The Art of Mathematical Proof

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1 Introduction

What is a math proof? That is, after all, why you are reading this article -
to answer this innocent looking question. Well I'm not going to answer that,
at least not right away. Instead, I want you to think about the following
statements:

- The sum of two even integers is even.

- The sum of two odd integers is even.

- The sum of an even and an odd integer is odd.

You know these statements to be true, but how do you know they are true,
why are they true? Maybe you quickly added some examples in your head,
each example affirming your belief in these statements. That may be good
enough to have satiated any doubt you had; after all, patterns often arise
in numbers - no doubt something you noticed in your math courses growing
up. However, this form of confirmation doesn't answer the more important
question - *why* are these statements true? All of mathematical theory is
focused around answering the question of *why*: *why* is this statement true;
why is this impossible; *why* must this object exist; *why* will this process
always yeild a correct result... you get the point. In order to answer every
why we encounter, mathematics employs the mathematical proof. Quite
simply, a mathematical proof is the 'proof' that something must be, or can't
be possibly be true.

We are all famaliar with the stereotypical child that is constantly asking
why: *why* do I have to go to school; *why* is it important to get a job; *why*
do we need money; etc. Mathematicians share many things in common
with this stereotype. They are constantly questioning why, and if the given
explanation is not to their satisfaction, they will then question why the very

explanation must be true, and continue on doing so until you've eliminated all doubt in their minds. However, unlike our imaginary child, it is possible to eliminate a mathematician's doubt. But in order for us to do so, we must present our mathematician with a logical argument that demonstrates the validaty of our statement without leaving any possibility unaddressed. This logical argument is what mathematics refers to as a proof.

Now that you have an idea of what a math proof is, you're probably wondering what one looks like. A proof starts with the facts at hand "I have two even integers, therefore I can represent the first number as $2 * j$ and the second as $2 * k$, where j and k are integers because even numbers are defined to be 2 times some integer.", then deduces what must be true given those facts,"Therefor, I can represent their sum as $(2 * j + 2 * k)$". With the new information, more facts can then be uncovered "We can then factor out the 2 from this representation: $(2 * j + 2 * k) = 2 * (j + k)$. The number $(j + k)$ must be an integer because the sum of two integers is an integer." This process continues on until the truth of the statement is uncovered, "I've shown that the sum of the two even numbers is equal to 2 times the integer $(j + k)$, which means that this number is even by definition.".

Don't worry if you don't see why the argument above does in fact prove that the sum of two integers is even, we'll come back to it soon. Instead, focus on the nature of the argument. Notice that there's no references to a specific example; that each step in the argument was coupled with a justification; that eveness was given an explicit definition - these are the hallmarks of a proof.

Okay, that's great, Will, can you now tell me exactly what a proof is, and how I can go about writing one? Well I can't... there's no easy way to define what a proof is, nor a simple set of rules I can give for writing one - it'd be like defining the color red, sure I could tell you its wavelength, but that doesn't come close to explaining it. What I can do however is teach you how to recognize what is a proof and what's not, as well as the tools to begin to start proving things on your own.

2   The Fundamentals of Theory

In mathematics, definitions are the building blocks of theory. A definition unambiguously and efficiently describes an 'object'. With an object defined, the rest of theory springs forth. First, the properties that are inherent to the definition of the object are proved. From there, more complex facts are uncovered, and then more, and more, and more (you get where I'm going). Each new fact uses the previous facts and definitions to show that it too

must also be true. This process of building up facts endows mathematics with enormous accountability. By the very nature of our exploration, we can always be confident that the facts we discover are in fact true, no matter how far-fetched or complicated they appear.

Let's start with one of the most simple definitions - the set of integers. We'll define the set of integers $\mathbb{Z}$ to be the counting numbers $1, 2, 3, ...$ along with 0, and the negative numbers $-1, -2, -3....$ We can represent the integers in set notation:

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$.

and we'll define an integer to be any number within the set of integers. We'll also equip the integers with the familiar operations of addition, subtraction, and multiplication.

