + \omega q_1}{K^2 + \omega^2} \\
0 &= -\nu\omega + \pi g r a_1 = -\nu\omega + \pi g r \frac{Kp_1 + \omega q_1}{K^2 + \omega^2} \\
&= -\nu\omega^3 - K^2 \nu\omega + \pi g r K p_1 + \pi g r q_1 \omega \\
0 &= \nu\omega^3 + (\nu K^2 - \pi g r q_1)\omega - \pi g r K p_1 \\
&= \omega(\nu\omega^2 + \nu K^2 - \pi g r q_1) - \pi g r K p_1
\end{aligned}$$

Looking at this, we can see the same solutions as before.

$$p_1 = 0 \Rightarrow 0 = \omega(\nu\omega^2 + \nu K^2 - \pi g r q_1) \Rightarrow \omega = 0, \pm\sqrt{\frac{\pi g r q_1}{\nu} - K^2}$$

This creates a pitchfork bifurcation. However, for $p_1 \neq 0$ along with proper values for the other coefficients, the pitchfork bifurcation is ruined and replaced with an imperfect bifurcation similar to that in Figure 3.6.3 in the text.

9.2 Simple Properties of the Lorenz Equations

9.2.1

a)

$$\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= rx - xz - y \\
\dot{z} &= xy - bz
\end{aligned}$$

$$C^{\pm} = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$$

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}_{C^{\pm}} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b \end{pmatrix}$$

$$\det\begin{pmatrix} -\sigma-\lambda & \sigma & 0 \\ 1 & -1-\lambda & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b-\lambda \end{pmatrix}$$

$$= (-\sigma - \lambda)((-1-\lambda)(-b-\lambda) + b(r-1)) - \sigma((-b-\lambda) + b(r-1))$$
$$= -(\sigma + \lambda)((1+\lambda)(b+\lambda) + b(r-1)) + \sigma(b + \lambda - b(r-1))$$
$$= -(\sigma + \lambda)(b + (1+b)\lambda + \lambda^2 + br - b) + \sigma(b - br + b) + \sigma\lambda$$
$$= -(\sigma + \lambda)(\lambda^2 + (1+b)\lambda + br) + \sigma(2b - br) + \sigma\lambda$$
$$= -\sigma\lambda^2 - \sigma(1+b)\lambda - \sigma br - \lambda^3 - (1+b)\lambda^2 - br\lambda + 2\sigma b - b\sigma r + \sigma\lambda$$
$$= -\lambda^3 - (\sigma + 1 + b)\lambda^2 - b(r+\sigma)\lambda - 2b\sigma(r-1) = 0$$
$$= \lambda^3 + (\sigma + 1 + b)\lambda^2 + b(r+\sigma)\lambda + 2b\sigma(r-1) = 0$$

b)

Assuming a root of the form $\lambda = iw$

$$\lambda^3 + (\sigma + 1 + b)\lambda^2 + b(r+\sigma)\lambda + 2b\sigma(r-1) = 0$$
$$(iw)^3 + (\sigma + 1 + b)(iw)^2 + b(r+\sigma)(iw) + 2b\sigma(r-1) = 0$$
$$-iw^3 - (\sigma + 1 + b)w^2 + ib(r+\sigma)w + 2b\sigma(r-1) = 0$$

Both the real and imaginary parts have to separately equal 0.

$$-(\sigma + 1 + b)w^2 + 2b\sigma(r-1) = 0 \Rightarrow w = \pm\sqrt{\frac{2b\sigma(r-1)}{(\sigma+1+b)}}$$
$$-iw^3 + ib(r+\sigma)w = -iw(w^2 - b(r+\sigma)) = 0 \Rightarrow w = 0, \pm\sqrt{b(r+\sigma)}$$

Equating the two roots for w and solving for r

$$\pm\sqrt{\frac{2b\sigma(r-1)}{(\sigma+1+b)}} = \pm\sqrt{b(r+\sigma)}$$
$$\frac{2b\sigma(r-1)}{(\sigma+1+b)} = b(r+\sigma)$$
$$2b\sigma(r-1) = b(r+\sigma)(\sigma+1+b)$$
$$2b\sigma r - 2b\sigma = br(\sigma+1+b) + b\sigma(\sigma+1+b)$$
$$2b\sigma r - br(\sigma+1+b) = b\sigma(\sigma+1+b) + 2b\sigma$$
$$r(2b\sigma - b(\sigma+1+b)) = b\sigma(\sigma+1+b) + 2b\sigma$$
$$rb(\sigma - 1 - b) = b\sigma(\sigma + 3 + b)$$
$$r = \sigma\frac{(\sigma+3+b)}{(\sigma-1-b)}$$

We need $b + 1 < \sigma$ in order for the derivation to work. We solved for

$$1 < r < r_H = \sigma\frac{(\sigma+3+b)}{(\sigma-1-b)} = \frac{\sigma}{(\sigma-(1+b))}(\sigma+3+b)$$

σ and b are positive, so $1 < (\sigma + 3 + b)$, and as long as $0 < \sigma - (1+b) < \sigma \Rightarrow 1 + b < \sigma$, we ensure that $1 < r$.

c)

$$w = \sqrt{b(r+\sigma)}$$

$$\begin{aligned}(\lambda + iw)(\lambda - iw)(\lambda - \lambda_3)\\ (\lambda^2 + w^2)(\lambda - \lambda_3)\\ (\lambda^2 + b(r+\sigma))(\lambda - \lambda_3)\\ \lambda^3 - \lambda_3\lambda^2 + \dots\end{aligned}$$

$$\lambda^3 + (\sigma + 1 + b)\lambda^2 + b(r+\sigma)\lambda + 2b\sigma(r-1) = \lambda^3 - \lambda_3\lambda^2 + \dots$$

$$\lambda_3 = -(\sigma + 1 + b)$$

9.2.3

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= rx - xz - y\\ \dot{z} &= xy - bz\end{aligned}$$

$$\begin{aligned}V &= x^2 + y^2 + (z-r-\sigma)^2\\ \dot{V} &= 2x\dot{x} + 2y\dot{y} + 2(z-r-\sigma)\dot{z}\\ &= 2x\sigma(y-x) + 2y(rx - xz - y) + 2(z-r-\sigma)(xy - bz)\\ \frac{1}{2}\dot{V} &= \sigma xy - \sigma x^2 + rxy - xyz - y^2 + xyz - bz^2 - (r+\sigma)xy + b(r+\sigma)z\\ &= -\sigma x^2 - y^2 - bz^2 + b(r+\sigma)z\\ &= -\sigma x^2 - y^2 - b(z^2 - (r+\sigma)z)\\ &= -\sigma x^2 - y^2 - b\left(z^2 - (r+\sigma)z + \frac{(r+\sigma)^2}{4} - \frac{(r+\sigma)^2}{4}\right)\\ &= -\sigma x^2 - y^2 - b\left(\left(z - \frac{r+\sigma}{2}\right)^2 - \frac{(r+\sigma)^2}{4}\right)\\ &= -\sigma x^2 - y^2 - b\left(z - \frac{r+\sigma}{2}\right)^2 + b\frac{(r+\sigma)^2}{4}\end{aligned}$$

We need $\dot{V} < 0$ for the trajectories to point inward on the surface.

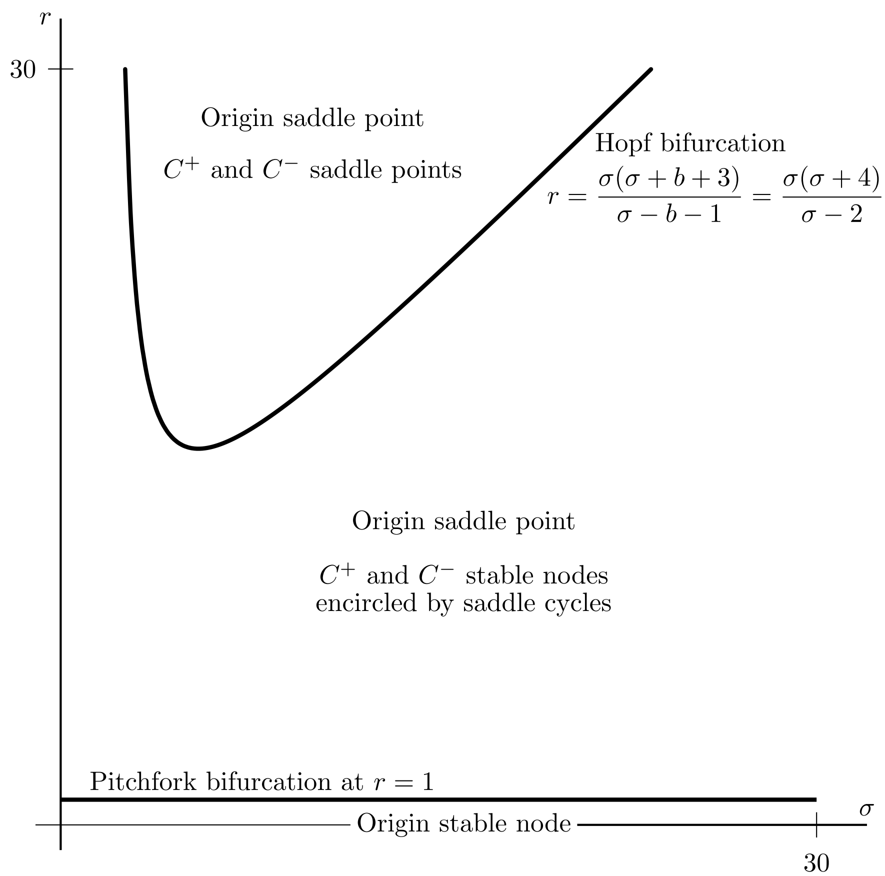
$$\begin{aligned}\frac{1}{2}\dot{V} = -\sigma x^2 - y^2 - b\left(z - \frac{r+\sigma}{2}\right)^2 + b\frac{(r+\sigma)^2}{4} &< 0\\ b\frac{(r+\sigma)^2}{4} &< \sigma x^2 + y^2 + b\left(z - \frac{r+\sigma}{2}\right)^2\\ 1 < \frac{4\sigma}{b(r+\sigma)^2}x^2 + \frac{4}{b(r+\sigma)^2}y^2 + \frac{4}{(r+\sigma)^2}&\left(z - \frac{r+\sigma}{2}\right)^2\end{aligned}$$

This last inequality describes all of $\mathbb{R}^3$ minus an ellipsoid with center $\left(0, 0, \dfrac{r+\sigma}{2}\right)$, with the maximum distance from the center in x, y, and z as $\sqrt{\dfrac{b(r+\sigma)^2}{4\sigma}}$, $\sqrt{\dfrac{b(r+\sigma)^2}{4}}$, and $\sqrt{\dfrac{(r+\sigma)^2}{4}}$ respectively.

Thus we can enclose this ellipsoid with a sphere centered at $(0, 0, r+\sigma)$ by picking a sufficiently large radius. However, the radius is a bit annoying to calculate because the center of the ellipsoid is not the center of the sphere.

9.2.5

For $b = 1$ we have the following stability diagram for the Lorenz equations.



9.3 Chaos on a Strange Attractor

9.3.1

a)

The system is periodic, but it is not chaotic because it's not sensitive to initial conditions, which is one of the requirements for chaos.

b)

In fact, the error never changes.

$$\dot{\theta}_1 = w_1 \Rightarrow \theta_1(t) = w_1 t + C_1, \quad \theta_1(0) = C_1$$
$$\dot{\theta}_2 = w_2 \Rightarrow \theta_2(t) = w_2 t + C_2, \quad \theta_2(0) = C_2$$

Adding error to the initial conditions makes a blob around the initial conditions, but the blob moves together and doesn't distort.

The system can be interpreted as movement of a dot on a sphere. w_1 controls how fast the dot moves in the latitudinal direction and w_2 how fast the dot moves in the longitudinal direction.

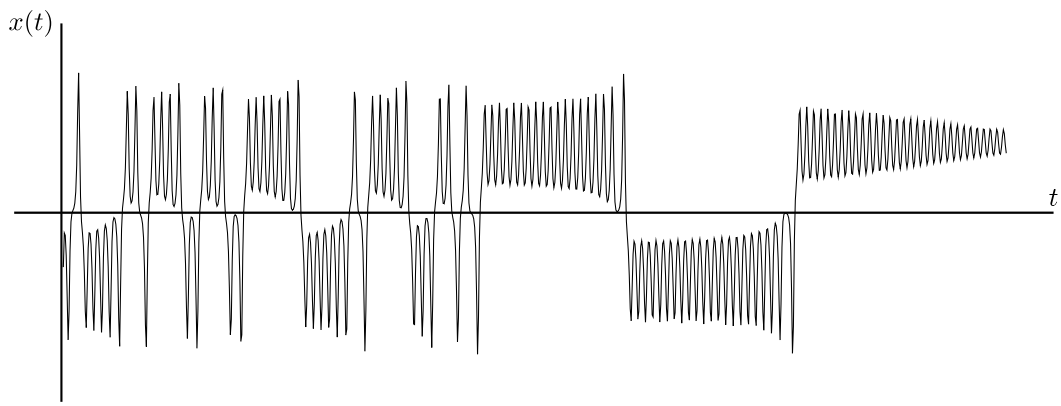
$$||(\vec{\theta}(t) + \vec{\delta}(t)) - \vec{\theta}(t)|| = ||\vec{\delta}(t)|| = ||(\delta_1, \delta_2)|| = ||(\delta_1, \delta_2)||e^{0t}$$

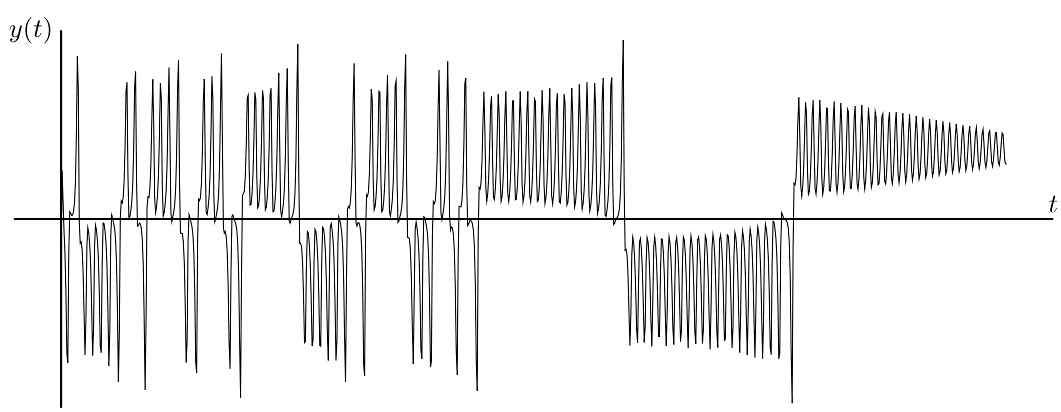
So the largest Liapunov exponent is 0.

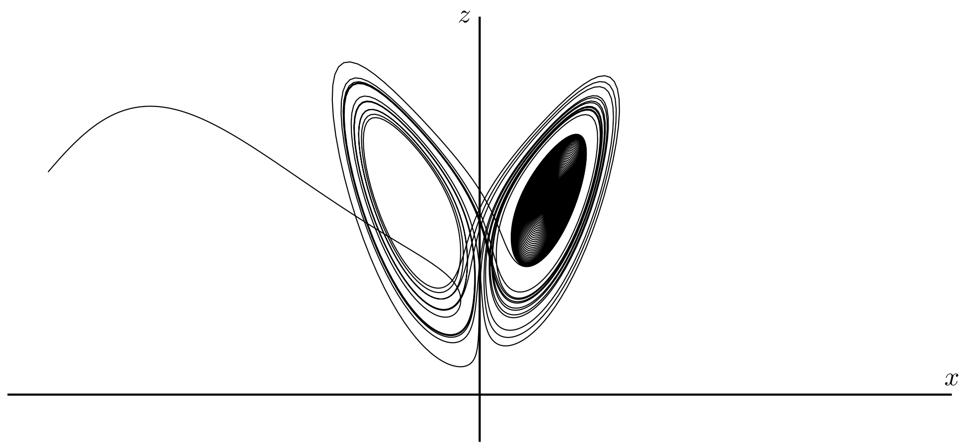
9.3.3

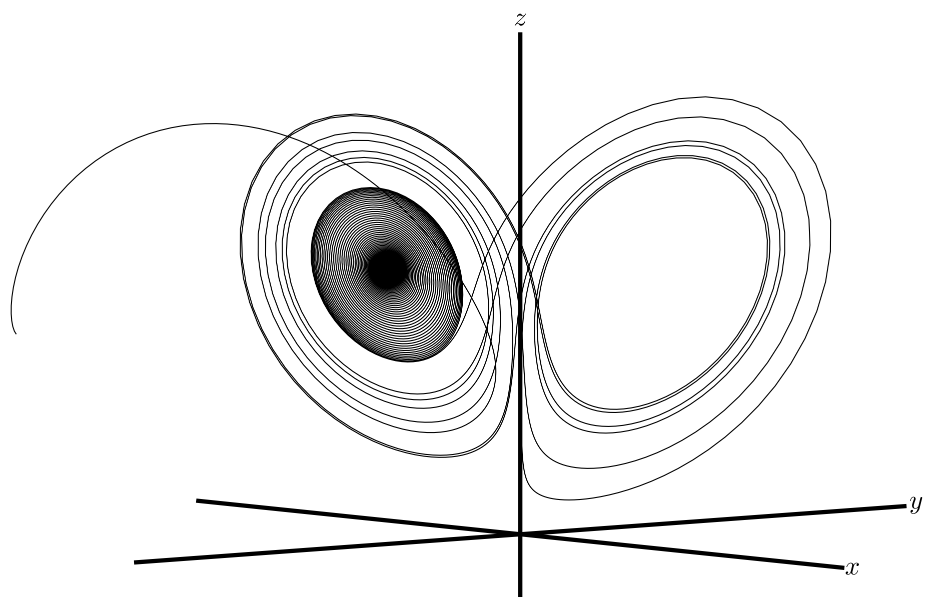
With the parameters $r = 22$, $\sigma = 10$, and $b = \frac{8}{3}$, some of the trajectories can be chaotic for a little while but eventually stay in the middle of one of the pieces. (Trajectories can also go directly to the middle of a piece with no chaos, but those are not shown.) The two simulations were run until $t = 100$.

Initial condition $(x, y, z) = (-58.26, -3.3144, 12.221)$

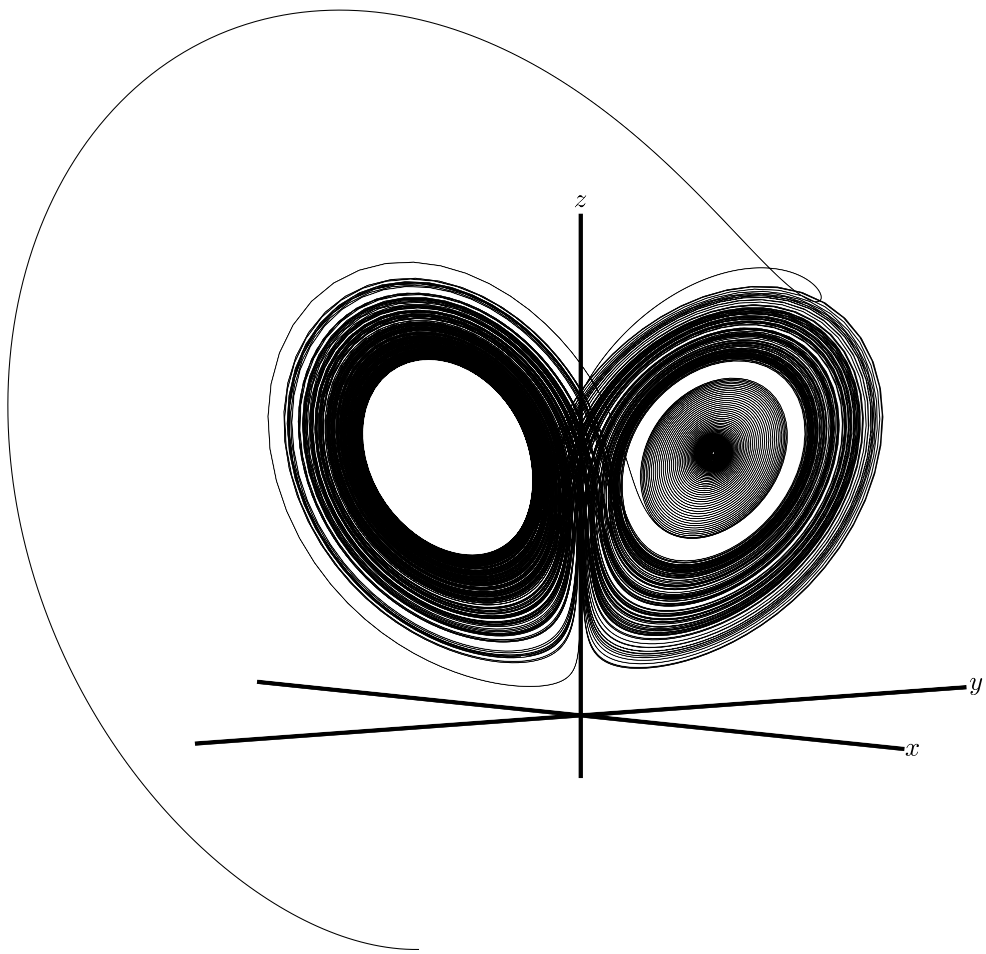








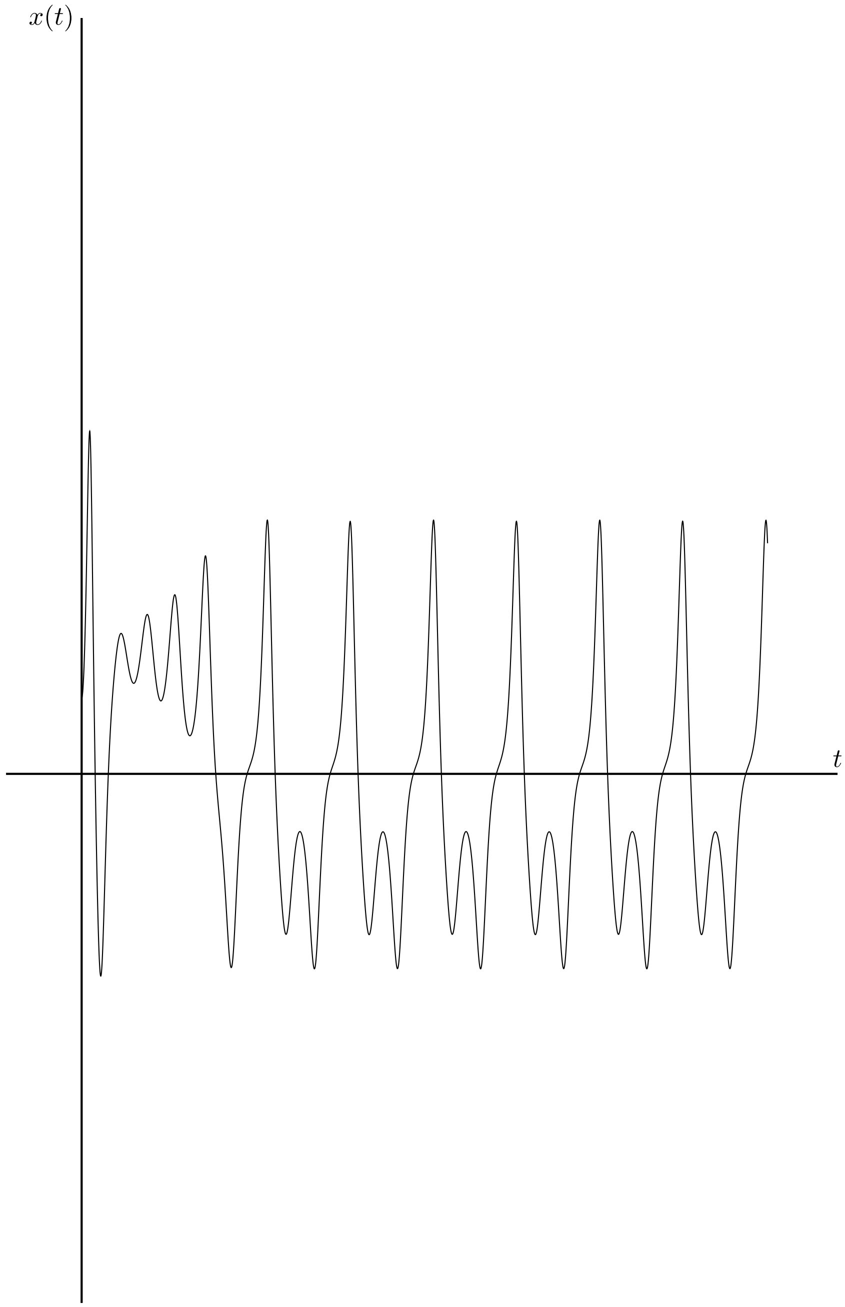
This other trajectory has an initial condition $(x, y, z) = (-20, 0, -20)$, but the simulation had to be run for much, much longer. We show only the final graph due to the simulation length.

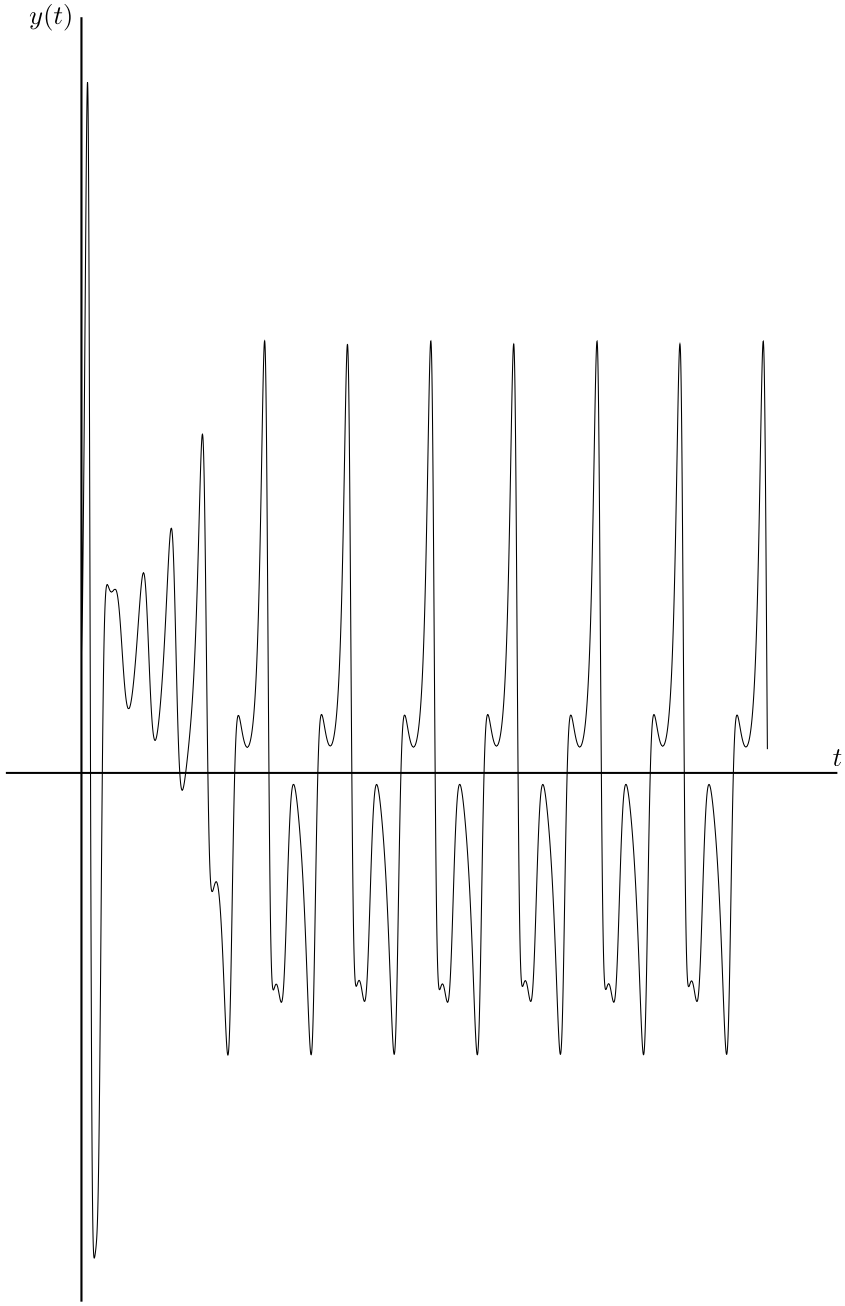

z
y
x

9.3.5

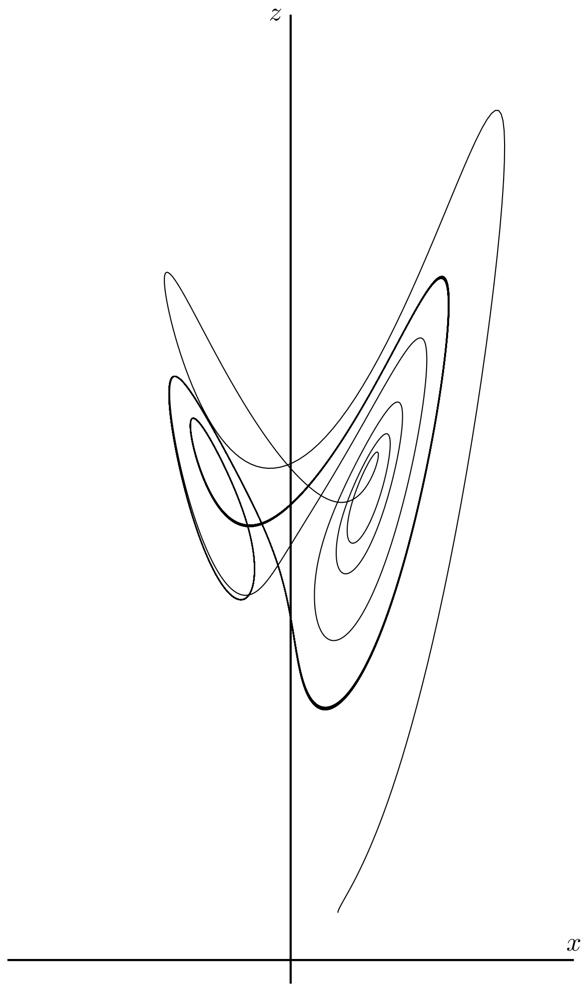
With the parameters $r = 100$, $\sigma = 10$, and $b = \frac{8}{3}$, the trajectories go to one of two stable limit cycles.

Initial condition $(x, y, z) = (10, 10, 10)$

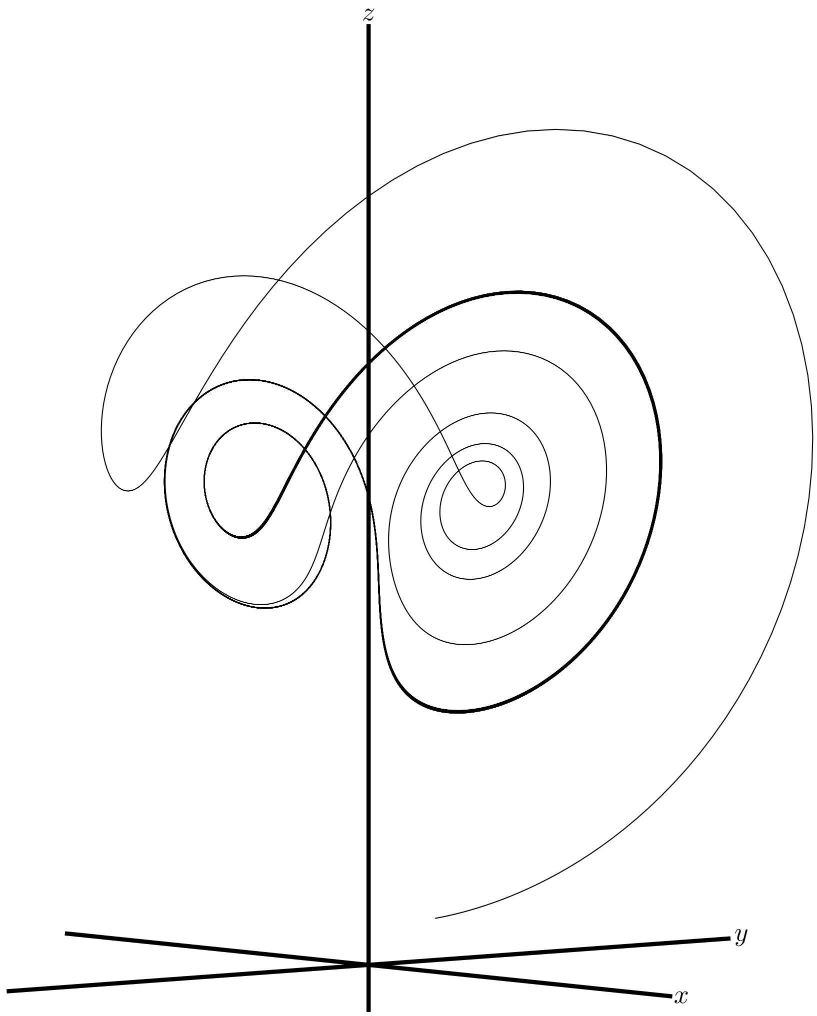




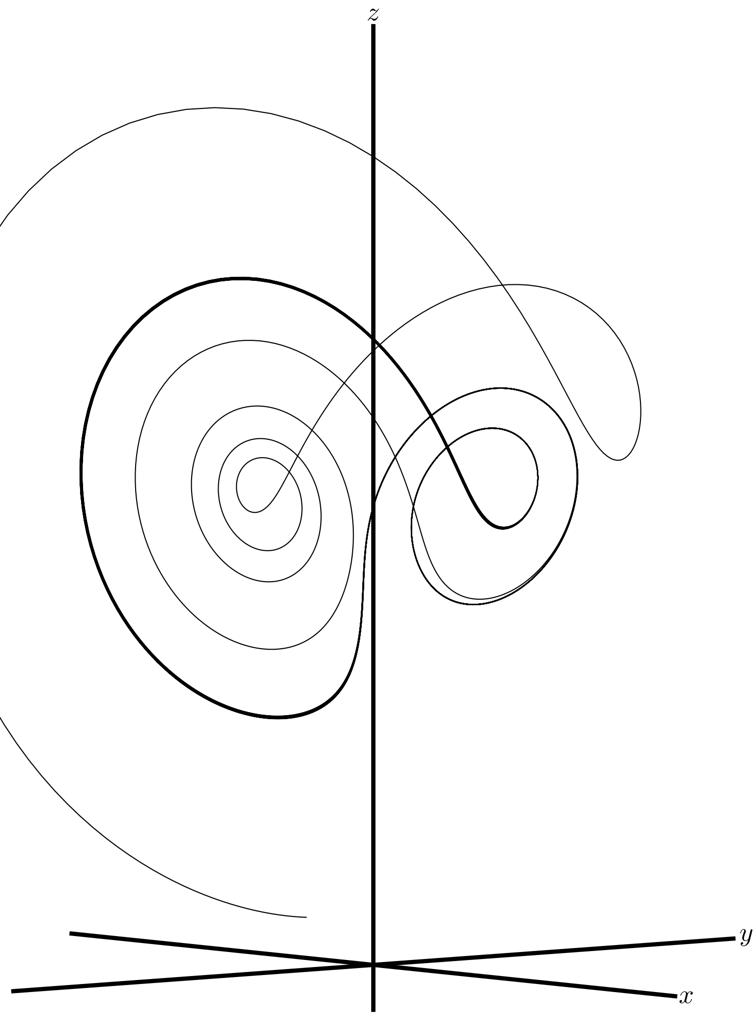
$y(t)$
t



z
x



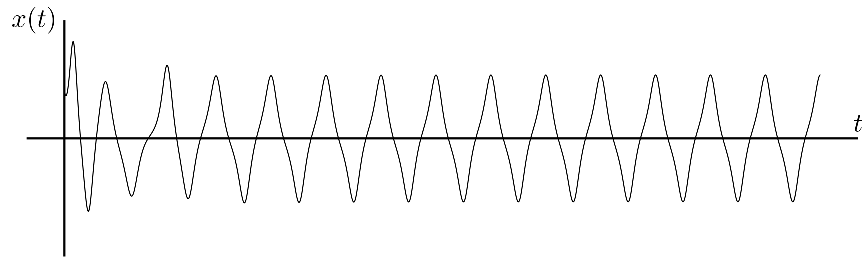
The other limit cycle is symmetric and starts at initial condition $(x, y, z) = (-10, -10, 10)$.

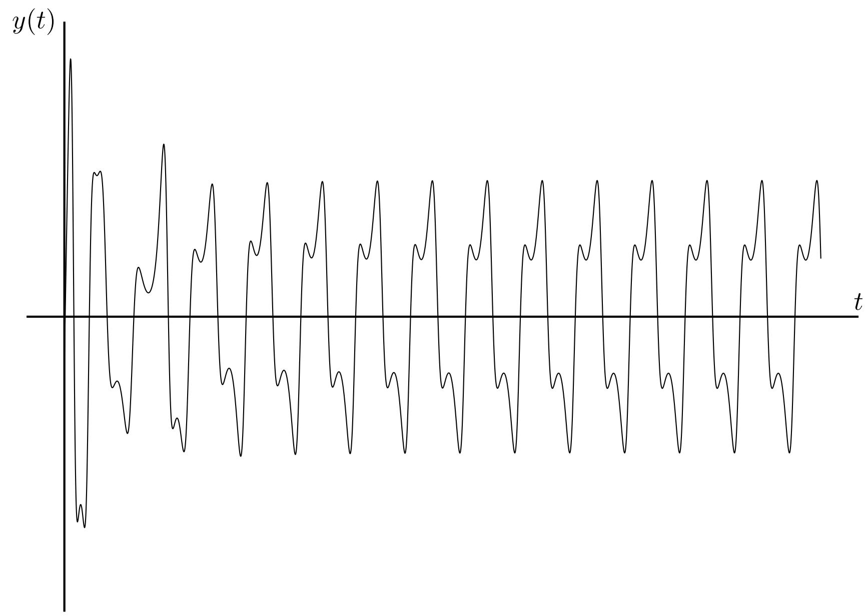

z
y
x

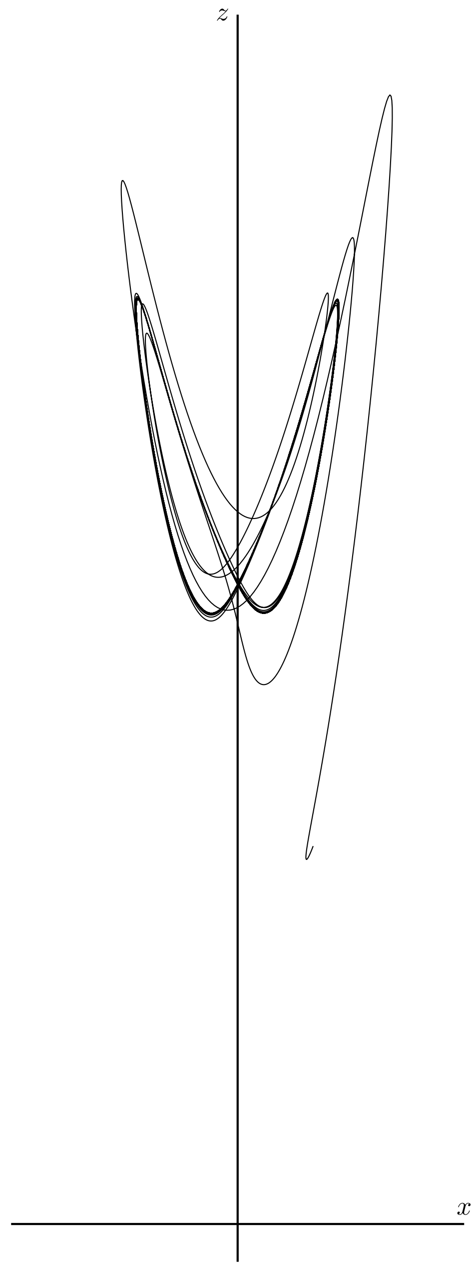
9.3.7

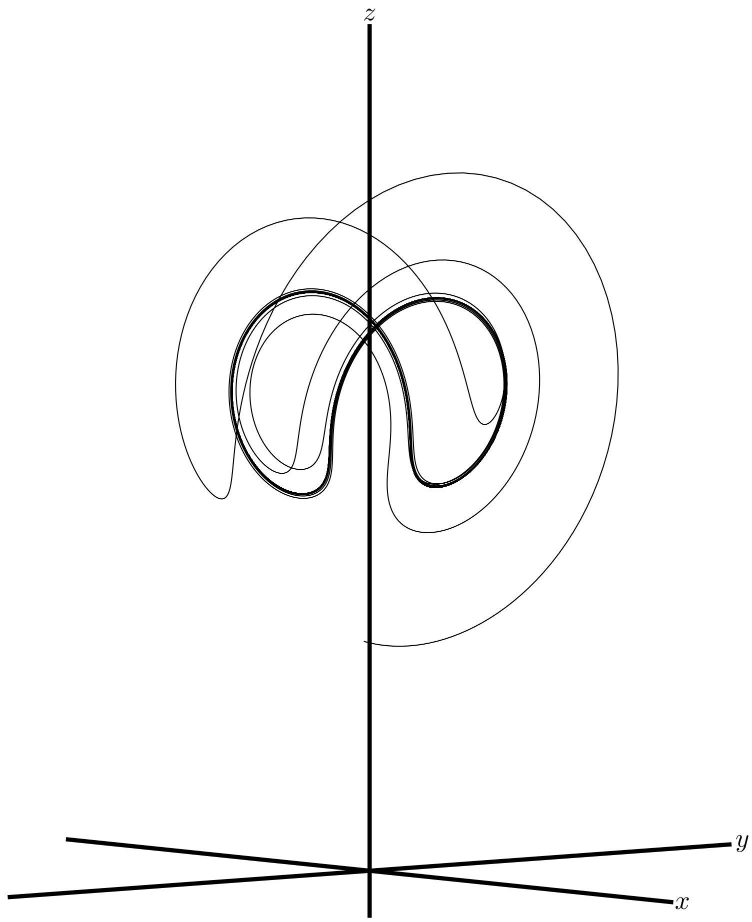
With the parameters $r = 400$, $\sigma = 10$, and $b = \frac{8}{3}$, trajectories go towards a single stable limit cycle.

