Collection of Problems

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Contents

§ PROBLEM 1. (Analysis) *If for a function $f : \mathbb{R} \to \mathbb{R}$ image of each compact set is compact then f is continous. T/F.*

Solution. No, we can take the function

$$f = \begin{cases} \sin\left(\frac{1}{x}\right) \text{ if } x \neq 0, \\ 0 \text{ else.} \end{cases}$$

This function is discontinous at 0. □

§ PROBLEM 2. *Existence of the limit* $\lim_{n\to\infty} \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$.

Solution. Let $x_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} - log n$. Then $x_{n+1} - x_n = \frac{1}{n+1} - log(\frac{n+1}{n})$. But $\log(1+x) \geq \frac{x}{x+1}$. Thus the sequence is decreasing and we can show(!) that it is bounded below. □

§ PROBLEM 3. *What is the smallest positive real numer c such that* $||x||_1 \leq c||x||_\infty$ *for all* $x \in \mathbb{R}^n$.

Solution. Clearly $||x||_1 \leq n||x||_\infty$. Now, we claim that $c = n$. Let if possible $||x||_1 \leq (n-\epsilon)||x||_\infty$ for some $\epsilon > 0$, for all $x \in \mathbb{R}^n$. But for $x = (1, 1, \ldots, 1)$ we will have $||x||_1 = n, ||x||_\infty = 1$ and hence $||x|| > ||x||_\infty$. □

§ PROBLEM 4. *If a group is finitely generated then show that there exist atmost finitely many subgroup of any given index.*

Solution. Let us consider G be the group and H be its subgroup such that $[G : H] = n$. The group acts on the cosets $\{H, g_2H, \ldots, g_nH\} = \{1, 2, 3, \ldots, n\}$ and it induces a homomorphism

$$\varphi_H : G \to S_n \text{ such that } g \xrightarrow{\varphi_H} \sigma_g.$$

Now the stabilizer of the element H in G/H can be identified as $\{g \in G \mid \sigma_g = 1\}$ i.e., $\{g \in G \mid gg_iH = g_iH, 1 \leq i \leq n\}$ i.e., H. We claim that different subgroups H and H' will induce different maps. For $h \in H, h \notin H'$ we have $\varphi_H(h) = 1$ but $\varphi_{H'}(h) \neq 1$. Again there are atmost finitely many maps from G to S_n and hence as a result there can exist only finite many subgroups of index n. □

§ PROBLEM 5. *For primes $p > q > 2$, group of order pq^2 contains a subgroup of ordre pq.*

Solution. The number of sylow p subgroup n_p divides q^2 as well as $p \mid n_p - 1$. Now n_p is odd if it is equal to q or q^2. Since p is also an odd prime we can not have $p \mid n_p - 1$ in this case. Thus we must have $n_p = 1$ i.e., the sylow$-p$ subgroup, H in G is normal and has order p. Now by Cauchy's theorem there exists $b \in G$ of order q. Let $K =< b >$. Then HK is the desired subgroup of G. □

§ PROBLEM 6. *SL_n is a product of matrices of the form $E_{ij}(a) = I + a\delta_{ij}, 1 \leq i \neq j \leq n$.*

Solution. Clearly $E_{ij}(a) \in SL_n$ and

$$\delta_{ij}\delta_{kl} = \begin{cases} \delta_{il} & \text{if } j = k, \\ 0 & \text{else.} \end{cases}$$

implies

$$\begin{aligned} E_{ij}(a)E_{ij}(-a) &= (I + a\delta_{ij})(I - a\delta_{ij}) \\ &= I - a^2\delta_{ij}\delta_{ij} \\ &= I. \end{aligned}$$

For $A \in SL_n$, since not all entries in the first column can be zero we must have $a_{i1} \neq 0$ and $E_{1i}(1)A = (I + \delta_{1i})A = A+$ □

§ PROBLEM 7. *X be a compact metric space with atleast two points and $a \in X$. Then*

1. *either $X \setminus \{a\}$ is compact or X is connected,*
2. *but not both.*

Solution.

1. Let us assume that $A = X \setminus \{a\}$ is not compact then we know A is not closed.
2. Let us assume that X is connected and if possible $X \setminus \{a\}$ is compact. Then $X \setminus \{a\}$ is closed. Also $\{a\}$ is a closed subset of X. This contradicts that $X = (X \setminus \{a\}) \cup \{a\}$ is connected.

   Conversely if $A = X \setminus \{a\}$ is compact then it will be closed in X and we will have $X = A \cup B$, for $B = \{a\}$. Thus X is not connected.

