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Problem 1. (Analysis) If for a function $f : \mathbb{R} \rightarrow \mathbb{R}$ image of each compact set is compact then f is continuous. T/F.

Solution. No, we can take the function

$$f = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{else.} \end{cases}$$

This function is discontinuous at 0. □

Problem 2. Existence of the limit $\lim_{n \rightarrow \infty} \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} - \log n$.

Solution. Let $x_n = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} - \log n$. Then $x_{n+1} - x_n = \frac{1}{n+1} - \log(\frac{n+1}{n})$. But $\log(1+x) \geq \frac{x}{x+1}$. Thus the sequence is decreasing and we can show(!) that it is bounded below. □

Problem 3. What is the smallest positive real number c such that $\|x\|_1 \leq c\|x\|_\infty$ for all $x \in \mathbb{R}^n$.

Solution. Clearly $\|x\|_1 \leq n\|x\|_\infty$. Now, we claim that $c = n$. Let if possible $\|x\|_1 \leq (n - \epsilon)\|x\|_\infty$ for some $\epsilon > 0$, for all $x \in \mathbb{R}^n$. But for $x = (1, 1, \dots, 1)$ we will have $\|x\|_1 = n, \|x\|_\infty = 1$ and hence $\|x\|_1 > \|x\|_\infty$. □

Problem 4. If a group is finitely generated then there exist atmost finitely many subgroup of any index.

Solution. Let us consider G be the group and H be its subgroup such that $[G : H] = n$. The group acts on the cosets $\{H, g_2H, \dots, g_nH\} = \{1, 2, 3, \dots, n\}$ and it induces a homomorphism

$$\varphi_H : G \rightarrow S_n \text{ such that } g \mapsto_{\varphi_H} \sigma_g.$$

Now the stabilizer of the element H in G/H can be identified as $\{g \in G \mid \sigma_g = 1\}$ i.e., $\{g \in G \mid gg_iH = g_iH, 1 \leq i \leq n\}$ i.e., H . We claim that different subgroups H and H' will induce different maps. For $h \in H, h \notin H'$ we have $\varphi_H(h) = 1$ but $\varphi_{H'}(h) \neq 1$. Again there are atmost finitely many maps from G to S_n and hence as a result there can exist only finite many subgroups of index n . □

Problem 5. For primes $p > q > 2$, group of order pq^2 contains a subgroup of order pq .

Solution. The number of sylow p subgroup n_p divides q^2 as well as $p \mid n_p - 1$. Now n_p is odd if it is equal to q or q^2 . Since p is also an odd prime we can not have $p \mid n_p - 1$ in this case. Thus we must have $n_p = 1$ i.e., the sylow- p subgroup, H in G is normal and has order p . Now by Cauchy's theorem there exists $b \in G$ of order q . Let $K = \langle b \rangle$. Then HK is the desired subgroup of G . \square

Problem 6. SL_n is a product of matrices of the form $E_{ij}(a) = I + a\delta_{ij}, 1 \leq i \neq j \leq n$.

Solution. Clearly $E_{ij}(a) \in SL_n$ and

$$\delta_{ij}\delta_{kl} = \begin{cases} \delta_{il} & \text{if } j = k, \\ 0 & \text{else.} \end{cases}$$

implies

$$\begin{aligned} E_{ij}(a)E_{ij}(-a) &= (I + a\delta_{ij})(I - a\delta_{ij}) \\ &= I - a^2\delta_{ij}\delta_{ij} \\ &= I. \end{aligned}$$

For $A \in SL_n$, since not all entries in the first column can be zero we must have $a_{i1} \neq 0$ and $E_{1i}(1)A = (I + \delta_{1i})A = A +$ \square

Problem 7. X be a compact metric space with atleast two points and $a \in X$. Then

1. either $X \setminus \{a\}$ is compact or X is connected,
2. but not both.

Solution.

1. Let us assume that $A = X \setminus \{a\}$ is not compact then we know A is not closed.
2. Let us assume that X is connected and if possible $X \setminus \{a\}$ is compact. Then $X \setminus \{a\}$ is closed. Also $\{a\}$ is a closed subset of X . This contradicts that $X = (X \setminus \{a\}) \cup \{a\}$ is connected.

Conversely if $A = X \setminus \{a\}$ is compact then it will be closed in X and we will have $X = A \cup B$, for $B = \{a\}$. Thus X is not connected.

□

Problem 8. $GL_n^+(\mathbb{R})$ and $GL_n^-(\mathbb{R})$ are homeomorphic.

Solution. We can define $\psi : GL_n^+(\mathbb{R}) \rightarrow GL_n^-(\mathbb{R})$ such that $\psi(M) = AM$, where A is a diagonal matrix such that $a_{11} = -1$ and $a_{ii} = 1$ for $1 < i \leq n$.

□

Problem 9. Show that the General Linear group with positive determinant, $GL_n^+(\mathbb{R})$ is connected.

Solution. We know that $GL_n^+(\mathbb{R}) = \det^{-1}((0, \infty))$ and hence it is open. If we can show that this there is some kind of homeomorphism we are through.

□

Problem 10. (Matrix, Topology) Show that $SL_2(\mathbb{R})$ is connected.

Solution. Here we will use the fact that the General Linear group with positive determinant, $GL_n^+(\mathbb{R})$ is path connected. With the help of this fact we can define a continuous map

$$\phi : GL_n^+(\mathbb{R}) \rightarrow SL_n(\mathbb{R})$$

such that

$$\phi(A) = \frac{A}{(\det(A))^{\frac{1}{n}}}.$$

Clearly this is a surjection and hence $SL_n(\mathbb{R})$ is connected. □

Problem 11. $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Then show that f is open iff it is strictly monotone.

Solution. Let us assume that f is open and if possible there exist $a < b < c$ such that $f(a) < f(b) > f(c)$. Now if we restrict f to the interval $[a, c]$, then its supremum, M will exist and M will strictly be greater than $f(a), f(c)$ i.e., $f([a, c]) = [m, M]$. Therefore $f((a, c))$ will be a half closed interval i.e., either $f((a, c)) = [m, M]$ or $f((a, c)) = (m, M]$, contradicting our assumption that the map f is open.

Conversely WLOG let us assume that f is strictly increasing. It is sufficient to show that f maps open interval to open sets. Now, f being continuous and strictly increasing implies $f((a, b)) = (f(a), f(b))$. □