

# Awkward Number Series

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## 1 Notation and Assumed Knowledge

### 1.1 Notation

#### Notation

- $\mathbb{Z}$  is defined to be the set of integers.
- $\mathbb{N} \subset \mathbb{Z}$  is defined to be the set of natural numbers, including 0.
- $\mathbb{N}^+ \subset \mathbb{N}$  is defined to be the set of positive integers.
- For any  $x \in \mathbb{N}^+$ ,  $[x] = \{ j \in \mathbb{N} \mid j < x \}$ .
- $\mathbb{Q}$  is defined to be the set of rational numbers.

### 1.2 Assumed Knowledge

#### 1.2.1 Remainders & Divisibility

##### Remainder Theorem

For any natural number  $x$ , for any positive integer  $y$ , there exists a unique integers  $z \in \mathbb{N}$  and  $r \in [y]$  such that  $x = zy + r$ . We call  $r$  the *remainder* of  $x$  when divided by  $y$ .

**Definition**

For any natural number  $x$ , for any positive integer  $y$ , the remainder function  $\rho : (\mathbb{N} \times \mathbb{N}^+) \rightarrow \mathbb{N}$ ,  $\rho(x, y)$  is defined to be the remainder of  $x$  when divided by  $y$ .

**Remainder Function Properties**

The remainder function has the following properties:

- For any  $i \in \mathbb{N}^+$ , for any  $j \in [i]$ ,  $\rho(j, i) = j$ .
- For any  $i \in \mathbb{N}^+$ , for any  $j \in \mathbb{Z}$ ,  $\rho(ij, j) = 0$ .
- For any  $j, k \in \mathbb{N}$ ,  $i \in \mathbb{N}^+$ ,  $\rho(kj, ki) = k\rho(j, i)$ .
- For any  $j, k \in \mathbb{N}$ ,  $i \in \mathbb{N}^+$ ,  $\rho(j + k, i) = \rho(\rho(j, i) + \rho(k, i), i)$ .
- For any  $j, k \in \mathbb{N}$ ,  $i \in \mathbb{N}^+$ ,  $\rho(k, i) = k - ji$  whenever  $ji \leq k < (j + 1)i$ .

**Definition**

For any natural number  $x$ , for any positive integer  $y$ , if  $\rho(x, y) = 0$ , then we say that  $x$  is *divisible* by  $y$ , that  $x$  is a *multiple* of  $y$ , and that  $y$  is a *divisor* of  $x$ .

**Definition**

For any set of integers  $X \subset \mathbb{N}^+$ , an integer  $z$  is called a *common multiple* of the elements of  $X$  whenever  $z$  is a multiple of every element of  $X$ .

### 1.2.2 Miscellaneous

**Definition**

For any  $q \in \mathbb{Q}$ , the *ceiling function*  $\lceil q \rceil = z$ , where  $z$  is the integer such that  $z - 1 < q \leq z$ .

**Lemma**

For any  $q = \frac{a}{b} \in \mathbb{Q}$ ,  $a, b \in \mathbb{Z}$

- $\lceil q \rceil = q$  whenever  $\rho(a, b) = 0$ .
- $\lceil q \rceil = \frac{c}{b}$ , where  $c = a + b - \rho(a, b)$  whenever  $\rho(a, b) > 0$ .

**Definition**

For any  $x, y \in \mathbb{N}$ , the function  $\gcd(x, y)$  is defined to be the greatest common divisor of  $x$  and  $y$ .

**Definition**

Any integer  $p > 1$  is called *prime* if its only divisors are one and itself.

## 2 Awkward Number Series

### 2.1 Definition & Basic Properties

#### Definition

For any positive integers  $a, n$ , the *awkward number series*,  $S_{a,n}$  is defined as:

- An initial element  $s_0 = a + n$
- For any  $i > 0$ ,  $s_i$  is defined to be the least greatest integer such that  $s_i > s_{i-1}$  and  $\rho(s_i, s_k) \geq a$  so all  $k < i$ .

We say that the awkward number series  $S_{a,n}$  has  $a$  *activators*, and  $n$  *initial non-activators*.

#### Lemma

For any awkward number series  $S_{a,n}$ , for any  $x \in \mathbb{N}$  such that  $x > a + n$ ,  $x$  is either an element of  $S_{a,n}$  or there exists some  $s_j \in S_{a,n}$  such that  $x > s_j$  and  $\rho(x, s_j) < a$ .

#### Proof

Let  $S_{a,n}$  be any awkward number series.

Let  $x \in \mathbb{N}$  be any natural number such that  $x > a + n$ .

Assume that there does not exist an  $s_j \in S_{a,n}$  such that  $s_j < x$  and  $\rho(x, s_j) < a$ .

As such, for all  $s_j \in S_{a,n}$  such that  $s_j < x$ ,  $\rho(x, s_j) \geq a$  must be the case.

