Collection of Problems

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Contents

Problem 1. (Analysis) *If for a function $f : \mathbb{R} \to \mathbb{R}$ image of each compact set is compact then f is continous. T/F.*

Solution. No, we can take the function

$$f = \begin{cases} sin(\frac{1}{x}) \text{ if } x \neq 0, \\ 0 \text{ else.} \end{cases}$$

This function is discontinous at 0. $\square$

Problem 2. *Existence of the limit* $\lim_{n\to\infty} \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} - logn$.

Solution. Let $x_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} - logn$. Then $x_{n+1} - x_n = \frac{1}{n+1} - log(\frac{n+1}{n})$. But $log(1+x) \geq \frac{x}{x+1}$. Thus the sequence is decreasing and we can show(!) that it is bounded below. $\square$

Problem 3. *What is the smallest positive real numer c such that $||x||_1 \leq c||x||_\infty$ for all $x \in \mathbb{R}^n$.*

Solution. Clearly $||x||_1 \leq n||x||_\infty$. Now, we claim that $c = n$. Let if possible $||x||_1 \leq (n-\epsilon)||x||_\infty$ for some $\epsilon > 0$, for all $x \in \mathbb{R}^n$. But for $x = (1, 1, \ldots, 1)$ we will have $||x||_1 = n, ||x||_\infty = 1$ and hence $||x|| > ||x||_\infty$. $\square$

Problem 4. *If a group is finitely generated then show that there exist atmost finitely many subgroup of any given index.*

Solution. Let us consider G be the group and H be its subgroup such that $[G : H] = n$. The group acts on the cosets $\{H, g_2H, \ldots, g_nH\} = \{1, 2, 3, \ldots, n\}$ and it induces a homomorphism

$$\varphi_H : G \to S_n \text{ such that } g \mapsto_{\varphi_H} \sigma_g.$$

Now the stabilizer of the element H in G/H can be identified as $\{g \in G \mid \sigma_g = 1\}$ i.e., $\{g \in G \mid gg_iH = g_iH, 1 \leq i \leq n\}$ i.e., H. We claim that different subgroups H and H' will induce different maps. For $h \in H, h \notin H'$ we have $\varphi_H(h) = 1$ but $\varphi_{H'}(h) \neq 1$. Again there are atmost finitely many maps from G to S_n and hence as a result there can exist only finite many subgroups of index n. $\square$

Problem 5. *For primes $p > q > 2$, group of order pq^2 contains a subgroup of ordre pq.*

Solution. The number of sylow p subgroup n_p divides q^2 as well as $p \mid n_p - 1$. Now n_p is odd if it is equal to q or q^2. Since p is also an odd prime we can not have $p \mid n_p - 1$ in this case. Thus we must have $n_p = 1$ i.e., the sylow$-p$ subgroup, H in G is normal and has order p. Now by Cauchy's theorem there exists $b \in G$ of order q. Let $K =< b >$. Then HK is the desired subgroup of G. □

Problem 6. *SL_n is a product of matrices of the form $E_{ij}(a) = I + a\delta_{ij}, 1 \leq i \neq j \leq n$.*

Solution. Clearly $E_{ij}(a) \in SL_n$ and

$$\delta_{ij}\delta_{kl} = \begin{cases} \delta_{il} & \text{if } j = k, \\ 0 & \text{else.} \end{cases}$$

implies

$$\begin{aligned} E_{ij}(a)E_{ij}(-a) &= (I + a\delta_{ij})(I - a\delta_{ij}) \\ &= I - a^2\delta_{ij}\delta_{ij} \\ &= I. \end{aligned}$$

For $A \in SL_n$, since not all entries in the first column can be zero we must have $a_{i1} \neq 0$ and $E_{1i}(1)A = (I + \delta_{1i})A = A+$ □

Problem 7. *X be a compact metric space with atleast two points and $a \in X$. Then*

1. *either $X \setminus \{a\}$ is compact or X is connected,*
2. *but not both.*

Solution.

1. Let us assume that $A = X \setminus \{a\}$ is not compact then we know A is not closed.
2. Let us assume that X is connected and if possible $X \setminus \{a\}$ is compact. Then $X \setminus \{a\}$ is closed. Also $\{a\}$ is a closed subset of X. This contradicts that $X = (X \setminus \{a\}) \cup \{a\}$ is connected.

   Conversely if $A = X \setminus \{a\}$ is compact then it will be closed in X and we will have $X = A \cup B$, for $B = \{a\}$. Thus X is not connected.