The attentive reader may have noticed that in the introduction proof, I justified that the sum of j and k was an integer because both j and k are integers. We'll accept this statement as fact without proving it to be true. Although simply accepting something to be true seems perilous and illogical, there are times when we cannot escape having to do so. When we accept a statement as fact without proving it, we call it an axiom. An axiom does not need to be proven; instead, the truth of an axiom is self-evident. As such, we never arbitrarily perscribe something axiomatic; rather, we only do so when it is necessary and appropriate.

From our axioms and definitions, we then derive lemmas and theorems. A lemma is simply a statement that has been proven to be true, such as: "The sum of two even integers is even." Theorems are no different than lemmas in their design; however, they're generally attributed to more important or complex statements. You can think of lemmas as the stepping stones to theorems; in fact, lemmas are often proven for the express purpose of being used within the proof of a theorem.

That's all there is to math theory. We start with some self evident truths - our axioms. We then define an 'object' or two and explore all the truths held within it, spitting out lemmas and theorems along the way.

3 Evenness - What is it?

Before we can take a closer look at the proof from the introduction, we'll need to start with definition of what it means to be even:

Definition

An integer, n, is even if $n = 2 * j$ for some integer j.

Let's take the time to examine this definition in some detail. There are several things of note contained in the simple sentence above. First, we see that this definition is defining a specific type of integer, specifically, an even integer. How does it do this? Our definition states a property that an arbitray integer, n, must have for it to be even; in particular, that there must exist another integer, j, such that $n = 2 * j$. What exactly does this mean? In a simplified perspective, you can think of the definition as a test of evenness that could be applied to any specific integer. Is 6 even? Of course it is. Why? Because we can write $6 = 2 * 3$. Whenever we look at a specific example, we simply replace the symbols in our definition with the specifics. In this last example, we substituted 6 for n, and then were able to see that j must be 3. While it is possible to think of a definition like a test, it is so much more than that. A definition describes the essense of a mathematical object. It is efficient in the sense that it gives no more detail than necessary. For instance, we did not include in our definition that the sum of two even integers is even. While this is true, it is consequential of the originally stated property, and thus excessive to our understanding of evenness.

If you're wondering if there's anything special about the letters n and j used within the definition - there's not. We must often speak abstactly in mathematics, so it helps to ascribe symbols to the objects which we are refering. The symbols we choose are arbitrary. For example, our defintion of evenness could have easily been stated using the letters m and z, or greek characters α and ω, or even the words $flamigo$ and $shrimp$. It is also important to note the we don't have to refer to every even integer as n just because that was the symbol used within the definition. In fact, in most cases it would be impossible to do so. For instance, if I want to reference both 4 and 6 symbolically, then I couldn't possibly call them both n. Otherwise $4 = n = 6$, and that's just nonsense! Instead, we can choose new symbols whenever necessary, such as $\alpha = 4$ and $x = 6$. As long as we tell our audience what a new symbol refers to when we introduce it, we can confidently use it in place of its original reference.

4   Algebasics

Although it is somewhat tedious to do so, we must engage in a quick overview of the laws of algebra before we can dive fully into our first proof. Keep in mind, these are the same rules that you've been familiar with since elementary school. They've merely been stated in a more rigorous fashion below.

Associative Property
As we all know, the order that we add integers does not change the resulting sum. For example, $(3 + 5) + 9 = 8 + 9 = 17 = 3 + 14 = 3 + (5 + 9)$. Formally, for any integers a, b and c, it is the case that $(a+b)+c = a+(b+c)$.

Commutative Property
Simiraly, we can change the positions of the integers in our sum without changing the result. For example, $12 + 4 = 16 = 4 + 12$. Formally, for any integers a and b, it holds that $a + b = b + a$.

Identity
The integer 0 is the identity under addition. It is called the identity because the sum of 0 and any other integer is always the other integer. Formally, for any integer a, $a + 0 = a$.