By definition  $x$  must be an element of  $S_{a,n}$ .

Now let us assume there exists some element  $s_j \in S_{a,n}$  such that  $s_j < x$  and  $\rho(s_j, x) < a$ .

As such, it is not the case that for all  $s_j \in S_{a,n}$  such that  $s_j < x$ ,  $\rho(x, s_j) \geq a$ .

By definition  $x$  cannot be an element of  $S_{a,n}$ .

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**Lemma**

For any awkward number series  $S_{a,n}$ , for any  $s_i, s_j \in S_{a,n}$ ,  $\rho(s_i, s_j) < a$  if and only if  $s_i = s_j$ .

*Proof*

Let  $S_{a,n}$  be any awkward number series.

Let  $s_i \in S_{a,n}$  be any element in the series.

$s_i = s_i + 0$ . As such,  $\rho(s_i, s_i) = 0$  by definition of the remainder.

By definition of an awkward number series,  $a \geq 1 > 0$ .

Let  $s_j \in S_{a,n}$  be any element of the series such that  $s_j < s_i$ .

By definition of an awkward number series,  $\rho(s_i, s_j) \geq a$ .

As such,  $\rho(s_i, s_j) < a$  cannot be the case.

Let  $s_k \in S_{a,n}$  be any element of the series such that  $s_k > s_i$ .

$s_i = 0s_k + s_i$ . As such,  $\rho(s_i, s_k) = s_i$  by definition of the remainder.

By definition of an awkward number series,  $s_i \geq s_0 = a + n$ .

As such,  $\rho(s_i, s_k) = s_i \geq a + n \geq a$ .

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**Corollary**

For any awkward number series  $S_{a,n}$ , for any  $s_i \in S_{a,n}$ , it is the case that  $s_{i+1} \geq s_i + a$ .

*Proof*

Let  $S_{a,n}$  be any awkward number series.

Let  $s_i \in S_{a,n}$  be any element within the series.

Assume  $s_{i+1} < s_i + a$ .

Subtracting  $s_i$  from both sides yields,  $s_{i+1} - s_i < a$ .

By definition,  $s_i < s_{i+1}$ . As such,  $s_{i+1} - s_i > 0$ .

Let  $r = s_{i+1} - s_i$ . Then  $0 < r < a < s_i$ .

Furthermore,  $s_{i+1} = s_i + (s_{i+1} - s_i) = s_i + r$ .

By definition,  $r$  must be the remainder of  $s_{i+1}$  when divided by  $s_i$ .

As such,  $\rho(s_{i+1}, s_i) = r < a$ . However, this contradicts [the previous lemma](#).

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### **Awkward Infinity Theorem**

Every awkward number series contains an infinite number of elements.

*Proof*

Let  $S_{a,n}$  be any awkward number series.

Assume that  $S_{a,n}$  contains a finite number of elements.

Let  $s_{max}$  be the greatest element within  $S_{a,n}$ .

Let  $m$  be any common multiple of the elements of  $S_{a,n}$  such that  $m > s_{max}$ .

Consider the value  $m + a$ .

By assumption  $S_{a,n}$  is finite, as such, for any integer  $x > s_{max}$ , there exists some  $s_j \in S_{a,n}$  such that  $\rho(m + a, s_j) < a$  [by previous lemma](#).

Let  $\rho(m + a, s_j) = b < a$ .

By the [remainder theorem](#), there exists some integer  $x$  such that  $m + a = xs_j + b$ .

Since  $m$  is a common multiple of all the elements of  $S_{a,n}$ , then  $\frac{m}{s_j} \in \mathbb{N}$ .

Let  $y = \frac{m}{s_j}$ . Then  $m = ys_j$ .

Consider the equation  $a = (m + a) - m$ .

Substituting  $xs_j + b$  for  $m + a$  yields  $a = xs_j + b - m$ .

Substituting  $ys_j$  for  $m$  yields  $a = xs_j + b - ys_j$ .

Applying the distributive property yields  $a = (x - y)s_j + b$ .

Since  $b < a < s_j$ , then  $b$  must be the remainder of  $a$  when divided by  $s_j$  [by definition](#), as such  $\rho(a, s_j) = b$ .

Furthermore,  $a < s_j$ , as such  $\rho(a, s_j) = a = b$  [by properties of  \$\rho\$](#) .

However,  $b < a$  by assumption. As such, we have reached a contradiction.

Therefore, it must be the case that either  $m + a$  is an element of  $S_{a,n}$ , or there exists some other element in  $S_{a,n}$  less than  $m + a$  that was not accounted for. In either case,  $S_{a,n}$  cannot be finite.

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## 2.2 Prime Numbers

### Lemma

Every element of  $S_{1,1}$  is prime.

*Proof*

Assume there exists  $s_i \in S_{1,1}$  such that  $s_i$  is not prime.

Then there exists integers  $u, v$  such that  $1 < u \leq v < s_i$ ,  $s_i = uv$ .

