Awkward Number Series

Will Dengler

May 7, 2021

1 Notation and Assumed Knowledge

Notation

- $\mathbb{Z}$ is defined to be the set of integers.
- $\mathbb{N} \subset \mathbb{Z}$ is defined to be the set of natural numbers, including 0.
- $\mathbb{N}^+ \subset \mathbb{N}$ is defined to be the set of positive integers.
- For any $x \in \mathbb{N}^+$, $[x] = \{ \ j \in \mathbb{N} \mid j < x \ \}$.
- $\mathbb{Q}$ is defined to be the set of rational numbers.

Remainder Theorem
For any natural number x, for any positive integer y, there exists a unique integers $z \in \mathbb{N}$ and $r \in [y]$ such that $x = zy + r$. We call r the *remainder* of x when divided by z.

Definition
For any natural number x, for any positive integer y, the remainder function $\rho : (\mathbb{N} \times \mathbb{N}^+) \to \mathbb{N}$, $\rho(x, y)$ is defined to be the remainder of x when divided by y.

Definition
For any natural number x, for any positive integer y, if $\rho(x, y) = 0$, then we say that x is *divisible* by y or that x is a *multiple* of y.

Definition
For any set of integers $X \subset \mathbb{N}^+$, an integer z is called a *common multiple* of the elements of X whenever z is a multiple of every element of X.

Lemma
For any set of integers $X \subset \mathbb{N}^+$, if $z \in \mathbb{N}^+$ is a common multiple of the elements of X, then for all $x \in X$, $z \geq x$.

Lemma
For any set of integers $X \subset \mathbb{N}^+$, such that y is the greatest element of X. If there exists $x \in X$ such that y is not a multiple of x, then for every positive common multiple $z \in \mathbb{N}^+$ of the elements of X, $z > y$.

Remainder Function Properties
The remainder function has the following properties:

- For any $i \in \mathbb{N}^+$, for any $j \in [i]$, $\rho(j, i) = j$.
- For any $i \in \mathbb{N}^+$, for any $j \in \mathbb{Z}$, $\rho(ij, j) = 0$.
- For any $j, k \in \mathbb{N}, i \in \mathbb{N}^+$, $\rho(kj, ki) = k\rho(j, i)$.
- For any $j, k \in \mathbb{N}, i \in \mathbb{N}^+$, $\rho(j + k, i) = \rho(\rho(j, i) + \rho(k, i), i)$.
- For any $j, k \in \mathbb{N}, i \in \mathbb{N}^+$, $\rho(k, i) = k - ji$ whenever $ji \leq k < (j + 1)i$.

Definition
For any $q \in \mathbb{Q}$, the *ceiling function* $\lceil q \rceil = z$, where z is the integer such that $z - 1 < q \leq z$.

Lemma
For any $q = \frac{a}{b} \in \mathbb{Q}$:

- $\lceil q \rceil = q$ whenever $\rho(a, b) = 0$.
- $\lceil q \rceil = \frac{c}{b}$, where $c = a + b - \rho(a, b)$ whenever $\rho(a, b) > 0$.

Definition
For any $x, y \in \mathbb{N}$, the function $gcd(x, y)$ is defined to be the greatest common divisor of x and y.

Assumed Knowledge
The prime numbers can be recursively defined as the series:

- $p_0 = 2$ is the first element in the series.
- For all $k \in \mathbb{N}^+$, p_k is the least greatest integer such that $p_k > p_{k-1}$, and for all $j < k$, p_k is not divisible by p_j.

2 Awkward Number Series

Definition
For any positive integers a, n, the *awkward number series*, $S_{a,n}$ is defined as:

- An initial element $s_0 = a + n$
- For any $i > 0$, s_i is defined to be the least greatest integer such that $s_i > s_{i-1}$ and $\rho(s_i, s_k) \geq a$ so all $k < i$.

We say that the awkward number series $S_{a,n}$ has a *activators*, and n *initial non-activators.*

Lemma
The awkward number series $S_{1,1}$ is equal to the set of prime numbers.

Proof
TODO

Lemma
For any awkward number series $S_{a,n}$, for any $x \in \mathbb{N}$ such that $x > a+n$, x is either an element of $S_{a,n}$ or there exists some $s_j \in S_{a,n}$ such that $x > s_j$ and $\rho(x, s_j) < a$.

