Awkward Number Series

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1 Abstract

This paper defines and explores awkward number series - a generalization of the prime number series. In doing so, we shall find that the prime number series is an awkward number series itself. Not only that, but several of the basic proofs about the prime numbers can instead be proven about awkward number series as a whole. In particular, we shall find that every awkward number series contains an infinite number of elements; that every positive integer can be awkwardly factorized - a generalization on prime factorization - within a special class of awkward number series; and that every awkward number series can be produced by a state machine called an awkward state machine.

Awkward number series are born out of the less-used definition of the series of prime numbers: that the first prime number is the integer 2, and the next prime number in the series is the least greatest integer that is greater than the previous prime number, but not divisible by any of the previous prime numbers. Awkward numbers series apply two generalizations to this definition of primes. Rather than starting with the integer 2, what if we were to start with the integer 3 or 7 or 4353460983? Secondly, rather than the next element be simply not divisible by the previous elements, what would happen if we insisted that the next element cannot have a remainder greater than 2 or 8 or 13453411 when divided by any of the previous elements?

This paper does not claim to be groundbreaking; nevertheless, reading through the generalizations of Euclid's proofs on primes should be of interest to many in the mathematics community. Perhaps most interestingly, this paper will prove that the prime number series is an awkward number series, as such, any result about awkward number series can be immediately applied to the

primes, and many unanswered questions about primes can be restated about awkward number series.

2 Notation and Assumed Knowledge

This paper does not require a great deal of prior knowledge in order to be understood. In fact, anyone who is familiar with how to prove things should have the background knowledge required to read through this paper. In this section we shall lay out the notation we shall be using, and any previous knowledge we expect the reader to have. For the most part, this paper will stick to the standard notation used within mathematics today.

2.1 Notation

Notation

- $\mathbb{Z}$ is defined to be the set of integers.
- $\mathbb{N} \subset \mathbb{Z}$ is defined to be the set of natural numbers, including 0.
- $\mathbb{N}^+ \subset \mathbb{N}$ is defined to be the set of positive integers.
- For any $x \in \mathbb{N}^+$, $[x] = \{ j \in \mathbb{N} \mid j < x \}$.
- $\mathbb{Q}$ is defined to be the set of rational numbers.

2.2 Assumed Knowledge

2.2.1 Remainders & Divisibility

Remainder Theorem
For any natural number x, for any positive integer y, there exists a unique integers $z \in \mathbb{N}$ and $r \in [y]$ such that $x = zy + r$. We call r the *remainder* of x when divided by z.

Definition
For any natural number x, for any positive integer y, the remainder function $\rho : (\mathbb{N} \times \mathbb{N}^+) \to \mathbb{N}$, $\rho(x, y)$ is defined to be the remainder of x when divided by y.

Remainder Function Properties
The remainder function has the following properties:

- For any $i \in \mathbb{N}^+$, for any $j \in [i]$, $\rho(j, i) = j$.
- For any $i \in \mathbb{N}^+$, for any $j \in \mathbb{Z}$, $\rho(ij, j) = 0$.
- For any $j, k \in \mathbb{N}, i \in \mathbb{N}^+$, $\rho(kj, ki) = k\rho(j, i)$.
- For any $j, k \in \mathbb{N}, i \in \mathbb{N}^+$, $\rho(j + k, i) = \rho(\rho(j, i) + \rho(k, i), i)$.
- For any $j, k \in \mathbb{N}, i \in \mathbb{N}^+$, $\rho(k, i) = k - ji$ whenever $ji \leq k < (j + 1)i$.

Definition
For any natural number x, for any positive integer y, if $\rho(x, y) = 0$, then we say that x is *divisible* by y, that x is a *multiple* of y, and that y is a *divisor* of x.

Lemma
For any positive integers $x, y \geq 2$, such that $\rho(x, y) = 0$, it is the case that there exists some integers $z, p \in \mathbb{N}^+$ such that $x = zy^p$, $z < x$ and $\rho(z, y) > 0$.

Go to proof

Definition
For any set of integers $X \subset \mathbb{N}^+$, an integer z is called a *common multiple* of the elements of X whenever z is a multiple of every element of X.

2.2.2 Miscellaneous

Definition
For any $q \in \mathbb{Q}$, the *ceiling function* $\lceil q \rceil = z$, where z is the integer such that $z - 1 < q \leq z$.

Lemma
For any $q = \frac{a}{b} \in \mathbb{Q}$, $a, b \in \mathbb{Z}$

- $\lceil q \rceil = q$ whenever $\rho(a, b) = 0$.
- $\lceil q \rceil = \frac{c}{b}$, where $c = a + b - \rho(a, b)$ whenever $\rho(a, b) > 0$.

Definition
For any $x, y \in \mathbb{N}$, the function $gcd(x, y)$ is defined to be the greatest common divisor of x and y.

Definition
Any integer $p > 1$ is called *prime* if its only divisors are one and itself.

3 Awkward Number Series

In our first section, we will lay out the formal definition of an awkward number series. We will then very quickly prove some basic results directly from our definition. Finally, we shall conclude with our first theorem, the awkward infinity theorem, that states that every awkward number series contains an infinite number of elements. Most interestingly, we shall see that our proof for the awkward infinity theorem is essentially a copy of Euclid's original proof that there are an infinite number of primes.

3.1 Definition & Basic Properties

Definition
For any positive integers a, n, the *awkward number series*, $S_{a,n}$ is defined as:

- An initial element $s_0 = a + n$
- For any $i > 0$, s_i is defined to be the least greatest integer such that $s_i > s_{i-1}$ and $\rho(s_i, s_k) \geq a$ so all $k < i$.

We say that the awkward number series $S_{a,n}$ has *a activators*, and n *initial non-activators*.

Lemma
For any awkward number series $S_{a,n}$, for any $x \in \mathbb{N}$ such that $x > a+n$, x is either an element of $S_{a,n}$ or there exists some $s_j \in S_{a,n}$ such that $x > s_j$ and $\rho(x, s_j) < a$.

Proof
Let $S_{a,n}$ be any awkward number series.

Let $x \in \mathbb{N}$ be any natural number such that $x > a + n$.

Assume that there does not exist an $s_j \in S_{a,n}$ such that $s_j < x$ and $\rho(x, s_j) < a$.

As such, for all $s_j \in S_{a,n}$ such that $s_j < x$, $\rho(x, s_j) \geq a$ must be the case.

By definition x must be an element of $S_{a,n}$.

Now let us assume there exists some element $s_j \in S_{a,n}$ such that $s_j < x$ and $\rho(x, s_j) < a$.

As such, it is not the case that for all $s_j \in S_{a,n}$ such that $s_j < x$, $\rho(x, s_j) \geq a$.

By definition x cannot be an element of $S_{a,n}$.

> **Lemma**
> For any awkward number series $S_{a,n}$, for any $s_i, s_j \in S_{a,n}$, $\rho(s_i, s_j) < a$ if and only if $s_i = s_j$.

Proof
Let $S_{a,n}$ be any awkward number series.