□

§ PROBLEM 8. *$GL_n^+(\mathbb{R})$ and $GL_n^-(\mathbb{R})$ are homeomorphic.*

Solution. We can define $\psi : GL_n^+(\mathbb{R}) \to GL_n^-(\mathbb{R})$ such that $\psi(M) = AM$, where A is a diagonal matrix such that $a_{11} = -1$ and $a_{ii} = 1$ for $1 < i \leq n$. □

§ PROBLEM 9. *Show that the General Linear group with positive determinant, $GL_n^+(\mathbb{R})$ is connected.*

Solution. We know that $GL_n^+(\mathbb{R}) = \det^{-1}((0,\infty))$ and hence it is open. If we can show that this there is some kind of homeomorphism we are through. □

§ PROBLEM 10. (Matrix, Topology) *Show that $SL_2(\mathbb{R})$ is connected.*

Solution. Here we will use the fact that the General Linear group with positive determinant, $GL_n^+(\mathbb{R})$ is path connected. With the help of this fact we can define a continous map

$$\phi : GL_n^+(\mathbb{R}) \to SL_n(\mathbb{R})$$

such that

$$\phi(A) = \frac{A}{(\det(A))^{\frac{1}{n}}}.$$

Clearly this is a surjection and hence $SL_n(\mathbb{R})$ is connected. □

§ PROBLEM 11. *$f : \mathbb{R} \to \mathbb{R}$ is continous. Then show that f is open iff it is strictly monotone.*

Solution. Let us assume that f is open and if possible there exist $a < b < c$ such that $f(a) < f(b) > f(c)$. Now if we restrict f to the interval $[a,c]$, then its supremum, M will exist and M will strictly be greater than $f(a), f(c)$ i.e., $f([a,c]) = [m,M]$. Therefore $f((a,c))$ will be a half closed interval i.e., either $f((a,c)) = [m,M]$ or $f((a,c)) = (m.M])$, contradicting our assumption that the map f is open.

Conversely WLOG let us assume that f is strictly increasing. It is sufficient to show that f maps open intetrval to open sets. Now, f being continous and strictly increasing implies $f((a,b)) = (f(a), f(b))$. □

§ PROBLEM 12. (Group Theory, Sylow Theorems) *What is the number of sylow$-p$ subgroups in $GL_n(\mathbb{F}_p)$.*

Solution. We have $|G| = |GL_n(\mathbb{F}_p)| = (p^n - 1)(p^n - p)\dots(p^n - p^{n-1})$. Therefore the cardinality of a $sylow-p$ subgroup in G is $p^{1+2+\dots+(n-1)} = p^{\frac{(n-1)n}{2}}$. Now the subgroup H of G consisting of the upper triangular matrices with diagonal entries 1 is a $sylow-p$ subgroup of G. Thus the number of $sylow-p$ subgroup is same as the index of the normalizer of H in G. We claim

$$N = \{A \in G \mid a_{ii} \neq 0, a_{ij} = 0 \text{ for } \ i < j\}$$

is equal to $N_H(G)$. $N \subseteq N_H(G)$ is obvious.

To proof the other direction we have to do some work. We have

$$N = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,n-1} & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2,n-1} & a_{2n} \\ & & & \ddots & & \\ 0 & 0 & 0 & \dots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 & a_{nn} \end{bmatrix} \mid a_{ij} \in F_p, a_{ii} \neq 0 \right\}.$$

Let us consider the subspace $V_i = \langle e_1, e_2, \dots, e_i \rangle$. It is clear that $HV_i \subseteq V_i$. **First** we claim that this are the only subspaces such that $HU \subseteq U$. If $u = (u_1, u_2, \dots, u_n)^t$ is some basis vector of U with say $u_i \neq 0$. WLOG we can assume $u_i = 1$. Now for $j \leq i$

$$(I + \delta_{ji})u = (u_1, u_2, \dots, u_j + u_i, \dots, u_n)^t.$$

Thus $(u_1, u_2, \dots, u_j + u_i, \dots, u_n) - (u_1, u_2, \dots, u_j, \dots, u_n) = (0, 0, \dots, u_i, \dots, 0) = e_j$ is contained in U. Therefore we can conclude that $U = V_j$, where j is largest index such that a basis vector has a nonzero jth entry.