Initial condition $(x, y, z) = (40, -40, 200)$



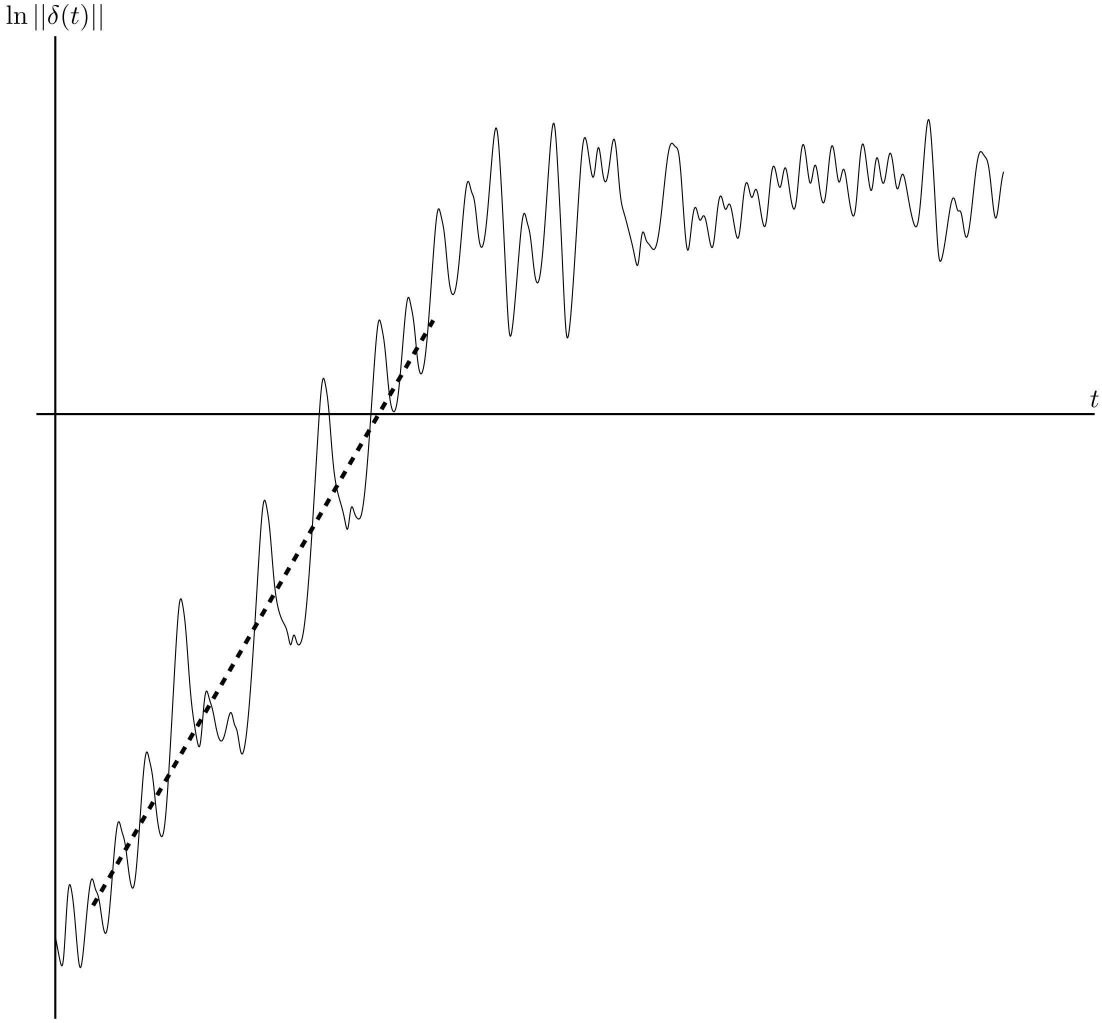



z
x



9.3.9

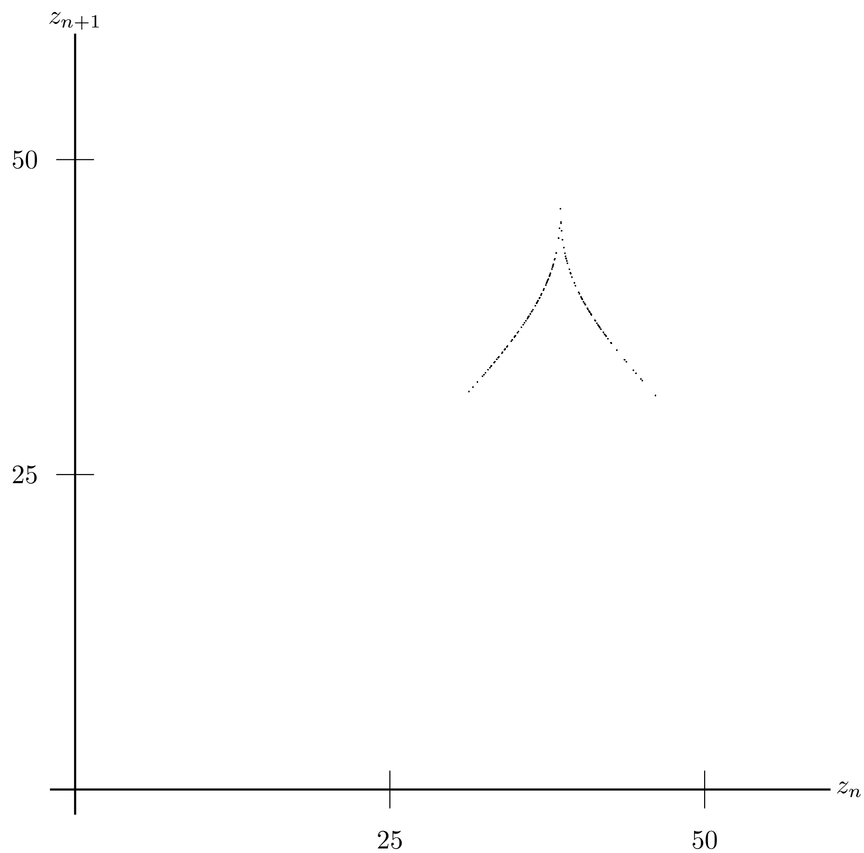
The following graph was made by running an arbitrary initial condition until the trajectory was on the Lorenz attractor at $(x, y, z) = (0.8961, 1.5605, 11.246)$. Then the trajectory and a perturbed trajectory of $(x, y, z) = (0.8961, 1.5605, 11.246 + 0.001)$ were simulated until $t = 25$ and the error computed in the following graph. Then we used a linear fit of the error between $t \in [1, 10]$ so as to avoid the initial transient and leveling off of the error. The slope is an estimate of the largest Liapunov exponent, $\lambda = 0.8623 \approx 0.906$. The latter is the "real" value calculated with more advanced methods.



$\ln||\delta(t)||$
t

9.4 Lorenz Map

9.4.1

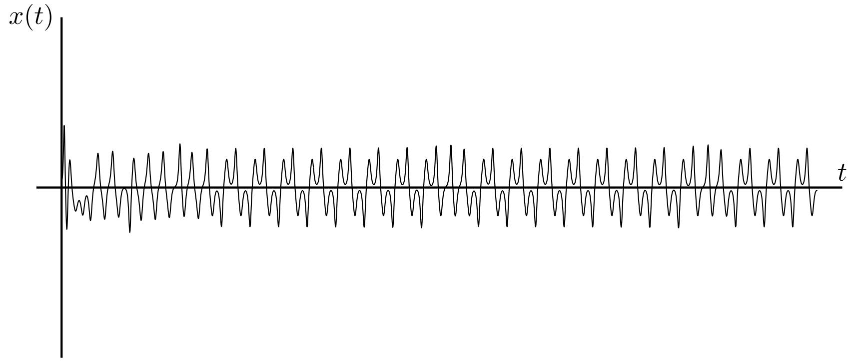


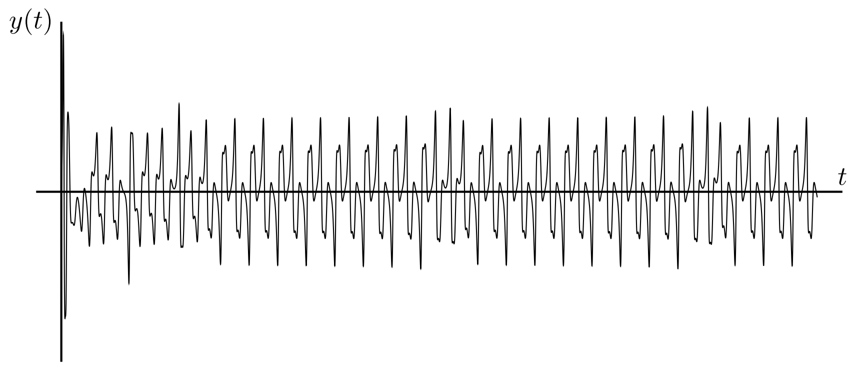
9.5 Exploring Parameter Space

9.5.1

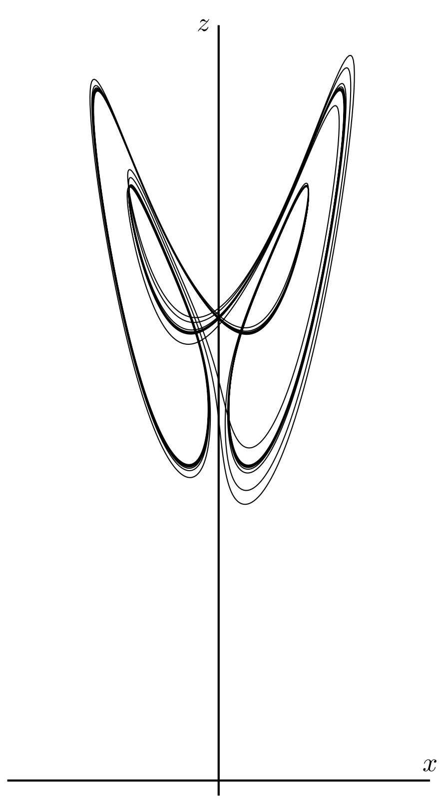
With the parameters $r = 166.3$, $\sigma = 10$, and $b = \frac{8}{3}$, the simulation was run until $t = 30$, showing intermittent chaos.

Initial condition $(x, y, z) = (10, -10, 0)$





This graph is for $t \in [20, 30]$, and you can see the mostly periodic behavior of the grouped lines and the trajectory splitting off at some point.



9.5.3

Parameters $145 < r < 166 \quad \sigma = 10 \quad b = \frac{8}{3}$

This range of r is a period doubling region for the Lorenz system.

All these inequalites are approximate.

$166.07 < r$

No limit cycle, intermittent chaos

$r = 166.07$

Single stable symmetric limit cycle born

$r = 154.4$

Single stable symmetric limit cycle splits into two asymmetric limit cycles

$r = 148.2$

Each of two stable asymmetric limit cycles split, creating a total of four asymmetric limit cycles with double the period

$145 < r < 148.2$

An infinite number of period double bifurcations occurs in this region

$r = 145$

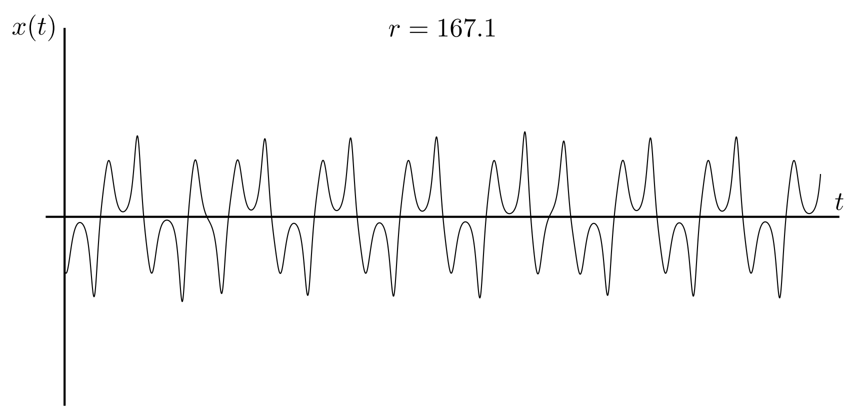
Chaos

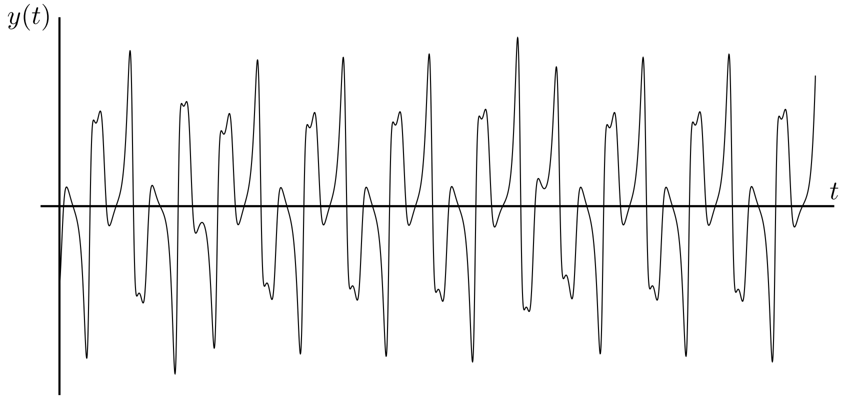
A sampling of r values

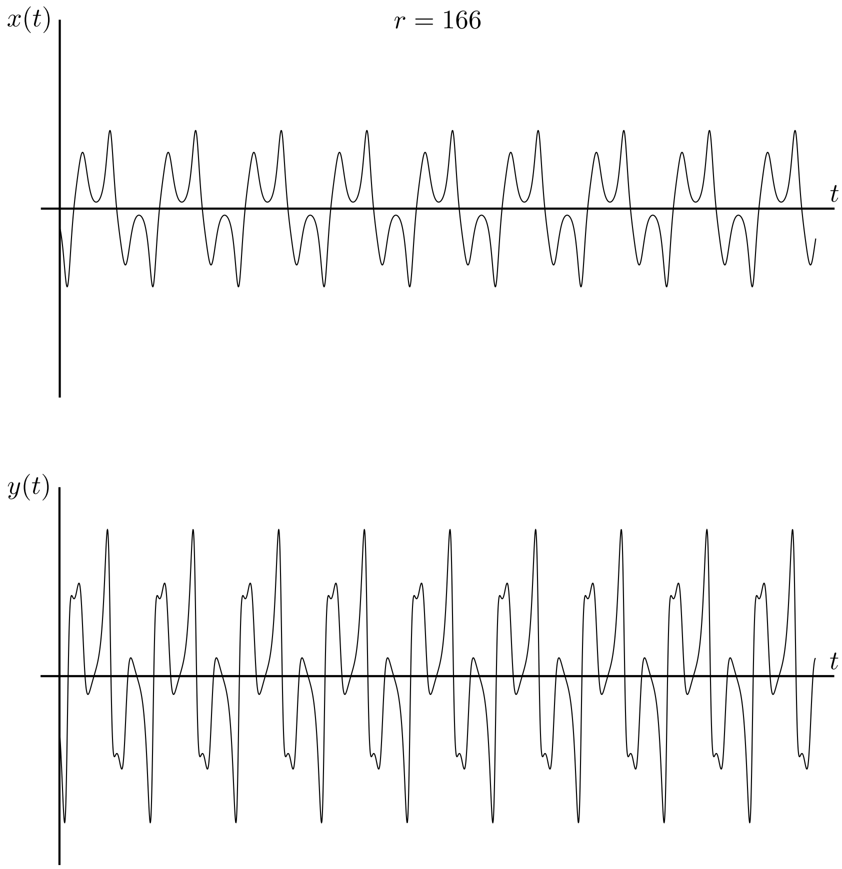
$$167.1 \quad 166 \quad 160 \quad 155 \quad 154 \quad 148.5 \quad 148 \quad 147.5 \quad 145$$

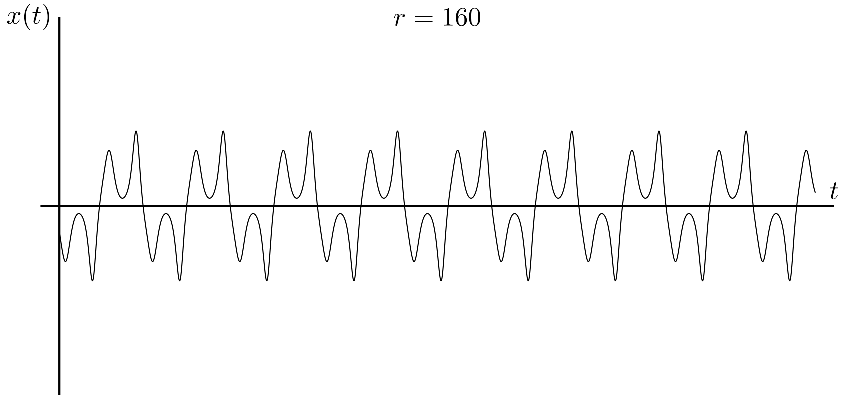
in this order were simulated to show changes in behavior.

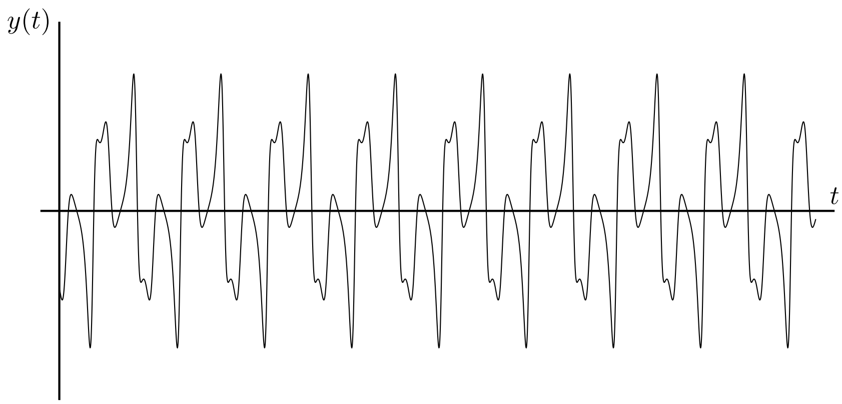
Unless otherwise noted, each simulation was run until $t = 10$ after reaching the limit cycle.

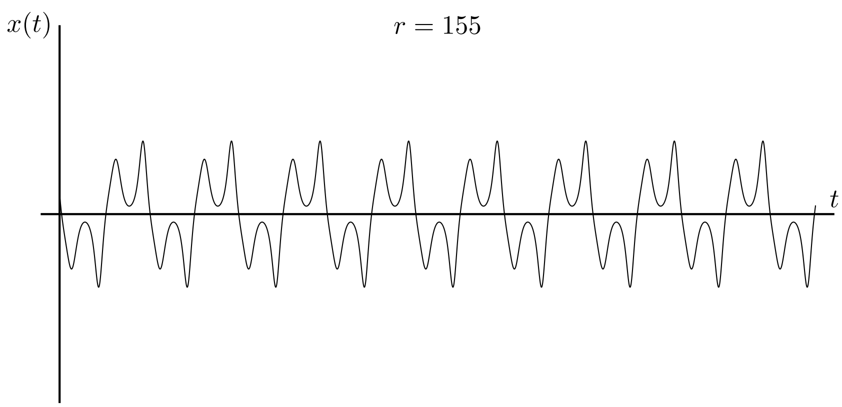


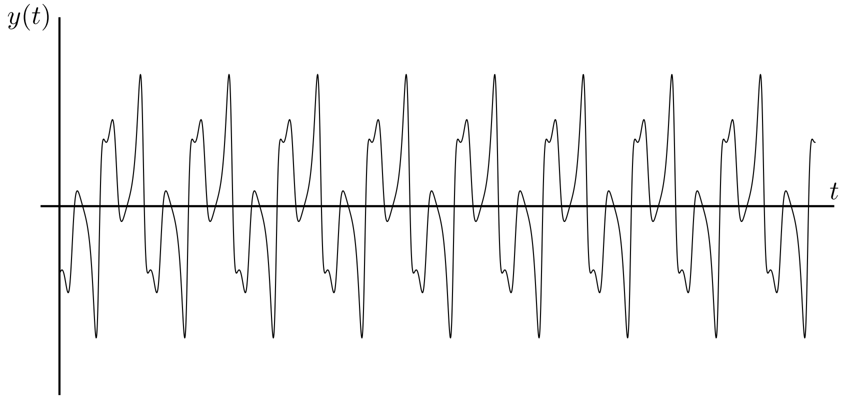

$y(t)$
t

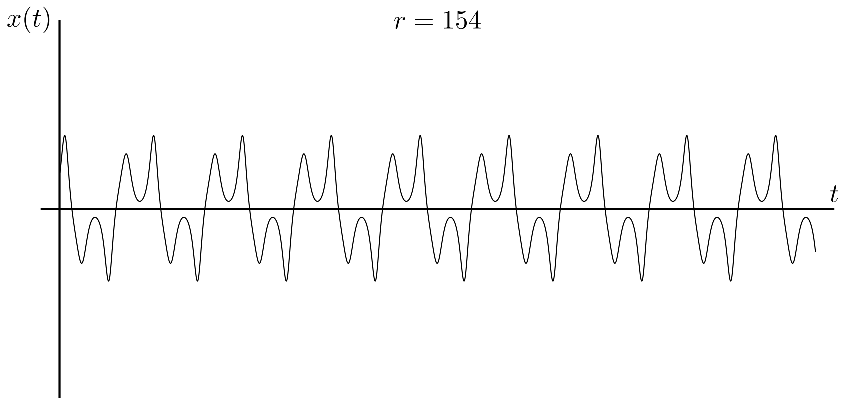

$x(t)$
$r = 166$
t
$y(t)$
t

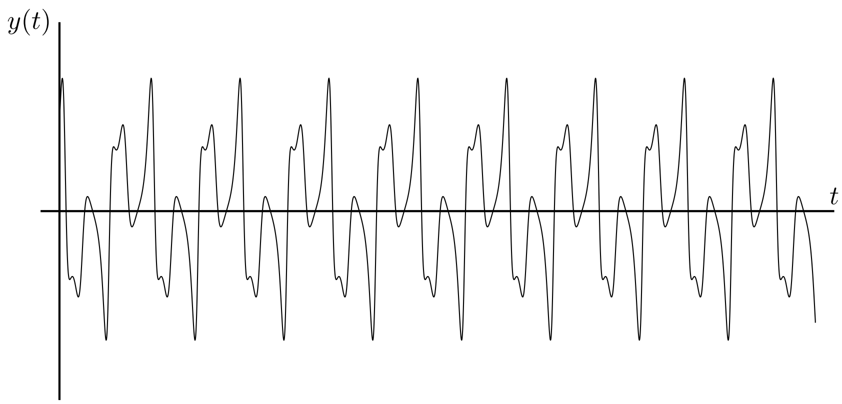

$x(t)$
$r = 160$
t

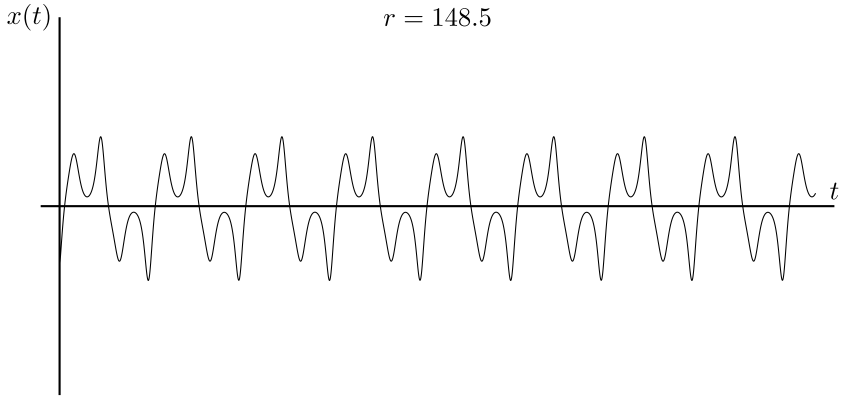

$y(t)$
t

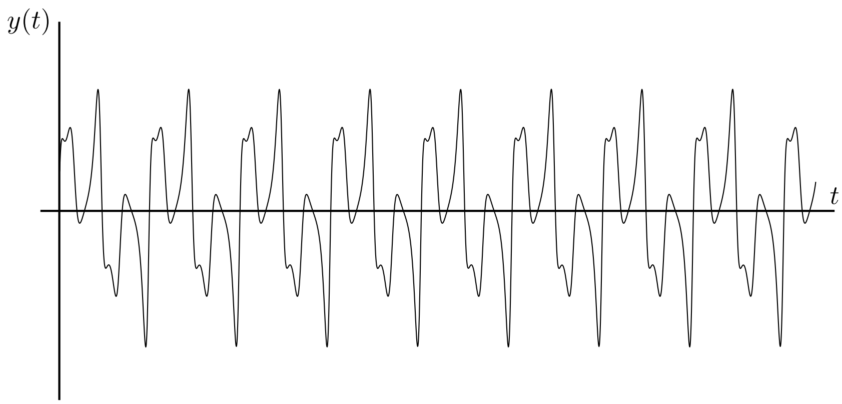

$x(t)$
$r = 155$
t

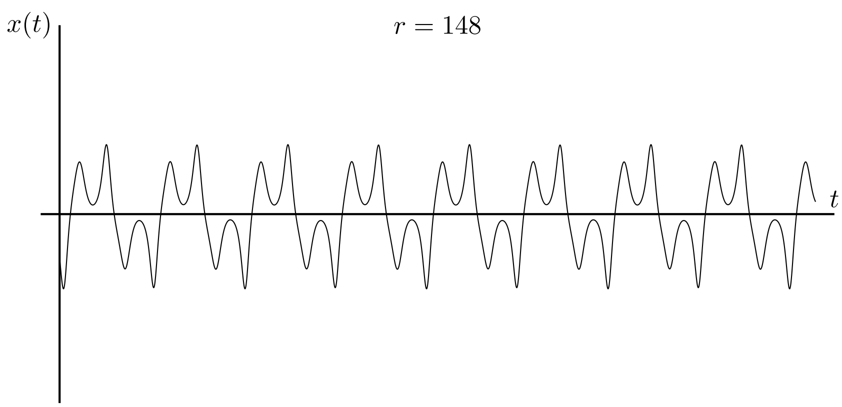

$y(t)$
t

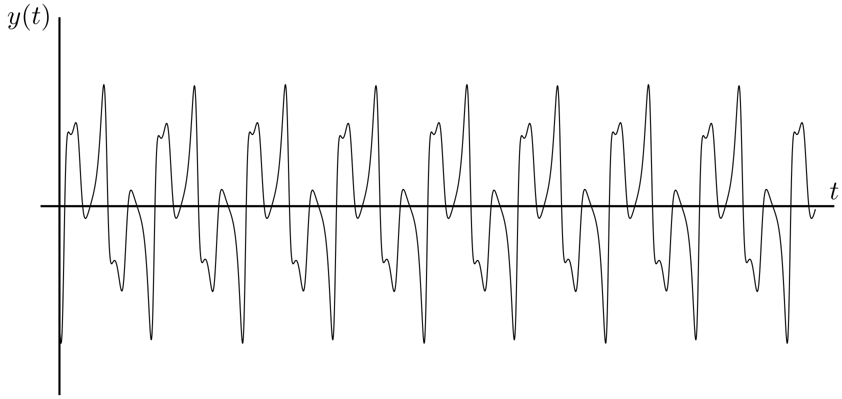

$x(t)$
$r = 154$
t

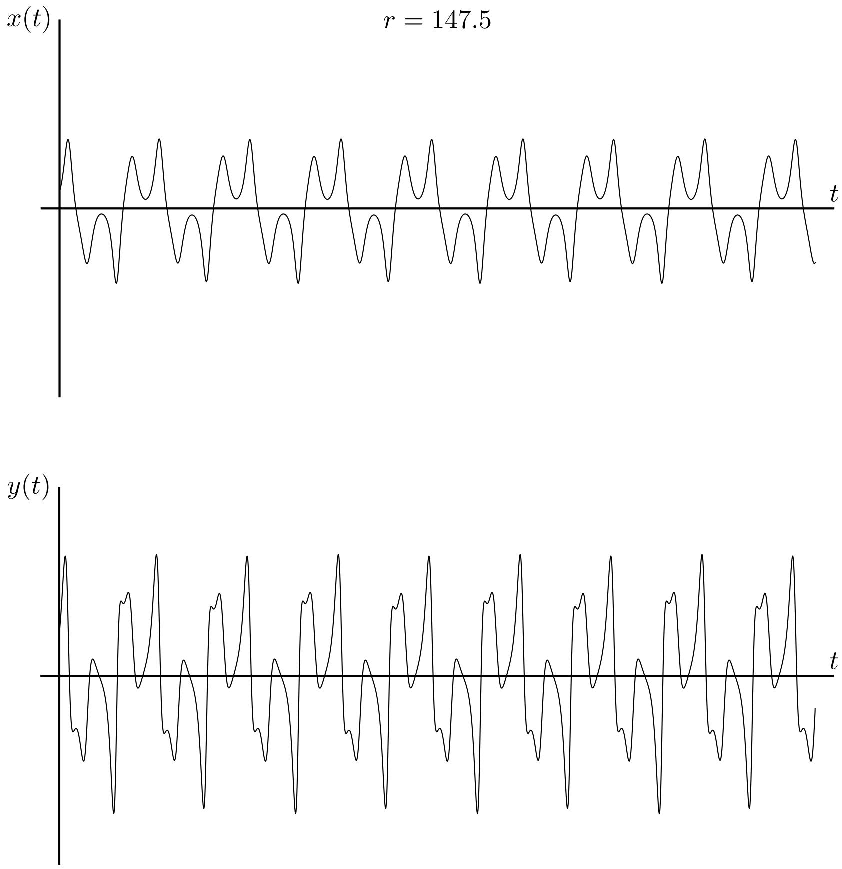

$y(t)$
t

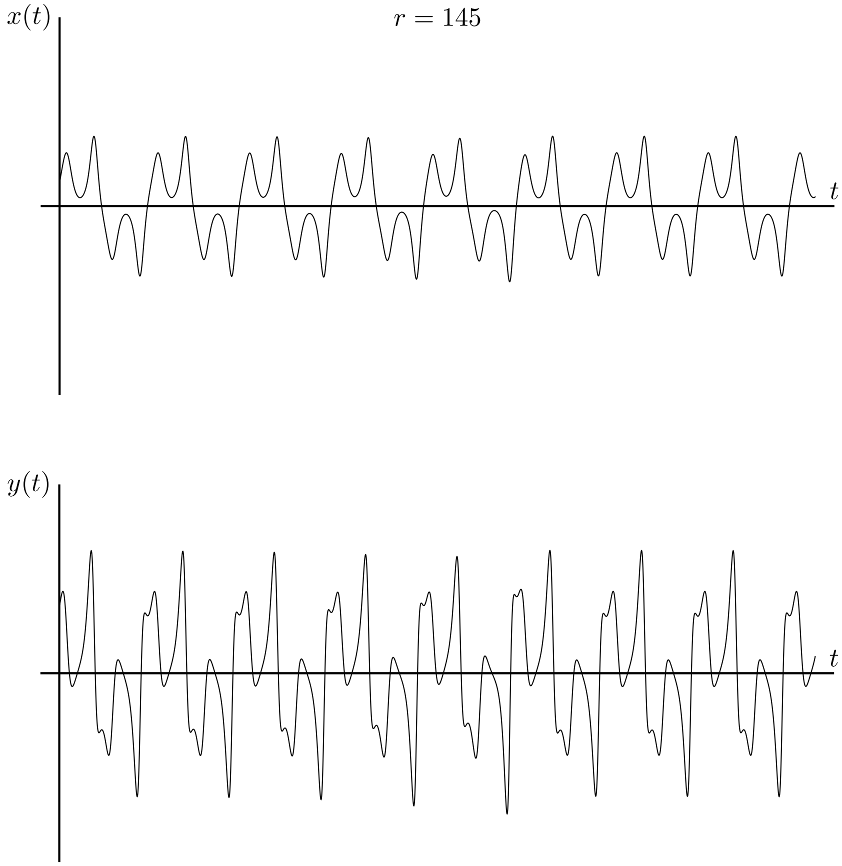

$x(t)$
$r = 148.5$
t

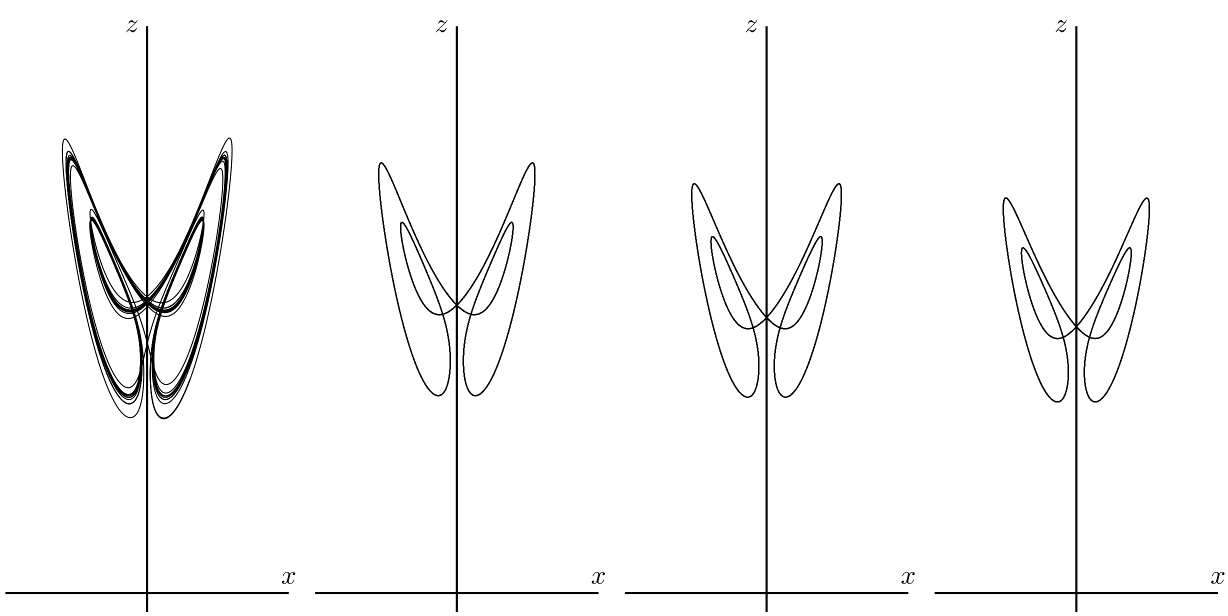

$y(t)$
t


$x(t)$
$r = 148$
t

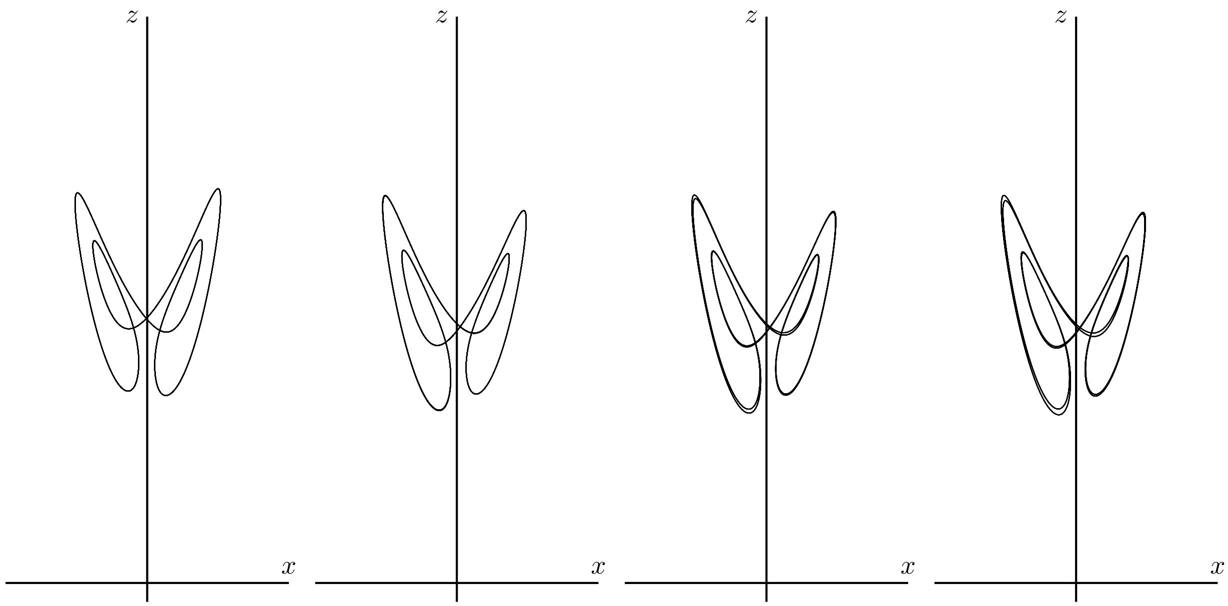

$y(t)$
t


$x(t)$
$r = 147.5$
t
$y(t)$
t



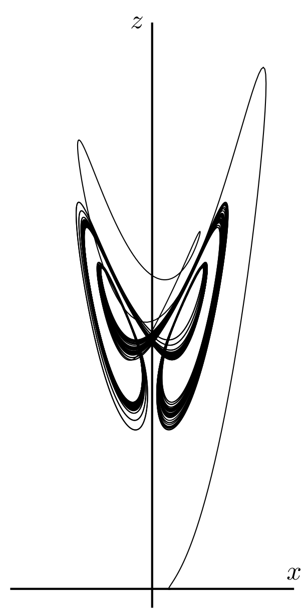


$r = 167.1$ $r = 166$ $r = 160$ $r = 155$



$r = 154$ $\quad$ $r = 148.5$ $\quad$ $r = 148$ $\quad$ $r = 147.5$

Initial condition $(x, y, z) = (10, -10, 0) \quad t \in [0, 50]$



$r = 145$

9.5.5

$$\dot{x} = \sigma(y - x) \qquad \dot{y} = rx - xz - y \qquad \dot{z} = xy - bz$$

a)

$$X = \epsilon x \quad Y = \epsilon^2 \sigma y \quad Z = \sigma(\epsilon^2 z - 1) \quad \tau = \frac{t}{\epsilon} \quad \epsilon = \frac{1}{\sqrt{r}}$$
$$\dot{x} = \frac{1}{\epsilon^2}\frac{\mathrm{d}X}{\mathrm{d}\tau} = \sigma(y-x) = \sigma\left(\frac{Y}{\epsilon^2\sigma} - \frac{X}{\epsilon}\right)$$
$$\dot{y} = \frac{1}{\epsilon^3\sigma}\frac{\mathrm{d}Y}{\mathrm{d}\tau} = rx - xz - y = r\frac{X}{\epsilon} - \frac{X}{\epsilon}\frac{1}{\epsilon^2}\left(\frac{Z}{\sigma}+1\right) - \frac{Y}{\epsilon^2\sigma}$$
$$\dot{z} = \frac{1}{\epsilon^2}\left(\frac{1}{\sigma\epsilon}+1\right)\frac{\mathrm{d}Z}{\mathrm{d}\tau} = xy - bz = \frac{X}{\epsilon}\frac{Y}{\epsilon^2\sigma} - b\frac{1}{\epsilon^2}\left(\frac{Z}{\sigma}+1\right)$$

$$\begin{aligned}
\frac{1}{\epsilon^2}\frac{\mathrm{d}X}{\mathrm{d}\tau} &= \sigma\left(\frac{Y}{\epsilon^2\sigma} - \frac{X}{\epsilon}\right) \\
\Rightarrow \frac{\mathrm{d}X}{\mathrm{d}\tau} &= Y - \sigma\epsilon X \\
\frac{1}{\epsilon^3\sigma}\frac{\mathrm{d}Y}{\mathrm{d}\tau} &= r\frac{X}{\epsilon} - \frac{X}{\epsilon}\frac{1}{\epsilon^2}\left(\frac{Z}{\sigma}+1\right) - \frac{Y}{\epsilon^2\sigma} \\
\Rightarrow \frac{\mathrm{d}Y}{\mathrm{d}\tau} &= r\epsilon^2\sigma X - XZ - \sigma X - \epsilon Y \\
&= \sigma X - XZ - \sigma X - \epsilon Y \\
&= -XZ - \epsilon Y \\
\frac{1}{\epsilon^2}\left(\frac{1}{\sigma\epsilon}+1\right)\frac{\mathrm{d}Z}{\mathrm{d}\tau} &= \frac{X}{\epsilon}\frac{Y}{\epsilon^2\sigma} - b\frac{1}{\epsilon^2}\left(\frac{Z}{\sigma}+1\right) \\
\Rightarrow (1+\epsilon\sigma)\frac{\mathrm{d}Z}{\mathrm{d}\tau} &= XY - b\epsilon\sigma\left(\frac{Z}{\sigma}+1\right)
\end{aligned}$$

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{\mathrm{d}X}{\mathrm{d}\tau} &= Y \\
\lim_{\epsilon\to 0}\frac{\mathrm{d}Y}{\mathrm{d}\tau} &= -XZ \\
\lim_{\epsilon\to 0}(1+\epsilon\sigma)\frac{\mathrm{d}Z}{\mathrm{d}\tau} &= \frac{\mathrm{d}Z}{\mathrm{d}\tau} = XY
\end{aligned}$$

b)

$$\begin{aligned}
(Y^2+Z^2)' &= 2YY' + 2ZZ' = 2Y(-XZ) + 2Z(XY) \\
&= -2XYZ + 2XYZ = 0 \\
(X^2 - 2Z)' &= 2XX' - 2Z' = 2XY - 2XY = 0
\end{aligned}$$

c)

$$\begin{aligned}
\dot{V} &= \int \nabla\cdot\mathbf{f}\ \mathrm{d}V = \int_V \nabla\cdot\langle X', Y', Z'\rangle\ \mathrm{d}V = \int_V \nabla\cdot\langle Y, -XZ, XY\rangle\ \mathrm{d}V \\
&= \int_V 0\ \mathrm{d}V = 0
\end{aligned}$$

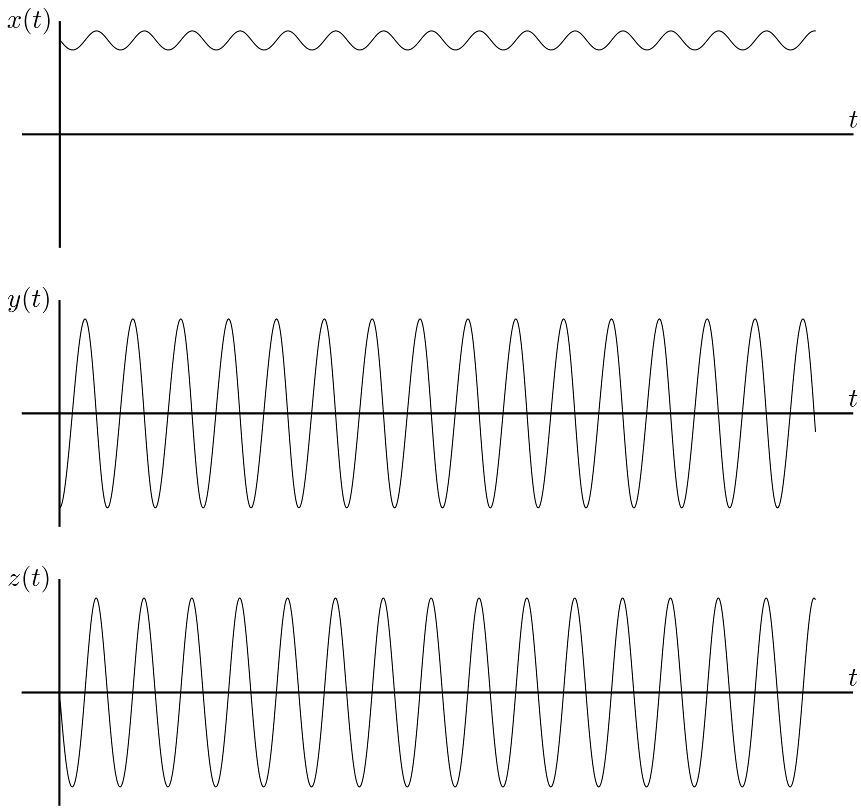
So the set of equations has a rate of change of zero on a volume, otherwise known as *volume preserving*.

d)

As discussed in the text, the Rayleigh number r is a dimensionless ratio. Its numerator includes the effects driving the system. Its denominator includes the two sources of energy dissipation (for the waterwheel, these were the leakage of water through the holes and the viscous damping due to the brake). The limit of infinite r means that the dissipation terms are tending to zero, in a relative sense. And without dissipation, the system would be expected to act conservatively.

e)

Initial condition $(x, y, z) = (10, -10, 0) \quad t \in [0, 10]$



The behavior here is a stable limit cycle. Numerical solutions of the Lorenz equations for large r also show a single symmetric limit cycle, so the Lorenz equations and this approximate system are in agreement.