Let $s_i \in S_{a,n}$ be any element in the series.

$s_i = s_i + 0$. As such, $\rho(s_i, s_i) = 0$ by definition of the remainder.

By definition of an awkward number series, $a \geq 1 > 0$.

Let $s_j \in S_{a,n}$ be any element of the series such that $s_j < s_i$.

By definition of an awkward number series, $\rho(s_i, s_j) \geq a$.

As such, $\rho(s_i, s_j) < a$ cannot be the case.

Let $s_k \in S_{a,n}$ be any element of the series such that $s_k > s_i$.

$s_i = 0s_k + s_i$. As such, $\rho(s_i, s_k) = s_i$ by definition of the remainder.

By definition of an awkward number series, $s_i \geq s_0 = a + n$.

As such, $\rho(s_i, s_k) = s_i \geq a + n \geq a$.

Corollary
For any awkward number series $S_{a,n}$, for any $s_i \in S_{a,n}$, it is the case that $s_{i+1} \geq s_i + a$.

Proof
Let $S_{a,n}$ be any awkward number series.

Let $s_i \in S_{a,n}$ be any element within the series.

Assume $s_{i+1} < s_i + a$.

Subtracting s_i from both sides yields, $s_{i+1} - s_i < a$.

By definition, $s_i < s_{i+1}$. As such, $s_{i+1} - s_i > 0$.

Let $r = s_{i+1} - s_i$. Then $0 < r < a < s_i$.

Furthermore, $s_{i+1} = s_i + (s_{i+1} - s_i) = s_i + r$.

By definition, r must be the remainder of s_{i+1} when divided by s_i.

As such, $\rho(s_{i+1}, s_i) = r < a$. However, this contradicts the previous lemma.

Awkward Infinity Theorem
Every awkward number series contains an infinite number of elements.

Proof
Let $S_{a,n}$ be any awkward number series.

Assume that $S_{a,n}$ contains a finite number of elements.

Let s_{max} be the greatest element within $S_{a,n}$.

Let m be any common multiple of the elements of $S_{a,n}$ such that $m > s_{max}$.

Consider the value $m + a$.

By assumption $S_{a,n}$ is finite, as such, for any integer $x > s_{max}$, there exists some $s_j \in S_{a,n}$ such that $\rho(m + a, s_j) < a$ by previous lemma.

Let $\rho(m + a, s_j) = b < a$.

By the remainder theorem, there exists some integer x such that $m + a = xs_j + b$.

Since m is a common multiple of all the elements of $S_{a,n}$, then $\frac{m}{s_j} \in \mathbb{N}$.

Let $y = \frac{m}{s_j}$. Then $m = ys_j$.

Consider the equation $a = (m + a) - m$.

Substituting $xs_j + b$ for $m + a$ yields $a = xs_j + b - m$.

Substituting ys_j for m yields $a = xs_j + b - ys_j$.

Applying the distributive property yields $a = (x - y)s_j + b$.

Since $b < a < s_j$, then b must be the remainder of a when divided by s_j by definition, as such $\rho(a, s_j) = b$.

Furthermore, $a < s_j$, as such $\rho(a, s_j) = a = b$ by properties of ρ.

However, $b < a$ by assumption. As such, we have reached a contradiction.

Therefore, it must be the case that either $m+a$ is an element of $S_{a,n}$, or there exists some other element in $S_{a,n}$ less than $m+a$ that was not accounted for. In either case, $S_{a,n}$ cannot be finite.

3.2 Prime Numbers

Now that we have wet our feet with awkward number series, it is a worthwhile exercise to tie them to the prime numbers. Doing so should strengthen our understanding of awkward number series as a whole. Not only that, we shall also then be able to restate any major results about awkward number series in terms of the primes. To accomplish this, this section will prove that the awkward number series $S_{1,1}$ is in fact equal to the prime number series.

> **Lemma**
> Every element of $S_{1,1}$ is prime.

Proof
Assume there exists $s_i \in S_{1,1}$ such that s_i is not prime.

Then there exists integers u, v such that $1 < u \leq v < s_i$, $s_i = uv$.

Assume there exists $s_j < s_i$ such that s_j divides either u or v.

Then $u = ts_j$ or $v = ts_j$ for some integer t.

As such, $s_i = ts_jv$ or $s_i = uts_j$.

In either case, $\rho(s_i, s_j) = 0$.

However, by definition, $\rho(s_i, s_j) > 0$.

As such, it must be the case that $\rho(u, s_k) \geq 1$ for all $s_k < s_i$.

Let $s_j \in S_{1,1}$ be the element such that $s_j < u < s_{j+1}$.

However, s_{j+1} is the least greatest integer greater than s_j with the property that $\rho(s_{j+1}, s_k) \geq 1$ for all $s_k \leq s_j$.

As such, u cannot exist. Therefore, the only divisors of s_i are 1 and itself.

Thus, s_i is prime by definition.

Lemma
For any $n \in \mathbb{N}^+$, for any prime $p \geq 1 + n$, it is the case that $p \in S_{1,n}$.

Proof
Let n be any positive integer. Let p be any prime such that $p \geq 1 + n$.

Note that $1 + n = s_0 \in S_{1,n}$ by definition of an awkward number series.

Let $s_i \in S_{1,n}$ such that $s_i \leq p < s_{i+1}$.

Assume $p > s_i$.

By definition of prime, the only factors of p are 1 and p.

As such, $\rho(p, s_j) \geq 1$ for all $s_j \in S_{1,n}$ such that $s_j < p$.

By assumption, $s_i < p$, as such, $\rho(p, s_j) \geq 1$ for all $s_j \leq s_i$.

By definition of an awkward number series, s_{i+1} is the least greatest integer greater than s_i such that $\rho(p, s_j) \geq 1$ for all $s_j \leq s_i$.

As such, $p = s_{i+1}$ must be the case.

However, s_{i+1} was chosen such that $p < s_{i+1}$. As such, we have reached a contradiction. Therefor, it must be the case that $p = s_i$.

Lemma
The awkward number series $S_{1,1}$ is equal to the set of prime numbers.

Proof

By the previous lemma, we know that the elements of $S_{1,1}$ are a subset of the prime numbers. As such, we need to show that every prime is an element of $S_{1,1}$.

By previous lemma, $S_{1,1}$ contains every prime greater than or equal to $1 + 1 = 2$.

By definition of primes, prime numbers are integers strictly greater than 1. As such, every prime is greater than or equal to 2.

As such, $S_{1,1}$ contains every prime number.

Corollary
There are an infinite number of prime numbers.

Proof
The set of prime numbers is equal to the elements of the awkward number series $S_{1,1}$ by previous lemma.

Every awkward number series contains an infinite number of elements by the awkward infinity theorem.

3.3 Dimension, Staples, and Basis

Definition
For any awkward number series $S_{a,n}$, the value $\lceil \frac{n}{a} \rceil + 1$ is called the *dimension* of the series, denoted $dim(S_{a,n})$.

