Awkward Number Series

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1 Notation and Assumed Knowledge

1.1 Notation

Notation

- $\mathbb{Z}$ is defined to be the set of integers.
- $\mathbb{N} \subset \mathbb{Z}$ is defined to be the set of natural numbers, including 0.
- $\mathbb{N}^+ \subset \mathbb{N}$ is defined to be the set of positive integers.
- For any $x \in \mathbb{N}^+$, $[x] = \{ \ j \in \mathbb{N} \mid j < x \ \}$.
- $\mathbb{Q}$ is defined to be the set of rational numbers.

1.2 Assumed Knowledge

1.2.1 Remainders & Divisibility

Remainder Theorem
For any natural number x, for any positive integer y, there exists a unique integers $z \in \mathbb{N}$ and $r \in [y]$ such that $x = zy + r$. We call r the *remainder* of x when divided by z.

Definition
For any natural number x, for any positive integer y, the remainder function $\rho : (\mathbb{N} \times \mathbb{N}^+) \to \mathbb{N}$, $\rho(x, y)$ is defined to be the remainder of x when divided by y.

Remainder Function Properties
The remainder function has the following properties:

- For any $i \in \mathbb{N}^+$, for any $j \in [i]$, $\rho(j, i) = j$.
- For any $i \in \mathbb{N}^+$, for any $j \in \mathbb{Z}$, $\rho(ij, j) = 0$.
- For any $j, k \in \mathbb{N}, i \in \mathbb{N}^+$, $\rho(kj, ki) = k\rho(j, i)$.
- For any $j, k \in \mathbb{N}, i \in \mathbb{N}^+$, $\rho(j + k, i) = \rho(\rho(j, i) + \rho(k, i), i)$.
- For any $j, k \in \mathbb{N}, i \in \mathbb{N}^+$, $\rho(k, i) = k - ji$ whenever $ji \leq k < (j + 1)i$.

Definition
For any natural number x, for any positive integer y, if $\rho(x, y) = 0$, then we say that x is *divisible* by y, that x is a *multiple* of y, and that y is a *divisor* of x.

Definition
For any set of integers $X \subset \mathbb{N}^+$, an integer z is called a *common multiple* of the elements of X whenever z is a multiple of every element of X.

1.2.2 Miscellaneous

Definition
For any $q \in \mathbb{Q}$, the *ceiling function* $\lceil q \rceil = z$, where z is the integer such that $z - 1 < q \leq z$.

Lemma
For any $q = \frac{a}{b} \in \mathbb{Q}$, $a, b \in \mathbb{Z}$

- $\lceil q \rceil = q$ whenever $\rho(a, b) = 0$.
- $\lceil q \rceil = \frac{c}{b}$, where $c = a + b - \rho(a, b)$ whenever $\rho(a, b) > 0$.

Definition
For any $x, y \in \mathbb{N}$, the function $gcd(x, y)$ is defined to be the greatest common divisor of x and y.

Definition
Any integer $p > 1$ is called *prime* if its only divisors are one and itself.

2 Awkward Number Series

2.1 Definition & Basic Properties

Definition
For any positive integers a, n, the *awkward number series*, $S_{a,n}$ is defined as:

- An initial element $s_0 = a + n$
- For any $i > 0$, s_i is defined to be the least greatest integer such that $s_i > s_{i-1}$ and $\rho(s_i, s_k) \geq a$ so all $k < i$.

We say that the awkward number series $S_{a,n}$ has a *activators*, and n *initial non-activators*.

Lemma
For any awkward number series $S_{a,n}$, for any $x \in \mathbb{N}$ such that $x > a+n$, x is either an element of $S_{a,n}$ or there exists some $s_j \in S_{a,n}$ such that $x > s_j$ and $\rho(x, s_j) < a$.

Proof
Let $S_{a,n}$ be any awkward number series.

Let $x \in \mathbb{N}$ be any natural number such that $x > a + n$.

Assume that there does not exist an $s_j \in S_{a,n}$ such that $s_j < x$ and $\rho(x, s_j) < a$.

As such, for all $s_j \in S_{a,n}$ such that $s_j < x$, $\rho(x, s_j) \geq a$ must be the case.

By definition x must be an element of $S_{a,n}$.

