Awkward Number Series

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1 Notation and Assumed Knowledge

Notation

- $\mathbb{Z}$ is defined to be the set of integers.
- $\mathbb{N} \subset \mathbb{Z}$ is defined to be the set of natural numbers, including 0.
- $\mathbb{N}^+ \subset \mathbb{N}$ is defined to be the set of positive integers.
- For any $x \in \mathbb{N}^+$, $[x] = \{ j \in \mathbb{N} \mid j < x \}$.
- $\mathbb{Q}$ is defined to be the set of rational numbers.
- For any $q \in \mathbb{Q}$, the *ceiling function* $\lceil q \rceil = z$, where z is the integer such that $z - 1 < q \leq z$.
- For any $x, y \in \mathbb{N}$, the function $gcd(x, y)$ is defined to be the greatest common divisor of x and y.

Definition
For any natural number x, for any positive integer y, the remainder function $\rho : (\mathbb{N} \times \mathbb{N}^+) \to \mathbb{N}$, $\rho(x, y)$ is defined to be the remainder of x when divided by y.

Assumed Knowledge
For any natural number x, for any positive integer y, there exists a unique integer $z \in \mathbb{N}$ such that $x = zy + \rho(x, y)$.

Definition
For any natural number x, for any positive integer y, if $\rho(x, y) = 0$, then we say that x is *divisible* by y.

Assumed Knowledge
The prime numbers can be recursively defined as the series:

- $p_0 = 2$ is the first element in the series.
- For all $k \in \mathbb{N}^+$, p_k is the least greatest integer such that $p_k > p_{k-1}$, and for all $j < k$, p_k is not divisible by p_j.

2 Awkward Number Series

Definition
For any positive integers a, n, the *awkward number series*, $S_{a,n}$ is defined as:

- An initial element $s_0 = a + n$
- For any $i > 0$, s_i is defined to be the least greatest integer such that $s_i > s_{i-1}$ and $\rho(s_i, s_k) \geq a$ so all $k < i$.

We say that the awkward number series $S_{a,n}$ has a *activators*, and n *initial non-activators*.

Lemma
The awkward number series $S_{1,1}$ is equal to the set of prime numbers.

Proof
TODO

Awkward Infinity Theorem
Every awkward number series contains an infinite number of elements.

Proof
Let $S_{a,n}$ be any awkward number series.

Assume that $S_{a,n}$ contains a finite number of elements.

Let s_i be the greatest element within $S_{a,n}$.

Let m be any positive common multiple of the elements of $S_{a,n}$.

Notice that $m > s_i$ since m is a multiple of s_i, but s_i is not a multiple of any s_j such that $j < i$.

Consider the value $m + a$.

Since $S_{a,n}$ is finite, there must exist some element, $s_j \in S_{a,n}$ such that $\rho(m + a, s_j) < a$. Otherwise, there is some element smaller than $m + a$ that has not been accounted for, or $m + a$ would be an element of $S_{a,n}$ that has not been accounted for.

Let $\rho(m + a, s_j) = b$.

There exists some integer x such that $m + a = xs_j + b$.

Since m is a common multiple of all the elements of $S_{a,n}$, then $\frac{m}{s_j} \in \mathbb{N}$.

Let $y = \frac{m}{s_j}$. Then $m = ys_j$.

Consider the equation $a = (m + a) - m$.

Substituting $xs_j + b$ for $m + a$ yields $a = xs_j + b - m$.

Substituting ys_j for m yields $a = xs_j + b - ys_j$.

Applying the distributive property yields $a = (x - y)s_j + b$.

If $x < y$, then $(x - y)s_j \leq -s_j$.

Since $0 \leq b < a < s_j$, then $(x - y)s_j + b < 0$ if $x < y$.

However, $a > 0$, as such, $x < y$ cannot be the case.

If $x > y$, then $(x - y)s_j \geq s_j$.

Since $0 \leq b$ and $a < s_j$, then $(x - y)s_j + b \geq s_j$ if $x > y$.

However, $a < s_j$, as such, $x > y$ cannot be the case.

As such, $x = y$ must be the case.

Substituting x for y yields $a = (x - x)s_j + b = b$.

By assumption, $b < a$, as such we have reached a contradiction.