Definition
For any awkward number series $S_{a,n}$, for $i \in [dim(S_{a,n})]$, s_i is called a *basis* of the awkward number series.

Lemma
For any awkward number series $S_{a,n}$, it is the case that $dim(S_{a,n}) \geq 2$.

Proof
Assume there exists an awkward number series $S_{a,n}$ such that $dim(S_{a,n}) < 2$.

By definition, $dim(S_{a,n}) = \lceil \frac{n}{a} \rceil + 1$.

By definition, $a, n \in \mathbb{N}^{+}$. As such, $\lceil \frac{n}{a} \rceil > 0$.

Adding 1 to both sides yields $\lceil \frac{n}{a} \rceil + 1 = dim(S_{a,n}) > 1$.

By definition, $\lceil \frac{n}{a} \rceil \in \mathbb{Z}$. As such, $dim(S_{a,n}) \geq 2$.

Lemma
For any awkward number series $S_{a,n}$, for any basis s_i of the series, it is the case that $s_i = a(i+1) + n$.

Proof
Let $S_{a,n}$ be any awkward number series. We shall complete this proof via induction on the index of the first $\lceil \frac{n}{a} \rceil + 1$ elements.

Base Case
By definition, the initial element is $s_0 = a + n = a(0+1) + n$.

Inductive Hypothesis
Assume for the some integer k such that $0 \leq k < \lceil \frac{n}{a} \rceil$, that $s_j = a(j+1) + n$ for all $j \leq k$.

Inductive Step

Let s_j be any element such that $s_j \leq s_k$.

By the inductive hypothesis, $s_k = a(k+1) + n$ and $s_j = a(j+1) + n$.

Redistributing the a term in s_k yields $a(k+1)+n = a(j+1)+(k-j)a+n$.

As such, $s_k = s_j + (k-j)a$ by substitution.

Adding a to both sides yields $s_k + a = s_j + (k-j+1)a$.

By the inductive hypothesis, $k < \lceil \frac{n}{a} \rceil$.

As such, $k - j + 1 < \lceil \frac{n}{a} \rceil - j + 1$.

Since $j > 0$, then $\lceil \frac{n}{a} \rceil - j + 1 < \lceil \frac{n}{a} \rceil + 1 \leq \lceil \frac{n}{a} \rceil$.

As such, $(k-j+1)a \leq a\lceil \frac{n}{a} \rceil$.

If $\rho(n,a) > 0$, then $a\lceil \frac{n}{a} \rceil = \frac{c}{a}$ where $c = n + a - \rho(n,a)$ by previous lemma.

Since $\rho(n,a) > 0$, then $c < n + a = s_0$ by definition.

If $\rho(n,a) = 0$, then $a\lceil \frac{n}{a} \rceil = n < s_0$ by previous lemma.

In either case, $a\lceil \frac{n}{a} \rceil < s_0$.

As such, $(k-j+1)a < s_0$, therefore, $(k-j+1)a \in [s_j]$.

As such, since $s_{k+1} = s_j + (k-j+1)a$, then $\rho(s_{k+1}, s_j) = (k-j+1)a$.

Since $j \leq k$, then $(k-j+1)a \geq (k-k+1)a = a$.

As such, $\rho(s_k + a, s_j) \geq a$ for any $j \leq k$.

By previous corollary, $s_{k+1} \geq s_k + a$.

As such, $s_k + a$ is the least greatest integer greater than s_k with the property that $\rho(s_k + a, s_j) \geq a$ for all $j \leq k$. Therefor, $s_{k+1} = s_k + a$ by definition.

Substituting for s_k yields, $s_{k+1} = a(k+1) + n + a = a(k+2) + n$. As such, we have completed the inductive step.

Definition
For any awkward number series $S_{a,n}$, $s_i \in S_{a,n}$ is called a *staple* whenever $s_i = s_{i-1} + a$.

Lemma
For any awkward number series $S_{a,n}$, for any integer $0 < i < dim(S_{a,n})$, the element $s_i \in S_{a,n}$ is a staple.

Proof
Let $S_{a,n}$ be any awkward number series.

Let i be any integer such that $0 < i < dim(S_{a,n})$.

By previous lemma, $s_i = a(i+1) + n$ and $s_{i-1} = ai + n$.

Consider the difference $s_i - s_{i-1}$.

Substituting $a(i+1)+n$ for s_i yields $s_i - s_{i-1} = a(i+1) + n - s_{i-1}$.

Substituting $ai+n$ for s_{i-1} yields $a(i+1)+n-s_{i-1} = a(i+1)+n-(ai+n)$.

Distributing the -1 yields $a(i+1) + n - (ai+n) = a(i+1) + n - ai - n$.

Adding the n terms yields, $a(i+1) + n - ai - n = a(i+1) - ai$.

Factoring the a yields, $a(i+1) - ai = a(i+1-i) = a(1) = a$.

As such, $s_i - s_{i-1} = a$.

Adding s_{i-1} to both sides yields $s_i = s_{i-1} + a$.

Thus, s_i is a staple by definition.

Lemma
For any awkward number series $S_{a,n}$ such that $a \geq n$, it is the case that $dim(S_{a,n}) = 2$.

Proof
Let $S_{a,n}$ be an awkward number series such that $a \geq n$.

By definition of dimension, $dim(S_{a,n}) = \lceil \frac{n}{a} \rceil + 1$.

Since $a, n \in \mathbb{N}^+$ by definition of an awkward number series, then $\frac{n}{a} > 0$.

Since $n \leq a$, then $\frac{n}{a} \leq \frac{a}{a} = 1$.

As such, $0 < \frac{n}{a} \leq 1$, therefore, $\lceil \frac{n}{a} \rceil = 1$ by definition of the ceiling function.

Adding 1 to both sides yields $\lceil \frac{n}{a} \rceil + 1 = 2$.

Substituting in $dim(S_{a,n})$ yields $dim(S_{a,n}) = 2$.

Lemma
For any awkward number series $S_{a,n}$ such that $a \geq n$, the series only contains a single staple which is s_1.

Proof
Let $S_{a,n}$ be an awkward number series such that $a \geq n$.

By previous lemma, $dim(S_{a,n}) = 2$ since $a \geq n$.

Since $1 \in [2] = [dim(S_{a,n})]$, then s_1 is a staple by previous lemma.

Now we must show that there can be no element $s_1 < s_i \in S_{a,n}$ that is also a staple.

Assume there exists some staple $s_i > s_1$.

By definition of a staple, $s_i = s_{i-1} + a$.

Since $s_i > s_1$, then $s_{i-1} \geq s_1 > s_0$. As such, by definition of an awkward number series, $\rho(s_{i-1}, s_0) \geq a$.

Furthermore, there exists some integer $t \in \mathbb{N}$ such that $s_{i-1} = ts_0 + \rho(s_{i-1}, s_0)$ by the remainder theorem.

Adding a to both sides yields $s_{i-1} + a = ts_0 + \rho(s_{i-1}, s_0) + a$.