Now let us assume there exists some element $s_j \in S_{a,n}$ such that $s_j < x$ and $\rho(s_j, x) < a$.

As such, it is not the case that for all $s_j \in S_{a,n}$ such that $s_j < x$, $\rho(x, s_j) \geq a$.

By definition x cannot be an element of $S_{a,n}$.

Lemma
For any awkward number series $S_{a,n}$, for any $s_i, s_j \in S_{a,n}$, $\rho(s_i, s_j) < a$ if and only if $s_i = s_j$.

Proof
Let $S_{a,n}$ be any awkward number series.

Let $s_i \in S_{a,n}$ be any element in the series.

$s_i = s_i + 0$. As such, $\rho(s_i, s_i) = 0$ by definition of the remainder.

By definition of an awkward number series, $a \geq 1 > 0$.

Let $s_j \in S_{a,n}$ be any element of the series such that $s_j < s_i$.

By definition of an awkward number series, $\rho(s_i, s_j) \geq a$.

As such, $\rho(s_i, s_j) < a$ cannot be the case.

Let $s_k \in S_{a,n}$ be any element of the series such that $s_k > s_i$.

$s_i = 0s_k + s_i$. As such, $\rho(s_i, s_k) = s_i$ by definition of the remainder.

By definition of an awkward number series, $s_i \geq s_0 = a + n$.

As such, $\rho(s_i, s_k) = s_i \geq a + n \geq a$.

Corollary
For any awkward number series $S_{a,n}$, for any $s_i \in S_{a,n}$, it is the case that $s_{i+1} \geq s_i + a$.

Proof
Let $S_{a,n}$ be any awkward number series.

Let $s_i \in S_{a,n}$ be any element within the series.

Assume $s_{i+1} < s_i + a$.

Subtracting s_i from both sides yields, $s_{i+1} - s_i < a$.

By definition, $s_i < s_{i+1}$. As such, $s_{i+1} - s_i > 0$.

Let $r = s_{i+1} - s_i$. Then $0 < r < a < s_i$.

Furthermore, $s_{i+1} = s_i + (s_{i-1} - s_i) = s_i + r$.

By definition, r must be the remainder of s_{i+1} when divided by s_i.

As such, $\rho(s_{i+1}, s_i) = r < a$. However, this contradicts the previous lemma.

Awkward Infinity Theorem
Every awkward number series contains an infinite number of elements.

Proof
Let $S_{a,n}$ be any awkward number series.

Assume that $S_{a,n}$ contains a finite number of elements.

Let s_{max} be the greatest element within $S_{a,n}$.

Let m be any common multiple of the elements of $S_{a,n}$ such that $m > s_{max}$.

Consider the value $m + a$.

By assumption $S_{a,n}$ is finite, as such, for any integer $x > s_{max}$, there exists some $s_j \in S_{a,n}$ such that $\rho(m + a, s_j) < a$ by previous lemma.

Let $\rho(m + a, s_j) = b < a$.

By the remainder theorem, there exists some integer x such that $m + a = xs_j + b$.

Since m is a common multiple of all the elements of $S_{a,n}$, then $\frac{m}{s_j} \in \mathbb{N}$.

Let $y = \frac{m}{s_j}$. Then $m = ys_j$.

Consider the equation $a = (m + a) - m$.

Substituting $xs_j + b$ for $m + a$ yields $a = xs_j + b - m$.

Substituting ys_j for m yields $a = xs_j + b - ys_j$.

Applying the distributive property yields $a = (x - y)s_j + b$.

Since $b < a < s_j$, then b must be the remainder of a when divided by s_j by definition, as such $\rho(a, s_j) = b$.

Furthermore, $a < s_j$, as such $\rho(a, s_j) = a = b$ by properties of ρ.

However, $b < a$ by assumption. As such, we have reached a contradiction.

Therefore, it must be the case that either $m+a$ is an element of $S_{a,n}$, or there exists some other element in $S_{a,n}$ less than $m+a$ that was not accounted for. In either case, $S_{a,n}$ cannot be finite.

2.2 Prime Numbers

Lemma
Every element of $S_{1,1}$ is prime.

Proof
Assume there exists $s_i \in S_{1,1}$ such that s_i is not prime.