Therefore, it must be the case that either $m+a$ is an element of $S_{a,n}$, or there exists some other element in $S_{a,n}$ less than $m+a$ that was not accounted for. In either case, $S_{a,n}$ cannot be finite.

Corollary
There are an infinite number of prime numbers.

Proof
The prime numbers are an awkward number series and every awkward number series contains an infinite number of elements.

Lemma
For any awkward number series $S_{a,n}$, the first $\lceil \frac{n}{a} \rceil + 1$ elements are given by $s_i = a(i+1) + n$.

Proof
Let $S_{a,n}$ be any awkward number series. We shall complete this proof via induction on the index of the first $\lceil \frac{n}{a} \rceil + 1$ elements.

Base Case
By definition, the initial element is $s_0 = a + n = a(0+1) + n$.

Inductive Hypothesis
Assume for the first $0 \leq j < \lceil \frac{n}{a} \rceil$, that $s_j = a(j+1) + n$.

Inductive Step
For all $x \in [a]$, $\rho(s_j + x, s_j) = x < a$. As such, $s_{j+1} \geq s_j + a$.

If we can show that $\rho(s_j + a, s_k) \geq a$ for all $k < j$, then $s_{j+1} = s_j + a$.

Furthermore, $s_j + a = a(j+1) + n + a = a(j+2) + n$, thus we will completed our proof.

Let $0 \leq k < j$. Then $s_j + a = s_k + a(j - k + 1)$ according to the inductive hypothesis.

As such, $\rho(s_j + a, s_k) = a(j - k + 1)$ as long as $a(j - k + 1) < s_k$.

Since $j < \lceil \frac{n}{a} \rceil$ and $j \in \mathbb{N}$, then $j \leq \lceil \frac{n}{a} \rceil - 1$.

As such, $a(j - k + 1) \leq a(\lceil \frac{n}{a} \rceil - 1 - k + 1) = a(\lceil \frac{n}{a} \rceil - k) = a\lceil \frac{n}{a} \rceil - ak$.

First, let us consider the case where $a \mid n$.

We will then have $a\lceil \frac{n}{a} \rceil = n$.

As such, $a(j - k + 1) \leq n - ak < s_k$.

Now let us consider the case where $\rho(n, a) \geq 1$.

Then $a\lceil\frac{n}{a}\rceil = a\frac{n+a-\rho(n,a)}{a} = n + a - \rho(n,a) = s_0 - \rho(n,a) < s_0 \leq s_k$.

As such, $a(j-k+1) < s_k - ak \leq s_k$.

Therefore, $\rho(s_j + a, s_k) = a(j-k+1)$ does in fact hold.

As such, we now need to show that $a(j-k+1) \geq a$.

We chose $k < j$, as such, $a(j-k+1) \geq a(j-j+1) = a$.

We have shown that $s_j + a = a(j+2)+n$ is the least greatest integer greater than s_j such that $\rho(s_j+a, s_k) \geq a$ for all $k \leq j$. Therefore, $s_{j+1} = a(j+2)+n$.

Definition
For any awkward number series $S_{a,n}$, the value $\lceil\frac{n}{a}\rceil + 1$ is called the *dimension* of the series.

Awkward Uniqueness Theorem
For any two awkward number series $S_{a,b}$ and $S_{c,d}$, $S_{a,b} = S_{c,d}$ if and only if $a = c$ and $b = d$.

In other words, no two awkward series contain the same set of elements.

Proof
TODO

Definition
An awkward number series, $S_{a,n}$ is called *simple* if the $gcd(a,n) = 1$. Otherwise the awkward number series is said to be *redundant*.

Definition
Any two awkward number series $S_{a,b}$ and $S_{c,d}$ are called *similar* whenever $\frac{a}{gcd(a,b)} = \frac{c}{gcd(c,d)}$ and $\frac{b}{gcd(a,b)} = \frac{d}{gcd(c,d)}$. Otherwise the series are said to be *dissimilar*.

Awkward Similarity Theorem
For any simple awkward number series $S_{a,n}$, for any positive integer x, the elements of the awkward number series $S_{xa,xn} = \{\ xs_i \mid s_i \in S_{a,n}\ \}$.

Proof
TODO