9.6 Using Chaos to Send Secret Messages

9.6.1

a)

From the text

$$\dot{V} = -e_2^2 - 4be_3^2 < -k\left(\frac{1}{2}e_2^2 + 2e_3^2\right) = -kV$$

We can solve the inequality for each component separately.

$$-e_2^2 < -\frac{1}{2}ke_2^2 \Rightarrow k < 2$$

$$-4be_3^2 < -2ke_3^2 \Rightarrow k < 2b$$

Therefore, as long as $k < \min\{2, 2b\}$, the inequality will hold. Integration then yields

$$0 \leq V(t) \leq V_0 e^{-kt}$$

b)

$$\frac{1}{2}e_2^2 \leq V < V_0 e^{-kt} \Rightarrow e_2(t) < \sqrt{2V_0}e^{-\frac{kt}{2}}$$

Similarly,

$$2e_3^2 \leq V < V_0 e^{-kt} \Rightarrow e_3(t) < \sqrt{\frac{V_0}{2}}e^{-\frac{kt}{2}}$$

c)

First take the $\dot{e}_1$ equation, rearrange it, bound it,

$$\dot{e}_1 = \sigma(e_2 - e_1) \Rightarrow \dot{e}_1 + \sigma e_1 = \sigma e_2 < \sigma\sqrt{2V_0}e^{-\frac{kt}{2}}$$

and then use an integrating factor.

$$e_1(t) = Ce^{-\sigma t} - \frac{2\sigma\sqrt{2V_0}}{k - 2\sigma}e^{-\frac{kt}{2}}$$

So all components of $\mathbf{e}(t)$ decay exponentially fast.

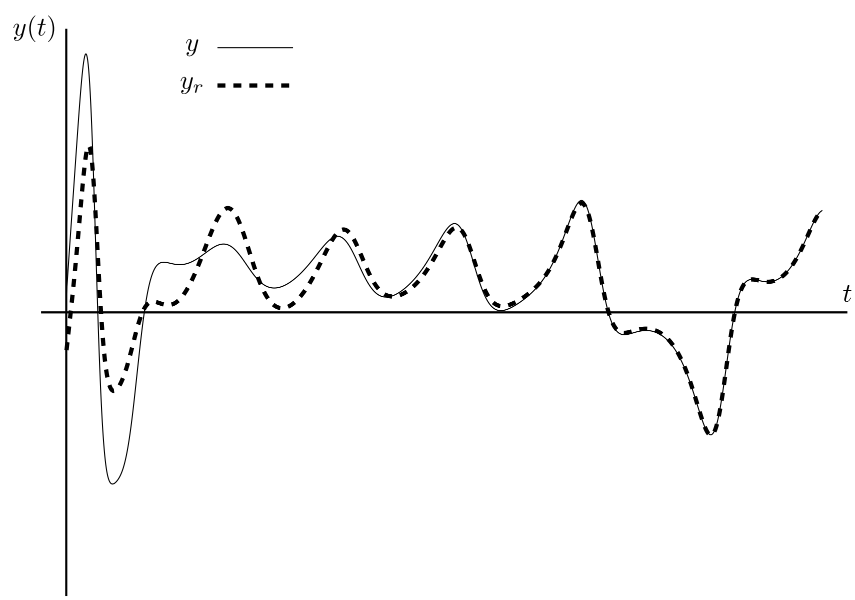
9.6.3

a)

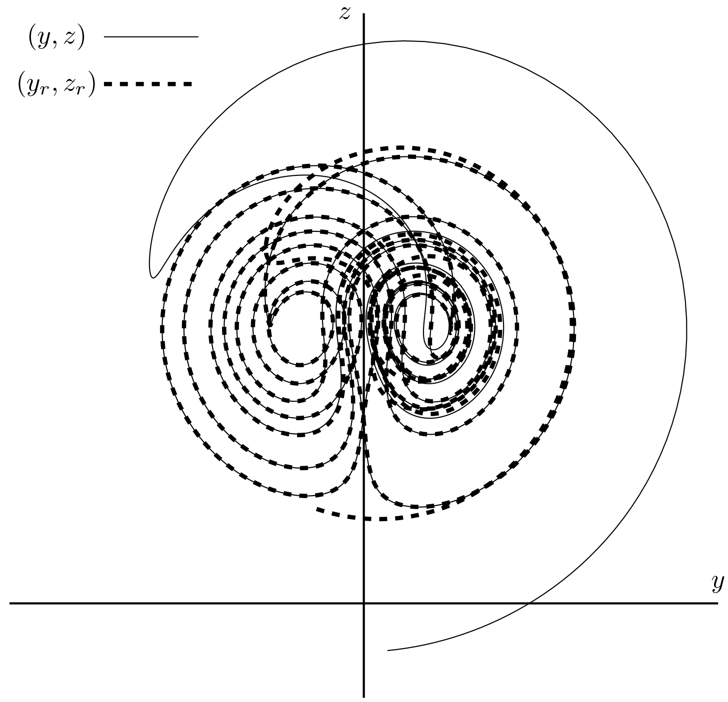
Initial conditions $\quad t \in [0, 3]$

$(x, y, z) = (15, 5, -10)$

$(x_r, y_r, z_r) = (x, -10, 20)$



b)

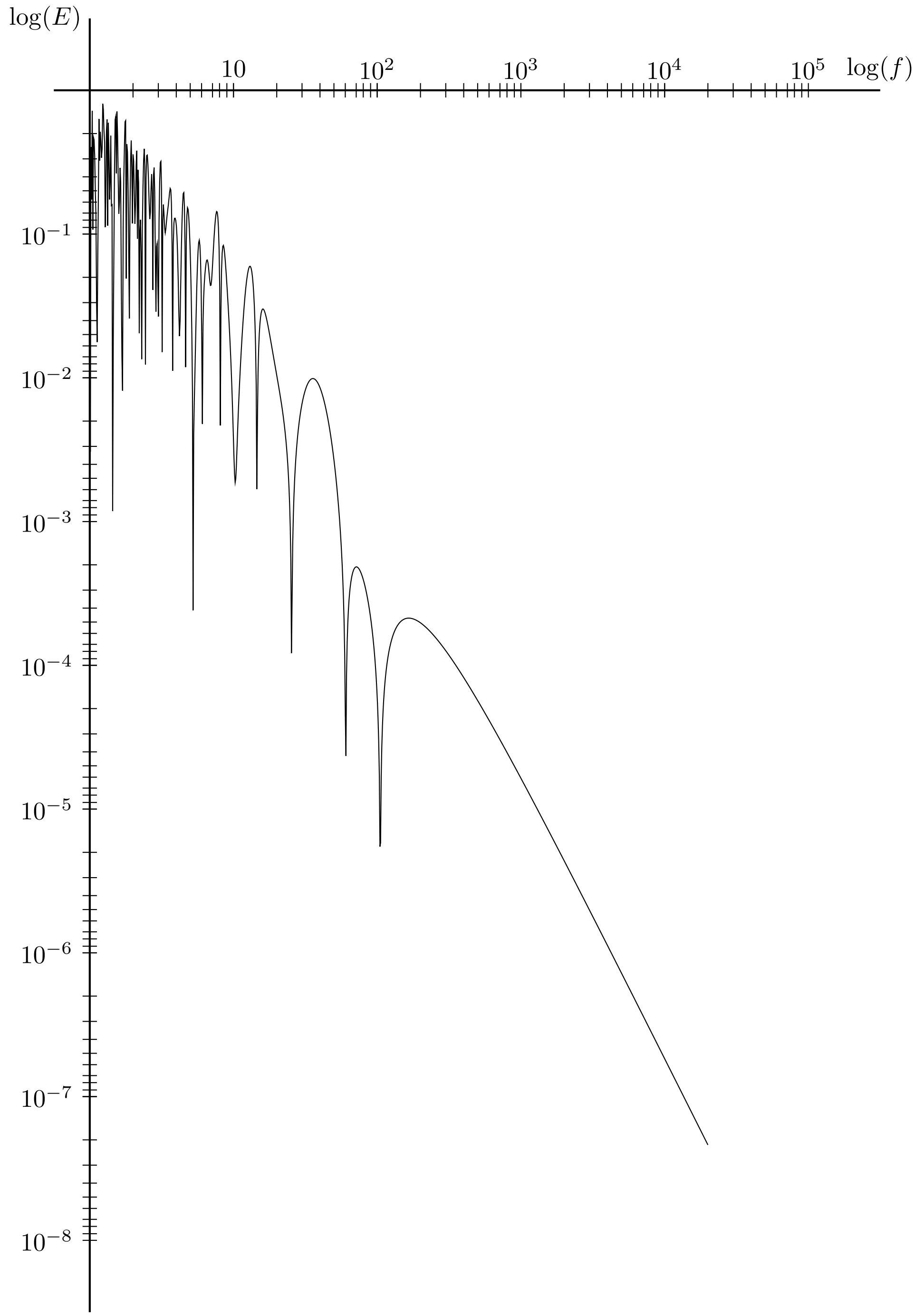


9.6.5

The idea here is that the receiver and transmitter will still synchronize even though we aren't transmitting $x(t)$ but $s(t) = x(t) + m(t)$ instead. As long as $m(t)$ has a small amplitude relative to $x(t)$ on the strange attractor, then $s(t) \approx x(t)$ and the receiver $x_r(t)$ should still synchronize to the transmitter's $x(t)$ pretty closely.

In all the following simulations, we start the transmitter and receiver variables at $(x, y, z) = (x_r, y_r, z_r) = (7.6200, 9.6101, 48.3368)$, which is already close to the Lorenz attractor since we assume the receiver and transmitter will synchronize. Then we add a sine wave $A \sin(2\pi f t)$ ($A = 0.1$ and f is varied) to x_r and simulate for ten periods of the sine wave. Lastly, we simulate for ten more periods of the sine wave, record $E(f) = \max|m(t) - \hat{m}(t)|$ for this time interval, and graph the results from 1 Hz to 20,000 Hz.

Parameters $r = 60 \quad \sigma = 10 \quad b = \frac{8}{3}$



The error is rather erratic at low frequencies, but then at about 1000 Hz the error just keeps going down and down.

10

One-Dimensional Maps

10.1 Fixed Points and Cobwebs

10.1.1

$$x_{n+1} = \sqrt{x_n}$$

The fixed points occur when

$$x = \sqrt{x}$$
$$x^2 = x$$
$$x(x-1) = 0$$

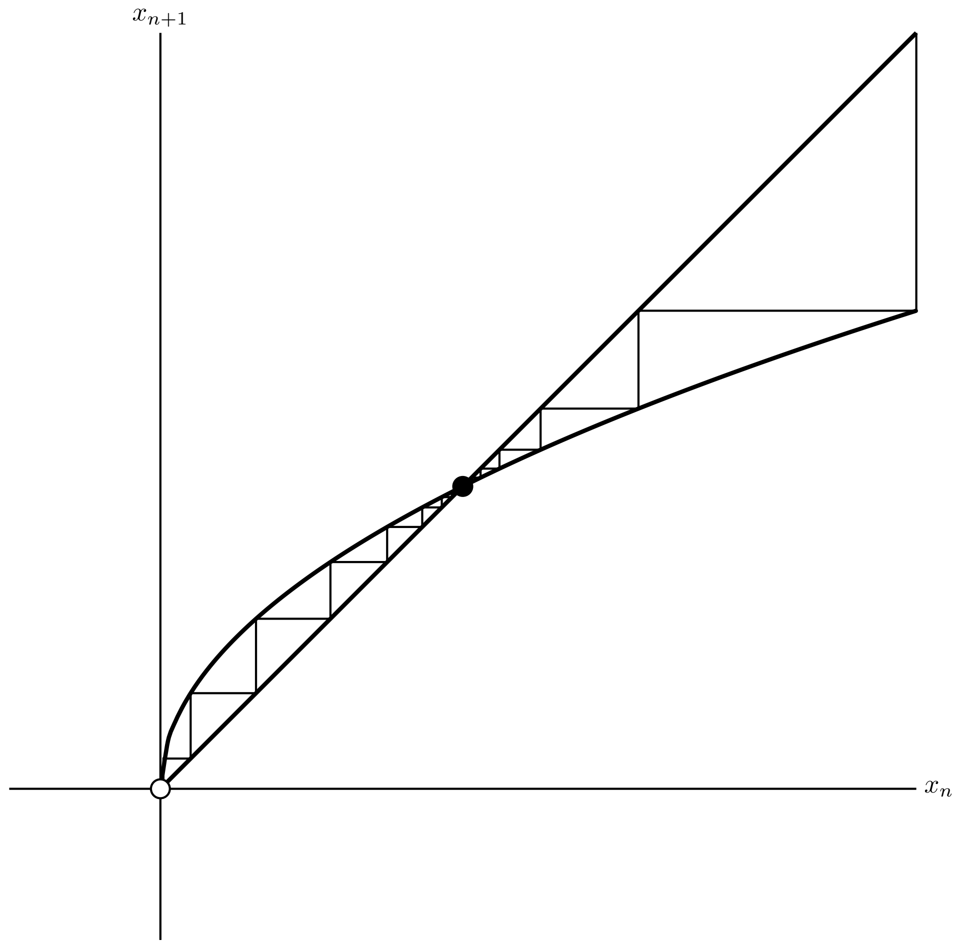
So the fixed points are $x^* = 0, 1$.

As for stability,

$$f(x) = \sqrt{x}$$
$$f'(x) = \frac{1}{2\sqrt{x}}$$
$$|f'(0)| = \infty \Rightarrow \text{ unstable}$$
$$f'(1) = \frac{1}{2} \Rightarrow \text{ stable}$$

Here is some numerical and graphical confirmation as well.

x_0	0	0.5	1	2
x_1	0	0.707107	1	1.41421
x_2	0	0.840896	1	1.18921
x_3	0	0.917004	1	1.09051
x_4	0	0.957603	1	1.04427
x_5	0	0.978572	1	1.0219
x_6	0	0.989228	1	1.01089
x_7	0	0.994599	1	1.00543
x_8	0	0.997296	1	1.00271
x_9	0	0.998647	1	1.00135



10.1.3

$$x_{n+1} = \exp(x_n)$$

The fixed points occur when

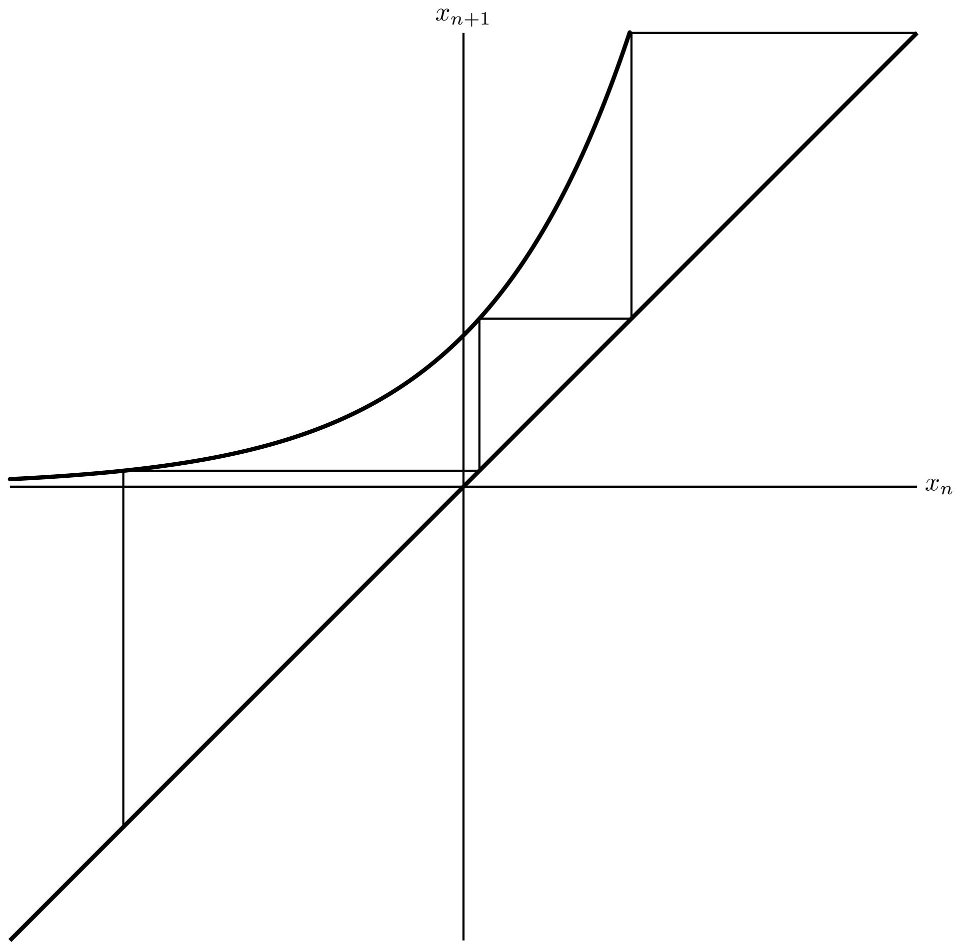
$$x = e^x$$

and there are none.

As for stability, there are no fixed points, but the sequence should always be increasing.

Here is some numerical and graphical confirmation as well.

x_0	-3
x_1	0.0497871
x_2	1.05105
x_3	2.86065
x_4	17.4728



10.1.5

$$x_{n+1} = \cot(x_n)$$

The fixed points occur when

$$x = \cot(x)$$

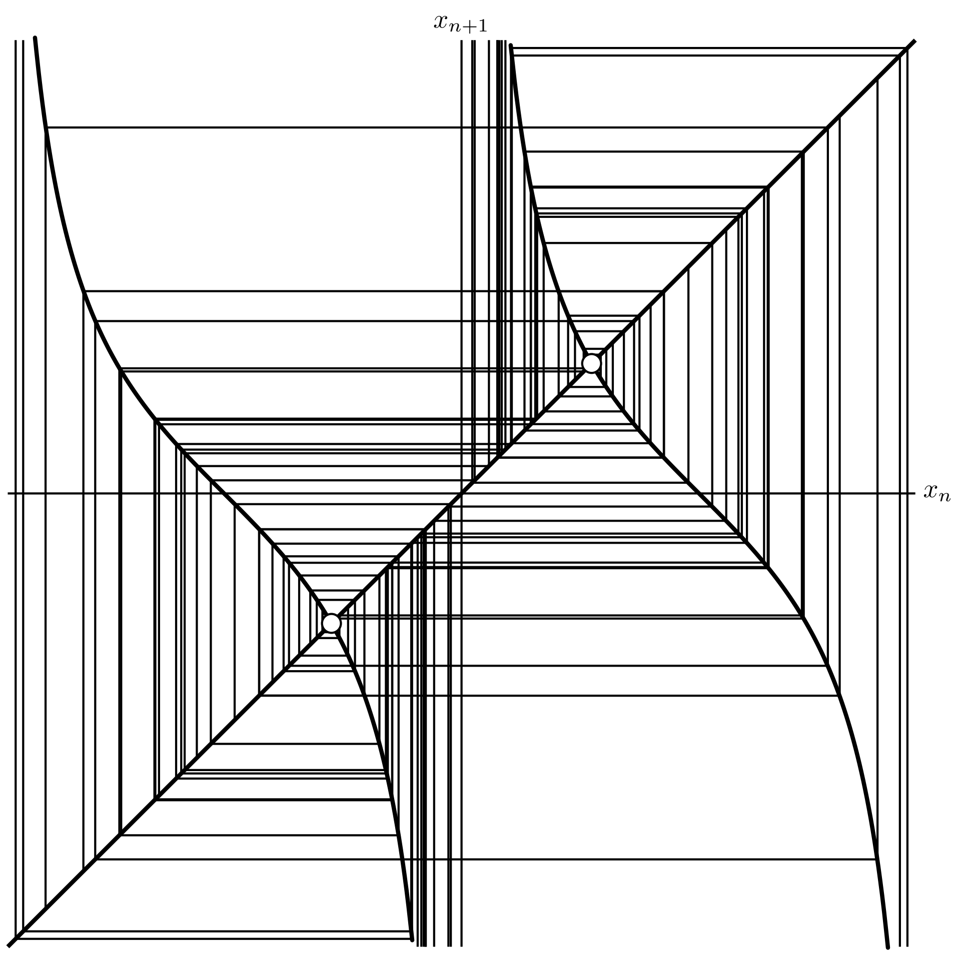
So the fixed points are $x^* = -1, 1$.

As for stability,

$$f(x) = \cot(x)$$
$$f'(x) = -\csc^2(x)$$
$$f'(1) \approx -1.41 \Rightarrow \text{ unstable}$$
$$f'(-1) \approx -1.41 \Rightarrow \text{ unstable}$$

Here is some numerical and graphical confirmation as well.

x_0	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
x_1	0.45766	-0.07091	-0.64209	-1.83049	?	1.83049	0.64209	0.07091	-0.45766
x_2	2.03031	-14.07777	-1.33725	0.26569		-0.26569	1.33725	14.07777	-2.03031
x_3	-0.49485		-0.23788	3.67479		-3.67479	0.23788		0.49485
x_4	-1.85312		-4.12414	1.69430		-1.69430	4.12414		1.85312
x_5	0.29007		-0.66703	-0.12414		0.12414	0.66703		-0.29007
x_6	3.35023		-1.26996	-8.01428		8.01428	1.26996		-3.35023
x_7	4.72318		-0.31026				0.31026		-4.72318
x_8	-0.01079		-3.11906				3.11906		0.01079
x_9	-92.66893		44.37344				-44.37344		92.66893



10.1.7

$$x_{n+1} = \sinh(x_n)$$

The fixed points occur when

$$x = \sinh(x)$$

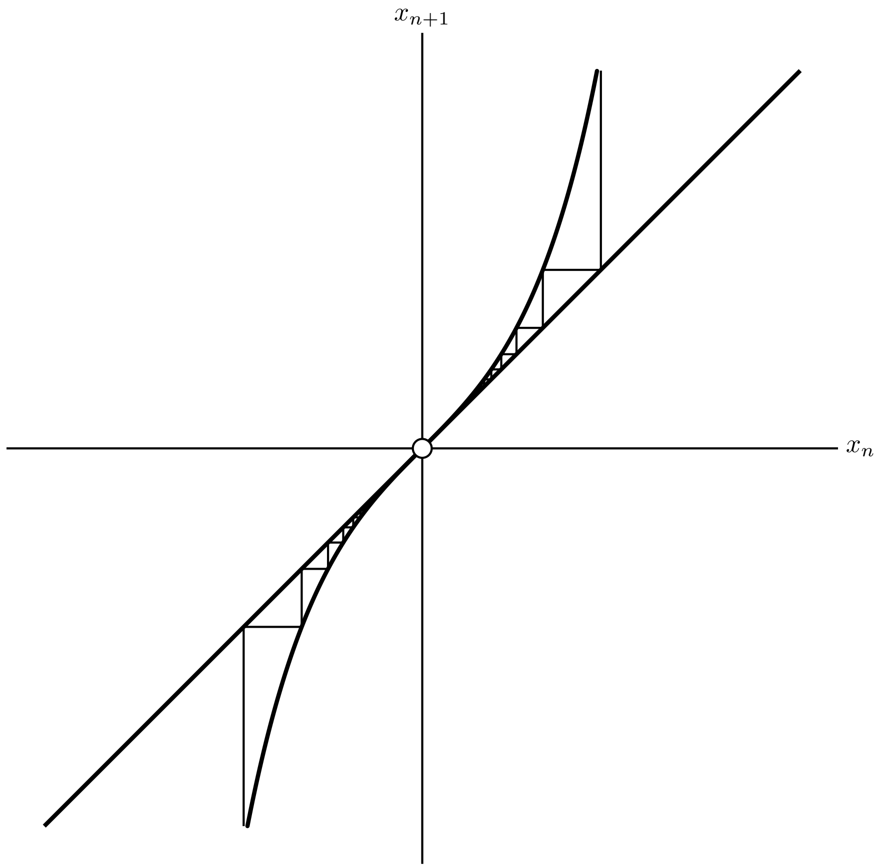
So the fixed point is $x^* = 0$.

As for stability,

$$f(x) = \sinh(x)$$
$$f'(x) = \cosh(x)$$
$$f'(0) = 1 \Rightarrow \text{ inconclusive, but the graph shows it's unstable}$$

Here is some numerical and graphical confirmation as well.

x_0	-0.50000	0.50000
x_1	-0.52110	0.52110
x_2	-0.54500	0.54500
x_3	-0.57238	0.57238
x_4	-0.60415	0.60415
x_5	-0.64158	0.64158
x_6	-0.68651	0.68651
x_7	-0.74173	0.74173
x_8	-0.81163	0.81163
x_9	-0.90372	0.90372
x_{10}	-1.03186	1.03186
x_{11}	-1.22497	1.22497
x_{12}	-1.55515	1.55515
x_{13}	-2.26231	2.26231
x_{14}	-4.75059	4.75059
x_{15}	-57.82220	57.82220



10.1.9

$$x_{n+1} = \frac{2x_n}{1+x_n}$$

The fixed points occur when

$$x = \frac{2x}{1+x}$$

$$x(1+x) = 2x$$

$$x(x-1) = 0$$

So the fixed points are $x^* = 0, 1$.

As for stability,

$$f(x) = \frac{2x}{1+x}$$

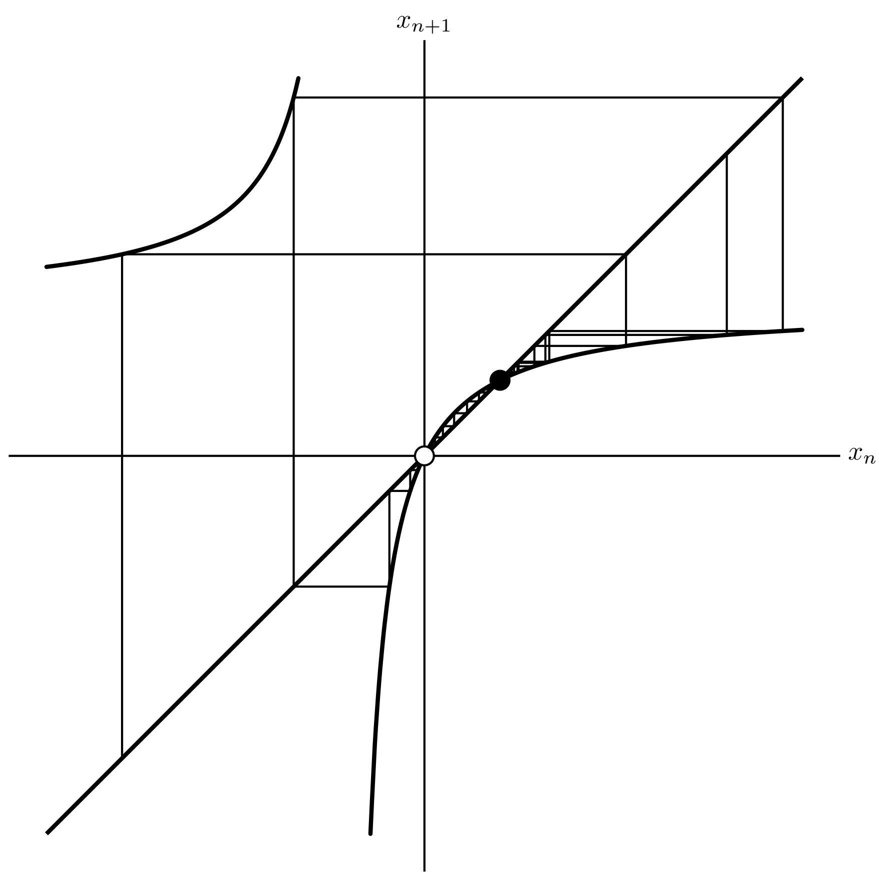
$$f'(x) = \frac{2(1-x) - 2x(-1)}{(1+x)^2} = \frac{2}{(1+x)^2}$$

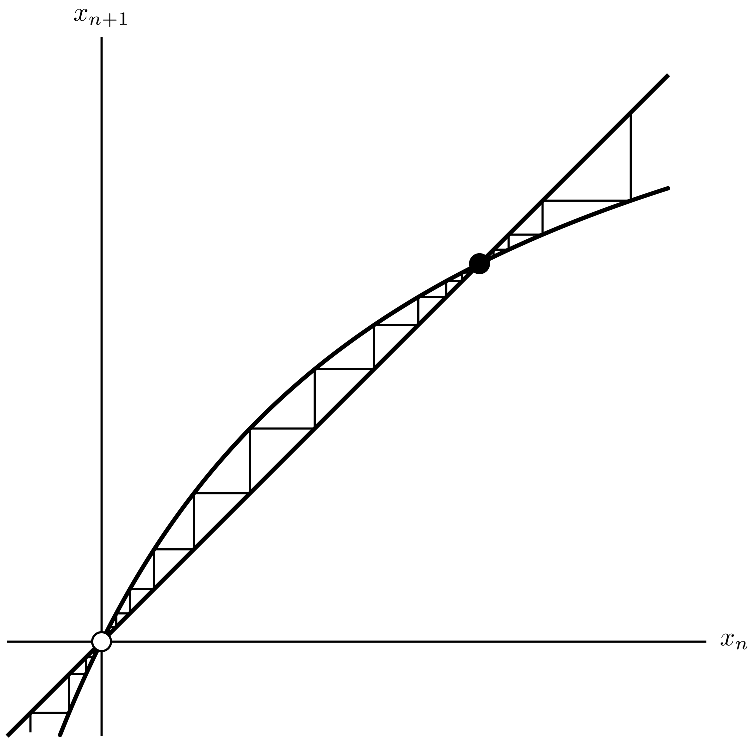
$$f'(0) = 2 \Rightarrow \text{ unstable}$$

$$f'(1) = \frac{1}{2} \Rightarrow \text{ stable}$$

Here is some numerical and graphical confirmation as well.

x_0	-4	-0.01	0.01	1.5	4
x_1	2.666667	-0.020202	0.019802	1.200000	1.600000
x_2	1.454545	-0.041237	0.038835	1.090909	1.230769
x_3	1.185185	-0.086022	0.074766	1.043478	1.103448
x_4	1.084746	-0.188235	0.139130	1.021277	1.049180
x_5	1.040650	-0.463768	0.244275	1.010526	1.024000
x_6	1.019920	-1.729730	0.392638	1.005236	1.011858
x_7	1.009862	4.740741	0.563877	1.002611	1.005894
x_8	1.004907	1.651613	0.721127	1.001304	1.002938
x_9	1.002447	1.245742	0.837971	1.000651	1.001467





10.1.11

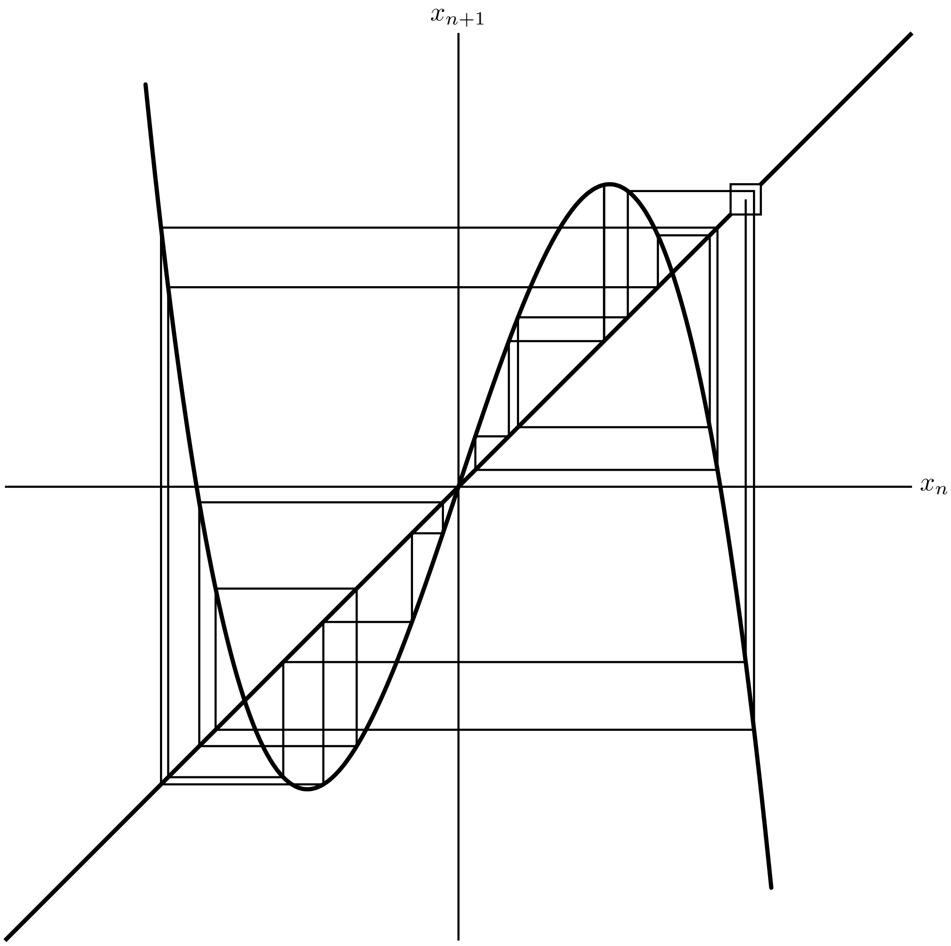
$$x_{n+1} = 3x_n - x_n^3$$

a)

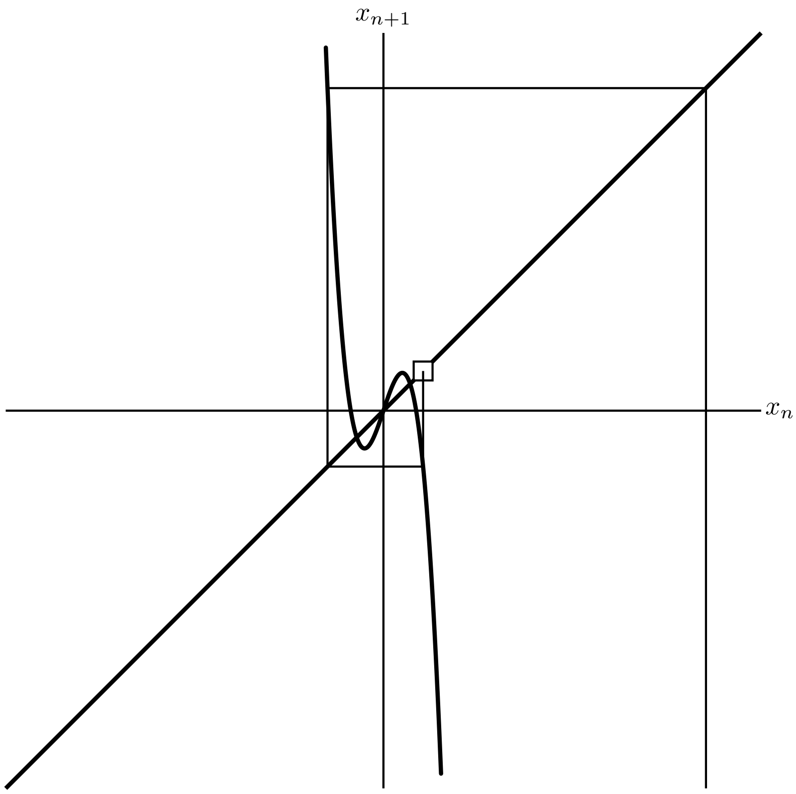
$$x = 3x - x^3 \Rightarrow 0 = 2x - x^3 = x(x + \sqrt{2})(x - \sqrt{2})$$

So the fixed points are $x = 0, \pm\sqrt{2}$.

b)

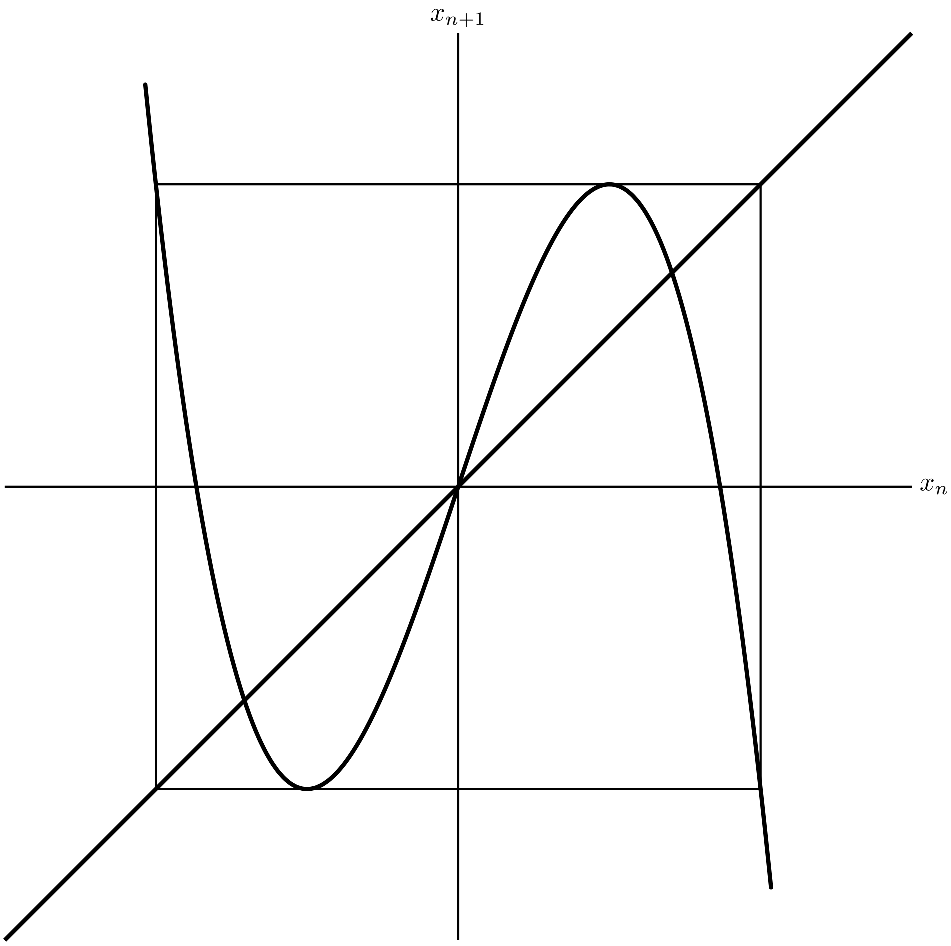


c)



d)

There exists a 2-cycle $x_n = \pm 2 \Rightarrow x_{n+1} = 3(\pm 2) - (\pm 2)^3 = \mp 2$ and $2 \to -2 \to 2 \to \cdots$, which is the reason for the change in behavior from $x_0 = 1.9$ to $x_0 = 2.1$.



If $-2 < x_n < 2$ then $-2 < x_{n+1} = 3x_n - x_n^3 < 2$ simply because the graph fits in the box made by the 2-cycle and thus every point maps to a point in the box.

As for starting outside the 2-cycle box,

$$|x_n| = 2 + \epsilon \quad \epsilon > 0$$

$$\begin{aligned}|x_1| &= |3x_0 - x_0^3| = |x_0||3 - x_0^2| = |x_0||3 - (2+\epsilon)^2| \\ &= |x_0||3 - (4 + 2\epsilon + \epsilon^2)| = |x_0|| - 1 - 2\epsilon - \epsilon^2)| \\ &= |x_0|(1 + 2\epsilon + \epsilon^2)\end{aligned}$$

$$|x_n| = |x_0|(1 + 2\epsilon + \epsilon^2)^n \Rightarrow \lim_{n\to\infty} |x_n| = \infty$$

So the iteration map will always be increasing in magnitude and spiraling away from the 2-cycle.

10.1.13

$$x_{n+1} = f(x_n) = x_n - \frac{g(x_n)}{g'(x_n)}$$
$$f'(x) = 1 - \frac{g'(x)}{g'(x)} + \frac{g(x)g''(x)}{(g'(x))^2}$$

If $g'(x^*) \neq 0$, then along with $g(x^*) = 0$,

$$f'(x) = 1 - 1 + 0\frac{g''(x)}{(g'(x))^2} = 0$$

is implied. The multiplier is zero and x^* is a superstable fixed point.

10.2 Logistic Map: Numerics

10.2.1

a)

$$\begin{aligned}
&1 < x_n \\
&x_{n+1} = rx_n(1 - x_n) < 0 \\
&x_{n+2} = rx_{n+1}(1 - x_{n+1}) < 0 \\
&\vdots \\
&x_\infty < 0
\end{aligned}$$

But that's not enough to show that the sequence diverges since it might have some convergent negative value.

$$\begin{aligned}
0 &< \epsilon \\
x_n &= 1 + \epsilon \\
x_{n+1} &= r(1+\epsilon)(1 - (1+\epsilon)) = -r\epsilon(1+\epsilon) \\
x_{n+1} - x_n &= -r\epsilon(1+\epsilon) - (1+\epsilon) = -(r+\epsilon)(1+\epsilon) < -r
\end{aligned}$$

So the sequence decreases by at least r every iteration. Since $r \neq 0$,

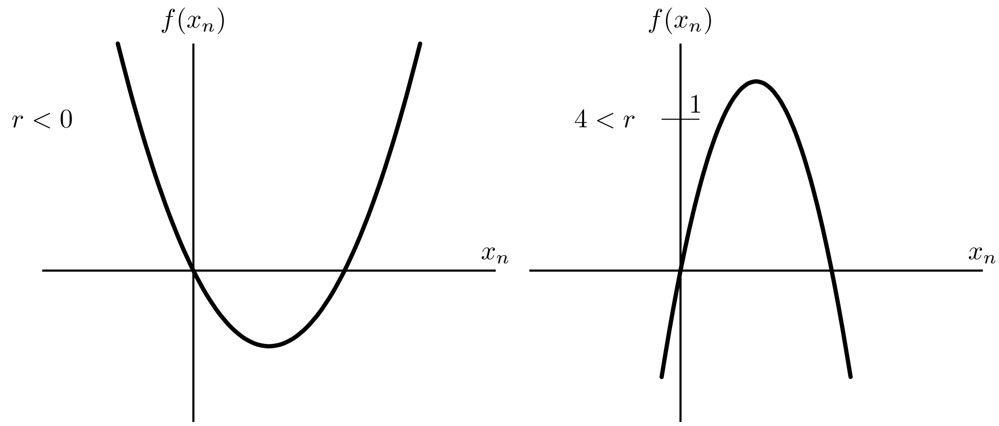
$$\lim_{n\to\infty} x_n \to -\infty$$

b)

It's a good idea to have $x_0 \in [0,1]$, otherwise $x_1 < 0$ and the sequence would diverge.