Inverses
For every integer, a, there exists another integer, $-a$, that is a's additive inverse. $-a$ has the special property that $a + -a = 0$. For example, the additive inverse of 2 is -2, the inverse of -71 is 71, and the inverse of 0 is 0.

Subtraction
If we want to subtract one value from the other, we could instead add the inverse of the second value to the first. While this may only seem useful if you were to have a calculator with a broken subtraction button, it is actually very useful to us. Using this 'trick', we can define subtraction using only addition. Now, why exactly is this so great, Will? Because we will be able to convert any subtraction problem to a problem of addition. Allowing us to prove all the properties of subtraction using the properties of addition; and freeing us from the burden of introducing new axioms! As such, let us define the difference of the integers, a and b, to be the sum of a with b's additive inverse. Therefore, if d is the difference of a and b, then $d = a + -b$. For simplicity, we write the difference as $d = a - b$. For example, if we want to subtract 8 from 6, we just add -8 to 6 giving us $6 - 8 = 6 + -8 = -2$. Like addition, subtraction obeys the associative property; however, it does not obey the commutative property. For example, $10 - 8 = 2 \neq 8 - 10 = -2$.

Multiplication
Just as we did with subtraction, we'll also define mutliplication in terms of addition. For any integer, a, the following three rules define multiplication:

1. For any positive integer b, the product of a and b is equal to a added to itself b times. As such, $a * b = a + a + ... + a = \sum_{n=1}^{b} a$. For example, $3 * 4 = 3 + 3 + 3 + 3 = 12$.

2. For any negative integer b, the product of a and b is equal to the inverse of a added to itself b inverse times. As such, $a * b = -a + -a... + -a = \sum_{n=1}^{-b} -a$. For example, $6 * -3 = -6 + -6 + -6 = -18$

3. Lastly, the product of a and 0 is always 0. As such, $a * 0 = 0$.

Just like addition, multiplication also respects associativity and commutativity.

Distributive Property
Lastly, the distributive property states that the product of an integer a with the sum of the integers b and c is equal to the sum of the products of a with b and a with c. In other words, $a * (b + c) = a * b + a * c$.

Further on in this article, we will prove the associative, communtative, and distributive properties of multiplication using the properties of addition. For now, however, we shall accept them as psuedo-axiomatic to prevent ourselves from getting too lost in the details.

5   The Sum of Two Even Integers is Even

Now that we have our definition of evenness and the basic laws of algebra, we can begin to tackle this proof. But in order to begin, we first need to determine what exactly it means to show that the sum of two integers is indeed even. Not only that, our argument has to be structured in a way such that regardless of the two even integers chosen, we clearly demonstrate that their sum is even. Now if you're wondering how we can possibly show that it's true for every pair of even integers, or why the proof in the introduction does fulfill this requirement, don't worry - you are not alone! The solution to our problem is actually quite simple, all we have to do is look back to what an even integer is defined to be. If you're wondering how such an abstract definition can help us, again, don't worry, I'm about to explain. Our problem is that we need to show for any two even integers, their sum is even. Well to do that, we need two even integers (obviously!), but we don't want to pick two specific integers. Instead, we need a description of two even integers that could apply to any specific pair. Here we go...

$\rightarrow$ Assume I have two even integers n and m.

Will, really? Yes, really! We need a description that fits any two even integers, and that description certainly does so. If you're wondering how this statement could possibly be helpful, just look back to your definition of an even integer. You'll find that the statement above contains a bit more information than it lets on at first:

$\rightarrow$ Since n is even, there exists some integer j such that $n = 2 * j$.

$\rightarrow$ Similarly, since m is also even, there exists some integer k such that $m = 2 * k$.

The specific values of j and k aren't relevant, all that's important is that they exist, and that we can represent our even integers n and m using them. Let's circle back now to the real problem, how do we show that the sum of n and m is even as well. To do this, let's give the sum a name - let's call it s. More formally:

$\rightarrow$ let $s = n + m$.