Then there exists integers u, v such that $1 < u \leq v < s_i$, $s_i = uv$.

Assume there exists $s_j < s_i$ such that s_j divides either u or v.

Then $u = ts_j$ or $v = ts_j$ for some integer t.

As such, $s_i = ts_jv$ or $s_i = uts_j$.

In either case, $\rho(s_i, s_j) = 0$.

However, by definition, $\rho(s_i, s_j) > 0$.

As such, it must be the case that $\rho(u, s_k) \geq 1$ for all $s_k < s_i$.

Let $s_j \in S_{1,1}$ be the element such that $s_j < u < s_{j+1}$.

However, s_{j+1} is the least greatest integer greater than s_j with the property that $\rho(s_{j+1}, s_k) \geq 1$ for all $s_k \leq s_j$.

As such, u cannot exist. Therefore, the only divisors of s_i are 1 and itself.

Thus, s_i is prime by definition.

Lemma
The awkward number series $S_{1,1}$ is equal to the set of prime numbers.

Proof
By the previous lemma, we know that the elements of $S_{1,1}$ are a subset of the prime numbers. As such, we need to show that every prime is an element of $S_{1,1}$.

Assume there exists some prime number p that is not an element of $S_{1,1}$.

Let $s_k \in S_{1,1}$ such that $s_k < p < s_{k+1}$.

Since p is prime, its only divisors are 1 and itself by definition.

As such, for all $s_i \in S_{1,1}$ such that $s_i < p$, it is the case that $\rho(p, s_i) > 0$.

However, s_{k+1} is the least greatest integer greater than s_k with the property that $\rho(p, s_i) > 0$ by definition.

As such, p cannot exist. Thus every prime number must be an element of $S_{1,1}$.

Corollary
There are an infinite number of prime numbers.

Proof
The set of prime numbers is equal to the elements of the awkward number series $S_{1,1}$ by previous lemma.

Every awkward number series contains an infinite number of elements by the awkward infinity theorem.

2.3 Dimension, Staples, and Basis

Definition
For any awkward number series $S_{a,n}$, the value $\lceil \frac{n}{a} \rceil + 1$ is called the *dimension* of the series, denoted $dim(S_{a,n})$.

Definition
For any awkward number series $S_{a,n}$, for $i \in [dim(S_{a,n})]$, s_i is called a *basis* of the awkward number series.

Lemma
For any awkward number series $S_{a,n}$, it is the case that $dim(S_{a,n}) \geq 2$.

Proof
Assume there exists an awkward number series $S_{a,n}$ such that $dim(S_{a,n}) < 2$.

By definition, $dim(S_{a,n}) = \lceil \frac{n}{a} \rceil + 1$.

By definition, $a, n \in \mathbb{N}^+$. As such, $\lceil \frac{n}{a} \rceil > 0$.

Adding 1 to both sides yields $\lceil \frac{n}{a} \rceil + 1 = dim(S_{a,n}) > 1$.

By definition, $\lceil \frac{n}{a} \rceil \in \mathbb{Z}$. As such, $dim(S_{a,n}) \geq 2$.

Lemma
For any awkward number series $S_{a,n}$, for any basis s_i of the series, it is the case that $s_i = a(i+1) + n$.

Proof
Let $S_{a,n}$ be any awkward number series. We shall complete this proof via induction on the index of the first $\lceil \frac{n}{a} \rceil + 1$ elements.

Base Case
By definition, the initial element is $s_0 = a + n = a(0+1) + n$.

Inductive Hypothesis
Assume for the some integer k such that $0 \leq k < \lceil \frac{n}{a} \rceil$, that $s_j = a(j+1) + n$ for all $j \leq k$.

Inductive Step

Let s_j be any element such that $s_j \leq s_k$.

By the inductive hypothesis, $s_k = a(k+1) + n$ and $s_j = a(j+1) + n$.

Redistributing the a term in s_k yields $a(k+1) + n = a(j+1) + (k-j)a + n$.

As such, $s_k = s_j + (k-j)a$ by substitution.

Adding a to both sides yields $s_k + a = s_j + (k-j+1)a$.

By the inductive hypothesis, $k < \lceil \frac{n}{a} \rceil$.