$r \in [0,4]$ is necessary to ensure that the sequence is trapped within [0,1] if the sequence starts in that interval. $r < 0$ would have the next iterate map to a negative value; and $4 < r$ would have the next iterate able to map outside [0,1] around the maximum of the parabola, but

$$x_0 \in [0,1] \text{ and } r \in [0,4] \Rightarrow rx_n(1-x_n) \in [0,1]$$



10.2.3, 10.2.5, and 10.2.7

Good black-and-white orbit diagrams of Exercises 10.2.3, 10.2.5, and 10.2.7 can be made with the following pseudocode.

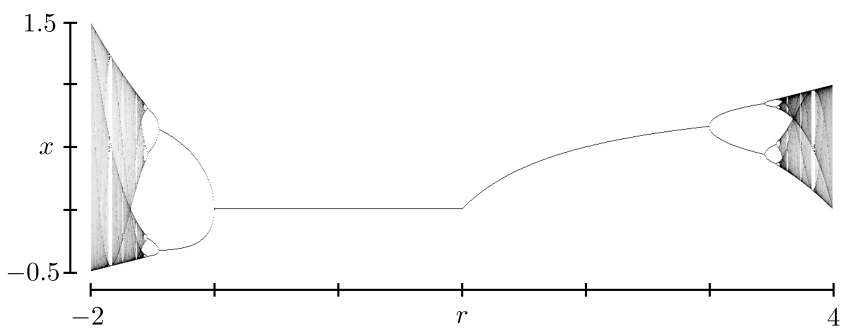
```
begin function orbit_diagrammer()
#create the bounds of the orbit diagram
  r_min = ???
  r_max = ???
  x_min = ???
  x_max = ???
#what is the r sample resolution
  r_step = ???
#how many sample points per r value
  x_samples = ???

#loops through all sampled r values
  for r from r_min to r_max by increment r_step
#initial x value to start iterations
    x = random number between x_min and x_max
#loop to pass through transient behavior until the orbits are reached
```

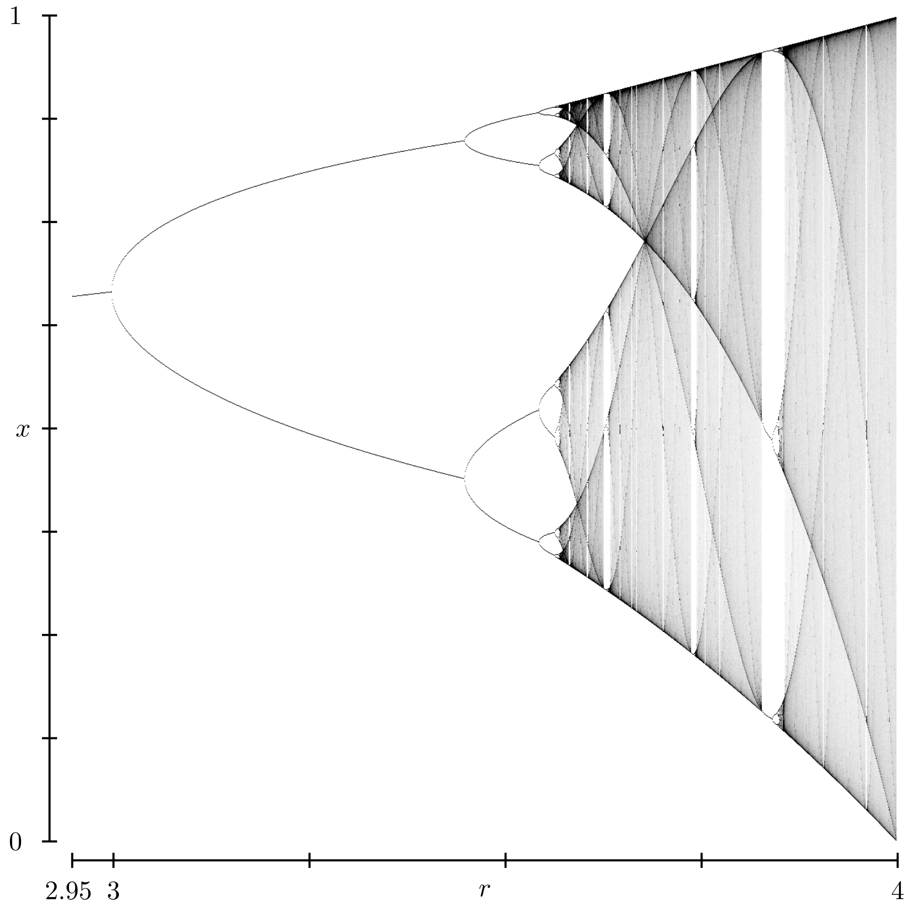
```
    for n = 1 to 1000
#the iteration map
      x = f(r,x)
    end for
#loop to calculate orbits
    for n = 1 to x_samples
      x = f(r,x)
      draw (r,x) coordinate
    end for
  end for
end function orbit_diagrammer()
```

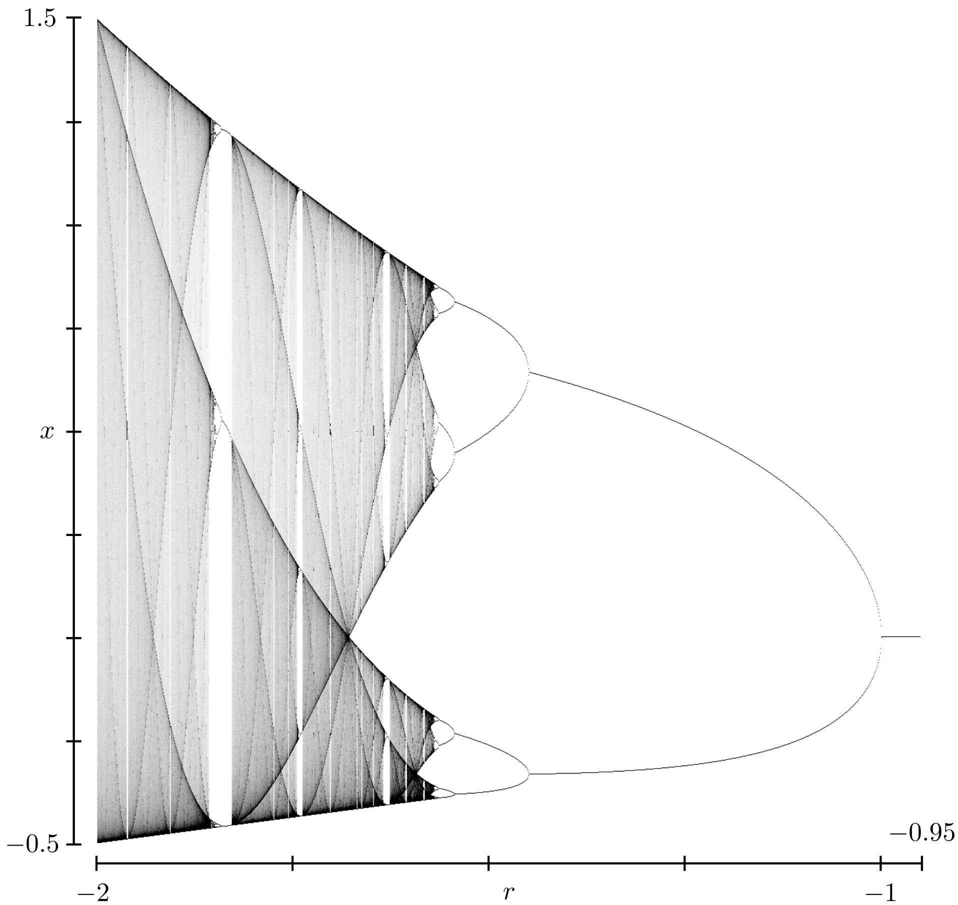
However, the following orbit diagrams use gray scale to show how often each pixel region was visited in the simulations.

10.2.3

$$x_{n+1} = rx_n(1 - x_n)$$

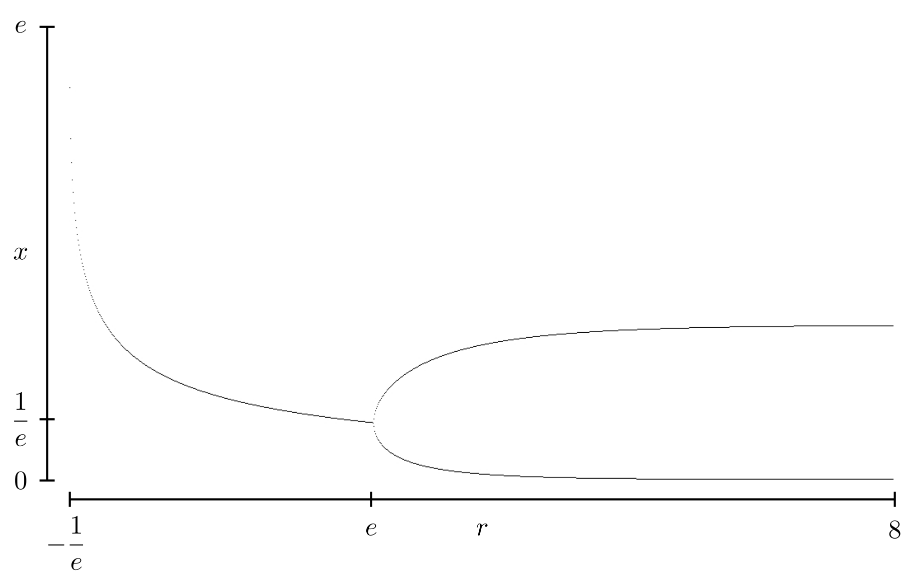



1
x
0
2.95
3
r
4



10.2.5

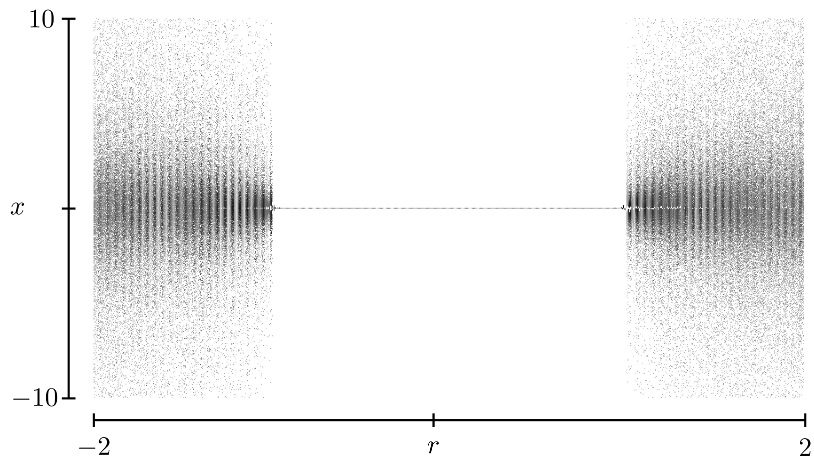
$$x_{n+1} = e^{-rx_n}$$



Look up the Lambert W function for those who are intrigued by all the e's.

10.2.7

$$x_{n+1} = r\tan(x_n)$$



10.3 Logistic Map: Analysis

10.3.1

$$
\begin{aligned}
x = f(x) &= rx(1-x) = rx - rx^2 \\
rx^2 + (1-r)x &= x(rx + 1 - r) = 0 \Rightarrow x^* = 0, \frac{r-1}{r} \\
f'(x) &= r(1-2x) \\
f'(0) &= r = 0 \\
&\quad x^* = 0 \text{ is a superstable fixed point when } r = 0 \\
f'\left(\frac{r-1}{r}\right) &= r\left(1 - 2\frac{r-1}{r}\right) = 0 \Rightarrow r = 0, 2 \\
&\quad x^* = \frac{r-1}{r} \text{ is a superstable fixed point when } r = 2
\end{aligned}
$$

$x^* = 0$ is superstable if $r = 0$ because $x_1 = 0$ no matter what x_0 is. The other fixed point is superstable when $r = 2$ but not when $r = 0$ because $x_1 = 0$ as before.

10.3.3

$$x_{n+1} = \frac{rx_n}{1 + x_n^2}$$

The fixed points occur when

$$\begin{aligned}
&x = \frac{rx}{1+x^2} \\
&x(1+x^2) = rx \\
&x(1+x^2) - rx = x(1-r+x^2) = 0 \\
&x = 0, \pm\sqrt{1-r}
\end{aligned}$$

So there are three fixed points if $r \leq 1$ and one fixed point if $1 < r$.

$$\begin{aligned}
&f(x) = \frac{rx}{1+x^2} \\
&f'(x) = \frac{r(1+x^2) - 2rx^2}{(1+x^2)^2} = \frac{r(1-x^2)}{(1+x^2)^2} \\
&f'(0) = r \qquad f'(\pm\sqrt{1-r}) = \frac{2}{r} - 1 \\
&r < 1 \Rightarrow x^* = 0 \text{ is stable, and } x^* = \pm\sqrt{1-r} \text{ are unstable} \\
&r = 1 \Rightarrow f'(0) = 1 \text{ is inconclusive} \\
&\qquad |x_{n+1}| = \left|\frac{x_n}{1+x_n^2}\right| < \left|\frac{x_n}{1}\right| = |x_n| \Rightarrow x^* = 0 \text{ is stable} \\
&1 < r \Rightarrow x^* = 0 \text{ is stable}
\end{aligned}$$

As for the existence of periodic orbits, we already proved that the sequence $|x_n|$ is monotonically decreasing when $r \leq 1$ and $x_n \neq 0$, which rules out any periodic orbits.

It turns out that there are no periodic orbits for $1 < r$ either, and never any chaos. We can prove this by showing that the sequence $|x_n|$ is monotonically increasing, monotonically decreasing, or constant for any positive r value.

$$\begin{aligned}
&|x| < \left|\frac{rx}{1+x^2}\right| = \frac{r|x|}{1+x^2} \\
&|x|(1+x^2) < r|x| \\
&|x|(1-r+x^2) < 0 \Rightarrow x^2 < r-1 \Rightarrow |x| < \sqrt{r-1}
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{rx}{1+x^2}\right| = \frac{r|x|}{1+x^2} < |x| \\
&r|x| < |x|(1+x^2) \\
&0 < |x|(1-r+x^2) \Rightarrow r-1 < x^2 \Rightarrow \sqrt{r-1} < |x|
\end{aligned}$$

So the last bunch of equations shows that the sequence $|x_n|$ is monotonically increasing or monotonically decreasing, and constant only if the sequence starts at a fixed point. Essentially the iterations always travel away from the origin, always travel towards the origin, or stay at a fixed point.

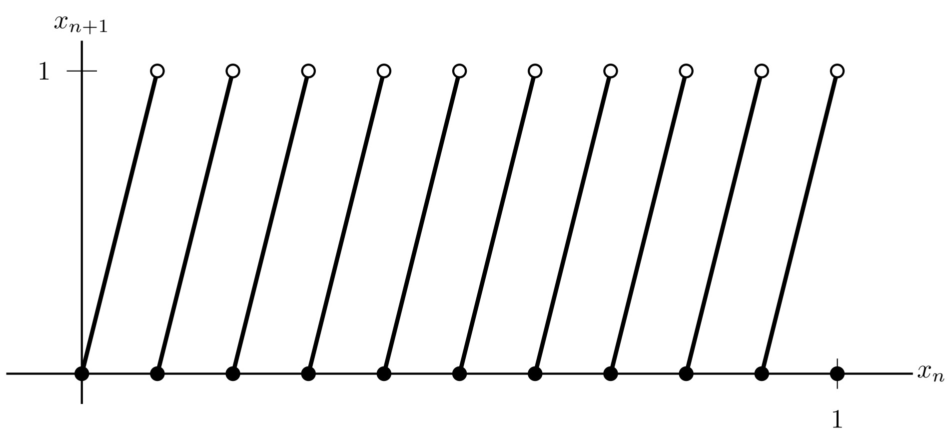
10.3.5

$$x_{n+1} = rx_n(1 - x_n) \quad y_{n+1} = y_n^2 + c \qquad x_n = ay_n + b$$
$$x_{n+1} = ay_{n+1} + b = a(y_n^2 + c) + b$$
$$x_{n+1} = r(ay_n + b)(1 - ay_n - b)$$
$$a(y_n^2 + c) + b = r(ay_n + b)(1 - ay_n - b)$$
$$ay_n^2 + ac + b = -a^2ry_n^2 + ar(1 - 2b)y_n + rb(1 - b)$$
$$a = -\frac{1}{r} \quad b = \frac{1}{2} \quad c = \frac{r(2 - r)}{4}$$

10.3.7

$$x_{n+1} = 10x_n \pmod 1$$

a)



b)

$x = 0, 0.\bar{1}, 0.\bar{2}, 0.\bar{3}, 0.\bar{4}, 0.\bar{5}, 0.\bar{6}, 0.\bar{7}$, and $0.\bar{8}$ are the fixed points. It looks like we're missing $x = 0.\bar{9}$, but that's actually a fancy way to write $x = 1$, which will get mapped to $x = 0$ after the first iteration.

c)

Any $x \in I$ with a p long repeating decimal expansion is a p-cycle of the map.

To find the stability of the p-cycles, we take the derivative of

$$\frac{\mathrm{d}}{\mathrm{d}x} f^p(x) = \frac{\mathrm{d}}{\mathrm{d}x} 10^p x \pmod 1 = 10^p > 1$$

so they're all unstable.

d)

Any orbit starting at an irrational number x_0 will be aperiodic, since the decimal expansion of an irrational number never repeats. There are an infinite number of irrational numbers in [0,1], so there are an infinite number of aperiodic orbits.

e)

The definition of the Liapunov exponent is

$$\lambda = \lim_{n\to\infty} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| f'(x_i) \right| \right\} = \lim_{n\to\infty} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} \ln(10) \right\} = \ln(10) > 0$$

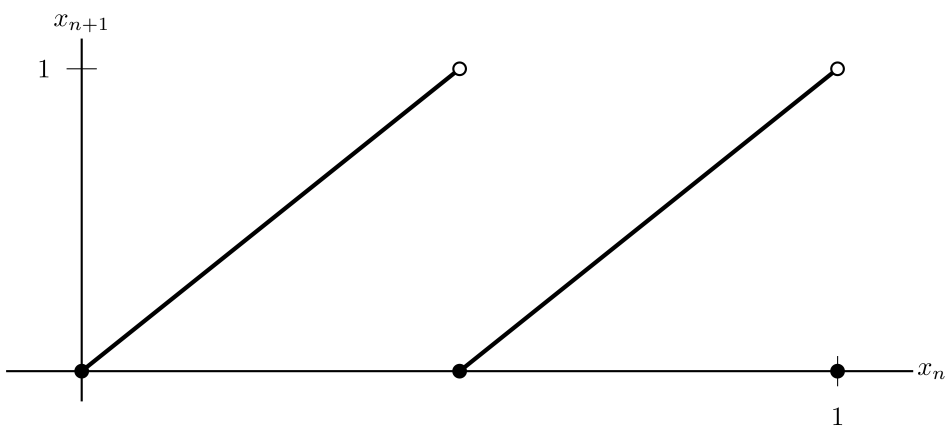
The Liapunov exponent is positive, so the decimal shift map has sensitive dependence on initial conditions.

10.3.9

$$x_{n+1} = 2x_n \pmod 1 \text{ in decimal } \equiv 10x_n \pmod 1 \text{ in binary}$$

We'll be using binary representation in this problem, as the binary shift map applies easily to binary numbers.

a)



b)

$x = 0$ is the only fixed point. It looks like $x = 0.\bar{1}$ is also a fixed point, but that's actually a fancy way to write $x = 1$, which will get mapped to $x = 0$ after the first iteration.

c)

Any $x \in I$ with a p long repeating decimal expansion is a p-cycle of the map.

To find the stability of the p-cycles, we take the derivative of

$$\frac{\mathrm{d}}{\mathrm{d}x} f^p(x) = \frac{\mathrm{d}}{\mathrm{d}x} 10^p x \pmod 1 = 10^p > 1 \qquad 2^p > 1 \text{ in decimal}$$

so they're all unstable.

d)

Any orbit starting at an irrational number x_0 will be aperiodic, since the binary expansion of an irrational number never repeats. There are an infinite number of irrational numbers in [0,1], so there are an infinite number of aperiodic orbits.

e)

The definition of the Liapunov exponent is

$$\lambda = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{i=0}^{n-1}\ln\left|f'(x_i)\right|\right\} = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{i=0}^{n-1}\ln(10)\right\} = \ln(10) > 0$$

$$\ln(2) > 0 \text{ in decimal}$$

The Liapunov exponent is positive, so the decimal shift map has sensitive dependence on initial conditions.

This one is a bit of a trick. Start the construction as zero. Then append all the number sequences of length one. Then append all number sequences of length two, then of length three, then four, etc.

$$x = 0$$

$$0, 1$$

$$x = 0.01$$

$$00, 01, 10, 11$$

$$x = 0.0100011011$$

$$x = 0.0100011011000001010011100101110111$$

$$\vdots$$

This number x contains all numbers in [0,1] with terminating binary expansions. Therefore this number will be arbitrarily close to any particular number in [0,1] within a finite number of iterations of the binary shift map. We need only find a terminating binary number within ϵ distance of the number of interest and apply the binary shift map to x until the terminating binary number is at the front, and then a little bit of error attached to the end.

Note: It is necessary to append enough zeros onto the terminating binary representation in order to achieve the ϵ tolerance. For example, if we want to approximate 0.1101 within 0.0001 tolerance, then a suitable number would be $0.1101\ldots$ and we could iterate the map on x until 0.1101 is at the front. But if we want a tolerance of 0.0000001, a suitable number would be $0.1101000\ldots$ and we could iterate the map on x until 0.1101000 is at the front.

10.3.11

$$x_{n+1} = f(x_n) = -(1+r)x_n - x_n^2 - 2x_n^3$$

a)

$$f'(x) = -(1+r) - 2x - 6x^2$$
$$f'(0) = -(1+r)$$
$$-1 < -(1+r) < 1 \Rightarrow 0 < r < 2$$

$$0 < r < 2 \Rightarrow \text{stable} \qquad r < 0 \text{ or } 2 < r \Rightarrow \text{unstable}$$

b)

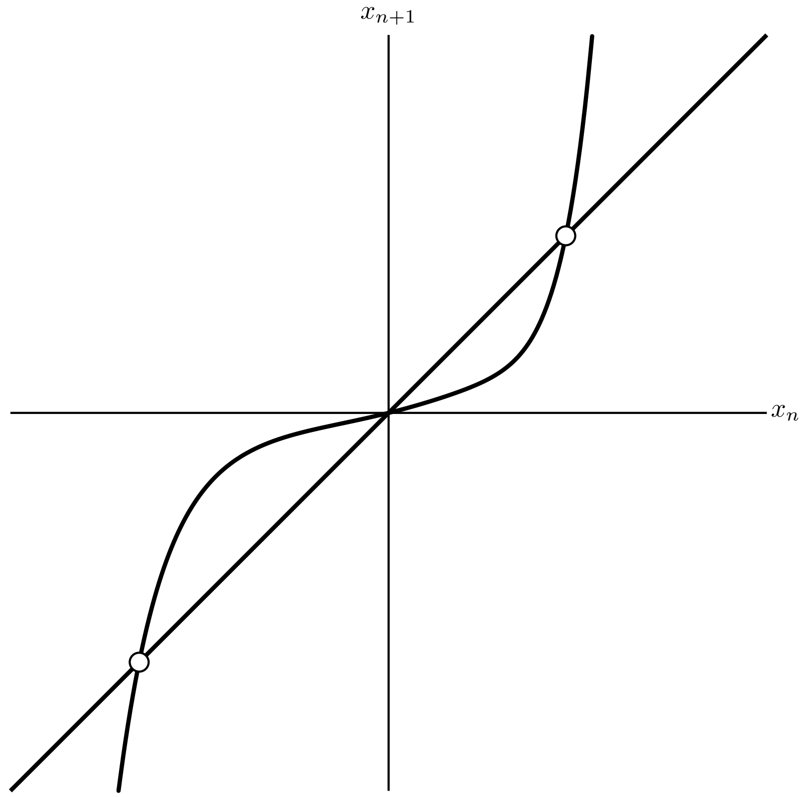
A flip bifurcation will occur at $x = 0$ and $r = 0$ if the stability changes on either side of the r value.

$$f(0) = 0$$
$$r < 0 \Rightarrow f'(0) = -(1+r) < -1 \Rightarrow \text{unstable}$$
$$r = 0 \Rightarrow f'(0) = -(1+r) = -1 \Rightarrow \text{inconclusive}$$
$$0 < r \Rightarrow f'(0) = -(1+r) > -1 \Rightarrow \text{stable}$$

Hence at $x = 0$ and $r = 0$ a flip bifurcation occurs.

c)

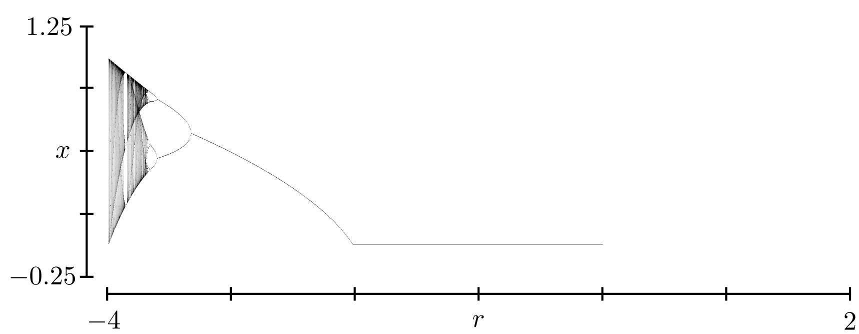
The following graph shows the plot at $r = -0.5$.

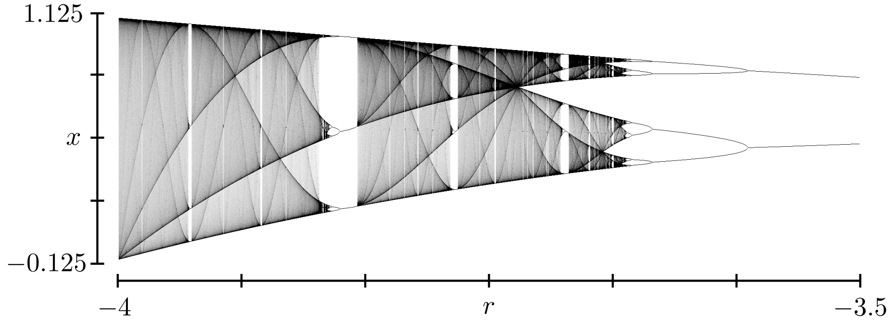


The origin is a fixed point of the original map, but the other two intersections are the 2-cycle points. The slope of the 2-cycle intersections with the diagonal have a magnitude greater than one, meaning the 2-cycle is unstable. Trajectories will be drawn towards the origin or repelled to infinity if the initial conditions start between the 2-cycle points or the outside respectively. Also, increasing r towards 0 will move the 2-cycle points closer to the origin until all three coalesce.

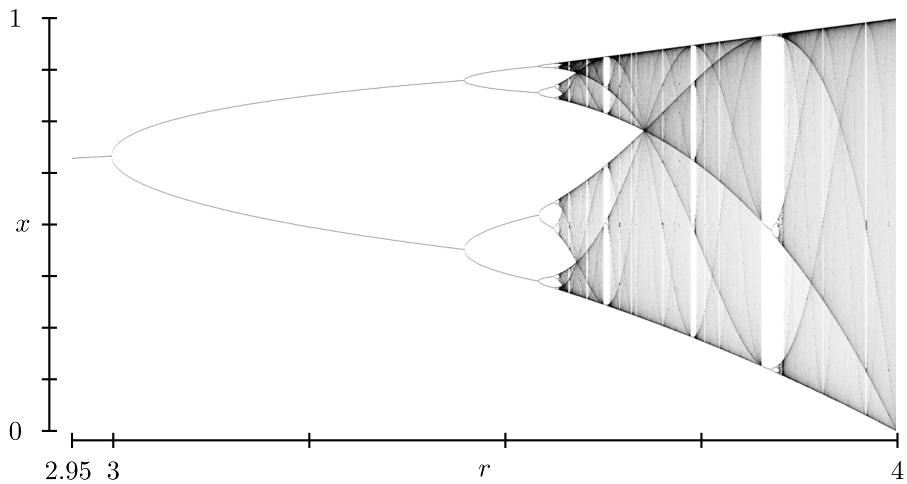
d)

The long-term behavior for orbits that start near $x^* = 0$ diverges for $r > 0$. For $-2 < r < 0$, the origin $x^* = 0$ is stable, but then there is a series of period-doubling bifurcations leading to chaos for $r < -2$, and the periodic orbits and chaos disappear shortly before $r = -4$.



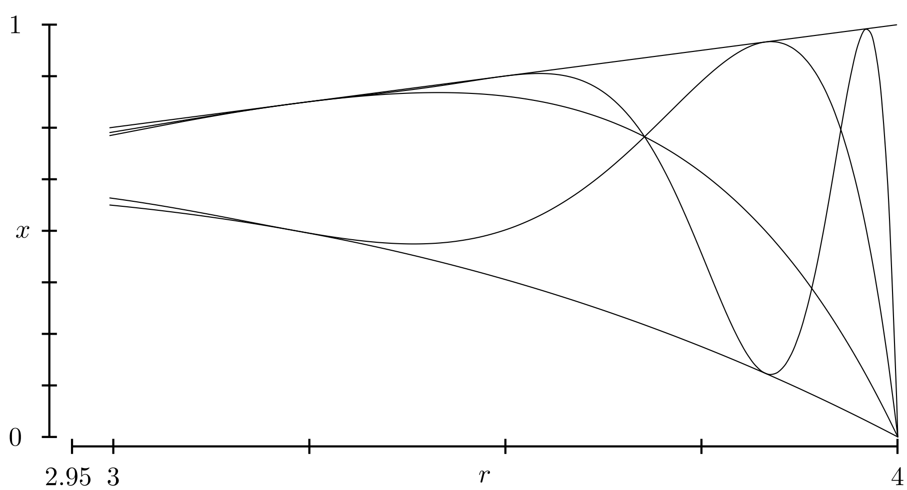


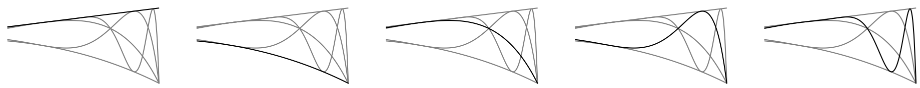
10.3.13



a)

The curves being referred to are $\left(r, f^k\left(\frac{1}{2}, r\right)\right)$, a graph of which appears below for $k = 1, 2, 3, 4, 5$. (Keep in mind that the curves are extended backwards to r values that don't correspond to periodic orbits.)





The reason these curves are darker in the high-resolution orbit diagram is that the slope of the map is zero when $x = \frac{1}{2}$ for all the iterated maps.

$$\frac{\mathrm{d}}{\mathrm{d}x} f^k(x) = f'\big(f^{k-1}(x)\big) f'\big(f^{k-2}(x)\big) f'\big(f^{k-3}(x)\big) \cdots f'\big(f(x)\big) f'(x)$$
$$f'\left(\frac{1}{2}\right) = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x} f^k \left(\frac{1}{2}\right) = 0$$

So all the points near $x = \frac{1}{2}$ are mapped to approximately the same value after an iteration of the map since the graph is flat there. Hence those regions are darker because many iterations are required to escape the region once a trajectory passes nearby.

b)

The corner of the "big wedge" occurs when $f^3\left(\frac{1}{2}, r\right) = f^4\left(\frac{1}{2}, r\right) = f^5\left(\frac{1}{2}, r\right)$, using the hint and the graphs of part (a). However, the first equality is enough to solve for the big wedge r value. Now being a bit clever,

$$u = f^3\left(\frac{1}{2}, r\right) = f^4\left(\frac{1}{2}, r\right) = f\left(f^3\left(\frac{1}{2}, r\right), r\right) = f\left(u, r\right)$$

The r value at the big wedge must be a fixed point of the original function, which we've called u. There are two fixed points of the logistic map, 0 and $1 - \frac{1}{r}$, and the vertical coordinate definitely isn't 0, so $u = 1 - \frac{1}{r}$. This means

$$f^3\left(\frac{1}{2}, r\right) = 1 - \frac{1}{r}$$
$$1 - r + \frac{1}{4}r^4 - \frac{1}{16}r^5 - \frac{1}{16}r^6 + \frac{1}{32}r^7 - \frac{1}{256}r^8 = 0$$

Normally you would use a computer to estimate the roots and pick the one that fits, but luckily this has some nice roots.

$$(r-2)^4(r+2)(r^3 - 2r^2 - 4r - 8) = 0$$

from which the cubic formula yields our root of interest. (The other roots are too far off or imaginary.)

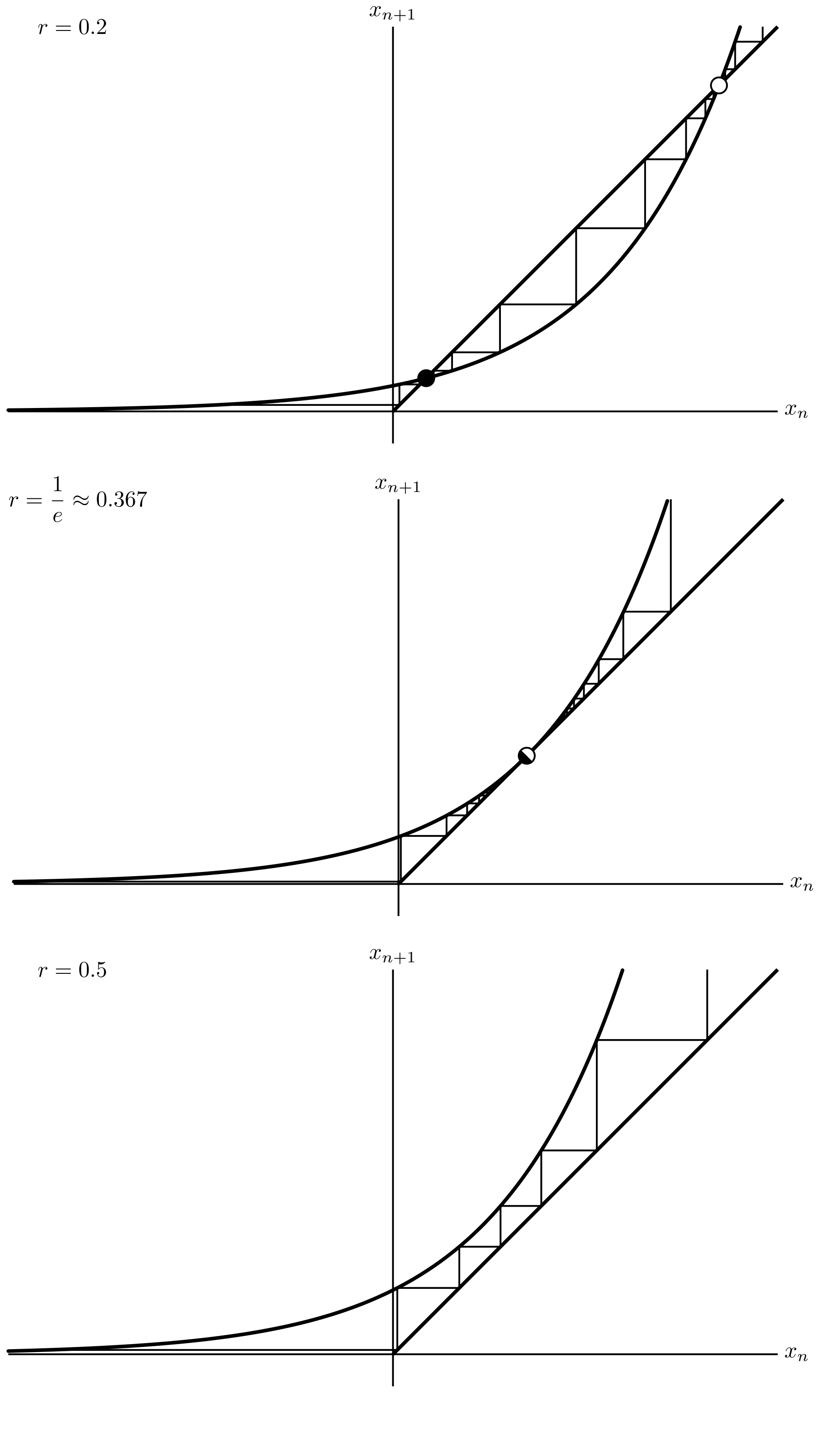
$$r = \frac{2}{3} + \frac{8}{3}\big(19 + \sqrt{297}\big)^{-\frac{1}{3}} + \frac{2}{3}\big(19 + \sqrt{297}\big)^{\frac{1}{3}} = 3.67857...$$

10.4 Periodic Windows

10.4.1

$$x_{n+1} = r\exp(x_n) \quad r > 0$$

a)

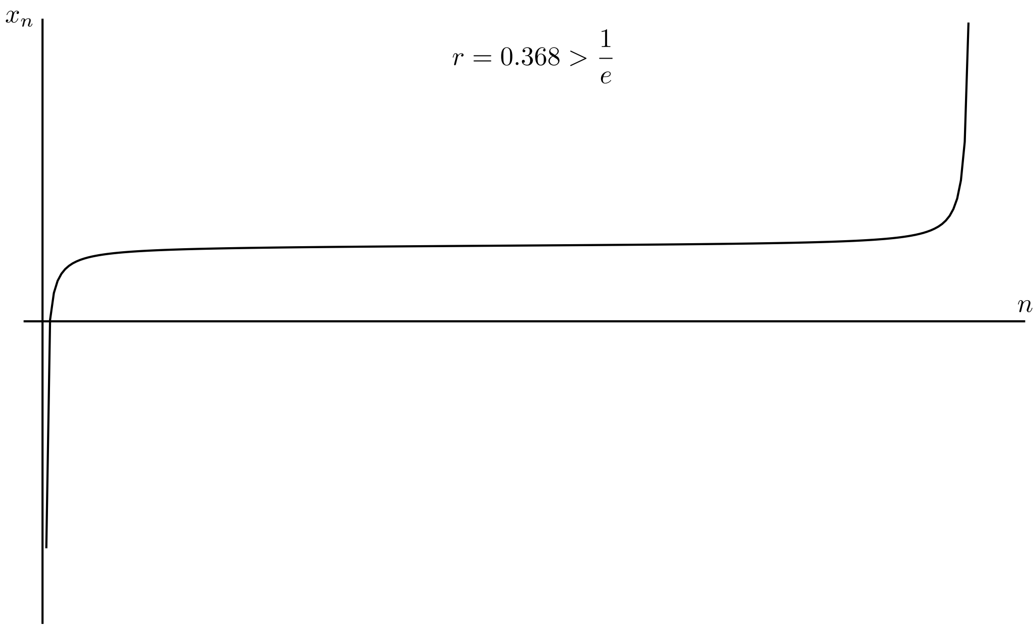


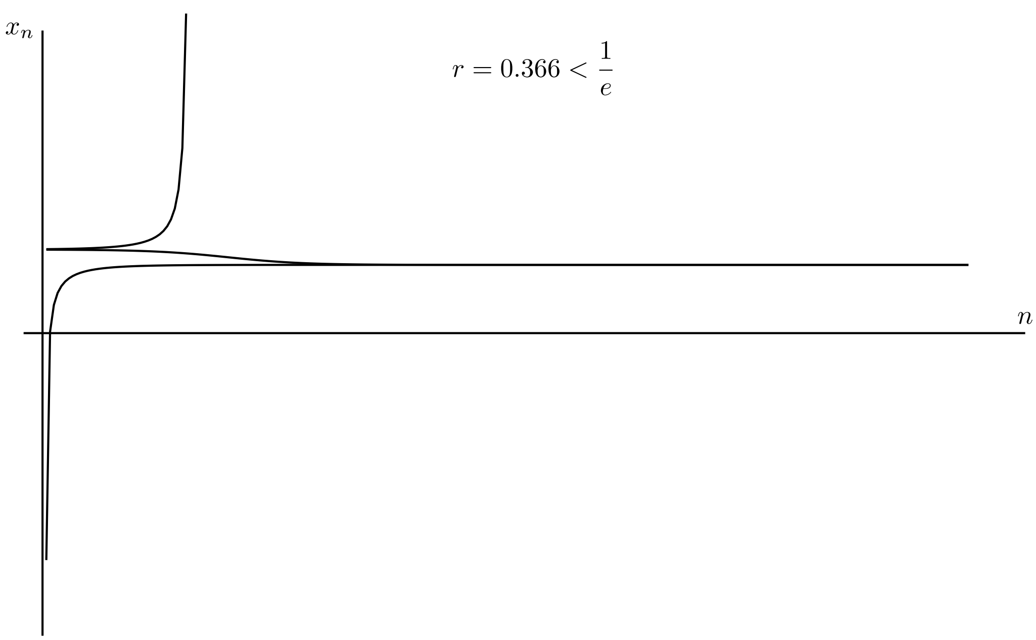
b)

A tangent bifurcation does occur at $r = \frac{1}{e}$ as is evident in the graphs. We can verify by confirming that the graph of the iteration function lies tangent to the diagonal line.

$$r = \frac{1}{e} \quad x = 1 \Rightarrow re^x = x \quad \frac{\mathrm{d}}{\mathrm{d}x} re^x = 1$$

c)





Here you can see the ghost of the tangent bifurcation. The first graph takes quite a while for the trajectory to iterate through, but the second graph still has the unstable and stable fixed points in existence.

Note: The graphs look continuous, but that really is an artifact of so many points near the ghost of the tangent bifurcation and connecting the dots.

10.4.3

A superstable 3-cycle

The iteration map

$$x_{n+1} = f(x_n) = 1 - rx_n^2$$

will have a superstable 3-cycle if

$$\frac{\mathrm{d}}{\mathrm{d}x} f(f(f(x))) = 0$$

and

$$x = f(f(f(x)))$$

for some x

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} f(f(f(x))) &= f'(f(f(x)))f'(f(x))f'(x) \\ &= (-2rf(f(x)))(-2rf(x))(-2rx) \\ &= -8r^3 f(f(x))f(x)x = 0 \\ &\Rightarrow x = 0 \text{ or } f(x) = 0 \text{ or } f(f(x)) = 0\end{aligned}$$

Now we need to check if any of these potential x values are 3-cycles, but we only need to check one of them because it's a cycle. The cycle, if it exists, will consist of the points

$$\ldots \to x_n \to x_{n+1} \to x_{n+2} \to x_{n+3} = x_n \to \ldots$$

where $f(f(x_n)) = 0$, $f(x_{n+1}) = 0$, and $x_{n+2} = 0$. So let's just pick the easiest equation to use, which is $f(f(x)) = 0$.