Proof
Let $S_{a,n}$ be any awkward number series.

Let $x \in \mathbb{N}$ be any natural number such that $x > a + n$.

Assume that there does not exist an $s_j \in S_{a,n}$ such that $s_j < x$ and $\rho(x, s_j) < a$.

As such, for all $s_j \in S_{a,n}$ such that $s_j < x$, $\rho(x, s_j) \geq a$ must be the case.

By definition x must be an element of $S_{a,n}$.

Now let us assume there exists some element $s_j \in S_{a,n}$ such that $s_j < x$ and $\rho(s_j, x) < a$.

As such, it is not the case that for all $s_j \in S_{a,n}$ such that $s_j < x$, $\rho(x, s_j) \geq a$.

By definition x cannot be an element of $S_{a,n}$.

Lemma
For any awkward number series $S_{a,n}$, for any $s_i, s_j \in S_{a,n}$, $\rho(s_i, s_j) < a$ if and only if $s_i = s_j$.

Proof
Let $S_{a,n}$ be any awkward number series.

Let $s_i \in S_{a,n}$ be any element in the series.

$s_i = s_i + 0$. As such, $\rho(s_i, s_i) = 0$ by definition of the remainder.

By definition of an awkward number series, $a \geq 1 > 0$.

Let $s_j \in S_{a,n}$ be any element of the series such that $s_j < s_i$.

By definition of an awkward number series, $\rho(s_i, s_j) \geq a$.

As such, $\rho(s_i, s_j) < a$ cannot be the case.

Let $s_k \in S_{a,n}$ be any element of the series such that $s_k > s_i$.

$s_i = 0s_k + s_i$. As such, $\rho(s_i, s_k) = s_i$ by definition of the remainder.

By definition of an awkward number series, $s_i \geq s_0 = a + n$.

As such, $\rho(s_i, s_k) = s_i \geq a + n \geq a$.

Awkward Infinity Theorem
Every awkward number series contains an infinite number of elements.

Proof
Let $S_{a,n}$ be any awkward number series.

Assume that $S_{a,n}$ contains a finite number of elements.

Let s_i be the greatest element within $S_{a,n}$.

Let m be any positive common multiple of the elements of $S_{a,n}$.

By previous lemma, $\rho(s_i, s_j) \geq a$ for all $s_j < s_i$.

As such, s_i is not a multiple of any of the preceding elements.

As such, $m > s_i$ by lemma.

Consider the value $m + a$.

By assumption $S_{a,n}$ is finite, as such, for any integer $x > s_i$, there exists some $s_j \in S_{a,n}$ such that $\rho(m + a, s_j) < a$ by previous lemma.

Let $\rho(m + a, s_j) = b < a$.

By the remainder theorem, there exists some integer x such that
$m + a = xs_j + b$.

Since m is a common multiple of all the elements of $S_{a,n}$, then $\frac{m}{s_j} \in \mathbb{N}$.

Let $y = \frac{m}{s_j}$. Then $m = ys_j$.

Consider the equation $a = (m + a) - m$.

Substituting $xs_j + b$ for $m + a$ yields $a = xs_j + b - m$.

Substituting ys_j for m yields $a = xs_j + b - ys_j$.

Applying the distributive property yields $a = (x - y)s_j + b$.

Since $b < a < s_j$, then b must be the remainder of a when divided by s_j by definition, as such $\rho(a, s_j) = b$.

Furthermore, $a < s_j$, as such $\rho(a, s_j) = a = b$ by properties of ρ.

However, $b < a$ by assumption. As such, we have reached a contradiction.

Therefore, it must be the case that either $m+a$ is an element of $S_{a,n}$, or there exists some other element in $S_{a,n}$ less than $m+a$ that was not accounted for. In either case, $S_{a,n}$ cannot be finite.

Corollary
There are an infinite number of prime numbers.

Proof
The prime numbers are an awkward number series and every awkward number series contains an infinite number of elements.

Lemma
For any awkward number series $S_{a,n}$, the first $\lceil \frac{n}{a} \rceil + 1$ elements are given by $s_i = a(i + 1) + n$.

Proof
Let $S_{a,n}$ be any awkward number series. We shall complete this proof via induction on the index of the first $\lceil \frac{n}{a} \rceil + 1$ elements.