Once again, we do not care about the specific value of s, we only care that it is the sum of n and m. Now that we have s, we must ask ourselves a very important question - what does it mean to show that s is even. Once again, I refer you to the definition of what it means to be even - s is even if there exists some integer p such that $s = 2*p$. Let's take a recap of all the things we know:

$\rightarrow$ We have two even integers $n = 2 * j$ and $m = 2 * k$, for some integers j and k.

$\rightarrow$ We defined the sum of n and m as the integer $s = n + m$.

$\rightarrow$ s is even if we can find another integer, p, such that $s = 2 * p$.

So what next? Well, let's try substituting $2 * j$ for n, and $2 * k$ for m in our representation of s:

$\rightarrow$ $s = n + m = 2 * j + m = 2 * j + 2 * k$.

Why can we do this? Since n is equal to $2 * j$ we can use them interchangeably - they refer to the same exact value. You can think of $2 * j$ as

a nickname for n - while they may look different, they actually refer to the same thing. Now we all should know what's coming next (after all, I told you only a few paragraphs ago), we are going to factor out the 2:

$\rightarrow s = 2 * j + 2 * k = 2 * (j + k)$.

It was no accident that I choose the sum of two integers is an integer as our example axiom; I needed it for what comes next:

$\rightarrow$ Since both j and k are integers, their sum, $j + k = p$, is also an integer.

What's the value of p? Once again, it's not important, all we need to know is that p is an integer. Finally, we can put it all together by substituting p for $j + k$:

$\rightarrow s = 2 * (j + k) = 2 * p$.

$\rightarrow$ Therefor, since p is an integer, and s is equal to $2*p$, s is even by definition.

$\rightarrow$ Furthermore, s is not only even, it's also the sum of n and m - which were represented in such a way that our argument could apply to any pair of even integers!

And that's all there is to it - we've now proven our statement that the sum of two even integers is even. If you're still a little unsure why the above proves our statement, don't worry - they'll be a few more proofs before the end of this article. If you are uneasy, try re-reading the proof and writing out each step along the way. As you write down each step, ask yourself: why does this argument apply to any integer(s); why am I able to deduce this information from the previous step(s); why is this particular statement true; and what am I accomplishing with this step? You might also find it helpful to read the following condensed version of the proof:

Lemma
The sum of two even integers is also even.

Proof
Assume I have two even integers n and m.
Since n is even, there exists some integer j such that $n = 2 * j$.
Similarly, since m is also even, there exists some integer k such that $m = 2*k$.

let $s = n + m$.

Then, $s = n + m = 2 * j + m = 2 * j + 2 * k$.

Then, $s = 2 * j + 2 * k = 2 * (j + k)$.

Since both j and k are integers, their sum $(j + k) = p$ is also an integer.
Thus, we can represent s as $s = 2 * (j + k) = 2 * p$.

Since p is an integer, and s is equal to $2 * p$, s is even by definition.
Furthermore, s is not only even, it's also the sum of n and m.
Thus we have completed our proof.
Q.E.D

6 The Oddities of Oddness

Definition
An integer, n is odd if $n = 2 * j + 1$ for some integer j.

Look familiar? It shouldn't come as much surprise that the definition of oddness is incredibly similar to that of evenness; after all, there seems to be a natural duality between the two concepts. Now, let's take a deeper look at this defintion. First, let's confirm that it makes sense with a few example: $3 = 2 * 1 + 1$, $5 = 2 * 2 + 1$, $-39 = 2 * -40 + 1$. Great, everything seems to check out. Now let's dig a little deeper. What if I were instead define an odd integer as follows:

Definition
An integer, n, is odd if $n = 2 * j + 3$ for some integer j.

Despite this definition looking different from our original, it isn't actually any less correct. Again, let's look at our examples: $3 = 0 * 2 + 3$, $5 = 1 * 2 + 3$, $-39 = 2 * -42 + 3$. Everything seems to work out fine once again. But how can this be; could we be overlooking something? We are not, and in order to make certain of this, we are going to prove that the two defintions actually define the same thing. In particular, we are going to prove that any integer written with our first defintion can be written in the form of our second defintion, and vice versa.