The x satisfying $f(f(x)) = 0$ should be a point of the 3-cycle, so

$$\begin{gathered} f(f(f(x))) = x \text{ and } f(f(x)) = 0 \\ \Rightarrow f(f(f(x))) = f(0) = 1 = x \end{gathered}$$

Now we plug $x = 1$ into $f(f(x)) = 0$ to derive an equation for r.

$$f(f(1)) = f(1 - r) = 1 - r(1 - r)^2 = 0$$

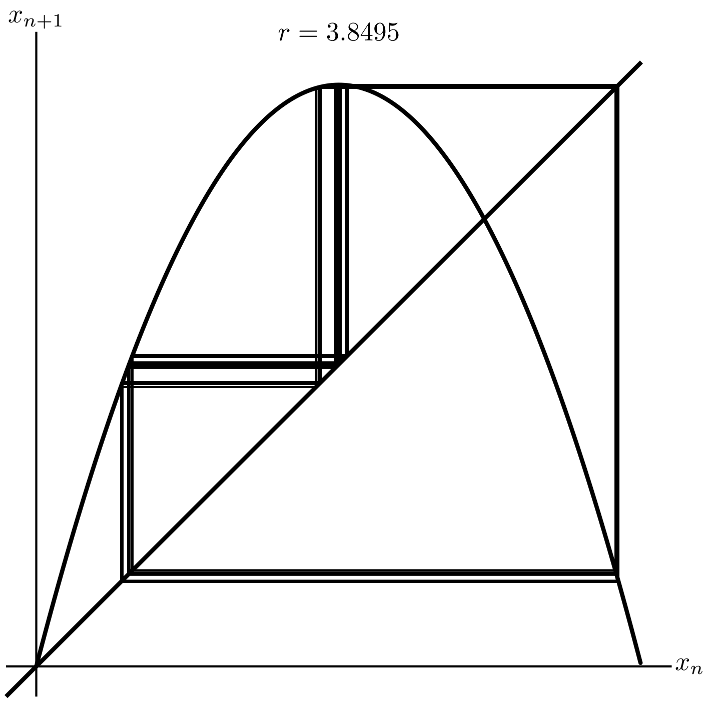
So as long as r satisfies

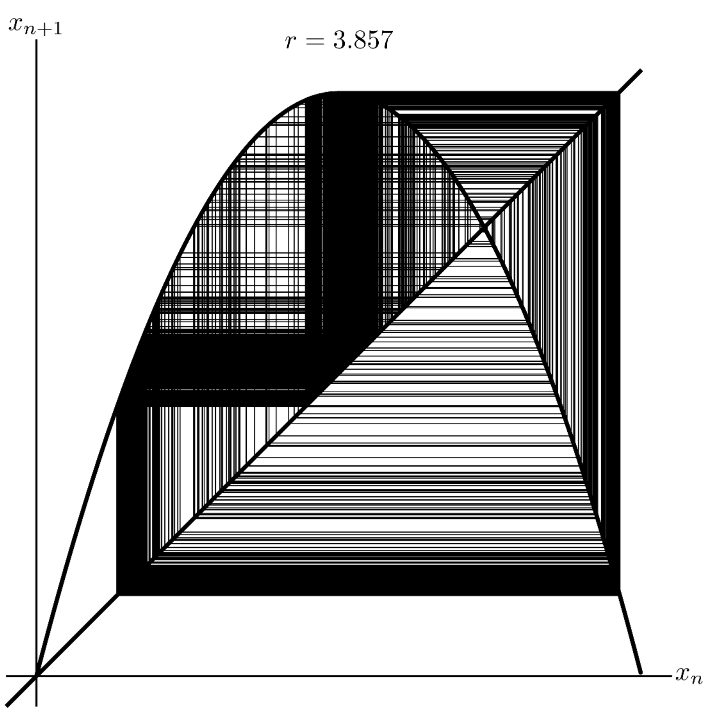
$$1 - r(1 - r)^2 = 0$$

there will exist a superstable 3-cycle or the iteration map.

10.4.5

A large sampling of initial values was iterated in the logistic map 300 times, but only the last 100 iterations were plotted in both cases. Clearly something happened after increasing r slightly.





10.4.7

a)

$$
\begin{aligned}
&x_0 = \frac{1}{2} \qquad r = 1 + \sqrt{5} \\
&x_1 = r x_0 (1 - x_0) = \frac{r}{4} = \frac{1 + \sqrt{5}}{4} \approx 0.8 \quad \text{R} \\
&x_2 = r x_1 (1 - x_1) = \frac{r^2}{4}\left(1 - \frac{r}{4}\right) = \frac{4r^2 - r^3}{16} = \frac{1}{2} \\
&r > 1 + \sqrt{5} \Rightarrow x_2 < \frac{1}{2} \quad \text{L}
\end{aligned}
$$

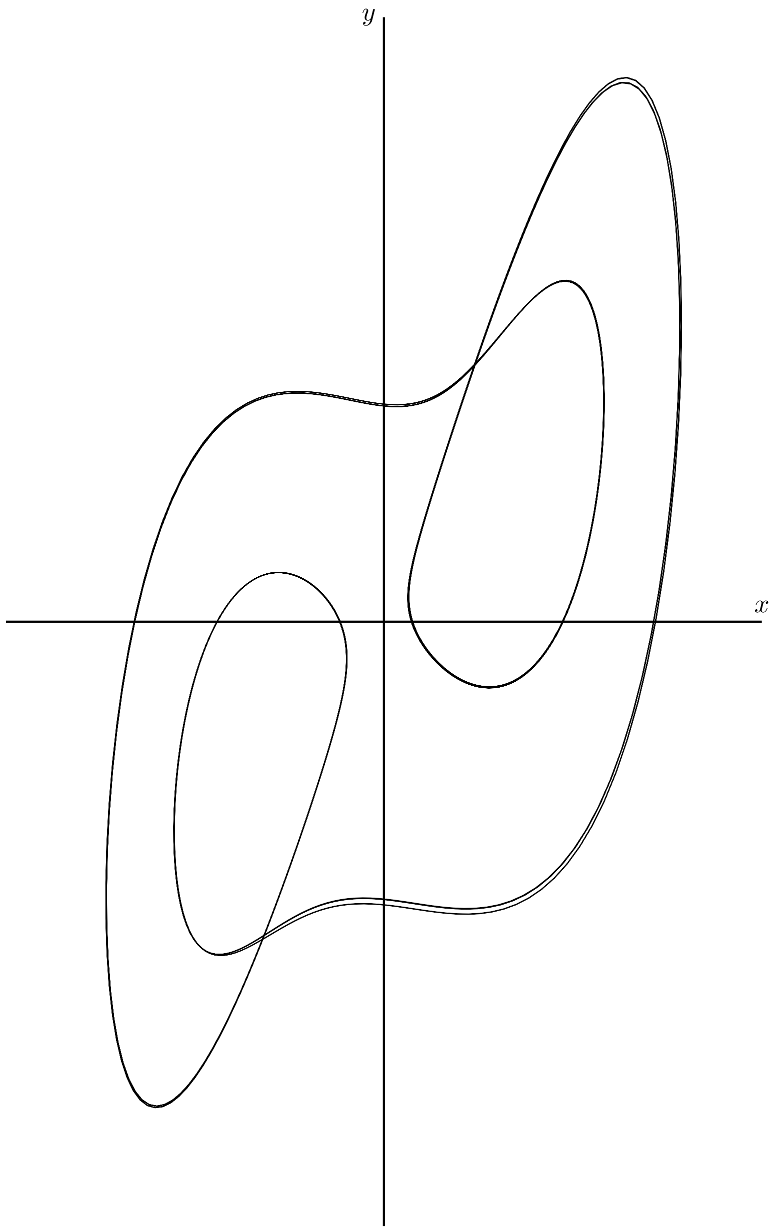
b)

RLRR

10.4.9

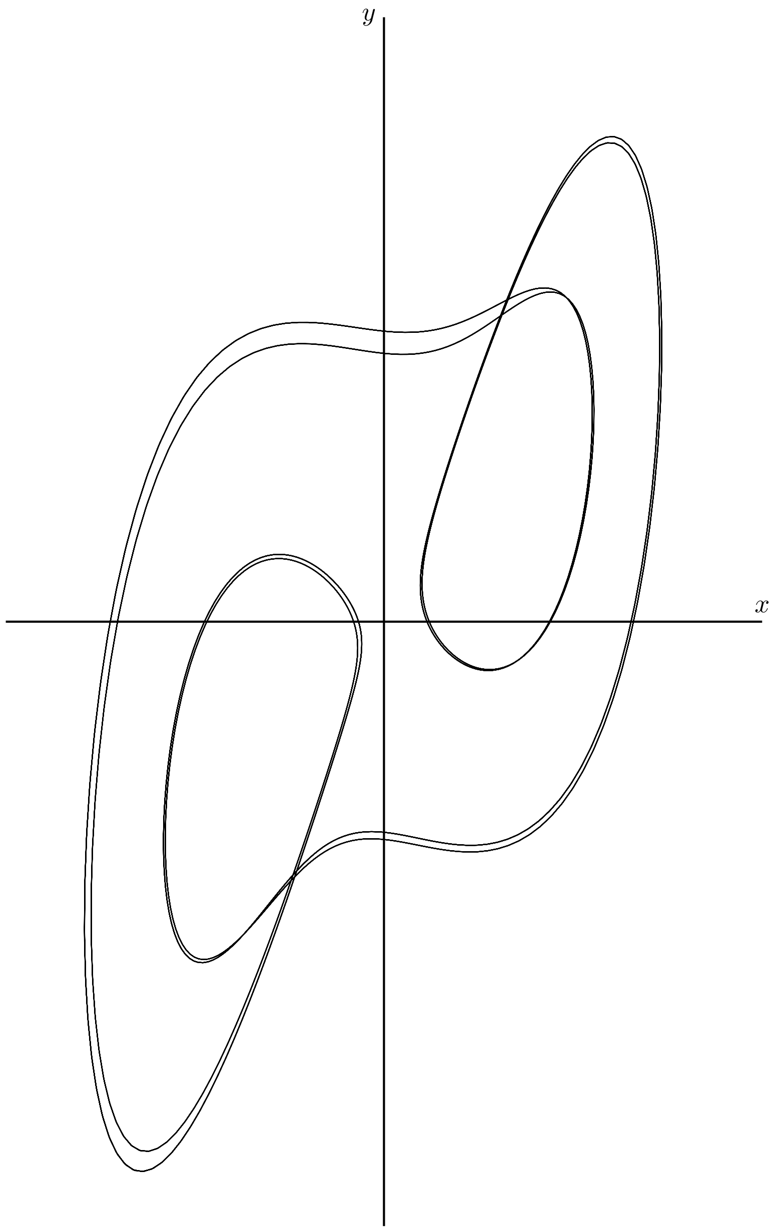
Parameters $r = 148.5 \quad \sigma = 10 \quad b = \frac{8}{3}$

Initial condition $(x, y, z) = (15, 5, -10)$



Parameters $r = 147.5 \quad \sigma = 10 \quad b = \frac{8}{3}$

Initial condition $(x, y, z) = (15, 5, -10)$



10.4.11

$x_{n+1} = a^{x_n}$	$x_1 = a > 0$
$0 < a < e^{-e}$	x_n tends to a stable 2-cycle
$e^{-e} < a < 1$	$x_n \to x^*$ where x^* is the unique root of $x^* = a^{x^*}$
$1 < a < e^{\frac{1}{e}}$	x_n tends to the smaller root of $x = a^x$
$a > e^{\frac{1}{e}}$	$x_n \to \infty$

10.5 Liapunov Exponent

10.5.1

$$x_{n+1} = rx_n$$

The explicit formula in terms of the initial condition is not hard to find, and is

$$x_n = r^n x_0$$

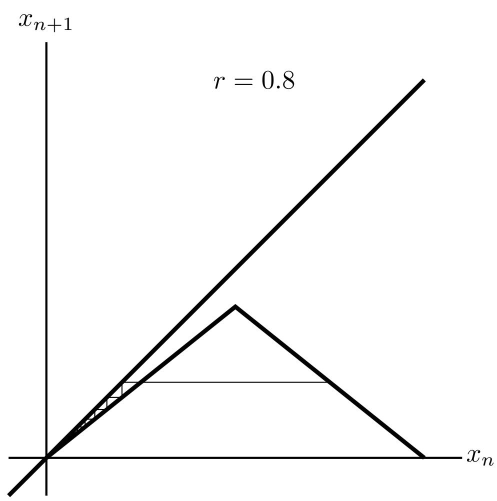
Now we can add in the error term and compute the Liapunov component exactly.

$$\lambda = \frac{1}{n}\left|\frac{\delta_n}{\delta_0}\right| = \frac{1}{n}\ln\left|\frac{r^n\delta_0}{\delta_0}\right| = \ln|r|$$

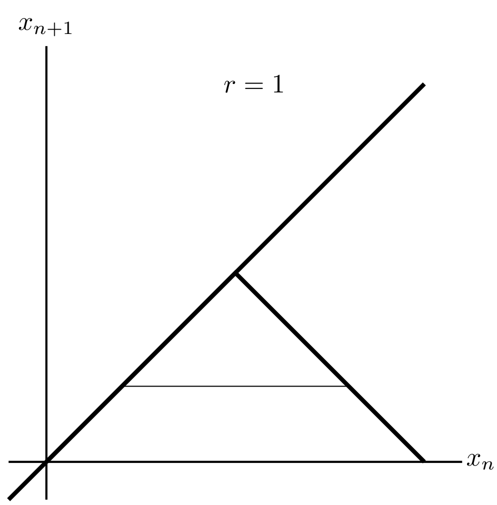
So the Liapunov exponent $\lambda = \ln|r|$.

10.5.3

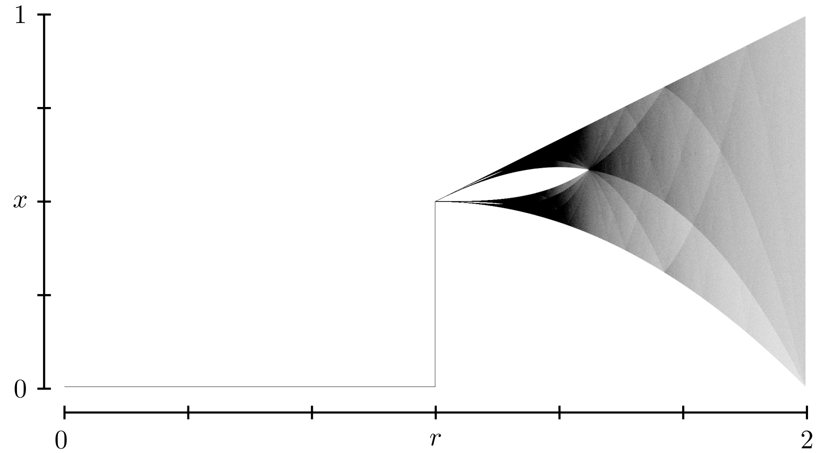
The origin is globally stable for $r < 1$, by cobwebbing.

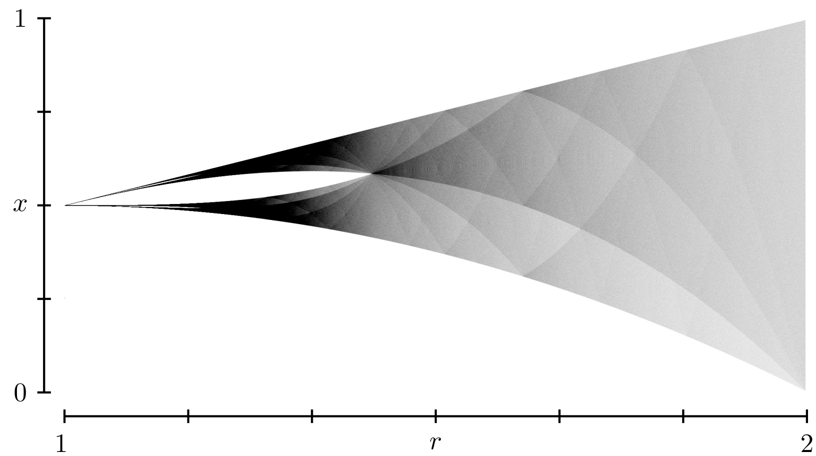


There is an interval of marginally stable fixed points for $r = 1$.



10.5.5





10.5.7

Suppose x^* is a fixed point of f undergoing a period-doubling bifurcation. Then $f(x^*) = -1$. Regarding x^* as x_0, we see $x_1 = x_2 = x_3 = \cdots = x^*$ since x^* is fixed. Hence,

$$\begin{aligned}
\lambda &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln\left|f'(x_i)\right| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln\left|f'(x^*)\right| \quad n \text{ terms} \\
&= \lim_{n\to\infty} \frac{1}{n} n \ln\left|f'(x^*)\right| = \lim_{n\to\infty} \ln\left|f'(x^*)\right| = \ln\left|f'(x^*)\right| = \ln\left|-1\right| \\
&= \ln(1) = 0
\end{aligned}$$

This shows $\lambda = 0$ at $r = r_1$ for the logistic map.

Next, consider where period-2 bifurcates to period-4: At $r = r_2$, we have period-2 points p, q with $f(p) = q$, $f(q) = p$, $\left(f^2\right)'(p) = -1$, and $\left(f^2\right)'(q) = -1$, so $f'\big(f(p)\big)f'(q) = -1$; i.e., $f'(p)f'(q) = -1$.

Consider the sequence $\{a_n\}$ in the definition of the Liapunov exponent:

$$\lim_{n\to\infty} a_n = \lambda \qquad \begin{aligned} a_n &= \frac{1}{n}\ln\left(\prod_{i=0}^{n-1}\left|f'(x_i)\right|\right) \\ a_1 &= \frac{1}{1}\ln\Big(\left|f'(x_0)\right|\Big) = \ln\Big(\left|f'(p)\right|\Big) \\ a_2 &= \frac{1}{2}\ln\Big(\left|f'(x_0)\right|\left|f'(x_1)\right|\Big) = \frac{1}{2}\ln\Big(\left|f'(p)f'(q)\right|\Big) \\ &= \frac{1}{2}\ln\big(|-1|\big) = 0 \quad \text{Since } f'(p)f'(q) = -1 \text{ when } r = r_2 \\ a_3 &= \frac{1}{3}\ln\Big(\left|f'(x_0)f'(x_1)f'(x_2)\right|\Big) \\ &= \frac{1}{3}\ln\Big(\left|f'(p)f'(q)f'(p)\right|\Big) = \frac{1}{3}\ln\Big(\left|(-1)f'(p)\right|\Big) \\ &= \frac{1}{3}\ln\left|f'(p)\right| \end{aligned}$$

In general

$$\begin{cases} a_{2k+1} = \frac{1}{2k+1}\ln\left|f'(p)\right| \\ a_{2k} = 0 \end{cases}$$

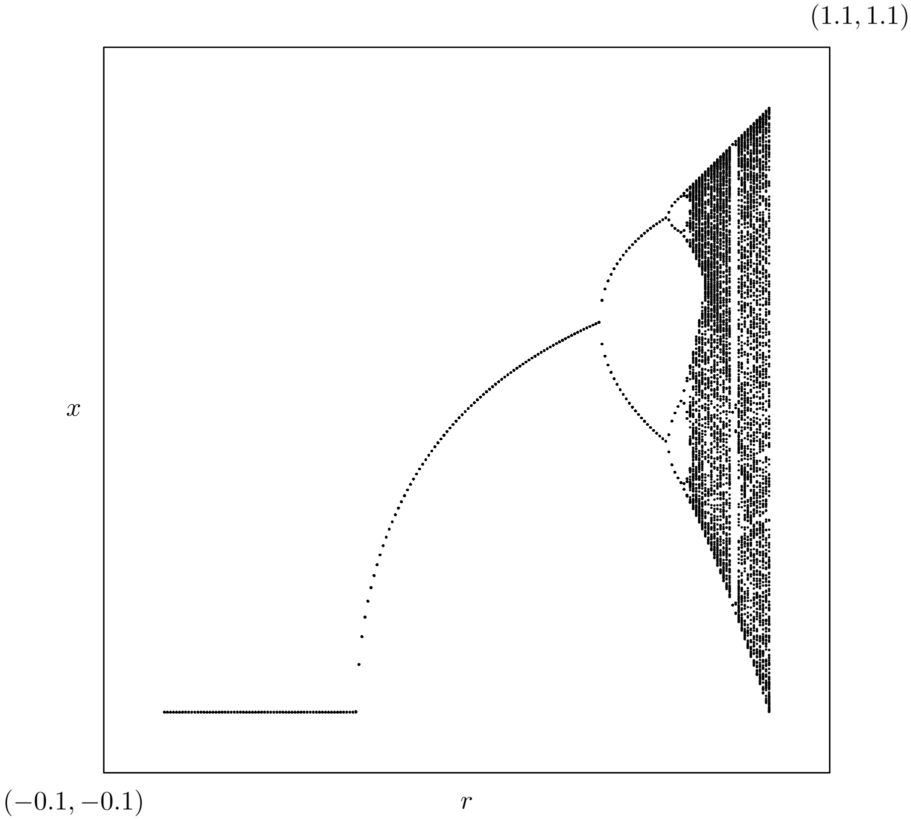
and as $k \to \infty$, this means $a_{2k+1} \to 0$ and therefore $a_n \to 0$ as $n \to \infty \Rightarrow \lambda = 0$ at $r = r_2$.

The same idea works at higher r_m.

10.6 Universality and Experiments

10.6.1

a)



See the solution to Exercise 10.2.3 for the psuedocode.

b)

Zooming in on the orbit diagram further and further with many more iterations gives the approximate r_i values.

$$r_1 \approx 0.71994 \qquad r_2 \approx 0.83326 \qquad r_3 \approx 0.85861$$
$$r_4 \approx 0.86408 \qquad r_5 \approx 0.86526 \qquad r_6 \approx 0.86551$$

c)

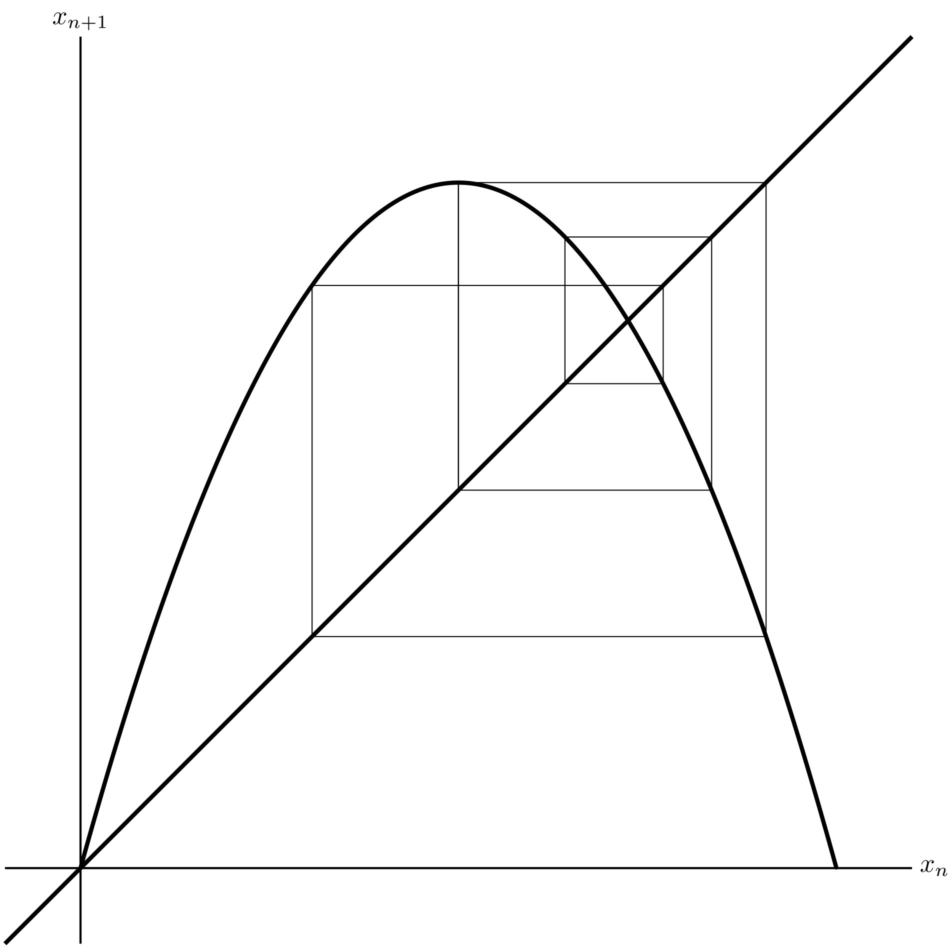
$$\frac{r_2 - r_1}{r_3 - r_2} \approx 4.47022$$
$$\frac{r_3 - r_2}{r_4 - r_3} \approx 4.63437$$
$$\frac{r_4 - r_3}{r_5 - r_4} \approx 4.63559$$
$$\frac{r_5 - r_4}{r_6 - r_5} \approx 4.72$$

The first few numbers look like they're converging to Feigenbaum $\delta = 4.669201609\ldots$, except the last one because five significant figures are not enough since the r_i values are getting so close together.

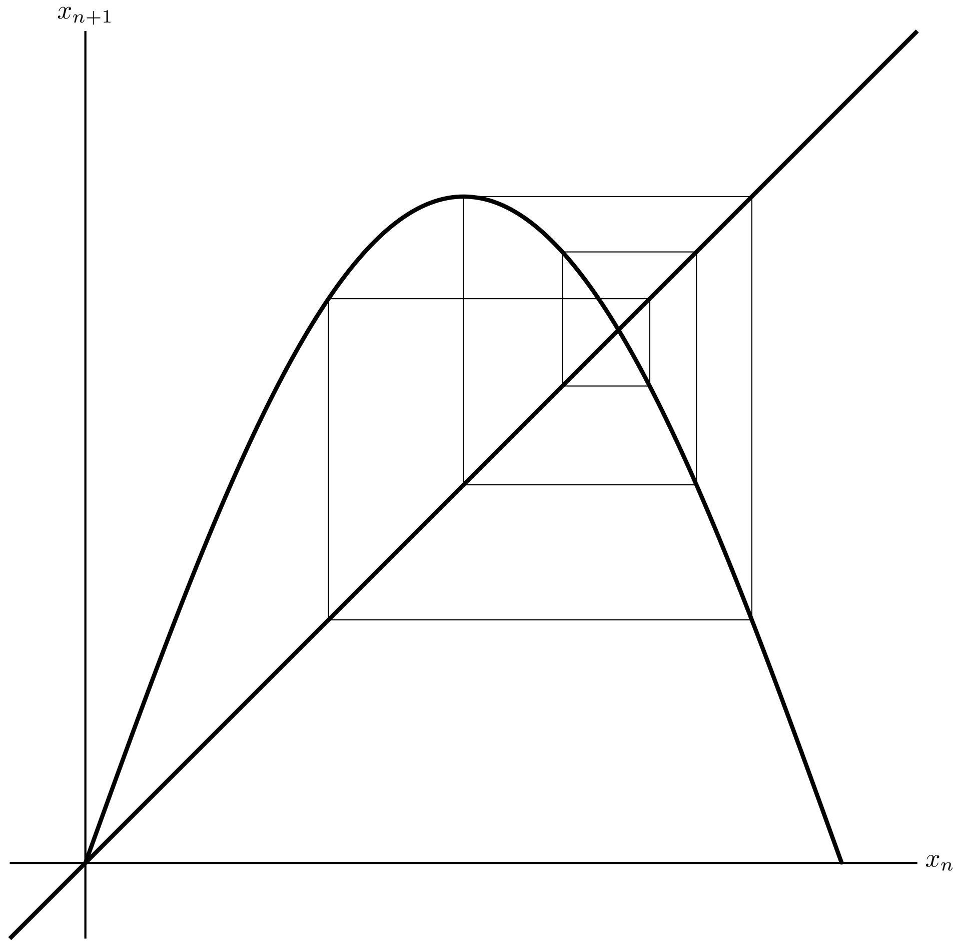
10.6.3

a)

Logistic map with $r = 3.6275575$



Sine map with $r = 0.8811406$



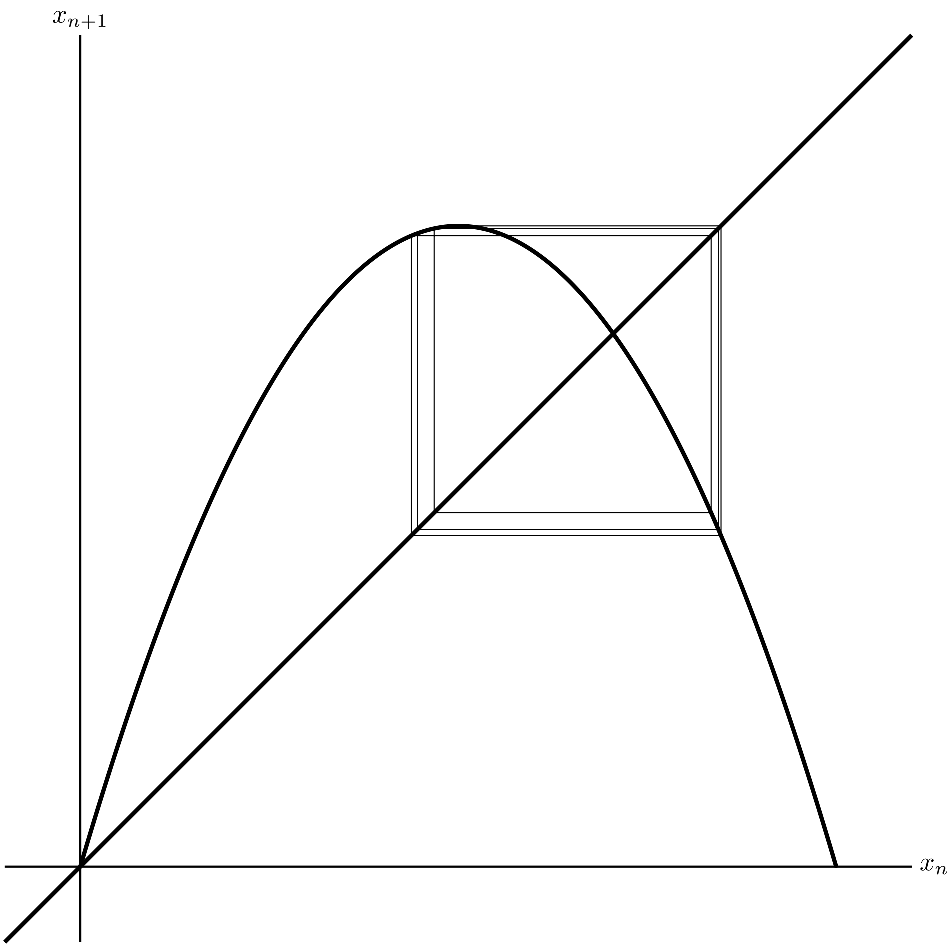
b)

The iteration pattern for both cycles is RLRRR.

10.6.5

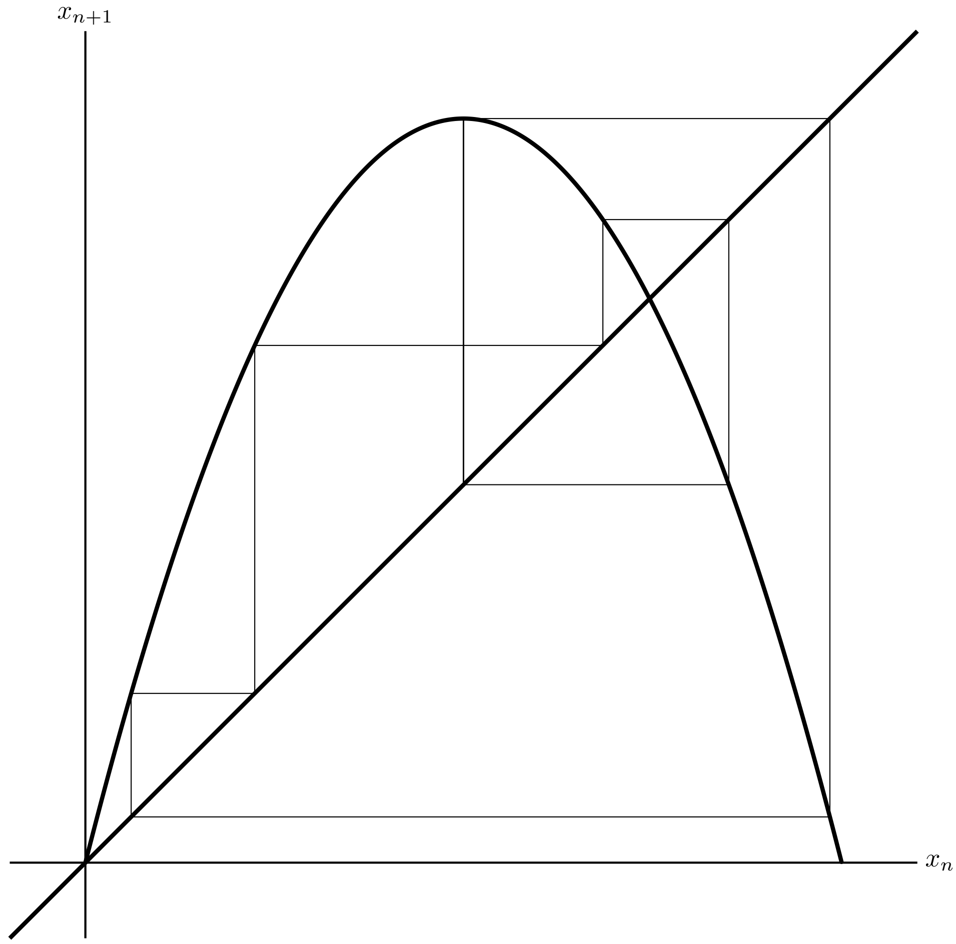
a)

Logistic map with $r = 3.39057065$ 5-cycle

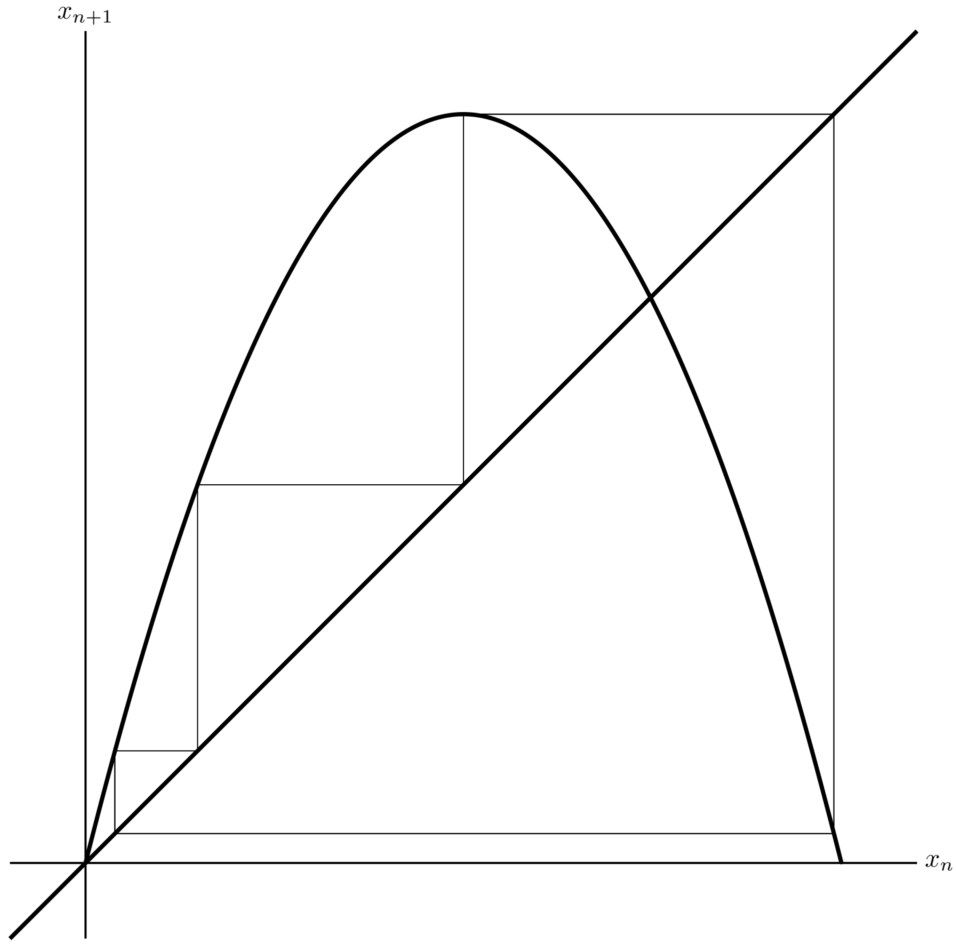


If you look very carefully you can see that this one doesn't complete. See part (b).

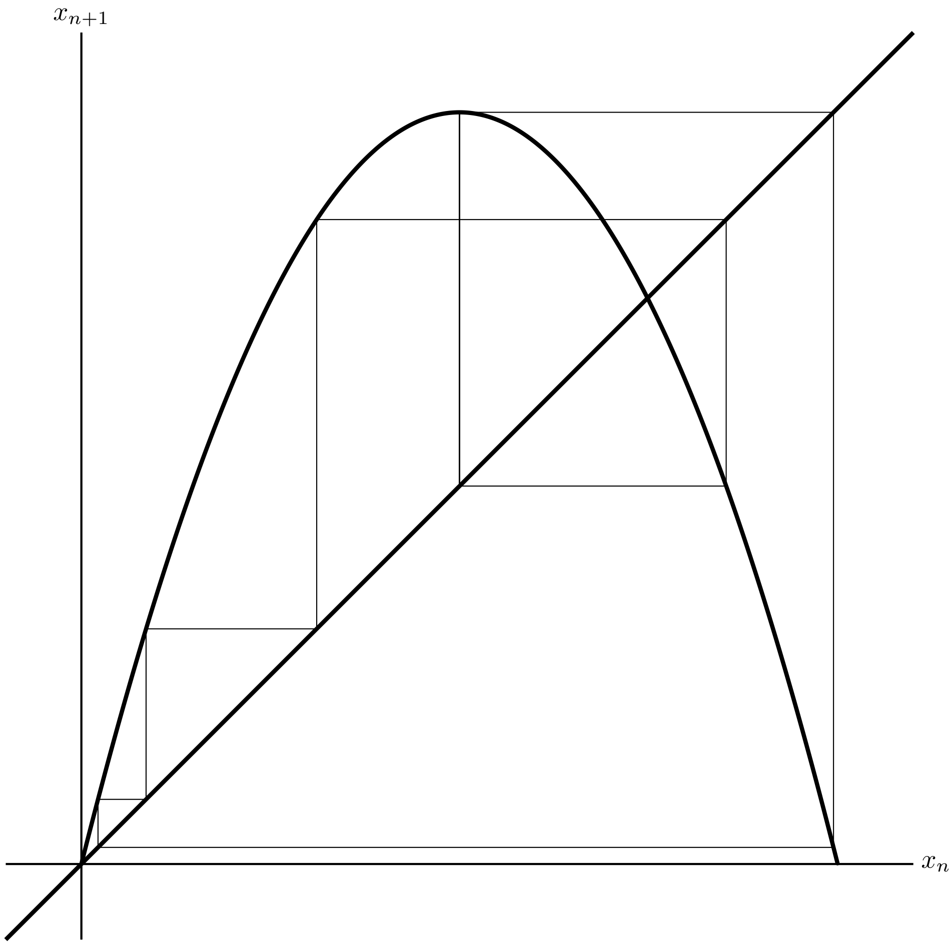
Logistic map with $r = 3.9375364$ 6-cycle



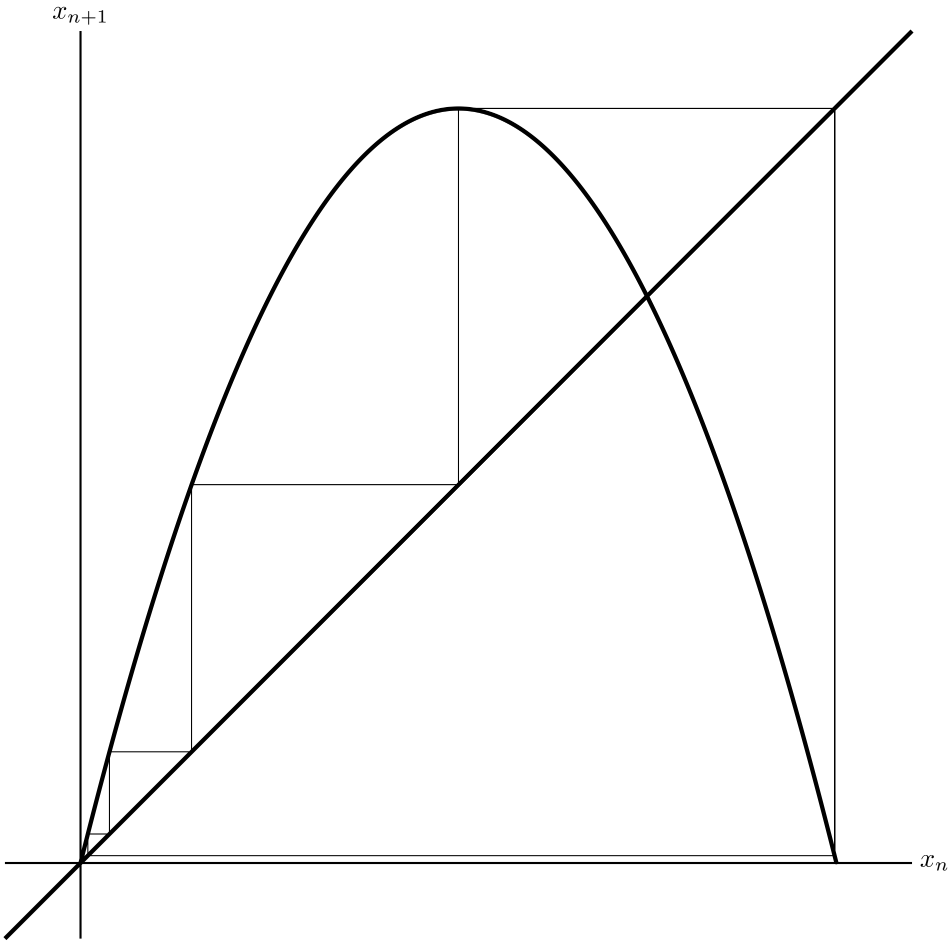
Logistic map with $r = 3.9602701$ 4-cycle