Base Case
By definition, the initial element is $s_0 = a + n = a(0 + 1) + n$.

Inductive Hypothesis
Assume for the first $0 \leq j < \lceil \frac{n}{a} \rceil$, that $s_j = a(j+1) + n$.

Inductive Step
For all $x \in [a]$, $\rho(s_j + x, s_j) = x < a$. As such, $s_{j+1} \geq s_j + a$.

If we can show that $\rho(s_j + a, s_k) \geq a$ for all $k < j$, then $s_{j+1} = s_j + a$.

Furthermore, $s_j + a = a(j+1) + n + a = a(j+2) + n$, thus we will completed our proof.

Let $0 \leq k < j$. Then $s_j + a = s_k + a(j - k + 1)$ according to the inductive hypothesis.

As such, $\rho(s_j + a, s_k) = a(j - k + 1)$ as long as $a(j - k + 1) < s_k$.

Since $j < \lceil \frac{n}{a} \rceil$ and $j \in \mathbb{N}$, then $j \leq \lceil \frac{n}{a} \rceil - 1$.

As such, $a(j - k + 1) \leq a(\lceil \frac{n}{a} \rceil - 1 - k + 1) = a(\lceil \frac{n}{a} \rceil - k) = a\lceil \frac{n}{a} \rceil - ak$.

First, let us consider the case where $a \mid n$.

We will then have $a\lceil \frac{n}{a} \rceil = n$.

As such, $a(j - k + 1) \leq n - ak < s_k$.

Now let us consider the case where $\rho(n, a) \geq 1$.

Then $a\lceil \frac{n}{a} \rceil = a\frac{n+a-\rho(n,a)}{a} = n + a - \rho(n, a) = s_0 - \rho(n, a) < s_0 \leq s_k$.

As such, $a(j - k + 1) < s_k - ak \leq s_k$.

Therefore, $\rho(s_j + a, s_k) = a(j - k + 1)$ does in fact hold.

As such, we now need to show that $a(j - k + 1) \geq a$.

We chose $k < j$, as such, $a(j - k + 1) \geq a(j - j + 1) = a$.

We have shown that $s_j + a = a(j+2) + n$ is the least greatest integer greater than s_j such that $\rho(s_j + a, s_k) \geq a$ for all $k \leq j$. Therefore, $s_{j+1} = a(j+2) + n$.

Definition
For any awkward number series $S_{a,n}$, the value $\lceil \frac{n}{a} \rceil + 1$ is called the *dimension* of the series, denoted $dim(S_{a,n})$.

Definition
For any awkward number series $S_{a,n}$, $s_i \in S_{a,n}$ is called a *staple* whenever $s_i = s_{i-1} + a$.

Lemma
For any awkward number series $S_{a,n}$, for any integer $0 < i \leq dim(S_{a,n})$, the element $s_i \in S_{a,n}$ is a staple.

Proof
Let $S_{a,n}$ be any awkward number series.

Let i be any integer such that $0 < i \leq dim(S_{a,n})$.

By previous lemma, $s_i = a(i+1) + n$ and $s_{i-1} = ai + n$.

Consider the difference $s_i - s_{i-1}$.

Substituting $a(i+1) + n$ for s_i yields $s_i - s_{i-1} = a(i+1) + n - s_{i-1}$.

Substituting $ai+n$ for s_{i-1} yields $a(i+1)+n-s_{i-1} = a(i+1)+n-(ai+n)$.

Distributing the -1 yields $a(i+1) + n - (ai+n) = a(i+1) + n - ai - n$.

Adding the n terms yields, $a(i+1) + n - ai - n = a(i+1) - ai$.

Factoring the a yields, $a(i+1) - ai = a(i+1-i) = a(1) = a$.

As such, $s_i - s_{i-1} = a$.

Subtracting s_{i-1} from both sides yields $s_i = s_{i-1} + a$.

Thus, s_i is a staple by definition.

Definition
For any awkward number series $S_{a,n}$, for $i \in [dim(S_{a,n})]$, s_i is called a *basis* of the awkward number series.

Lemma
For any awkward number series $S_{a,n}$, for any $i > 0$, there exists $s_j < s_i$ such that $\rho(s_i, s_j) = a$.