Assume there exists  $s_j < s_i$  such that  $s_j$  divides either  $u$  or  $v$ .

Then  $u = ts_j$  or  $v = ts_j$  for some integer  $t$ .

As such,  $s_i = ts_jv$  or  $s_i = uts_j$ .

In either case,  $\rho(s_i, s_j) = 0$ .

However, [by definition](#),  $\rho(s_i, s_j) > 0$ .

As such, it must be the case that  $\rho(u, s_k) \geq 1$  for all  $s_k < s_i$ .

Let  $s_j \in S_{1,1}$  be the element such that  $s_j < u < s_{j+1}$ .

However,  $s_{j+1}$  is the least greatest integer greater than  $s_j$  with the property that  $\rho(s_{j+1}, s_k) \geq 1$  for all  $s_k \leq s_j$ .

As such,  $u$  cannot exist. Therefore, the only divisors of  $s_i$  are 1 and itself.

Thus,  $s_i$  is prime [by definition](#).

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### **Lemma**

The awkward number series  $S_{1,1}$  is equal to the set of prime numbers.

*Proof*

[By the previous lemma](#), we know that the elements of  $S_{1,1}$  are a subset of the prime numbers. As such, we need to show that every prime is an element of  $S_{1,1}$ .



Assume there exists some prime number  $p$  that is not an element of  $S_{1,1}$ .

Let  $s_k \in S_{1,1}$  such that  $s_k < p < s_{k+1}$ .

Since  $p$  is prime, its only divisors are 1 and itself [by definition](#).

As such, for all  $s_i \in S_{1,1}$  such that  $s_i < p$ , it is the case that  $\rho(p, s_i) > 0$ .

However,  $s_{k+1}$  is the least greatest integer greater than  $s_k$  with the property that  $\rho(p, s_i) > 0$  [by definition](#).

As such,  $p$  cannot exist. Thus every prime number must be an element of  $S_{1,1}$ .

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### Corollary

There are an infinite number of prime numbers.

### Proof

The set of prime numbers is equal to the elements of the awkward number series  $S_{1,1}$  [by previous lemma](#).

Every awkward number series contains an infinite number of elements [by the awkward infinity theorem](#).

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## 2.3 Dimension, Staples, and Basis

### Definition

For any awkward number series  $S_{a,n}$ , the value  $\lceil \frac{n}{a} \rceil + 1$  is called the *dimension* of the series, denoted  $\dim(S_{a,n})$ .

**Definition**

For any awkward number series  $S_{a,n}$ , for  $i \in [\dim(S_{a,n})]$ ,  $s_i$  is called a *basis* of the awkward number series.

**Lemma**

For any awkward number series  $S_{a,n}$ , it is the case that  $\dim(S_{a,n}) \geq 2$ .

*Proof*

Assume there exists an awkward number series  $S_{a,n}$  such that  $\dim(S_{a,n}) < 2$ .

By definition,  $\dim(S_{a,n}) = \lceil \frac{n}{a} \rceil + 1$ .

By definition,  $a, n \in \mathbb{N}^+$ . As such,  $\lceil \frac{n}{a} \rceil > 0$ .

Adding 1 to both sides yields  $\lceil \frac{n}{a} \rceil + 1 = \dim(S_{a,n}) > 1$ .

By definition,  $\lceil \frac{n}{a} \rceil \in \mathbb{Z}$ . As such,  $\dim(S_{a,n}) \geq 2$ .

**Lemma**

For any awkward number series  $S_{a,n}$ , for any basis  $s_i$  of the series, it is the case that  $s_i = a(i + 1) + n$ .

*Proof*

Let  $S_{a,n}$  be any awkward number series. We shall complete this proof via induction on the index of the first  $\lceil \frac{n}{a} \rceil + 1$  elements.

*Base Case*

By definition, the initial element is  $s_0 = a + n = a(0 + 1) + n$ .

*Inductive Hypothesis*

Assume for the some integer  $k$  such that  $0 \leq k < \lceil \frac{n}{a} \rceil$ , that  $s_j = a(j + 1) + n$  for all  $j \leq k$ .

*Inductive Step*

Let  $s_j$  be any element such that  $s_j \leq s_k$ .

By the inductive hypothesis,  $s_k = a(k+1) + n$  and  $s_j = a(j+1) + n$ .

Redistributing the  $a$  term in  $s_k$  yields  $a(k+1) + n = a(j+1) + (k-j)a + n$ .

As such,  $s_k = s_j + (k-j)a$  by substitution.

Adding  $a$  to both sides yields  $s_k + a = s_j + (k-j+1)a$ .

By the inductive hypothesis,  $k < \lceil \frac{n}{a} \rceil$ .

As such,  $k-j+1 < \lceil \frac{n}{a} \rceil - j + 1$ .