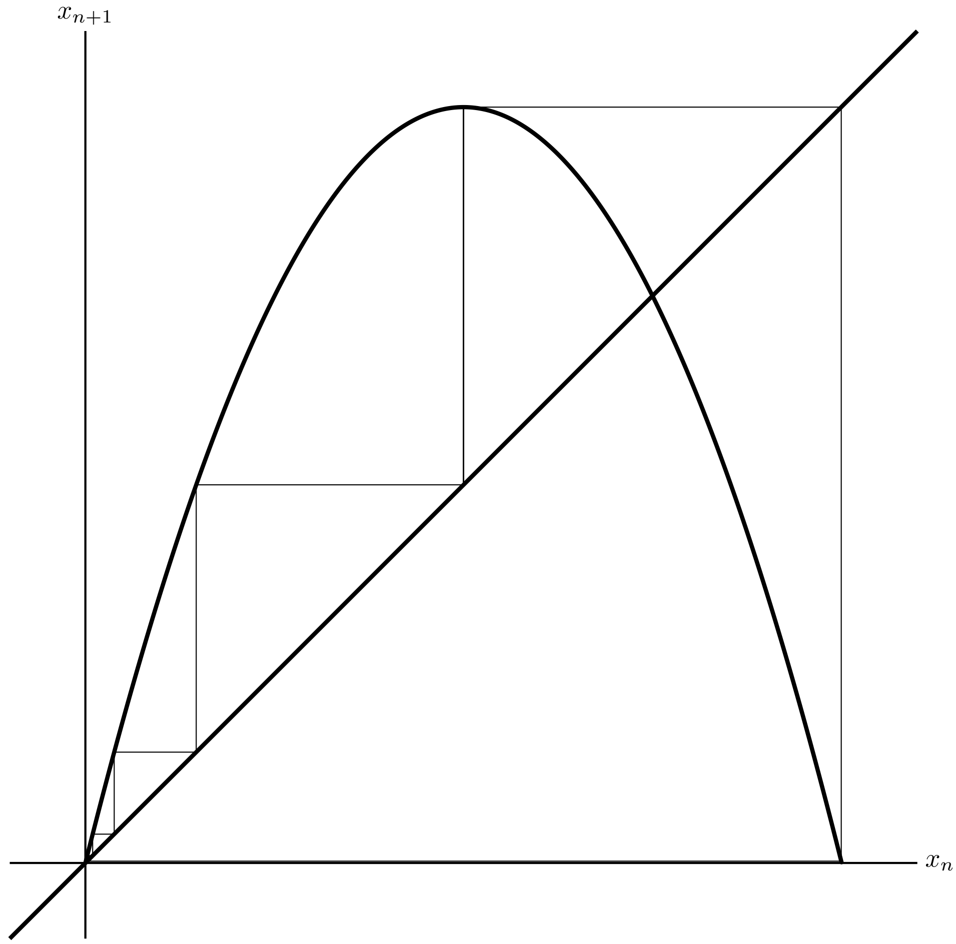
Logistic map with $r = 3.9777664$ 6-cycle



Logistic map with $r = 3.9902670$ 5-cycle

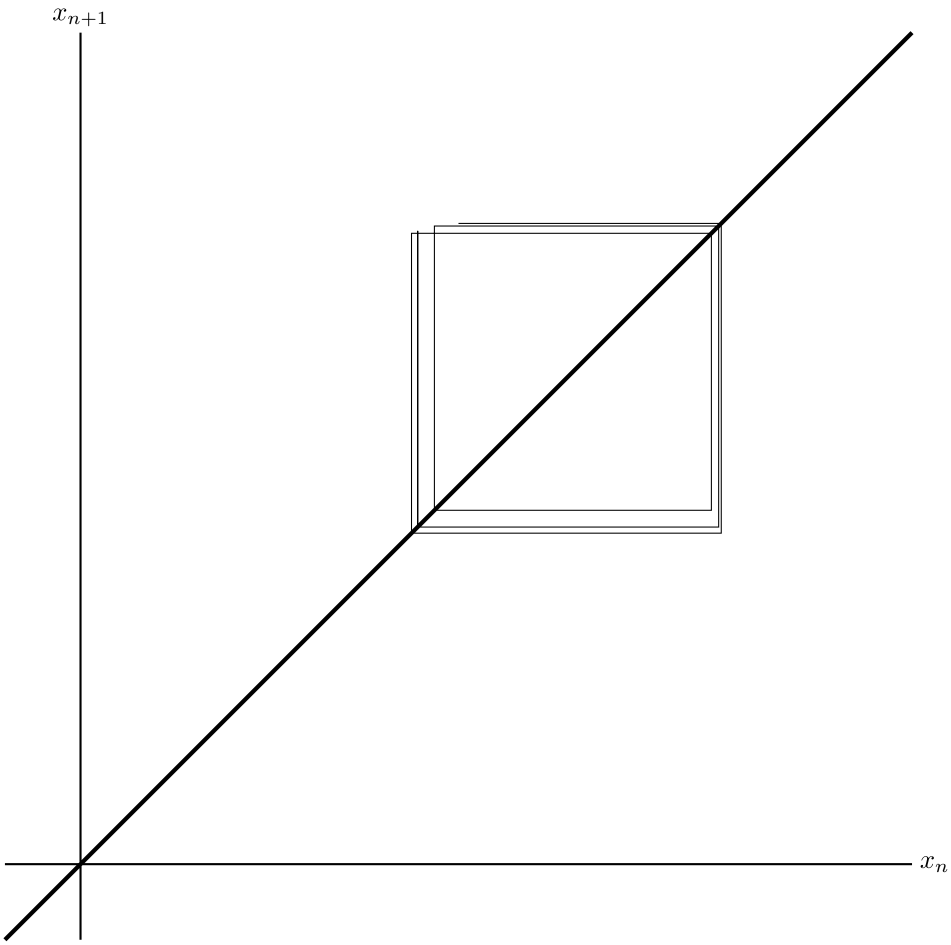


Logistic map with $r = 3.9975831$ 6-cycle

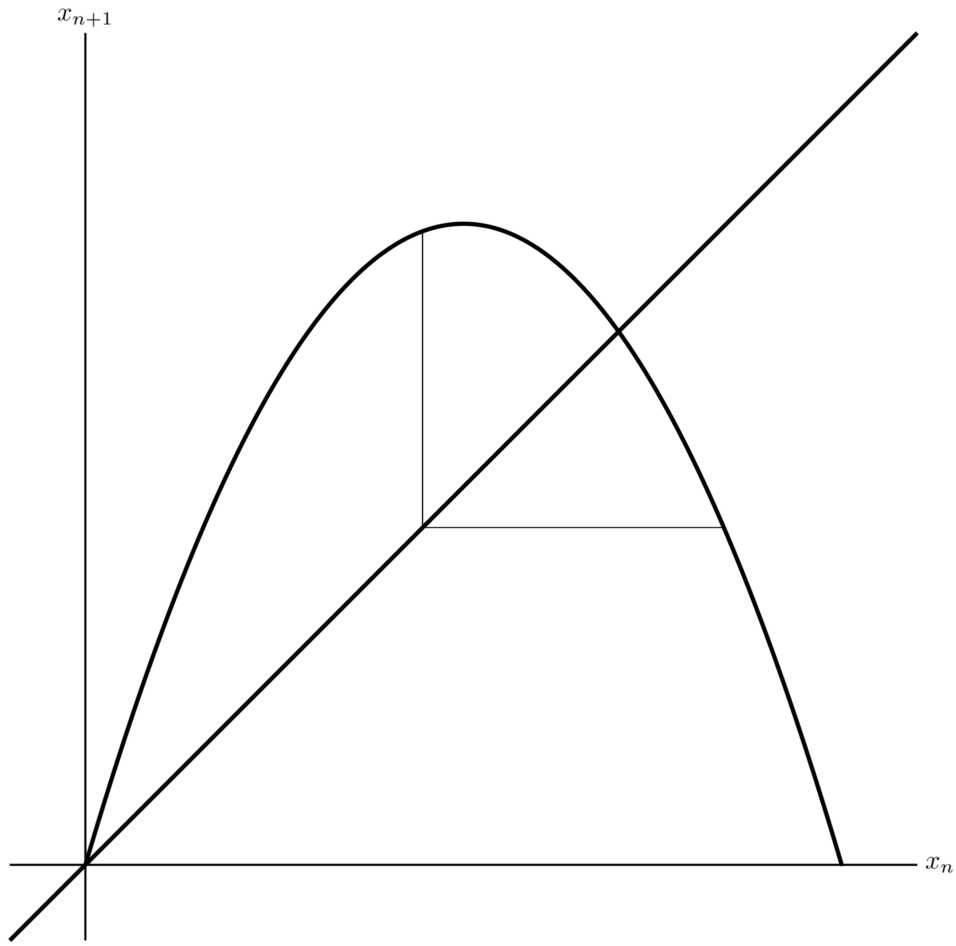


b)

The logistic map with $r = 3.39057065$ was supposed to have a 5-cycle when we plotted the cobweb, but the numerical simulation didn't complete.



The cobweb wasn't truncated in this graph. Instead the trajectory got stuck in the following pattern.



The trajectory alternates from $x \approx 0.454357$ to $x \approx 0.840579$, which turns out to be a 2-cycle of the map. The derivative evaluates to approximately $\Big(f\big(f(x)\big)\Big)' = -0.71503$ at the two alternating x coordinates, so this is a stable 2-cycle. The 5-cycle passes very close by this stable 2-cycle and was trapped due to roundoff error.

10.7 Renormalization

10.7.1

a)

$$
\begin{aligned}
&g(x) = 1 + c_2 x^2 \\
&g(x) = \alpha g^2\left(\frac{x}{\alpha}\right) \\
&1 + c_2 x^2 = \alpha(1 + c_2) + \frac{2c_2^2 x^2}{\alpha} + O(x^3) \\
&1 = \alpha(1 + c_2) \quad c_2 = \frac{2c_2^2}{\alpha}
\end{aligned}
$$

$$\Rightarrow (\alpha, c_2) = \begin{cases} (1, 0) \\ \left(-1-\sqrt{3}, \frac{-1-\sqrt{3}}{2}\right) \approx (-2.73205, -1.36603) \\ \left(-1+\sqrt{3}, \frac{-1+\sqrt{3}}{2}\right) \approx (0.732051, 0.366025) \end{cases}$$

We want the second solution since that has α closest to Feigenbaum $\alpha = -2.5...$

b)

$$\begin{aligned} g(x) &= 1 + c_2x^2 + c_4x^4 \\ g(x) &= \alpha g^2\left(\frac{x}{\alpha}\right) \\ 1 + c_2x^2 + c_4x^4 &= \alpha(1 + c_2 + c_4) + \frac{2(c_2^2 + 2c_2c_4)x^2}{\alpha} \\ &\quad + \frac{(c_2^3 + 6c_2^2c_4 + 2c_2c_4 + c_4^2)x^4}{\alpha^3} + O(x^5) \end{aligned}$$

$$1 = \alpha(1 + c_2 + c_4) \quad c_2 = \frac{2(c_2^2 + 2c_2c_4)}{\alpha} \quad c_4 = \frac{(c_2^3 + 6c_2^2c_4 + 2c_2c_4 + c_4^2)}{\alpha^3}$$

We can use the first and second equations to solve for c_2 and c_4 in terms of α.

$$c_2 = -2 + \frac{2}{\alpha} - \frac{\alpha}{2} \quad c_4 = 1 - \frac{1}{\alpha} + \frac{\alpha}{2}$$

Then we can substitute into the third equation to obtain a polynomial in α.

$$4\alpha^6 + 3\alpha^5 - 60\alpha^4 - 104\alpha^3 + 168\alpha^2 + 240\alpha - 384 + \frac{128}{\alpha} = 0$$

The root that is closest to the Feigenbaum $\alpha = -2.534...$ with corresponding c_2 and c_4 is

$$\alpha = -2.53403... \quad c_2 = -1.52224... \quad c_4 = 0.12761...$$

10.7.3

Suppose $g(x) = \alpha g\left(g\left(\frac{x}{\alpha}\right)\right)$ for all x. Then replace x with $\frac{x}{\mu}$ to get

$$g\left(\frac{x}{\mu}\right) = \alpha g\left(g\left(\frac{x}{\mu\alpha}\right)\right)$$

Multiply both sides by μ to get

$$\mu g\left(\frac{x}{\mu}\right) = \mu\alpha g\left(g\left(\frac{x}{\mu\alpha}\right)\right) \tag{1}$$

Let

$$h(x) = \mu g\left(\frac{x}{\mu}\right) \tag{2}$$

We want to show that h satisfies the same equation as g with the same α.

From Equation (1), we get (after using $\mu\alpha = \alpha\mu$)

$$h(x) = \mu g\left(\frac{x}{\mu}\right) = \mu\alpha g\left(g\left(\frac{x}{\mu\alpha}\right)\right) = \alpha\mu g\left(g\left(\frac{x}{\mu\alpha}\right)\right) \tag{3}$$

Compare $\alpha h\left(h\left(\frac{x}{\alpha}\right)\right)$, which gives

$$\alpha h\left(h\left(\frac{x}{\alpha}\right)\right) = \alpha h\left(\mu g\left(\frac{x}{\alpha\mu}\right)\right) \qquad \text{inner argument equality from Equation (2)}$$

$$= \alpha\mu g\left(\frac{\mu g\left(\frac{x}{\alpha\mu}\right)}{\mu}\right) \qquad \text{again from Equation (2)}$$

$$= \alpha\mu g\left(\frac{\mu g\left(\frac{x}{\alpha\mu}\right)}{\mu}\right) \qquad \text{cancel } \mu \text{ in fraction}$$

$$= h(x) \qquad \text{from Equation (3)}$$

So h satisfies the functional equation with the same α.

10.7.5

a)

From Example 10.7.1, we know that for $f(x,r) = r - x^2$ we have $R_0 = 0$ and $R_1 = 1$. Thus

$$f(x, R_0) = -x^2 \qquad f(x, r_1) = 1 - x^2$$

Hence

$$\alpha f\left(f\left(\frac{x}{\alpha}, 1\right), 1\right) = \alpha\left(\left[f\left(\frac{x}{\alpha}, 1\right)\right]^2\right) = \alpha\left[1 - \left(1 - \left[\frac{x}{\alpha}\right]^2\right)^2\right]$$

$$= \alpha\left[1 - \left(1 - \frac{2x^2}{\alpha^2} + \frac{x^4}{\alpha^4}\right)\right] = \frac{2x^2}{\alpha} - \frac{x^4}{\alpha^3}$$

b)

Matching $O(x^2)$ terms

$$f(x, r_0) = -x^2 \qquad \alpha f^2\left(\frac{x}{\alpha}, 1\right) = \frac{2x^2}{\alpha} + O(x^4) \Rightarrow -1 = \frac{2}{\alpha}$$

and hence $\alpha = -2$.

10.7.7

Redoing Exercise 10.7.1 part (a)

$$g(x) = \alpha g\left(g\left(\frac{x}{\alpha}\right)\right)$$

$$1 + c_1 x^4 = \alpha(1 + c_1) + \frac{4c_1^2 x^4}{\alpha^3} + O(x^5)$$

$$1 = \alpha(1 + c_1) \qquad c_1 = \frac{4c_1^2}{\alpha^3} \Rightarrow \alpha = -1.83509\ldots \quad c_1 = -1.54493\ldots$$

We chose the α and c_1 negative roots as found in the quadratic case.

Redoing Exercise 10.7.1 part (b)

$$g(x) = \alpha g\left(g\left(\frac{x}{\alpha}\right)\right)$$

$$1 + c_1 x^4 + c_2 x^8 = \alpha(1 + c_1 + c_2) + \frac{4(c_1^2 + 2c_1c_2)x^4}{\alpha^3} + \frac{2(3c_1^3 + 14c_1^2c_2 + 2c_1c_2 + 4c_2^2)x^8}{\alpha^7} + O(x^9)$$

$$1 = \alpha(1 + c_1 + c_2) \qquad c_1 = \frac{4(c_1^2 + 2c_1c_2)}{\alpha^3}$$

$$c_2 = \frac{2(3c_1^3 + 14c_1^2c_2 + 2c_1c_2 + 4c_2^2)}{\alpha^7}$$

Solving for α in the first equation gives

$$\alpha = \frac{1}{1 + c_1 + c_2}$$

Solving for c_1 and c_2 in the second and third equations gives

$$c_1 = -2 + \frac{2}{\alpha} - \frac{\alpha^3}{4} \qquad c_2 = 1 - \frac{1}{\alpha} + \frac{\alpha^3}{4}$$

And then plug in to obtain a polynomial in α.

$$\frac{8}{\alpha}\left(256 + 768\alpha^2 + 308\alpha^5 + 26\alpha^8 + 4\alpha^{10} + \alpha^{13}\right) = 6144 + 2048\alpha^2 + 1216\alpha^3 + 1248\alpha^5 + 248\alpha^8 + 11\alpha^{11}$$

which has relevant root and corresponding coefficients

$$\alpha = -1.732354430\ldots \qquad c_1 = -1.85478\ldots \qquad c_2 = 0.277528\ldots$$

Redoing Exercise 10.7.5 part (a)

$$f(x) = r - x^4$$

Clearly $R_0 = 0$ since $f(0) = 0$ when $r = 0$. Now we can find R_1 by demanding $f(f(0)) = 0$, which has relevant root $r = 1$. So $R_1 = 1$ as in the quadratic case.

Redoing Exercise 10.7.5 part (b), we match the leading order terms in

$$f(x, R_0) = \alpha f^2\left(\frac{x}{\alpha}, R_1\right)$$

$$-x^4 = \frac{4x^4}{\alpha^3} + O(x^5) \Rightarrow \alpha = -2^{\frac{2}{3}} = -1.58740\ldots$$

Numerically solving

$$f^{2^n}(x, R_n)\Big|_{x=0} = 0$$

gives

$$R_0 = 0 \quad R_1 = 1 \quad R_2 = 1.14571\ldots \quad R_3 = 1.16527\ldots$$

$$R_4 = 1.16794\ldots \quad R_5 = 1.16831\ldots$$

And using these to estimate δ

$$\delta \approx \frac{R_4 - R_3}{R_5 - R_4} \approx 7.30235$$

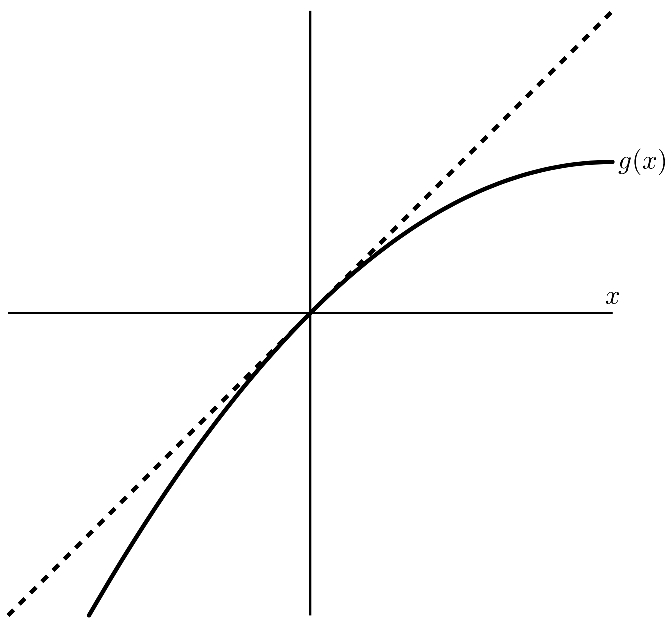
For the fourth-degree case, Briggs (1991) finds numerically that the true $\delta = 7.2846862171\ldots \quad \alpha = -1.6903029714\ldots$, which is close to our calculation.

10.7.9

Following Exercise 10.7.8, we rescale x by a factor of α and observe that the second iterate f^2 resembles f near the bifurcation point $x = 0$, $r = 0$. This gives

$$g(x) = \alpha g\left(g\left(\frac{x}{\alpha}\right)\right)$$

The boundary conditions are $g(0) = 1$ and $g'(0) = 1$ because of this tangency condition at the bifurcation.



a)

Try

$$g(x) = \frac{x}{1 + ax}$$

Then

$$\alpha g\left(g\left(\frac{x}{\alpha}\right)\right) = \frac{x}{\left(1 + \frac{ax}{\alpha}\right)\left(1 + \frac{ax}{\left(1+\frac{ax}{\alpha}\right)\alpha}\right)} = \frac{x\alpha}{2ax + \alpha}$$

which has the same form as $g(x)$ and equals $g(x)$ when

$$g(x) = \frac{x}{1 + ax} = \frac{x\alpha}{2ax + \alpha} = \alpha g\left(g\left(\frac{x}{\alpha}\right)\right) \Rightarrow \alpha = 2$$

which works for all a.

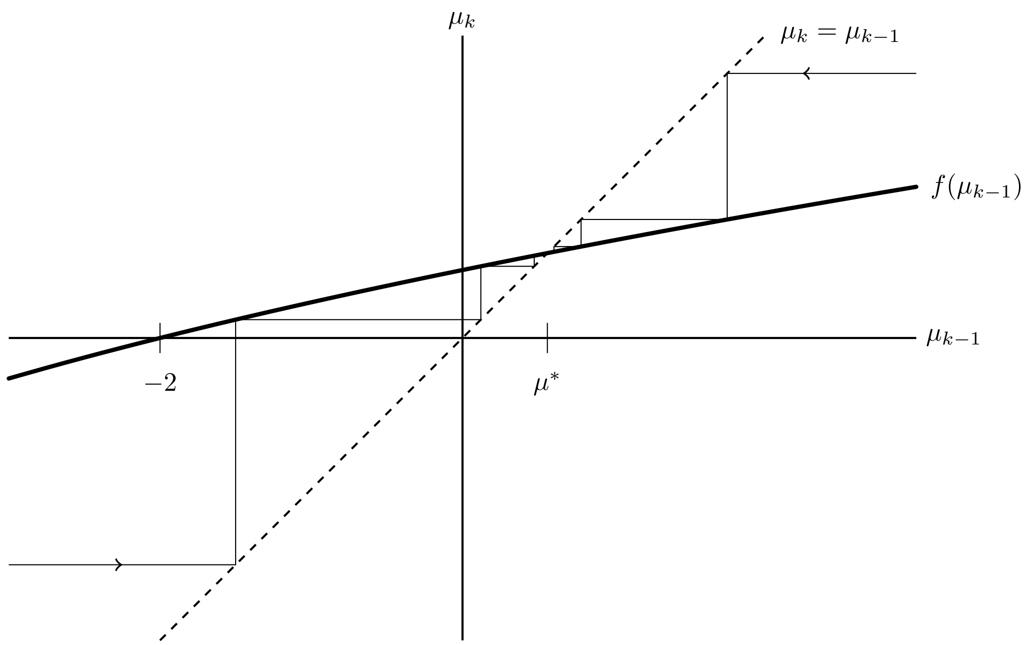
b)

$\alpha = 2$ is (almost) obvious because the steps to find g^2 are twice as long as those for g, as discussed in Exercise 10.7.8 part (c). Otherwise g and g^2 have the same dynamics—namely, slow passage through a bottleneck.

10.7.11

$$\mu_k = -2 + \sqrt{6 + \mu_{k-1}} = f(\mu_{k-1})$$

The graph and cobweb of f are



If we start from $\mu_1 = 0$, we see $\mu_k \to \mu^* > 0$, where

$$\mu^* = -2 + \sqrt{6 + \mu^*} \Rightarrow \mu^* = \frac{-3 + \sqrt{17}}{2}$$

as in the text.

11
Fractals

11.1 Countable and Uncountable Sets

11.1.1

The same argument is not valid because the constructed r is not necessarily rational. What if we pick the order of our list to be such that $r = \frac{1}{\sqrt{2}}$ when we're done?

$$\frac{1}{\sqrt{2}} = 0.707106\dots$$
$$x_1 = 0.6x_{1,2}x_{1,3}x_{1,4}x_{1,5}x_{1,6}\cdots$$
$$x_2 = 0.x_{2,1}9x_{2,3}x_{2,4}x_{2,5}x_{2,6}\cdots$$
$$x_3 = 0.x_{3,1}x_{3,2}6x_{3,4}x_{3,5}x_{3,6}\cdots$$
$$x_4 = 0.x_{4,1}x_{4,2}x_{4,3}0x_{4,5}x_{4,6}\cdots$$
$$x_5 = 0.x_{5,1}x_{5,2}x_{5,3}x_{5,4}9x_{5,6}\cdots$$
$$x_6 = 0.x_{6,1}x_{6,2}x_{6,3}x_{6,4}x_{6,5}5\dots$$
$$\vdots$$

One possible $r = 0.707106\dots = \frac{1}{\sqrt{2}}$, which is irrational and thus outside the set x_i. So r may not be in the list, but sometimes it's not supposed to be in the list anyway.

11.1.3

We can prove that the irrational numbers are uncountable by contradiction. We know the real numbers are uncountable, and taking away the countable set of rational numbers leaves the irrational numbers. Therefore the irrational numbers must be uncountable since they were made by starting with an uncountable set and then removing a countable set.

11.1.5

The bijective function $f : \mathbb{N} \to \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is very difficult to create, but we can convince ourselves the mapping exists by visualizing the points in Cartesian space.

The very first point is the origin. The next set of lattice points lies on the surface of the side length 2 cube centered at the origin, which will have $3^3 - 1 = 26$ points since the middle is taken out. The next set of lattice points lies on the side length 5 cube centered at the origin, which will have $5^3 - 3^3 = 98$ points since the interior side length 3 cube is taken out. The next set of lattice points will lie on the side length 7 cube, and so on and so forth.

The point (no pun intended) is that the lattice points can be labeled first by cube size and then arbitrarily on the surface of the cube by a finite natural number. Hence, the set of lattice points in 3-dimensional space is countable.

11.1.7

$$x_{n+1} = 2x_n \pmod 1$$

The problem becomes a little more intuitive if we convert to binary.

$$b_{n+1} = 10b_n \pmod 1$$

No matter what value starts the sequence, the first iteration maps into the interval [0,1] and stays there forever due to the modulo operation. The periodic orbits are the rational numbers in the interval [0,1] since the decimal eventually repeats or terminates, which is a countable set. The aperiodic orbits of the map are the irrational numbers, which is an uncountable set.

11.2 Cantor Set

11.2.1

The Cantor set has measure zero. The first iteration removes one interval of length $\frac{1}{3}$, the second iteration removes two intervals of length $\frac{1}{9}$, the third iteration removes four intervals of length $\frac{1}{27}$, etc.

The interval we start with has length one, and the length of the set after the iterated removals is

$$\begin{aligned} 1 - \left(\frac{1}{3} + 2\frac{1}{9} + 4\frac{1}{27} + \cdots\right) &= 1 - \frac{1}{3}\left(1 + \frac{2}{3} + \frac{4}{9} + \cdots\right) \\ &= 1 - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right) \\ &= 1 - \frac{1}{3}\frac{1}{1-\frac{2}{3}} = 1 - \frac{1}{3}\frac{1}{\frac{1}{3}} \\ &= 1 - \frac{1}{3}3 = 1 - 1 = 0 \end{aligned}$$

Therefore the measure of the set left over after completing the infinite number of iterated removals (the Cantor set) is zero.

11.2.3

The plan for this problem is to upper bound the measure of this countable subset of the real numbers with another set that we can shrink to zero measure.

First, this set of numbers is countable, so we can list them as $\{x_1, x_2, x_3, x_4, \ldots\}$.

We can make an interval centered around every x_n with length $\epsilon(2^{1-n})$ where $\epsilon > 0$.

$$I_n = \left(x_n - \frac{\epsilon}{2^n}, x_n + \frac{\epsilon}{2^n}\right)$$

This set of intervals is countable, and the union of this countable number of intervals contains $\{x_1, x_2, x_3, x_4, \ldots\}$ and then some. Therefore the measure of the countable set of intervals is an upper bound for the measure of our countable subset.

$$0 \leq \mu(\{x_1, x_2, x_3, x_4, \ldots\}) < \mu\left(\bigcup_{n=1}^{\infty} I_n\right) = \sum_{n=1}^{\infty} \epsilon(2^{1-n}) = 2\epsilon$$

We never explicitly chose ϵ, so we can scale the size of I_n and consequently the upper bound for the measure of $\{x_1, x_2, x_3, x_4, \ldots\}$ to as small as we want, meaning that the measure of $\{x_1, x_2, x_3, x_4, \ldots\}$ has to be 0.

11.2.5

a)

We'll attack this by pretending we know the ternary representation and multiplying by 3.

$$[0.5]_{10} = [0.t_1t_2t_3\ldots]_3$$
$$[3]_{10} \cdot [0.5]_{10} = [10]_3 \cdot [0.t_1t_2t_3\ldots]_3$$
$$[1.5]_{10} = [t_1.t_2t_3\ldots]_3$$
$$[1]_{10} + [0.5]_{10} = [t_1]_3 + [0.t_2t_3\ldots]_3$$

From this we know that we can replace t_1 with 1, and then if we iterate we can replace t_2 with 1 and t_3 with 1 and so on and so forth. Therefore the value decimal 0.5 is $0.\bar{1}$ in ternary.

b)

The Cantor set is all ternary numbers in [0,1] that do not contain any 1's in their ternary representation. This satisfies every point in the Cantor set corresponding to a point in [0,1], but not vice versa since any value with a 1 in its ternary representation is not mapped into the Cantor set.

However, we can use binary numbers to uniquely pair every point in [0,1] with a point in the Cantor set, and vice versa. Basically we read the digit in the binary number and pick the left or right third for that subinterval if the digit is 0 or 1 respectively. Then we repeat for the next digit and so on, and the Cantor set point is the left endpoint of the subinterval we end in. For example, the binary number 0.110101 directs us to pick right right left right left right and then the left endpoint of that subinterval, which is

$$[0,1] \underset{R}{\longrightarrow} \left[\frac{2}{3},1\right] \underset{R}{\longrightarrow} \left[\frac{8}{9},1\right] \underset{L}{\longrightarrow} \left[\frac{24}{27},\frac{25}{27}\right] \underset{R}{\longrightarrow} \left[\frac{74}{81},\frac{75}{81}\right] \underset{L}{\longrightarrow} \left[\frac{222}{243},\frac{223}{243}\right] \underset{L}{\longrightarrow} \frac{222}{243}$$

c)

Endpoints in the Cantor set are ternary numbers made entirely of 0's and 2's that either eventually reach endlessly repeating 2's or terminate, which is the same as endlessly repeating 0's. Hence, a number in the Cantor set that is not an endpoint must not terminate and must not eventually become endlessly repeating 2's, an example of which is $0.\overline{02}$ in ternary.

11.3 Dimension of Self-Similar Fractals

11.3.1

a)

Taking out the middle half of the [0,1] interval set leaves two intervals that will each contain the middle-halves Cantor set scaled by a factor of four. Then the similarity dimension is

$$2 = 4^d \Rightarrow d = \frac{\ln(2)}{\ln(4)} = \frac{\ln(2)}{2\ln(2)} = \frac{1}{2}$$

b)

The first iteration removes one interval of length $\frac{1}{2}$, the second iteration removes two intervals of length $\frac{1}{8}$, the third iteration removes four intervals of length $\frac{1}{32}$, etc.

The interval we start with has length one, so the the length of the set after the iterated removals is

$$\begin{aligned}
1-\left(\frac{1}{2}+2\frac{1}{8}+4\frac{1}{32}+\cdots\right) &= 1-\frac{1}{2}\left(1+\frac{2}{4}+\frac{4}{16}+\cdots\right)\\
&= 1-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right)\\
&= 1-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)\\
&= 1-\frac{1}{2}\frac{1}{1-\frac{1}{2}} = 1-\frac{1}{2}\frac{1}{\frac{1}{2}}\\
&= 1-\frac{1}{2}2 = 1-1 = 0
\end{aligned}$$

Hence, the measure of the middle-halves Cantor set is zero.

You can also see this intuitively by taking away the left half of the remaining interval every time instead of the middle half of each remaining interval. $[0,1] \to [0,0.5] \to [0,0.25] \to [0,0.125] \to \cdots \to [0,2^{-n}]$, so the length keeps getting closer and closer to 0.

11.3.3

a)

One iteration of the even-sevenths Cantor set leaves four intervals that will each contain the even-sevenths Cantor set scaled by a factor of seven. Then the similarity dimension is

$$4 = 7^d \Rightarrow d = \frac{\ln(4)}{\ln(7)}$$

b)

For odd n, one iteration of the even-$n$ths Cantor set leaves $\dfrac{n+1}{2}$ intervals that will each contain the even-$n$ths Cantor set scaled by a factor of n. Then the similarity dimension is

$$\frac{n+1}{2} = n^d \Rightarrow d = \frac{\ln(n+1) - \ln(2)}{\ln(n)}$$

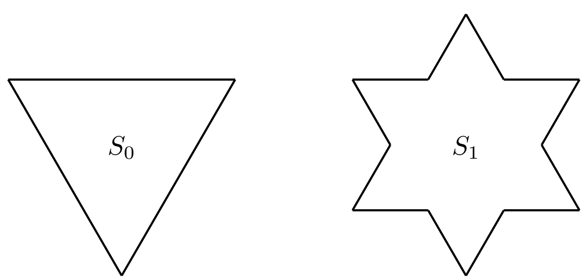
11.3.5

One iteration of this no-8's Cantor set will remove the interval [0.8,0.9) from [0,1] and leave nine intervals that will each contain this no-8's Cantor set scaled by a factor of ten. Then the similarity dimension is

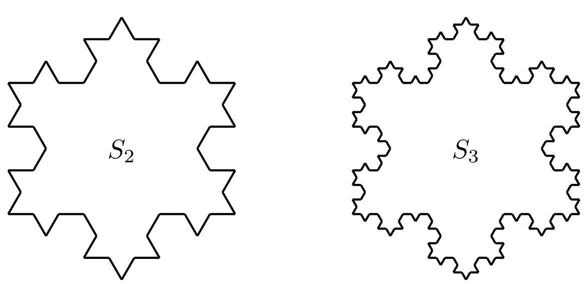
$$9 = 10^d \Rightarrow d = \frac{\ln(9)}{\ln(10)}$$

11.3.7

a)



b)



c)

The arc length of $S_0 = 3$ and the arc length are multiplied by $\frac{4}{3}$ each iteration.

Then the arc length of $S_\infty = 3\left(\frac{4}{3}\right)^\infty = \infty$.

d)

S_0 has side length one with area $\frac{\sqrt{3}}{4}$ and every iteration adds a new equilateral triangle onto each side that is $\frac{1}{3}$ the length of the side.

$$\text{Side length of } S_n = \left(\frac{1}{3}\right)^n$$

$$\text{Number of sides of } S_n = 3\,(4)^n$$

$$\begin{aligned}
\text{Area added from } S_{n-1} \text{ to } S_n &= (\text{Number of sides of } S_{n-1})\frac{\sqrt{3}}{4}\left(\frac{1}{3}(\text{Side length of } S_{n-1})\right)^2 \\
&= 3\,(4)^{n-1}\,\frac{\sqrt{3}}{4}\left(\frac{1}{3}\left(\frac{1}{3}\right)^{n-1}\right)^2 \\
&= 3\,(4)^{n-1}\,\frac{\sqrt{3}}{4}\left(\left(\frac{1}{3}\right)^{n}\right)^2 \\
&= \frac{3\left(4^{n-1}\right)\sqrt{3}}{4\left(3^{2n}\right)} = \frac{3\sqrt{3}}{16}\left(\frac{4}{9}\right)^n
\end{aligned}$$

If we sum the area of S_0 and the area added after each iteration we get

$$\begin{aligned}
\frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\sum_{n=1}^{\infty}\left(\frac{4}{9}\right)^n &= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\frac{\frac{4}{9}}{1-\frac{4}{9}} = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\frac{4}{9}\frac{9}{5} \\
&= \frac{\sqrt{3}}{4}\left(1 + \frac{3}{4}\frac{4}{9}\frac{9}{5}\right) = \frac{\sqrt{3}}{4}\left(1+\frac{3}{5}\right) \\
&= \frac{\sqrt{3}}{4}\frac{8}{5} = \frac{2\sqrt{3}}{5}
\end{aligned}$$

e)

The Koch snowflake is made of three Koch curves. Each Koch curve requires four copies of itself scaled by a factor of three in order to recreate itself. Therefore the Koch snowflake can be made by taking four complete copies of itself and scaling each one by a factor of three. Then each scaled Koch snowflake is cut into thirds and placed onto the full-size Koch snowflake.

Hence, the similarity dimension of the Koch snowflake is

$$4 = 3^d \Rightarrow d = \frac{\ln(4)}{\ln(3)}$$

11.3.9

At each iteration of the Menger sponge construction, the number of copies is $m = 27-6-1 = 20$ (six smaller cubes for the faces, and one smaller cube from the center) and each copy is scaled by a factor of $r = 3$ in each dimension. Therefore the similarity dimension is

$$d = \frac{\ln(m)}{\ln(r)} = \frac{\ln(20)}{\ln(3)}$$

As for the Menger hypersponge in N dimensions, each iteration of the construction makes 3^N smaller N-dimensional cube regions, and then we delete some. To find how many we delete, we need a pattern for the construction of the N-dimensional fractal. The pattern turns out to be straightforward.

For the first construction iteration of the 2-dimensional Sierpinski carpet, label the squares of the 3x3 grid as

(1,3)	(2,3)	(3,3)
(1,2)	(2,2)	(3,2)
(1,1)	(2,1)	(3,1)

and then we delete the square with at least two 2's in the coordinate, which is the middle square. Then we descend into the remaining eight squares, each with a side length reduced by a factor of three.

For the first construction iteration of the 3-dimensional Menger sponge, label the cubes of the 3x3x3 grid as (a, b, c) with $a, b, c \in \{1, 2, 3\}$. Then delete all the cubes with at least two 2's in the coordinate, which makes one cube for each surface and the middle cube. Then we descend into the remaining $3^3 - 6 - 1 = 20$ cubes, each with a side length reduced by a factor of three.

For the first construction iteration of the 4-dimensional Menger hypersponge, label the hypercubes of the 3x3x3x3 grid as (a, b, c, d) with $a, b, c, d \in \{1, 2, 3\}$ and then delete all the hypercubes with at least two 2's in the coordinate. Combinatorially counting the number of hypercubes with less than two 2's in the coordinate gives the same answer and has an easier formula.

$$\binom{4}{1}2^3 + \binom{4}{0}2^4 = 48$$

The number of ways to choose one coordinate out of four to place a 2 with 1's or 3's in the remaining three slots plus the number of ways to place zero 2's with 1's or 3's in the remaining four slots.

Hence, for the 4-dimensional Menger hypersponge, there are $m = 48$ copies after each iteration and each copy is scaled by a factor of $r = 3$. Therefore the similarity dimension is

$$d = \frac{\ln(m)}{\ln(r)} = \frac{\ln(48)}{\ln(3)}$$

For the N-dimensional Menger hypersponge, the similarity dimension is

$$d = \frac{\ln(m)}{\ln(r)} = \frac{\ln\left(N2^{N-1} + 2^N\right)}{\ln(3)}$$

Just to be safe, let's check the N-dimensional formula against the results for the Sierpinski carpet

$$N = 2 \Rightarrow d = \frac{\ln\left((2)(2^{2-1}) + 2^2\right)}{\ln(3)} = \frac{\ln(8)}{\ln(3)} \qquad \checkmark$$

and Menger sponge

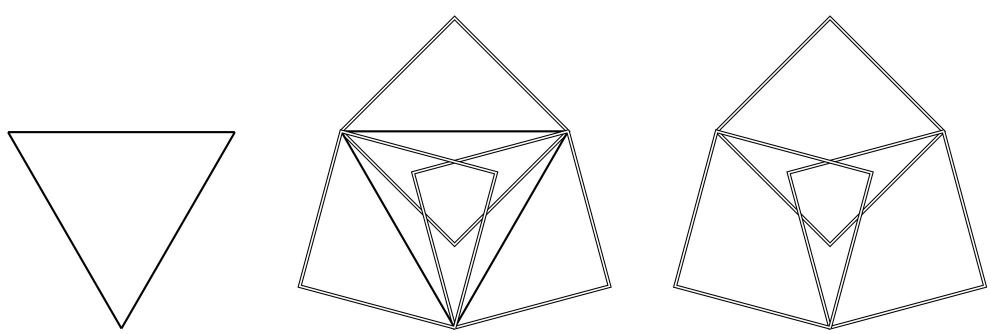
$$N = 3 \Rightarrow d = \frac{\ln\left((3)(2^{3-1}) + 2^3\right)}{\ln(3)} = \frac{\ln(20)}{\ln(3)} \qquad \checkmark$$

11.4 Box Dimension

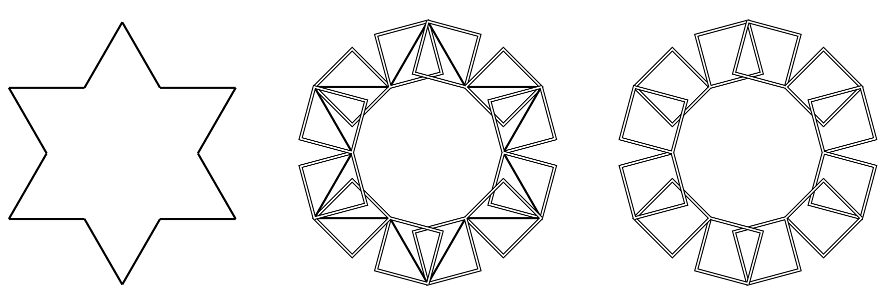
11.4.1

Iterations of the Koch snowflake

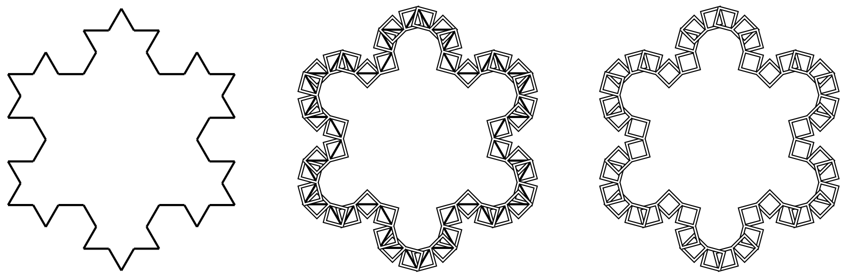
n = 0



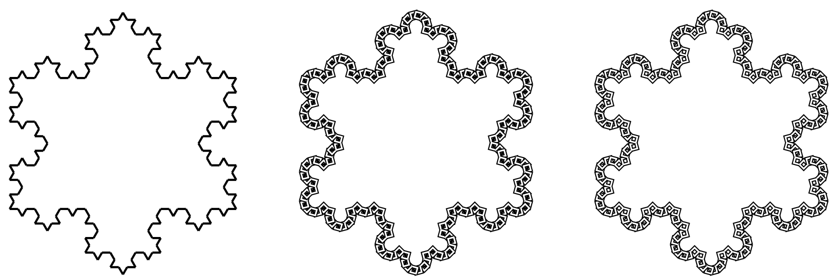
n = 1



n = 2



n = 3



If the box has side length $\epsilon = \dfrac{1}{3^n\sqrt{2}}$, the number of boxes needed is $N(\epsilon) = 3\,(4^n)$.

$$\begin{aligned} d &= \lim_{\epsilon\to 0} \frac{\ln\big(N(\epsilon)\big)}{\ln(\frac{1}{\epsilon})} = \lim_{n\to\infty} \frac{\ln\left(3\,(4^n)\right)}{\ln\left(3^n\sqrt{2}\right)} \\ &= \lim_{n\to\infty} \frac{\ln\left(4^n\right)+\ln(3)}{\ln(3^n)+\ln(\sqrt{2})} = \frac{\ln(4)}{\ln(3)} \end{aligned}$$

11.4.3

Upon each iteration, the box to be descended into is divided into twenty new boxes with the side length reduced by a factor of three. From this we can calculate

$$\epsilon = \left(\frac{1}{3}\right)^n \qquad N(\epsilon) = 20^n$$

and the box dimension is

$$d = \lim_{\epsilon\to 0} \frac{\ln\big(N(\epsilon)\big)}{\ln\left(\frac{1}{\epsilon}\right)} = \lim_{n\to\infty} \frac{\ln(20^n)}{\ln(3^n)} = \frac{\ln(20)}{\ln(3)}$$

11.4.5

For the 4-dimensional Menger hypersponge, each iteration of the construction descends into 48 smaller 4-dimensional hypercubes with the side length reduced by a factor of three. (See Exercise 11.3.9 for the derivation.) From this we can calculate

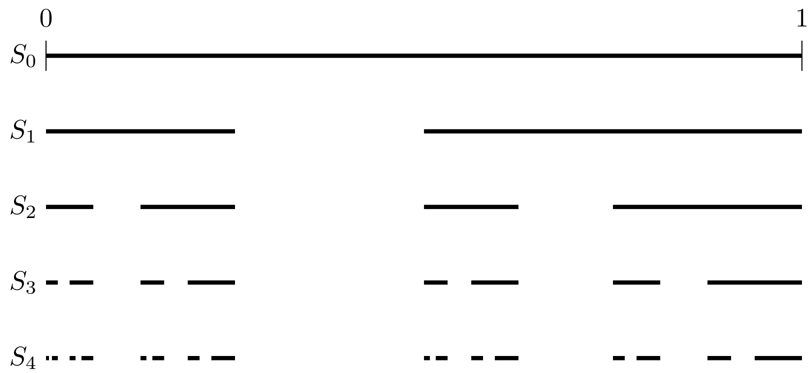
$$\epsilon = \left(\frac{1}{3}\right)^n \qquad N(\epsilon) = 48^n$$

and the box dimension is

$$d = \lim_{\epsilon \to 0} \frac{\ln\left(N(\epsilon)\right)}{\ln\left(\frac{1}{\epsilon}\right)} = \lim_{n \to \infty} \frac{\ln(48^n)}{\ln(3^n)} = \frac{\ln(48)}{\ln(3)}$$

11.4.7

a)



b)

We will take the covering interval to be $\frac{1}{4}$ for S_1 requiring 3 boxes, $\frac{1}{4^2}$ for S_2 requiring 3^2 boxes, $\frac{1}{4^3}$ for S_3 requiring 3^3 boxes, and so on and so forth. From this we can calculate

$$\epsilon = \left(\frac{1}{4}\right)^n \quad N(\epsilon) = 3^n$$

and the box dimension is

$$d = \lim_{\epsilon \to 0} \frac{\ln\left((N(\epsilon)\right)}{\ln\left(\frac{1}{\epsilon}\right)} = \lim_{n \to \infty} \frac{\ln(3^n)}{\ln(4^n)} = \frac{\ln(3)}{\ln(4)}$$

c)

S_∞ is self-similar. However, its self-similar pieces are not scaled by the same amount, unlike most of the other fractals we've seen.