As such, $k - j + 1 < \lceil \frac{n}{a} \rceil - j + 1$.

Since $j > 0$, then $\lceil \frac{n}{a} \rceil - j + 1 < \lceil \frac{n}{a} \rceil + 1 \leq \lceil \frac{n}{a} \rceil$.

As such, $(k-j+1)a \leq a\lceil \frac{n}{a} \rceil$.

If $\rho(n, a) > 0$, then $a\lceil \frac{n}{a} \rceil = \frac{c}{a}$ where $c = n + a - \rho(n, a)$ by previous lemma.

Since $\rho(n, a) > 0$, then $c < n + a = s_0$ by definition.

If $\rho(n, a) = 0$, then $a\lceil \frac{n}{a} \rceil = n < s_0$ by previous lemma.

In either case, $a\lceil \frac{n}{a} \rceil < s_0$.

As such, $(k-j+1)a < s_0$, therefore, $(k-j+1)a \in [s_j]$.

As such, since $s_{k+1} = s_j + (k-j+1)a$, then $\rho(s_{k+1}, s_j) = (k-j+1)a$.

Since $j \leq k$, then $(k-j+1)a \geq (k-k+1)a = a$.

As such, $\rho(s_k + a, s_j) \geq a$ for any $j \leq k$.

By previous corollary, $s_{k+1} \geq s_k + a$.

As such, $s_k + a$ is the least greatest integer greater than s_k with the property that $\rho(s_k + a, s_j) \geq a$ for all $j \leq k$. Therefor, $s_{k+1} = s_k + a$ by definition.

Substituting for s_k yields, $s_{k+1} = a(k+1) + n + a = a(k+2) + n$. As such, we have completed the inductive step.

Definition
For any awkward number series $S_{a,n}$, $s_i \in S_{a,n}$ is called a *staple* whenever $s_i = s_{i-1} + a$.

Lemma
For any awkward number series $S_{a,n}$, for any integer $0 < i < dim(S_{a,n})$, the element $s_i \in S_{a,n}$ is a staple.

Proof
Let $S_{a,n}$ be any awkward number series.

Let i be any integer such that $0 < i < dim(S_{a,n})$.

By previous lemma, $s_i = a(i+1) + n$ and $s_{i-1} = ai + n$.

Consider the difference $s_i - s_{i-1}$.

Substituting $a(i+1) + n$ for s_i yields $s_i - s_{i-1} = a(i+1) + n - s_{i-1}$.

Substituting $ai + n$ for s_{i-1} yields $a(i+1) + n - s_{i-1} = a(i+1) + n - (ai + n)$.

Distributing the -1 yields $a(i+1) + n - (ai + n) = a(i+1) + n - ai - n$.

Adding the n terms yields, $a(i+1) + n - ai - n = a(i+1) - ai$.

Factoring the a yields, $a(i+1) - ai = a(i+1-i) = a(1) = a$.

As such, $s_i - s_{i-1} = a$.

Adding s_{i-1} to both sides yields $s_i = s_{i-1} + a$.

Thus, s_i is a staple by definition.

Lemma
For any awkward number series $S_{a,n}$ such that $a \geq n$, the series only contains a single staple which is s_1.

Proof
Let $S_{a,n}$ be an awkward number series such that $a \geq n$.

By definition of dimension, $dim(S_{a,n}) = \lceil \frac{n}{a} \rceil + 1$.

Since $a, n \in \mathbb{N}^+$ by definition of an awkward number series, then $\frac{n}{a} > 0$.

Since $n \leq a$, then $\frac{n}{a} \leq \frac{a}{a} = 1$.

As such, $0 < \frac{n}{a} \leq 1$, therefore, $\lceil \frac{n}{a} \rceil = 1$ by definition of the ceiling function.

Adding 1 to both sides yields $\lceil \frac{n}{a} \rceil + 1 = 2$.

Substituting in $dim(S_{a,n})$ yields $dim(S_{a,n}) = 2$.

Since $1 \in [2]$, then s_1 is a staple by previous lemma,.

Now we must show that there can be no element $s_1 < s_i \in S_{a,n}$ that is also a staple.

Assume there exists some staple $s_i > s_1$.