Since  $j > 0$ , then  $\lceil \frac{n}{a} \rceil - j + 1 < \lceil \frac{n}{a} \rceil + 1 \leq \lceil \frac{n}{a} \rceil$ .

As such,  $(k-j+1)a \leq a\lceil \frac{n}{a} \rceil$ .

If  $\rho(n, a) > 0$ , then  $a\lceil \frac{n}{a} \rceil = \frac{c}{a}$  where  $c = n + a - \rho(n, a)$  by previous lemma.

Since  $\rho(n, a) > 0$ , then  $c < n + a = s_0$  by definition.

If  $\rho(n, a) = 0$ , then  $a\lceil \frac{n}{a} \rceil = n < s_0$  by previous lemma.

In either case,  $a\lceil \frac{n}{a} \rceil < s_0$ .

As such,  $(k-j+1)a < s_0$ , therefore,  $(k-j+1)a \in [s_j]$ .

As such, since  $s_{k+1} = s_j + (k-j+1)a$ , then  $\rho(s_{k+1}, s_j) = (k-j+1)a$ .

Since  $j \leq k$ , then  $(k-j+1)a \geq (k-k+1)a = a$ .

As such,  $\rho(s_k + a, s_j) \geq a$  for any  $j \leq k$ .

By previous corollary,  $s_{k+1} \geq s_k + a$ .

As such,  $s_k + a$  is the least greatest integer greater than  $s_k$  with the property that  $\rho(s_k + a, s_j) \geq a$  for all  $j \leq k$ . Therefor,  $s_{k+1} = s_k + a$  by definition.

Substituting for  $s_k$  yields,  $s_{k+1} = a(k+1) + n + a = a(k+2) + n$ . As such, we have completed the inductive step.

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**Definition**

For any awkward number series  $S_{a,n}$ ,  $s_i \in S_{a,n}$  is called a *staple* whenever  $s_i = s_{i-1} + a$ .

**Lemma**

For any awkward number series  $S_{a,n}$ , for any integer  $0 < i < \dim(S_{a,n})$ , the element  $s_i \in S_{a,n}$  is a staple.

*Proof*

Let  $S_{a,n}$  be any awkward number series.

Let  $i$  be any integer such that  $0 < i < \dim(S_{a,n})$ .

By previous lemma,  $s_i = a(i+1) + n$  and  $s_{i-1} = ai + n$ .

Consider the difference  $s_i - s_{i-1}$ .

Substituting  $a(i+1) + n$  for  $s_i$  yields  $s_i - s_{i-1} = a(i+1) + n - s_{i-1}$ .

Substituting  $ai + n$  for  $s_{i-1}$  yields  $a(i+1) + n - s_{i-1} = a(i+1) + n - (ai + n)$ .

Distributing the  $-1$  yields  $a(i+1) + n - (ai + n) = a(i+1) + n - ai - n$ .

Adding the  $n$  terms yields,  $a(i+1) + n - ai - n = a(i+1) - ai$ .

Factoring the  $a$  yields,  $a(i+1) - ai = a(i+1-i) = a(1) = a$ .

As such,  $s_i - s_{i-1} = a$ .

Adding  $s_{i-1}$  to both sides yields  $s_i = s_{i-1} + a$ .

Thus,  $s_i$  is a staple by definition.

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**Lemma**

For any awkward number series  $S_{a,n}$  such that  $a \geq n$ , it is the case that  $\dim(S_{a,n}) = 2$ .

*Proof*

Let  $S_{a,n}$  be an awkward number series such that  $a \geq n$ .

By definition of dimension,  $\dim(S_{a,n}) = \lceil \frac{n}{a} \rceil + 1$ .

Since  $a, n \in \mathbb{N}^+$  by definition of an awkward number series, then  $\frac{n}{a} > 0$ .

Since  $n \leq a$ , then  $\frac{n}{a} \leq \frac{a}{a} = 1$ .

As such,  $0 < \frac{n}{a} \leq 1$ , therefore,  $\lceil \frac{n}{a} \rceil = 1$  by definition of the ceiling function.

Adding 1 to both sides yields  $\lceil \frac{n}{a} \rceil + 1 = 2$ .

Substituting in  $\dim(S_{a,n})$  yields  $\dim(S_{a,n}) = 2$ .

**Lemma**

For any awkward number series  $S_{a,n}$  such that  $a \geq n$ , the series only contains a single staple which is  $s_1$ .

*Proof*

Let  $S_{a,n}$  be an awkward number series such that  $a \geq n$ .

By previous lemma,  $\dim(S_{a,n}) = 2$  since  $a \geq n$ .

Since  $1 \in [2] = [\dim(S_{a,n})]$ , then  $s_1$  is a staple by previous lemma.

Now we must show that there can be no element  $s_1 < s_i \in S_{a,n}$  that is also a staple.

Assume there exists some staple  $s_i > s_1$ .

By definition of a staple,  $s_i = s_{i-1} + a$ .