Now for any $g \in N_G(H)$ and $h \in H$, $ghg^{-1} \in H$. Therefore $gh = h'g$ for some $h' \in H$. Again we claim $hV_i = V_i$ foe each i. Since $he_i = (h_{1i}, h_{2i}, \dots, h_{ni})^t$, $he_1 = (h_{1i}, 0, \dots, 0) = e_1$. Again $he_2 = (h_{12}, 1, \dots, 0)^t = h_{12}e_1 + e_2$ i.e.,

$$he_2 - h(h_{12}e_1) = e_2 \in hV_i.$$

By this way we have $hV_i = V_i$. Therefore $ghV_i = gV_i = h(gV_i)$ i.e. $h(gV_i) \subseteq gV_i$ and $H(gV_i) \subseteq gV_i$. From our first claim we have $gV_i = V_j$ for some $1 \leq j \leq n$. Since g is invertible and it preserves rank we must have $gV_i = V_i$ for each $1 \leq i \leq n$. Thus we have $g \in N$ by simple observation. $\square$

§ PROBLEM 13. (Compelx Analysis) *Find the entire functions $f : \mathbb{C} \to \mathbb{R}$.*

Solution. If such an entire function $f(z)$ exists then the function $if(z)$ is also entire and so is $\exp^{if(z)}$. This gives us $\mid \exp^{if(z)} \mid = 1$ and by Liouville's theorem it is constant. Consequently $f(z)$ must be constant. $\square$

§ PROBLEM 14. *A subgroup H of index 5 in an odd order group G is normal.*

Solution. Since $|G:H|=5$, we get a homomorphism $\varphi: G \to S_5$ and $K=\ker(\varphi)\subseteq H$. Thus $|G:K|\geq 5$. The subgroups of odd order in S_5 can have order $3,5$ or 15. Now $|G:K|=5$ implies $H=K$ and hence $H \trianglelefteq G$. Otherwise G/K is a group of order 15. We know that any subgroup of S_n either contains all even permutations or exactly half of them. If there are exactly half of the elements in $G/K \cong P \subseteq S_5$ are even permutations then $\sigma: P \to \{1,-1\}$ is a surjection. This gives us $|P:\ker(\sigma)|=2$ i.e., $2 \mid |P|$, which is a clear contradiction to the fact that $|G|$ is odd. Thus all the elements in P are even permutations i.e., $G/K \subseteq A_5$. But A_5 has no subgroup of order 15, another contradiction. □

§ PROBLEM 15. *Let G be a finite group and H a subgroup of G of prime index p. If* $\gcd(|G|,p-1)=1$ *then* $G' \subseteq H$.

Solution. To be contd... □

§ PROBLEM 16. (Group Theory) *A finite simple group G does not have a normal subgroup of index n if $|G|$ does not divide $N!$.*

Solution. Let $|G:H|=n$ then we get a homomorphism $\varphi: G \to S_n$ induced by the action of G on the cosets of H. Now $K=\ker(\varphi)\subseteq H$ and is a normal subgroup of G. G being a simple group implies that $K=1$ i.e., G is embedded in S_n. Thus G can be thought as a subgroup of S_n and hence $|G| \mid n!$.
Consequences- A_5 has no subgroup of order 15 and 20 since $15 \nmid 24$ and $20 \nmid 6$. □

§ PROBLEM 17. (Real Analysis, Continous Functions) *Periodic continous function $f:\mathbb{R}\to\mathbb{R}$ is uniformly continous.*

Solution. For ease of calculation we will consider the period of f to be 1. Now f is continous on $[0,2]$ and for given $\epsilon>0$ there exists $\delta>0$ such that whenever $x,y\in[0,2]$ with

$$|x-y|<\delta \text{ implies } |f(x)-f(y)|<\epsilon.$$

For any $x,y\in\mathbb{R}$ with $x>y(>0,$ say) there exist $n,m\in\mathbb{N}$ such that $x=n+r, y=m+s$ with $0\leq r,s<1$. For $\delta<1$ if $|x-y|<\delta$ we claim that $n=m$ or $n=m+1$. Otherwise let if possible $n\geq m+2$ then

$$\begin{aligned} x-y &= (n+r)-(m-s) \\ &\geq 2+(r-s). \end{aligned}$$

But $0 \leq r < 1$ and $0 \leq s < 1$ give us $-1 \leq r - s \leq 1$ i.e., $x - y \geq 2 - 1 = 1$. This contradicts $|x - y| < \delta < 1$. Therefore if we choose $\delta' = \min\{\delta, 1\}$ then $|f(x) - f(y)| < \epsilon$ whenever $|x - y| \leq \delta'$. $\square$

§ PROBLEM 18. (Topology, Metric Space) *The complement of a proper subspace W of $\mathbb{R}^n$ is connected if and only if* $\dim(W) \leq n - 2$.