By definition of a staple, $s_i = s_{i-1} + a$.

By definition of an awkward number series, $\rho(s_{i-1}, s_0) \geq a$.

Furthermore, there exists some integer $t \in \mathbb{N}$ such that $s_{i-1} = ts_0 + \rho(s_{i-1}, s_0)$ by the remainder theorem.

Adding a to both sides yields $s_{i-1} + a = ts_0 + \rho(s_{i-1}, s_0) + a$.

Substituting in s_i for $s_{i-1} + a$ yields $s_i = ts_0 + \rho(s_{i-1}, s_0) + a$.

Since $\rho(s_{i-1}, s_0) \geq a$, then $\rho(s_{i-1}, s_0) + a \geq a + a$.

Furthermore, $n \leq a$, as such, $\rho(s_{i-1}, s_0) + a \geq a + a \geq a + n = s_0$.

Let $r = \rho(s_{i-1}, s_0) + a - s_0$.

Since $\rho(s_{i-1}, s_0) + a \geq s_0$, then $\rho(s_{i-1}, s_0) + a - s_0 \geq 0$ by subtracting s_0 from both sides.

Substituting in r yields $r \geq 0$.

Furthermore, both $\rho(s_{i-1}, s_0) < s_0$ and $a < s_0$, as such,
$\rho(s_{i-1}, s_0) + a < s_0 + s_0 = 2s_0$.

Subtracting s_0 from both sides yields, $\rho(s_{i-1}, s_0) + a - s_0 < s_0$.

Substituting r yields, $r < s_0$. As such, $0 \leq r < s_0$.

We have that $s_i = ts_0 + \rho(s_{i-1}, s_0) + a$.

Since $s_0 - s_0 = 0$, then $s_i = ts_0 + \rho(s_{i-1}, s_0) + a + (s_0 - s_0)$.

Substituting in r yields, $s_i = ts_0 + r + s_0$.

Factoring s_0 yields, $s_i = (t + 1)s_0 + r$.

Since $r \in [s_0]$, then $\rho(s_i, s_0) = r$ by definition of the remainder.

Since $\rho(s_{i-1}, s_0) < s_0$, then $r = \rho(s_{i-1}, s_0) + a - s_0 < s_0 + a - s_0 < a$ by substitution.

As such, $\rho(s_i, s_0) = r < a$. However, $\rho(s_i, s_0) \geq a$ by definition of an awkward number series. As such, we have reached a contradiction.

Therefor, our assumption that s_i is a staple must be false. As such, there can be no staple greater than s_1.

Corollary
For any awkward number series $S_{a,n}$ such that $a \geq n$, for any $s_i, s_j \in S_{a,n}$ such that $s_1 \leq s_i < s_j$, it is the case that $s_j \geq (j-i)(a+1) + s_i$.

Proof
We shall complete this proof by induction on the difference of i and j.
Let $S_{a,n}$ be any awkward number series such that $a \geq n$.

Base Case
Let $i \geq 1$ and $j = i + 1$.

By previous lemma, s_j cannot be a staple.

As such, $s_j \geq s_i + a + 1 = s_i + (1)(a+1) = s_i + (j-i)(a+1)$.

Inductive Hypothesis
Assume for some $1 \leq k \in \mathbb{N}$ that $s_j \geq s_i + (j-i)(a+1)$ whenever $s_j > s_i$ and $j - i \leq k$.

Inductive Step
Let $s_i \in S_{a,n}$ such that $s_1 \leq s_i$.

By the inductive hypothesis, the element $s_{i+k} \geq s_i + k(a+1)$.

By previous lemma, $s_{i+k+1} \geq s_{i+k} + a + 1$.

As such, $s_{i+k} + a \geq s_i + k(a+1) + a + 1 = s_i + (k+1)(a+1)$.

As such, we have shown that $s_{i+k+1} \geq s_i + (k+1)(a+1)$.

Furthermore, the difference $(i+k+1) - i = k + 1$.

2.4 Linearity Theorem

Lemma
For any awkward number series $S_{a,n}$, for any $i > 0$, there exists $s_j < s_i$ such that $\rho(s_i, s_j) = a$.