Substituting in s_i for $s_{i-1} + a$ yields $s_i = ts_0 + \rho(s_{i-1}, s_0) + a$.

Since $\rho(s_{i-1}, s_0) \geq a$, then $\rho(s_{i-1}, s_0) + a \geq a + a$.

Furthermore, $n \leq a$, as such, $\rho(s_{i-1}, s_0) + a \geq a + a \geq a + n = s_0$.

Let $r = \rho(s_{i-1}, s_0) + a - s_0$.

Since $\rho(s_{i-1}, s_0) + a \geq s_0$, then $\rho(s_{i-1}, s_0) + a - s_0 \geq 0$ by subtracting s_0 from both sides.

Substituting in r yields $r \geq 0$.

Furthermore, both $\rho(s_{i-1}, s_0) < s_0$ and $a < s_0$, as such,
$\rho(s_{i-1}, s_0) + a < s_0 + s_0 = 2s_0$.

Subtracting s_0 from both sides yields, $\rho(s_{i-1}, s_0) + a - s_0 < s_0$.

Substituting r yields, $r < s_0$. As such, $0 \leq r < s_0$.

We have that $s_i = ts_0 + \rho(s_{i-1}, s_0) + a$.

Since $s_0 - s_0 = 0$, then $s_i = ts_0 + \rho(s_{i-1}, s_0) + a + (s_0 - s_0)$.

Substituting in r yields, $s_i = ts_0 + r + s_0$.

Factoring s_0 yields, $s_i = (t+1)s_0 + r$.

Since $r \in [s_0]$, then $\rho(s_i, s_0) = r$ by definition of the remainder.

Since $\rho(s_{i-1}, s_0) < s_0$, then $r = \rho(s_{i-1}, s_0) + a - s_0 < s_0 + a - s_0 < a$ by substitution.

As such, $\rho(s_i, s_0) = r < a$. However, $\rho(s_i, s_0) \geq a$ by definition of an awkward number series. As such, we have reached a contradiction.

Therefor, our assumption that s_i is a staple must be false. As such, there can be no staple greater than s_1.

Corollary
For any awkward number series $S_{a,n}$ such that $a \geq n$, for any $s_i, s_j \in S_{a,n}$ such that $s_1 \leq s_i < s_j$, it is the case that $s_j \geq (j-i)(a+1) + s_i$.

Proof
Let $S_{a,n}$ be any awkward number series such that $a \geq n$.

We shall complete this proof by induction on the difference of indexes i and j for elements $s_i, s_j \in S_{a,n}$.

Base Case
Let $i \geq 1$ and $j = i + 1$.

As such, $j \geq 1 + 1 = 2$ by substitution.

Therefor, s_j cannot be a staple by previous lemma since $a \geq n$ and $j \geq 2$.

As such, $s_j > s_i + a$. Since $s_j \in \mathbb{Z}$, then $s_j \geq s_i + a + 1$.

Furthermore, $j - i = (i+1) - i = 1$ by substitution.

As such, $s_j \geq s_i + a + 1 = s_i + (1)(a+1) = s_i + (j-i)(a+1)$.

Inductive Hypothesis
Assume for some integer k such that $1 \leq k$, that $s_j \geq s_i + (j-i)(a+1)$ whenever $s_j > s_i$ and $j - i \leq k$.

Inductive Step
Let $s_i \in S_{a,n}$ such that $s_1 \leq s_i$.

Since $k = (k+i) - i$, then the element $s_{i+k} \geq s_i + k(a+1)$ by the inductive hypothesis.

Since $i, k \geq 1$, then $i + k + 1 \geq 1 + 1 + 1 = 3$ by substitution.

As such, $s_{i+k+1} \geq s_3 > s_1$. Therefore, s_{i+k+1} cannot be a staple by previous lemma.

As such, $s_{i+k+1} > s_{i+k} + a$.

Since $s_{i+k+1}, s_{i+k}, a \in \mathbb{Z}$, then $s_{i+k+1} \geq s_{i+k} + a + 1$.

Substituting out s_{i+k} yields, $s_{i+k+1} \geq k(a+1) + s_i + (a+1)$.

Factoring the $(a+1)$ yields $s_{i+k+1} \geq (k+1)(a+1) + s_i$.

Furthermore, $(i+k+1)-i = k+1$. As such, $s_{i+k+1} \geq ((i+k+1)-i)(a+1)+s_i$ by substitution.

Corollary
For any awkward number series $S_{a,n}$ such that $a \geq n$, for any $s_i \in S_{a,n}$ such that $s_1 \leq s_i$, it is the case that $s_i \geq (i-1)(a+1) + 2a + n$.

Proof
Let $S_{a,n}$ be any awkward number series such that $a \geq n$.

Let s_i be any element of $S_{a,n}$ such that $s_i \geq s_1$.

By the previous corollary, $s_i \geq (i-1)(a+1) + s_1$ since $s_i \geq s_1$.

By previous lemma, $dim(S_{a,n}) = 2$ since $a \geq n$.

As such, s_1 is a basis by definition.

As such, $s_1 = (1+1)a + n = 2a + n$ by previous lemma since s_1 is a basis.

Substituting out s_1 yields, $s_i \geq (i-1)(a+1) + 2a + n$.

3.4 Linearity Theorem

It is natural to ask if there is any format to the elements of awkward number series. In this section we shall provide an answer to this question via the awkward linearity theorem, which states that every element of an awkward number series can be expressed as a linearly combination of the activators and initial non-activators using positive integer coefficients.

Lemma
For any awkward number series $S_{a,n}$, for any $i > 0$, there exists $s_j < s_i$ such that $\rho(s_i, s_j) = a$.

Outline
This will be a proof by contradiction. We will assume that there exists some element $s_i \in S_{a,n}$, $s_0 < s_i$ such that $\rho(s_i, s_j) \neq a$ for all $s_j < s_i$. We will see this must mean that $s_{i-1} = s_i - 1$. Finally we will find that this implies that $\rho(s_{i-1}, s_i) \leq a$ which contradicts the definition of an awkward number series.

Proof
Let $S_{a,n}$ be any awkward number series.

Assume that there exists $s_i \in S_{a,n}$, $s_0 < s_i$ such that for all $s_j < s_i$, $\rho(s_i, s_j) \neq a$.

By definition, we know that $\rho(s_i, s_j) \geq a$.

As such, it must be the case that $\rho(s_i, s_j) > a$ since $\rho(s_i, s_j) \neq a$ by assumption.

Let $\rho(s_i, s_j) = r$.

$s_i = ts_j + r$ for some integer $t \in \mathbb{N}$ by the remainder theorem.

Subtracting 1 from both sides yields $s_i - 1 = ts_j + (r - 1)$.

Since $a < r$ and $a \in \mathbb{Z}$, then $a \leq r - 1$.

Furthermore, $r - 1 < r < s_j$, thus $r - 1 \in [s_j]$.

By definition, $r - 1$ must be the remainder of s_i when divided by s_j.

As such, for all $s_j < s_i$, $\rho(s_i, s_j) = r - 1 \geq a$.

