

Question 1. If for a function $f : \mathbb{R} \rightarrow \mathbb{R}$ image of each compact set is compact then f is continuous. T/F.

Solution. No, we can take the function

$$f = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{else.} \end{cases}$$

This function is discontinuous at 0. □

Question 2. Existence of the limit $\lim_{n \rightarrow \infty} \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} - \log n$.

Solution. Let $x_n = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} - \log n$. Then $x_{n+1} - x_n = \frac{1}{n+1} - \log(\frac{n+1}{n})$. But $\log(1+x) \geq \frac{x}{x+1}$. Thus the sequence is decreasing and we can show(!) that it is bounded below. □

Question 3. What is the smallest positive real number c such that $\|x\|_1 \leq c\|x\|_\infty$ for all $x \in \mathbb{R}^n$.

Solution. Clearly $\|x\|_1 \leq n\|x\|_\infty$. Now, we claim that $c = n$. Let if possible $\|x\|_1 \leq (n-\epsilon)\|x\|_\infty$ for some $\epsilon > 0$, for all $x \in \mathbb{R}^n$. But for $x = (1, 1, \dots, 1)$ we will have $\|x\|_1 = n, \|x\|_\infty = 1$ and hence $\|x\|_1 > \|x\|_\infty$. □

Question 4. If a group is finitely generated then there exist at most finitely many subgroups of any index.

Solution. Let us consider G be the group and H be its subgroup such that $[G : H] = n$. The group acts on the cosets $\{H, g_2H, \dots, g_nH\} = \{1, 2, 3, \dots, n\}$ and it induces a homomorphism

$$\varphi_H : G \rightarrow S_n \text{ such that } g \mapsto_{\varphi_H} \sigma_g.$$

Now the stabilizer of the element H in G/H can be identified as $\{g \in G \mid \sigma_g = 1\}$ i.e., $\{g \in G \mid gg_iH = g_iH, 1 \leq i \leq n\}$ i.e., H . We claim that different subgroups H and H' will induce different maps. For $h \in H, h \notin H'$ we have $\varphi_H(h) = 1$ but $\varphi_{H'}(h) \neq 1$. Again there are at most finitely many maps from G to S_n and hence as a result there can exist only finite many subgroups of index n . □

Question 5. For primes $p > q > 2$, group of order pq^2 contains a subgroup of order pq .

Solution. The number of Sylow p subgroup n_p divides q^2 as well as $p \mid n_p - 1$. Now n_p is odd if it is equal to q or q^2 . Since p is also an odd prime we can not have $p \mid n_p - 1$ in this case. Thus we must have $n_p = 1$ i.e., the Sylow- p subgroup, H in G is normal and has order p . Now by Cauchy's theorem there exists $b \in G$ of order q . Let $K = \langle b \rangle$. Then HK is the desired subgroup of G . □

Question 6. SL_n is a product of matrices of the form $E_{ij}(a) = I + a\delta_{ij}$, $1 \leq i \neq j \leq n$.

Solution. Clearly $E_{ij}(a) \in SL_n$ and

$$\delta_{ij}\delta_{kl} = \begin{cases} \delta_{il} & \text{if } j = k, \\ 0 & \text{else.} \end{cases}$$

implies

$$\begin{aligned} E_{ij}(a)E_{ij}(-a) &= (I + a\delta_{ij})(I - a\delta_{ij}) \\ &= I - a^2\delta_{ij}\delta_{ij} \\ &= I. \end{aligned}$$

For $A \in SL_n$, since not all entries in the first column can be zero we must have $a_{i1} \neq 0$ and $E_{1i}(1)A = (I + \delta_{1i})A = A +$
 \square

Question 7. X be a compact metric space with atleast two points and $a \in X$. Then either $X \setminus \{a\}$ is compact or X is connected but not both.

Solution. Let us assume that X is connected and if possible $X \setminus \{a\}$ is compact. Then $X \setminus \{a\}$ is closed. Also $\{a\}$ is a closed subset of X . This contradicts that $X = (X \setminus \{a\}) \cup \{a\}$ is connected.

Conversely if $A = X \setminus \{a\}$ is compact then it will be closed in X and we will have $X = A \cup B$, for $B = \{a\}$. Thus X is not connected. \square