Lemma
For any integer, n, there exists an integer, j, such that $n = 2 * j + 1$ if and

only if there also exists an integer, k, such that $n = 2 * k + 3$.

Proof
Assume I have an odd integer, n, defined by our first definition.
Therefore, there exists an integer, j, such that $n = 2 * j + 1$.

Now, we must show that there exists some integer, k, such that $n = 2 * k + 3$. By doing this, we will have shown that for any integer defined using our first defintion, it also fits the description of our second defintion.

Since j is an integer, I can subtract 1 from it, let's call this difference k. Formally, let $k = j - 1$.

Now we will use basic algebraic manipulation to coax the second defintion from our first.

$$n = 2 * j + 1 = 2 * (j + 0) + 1 = 2 * (j + (-1 + 1)) + 1$$

The above may seem strange or unhelpful, but it is in fact the key to our proof. By expanding out multiplication by 2 to include $-1 + 1 = 0$, we haven't changed the meaning of our equation thanks to the laws of algebra. However, we have converted our equation to a more useful form as will be revealed in the next step:

$$n = 2 * (j - 1) + 2 * 1 + 1 = 2 * (j - 1) + 3.$$

Now we've got the 3 we need for our second defintion. What's more, we can substitute k for $j - 1$:

Then, $n = 2 * (j - 1) + 3 = 2 * k + 3$.
Therefore, n is also odd in accordance with our second defintion!

What exactly does this mean? We've now proved that if we start with our first defintion, then we meet the criteria of our second defintion. Does this mean we have shown that the two definition are actually the same? Not quite, we still need to show that if we start with the second defintion then we will meet the criteria for the first one. But, why do we also need to do this? Let's take a look at a less mathematical example to explain that. Imagine you always wear a hooded jacket when it's raining; however, you also wear a hooded jacket whenever it's below 40 degrees. Thus, if it is raining, we can logically argue that you must be wearing a hooded jacket. However, if we

see you are wearing a hooded jacket, we can't assume that it must be raining out, because it might cold instead. This example highlights why we must prove both directions. If we neglected to do so, we can not be confident that $n = 2 * k + 3$ implies that there exists an integer, j, such that $n = 2 * j + 1$; just like we can't assume that a hooded jacket implies rainy weather. But, Will, didn't you say that both definitions actually defined the same thing, so this hooded jacket example doesn't apply? Unlike our rainy day example, either direction in our lemma implies the other; however, we don't *know* this without first proving it. For all we know, $n = 2 * k + 3$ implies that $n = 37^j + 49 * j$. As such, let's eliminate our remaining doubts by completing the proof:

Assume I have an odd integer, m, defined by our second defintion.
Therefore, there exists an integer p such that $m = 2 * p + 3$.

Similar to the first part of the proof, we must this time show that there exists some integer, q, such that $m = 2 * q + 1$. Once again, we will use some clever algebra to show this to be the case.

$$m = 2 * p + 3 = 2 * (p + 1 - 1) + 3 = 2 * (p + 1) - 2 + 1 + 3 = 2 * (p - 1) + 1.$$

Hopefully you know what comes next:

Let $q = p - 1$.
Then, $n = 2 * q + 1$.
Thus, n is odd in accordance with our first defintion!
Q.E.D

Now we've completed our proof, and by doing so, removed all doubt as to whether our two different defintions were actually defining the same thing. In fact, we could have chosen any odd integer to add to our $2 * j$ term, and have accurately defined oddness (try proving it's works if we had chosen 7 instead).

As we've now seen, it's possible to have more than one valid definition for a mathematical object. However, it'd be strange and confusing to give an object more than one defintion. As such, we normally pick whichever definition we feel is the most fitting (which is sometimes just our favorite), and then prove all other definitions are *equivalent* through lemmas and theorems - just as we have done above. In this particular case, we will stick with the first definition of oddness given, and restate that above lemma as:

Lemma
An integer, n, is odd if and only if there exists an integer, j, such that $n = 2 * j + 3$.