This implies that $s_i - 1 = s_{i-1}$ by definition.

By assumption, s_i has a remainder strictly greater than a when divided by any element less than it. As such, $\rho(s_i, s_{i-1}) > a$ must be the case.

Substituting s_i with $s_{i-1} + 1$ yields $\rho(s_i, s_{i-1}) = \rho(s_{i-1} + 1, s_{i-1})$.

By remainder property, $\rho(s_{i-1} + 1, s_{i-1}) = \rho(\rho(s_{i-1}, s_{i-1}) + \rho(1, s_{i-1}), s_{i-1})$.

By remainder property, $\rho(s_{i-1}, s_{i-1}) = 0$ since s_{i-1} is a multiple of itself.

By remainder property, $\rho(1, s_{i-1})$ since $1 < s_{i-1}$.

As such, $\rho(s_i, s_{i-1}) = \rho(s_{i-1} + 1, s_{i-1}) = \rho(0 + 1, s_{i-1}) = \rho(1, s_{i-1}) = 1$.

By definition, $a \geq 1$. By assumption, $1 = \rho(s_i, s_{i-1}) > a \geq 1$ which is a contradiction.

Awkward Linearity Theorem
For any awkward number series $S_{a,n}$, for any $s_i \in S_{a,n}$, there exists integers $x, y \in \mathbb{N}^+$ such that $s_i = xa + yn$.

Proof
This shall be a proof by induction. Let $S_{a,n}$ be any awkward number series.

Base Case
By definition, $s_0 = a + n = 1a + 1n$.

Inductive Hypothesis
Assume for some $0 \leq k$, that $s_i = xa + yn$ for some $x, y \in \mathbb{N}^+$ whenever $i \leq k$.

Inductive Step
By previous lemma, there exists some $s_i < s_{k+1}$ and some $t \in \mathbb{N}^+$ such that $s_{k+1} = ts_i + a$.

By the inductive hypothesis, $s_i = xa + yn$ for some integers $x, y \in \mathbb{N}^+$.

Substituting for s_i yields, $s_{k+1} = t(xa+yn)+a = txa+a+yn = (tx+1)a+yn$.

Corollary
For any awkward number series $S_{a,n}$, for any $s_0 < s_i \in S_{a,n}$, there exists integers $t, r \in \mathbb{N}^+$ such that $s_i = ts_0 + ra$.

Proof
Let $S_{a,n}$ be any awkward number series. We shall complete this proof by induction.

Base Case
By previous lemma $s_1 = 2a + n = (a + n) + a = s_0 + a$.

Inductive Hypothesis
Assume for some $1 \leq k$, that $s_i = ts_0 + ra$ for some integers $t, r \in \mathbb{N}^+$ whenever $i \leq k$.

Inductive Step
By previous lemma, there exists some $s_i < s_{k+1}$ and some $t \in \mathbb{N}^+$ such that $s_{k+1} = ts_i + a$.

If $s_i = s_0$, then we would have $s_{k+1} = ts_0 + a$. As such, we would have nothing left to show.

Let us assume $s_i > s_0$.

By inductive hypothesis, $s_i = us_0 + va$ for some integers $u, v \in \mathbb{N}^+$.

Substituting for s_i yields, $s_{k+1} = t(us_0 + va) + a = tus_0 + a(tv + 1)$

Corollary
For any awkward number series $S_{a,n}$, for any $s_0 < s_i \in S_{a,n}$, there exists integers $t, r \in \mathbb{N}^+$ such that $s_i = (t + r)a + tn$.

Proof
Let $S_{a,n}$ be any awkward number series. Let $s_0 < s_i \in S_{a,n}$.

By previous corollary, $s_i = ts_0 + ra$ for some integers $t, r \in \mathbb{N}^+$.

Substituting for $a + n$ for s_0 yields, $s_i = t(a + n) + ra$.

Distributing t over $a + n$ yields, $s_i = ta + ra + tn = (t + r)a + tn$.

3.5 Uniqueness, Simplicity, and Similarity

Awkward Uniqueness Theorem
For any two awkward number series $S_{a,b}$ and $S_{c,d}$, $S_{a,b} = S_{c,d}$ if and only if $a = c$ and $b = d$.

In other words, no two awkward series contain the same set of elements.

Proof
Let $S_{a,n}$ be any awkward number series. Assume $S_{c,d} = S_{a,n}$ for some awkward number series $S_{c,d}$.

Let us refer to the elements of $S_{a,n}$ as $s_0, s_1, ...$, and the elements of $S_{c,d}$ by $s_0^*, s_1^*, ...$

By definition, $s_0 = a + n$, and $s_0^* = c + d$.

By assumption, $s_0 = s_0^*$. As such, $a + n = c + d$.

Solving for c yields, $c = a + n - d$.

By previous lemma, $s_1 = 2a + n$, and $s_1^* = 2c + d$.

By assumption, $s_1 = s_1^*$. As such, $2a + n = 2c + d$.

Substituting $c = a + n - d$ yields, $2a + n = 2(a + n - d) + d$.

Distributing the 2 yields, $2a + n = 2a + 2n - 2d + d = 2a + 2n - d$.

Subtracting the d from both sides yields, $2a + n + d = 2a + 2n$.

Subtracting the $2a$ from both sides yields $n + d = 2n$.

Subtracting n from both sides yields $d = n$.

Substituting n for d into $a + n = c + d$ yields $a + n = c + n$.

Subtracting n from both sides yields $a = c$.

Awkward Similarity Theorem
For any awkward number series $S_{a,n}$, for any positive integer x, the elements of the awkward number series $S_{xa,xn} = \{ \ xs_i \mid s_i \in S_{a,n} \ \}$.

Outline
This will be a proof by induction on the index of the elements.

Proof
Let $S_{a,n}$ be any awkward number series. Let j be any positive integer.

We shall denote the elements of $S_{a,n}$ as $s_0, s_1, \ldots$. We will denote the elements of $S_{ja,jn}$ as $s_0^*, s_1^*, \ldots$.

Base Case
By definition, the first element of $S_{ja,jn}$ is $s_0^* = ja + jn = j(a+n)$.

By definition, the first element of $S_{a,n}$ is $s_0 = a + n$.

As such, $s_0^* = j(a+n) = js_0$.

Inductive Hypothesis
Assume for all some integer $i \in \mathbb{N}$, that for all $k \in [i+1]$ that $s_k^* = js_k$.

Inductive Step
We shall start by showing that $\rho(js_{i+1}, s_k^*) \geq ja$ for all $k \leq i$. Afterwards, we will then show that js_{i+1} is the least greatest integer that is both greater than s_i^* with this property. As such, $s_{i+1}^* = js_{i+1}$ by definition.

By the inductive hypothesis, $s_k^* = js_k$ for all $k \leq i$.

As such, $\rho(js_{i+1}, s_k^*) = \rho(js_{i+1}, js_k)$.

By property of the remainder function, $\rho(js_{i+1}, js_k) = j\rho(s_{i+1}, s_k)$.