11.4.9

Upon each iteration we descended into $p^2 - m^2$ new boxes with the side length reduced by a factor of p. From this we can calculate

$$\epsilon = \left(\frac{1}{p}\right)^n \quad N(\epsilon) = p^2 - m^2$$

and the box dimension is

$$d = \lim_{\epsilon \to 0} \frac{\ln\left((N(\epsilon)\right)}{\ln\left(\frac{1}{\epsilon}\right)} = \lim_{n \to \infty} \frac{\ln(p^2 - m^2)}{\ln(p)}$$

11.5 Pointwise and Correlation Dimensions

11.5.1

This is a challenging problem, and consequently the topic of several publications. See Grassberger and Procaccia (1983) for guidance.

12

Strange Attractors

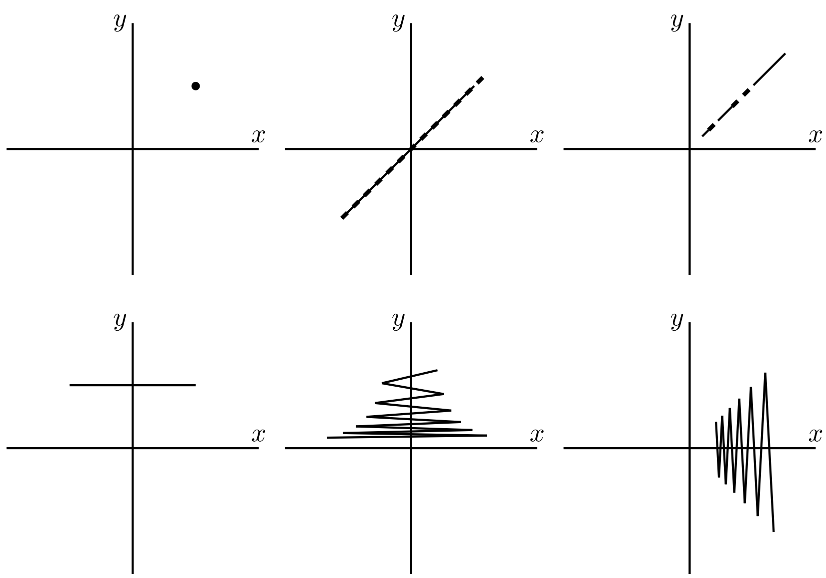
12.1 The Simplest Examples

12.1.1

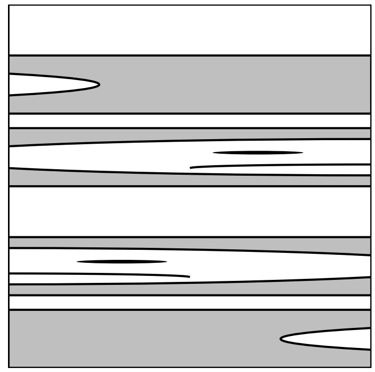
$$x_{n+1} = ax_n \qquad y_{n+1} = by_n$$

The equations are uncoupled, which makes things much simpler. The only thing that matters for convergence is the magnitude of the coefficients a and b. Less than 1, equal to 1, or greater than 1 will converge, remain a constant distance from or diverge from 0 along that coordinate respectively. The sign of a and b will determine whether or not the behavior is alternating.

Below are the types of graphs that can occur. Not every case is explicitly shown because some graphs look the same in forwards and backwards time for different a and b values.



12.1.3



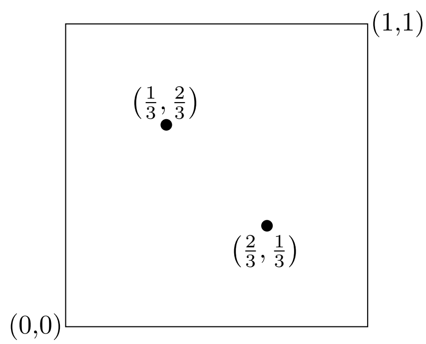
12.1.5

a)

$B(x,y) = (0.a_2a_3a_4..., 0.a_1b_1b_2b_3...)$ To describe the dynamics more transparently, associate the symbol $...b_3b_2b_1.a_1a_2a_3...$ with (x,y) simply by placing x and y back-to-back. Then $B(x,y) = ...b_3b_2b_1a_1.a_2a_3...$ in this notation. In other words, B just shifts the binary point one place to the right.

b)

In the notation above, ...1010.1010... and ...0101.0101... are the only period-2 points. They correspond to $(\frac{2}{3}, \frac{1}{3})$ and $(\frac{1}{3}, \frac{2}{3})$.



c)

The points with periodic orbits are when x is a repeating rational number and the digits of y are in the reverse order of x.

$$x = .\overline{a_1a_2a_3\ldots a_n} \qquad y = .\overline{a_n\ldots a_3a_2a_1}$$

$$(x,y) = \overline{a_1a_2a_3\ldots a_n}.\overline{a_1a_2a_3\ldots a_n}$$

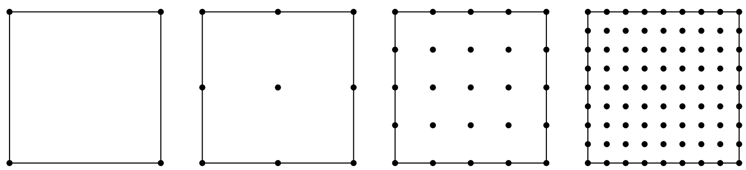
There is a countable number of repeating binary numbers, so there is a countable number of periodic orbits for the baker's map.

d)

Pick $x =$ irrational and $y =$ anything. The orbit will never repeat because irrationals never repeat. There is an uncountable number of irrational numbers, so there is an uncountable number of aperiodic orbits.

e)

There are dense orbits. The plan will be to fill out a grid of points where the coordinates are integer multiples of 2^0, then 2^{-1}, then 2^{-2}, then 2^{-3},



and so on.

Next, we'll need some notation. p and q are sequences of 0's and 1's. A point in the unit square can be represented by $(0.p, 0.q)$ for some p and q. $\tilde{q}$ is the reverse sequence of q.

Now take all the points in the first square and find their binary representation as $(0.p_1^1, 0.q_1^1)$, $(0.p_2^1, 0.q_2^1)$, $(0.p_3^1, 0.q_3^1)$, and $(0.p_4^1, 0.q_4^1)$, and set

$$x = 0.\tilde{q}_1^1 p_1^1 \tilde{q}_2^1 p_2^1 \tilde{q}_3^1 p_3^1 \tilde{q}_4^1 p_4^1$$

Now take all the points in the second square and find their binary representation as

$$(0.p_1^2, 0.q_1^2), (0.p_2^2, 0.q_2^2), \ldots, (0.p_9^2, 0.q_9^2)$$

Append to x with the same pattern as before.

$$x = 0.\tilde{q}_1^1 p_1^1 \ldots \tilde{q}_4^1 p_4^1 \tilde{q}_1^2 p_1^2 \ldots \tilde{q}_9^2 p_9^2$$

and so on.

Set $y = 0$ and we have a dense orbit. Applying the baker's map enough times

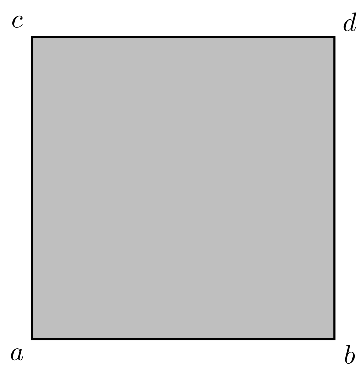
$$\begin{aligned} (x, y) &= 0.\tilde{q}_1^1 p_1^1 \ldots \tilde{q}_4^1 p_4^1 \tilde{q}_1^2 p_1^2 \ldots \tilde{q}_9^2 p_9^2 \ldots \\ B^n(x, y) &= \ldots \tilde{q}_i^j . p_i^j \ldots = (0.p_i^j \ldots, 0.q_i^j \ldots) \end{aligned}$$

will get to any grid point with some junk on the end of the binary representation. The distance between grid points goes to zero, so this orbit can get arbitrarily close to any point in the unit square within a finite number of steps. The orbit it dense.

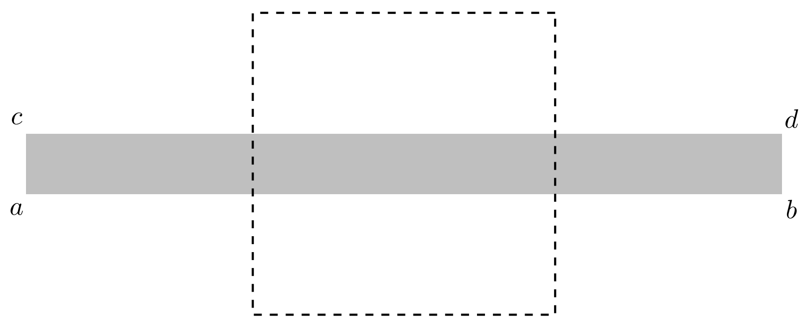
12.1.7

First, we need to make a function definition for the map.

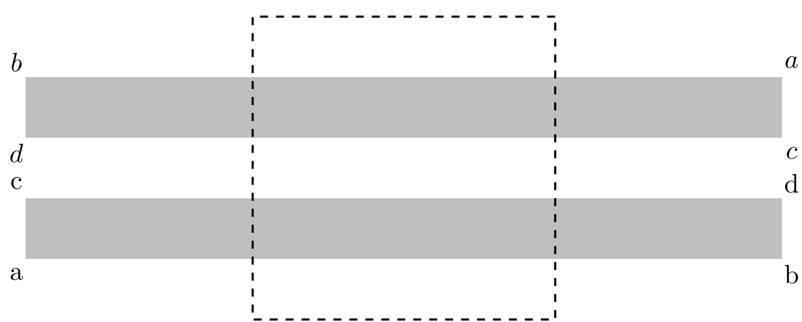
The square will have corners $(\pm 1, \pm 1)$ to make scaling and shifting easier.



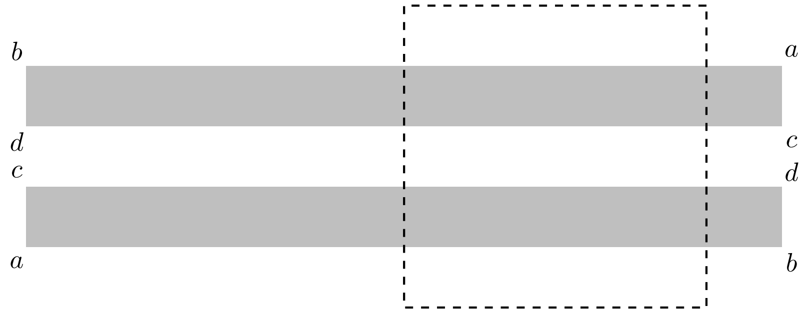
Scale the square by a factor of w in the x direction and h in the y direction.



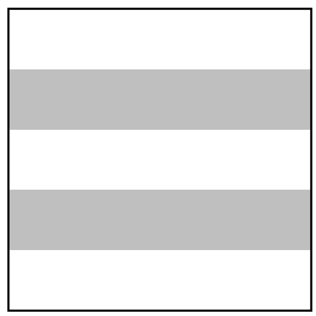
Instead of making the horseshoe shape, we're going to make two copies. The top and bottom copy correspond to $-1 \leq x < 0$ and $0 \leq x \leq 1$ respectively. We're also going to flip the top piece in both directions before shifting up.



Next, we shift both the pieces to the left.



And finally chop off the excess.



The series of steps

$$F(x,y) = \begin{cases} x' = wx \quad x'' = -x' \quad x''' = x'' - s_x & -1 \le x < 0 \\ y' = hy \quad y'' = -y' + s_y \quad y''' = y'' & \\ x' = wx \quad x'' = x' \quad x''' = x'' - s_x & 0 \le x \le 1 \\ y' = hy \quad y'' = y' - s_y \quad y''' = y'' & \end{cases}$$

Putting all the steps together

$$F(x,y) = \begin{cases} -wx - s_x & -1 \le x < 0 \\ -hy + s_y & \\ wx - s_x & 0 \le x \le 1 \\ hy - s_y & \end{cases}$$

But this is actually missing the chopping-off step. We can restrict the mapping to lie inside the square instead and not worry about the original. (This is assuming s_y isn't too big to chop off anything in the vertical direction.)

$$x_{n+1} = \pm w x_n - s_x \qquad y_{n+1} = \pm h y_n \mp s_y$$

$$-1 \le x_{n+1} = \pm w x_n - s_x \le 1$$

$$-1 \le w x_n - s_x \le 1 \qquad -1 \le w x_n + s_x \le 1$$

$$\frac{-1 + s_x}{w} \le x_n \le \frac{1 + s_x}{w} \qquad \frac{-1 - s_x}{w} \le x_n \le \frac{1 - s_x}{w}$$

It looks like the only points that survived from the original square are in two vertical strips.

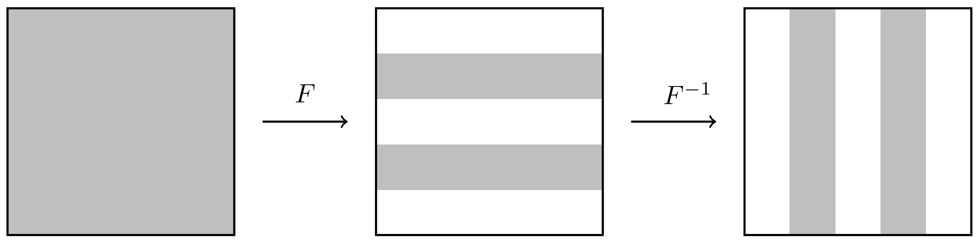
Next we can find the inverse map explicitly.

$$x_n = \frac{\pm x_{n+1} + s_x}{w} \qquad y_n = \frac{\pm y_{n+1} + s_y}{h}$$

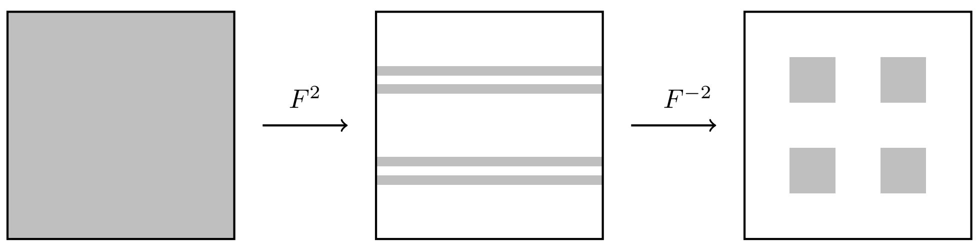
We picked $w = 5 \quad h = \frac{1}{5} \quad s_x = 2 \quad s_y = \frac{2}{5}$ so that the map came out symmetrically.

a)

Applying the map and then the inverse while remembering to chop off the excess gives

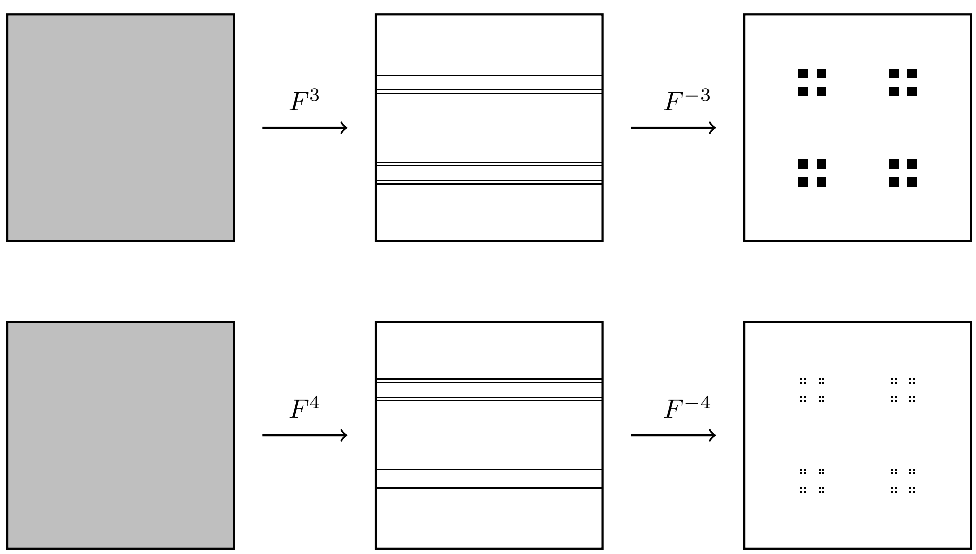


b)



c)

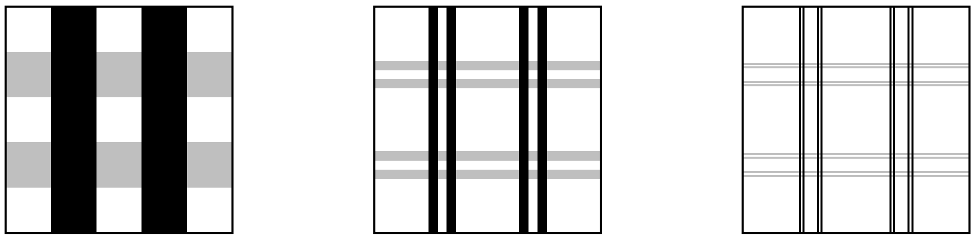
Continuing gives



Note: F^3 and F^4 are different, but the detail is too small to see. The color was also changed to black to show detail.

If we continue, remembering to chop off the excess at every application of F and F^{-1} to the square, we'll get the points that never get chopped off. The invariant set of the map is the Cartesian product of the Cantor set with itself, also known as *Cantor dust*.

Intuitively, this occurs because F and F^{-1} look just like each other but rotated one-quarter turn anti-clockwise. The intersection of these two sets is Cantor dust.



12.1.9

$$x_{n+1} = x_n + y_{n+1} = x_n + y_n + k\sin(x_n) \qquad y_{n+1} = y_n + k\sin(x_n)$$

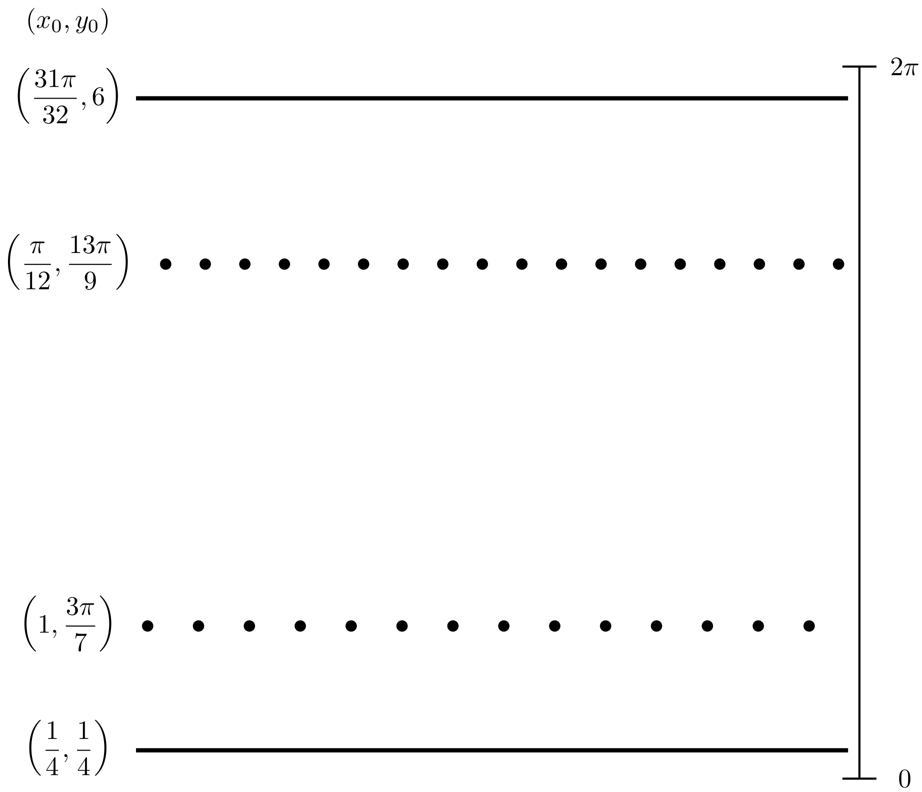
a)

$$J = \begin{pmatrix} 1 + k\cos(x) & 1 \\ k\cos(x) & 1 \end{pmatrix}$$

$$\det(J) = \big(1 + k\cos(x)\big)\big(1\big) - \big(1\big)\big(k\cos(x)\big) = 1$$

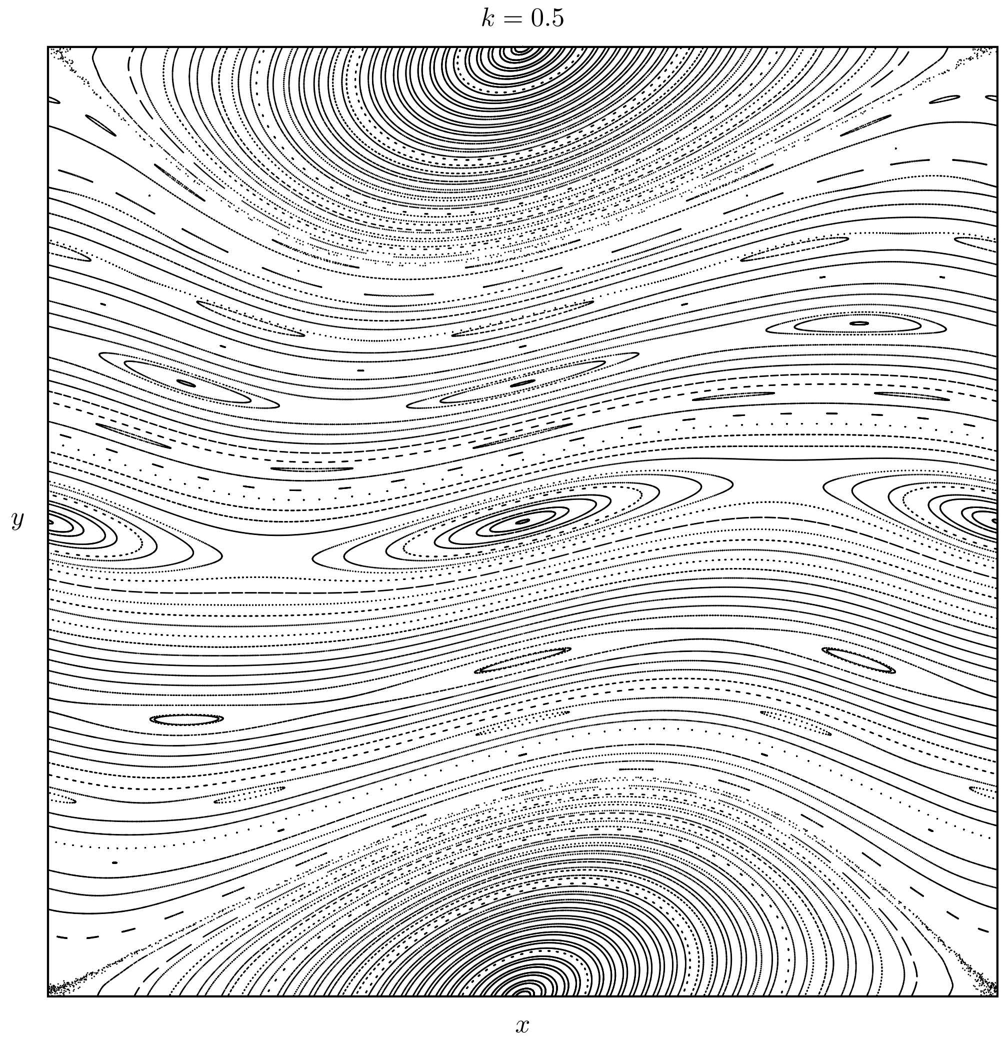
b)

$$k = 0 \Rightarrow x_{n+1} = x_n + y_{n+1} \quad y_{n+1} = y_n \Rightarrow x_n = x_0 + ny_0$$

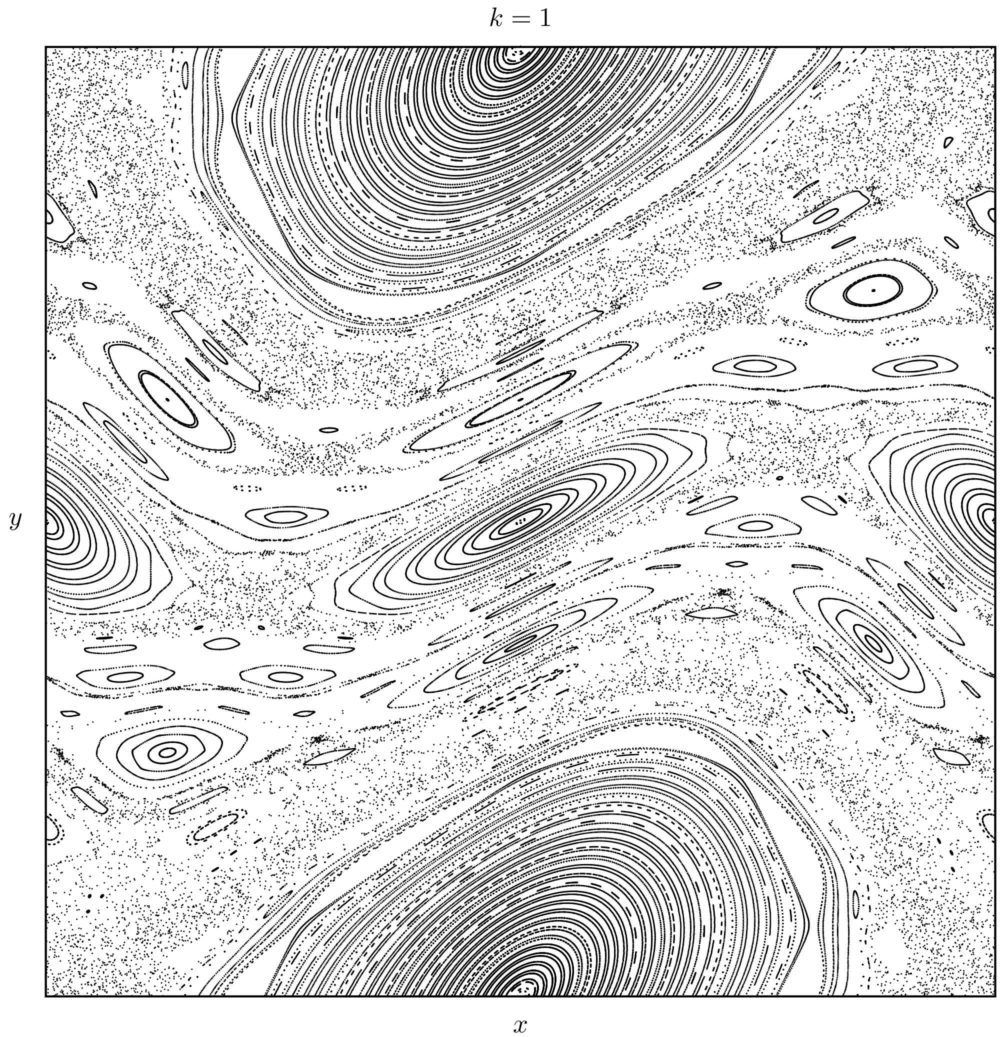


The orbits are periodic if y_0 is a rational multiple of π, otherwise the orbits are quasiperiodic.

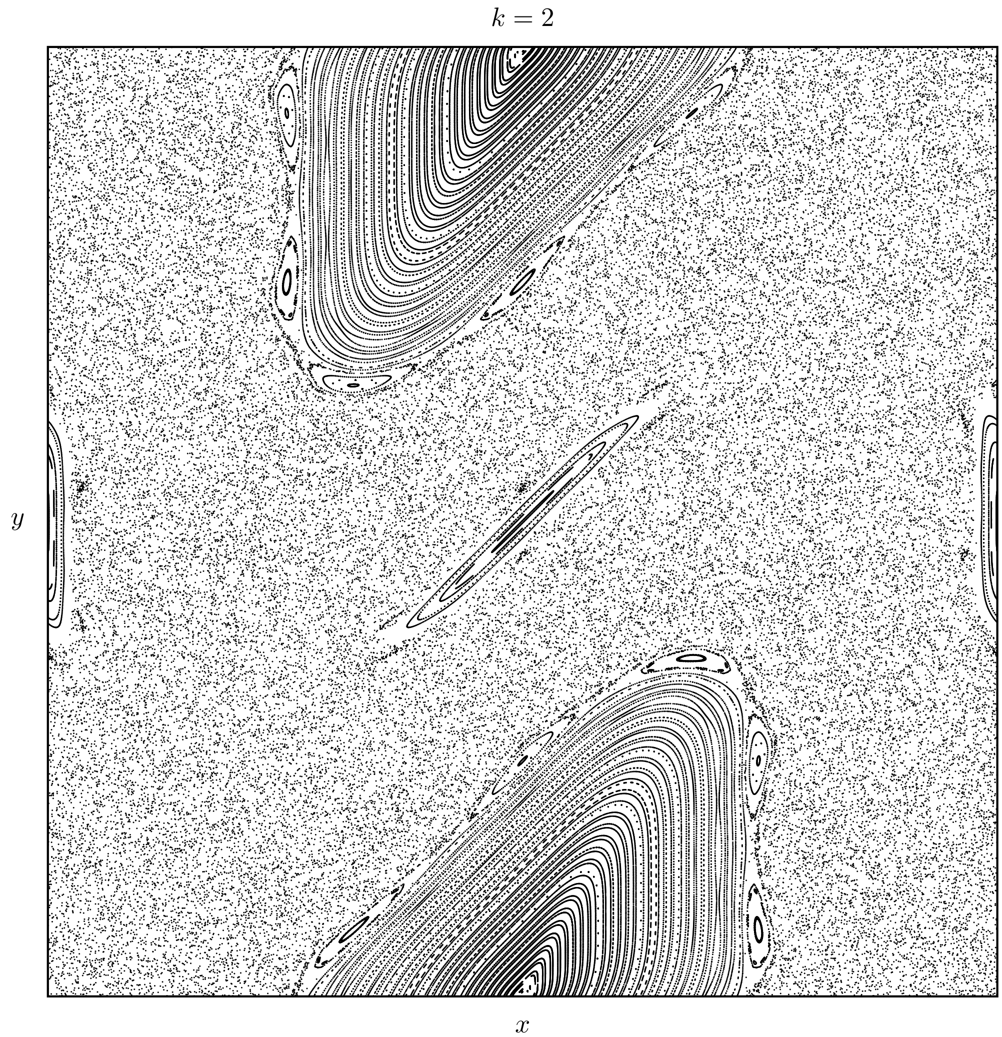
c)



d)



e)



12.2 Hénon Map

12.2.1

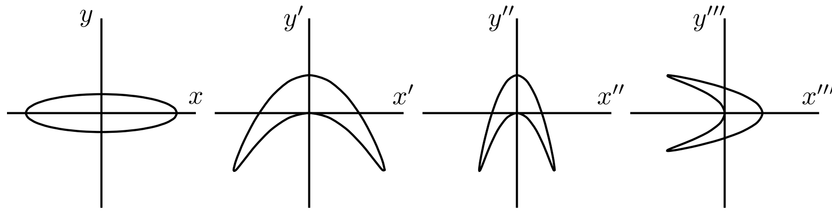
$$T'''(x,y) = (y,x) \qquad T''(x,y) = (bx,y) \qquad T'(x,y) = (x, 1+y-ax^2)$$

$$\begin{aligned} T'''(T''(T'(x,y))) &= T'''(T''(x,1+y-ax^2)) = T'''(bx,1+y-ax^2) \\ &= (1+y-ax^2, bx) \end{aligned}$$

12.2.3

a)

$a = 0.25 \quad b = 0.5$



(a) (b) (c) (d)

12.2.5

$$J = \begin{pmatrix} -2ax^* & 1 \\ b & 0 \end{pmatrix}$$

$$(-2ax^* - \lambda)(0 - \lambda) - b = 0$$

$$\lambda^2 + 2ax^*\lambda - b = 0$$

$$\lambda = \frac{-2ax^* \pm \sqrt{(2ax^*)^2 - 4(-b)}}{2(1)}$$

$$= -ax^* \pm \sqrt{(ax^*)^2 + b}$$

12.2.7

$$x_{n+2} = bx_n + 1 - a(y_n + 1 - ax_n^2)^2 \qquad y_{n+2} = b(y_n + 1 - ax_n^2)$$

$$x = bx + 1 - a(y + 1 - ax^2)^2 \qquad y = b(y + 1 - ax^2)$$

We can solve for y in the right-hand equation by rearranging a bit.

$$y = \frac{b(ax^2 - 1)}{b - 1}$$

We can then substitute for y in the left-hand equation.

$$x = bx + 1 - a(y + 1 - ax^2)^2 = bx + 1 - a\left(\frac{y}{b}\right)^2 = bx + 1 - \frac{a}{b^2}\left(\frac{b(ax^2 - 1)}{b - 1}\right)^2$$

Moving everything to one side and simplifying gives an equation without y.

$$0 = (b - 1)^3 x + (b - 1)^2 - a(ax^2 - 1)^2$$

The roots of this equation will be the x coordinates of the 2-cycle. However, two of the four roots to this equation will also be fixed points of the Hénon map $x = y + 1 - ax^2 = bx + 1 - ax^2$ and we can divide them out.

$$0 = \frac{(b-1)^3 x + (b-1)^2 - a(ax^2-1)^2}{bx + 1 - ax^2 - x} = a^2x^2 + a(b-1)x - a + (b-1)^2$$

which has solutions

$$x = \frac{1 - b \pm \sqrt{4a - 3(b-1)^2}}{2a} \Rightarrow y = \frac{b\big(1 - b \mp \sqrt{4a - 3(b-1)^2}\big)}{2a} \qquad a \neq 0$$

We have to throw out the $a = 0$ case since it's not a 2-cycle.

The coordinates must be real for the 2-cycle to exist, meaning

$$4a - 3(b-1)^2 > 0 \Rightarrow a > a_1 = \frac{3}{4}(1-b)^2$$

To determine stability, we'll have to linearize and find when all $|\lambda| < 1$.

$$J = \begin{pmatrix} \frac{\mathrm{d}x_{n+2}}{\mathrm{d}x_n} & \frac{\mathrm{d}x_{n+2}}{\mathrm{d}y_n} \\ \frac{\mathrm{d}y_{n+2}}{\mathrm{d}x_n} & \frac{\mathrm{d}y_{n+2}}{\mathrm{d}y_n} \end{pmatrix} = \begin{pmatrix} b + 4a^2x(y + 1 - ax^2) & -2a(y+1-ax^2) \\ -2abx & b \end{pmatrix}$$

$$J_{(x^*,y^*)} = \begin{pmatrix} b + \frac{4a^2x^*y^*}{b} & -\frac{2ay^*}{b} \\ -2abx^* & b \end{pmatrix}$$

where we simplified using the y equation for the 2-cycle.

$$\begin{aligned} \lambda_\pm &= \frac{2a^2x^*y^* + b^2 \pm 2\sqrt{a^2x^*y^*(a^2x^*y^* + b^2)}}{b} \\ &= \frac{2(a^2x^*y^*) + b^2 \pm 2\sqrt{(a^2x^*y^*)^2 + b^2(a^2x^*y^*)}}{b} \end{aligned}$$

We can use the equations from finding the 2-cycle to simplify.

$$a^2x^*y^* = a^2x^*\frac{b\big(a(x^*)^2 - 1\big)}{b-1} = \frac{abx^*\big(a^2(x^*)^2 - a\big)}{b-1}$$

$$0 = a^2(x^*)^2 + a(b-1)x^* - a + (b-1)^2 \qquad a^2(x^*)^2 - a = -(b-1)\big(ax^* + (b-1)\big)$$

$$a^2x^*y^* = -abx^*\big(ax^* + (b-1)\big) = -b\big(a^2(x^*)^2 + a(b-1)x^*\big)$$

$$0 = a^2(x^*)^2 + a(b-1)x^* - a + (b-1)^2 \qquad a^2(x^*)^2 + a(b-1)x^* = a - (b-1)^2$$

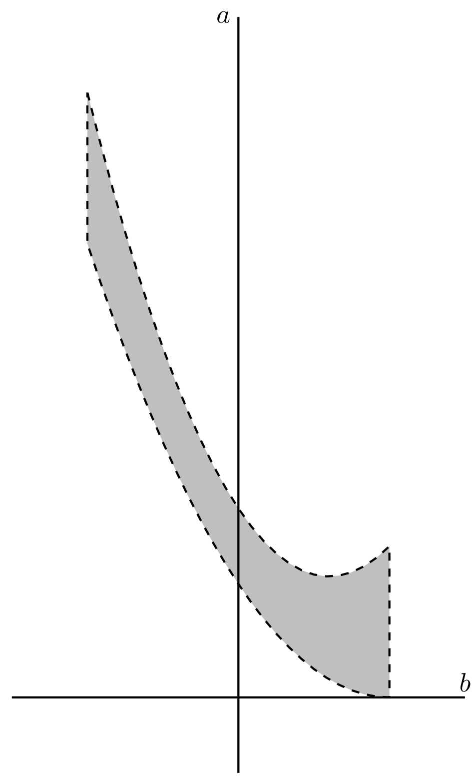
$$a^2x^*y^* = -b\big(a - (b-1)^2\big) = b\big((b-1)^2 - a\big)$$

$$\begin{aligned} \lambda_\pm &= \frac{2(a^2x^*y^*) + b^2 \pm 2\sqrt{(a^2x^*y^*)^2 + b^2(a^2x^*y^*)}}{b} \\ &= \frac{2b\big((b-1)^2 - a\big) + b^2 \pm 2\sqrt{b^2\big((b-1)^2 - a\big)^2 + b^3\big((b-1)^2 - a\big)}}{b} \\ &= 2\big((b-1)^2 - a\big) + b \pm 2\sqrt{\big((b-1)^2 - a\big)^2 + b\big((b-1)^2 - a\big)} \end{aligned}$$

Finally, we look at $|\lambda_{\pm}| < 1$ to determine stability. This is a bit of a headache because we have to worry about complex eigenvalues. Eventually we obtain the following by assuming a positive argument to the square root, a negative argument to the square root, and the inequality from the existence of the 2-cycle.

$$a < \frac{1}{4}(5b^2 - 6b + 5) \qquad -1 < b < 1 \qquad a > \frac{3}{4}(b-1)^2$$

The intersection of all three inequalities is plotted below.



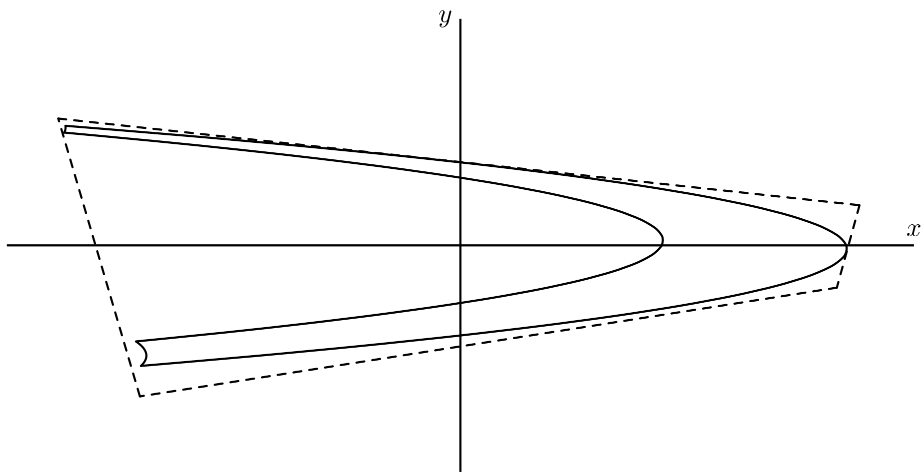
12.2.9

a)

$$L_1(t) = (-1.33 + 2.65t, 0.42 - 0.287t) \quad t \in [0,1]$$
$$L_2(t) = (1.32 - 0.075t, 0.133 - 0.273t) \quad t \in [0,1]$$
$$L_3(t) = (1.245 - 2.305t, -0.14 - 0.36t) \quad t \in [0,1]$$
$$L_4(t) = (-1.06 - 0.27t, -0.5 + 0.92t) \quad t \in [0,1]$$

$$H_1(t) = \big(0.42 - 0.287t + 1 - a(-1.33 + 2.65t)^2, b(-1.33 + 2.65t)\big) \quad t \in [0,1]$$
$$H_2(t) = \big(0.133 - 0.273t + 1 - a(1.32 - 0.075t)^2, b(1.32 - 0.075t)\big) \quad t \in [0,1]$$
$$H_3(t) = \big(-0.14 - 0.36t + 1 - a(1.245 - 2.305t)^2, b(1.245 - 2.305t)\big) \quad t \in [0,1]$$
$$H_4(t) = \big(-0.5 + 0.92t + 1 - a(-1.06 - 0.27t)^2, b(-1.06 - 0.27t)\big) \quad t \in [0,1]$$

$a = 1.4 \quad b = 0.5$



b)

For this part, it's easier to define the sides of the quadrilateral in Cartesian coordinates.

$$y_1 = -0.108302x + 0.275958$$
$$y_2 = 3.64x - 4.6718$$
$$y_3 = 0.156182x - 0.334447$$
$$y_4 = -3.40741x - 4.11185$$

We need to ensure that the top and bottom lines L_1 and L_3 bound the y value of the transformation for every given x value in the transformation.

$$\begin{aligned}
H_1 : x &= 0.42 - 0.287t + 1 - a(-1.33 + 2.65t)^2 \\
y_1 &= -0.108302(0.42 - 0.287t + 1 - 1.4(-1.33 + 2.65t)^2) + 0.275958 \\
&= 1.06477t^2 - 1.03771t + 0.390375 \\
y_3 &= 0.156182(0.42 - 0.287t + 1 - 1.4(-1.33 + 2.65t)^2) - 0.334447 \\
&= -1.5355t^2 + 1.49647t - 0.499447 \\
y_3 &\leq 0.3(-1.33 + 2.65t) \leq y_1 \\
y_3 &- 0.3(-1.33 + 2.65t) \leq 0 \leq y_1 - 0.3(-1.33 + 2.65t) \\
&- 1.5355t^2 + 0.70147t - 0.100447 \leq 0 \leq 1.06477t^2 - 1.83271t + 0.789375
\end{aligned}$$

We then would have to find the local minimum or maximum of these parabolas to determine if the inequality is satisfied, keeping in mind that $t \in [0, 1]$ for the parabolas. We would have to do this for the H_2, H_3, and H_4 curves as well.