Since  $s_i > s_1$ , then  $s_{i-1} \geq s_1 > s_0$ . As such, by definition of an awkward number series,  $\rho(s_{i-1}, s_0) \geq a$ .

Furthermore, there exists some integer  $t \in \mathbb{N}$  such that  $s_{i-1} = ts_0 + \rho(s_{i-1}, s_0)$  by the remainder theorem.

Adding  $a$  to both sides yields  $s_{i-1} + a = ts_0 + \rho(s_{i-1}, s_0) + a$ .

Substituting in  $s_i$  for  $s_{i-1} + a$  yields  $s_i = ts_0 + \rho(s_{i-1}, s_0) + a$ .

Since  $\rho(s_{i-1}, s_0) \geq a$ , then  $\rho(s_{i-1}, s_0) + a \geq a + a$ .

Furthermore,  $n \leq a$ , as such,  $\rho(s_{i-1}, s_0) + a \geq a + a \geq a + n = s_0$ .

Let  $r = \rho(s_{i-1}, s_0) + a - s_0$ .

Since  $\rho(s_{i-1}, s_0) + a \geq s_0$ , then  $\rho(s_{i-1}, s_0) + a - s_0 \geq 0$  by subtracting  $s_0$  from both sides.

Substituting in  $r$  yields  $r \geq 0$ .

Furthermore, both  $\rho(s_{i-1}, s_0) < s_0$  and  $a < s_0$ , as such,  $\rho(s_{i-1}, s_0) + a < s_0 + s_0 = 2s_0$ .

Subtracting  $s_0$  from both sides yields,  $\rho(s_{i-1}, s_0) + a - s_0 < s_0$ .

Substituting  $r$  yields,  $r < s_0$ . As such,  $0 \leq r < s_0$ .

We have that  $s_i = ts_0 + \rho(s_{i-1}, s_0) + a$ .

Since  $s_0 - s_0 = 0$ , then  $s_i = ts_0 + \rho(s_{i-1}, s_0) + a + (s_0 - s_0)$ .

Substituting in  $r$  yields,  $s_i = ts_0 + r + s_0$ .

Factoring  $s_0$  yields,  $s_i = (t + 1)s_0 + r$ .

Since  $r \in [s_0]$ , then  $\rho(s_i, s_0) = r$  [by definition of the remainder](#).

Since  $\rho(s_{i-1}, s_0) < s_0$ , then  $r = \rho(s_{i-1}, s_0) + a - s_0 < s_0 + a - s_0 < a$  by substitution.

As such,  $\rho(s_i, s_0) = r < a$ . However,  $\rho(s_i, s_0) \geq a$  [by definition of an awkward number series](#). As such, we have reached a contradiction.

Therefor, our assumption that  $s_i$  is a staple must be false. As such, there can be no staple greater than  $s_1$ .

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### Corollary

For any awkward number series  $S_{a,n}$  such that  $a \geq n$ , for any  $s_i, s_j \in S_{a,n}$  such that  $s_1 \leq s_i < s_j$ , it is the case that  $s_j \geq (j - i)(a + 1) + s_i$ .

### Proof

Let  $S_{a,n}$  be any awkward number series such that  $a \geq n$ .

We shall complete this proof by induction on the difference of indexes  $i$  and  $j$  for elements  $s_i, s_j \in S_{a,n}$ .

### Base Case

Let  $i \geq 1$  and  $j = i + 1$ .

As such,  $j \geq 1 + 1 = 2$  by substitution.

Therefor,  $s_j$  cannot be a staple [by previous lemma](#) since  $a \geq n$  and  $j \geq 2$ .

As such,  $s_j > s_i + a$ . Since  $s_j \in \mathbb{Z}$ , then  $s_j \geq s_i + a + 1$ .

Furthermore,  $j - i = (i + 1) - i = 1$  by substitution.

As such,  $s_j \geq s_i + a + 1 = s_i + (1)(a + 1) = s_i + (j - i)(a + 1)$ .

### *Inductive Hypothesis*

Assume for some integer  $k$  such that  $1 \leq k$ , that  $s_j \geq s_i + (j - i)(a + 1)$  whenever  $s_j > s_i$  and  $j - i \leq k$ .

### *Inductive Step*

Let  $s_i \in S_{a,n}$  such that  $s_1 \leq s_i$ .

Since  $k = (k + i) - i$ , then the element  $s_{i+k} \geq s_i + k(a + 1)$  by the inductive hypothesis.

Since  $i, k \geq 1$ , then  $i + k + 1 \geq 1 + 1 + 1 = 3$  by substitution.

As such,  $s_{i+k+1} \geq s_3 > s_1$ . Therefore,  $s_{i+k+1}$  cannot be a staple [by previous lemma](#).

As such,  $s_{i+k+1} > s_{i+k} + a$ .

Since  $s_{i+k+1}, s_{i+k}, a \in \mathbb{Z}$ , then  $s_{i+k+1} \geq s_{i+k} + a + 1$ .