Outline
This will be a proof by contradiction. We will assume that there exists some element $s_i \in S_{a,n}$, $s_0 < s_i$ such that $\rho(s_i, s_j) \neq a$ for all $s_j < s_i$. We will see this must mean that $s_{i-1} = s_i - 1$. Finally we will find that this implies that $\rho(s_{i-1}, s_i) \leq a$ which contradicts the definition of an awkward number series.

Proof
Let $S_{a,n}$ be any awkward number series.

Assume that there exists $s_i \in S_{a,n}$, $s_0 < s_i$ such that for all $s_j < s_i$, $\rho(s_i, s_j) \neq a$.

By definition, we know that $\rho(s_i, s_j) \geq a$.

As such, it must be the case that $\rho(s_i, s_j) > a$ since $\rho(s_i, s_j) \neq a$ by assumption.

Let $\rho(s_i, s_j) = r$.

$s_i = ts_j + r$ for some integer $t \in \mathbb{N}$ by the remainder theorem.

Subtracting 1 from both sides yields $s_i - 1 = ts_j + (r - 1)$.

Since $a < r$ and $a \in \mathbb{Z}$, then $a \leq r - 1$.

Furthermore, $r - 1 < r < s_j$, thus $r - 1 \in [s_j]$.

By definition, $r - 1$ must be the remainder of s_i when divided by s_j.

As such, for all $s_j < s_i$, $\rho(s_i, s_j) = r - 1 \geq a$.

This implies that $s_i - 1 = s_{i-1}$ by definition.

By assumption, s_i has a remainder strictly greater than a when divided by any element less than it. As such, $\rho(s_i, s_{i-1}) > a$ must be the case.

Substituting s_i with $s_{i-1} + 1$ yields $\rho(s_i, s_{i-1}) = \rho(s_{i-1} + 1, s_{i-1})$.

By remainder property, $\rho(s_{i-1} + 1, s_{i-1}) = \rho(\rho(s_{i-1}, s_{i-1}) + \rho(1, s_{i-1}), s_{i-1})$.

By remainder property, $\rho(s_{i-1}, s_{i-1}) = 0$ since s_{i-1} is a multiple of itself.

By remainder property, $\rho(1, s_{i-1})$ since $1 < s_{i-1}$.

As such, $\rho(s_i, s_{i-1}) = \rho(s_{i-1} + 1, s_{i-1}) = \rho(0 + 1, s_{i-1}) = \rho(1, s_{i-1}) = 1$.

By definition, $a \geq 1$. By assumption, $1 = \rho(s_i, s_{i-1}) > a \geq 1$ which is a contradiction.

Awkward Linearity Theorem
For any awkward number series $S_{a,n}$, for any $s_i \in S_{a,n}$, there exists integers $x, y \in \mathbb{N}^+$ such that $s_i = xa + yn$.

Proof
This shall be a proof by induction. Let $S_{a,n}$ be any awkward number series.

Base Case
By definition, $s_0 = a + n = 1a + 1n$.

Inductive Hypothesis
Assume for some $0 \leq k$, that $s_i = xa + yn$ for some $x, y \in \mathbb{N}^+$ whenever $i \leq k$.

Inductive Step
By previous lemma, there exists some $s_i < s_{k+1}$ and some $t \in \mathbb{N}^+$ such that $s_{k+1} = ts_i + a$.

By the inductive hypothesis, $s_i = xa + yn$ for some integers $x, y \in \mathbb{N}^+$.

Substituting for s_i yields, $s_{k+1} = t(xa+yn)+a = txa+a+yn = (tx+1)a+yn$.

Corollary
For any awkward number series $S_{a,n}$, for any $s_0 < s_i \in S_{a,n}$, there exists integers $t, r \in \mathbb{N}^+$ such that $s_i = ts_0 + ra$.

Proof
Let $S_{a,n}$ be any awkward number series. We shall complete this proof by induction.

Base Case
By previous lemma $s_1 = 2a + n = (a + n) + a = s_0 + a$.

Inductive Hypothesis
Assume for some $1 \leq k$, that $s_i = ts_0 + ra$ for some integers $t, r \in \mathbb{N}^+$ whenever $i \leq k$.

Inductive Step
By previous lemma, there exists some $s_i < s_{k+1}$ and some $t \in \mathbb{N}^+$ such that $s_{k+1} = ts_i + a$.