Outline
This will be a proof by contradiction. We will assume that there exists some element $s_i \in S_{a,n}$, $s_0 < s_i$ such that $\rho(s_i, s_j) \neq a$ for all $s_j < s_i$. We will see this must mean that $s_{i-1} = s_i - 1$. Finally we will find that this implies that $\rho(s_{i-1}, s_i) \leq a$ which contradicts the definition of an awkward number series.

Proof
Let $S_{a,n}$ be any awkward number series.

Assume that there exists $s_i \in S_{a,n}$, $s_0 < s_i$ such that for all $s_j < s_i$, $\rho(s_i, s_j) \neq a$.

By definition, we know that $\rho(s_i, s_j) \geq a$.

As such, it must be the case that $\rho(s_i, s_j) > a$ since $\rho(s_i, s_j) \neq a$ by assumption.

Let $\rho(s_i, s_j) = r$.

$s_i = ts_j + r$ for some integer $t \in \mathbb{N}$ by the remainder theorem.

Subtracting 1 from both sides yields $s_i - 1 = ts_j + (r-1)$.

Since $a < r$ and $a \in \mathbb{Z}$, then $a \leq r - 1$.

Furthermore, $r - 1 < r < s_j$, thus $r - 1 \in [s_j]$.

By definition, $r - 1$ must be the remainder of s_i when divided by s_j.

As such, for all $s_j < s_i$, $\rho(s_i, s_j) = r - 1 \geq a$.

This implies that $s_i - 1 = s_{i-1}$ by definition.

By assumption, s_i has a remainder strictly greater than a when divided by any element less than it. As such, $\rho(s_i, s_{i-1}) > a$ must be the case.

Substituting s_i with $s_{i-1} + 1$ yields $\rho(s_i, s_{i-1}) = \rho(s_{i-1} + 1, s_{i-1})$.

By remainder property, $\rho(s_{i-1} + 1, s_{i-1}) = \rho(\rho(s_{i-1}, s_{i-1}) + \rho(1, s_{i-1}), s_{i-1})$.

By remainder property, $\rho(s_{i-1}, s_{i-1}) = 0$ since s_{i-1} is a multiple of itself.

By remainder property, $\rho(1, s_{i-1})$ since $1 < s_{i-1}$.

As such, $\rho(s_i, s_{i-1}) = \rho(s_{i-1} + 1, s_{i-1}) = \rho(0 + 1, s_{i-1}) = \rho(1, s_{i-1}) = 1$.

By definition, $a \geq 1$. By assumption, $1 = \rho(s_i, s_{i-1}) > a \geq 1$ which is a contradiction.

Awkward Linearity Theorem
For any awkward number series $S_{a,n}$, for any $s_i \in S_{a,n}$, there exists integers $x, y \in \mathbb{N}^+$ such that $s_i = xa + yn$.

Proof
This shall be a proof by induction. Let $S_{a,n}$ be any awkward number series.

Base Case

By definition, $s_0 = a + n = 1a + 1n$.

Inductive Hypothesis
Assume for some $0 \leq k$, that $s_i = xa + yn$ for some $x, y \in \mathbb{N}^+$ whenever $i \leq k$.

Inductive Step
By previous lemma, there exists some $s_i < s_{k+1}$ and some $t \in \mathbb{N}^+$ such that $s_{k+1} = ts_i + a$.

By the inductive hypothesis, $s_i = xa + yn$ for some integers $x, y \in \mathbb{N}^+$.

Substituting for s_i yields, $s_{k+1} = t(xa+yn)+a = txa+a+yn = (tx+1)a+yn$.

Corollary
For any awkward number series $S_{a,n}$, for any $s_0 < s_i \in S_{a,n}$, there exists integers $t, r \in \mathbb{N}^+$ such that $s_i = ts_0 + ra$.

Proof
Let $S_{a,n}$ be any awkward number series. We shall complete this proof by induction.

Base Case
By previous lemma $s_1 = 2a + n = (a + n) + a = s_0 + a$.

Inductive Hypothesis
Assume for some $1 \leq k$, that $s_i = ts_0 + ra$ for some integers $t, r \in \mathbb{N}^+$ whenever $i \leq k$.