7   Split Personality

Now that we have definitions for both even and odd integers, we are going to prove that an integer cannot be both even and odd. Yes, we all 'know' that this statement is true, but our stubborn mathematicians seem to doubt anything we say without any proof (they're not always the easiest group to get along with). Unfortinately, they do have a point. If someone were to (incorrectly) state that a number could be both even and odd; we would be unable to refute their claims! As such, unless we prove this obvious statement, it would be improper for us to claim its truth.

Before we dive into our proof, let's first think about what we need to show in order to prove that an integer can't be both even and odd. To accomplish our goal, we are going to use a technique often termed *contradiction*. In a contradictory proof, we endow an object with two or more conflicting properties. In the problem at hand, our object will be an integer, and the conflicting properties will be even and oddness. We then 'innocently' explore all the truths that lead from our object endowed with conflicting properties. However, our exploration won't actually be so innocent. We know that we our exploring the land of the impossible, a place outside of logic. As we stumble about this false reality we've created, we will surely encounter many strange things, things that defy logic. It is our task to choose one of these strange encounters to explore in detail. Why? Because we are searching for proof that the very thing in front of us can't possibly exist. We may know we've stepped outside of the realm of possibility; unfortinately, our friends back at home aren't going to believe us without proof. So what exactly does proof you've stumbled into the realm of impossibility look like? It looks like an equation that when solved reveals that an integer is equal to two different values at the same time; or a set that is somehow both empty and nonempty; or an integer that is both less than and greater than some other integer. These are but a tiny few of all the absurdities we might run into, but they perfectly illustrate the type of information we are seeking. Each statement is something that we know can't be true; each statement is a *contradiction*. Once we find our contradiction, we can start to pull at the very fabric with which we've woven the impossible. How do we do this? We simply start working backwards from the contradiction until we've stepped back into reality. Once we've made it back to reality, we've identified the culprit

of our journey - it's the statement the sent us into the realm of impossibility in the first place! With the knowledge of the culprit statement, and proof that it leads to a contradiction, we can now confidently assert that culprit statement must actually be false, for we now know that it can't possibly be true.

This is exactly how we are going to prove that an integer can't be both even and odd at the same time. We'll start with an arbitrary integer, then we'll assume it's both even and odd. We'll stumble along until we find a contradiction. Then we'll trace that contradiction's origins back to our assumption that our integer was both even and odd. At that point, we will have shown that it's impossible for the integer to be both even and odd! Why? Because if it is both even and odd, then reality is falling apart in front of us! Without further ado, let us now prove this statement:

Lemma
An integer cannot be both even and odd.

Proof
Let n be any integer.
Assume n is both even and odd.

Now that we've made our assumption that will fling us out of reality, it's time to start exploring:

Then, there exists some integer, j, such that $n = 2 * j$ since n is even. Similarly, there exists some integer, k, such that $n = 2 * k + 1$ since n is also odd.

Let's take a closer look at the integer j...

Since $n = 2 * j$, we can divide both sides of this equation by 2 in order to solve for j, which gives us, $j = \frac{n}{2}$.

If you were wondering why we need to explore the realm of impossibility in order to find a contradiction, the answer should now be clear - our false reality appears to be real at first! So let's continue exploring, hopefully we don't have to go much further... Why don't we see what happens when we subsititue $2 * k + 1$ for n in our equation for j...

Substituting give us, $j = \frac{n}{2} = \frac{2*k+1}{2} = \frac{2*k}{2} + \frac{1}{2} = k + \frac{1}{2}$.

Subtracting k from both sides of this equation yeilds, $j - k = \frac{1}{2}$.
But j and k are both integers, so there difference, $d = j - k$, is also an integer (you should prove this for yourself).

Substituting once again gives us, $d = \frac{1}{2}$.

Wait a minute, d is an integer, and $\frac{1}{2}$ is not contained within the set of integers - this can't possibly be; this must be the contradiction we've been looking for!

Now it's time to work backwards from our contradiction...