Substituting out  $s_{i+k}$  yields,  $s_{i+k+1} \geq k(a + 1) + s_i + (a + 1)$ .

Factoring the  $(a + 1)$  yields  $s_{i+k+1} \geq (k + 1)(a + 1) + s_i$ .

Furthermore,  $(i + k + 1) - i = k + 1$ . As such,  $s_{i+k+1} \geq ((i + k + 1) - i)(a + 1) + s_i$  by substitution.

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## 2.4 Linearity Theorem

### **Lemma**

For any awkward number series  $S_{a,n}$ , for any  $i > 0$ , there exists  $s_j < s_i$  such that  $\rho(s_i, s_j) = a$ .

### *Outline*



This will be a proof by contradiction. We will assume that there exists some element  $s_i \in S_{a,n}$ ,  $s_0 < s_i$  such that  $\rho(s_i, s_j) \neq a$  for all  $s_j < s_i$ . We will see this must mean that  $s_{i-1} = s_i - 1$ . Finally we will find that this implies that  $\rho(s_{i-1}, s_i) \leq a$  which contradicts the definition of an awkward number series.

*Proof*

Let  $S_{a,n}$  be any awkward number series.

Assume that there exists  $s_i \in S_{a,n}$ ,  $s_0 < s_i$  such that for all  $s_j < s_i$ ,  $\rho(s_i, s_j) \neq a$ .

By definition, we know that  $\rho(s_i, s_j) \geq a$ .

As such, it must be the case that  $\rho(s_i, s_j) > a$  since  $\rho(s_i, s_j) \neq a$  by assumption.

Let  $\rho(s_i, s_j) = r$ .

$s_i = ts_j + r$  for some integer  $t \in \mathbb{N}$  by the remainder theorem.

Subtracting 1 from both sides yields  $s_i - 1 = ts_j + (r - 1)$ .

Since  $a < r$  and  $a \in \mathbb{Z}$ , then  $a \leq r - 1$ .

Furthermore,  $r - 1 < r < s_j$ , thus  $r - 1 \in [s_j]$ .

By definition,  $r - 1$  must be the remainder of  $s_i$  when divided by  $s_j$ .

As such, for all  $s_j < s_i$ ,  $\rho(s_i, s_j) = r - 1 \geq a$ .

This implies that  $s_i - 1 = s_{i-1}$  by definition.

By assumption,  $s_i$  has a remainder strictly greater than  $a$  when divided by any element less than it. As such,  $\rho(s_i, s_{i-1}) > a$  must be the case.

Substituting  $s_i$  with  $s_{i-1} + 1$  yields  $\rho(s_i, s_{i-1}) = \rho(s_{i-1} + 1, s_{i-1})$ .

By remainder property,  $\rho(s_{i-1} + 1, s_{i-1}) = \rho(\rho(s_{i-1}, s_{i-1}) + \rho(1, s_{i-1}), s_{i-1})$ .

By remainder property,  $\rho(s_{i-1}, s_{i-1}) = 0$  since  $s_{i-1}$  is a multiple of itself.

By remainder property,  $\rho(1, s_{i-1})$  since  $1 < s_{i-1}$ .

As such,  $\rho(s_i, s_{i-1}) = \rho(s_{i-1} + 1, s_{i-1}) = \rho(0 + 1, s_{i-1}) = \rho(1, s_{i-1}) = 1$ .

By definition,  $a \geq 1$ . By assumption,  $1 = \rho(s_i, s_{i-1}) > a \geq 1$  which is a contradiction.

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**Awkward Linearity Theorem**

For any awkward number series  $S_{a,n}$ , for any  $s_i \in S_{a,n}$ , there exists integers  $x, y \in \mathbb{N}^+$  such that  $s_i = xa + yn$ .

*Proof*

This shall be a proof by induction. Let  $S_{a,n}$  be any awkward number series.

*Base Case*

By definition,  $s_0 = a + n = 1a + 1n$ .

*Inductive Hypothesis*

Assume for some  $0 \leq k$ , that  $s_i = xa + yn$  for some  $x, y \in \mathbb{N}^+$  whenever  $i \leq k$ .

*Inductive Step*

By previous lemma, there exists some  $s_i < s_{k+1}$  and some  $t \in \mathbb{N}^+$  such that  $s_{k+1} = ts_i + a$ .

By the inductive hypothesis,  $s_i = xa + yn$  for some integers  $x, y \in \mathbb{N}^+$ .

Substituting for  $s_i$  yields,  $s_{k+1} = t(xa + yn) + a = txa + a + yn = (tx + 1)a + yn$ .

---

**Corollary**

For any awkward number series  $S_{a,n}$ , for any  $s_0 < s_i \in S_{a,n}$ , there exists integers  $t, r \in \mathbb{N}^+$  such that  $s_i = ts_0 + ra$ .