By definition of an awkward number series, $\rho(s_{i+1}, s_k) \geq a$ for all $k \leq i$.

As such, $\rho(js_{i+1}, s_k^*) = j\rho(s_{i+1}, s_k) \geq ja$.

Thus, we have shown that js_{i+1} is a viable element of $S_{ja,jn}$. We now must show that that js_{i+1} is the least greatest integer greater than s_i^* with the divisibility property.

Assume there exists some integer $s_i^* < x < js_{i+1}$ such that $\rho(x, s_k^*) \geq ja$ for all $k \leq i$.

By the remainder theorem, $x = tj + \rho(x, j)$, for some $t \in \mathbb{N}$.

Let $r = \rho(x, j)$. Then $x = tj + r$.

Let $k \in [i + 1]$. Then $\rho(x, s_k^*) = \rho(tj + r, s_k^*)$ by substitution.

By the inductive hypothesis, $s_k^* = js_k$.

As such, $\rho(tj + r, s_k^*) = \rho(tj + r, js_k)$ by substitution.

By property of the remainder function, $\rho(tj+r, js_k) = \rho(\rho(tj, js_k)+\rho(r, js_k), js_k)$.

By property of the remainder function, $\rho(tj, js_k) = j\rho(t, s_k)$.

Since $r < j < js_k$, then $\rho(r, js_k) = r$ by property of the remainder function.

As such, $\rho(\rho(tj, js_k) + \rho(r, js_k), js_k) = \rho(j\rho(t, s_k) + r, js_k)$ by substitution.

By definition of the remainder, $\rho(t, s_k) \in [s_k]$, as such, $\rho(t, s_k) \leq s_k - 1$.

As such, $j\rho(t, s_k) \leq j(s_k - 1)$ by substitution.

Adding r to both sides yields: $j\rho(t, s_k) + r \leq j(s_k - 1) + r$.

We also know that $r < j$.

As such, $j(s_k - 1) + r < j(s_k - 1) + j = j(s_k - 1 + 1) = js_k$ by substitution.

As such, $j\rho(t, s_k)+r < js_k$, thus $\rho(x, s_k^*) = \rho(j\rho(t, s_k)+r, js_k) = j\rho(t, s_k)+r$ by property of the remainder function.

By assumption, $\rho(x, s_k^*) \geq ja$.

As such, $j(\rho(t, s_k) + 1) = j\rho(t, s_k) + j > j\rho(t, s_k) + r = \rho(x, s_k^*) \geq ja$.

As such, $j(\rho(t, s_k) + 1) > ja$.

As such, $\rho(t, s_k) + 1 > a$ by dividing both sides by j.

Thus, $\rho(t, s_k) \geq a$.

Now if we can show that $s_i < t < s_{i+1}$, then t would have to be element $s_{i+1} \in S_{a,n}$ which would be a contradiction.

By assumption, $s_i^* < x = jt + r$.

$s_i^* = js_i$ by the inductive hypothesis.

As such, $js_i < jt + r < jt + j = j(t + 1)$ by substitution.

Dividing by j yields: $s_i < t + 1$. Furthermore, $s_i \in \mathbb{N}$, as such $s_i \leq t$.

However, we've shown that $\rho(t, s_i) \geq a > 0$. As such, $t \neq s_i$. Thus, $s_i < t$ must be the case.

Now we just need to show that $t < s_{i+1}$.

We know that $x = tj + r < js_{i+1}$ by assumption on x.

As such, $tj \leq tj + r < js_{i+1}$. Thus, $t < s_{i+1}$ by dividing by j.

But this would mean that t must be the $(i + 2)^{th}$ element of $S_{a,n}$, which is a contradiction.

Definition
Any two awkward number series $S_{a,b}$ and $S_{c,d}$ are called *similar* whenever $\frac{a}{gcd(a,b)} = \frac{c}{gcd(c,d)}$ and $\frac{b}{gcd(a,b)} = \frac{d}{gcd(c,d)}$. Otherwise the series are said to be *dissimilar*.

Corollary
For any similar awkward number series $S_{a,b}$ and $S_{c,d}$, it is the case that $S_{a,b} = \{ \ \frac{gcd(a,b)}{gcd(c,d)}(s_i) \mid s_i \in S_{c,d} \ \}$.

Proof
Let $S_{a,b}$ and $S_{c,d}$ be any two similar awkward number series.

Let $e = \frac{a}{gcd(a,b)}$ and $f = \frac{b}{gcd(a,b)}$.

Since $S_{a,b}$ and $S_{e,f}$ are similar, then $e = \frac{c}{gcd(c,d)}$ and $f = \frac{d}{gcd(c,d)}$.

As such, $gcd(a,b)(e) = \frac{gcd(a,b)(a)}{gcd(a,b)} = a$ and $gcd(a,b)(f) = \frac{gcd(a,b)(b)}{gcd(a,b)} = b$.

Furthermore, $gcd(c,d)(e) = \frac{gcd(c,d)(c)}{gcd(c,d)} = c$ and $gcd(c,d)(f) = \frac{gcd(c,d)(d)}{gcd(c,d)} = d$.

Consider the awkward number series $S_{e,f}$.

By the awkward similarity theorem, $S_{a,b} = \{ \ gcd(a,b)(s_i^*) \mid s_i^* \in S_{e,f} \ \}$.

Equivalently, for any $s_i \in S_{a,b}$, $s_i = gcd(a,b)(s_i^*)$ where $s_i^* \in S_{e,f}$.

By the awkward similarity theorem, $S_{c,d} = \{ \ gcd(c,d)(s_i^*) \mid s_i^* \in S_{e,f} \ \}$.

Equivalently, for any $r_i \in S_{c,d}$, $r_i = gcd(c,d)(s_i^*)$ where $s_i^* \in S_{e,f}$.

As such, $s_i^* = \frac{r_i}{gcd(c,d)}$.

As such, for any $s_i \in S_{a,b}$, $s_i = gcd(a,b)(s_i^*) = \frac{gcd(a,b)}{gcd(c,d)}(r_i)$ where $r_i \in S_{c,d}$ by substitution.

Corollary
For any awkward number series $S_{a,n}$, for any $s_i \in S_{a,n}$, it is the case that s_i is divisible by $gcd(a,n)$.

Proof
Let $S_{a,n}$ be any awkward number series. Let $gcd(a, n) = g$.

Let $b = \frac{a}{g}$ and $m = \frac{n}{g}$. As such $a = bg$ and $n = mg$.

Since g divides both a and n and $g, a, n \in \mathbb{N}^+$, then both b and m are positive integers.

As such, $S_{a,n} = S_{bg,mg} = \{ \ gs_i \mid s_i \in S_{b,m} \ \}$ by the awkward similarity theorem.

As such, every element of $S_{a,n}$ is a multiple of $g = gcd(a, n)$.

Definition
An awkward number series, $S_{a,n}$ is called *simple* if the $gcd(a, n) = 1$. Otherwise the awkward number series is said to be *redundant*.