If $s_i = s_0$, then we would have $s_{k+1} = ts_0 + a$. As such, we would have nothing left to show.

Let us assume $s_i > s_0$.

By inductive hypothesis, $s_i = us_0 + va$ for some integers $u, v \in \mathbb{N}^+$.

Substituting for s_i yields, $s_{k+1} = t(us_0 + va) + a = tus_0 + a(tv + 1)$

Corollary
For any awkward number series $S_{a,n}$, for any $s_0 < s_i \in S_{a,n}$, there exists integers $t, r \in \mathbb{N}^+$ such that $s_i = (t + r)a + tn$.

Proof
Let $S_{a,n}$ be any awkward number series. Let $s_0 < s_i \in S_{a,n}$.

By previous corollary, $s_i = ts_0 + ra$ for some integers $t, r \in \mathbb{N}^+$.

Substituting for $a + n$ for s_0 yields, $s_i = t(a + n) + ra$.

Distributing t over $a + n$ yields, $s_i = ta + ra + tn = (t + r)a + tn$.

2.5 Uniqueness, Simplicity, and Similarity

Awkward Uniqueness Theorem
For any two awkward number series $S_{a,b}$ and $S_{c,d}$, $S_{a,b} = S_{c,d}$ if and only if $a = c$ and $b = d$.

In other words, no two awkward series contain the same set of elements.

Proof
Let $S_{a,n}$ be any awkward number series. Assume $S_{c,d} = S_{a,n}$ for some awkward number series $S_{c,d}$.

Let us refer to the elements of $S_{a,n}$ as $s_0, s_1, ...$, and the elements of $S_{c,d}$ by $s_0^*, s_1^*, ...$.

By definition, $s_0 = a + n$, and $s_0^* = c + d$.

By assumption, $s_0 = s_0^*$. As such, $a + n = c + d$.

Solving for c yields, $c = a + n - d$.

By previous lemma, $s_1 = 2a + n$, and $s_1^* = 2c + d$.

By assumption, $s_1 = s_1^*$. As such, $2a + n = 2c + d$.

Substituting $c = a + n - d$ yields, $2a + n = 2(a + n - d) + d$.

Distributing the 2 yields, $2a + n = 2a + 2n - 2d + d = 2a + 2n - d$.

Subtracting the d from both sides yields, $2a + n + d = 2a + 2n$.

Subtracting the $2a$ from both sides yields $n + d = 2n$.

Subtracting n from both sides yields $d = n$.

Substituting n for d into $a + n = c + d$ yields $a + n = c + n$.

Subtracting n from both sides yields $a = c$.

Definition
An awkward number series, $S_{a,n}$ is called *simple* if the $gcd(a, n) = 1$. Otherwise the awkward number series is said to be *redundant*.

Definition
Any two awkward number series $S_{a,b}$ and $S_{c,d}$ are called *similar* whenever $\frac{a}{gcd(a,b)} = \frac{c}{gcd(c,d)}$ and $\frac{b}{gcd(a,b)} = \frac{d}{gcd(c,d)}$. Otherwise the series are said to be *dissimilar*.

Awkward Similarity Theorem
For any simple awkward number series $S_{a,n}$, for any positive integer x, the elements of the awkward number series $S_{xa,xn} = \{\ xs_i \mid s_i \in S_{a,n}\ \}$.

Outline
This will be a proof by induction on the index of the elements.

Proof
Let $S_{a,n}$ be any simple awkward number series. Let j be any positive integer.

We shall denote the elements of $S_{a,n}$ as $s_0, s_1, \ldots.$ We will denote the elements of $S_{ja,jn}$ as $s^*_0, s^*_1, \ldots.$

Base Case
By definition, the first element of $S_{ja,jn}$ is $s^*_0 = ja + jn = j(a + n)$.

By definition, the first element of $S_{a,n}$ is $s_0 = a + n$.

As such, $s^*_0 = j(a + n) = js_0$.

Inductive Hypothesis
Assume for all $0 \leq i$ that $s^*_i = js_i$.

