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Problem 1. If for a function $f : \mathbb{R} \to \mathbb{R}$ image of each compact set is compact then $f$ is continous. T/F.

*Solution.* No, we can take the function

$$f = \begin{cases} sin(\frac{1}{x}) \text{ if } x \neq 0, \\ 0 \text{ else.} \end{cases}$$

This function is discontinous at 0. ☐

Problem 2. Existence of the limit $\lim_{n\to\infty} \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$.

*Solution.* Let $x_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} - logn$. Then $x_{n+1} - x_n = \frac{1}{n+1} - log(\frac{n+1}{n})$. But $log(1+x) \geq \frac{x}{x+1}$. Thus the sequence is decreasing and we can show(!) that it is bounded below. ☐

Problem 3. What is the smallest positive real numer $c$ such that $||x||_1 \leq c||x||_\infty$ for all $x \in \mathbb{R}^n$.

*Solution.* Clearly $||x||_1 \leq n||x||_\infty$. Now, we claim that $c = n$. Let if possible $||x||_1 \leq (n-\epsilon)||x||_\infty$ for some $\epsilon > 0$, for all $x \in \mathbb{R}^n$. But for $x = (1, 1, \ldots, 1)$ we will have $||x||_1 = n, ||x||_\infty = 1$ and hence $||x|| > ||x||_\infty$. ☐

Problem 4. If a group is finitely generated then there exist atmost finitely many subgroup of any index.

*Solution.* Let us consider $G$ be the group and $H$ be its subgroup such that $[G : H] = n$. The group acts on the cosets $\{H, g_2H, \ldots, g_nH\} = \{1, 2, 3, \ldots, n\}$ and it induces a homomorphism

$$\varphi_H : G \to S_n \text{ such that } g \mapsto_{\varphi_H} \sigma_g.$$

Now the stabilizer of the element $H$ in $G/H$ can be identified as $\{g \in G \mid \sigma_g = 1\}$ i.e., $\{g \in G \mid gg_iH = g_iH, 1 \leq i \leq n\}$ i.e., $H$. We claim that different subgroups $H$ and $H'$ will induce different maps. For $h \in H, h \notin H'$ we have $\varphi_H(h) = 1$ but $\varphi_{H'}(h) \neq 1$. Again there are atmost finitely many maps from $G$ to $S_n$ and hence as a result there can exist only finite many subgroups of index $n$. ☐

Problem 5. For primes $p > q > 2$, group of order $pq^2$ contains a subgroup of ordre $pq$.

*Solution.* The number of sylow $p$ subgroup $n_p$ divides $q^2$ as well as $p \mid n_p - 1$. Now $n_p$ is odd if it is equal to $q$ or $q^2$. Since $p$ is also an odd prime we can not have $p \mid n_p - 1$ in this case. Thus we must have $n_p = 1$ i.e., the sylow$-p$ subgroup, $H$ in $G$ is normal and has order $p$. Now by Cauchy's theorem there exists $b \in G$ of order $q$. Let $K =< b >$. Then $HK$ is the desired subgroup of $G$. ☐

**Problem 6.** $SL_n$ is a product of matrices of the form $E_{ij}(a) = I + a\delta_{ij}, 1 \le i \ne j \le n$.

*Solution.* Clearly $E_{ij}(a) \in SL_n$ and

$$\delta_{ij}\delta_{kl} = \begin{cases} \delta_{il} & \text{if } j = k, \\ 0 & \text{else.} \end{cases}$$

implies

$$\begin{aligned} E_{ij}(a)E_{ij}(-a) &= (I + a\delta_{ij})(I - a\delta_{ij}) \\ &= I - a^2\delta_{ij}\delta_{ij} \\ &= I. \end{aligned}$$

For $A \in SL_n$, since not all entries in the first column can be zero we must have $a_{i1} \ne 0$ and $E_{1i}(1)A = (I+\delta_{1i})A = A+$

□

**Problem 7.** $X$ be a compact metric space with atleast two points and $a \in X$. Then

1. either $X \smallsetminus \{a\}$ is compact or $X$ is connected,
2. but not both.

*Solution.*

1. Let us assume that $A = X \smallsetminus \{a\}$ is not compact then we know $A$ is not closed.
2. Let us assume that $X$ is connected and if possible $X \smallsetminus \{a\}$ is compact. Then $X \smallsetminus \{a\}$ is closed. Also $\{a\}$ is a closed subset of $X$. This contradicts that $X = (X \smallsetminus \{a\}) \cup \{a\}$ is connected.

   Conversely if $A = X \smallsetminus \{a\}$ is compact then it will be closed in $X$ and we will have $X = A \cup B$, for $B = \{a\}$. Thus $X$ is not connected.

□

**Problem 8.** $GL_n^+(\mathbb{R})$ and $GL_n^-(\mathbb{R})$ are homeomorphic.

*Solution.* We can define $\psi : GL_n^+(\mathbb{R}) \to GL_n^-(\mathbb{R})$ such that $\psi(M) = AM$, where $A$ is a diagonal matrix such that $a_{11} = -1$ and $a_{ii} = 1$ for $1 < i \le n$. □

**Problem 9.** Show that the General Linear group with positive determinant, $GL_n^+(\mathbb{R})$ is connected.

*Solution.* We know that $GL_n^+(\mathbb{R}) = \det^{-1}((0, \infty))$ and hence it is open. If we can show that this there is some kind of homeomorphism we are through. □

Problem 10. Show that $SL_2(\mathbb{R})$ is connected.

*Solution.* Here we will use the fact that the General Linear group with positive determinant, $GL_n^+(\mathbb{R})$ is path connected. With the help of this fact we can define a continous map

$$\phi : GL_n^+(\mathbb{R}) \to SL_n(\mathbb{R})$$

such that

$$\phi(A) = \frac{A}{(\det(A))^{\frac{1}{n}}}.$$

Clearly this is a surjection and hence $SL_n(\mathbb{R})$ is connected. $\square$

Problem 11. $f : \mathbb{R} \to \mathbb{R}$ is continous. Then show that $f$ is open iff it is strictly monotone.

*Solution.* Let us assume that $f$ is open and if possible there exist $a < b < c$ such that $f(a) < f(b) > f(c)$. Now if we restrict $f$ to the interval $[a, c]$, then its supremum, $M$ will exist and $M$ will strictly be greater than $f(a), f(c)$ i.e., $f([a, c]) = [m, M]$. Therefore $f((a, c))$ will be a half closed interval i.e., either $f((a, c)) = [m, M]$ or $f((a, c)) = (m.M])$, contradicting our assumption that the map $f$ is open.

Conversely WLOG let us assume that $f$ is strictly increasing. It is sufficient to show that $f$ maps open intetrval to open sets. Now, $f$ being continous and strictly increasing implies $f((a, b)) = (f(a), f(b))$. $\square$