Inductive Step
By previous lemma, there exists some $s_i < s_{k+1}$ and some $t \in \mathbb{N}^+$ such that $s_{k+1} = ts_i + a$.

If $s_i = s_0$, then we would have $s_{k+1} = ts_0 + a$. As such, we would have nothing left to show.

Let us assume $s_i > s_0$.

By inductive hypothesis, $s_i = us_0 + va$ for some integers $u, v \in \mathbb{N}^+$.

Substituting for s_i yields, $s_{k+1} = t(us_0 + va) + a = tus_0 + a(tv + 1)$

Corollary
For any awkward number series $S_{a,n}$, for any $s_0 < s_i \in S_{a,n}$, there exists integers $t, r \in \mathbb{N}^+$ such that $s_i = (t + r)a + tn$.

Proof
Let $S_{a,n}$ be any awkward number series. Let $s_0 < s_i \in S_{a,n}$.

By previous corollary, $s_i = ts_0 + ra$ for some integers $t, r \in \mathbb{N}^+$.

Substituting for $a + n$ for s_0 yields, $s_i = t(a + n) + ra$.

Distributing t over $a + n$ yields, $s_i = ta + ra + tn = (t + r)a + tn$.

Awkward Uniqueness Theorem
For any two awkward number series $S_{a,b}$ and $S_{c,d}$, $S_{a,b} = S_{c,d}$ if and only if $a = c$ and $b = d$.

In other words, no two awkward series contain the same set of elements.

Proof

Let $S_{a,n}$ be any awkward number series. Assume $S_{c,d} = S_{a,n}$ for some awkward number series $S_{c,d}$.

Let us refer to the elements of $S_{a,n}$ as $s_0, s_1, ...$, and the elements of $S_{c,d}$ by $s_0^*, s_1^*, ...$

By definition, $s_0 = a + n$, and $s_0^* = c + d$.

By assumption, $s_0 = s_0^*$. As such, $a + n = c + d$.

Solving for c yields, $c = a + n - d$.

By previous lemma, $s_1 = 2a + n$, and $s_1^* = 2c + d$.

By assumption, $s_1 = s_1^*$. As such, $2a + n = 2c + d$.

Substituting $c = a + n - d$ yields, $2a + n = 2(a + n - d) + d$.

Distributing the 2 yields, $2a + n = 2a + 2n - 2d + d = 2a + 2n - d$.

Subtracting the d from both sides yields, $2a + n + d = 2a + 2n$.

Subtracting the $2a$ from both sides yields $n + d = 2n$.

Subtracting n from both sides yields $d = n$.

Substituting n for d into $a + n = c + d$ yields $a + n = c + n$.

Subtracting n from both sides yields $a = c$.

Definition
An awkward number series, $S_{a,n}$ is called *simple* if the $gcd(a, n) = 1$. Otherwise the awkward number series is said to be *redundant*.

Definition
Any two awkward number series $S_{a,b}$ and $S_{c,d}$ are called *similar* whenever $\frac{a}{gcd(a,b)} = \frac{c}{gcd(c,d)}$ and $\frac{b}{gcd(a,b)} = \frac{d}{gcd(c,d)}$. Otherwise the series are said to be *dissimilar*.

Awkward Similarity Theorem
For any simple awkward number series $S_{a,n}$, for any positive integer x, the elements of the awkward number series $S_{xa,xn} = \{ \ xs_i \mid s_i \in S_{a,n} \ \}$.

Outline
This will be a proof by induction on the index of the elements.

Proof
Let $S_{a,n}$ be any simple awkward number series. Let j be any positive integer.

We shall denote the elements of $S_{a,n}$ as $s_0, s_1, ...$ We will denote the elements of $S_{ja,jn}$ as $s^*_0, s^*_1, ...$

Base Case
By definition, the first element of $S_{ja,jn}$ is $s^*_0 = ja + jn = j(a + n)$.

By definition, the first element of $S_{a,n}$ is $s_0 = a + n$.

As such, $s^*_0 = j(a + n) = js_0$.

Inductive Hypothesis
Assume for all $0 \leq i$ that $s^*_i = js_i$.