3.6 Awkward Factorization

Definition
For any $n \in \mathbb{N}^+$, for any awkward number series $S_{1,n}$, for any positive integer $x \geq 1 + n$. Whenever there exists some subset, A, of the elements of $S_{1,n}$ such that $x = cp$ for some integers c and p such that $1 \leq c < 1 + n$ and $p = \prod_{s_a \in A} s_a^{p_a}$ where $p_a \in \mathbb{N}^+$, then we call cp an awkward factorization of x in $S_{1,n}$.

For simplicity, whenever $c = 1$ we may exclude it from the awkward factorization of x. Similarly, we may exclude any of the powers p_a from the awkward factorization if they are equal to 1.

Lemma
For any $n \in \mathbb{N}^+$, for any awkward number series $S_{1,n}$, for any element $s_i \in S_{a,n}$, it is the case that s_i is an awkward factorization of itself in $S_{1,n}$.

Proof
Let $n \in \mathbb{N}^+$, let $s_i \in S_{1,n}$, and let $A = \{s_i\}$.

Then $s_i = (1)(s_i)^1$ is an awkward factorization of $s_i \in S_{1,n}$ by definition.

Lemma
For any $n \in \mathbb{N}^+$, for any awkward number series $S_{1,n}$, for any positive integer $x \geq 1+n$, it is the case that there exists some integers $c, p \in \mathbb{N}^+$ and some element $s_t \in S_{1,n}$ such that $x = c(s_t)^p$, $c < x$, and $\rho(c, s_t) > 0$.

Proof
Let $n \in \mathbb{N}^+$, let x be any integer such that $x \geq 1 + n$.

If $x \in S_{1,n}$, then $x = 1(x) = 1(x)^1$ and we would be done since $\rho(1, x) = 1 > 0$ and $1 < 1 + n \leq x$.

As such, assume that x is not an element of $S_{1,n}$.

By previous lemma, there exists some $s_t \in S_{1,n}$ such that $\rho(x, s_t) < 1$.

Since $\rho(x, s_t) \in [s_t]$ by definition, and $\rho(x, s_j) < 1$, then it must be the case that $\rho(x, s_j) = 0$.

Since $n \in \mathbb{N}^+$, then $n \geq 1$. As such, $x \geq 1 + n \geq 1 + 1 = 2$ by substitution.

Since $s_t \in S_{1,n}$, then $s_t \geq s_0 = 1 + n$ by definition of an awkward number series.

Furthermore, $s_t \geq 1 + n \geq 1 + 1 = 2$ by substitution.

As such, there exists integers $c, p \in \mathbb{N}^+$ such that $x = c(s_t)^p$, $c < x$, and $\rho(c, s_t) > 0$ by previous lemma.

Lemma
For any $j \in \mathbb{N}^+$, if for all $i \in [j]$, $x_i = x_{i+1}(a_i)^{p_i}$ such that $x_i, x_{i+1}, a_i, p_i \in \mathbb{N}^+$, then for any $k, l \in [j]$ such that $k < l$, it is the case that $x_k = x_l \prod_{i=k}^{l-1} (a_i)^{p_i}$

Proof
We shall complete this proof by induction on the difference of the indexes.

Assume for all $i \in [j]$, for some integer $j \in \mathbb{N}^+$, that $x_i = x_{i+1}(a_i)^{p_i}$ with $x_i, x_{i+1}, a_i, p_i \in \mathbb{N}^+$.

Base Case
Let $k \in [j-1]$. By assumption, $x_k = x_{k+1}(a_k)^{p_k}$.

Furthermore, $\prod_{i=k}^{k} (a_i)^{p_i} = (a_k)^{p_k}$. As such, $x_k = x_{k+1} \prod_{i=k}^{k} (a_i)^{p_i}$ by substitution.

Inductive Hypothesis
Assume for some integer m such that $1 \leq m < j - 1$, that $x_k = x_l \prod_{i=k}^{l-1} (a_i)^{p_i}$ whenever $l - k \leq m$.

Inductive Step
Let $k, l \in [j]$ be any integers such that $k - l = m + 1$.

Since $k - l = m + 1$, then we can subtract 1 from both sides to yield $k - (l - 1) = m$. Furthermore, $m \geq 1$, so $k - (l - 1) \geq 1$ is also true.

As such, we can apply the inductive hypothesis: $x_k = x_{l-1} \prod_{i=k}^{l-2} (a_i)^{p_i}$.

By assumption, $x_{l-1} = x_l (a_{l-1})^{p_{l-1}}$.

As such, $x_k = x_l (a_{l-1})^{p_{l-1}} \prod_{i=k}^{l-2} (a_i)^{p_i} = x_l \prod_{i=k}^{l-1} (a_i)^{p_i}$ by substitution.

Lemma
For any $n \in \mathbb{N}^+$, for any awkward number series $S_{1,n}$ for any $j \in \mathbb{N}^+$, if for all $i \in [j]$, $x_i = x_{i+1}(a_i)^{p_i}$ where:

- $x_i, x_{i+1}, p_i \in \mathbb{N}^+$
- $x_i > x_{i+1}$
- $a_i \in S_{1,n}$ and $a_i = a_k$ if and only if $i = k$
- $\rho(x_{i+1}, a_k) > 0$ whenever $k \leq i$
- $x_j \geq 1 + n$

then $x_j = x_{j+1}(a_j)^{p_j}$ such that:

- $x_{j+1}, p_j \in \mathbb{N}^+$
- $x_j > x_{j+1}$
- $a_j \in S_{1,n}$ and $a_j \neq a_k$ for all $k < j$
- $\rho(x_{j+1}, a_k) > 0$ whenever $k \leq j$

Proof
Assume for some integer j that the properties described above hold.

Then there exists some integers $x, p \in \mathbb{N}^+$ and some element $a \in S_{1,n}$ such that $x_j = x(a)^p$, $x < x_j$, and $\rho(x, a) > 0$ by previous lemma.

If we can show that $\rho(x, a_k) > 0$ for all $k \in [j]$, and $a \neq a_k$ for all $k \in [j]$, then we will have shown that $x = x_{j+1}$, $a = a_j$, and $p = p_j$ and we will have completed our proof.

Let us begin by showing $\rho(x, a_k) > 0$ for all $k \in [j]$. We shall accomplish this by contradiction.

Assume that there exists $k \in [j]$ such that $\rho(x, a_k) = 0$.

As such, there exists some integer u such that $x = ua_k$ by the remainder theorem.

We shall now show that this leads to $\rho(x_{k+1}, a_k) = 0$. In order to do so, we need to express x_{k+1} as a multiple of a_k.

There are two cases to consider, when $k = j - 1$ and $k < j - 1$. Let us start with the case where $k = j - 1$.

We know $x_{k+1} = x_j = x(a)^p = ua_k(a)^p$ by substitution. Therefor, $\rho(x_j, a_k) = \rho(x_j, a_{j-1}) = 0$. However, this contradicts our original assumption that $\rho(x_j, a_{j-1}) > 0$. Therefor, $k < j - 1$ must be the case.