Solution. Let us consider a proper subspace W such that $\dim(W) > n-2$, therefore $\dim(W) = n-1$ and $W^\perp$ is of dimension 1. If $W^\perp = \operatorname{span}\{v\}$ then we can consider the continous function $g : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = \langle v, x \rangle$. In this case $f^{-1}(0) = W$ and $f^{-1}(\mathbb{R} \smallsetminus \{0\}) = \mathbb{R}^n \smallsetminus W$. Thus we obtain two open sets $A = f^{-1}((0, \infty))$ and $B = f^{-1}((-\infty, 0))$ such that $A \cup B = \mathbb{R}^n \smallsetminus W$ i.e., the complement W is not connected. Hence for the complement of a proper subspace of $\mathbb{R}^n$ to be connected we must have $\dim(W) \leq n - 2$.

Conversely let us assume that $\dim(W) \leq n-2$. We need to show that $\mathbb{R}^n \smallsetminus W$ is connected. The idea is to project any two vectors $x, y \in \mathbb{R}^n \smallsetminus W$ to $W^\perp$, which is path connected. By this we get the path $x \to x' \to y' \to y$.

$$\begin{tikzcd} & y \\ x \arrow[dd] & \\ & y' \arrow[uu] \\ x' \arrow[ur, bend right] & \end{tikzcd}$$

Let $\{e_1, e_2, \ldots, e_k\}$ is an orthonormal basis for W and $\{e_{k+1}, \ldots, e_n\}$is an orthonormal basis for $W^\perp$. The projection x' of a vector $x = \sum_{i=1}^n x_i e_i$ onto $W^\perp$ is given by $\sum_{i=k+1}^n \langle x, e_i \rangle e_i = \sum_{i=k+1}^n x_i e_i$. We claim that the straight line connecting x and x' lies on W^c. $\square$

§ PROBLEM 19. (Complex Analysis) *Entire function $f : \mathbb{C} \to \mathbb{C}$ with $\Im(f) > 0$ is constant.*

Solution. For an entire function f, $\exp^{-if(z)}$ is also an entire function and $|\exp^{-if(z)}| = |\exp^{\Im(f)}|$. Similarly $\exp^{if(z)}$ is entire and $|\exp^{if(z)}| = |\exp^{-\Im(f)}| < 1$. Therefore $\exp^{if(z)}$ is constant and so is $f(z)$ $\square$

§ PROBLEM 20. (Functional Analysis) *Let X, Y, Z are Banach spaces such that $A : X \to Y$ and $B : Y \to Z$ are linear maps. If BA, B are bounded and B is injective then A is also bounded.*

Solution. Let $x_n \to x$ and $A(x_n) \to y$. B being bounded implies $B(A(x_n)) \to B(y)$. Moreover $(BA)(x_n) \to (BA)(x)$ and B is injective. Therefore $BA(x) = B(y)$ implies $A(x) = y$ and hence A is a closed map. Hence A is a bounded linear operator. □

§ PROBLEM 21. *Evaluate the limit*

$$\pi \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \sin\left(k\frac{\pi}{n}\right)}{n}.$$

Solution. We know that for an integrable function $f : [a, b] \to \mathbb{R}$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

where $x_k = a + k\Delta x$ and $\Delta x = \frac{b-a}{n}$. Comparing with the given function with the standard result we get $a = 0$, $\frac{b-a}{n} = \frac{\pi}{n}$ i.e., $b = \pi$. Thus

$$\begin{aligned}
\pi \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \sin\left(k\frac{\pi}{n}\right)}{n} &= \lim_{n\to\infty} \sum_{k=1}^{n} \sin\left(k\frac{\pi}{n}\right)\frac{\pi}{n} \\
&= \int_0^\pi \sin(x)\,dx \\
&= [-\cos x]^{\pi}{}_{0} \\
&= 2.
\end{aligned}$$

□

§ PROBLEM 22. (Linear Algebra, Topology) *The set of rank two matrices in $M_{2\times 3}$ is open.*

Solution. The required set is the inverse image of $\mathbb{R}^3 \setminus (0,0,0)$, where $f : M_{2\times 3} \to \mathbb{R}$ is a continous map given by $f(A) = f(A_1, A_2, A_3) = (\det(A_1, A_2), \det(A_2, A_3), \det(A_3, A_1))$. Inverse image of $(0,0,0)$ is the set of all matrices of rank less than or equal to 1. Each $\det(A_i, A_j)$ map is continous because they are polynomials in the entries of A. Consequently by *mapping into products* the map f is continous. □

§ PROBLEM 23. (Linear Algebra, Topology) *The orthogonal matrices of size $n \times n$ over $\mathbb{R}$, $\mathcal{O}_n(\mathbb{R})$ is compact. Is $\mathcal{O}_n(\mathbb{C})$ compact?*