Inductive Step
We shall start by showing that $\rho(js_{i+1}, s^*_k) \geq ja$ for all $k \leq i$. Afterwards, we will then show that js_{i+1} is the least greatest integer that is both greater than s^*_i with this property. As such, $s^*i + 1 = js^{i+1}$ by definition.
By the inductive hypothesis, $s^*_k = js_k$ for all $k \leq i$.

As such, $\rho(js_{i+1}, s^*_k) = \rho(js_{i+1}, js_k)$.

By previous lemma (TODO), $\rho(js_{i+1}, js_k) = j\rho(s_{i+1}, s_k)$.

By definition, $\rho(s_{i+1}, s_k) \geq a$ for all $k \leq i$.

As such, $\rho(js_{i+1}, s_k^*) = j\rho(s_{i+1}, s_k) \geq ja$.

Thus, we have shown that js^{i+1} is a viable element of $S_{ja,jn}$. We now must show that that js^{i+1} is the least greatest integer greater than s_i^* with the divisibility property.

Assume there exists some integer $s_i^* < x < js^{i+1}$ such that $\rho(x, s_k^*) \geq ja$ for all $k \leq i$.

We know that $x = tj + \rho(x, j)$, for some $x \in \mathbb{N}$.
Let $r = \rho(x, j)$. Then $x = tj + r$.

Let $k \in [i+1]$. Then $\rho(x, s_k^*) = \rho(tj + r, s_k^*)$.

By the inductive hypothesis, $s_k^* = js_k$.

As such, $\rho(tj + r, s_k^*) = \rho(tj + r, js_k)$

By remainder property (TODO), $\rho(tj+r, js_k) = \rho(\rho(tj, js_k)+\rho(r, js_k), js_k)$.

By remainder property (TODO), $\rho(tj, js_k) = j\rho(t, s_k)$.

Since $r < j < js_k$, then $\rho(r, js_k) = r$.

As such, $\rho(\rho(tj, js_k) + \rho(r, js_k), js_k) = \rho(j\rho(t, s_k) + r, js_k)$.

By definition, $0 \leq \rho(t, s_k) < s_k$. Furthermore, $\rho(t, s_k) \in \mathbb{N}$. As such, $\rho(t, s_k) \leq s_k - 1$.

As such, $j\rho(t, s_k) \leq j(s_k - 1)$.

Thus, $j\rho(t, s_k) + r \leq j(s_k - 1) + r$.

We also know that $r < j$.

As such, $j(s_k - 1) + r < j(s_k - 1) + j = j(s_k - 1 + 1) = js_k$.

As such, $j\rho(t, s_k)+r < js_k$, thus $\rho(x, s_k^*) = \rho(j\rho(t, s_k)+r, js_k) = j\rho(t, s_k)+r$.

By assumption, $\rho(x, s_k^*) \geq ja$.

As such, $j\rho(t, s_k) + j) > j\rho(t, s_k) + r \geq ja$.

As such, $j(\rho(t, s_k) + 1) > ja$.

As such, $\rho(t, s_k) + 1 > a$.

Thus, $\rho(t, s_k) \geq a$.

Now if we can show that $s_i < t < s_{i+1}$, then t would have to be element $s_{i+1} \in S_{a,n}$ which would be a contradiction.

By assumption, $s_i^* < x = jt + r$.

$s_i^* = js_i$ by the inductive hypothesis.

As such, $js_i < jt + r < jt + j = j(t + 1)$

Thus $s_i < t + 1$. Since $s_i \in \mathbb{N}$, then $s_i \leq t$.

However, we've shown that $\rho(t, s_i) \geq a > 0$. As such, $t \neq s_i$. Thus, $s_i < t$ must be the case.

Now we just need to show that $t < js_{i+1}$.

We know that $x = tj + r < js_{i+1}$.

As such, $tj \leq tj + r < js_{i+1}$. Thus, $t < s_{i+1}$.

But this would mean that t must be the $(i + 2)^{th}$ element of $S_{a,n}$, which is a contradiction.

Lemma
For any awkward number series $S_{a,n}$ such that $a \geq n$ only contains a single staple which is s_1.

Proof
Let $S_{a,n}$ be an awkward number series such that $a \geq n$.

By previous lemma, for all $i \in [dim(S_{a,n})]$, $s_i = (i+1) + n$.

By definition, $dim(S_{a,n}) = \lceil \frac{n}{a} \rceil + 1$.