Similarly, we need to ensure that the left and right lines, L_4 and L_2 respectively, bound the x value of the transformation for every given y value in the transformation.

Verification of the other inequalities is left to the reader.

12.2.11

For $b = 0$, the map reduces to a one-dimensional map after one iteration.

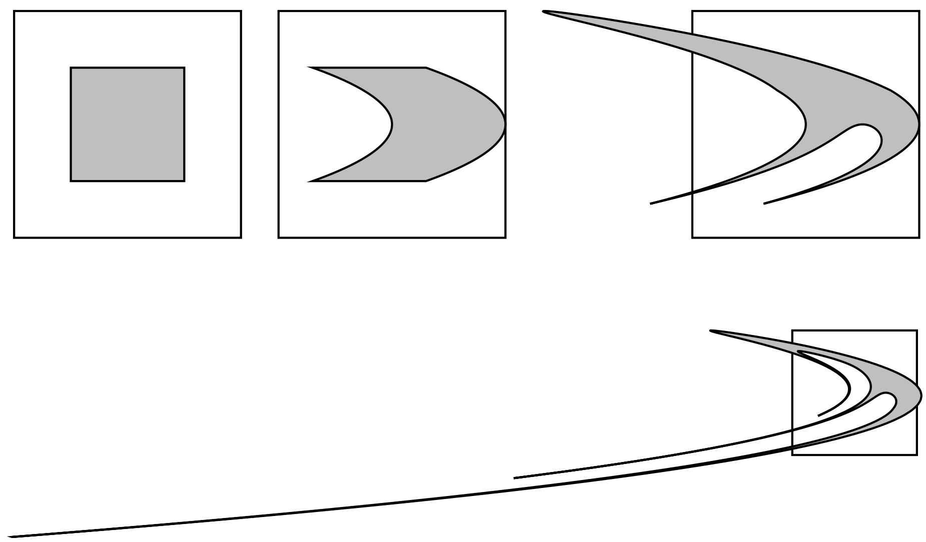
$$x_1 = y_0 + 1 - ax_n^2 \qquad y_1 = (0)x_0 = 0$$

$$x_{n+1} = y_n + 1 - ax_n^2 = 1 - ax_n^2 \qquad y_{n+1} = 0 \quad \text{for } n \neq 0$$

From then on, $y_n = 0$, making x_{n+1} effectively a function of x_n only.

12.2.13

A cursory investigation of the area preserving the Hénon map doesn't reveal much. Some iterations of a square centered at the origin are graphed below.



12.2.15

$$x_{n+1} = 1 + y_n - a|x_n|$$

$$y_{n+1} = bx_n$$

$$J = \begin{pmatrix} \pm a & 1 \\ b & 0 \end{pmatrix} \qquad \det(J) = -b$$

So the Lozi map contracts areas if $|\det(J)| = |-b| < 1 \Rightarrow -1 < b < 1$.

12.2.17

$$x_{n+2} = 1 + y_{n+1} - a|x_{n+1}| \qquad y_{n+2} = bx_{n+1}$$

$$x_{n+2} = 1 + bx_{n+1} - a\Big|1 + y_n - a|x_n|\Big| \qquad y_{n+2} = b\big(1 + y_n - a|x_n|\big)$$

$$x = 1 + bx - a\Big|1 + y - a|x|\Big| \qquad y = b\big(1 + y - a|x|\big)$$

2-cycle points

$$(x, y) = \left(\frac{1-a-b}{a^2+(b-1)^2}, \frac{b(1+a-b)}{a^2+(b-1)^2}\right) \qquad a > 1-b$$
$$= \left(\frac{1+a-b}{a^2+(b-1)^2}, \frac{b(1-a-b)}{a^2+(b-1)^2}\right) \qquad a > b-1$$

but along with the condition that $-1 < b < 1$, only $a > 1-b$ must be satisfied for the fixed points to exist.

Linearization

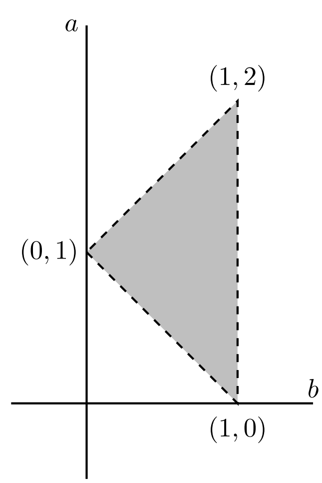
$$A = \begin{pmatrix} b + a^2 \ \mathrm{sgn}\big(x(1+y-a|x|)\big) & -a \ \mathrm{sgn} \\ -ab \ \mathrm{sgn}(x) & b \end{pmatrix}$$

$$A_{\left(\frac{1-a-b}{a^2+(b-1)^2}, \frac{b(1+a-b)}{a^2+(b-1)^2}\right)} = \begin{pmatrix} b-a^2 & -a \\ ab & b \end{pmatrix}$$

$$A_{\left(\frac{1+a-b}{a^2+(b-1)^2}, \frac{b(1-a-b)}{a^2+(b-1)^2}\right)} = \begin{pmatrix} b-a^2 & a \\ -ab & b \end{pmatrix}$$

Plugging in the coordinates and solving for $|\lambda_\pm| < 1$ along with the other conditions gives the stable region.

$$|\lambda_\pm| = \frac{-a^2 + 2b \pm a\sqrt{a^2-4b}}{2} < 1 \qquad -1 < b < 1 \qquad a > 1-b$$



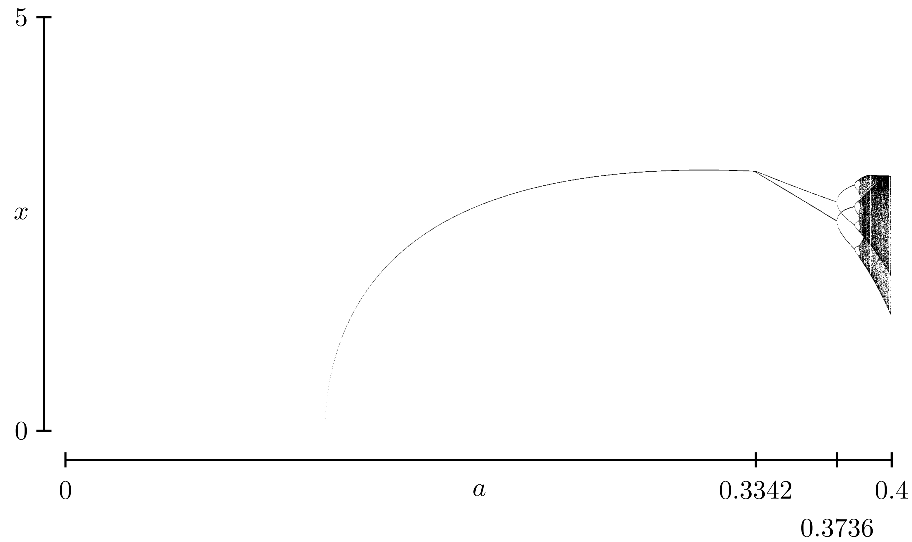
12.3 Rössler System

12.3.1

a)

Finding when the Hopf bifurcation occurs is most easily done with an orbit diagram.

$b = 2 \quad c = 4$



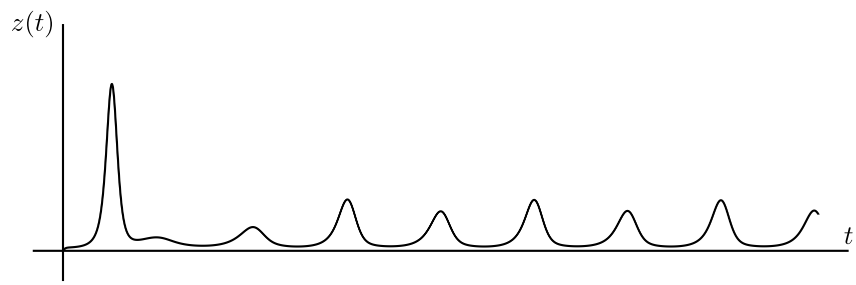
This graph is made by numerically integrating the system until the trajectory reaches the attractor; then the positive values of x are recorded every time y changes sign for each a.

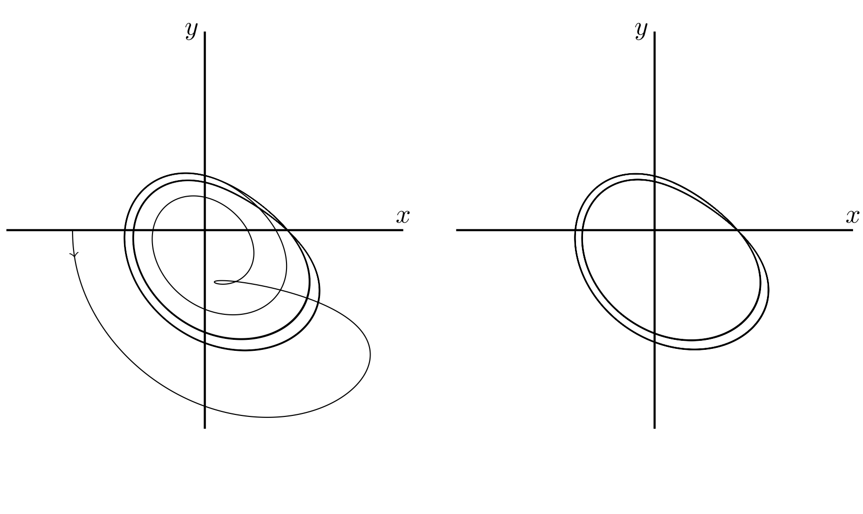
The first branching occurs at $a \approx 0.3342$, which is the Hopf bifurcation, and the next branching occurs at $a \approx 0.3736$, which is the period-doubling bifurcation.

b)

$a = 0.3342 \quad b = 2 \quad c = 4$

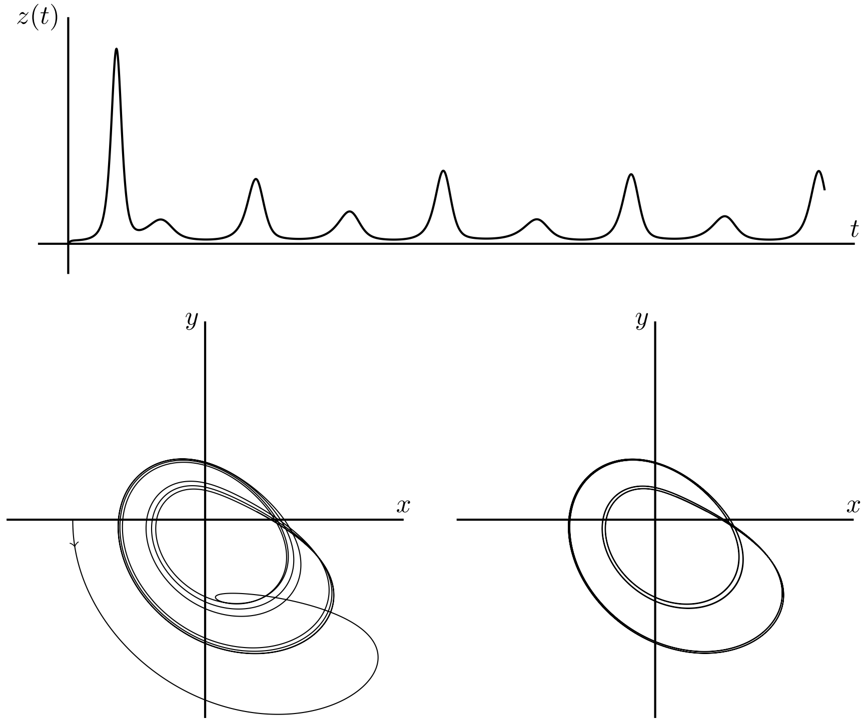
Initial condition $(x, y, z) = (-5, 0, 0) \qquad t \in [0, 50]$





$a = 0.3736 \quad b = 2 \quad c = 4$

Initial condition $(x, y, z) = (-5, 0, 0) \qquad t \in [0, 50]$



12.3.3

The Rössler system lacks the symmetry of the Lorenz system. The Lorenz equations are invariant under the transformation $(x, y, z) \to (-x, -y, z)$, so if $\big(x(t), y(t), z(t)\big)$ is a solution to the Lorenz system, then $\big(-x(t), -y(t), z(t)\big)$ is also a solution.

12.4 Chemical Chaos and Attractor Reconstruction

12.4.1

The general equation of an ellipse is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

such that A, B, C, D, E, and F are constants and (x, y) is a set of points. The ellipse is a non-degenerate real ellipse if $B^2 - 4AC < 0$ and

$$C \begin{vmatrix} A & \frac{B}{2} & \frac{D}{2} \\ \frac{B}{2} & C & \frac{E}{2} \\ \frac{D}{2} & \frac{E}{2} & F \end{vmatrix} = C\Delta < 0$$

Otherwise, if $C\Delta > 0$ then the ellipse is imaginary, and if $\Delta = 0$ then the ellipse is a point.

Hence, if $(x, y) = \big(\sin(t), \sin(t+\tau)\big)$ is an ellipse, then we should be able to find constants A, B, C, D, E, and F such that

$$A\sin^2(t) + B\sin(t)\sin(t+\tau) + C\sin^2(t+\tau) + D\sin(t) + E\sin(t+\tau) + F = 0$$

Simplifying using symbolic mathematics software gives a big mess.

$$\begin{aligned}
&A\sin^2(t) - A\cos^2(t) + A - B\cos(\tau)\cos^2(t) + B\cos(\tau)\sin^2(t)\\
&+ 2B\sin(\tau)\sin(t)\cos(t) + B\cos(\tau) - C\sin^2(\tau)\sin^2(t) - C\cos^2(\tau)\cos^2(t)\\
&+ C\sin^2(\tau)\cos^2(t) + C\cos^2(\tau)\sin^2(t) + 4C\sin(\tau)\cos(\tau)\sin(t)\cos(t)\\
&+ C + 2D\sin(t) + 2E\sin(\tau)\cos(t) + 2E\cos(\tau)\sin(t) + 2F = 0
\end{aligned}$$

But we know that all the like t terms must cancel, giving

$$\begin{array}{lcl}
\sin^2(t) & : & A + B\cos(\tau) - C\sin^2(\tau) + C\cos^2(\tau) = 0\\
\sin(t)\cos(t) & : & 2B\sin(\tau) + 4C\sin(\tau)\cos(\tau) = 0\\
\cos^2(t) & : & -A - B\cos(\tau) - C\cos^2(\tau) + C\sin^2(\tau) = 0\\
\sin(t) & : & 2D + 2E\cos(\tau) = 0\\
\cos(t) & : & 2E\sin(\tau) = 0\\
1 & : & A + B\cos(\tau) + C + 2F = 0
\end{array}$$

which is really a homogeneous matrix equation with solution

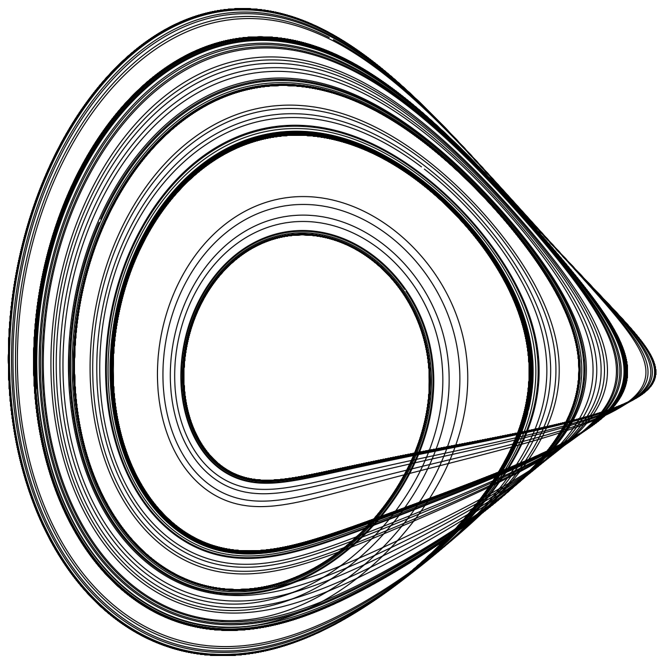
$$\begin{pmatrix} A\\ B\\ C\\ D\\ E\\ F \end{pmatrix} = \begin{pmatrix} 1\\ -2\cos(\tau)\\ 1\\ 0\\ 0\\ -\sin^2(\tau) \end{pmatrix}$$

Therefore the trajectory in Figure 12.4.5 is an ellipse.

12.4.3

Following is the delayed trajectory. The initial conditions were $(x, y, z) = (-5, 0, 0)$, and the simulation was run until $t = 1000$, with the initial transient discarded.

$a = 0.4 \quad b = 2 \quad c = 4 \qquad \tau = 1.5$



Trajectories circle around the middle of the attractor in about 6 units of time, so the delay is about one-quarter of this period.

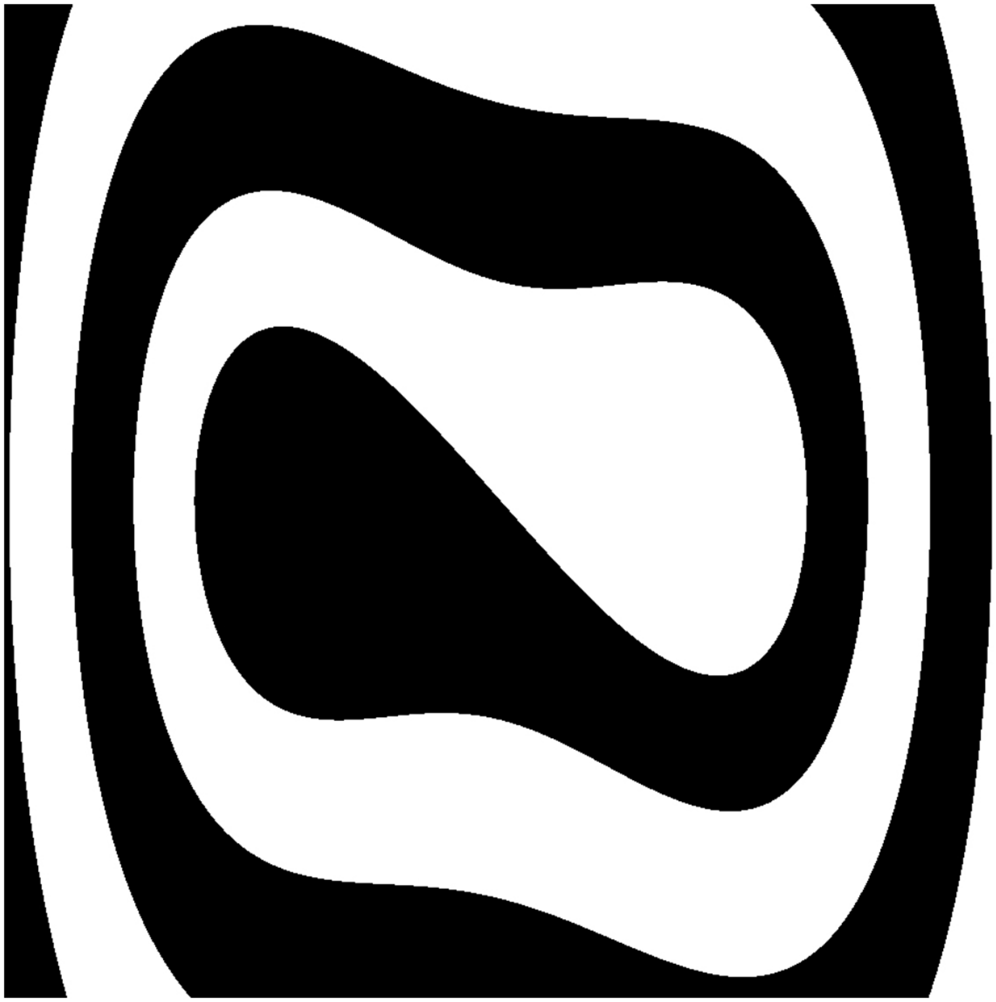
12.5 Forced Double-Well Oscillator

12.5.1

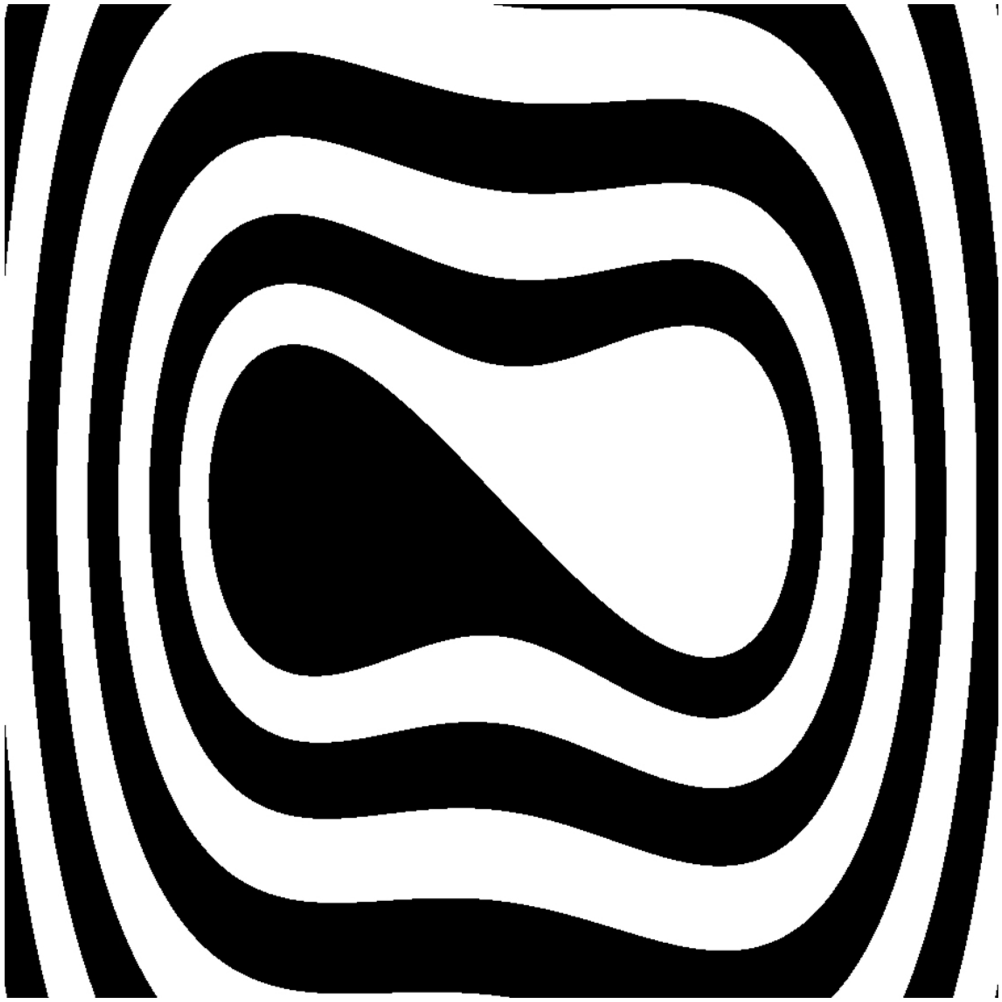
$$\dot{x} = y \qquad \dot{y} = -\delta y + x - x^3 + F\cos(\omega t) \qquad \omega = 1 \quad F = 0$$

The following simulations have intial conditions of $x, y \in [-2.5, 2.5]$ in a 900×900 pixel image. After simulating for 200π units of time (100 cycles)—which is a bit silly since $F = 0$ means there is no drive—points that were within 0.1 of the -1 and $+1$ basins with a speed less than 0.1 were considered to have settled in that basin. A black pixel corresponds to that initial condition settling into the -1 basin, and white corresponds to the $+1$ basin.

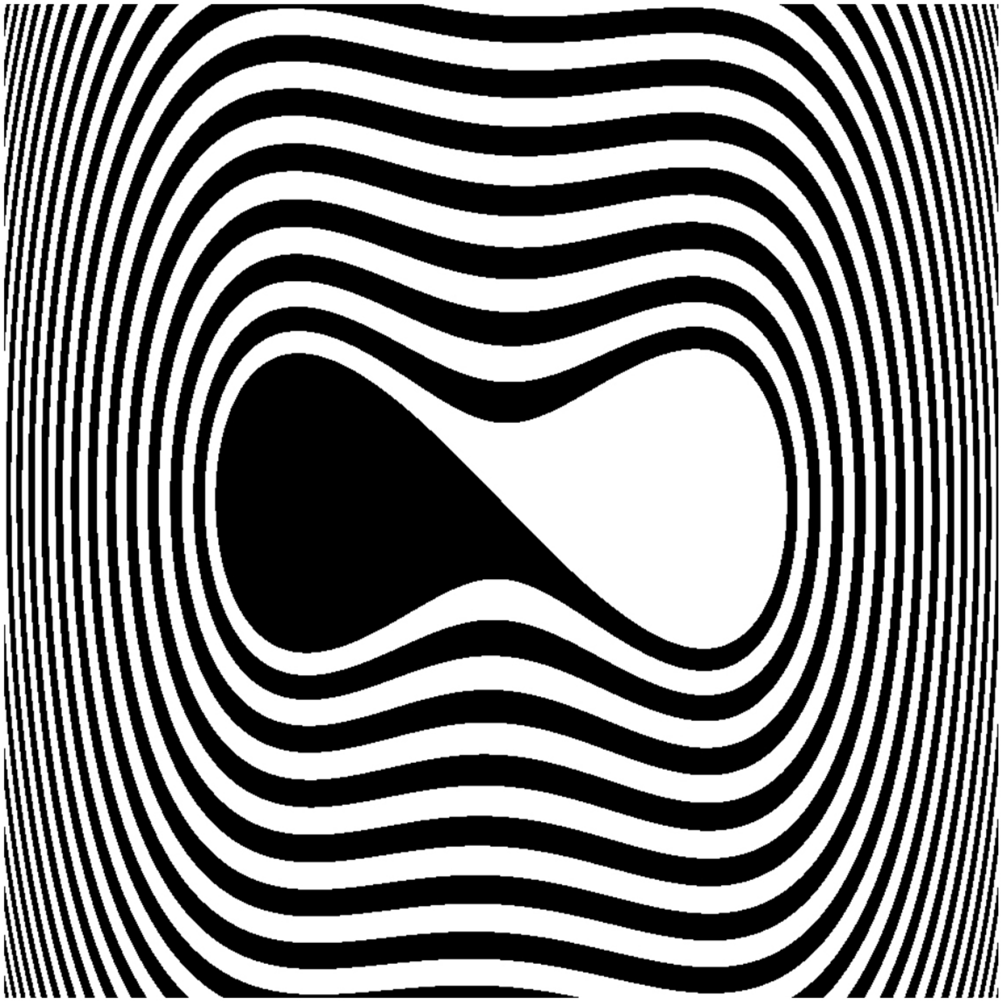
$\delta = 0.25$



$\delta = 0.125$



$\delta = 0.0625$

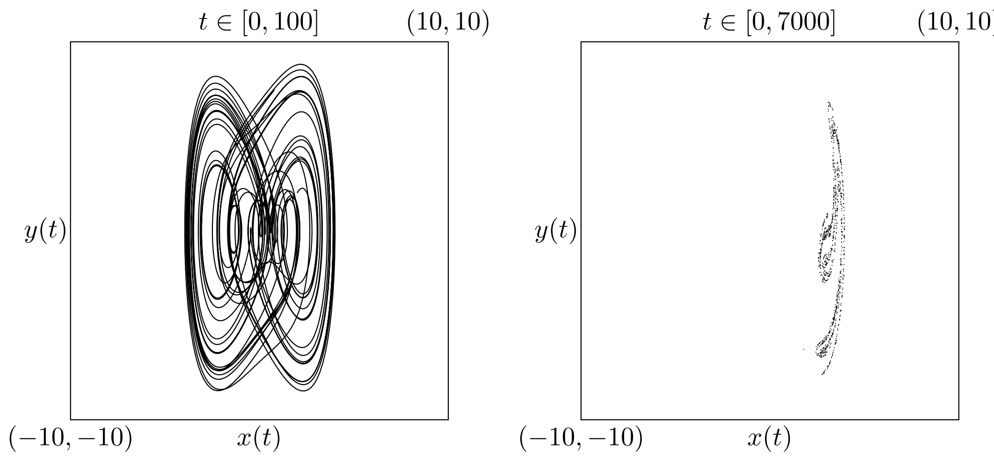


The basins become thinner as the damping decreases, and they appear to wrap around each other more and more as δ tends to zero. This will make for sensitive dependence on initial conditions, which means that accurate prediction of which basin an initial condition will settle into is virtually impossible unless the initial condition is practically settled into one of the basins already.

12.5.3

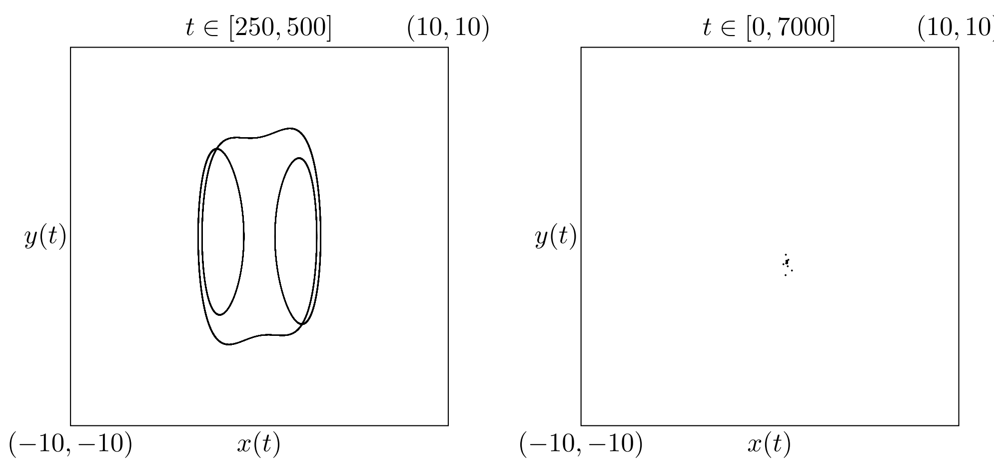
$$\dot{x} = y \qquad \dot{y} = -ky - x^3 + B\cos(t) \qquad k = 0.1 \quad B = 12$$

$(x, y) = (2, 2)$



There is also a limit cycle for this system.

$(x, y) = (1, 0)$



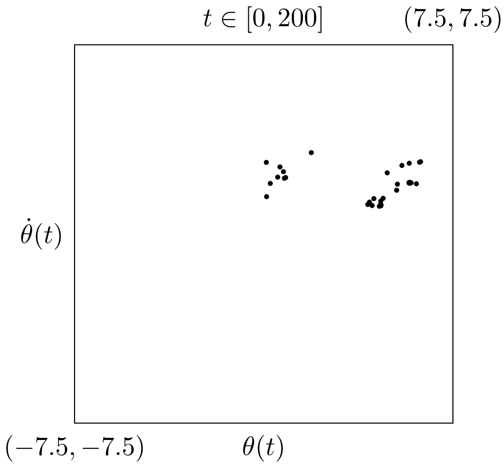
12.5.5

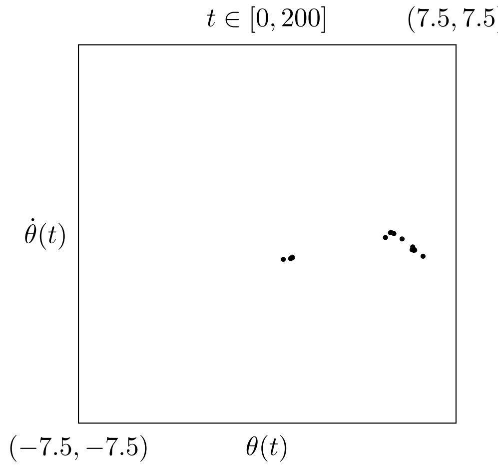
$$\ddot{\theta} + b\dot{\theta} - \sin(\theta) + F\cos(t) \qquad b = 0.2 \quad F = 2$$

Note: The θ value is taken modulo 2π.

a)

$(\theta, \dot{\theta}) = (5, 10) \quad (0.5, 0)$ on the left and right respectively

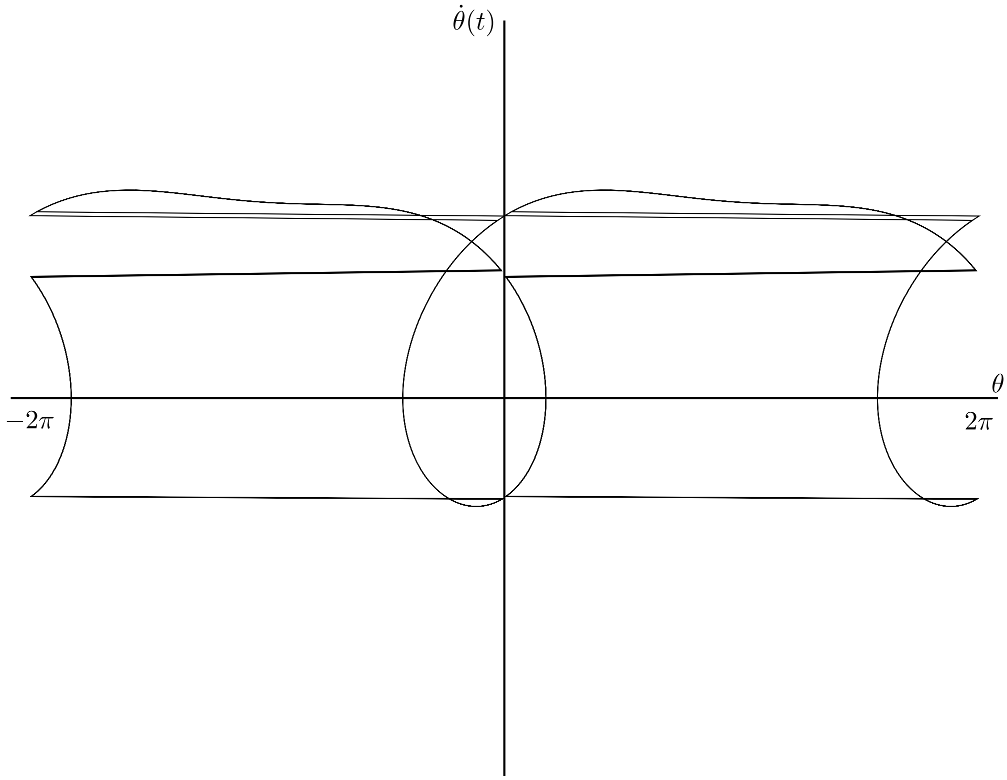




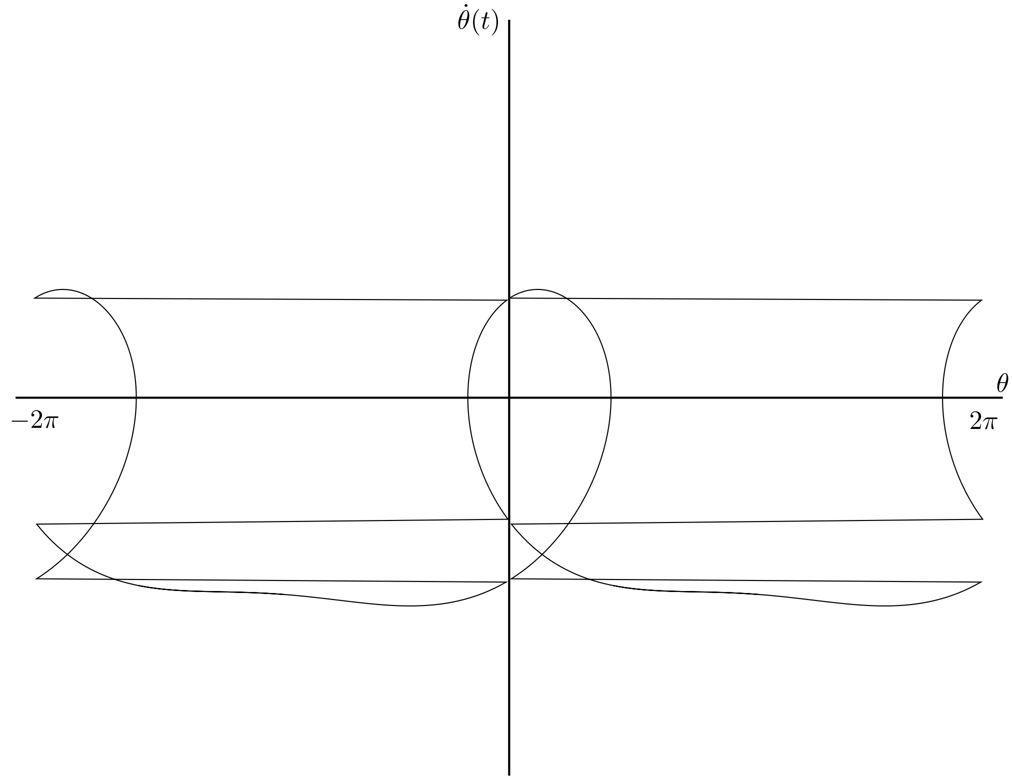
The left and right Poincaré section settles down to approximately (5.807,2.0338) and (5.806,-0.61238) respectively.

These fixed points correspond to period motion of the pendulum, shown below.

(5.807,2.0338)



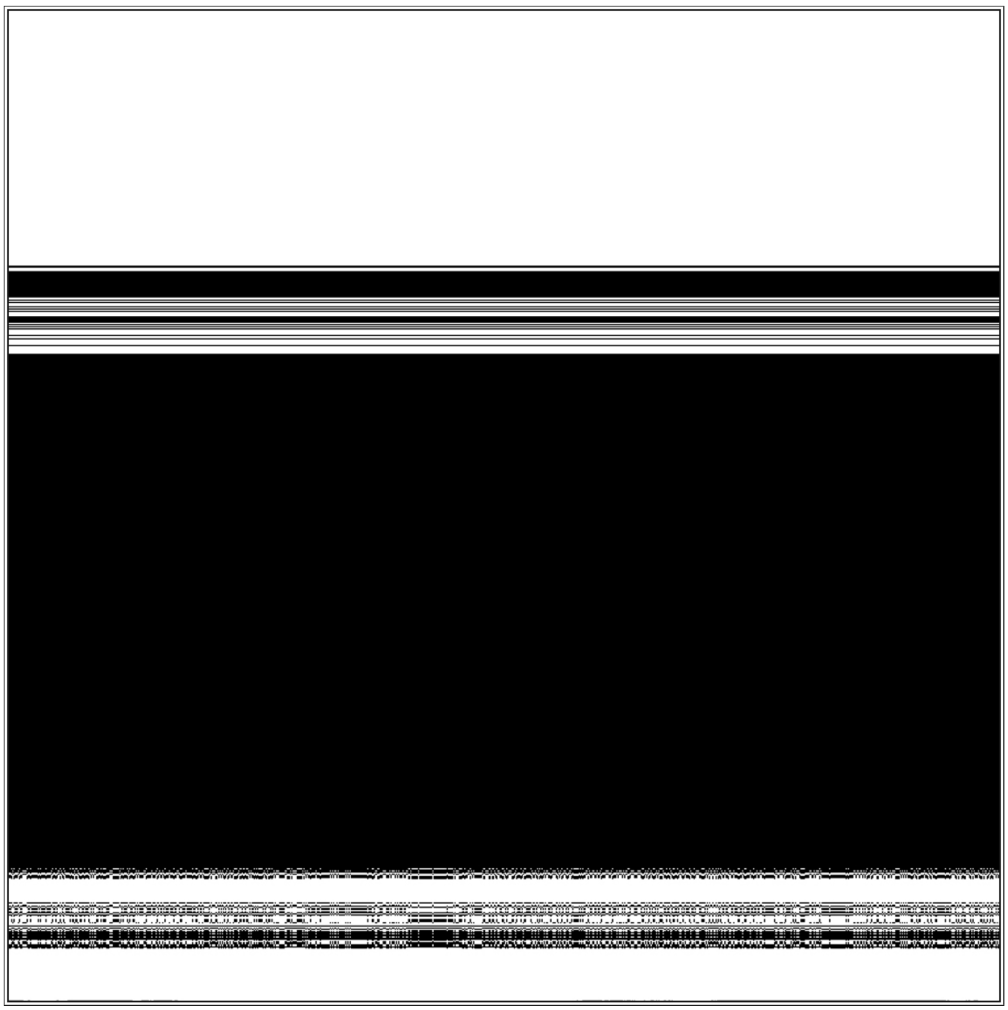
(5.806,-0.61238)



Note: The jumps in the middle are due to the modulo 2π operation.

b)

The following simulations have intial conditions of $\theta, \dot{\theta} \in [-2.5, 2.5]$ in a 900x900 pixel image. A point was determined to be in a basin if, when strobed at an integer multiple of 2π, the θ and $\dot{\theta}$ were within 0.1 and had a speed less than 0.1 of the fixed point of the Poincaré map (5.807,2.0338) and (5.806,-0.61238). A black pixel corresponds to that initial condition settling into the (5.806,-0.61238) basin, and white corresponds to the (5.807,2.0338) basin.



You can see the fractal aspect of the basins at the top of the image.