*Proof*

Let  $S_{a,n}$  be any awkward number series. We shall complete this proof by induction.

*Base Case*

By previous lemma  $s_1 = 2a + n = (a + n) + a = s_0 + a$ .

*Inductive Hypothesis*

Assume for some  $1 \leq k$ , that  $s_i = ts_0 + ra$  for some integers  $t, r \in \mathbb{N}^+$  whenever  $i \leq k$ .

*Inductive Step*

By previous lemma, there exists some  $s_i < s_{k+1}$  and some  $t \in \mathbb{N}^+$  such that  $s_{k+1} = ts_i + a$ .

If  $s_i = s_0$ , then we would have  $s_{k+1} = ts_0 + a$ . As such, we would have nothing left to show.

Let us assume  $s_i > s_0$ .

By inductive hypothesis,  $s_i = us_0 + va$  for some integers  $u, v \in \mathbb{N}^+$ .

Substituting for  $s_i$  yields,  $s_{k+1} = t(us_0 + va) + a = tus_0 + a(tv + 1)$

---

**Corollary**

For any awkward number series  $S_{a,n}$ , for any  $s_0 < s_i \in S_{a,n}$ , there exists integers  $t, r \in \mathbb{N}^+$  such that  $s_i = (t + r)a + tn$ .

*Proof*

Let  $S_{a,n}$  be any awkward number series. Let  $s_0 < s_i \in S_{a,n}$ .

By previous corollary,  $s_i = ts_0 + ra$  for some integers  $t, r \in \mathbb{N}^+$ .

Substituting for  $a + n$  for  $s_0$  yields,  $s_i = t(a + n) + ra$ .

Distributing  $t$  over  $a + n$  yields,  $s_i = ta + ra + tn = (t + r)a + tn$ .

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## 2.5 Uniqueness, Simplicity, and Similarity

### Awkward Uniqueness Theorem

For any two awkward number series  $S_{a,b}$  and  $S_{c,d}$ ,  $S_{a,b} = S_{c,d}$  if and only if  $a = c$  and  $b = d$ .

In other words, no two awkward series contain the same set of elements.

### *Proof*

Let  $S_{a,n}$  be any awkward number series. Assume  $S_{c,d} = S_{a,n}$  for some awkward number series  $S_{c,d}$ .

Let us refer to the elements of  $S_{a,n}$  as  $s_0, s_1, \dots$ , and the elements of  $S_{c,d}$  by  $s_0^*, s_1^*, \dots$ .

By definition,  $s_0 = a + n$ , and  $s_0^* = c + d$ .

By assumption,  $s_0 = s_0^*$ . As such,  $a + n = c + d$ .

Solving for  $c$  yields,  $c = a + n - d$ .

By previous lemma,  $s_1 = 2a + n$ , and  $s_1^* = 2c + d$ .

By assumption,  $s_1 = s_1^*$ . As such,  $2a + n = 2c + d$ .

Substituting  $c = a + n - d$  yields,  $2a + n = 2(a + n - d) + d$ .

Distributing the 2 yields,  $2a + n = 2a + 2n - 2d + d = 2a + 2n - d$ .

Subtracting the  $d$  from both sides yields,  $2a + n + d = 2a + 2n$ .

Subtracting the  $2a$  from both sides yields  $n + d = 2n$ .

Subtracting  $n$  from both sides yields  $d = n$ .

Substituting  $n$  for  $d$  into  $a + n = c + d$  yields  $a + n = c + n$ .

Subtracting  $n$  from both sides yields  $a = c$ .

---

**Definition**

An awkward number series,  $S_{a,n}$  is called *simple* if the  $\gcd(a, n) = 1$ . Otherwise the awkward number series is said to be *redundant*.

**Definition**

Any two awkward number series  $S_{a,b}$  and  $S_{c,d}$  are called *similar* whenever  $\frac{a}{\gcd(a,b)} = \frac{c}{\gcd(c,d)}$  and  $\frac{b}{\gcd(a,b)} = \frac{d}{\gcd(c,d)}$ . Otherwise the series are said to be *dissimilar*.

**Awkward Similarity Theorem**

For any simple awkward number series  $S_{a,n}$ , for any positive integer  $x$ , the elements of the awkward number series  $S_{xa, xn} = \{ xs_i \mid s_i \in S_{a,n} \}$ .

*Outline*

This will be a proof by induction on the index of the elements.

*Proof*

Let  $S_{a,n}$  be any simple awkward number series. Let  $j$  be any positive integer.

We shall denote the elements of  $S_{a,n}$  as  $s_0, s_1, \dots$ . We will denote the elements of  $S_{ja, jn}$  as  $s_0^*, s_1^*, \dots$ .

*Base Case*

By definition, the first element of  $S_{ja, jn}$  is  $s_0^* = ja + jn = j(a + n)$ .

By definition, the first element of  $S_{a,n}$  is  $s_0 = a + n$ .