Since $n \leq a$, then $\lceil \frac{n}{a} \rceil = 1$. As such, $dim(S_{a,n}) = 2$.

Thus, $s_1 = 2a + n = a + s_0$ is in fact a staple.

NOTE: prove that the basis (except s_0) are all staples before this and then use that instead.

Now we must show that there can be no element $s_1 < s_i \in S_{a,n}$ that is also a staple.

Assume there exists some staple $s_i > s_1$.

By definition, $s_i = s_{i-1} + a$.

By definition, $\rho(s_{i-1}, s_0) \geq a$.

Furthermore, there exists some integer $t \in \mathbb{N}$ such that $s_{i-1} = ts_0 + \rho(s_{i-1}, s_0)$.

Adding a to both sides yields $s_{i-1} + a = ts_0 + \rho(s_{i-1}, s_0) + a$.

Substituting in $s_i = s_{i-1} + a$ yields $s_i = s_{i-1} + a = ts_0 + \rho(s_{i-1}, s_0) + a$.

Consider $\rho(s_i, s_0) = \rho(ts_0 + \rho(s_{i-1}, s_0) + a, s_0)$.

By the property of ρ, $\rho(ts_0 + \rho(s_{i-1}, s_0) + a, s_0) = \rho(\rho(s_{i-1}, s_0) + a, s_0)$.

Since $\rho(s_{i-1}, s_0) \geq a \geq n$, then $\rho(s_{i-1}, s_0) + a \geq a + n$.

Since $\rho(s_{i-1}, s_0) < s_0$, then $\rho(s_{i-1}) + a < s_0 + a = 2a + n < 2a + 2n = 2s_0$.

As such, $s_0 \leq \rho(s_{i-1}, s_0) + a < 2s_0$.

By property of ρ, $\rho(\rho(s_{i-1}, s_0) + a, s_0) = \rho(s_{i-1}, s_0) + a - s_0$.

Substituting for s_0, $\rho(s_i, s_0) = \rho(s_{i-1}, s_0) + a - a - n = \rho(s_{i-1}, s_0) - n$.

By definition, $\rho(s_{i-1}, s_0) < s_0$, as such, $\rho(s_{i-1}, s_0) - n < s_0 - n$.

Substituting for s_0, $\rho(s_{i-1}, s_0) - n < s_0 - n = a$.

As such, $\rho(s_i, s_0) = \rho(s_{i-1}, s_0) - n < a$. However, since $s_i > s_1$, then $\rho(s_i, s_0) \geq a$ by definition. As such, we have reached a contradiction.

Corollary
For any awkward number series $S_{a,n}$ such that $a \geq n$, for any $s_i, s_j \in S_{a,n}$ such that $s_1 \leq s_i < s_j$, it is the case that $s_j \geq (j - i)(a + 1) + s_i$.

Proof
We shall complete this proof by induction on the difference of i and j.
Let $S_{a,n}$ be any awkward number series such that $a \geq n$.

Base Case
Let $i \geq 1$ and $j = i + 1$.

By previous lemma, s_j cannot be a staple.

As such, $s_j \geq s_i + a + 1 = s_i + (1)(a + 1) = s_i + (j - i)(a + 1)$.

Inductive Hypothesis
Assume for some $1 \leq k \in \mathbb{N}$ that $s_j \geq s_i + (j - i)(a + 1)$ whenever $s_j > s_i$ and $j - i \leq k$.

Inductive Step

Let $s_i \in S_{a,n}$ such that $s_1 \leq s_i$.

By the inductive hypothesis, the element $s_{i+k} \geq s_i + k(a+1)$.

By previous lemma, $s_{i+k+1} \geq s_{i+k} + a + 1$.

As such, $s_{i+k} + a \geq s_i + k(a+1) + a + 1 = s_i + (k+1)(a+1)$.

As such, we have shown that $s_{i+k+1} \geq s_i + (k+1)(a+1)$.

Furthermore, the difference $(i+k+1) - i = k+1$.

Definition
For any awkward number series $S_{a,n}$, $s_{i-1}, s_i \in S_{a,n}$ are called *twins* whenever $s_i = s_{i-1} + (a+1)$.

Awkward Twin Conjecture
For any awkward number series $S_{a,n}$, $S_{a,n}$ contains an infinite number of twins.