□

Problem 8. *$GL_n^+(\mathbb{R})$ and $GL_n^-(\mathbb{R})$ are homeomorphic.*

Solution. We can define $\psi : GL_n^+(\mathbb{R}) \to GL_n^-(\mathbb{R})$ such that $\psi(M) = AM$, where A is a diagonal matrix such that $a_{11} = -1$ and $a_{ii} = 1$ for $1 < i \leq n$. □

Problem 9. *Show that the General Linear group with positive determinant, $GL_n^+(\mathbb{R})$ is connected.*

Solution. We know that $GL_n^+(\mathbb{R}) = \det^{-1}((0,\infty))$ and hence it is open. If we can show that this there is some kind of homeomorphism we are through. $\square$

Problem 10. (Matrix, Topology) *Show that $SL_2(\mathbb{R})$ is connected.*

Solution. Here we will use the fact that the General Linear group with positive determinant, $GL_n^+(\mathbb{R})$ is path connected. With the help of this fact we can define a continous map

$$\phi : GL_n^+(\mathbb{R}) \to SL_n(\mathbb{R})$$

such that

$$\phi(A) = \frac{A}{(\det(A))^{\frac{1}{n}}}.$$

Clearly this is a surjection and hence $SL_n(\mathbb{R})$ is connected. $\square$

Problem 11. *$f : \mathbb{R} \to \mathbb{R}$ is continous. Then show that f is open iff it is strictly monotone.*

Solution. Let us assume that f is open and if possible there exist $a < b < c$ such that $f(a) < f(b) > f(c)$. Now if we restrict f to the interval $[a,c]$, then its supremum, M will exist and M will strictly be greater than $f(a), f(c)$ i.e., $f([a,c]) = [m,M]$. Therefore $f((a,c))$ will be a half closed interval i.e., either $f((a,c)) = [m,M]$ or $f((a,c)) = (m.M])$, contradicting our assumption that the map f is open.

Conversely WLOG let us assume that f is strictly increasing. It is sufficient to show that f maps open intetrval to open sets. Now, f being continous and strictly increasing implies $f((a,b)) = (f(a), f(b))$. $\square$

Problem 12. (Group Theory, Sylow Theorems) *What is the number of $sylow - p$ subgroups in $GL_n(\mathbb{F}_p)$.*

Solution. We have $|G| = |GL_n(\mathbb{F}_p)| = (p^n - 1)(p^n - p)\ldots(p^n - p^{n-1})$. Therefore the cardinality of a *sylow* $- p$ subgroup in G is $p^{1+2+\cdots+(n-1)} = p^{\frac{(n-1)n}{2}}$. Now the subgroup H of G consisting of the upper triangular matrices with diagonal entries 1 is a *sylow* $- p$ subgroup of G. Thus the number of *sylow* $- p$ subgroup is same as the index of the normalizer of H in G. We claim

$$N = \{A \in G \mid a_{ii} \neq 0, a_{ij} = 0 \text{ for } i < j\}$$

is equal to $N_H(G)$. $N \subseteq N_H(G)$ is obvious.

To proof the other direction we have to do some work. We have

$$N = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,n-1} & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2,n-1} & a_{2n} \\ & & & \ddots & & \\ 0 & 0 & 0 & \dots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 & a_{nn} \end{bmatrix} \mid a_{ij} \in F_p, a_{ii} \neq 0 \right\}.$$

Let us consider the subspace $V_i = \langle e_1, e_2, \dots, e_i \rangle$. It is clear that $HV_i \subseteq V_i$. **First** we claim that this are the only subspaces such that $HU \subseteq U$. If $u = (u_1, u_2, \dots, u_n)^t$ is some basis vector of U with say $u_i \neq 0$. WLOG we can assume $u_i = 1$. Now for $j \leq i$

$$(I + \delta_{ji})u = (u_1, u_2, \dots, u_j + u_i, \dots, u_n)^t.$$

Thus $(u_1, u_2, \dots, u_j + u_i, \dots, u_n) - (u_1, u_2, \dots, u_j, \dots, u_n) = (0, 0, \dots, u_i, \dots, 0) = e_j$ is contained in U. Therefore we can conclude that $U = V_j$, where j is largest index such that a basis vector has a nonzero jth entry.

Now for any $g \in N_G(H)$ and $h \in H$, $ghg^{-1} \in H$. Therefore $gh = h'g$ for some $h' \in H$. Again we claim $hV_i = V_i$ foe each i. Since $he_i = (h_{1i}, h_{2i}, \dots, h_{ni})^t$, $he_1 = (h_{1i}, 0, \dots, 0) = e_1$. Again $he_2 = (h_{12}, 1, \dots, 0)^t = h_{12}e_1 + e_2$ i.e.,

$$he_2 - h(h_{12}e_1) = e_2 \in hV_i.$$

By this way we have $hV_i = V_i$. Therefore $ghV_i = gV_i = h(gV_i)$ i.e. $h(gV_i) \subseteq gV_i$ and $H(gV_i) \subseteq gV_i$. From our first claim we have $gV_i = V_j$ for some $1 \leq j \leq n$. Since g is invertible and it preserves rank we must have $gV_i = V_i$ for each $1 \leq i \leq n$. Thus we have $g \in N$ by simple observation.

$\square$