How did we know d was an integer? We used the fact that difference of two integers is also an integer; however, since you proved this for yourself (right?), then we can all agree that this statement comes from reality, so it could not be what lead us to our contradiction.

Let's keep moving further back...

How did we come to $d = \frac{1}{2}$? Well we were able to do so using simple algebra, and simple algebra also comes from reality, so that too is not the culprit. What came before our algebra? We substituted $2 * k + 1$ for n in our representation of j. Substitution also comes from reality, so the act of it is not the culprit. However, the *reason* we could perform this substitution was! Why were we able to substitue $2 * k + 1$ for n in our representation of j? Because we had defined n to be odd; and where did we get our equation for j? From the fact that n is also even! If we had assumed n was only even or only odd, we would have never have been able to perfom the substitution. Thus, we'd never have been able to show that $\frac{1}{2}$ had somehow managed to sneak it's way into the set of integers.

With our contradiction in hand, we can conclude that it was incorrect of us to have assumed our integer both even and odd. Furthermore, since we chose n to be any integer, we now can confidently state that it's incorrect to assume any integer can be both even and odd!
Q.E.D

8 Tying Up Loose Ends

Now that we've got definitions for even and oddness, and we are confident that an integer can't both be even and odd, we can finally prove the other two statements from the introduction.

Lemma
The sum of two odd integers is even.

Proof
Assume n and m are odd integers.
Then there exists integers j and k such that $n = 2 * j + 1$ and $m = 2 * k + 1$.

Let $s = n + m$.
Substituting, $s = n + m = 2 * j + 1 + 2 * k + 1 = 2 * j + 2 * k + 2$.

We can now factor out the 2, $s = 2 * (j + k + 1)$.

Let $p = j + k + 1$.
We know that p must be an integer because the sum of j and k is an integer, and 1 plus that value is also an integer.

Thus, $s = n + m = 2 * (j + k + 1) = 2 * p$, and is therefor even by definition.
Q.E.D

Up to this point, I've only asked you to prove things in passing. However, it is now time for you to engage the material yourself. You can be sure that I will never ask you to prove something that requires a technique or knowledge that has yet to be presented to you. However, that doesn't mean I won't ask you to push yourself. Here's several tips to keep in mind when attempting a proof:

- Don't try to think of the entire proof before starting. A good math proof often requires you to hit a dead end or two before the correct path becomes clear. There's no shame in stumbling your way through a proof; you can always beautify your argument afterwards.

- If you don't know where to start, try 'expanding' every definition in your statement, i.e I have an even integer, n, which means there exists an integer, j, such that $n = 2 * j$.

- Before you start your proof, write down what you need to show in order to prove your statement. i.e To show that the sum, s, is even, I must show there exists an integer, p, such that $s = 2 * p$.

- If you keep hitting a dead end with the direct approach, try assuming your statement is false and then look for a contradiction.

- If you've been at a proof for awhile and feel like you aren't getting anywhere, try taking a break. You'll be surpised how effective a good night's sleep can be.

- Try looking over all the lemmas and thereoms that are related to the topics in your statement; they could be the key to your proof.

- Move on to a different proof and come back later. You might be inspired by a technique used elsewhere.

- Be creative! Mathematics may be rooted in logic, but that does not deprive it of expression. As long as you don't contradict the rules of logic, feel free to express your argument in whichever manner you feel is the most clear.

I now ask you to prove the following lemma. Look to our previous proofs for inspiration and guidance.

Lemma
The sum of an even and an odd integer is odd.

9   Some Handy Notations

Let's be honest, no one wants to spend the time to write every proof in as much detail as we have within this article. Especially considering that our first attempt at a proof is bound to be riddled with dead ends. To help ease the tedium, mathematicians have developed several notations that make writing proofs faster. I have included several of the most common notations below.

Symbol: $\mathbb{Z}$
Meaning: "The set of integers"
Example with symbol: For all n in $\mathbb{Z}$, n is odd if and only if there exists an integer, j, such that $n = 2 * j + 3$.

Example without symbol: For all n in the set of integers, n is odd if and only if there exists an integer, j, such that $n = 2 * j + 3$.