As such,  $s_0^* = j(a + n) = js_0$ .

*Inductive Hypothesis*

Assume for all  $0 \leq i$  that  $s_i^* = js_i$ .

*Inductive Step*

We shall start by showing that  $\rho(js_{i+1}, s_k^*) \geq ja$  for all  $k \leq i$ . Afterwards, we will then show that  $js_{i+1}$  is the least greatest integer that is both greater than  $s_i^*$  with this property. As such,  $s^*i + 1 = js^{i+1}$  by definition.

By the inductive hypothesis,  $s_k^* = js_k$  for all  $k \leq i$ .

As such,  $\rho(js_{i+1}, s_k^*) = \rho(js_{i+1}, js_k)$ .

By previous lemma (TODO),  $\rho(js_{i+1}, js_k) = j\rho(s_{i+1}, s_k)$ .

By definition,  $\rho(s_{i+1}, s_k) \geq a$  for all  $k \leq i$ .

As such,  $\rho(js_{i+1}, s_k^*) = j\rho(s_{i+1}, s_k) \geq ja$ .

Thus, we have shown that  $js^{i+1}$  is a viable element of  $S_{ja, jn}$ . We now must show that that  $js^{i+1}$  is the least greatest integer greater than  $s_i^*$  with the divisibility property.

Assume there exists some integer  $s_i^* < x < js^{i+1}$  such that  $\rho(x, s_k^*) \geq ja$  for all  $k \leq i$ .

We know that  $x = tj + \rho(x, j)$ , for some  $x \in \mathbb{N}$ .

Let  $r = \rho(x, j)$ . Then  $x = tj + r$ .

Let  $k \in [i + 1]$ . Then  $\rho(x, s_k^*) = \rho(tj + r, s_k^*)$ .

By the inductive hypothesis,  $s_k^* = js_k$ .

As such,  $\rho(tj + r, s_k^*) = \rho(tj + r, js_k)$

By remainder property (TODO),  $\rho(tj + r, js_k) = \rho(\rho(tj, js_k) + \rho(r, js_k), js_k)$ .

By remainder property (TODO),  $\rho(tj, js_k) = j\rho(t, s_k)$ .

Since  $r < j < js_k$ , then  $\rho(r, js_k) = r$ .

As such,  $\rho(\rho(tj, js_k) + \rho(r, js_k), js_k) = \rho(j\rho(t, s_k) + r, js_k)$ .

By definition,  $0 \leq \rho(t, s_k) < s_k$ . Furthermore,  $\rho(t, s_k) \in \mathbb{N}$ . As such,  $\rho(t, s_k) \leq s_k - 1$ .

As such,  $j\rho(t, s_k) \leq j(s_k - 1)$ .

Thus,  $j\rho(t, s_k) + r \leq j(s_k - 1) + r$ .

We also know that  $r < j$ .

As such,  $j(s_k - 1) + r < j(s_k - 1) + j = j(s_k - 1 + 1) = js_k$ .

As such,  $j\rho(t, s_k) + r < js_k$ , thus  $\rho(x, s_k^*) = \rho(j\rho(t, s_k) + r, js_k) = j\rho(t, s_k) + r$ .

By assumption,  $\rho(x, s_k^*) \geq ja$ .

As such,  $j\rho(t, s_k) + j > j\rho(t, s_k) + r \geq ja$ .

As such,  $j(\rho(t, s_k) + 1) > ja$ .

As such,  $\rho(t, s_k) + 1 > a$ .

Thus,  $\rho(t, s_k) \geq a$ .

Now if we can show that  $s_i < t < s_{i+1}$ , then  $t$  would have to be element  $s_{i+1} \in S_{a,n}$  which would be a contradiction.

By assumption,  $s_i^* < x = jt + r$ .

$s_i^* = js_i$  by the inductive hypothesis.

As such,  $js_i < jt + r < jt + j = j(t + 1)$

Thus  $s_i < t + 1$ . Since  $s_i \in \mathbb{N}$ , then  $s_i \leq t$ .

However, we've shown that  $\rho(t, s_i) \geq a > 0$ . As such,  $t \neq s_i$ . Thus,  $s_i < t$  must be the case.

Now we just need to show that  $t < js_{i+1}$ .

We know that  $x = tj + r < js_{i+1}$ .

As such,  $tj \leq tj + r < js_{i+1}$ . Thus,  $t < s_{i+1}$ .

But this would mean that  $t$  must be the  $(i+2)^{th}$  element of  $S_{a,n}$ , which is a contradiction.

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## 2.6 Twins

### Definition

For any awkward number series  $S_{a,n}$ ,  $s_{i-1}, s_i \in S_{a,n}$  are called *twins* whenever  $s_i = s_{i-1} + (a+1)$ .

### Awkward Twin Conjecture

For any awkward number series  $S_{a,n}$ ,  $S_{a,n}$  contains an infinite number of twins.