Now let us consider the case where $k < j - 1$.

We can apply the previous lemma to get $x_{k+1} = x(a)^p \displaystyle\prod_{i=k+1}^{j-1} (a_i)^{p_i}$.

As such, $x_{k+1} = ua_k(a)^p \displaystyle\prod_{i=k+1}^{j-1} (a_i)^p_i$ by substitution.

Therefor, $\rho(x_{k+1}, a_k) = 0$ by the definition of a remainder.

However, we have that $k \in [j]$, thus our assumption $\rho(x_{k+1}, a_k) > 0$ holds. As such, we have reached a contradiction by assuming $\rho(x, a_k) = 0$. Therefore, $\rho(x, a_k) > 0$ must actually be the case for all $k \in [j]$.

Now we are only left with showing that $a \neq a_k$ for all $k \in [j]$ to complete our proof. We shall once again use contradiction.

Assume there exists some integer $k \in [j]$ such that $a = a_k$.

We have $x_j = x(a)^p$. As such, $x_j = x(a)^p = x(a_k)^p$ by substitution.

Furthermore, $x_{k+1} = x(a)^p \prod_{i=k+1}^{j-1} (a_i)^{p_i} = x(a_k)^p \prod_{i=k+1}^{j-1} (a_i)^{p_i}$ whenever $k+1 < j$; and $x_{k+1} = x_j = x(a_k)^p$ by substitution when $k+1 = j$. In either case, $\rho(x_{k+1}, a_k) = 0$.

However, this contradicts our assumption that $\rho(x_{k+1}, a_k) > 0$. As such, our assumption that $a = a_k$ must have been incorrect. Therefore, $a \neq a_k$ for all $k \in [j]$ must hold.

As such, we have now shown that $x_j = x(a)^p$ where $x, p \in \mathbb{N}^+$, $x < x_j$, $a \in S_{1,n}$, $a \neq a_k$ for all $k \in [j]$, and $\rho(x, a_k) > 0$ as well as $\rho(x, a) > 0$.

Awkward Factorization Theorem
For any $n \in \mathbb{N}^+$, for any awkward number series $S_{1,n}$, for any positive integer x, it is the case that x has an awkward factorization in $S_{1,n}$.

Proof
Let n and x_0 be any positive integers.

If $x_0 \in [1+n]$, then x_0 is the awkward factorization of itself and we are done.

As such, let us assume $x_0 \geq 1+n$.

As such, there exists some integers $x_1, p_0 \in \mathbb{N}^+$ and some element $a_0 \in S_{1,n}$ such that $x_0 = x_1(a_0)^{p_0}$, $x_1 < x_0$, and $\rho(x_1, a_0) > 0$ by previous lemma.

If $x_1 < 1+n$, then $x_1(a_0)^{p_0}$ is an awkward factorization and we are done.

As such, let us assume $x_1 \geq 1+n$.

Applying the previous lemma yields $x_1 = x_2(a_1)^{p_1}$.

Once again, if $x_2 < 1 + n$ then $x = x_2(a_1)^{p_1}(a_0)^{p_0}$ would be an awkward factorization and we would be down.

Notice that we can repeatedly apply the previous lemma to yield $x_i = x_{i+1}(a_i)^{p_i}$ as long as $x_i > 1 + n$.

Let $s_t \in S_{1,n}$ be the element such that $s_t \leq x_0 < s_{t+1}$.

Since $x_0 \leq s_t$, then $a_i \leq s_t$ for all i. Furthermore, a_i are all unique elements of $S_{1,n}$. As such, there must exist some $j \leq t + 1$ such that $x_j < 1 + n$, otherwise we would have $t + 1$ unique elements when there is only t elements to choose from.

By previous lemma, $x_0 = x_j \displaystyle\prod_{i=0}^{j-1}(a_i)^{p_i}$. Furthermore, this is an awkward factorization of x in $S_{1,n}$.

Definition
For any $n \in \mathbb{N}^+$, for any awkward number series $S_{1,n}$, for any positive integer x, if k is the number of unique awkward factorizations of x in $S_{1,n}$, then we say that x is *k-expressible* in $S_{1,n}$.

3.7 Twins

Definition
For any awkward number series $S_{a,n}$, $s_{i-1}, s_i \in S_{a,n}$ are called *twins* whenever $s_i = s_{i-1} + (a + 1)$.

Awkward Twin Conjecture
For any awkward number series $S_{a,n}$, $S_{a,n}$ contains an infinite number of twins.

3.8 Awkward State Machines

3.9 Awkward Vectors

3.10 Uncategorized

Lemma
For any awkward number series $S_{a,n}$, for any element $s_i \in S_{a,n}$, it is the case that $s_{i+1} \leq a + l$, where l is the least common multiple of all elements $s_k \in S_{a,n}$ such that $s_k \leq s_i$.

Proof
TODO: proof follows directly from the infinite theorem.

3.11 Proofs for Assumed Knowledge

Lemma
For any positive integers $x, y \geq 2$, such that $\rho(x, y) = 0$, it is the case that there exists some integers $z, p \in \mathbb{N}^+$ such that $x = zy^p$, $z < x$ and $\rho(z, y) > 0$.

Proof
Let x, y be any integers such that $x, y \geq 2$ and $\rho(x, y) = 0$.

If $x = y^p = (1)y^p$ for some $p \in \mathbb{N}^+$, then we would be done since $\rho(1, y) = 1$ and $1 < x$.

As such, assume $x \neq y^p$ for any $p \in \mathbb{N}^+$.

Since $\rho(x, y) = 0$, then there exists some integer $t \in \mathbb{N}^+$ such that $x = ty = ty^1$ by the remainder theorem.

Assume that for all $p \in \mathbb{N}^+$, that $\rho(x, y^p) = 0$.

Let q be any integer such that $x < y^q$.

Then $\rho(x, y^q) = x$ since $x < y^q$ which contradicts $\rho(z, y^p) = 0$ for all $p \in \mathbb{N}^+$.

As such, there must exist some integer $p \in \mathbb{N}^+$ such that $\rho(x, y^p) = 0$ and $\rho(x, y^{p+1}) \neq 0$.

By the remainder theorem, there exists some integer $z \in \mathbb{N}^+$ such that $x = zy^p$.

Assume that $\rho(z, y) = 0$.

Then there exists some integer w such that $z = wy$ by the remainder theorem.

As such, $x = zy^p = wyy^p$ by substitution.

Furthermore, $x = wyy^p = wy^{p+1}$ by properties of powers.

As such, $\rho(x, y^{p+1}) = 0$ by definition of the remainder. However, we chose p such that $\rho(x, y^{p+1}) \neq 0$. Therefor, we have reached a contradiction and our assumption that $\rho(z, y) = 0$ must be false.

As such, $\rho(z, y) \neq 0$ must be the case.

Furthermore, since $y \geq 2$ then $y^p \geq 2 > 1$.

As such, $x = zy^p > z(1) = z$.