Symbol: $\forall$
Meaning: "for all", "for every", or "for each"
Example with symbol: $\forall n$ in $\mathbb{Z}$, n is odd if and only if there exists an integer, j, such that $n = 2 * j + 3$.
Example without symbol: For all n in the set of integers, n is odd if and only if there exists an integer, j, such that $n = 2 * j + 3$.

Symbol: $\exists$
Meaning: "there exists" or "exists"
Example with symbol: $\forall n$ in $\mathbb{Z}$, n is odd if and only if $\exists$ an integer, j, such that $n = 2 * j + 3$.
Example without symbol: For all n in the set of integers, n is odd if and only if there exists an integer, j, such that $n = 2 * j + 3$.

Symbol: $\leftrightarrow$
Meaning: "if and only if"
Example with symbol: $\forall n$ in $\mathbb{Z}$, n is odd $\leftrightarrow$ $\exists$ an integer, j, such that $n = 2 * j + 3$.
Example without symbol: For all n in the set of integers, n is odd if and only if there exists an integer, j, such that $n = 2 * j + 3$.

Symbol: s.t
Meaning: "such that"
Example with symbol: $\forall n$ in $\mathbb{Z}$, n is odd $\leftrightarrow$ $\exists$ an integer, j, s.t $n = 2*j+3$.
Example without symbol: For all n in the set of integers, n is odd if and only if there exists an integer, j, such that $n = 2 * j + 3$.

Symbol: $\in$
Meaning: "in" (generally referring to set membership)
Example with symbol: $\forall n \in \mathbb{Z}$, n is odd $\leftrightarrow$ $\exists$ an integer, j, s.t $n = 2*j+3$.
Example without symbol: For all n in the set of integers, n is odd if and only if there exists an integer, j, such that $n = 2 * j + 3$.

Symbol: !$\exists$
Meaning: "there exists unique"
Example with symbol: $\forall n \in \mathbb{Z}$, n is odd $\leftrightarrow$!$\exists$ integer, j, s.t $n = 2*j+3$.
Example without symbol: For all n in the set of integers, n is odd if and only if there exists a unique integer, j, such that $n = 2 * j + 3$.

Symbol: $\rightarrow$
Meaning: "then"
Example with symbol: If n and m are odd $\rightarrow$ their sum is even.
Example without symbol: If n and m are odd then their sum is even.

For the sake of clarity, I'll be conservative with my use of symbols throughout the remainder of this article. However, you should feel free to use them within your proofs as much or as little as you want - I think you'll find they save you a lot of time.

10 Logical Lessons (incomplete)

When we need to show that one statement implies another, we use an *if-then* statement. The most common structure for an *if-then* statement is quite obvious: "If *Statement-A*, then *Statement-B*." For example:

- If it is raining, then you will wear a hooded jacket.

- If n is odd, then there exists an integer k such that $n = 2 * k + 3$.

- If n and m are integers, then there sum is an integer.

The meaning of an *if-then* statement follows intuition - whenever *Statement-A* is true, then *Statement-B* is also true. As such, an *if-then* statement defines a consequential relationship between statements A and B; explicity, B is a consequence of A.

As we observed in THE ODDITIES OF ODDNESS, the *if-then* statement does not always work in reverse - it is not necessarily the case that A is also a consequence of B. However, we often encounter (and even seek out) statements that are consequential of one another. Statements involved in such a relationship hold a special interest to mathematicians because they are *equivalent*. When two statements are equivalent, we can effectively substitue equivalent either statement for the other. But meer substitution does not do the concept of equivalency justice...

11 Oops!

This article isn't finished yet, but updates are pushed daily. Come back soon!

12   Reccomended Reading

- "What is Mathematics?" - Richard Courant & Herbert Robbins

- "Linear Algebra and its Applications" - Davic C. Lay

- "A Book of Abstract Algebra" - Charles C. Pinter (advanced)

- "Abstract Algebra" - I.N. Herstein (advanced)

- More to come...