Inductive Step
We shall start by showing that $\rho(js_{i+1}, s^*_k) \geq ja$ for all $k \leq i$. Afterwards, we will then show that js_{i+1} is the least greatest integer that is both greater than s^*_i with this property. As such, $s^*i + 1 = js^{i+1}$ by definition.
By the inductive hypothesis, $s^*_k = js_k$ for all $k \leq i$.

As such, $\rho(js_{i+1}, s^*_k) = \rho(js_{i+1}, js_k)$.

By previous lemma (TODO), $\rho(js_{i+1}, js_k) = j\rho(s_{i+1}, s_k)$.

By definition, $\rho(s_{i+1}, s_k) \geq a$ for all $k \leq i$.

As such, $\rho(js_{i+1}, s^*_k) = j\rho(s_{i+1}, s_k) \geq ja$.

Thus, we have shown that js^{i+1} is a viable element of $S_{ja,jn}$. We now must show that that js^{i+1} is the least greatest integer greater than s^*_i with the divisibility property.

Assume there exists some integer $s^*_i < x < js^{i+1}$ such that $\rho(x, s^*_k) \geq ja$ for all $k \leq i$.

We know that $x = tj + \rho(x, j)$, for some $x \in \mathbb{N}$.
Let $r = \rho(x, j)$. Then $x = tj + r$.

Let $k \in [i + 1]$. Then $\rho(x, s^*_k) = \rho(tj + r, s^*_k)$.

By the inductive hypothesis, $s^*_k = js_k$.

As such, $\rho(tj + r, s_k^*) = \rho(tj + r, js_k)$

By remainder property (TODO), $\rho(tj+r, js_k) = \rho(\rho(tj, js_k)+\rho(r, js_k), js_k)$.

By remainder property (TODO), $\rho(tj, js_k) = j\rho(t, s_k)$.

Since $r < j < js_k$, then $\rho(r, js_k) = r$.

As such, $\rho(\rho(tj, js_k) + \rho(r, js_k), js_k) = \rho(j\rho(t, s_k) + r, js_k)$.

By definition, $0 \leq \rho(t, s_k) < s_k$. Furthermore, $\rho(t, s_k) \in \mathbb{N}$. As such, $\rho(t, s_k) \leq s_k - 1$.

As such, $j\rho(t, s_k) \leq j(s_k - 1)$.

Thus, $j\rho(t, s_k) + r \leq j(s_k - 1) + r$.

We also know that $r < j$.

As such, $j(s_k - 1) + r < j(s_k - 1) + j = j(s_k - 1 + 1) = js_k$.

As such, $j\rho(t, s_k)+r < js_k$, thus $\rho(x, s_k^*) = \rho(j\rho(t, s_k)+r, js_k) = j\rho(t, s_k)+r$.

By assumption, $\rho(x, s_k^*) \geq ja$.

As such, $j\rho(t, s_k) + j) > j\rho(t, s_k) + r \geq ja$.

As such, $j(\rho(t, s_k) + 1) > ja$.

As such, $\rho(t, s_k) + 1 > a$.

Thus, $\rho(t, s_k) \geq a$.

Now if we can show that $s_i < t < s_{i+1}$, then t would have to be element $s_{i+1} \in S_{a,n}$ which would be a contradiction.

By assumption, $s_i^* < x = jt + r$.

$s_i^* = js_i$ by the inductive hypothesis.

As such, $js_i < jt + r < jt + j = j(t+1)$

Thus $s_i < t + 1$. Since $s_i \in \mathbb{N}$, then $s_i \leq t$.

However, we've shown that $\rho(t, s_i) \geq a > 0$. As such, $t \neq s_i$. Thus, $s_i < t$ must be the case.

Now we just need to show that $t < js_{i+1}$.

We know that $x = tj + r < js_{i+1}$.

As such, $tj \leq tj + r < js_{i+1}$. Thus, $t < s_{i+1}$.

But this would mean that t must be the $(i+2)^{th}$ element of $S_{a,n}$, which is a contradiction.

2.6 Twins

Definition
For any awkward number series $S_{a,n}$, $s_{i-1}, s_i \in S_{a,n}$ are called *twins* whenever $s_i = s_{i-1} + (a+1)$.

Awkward Twin Conjecture
For any awkward number series $S_{a,n}$, $S_{a,n}$ contains an infinite number of twins.