Solution. For any $A \in \mathcal{O}_n(\mathbb{R})$ we have $AA^T = I_n$. Now $(AA^T)_{ij} = \sum_{j=1}^n {a_{ij}}^2$ i.e., for each i, j the term $|a_{ij}| \leq 1$. Thus the elements of the set are bounded above by n^2, since

$$||A|| = \sqrt{\sum_{i=1}^n \sum_{j=1}^n {a_{ij}}^2} \leq n^2.$$

To show that the given set is infact a closed set in $M_n(\mathbb{R})$ we consider the map $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ given by $f(A) = AA^T$. We claim that this map is continous. For a sequence $A_n \to A$ we must have ${a_{ij}}^{(n)} \to a_{ij}$ for each i, j. Therfore

$$f(A_n) = \left(A^{(n)}\left(A^{(n)}\right)^T\right)_{ij} = \sum_{j=1} {a_{ij}}^{(n)} {a_{ji}}^{(n)} \to \sum_{j=1} a_{ij}a_{ji} = (AA^T)_{ij}.$$

Thus f is continous and the inverse image of the closed set $\{I_n\}$(*singleton set in a metric space is closed*) is precisely $\mathcal{O}_n(\mathbb{R})$.

To show that $\mathcal{O}_2(\mathbb{C})$ is not compact we need to find an unbounded matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that $a^2 + b^2 = c^2 + d^2 = 1, ac + bd = 0$. We can consider $a = i\sqrt{n} = d, b = -c = \sqrt{n+1}$. In this case we get unbounded matrices in $\mathcal{O}_n(\mathbb{C})$ for $n = 1, 2, 3, \ldots$ because

$$\begin{aligned} AA^T &= \begin{bmatrix} i\sqrt{n} & \sqrt{n+1} \\ -\sqrt{n+1} & i\sqrt{n} \end{bmatrix} \begin{bmatrix} i\sqrt{n} & -\sqrt{n+1} \\ \sqrt{n+1} & i\sqrt{n} \end{bmatrix} \\ &= \begin{bmatrix} -n + (n+1) & i\sqrt{n^2+n} - i\sqrt{n^2+n} \\ -i\sqrt{n^2+n} + i\sqrt{n^2+n} & (n+1) - n \end{bmatrix} \\ &= I_n. \end{aligned}$$

□

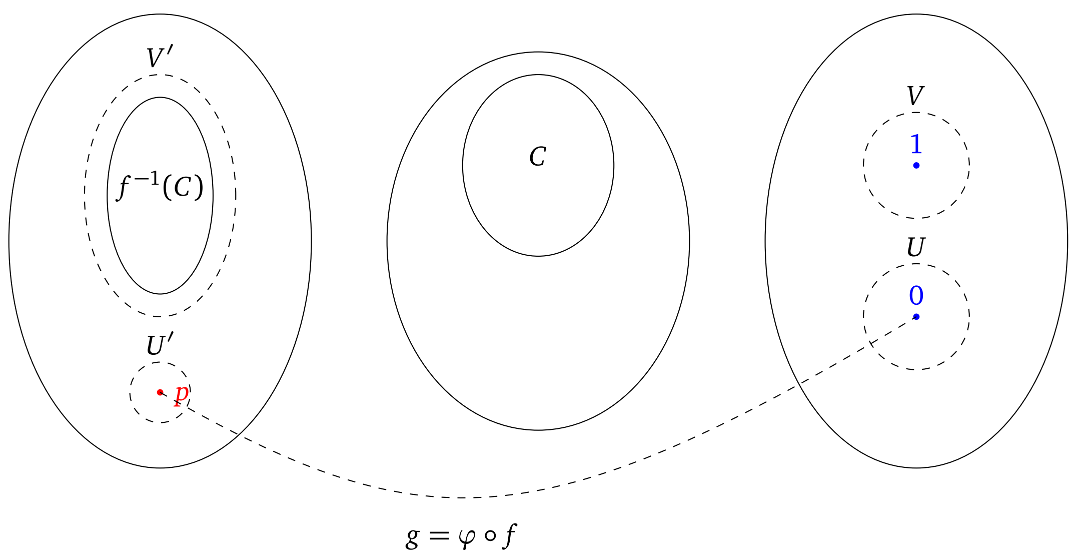
§ PROBLEM 24. *For a finite group G of order n with a subgroup H of order m, $\left(\frac{m}{n}\right)!| < 2n$ implies G is not simple.*

Solution. Let us consider $\varphi : G \to S_{\frac{n}{m}}$ induced by the action of G on the cosets of H. Now $K = \ker(\varphi) \trianglelefteq G$. If $K \neq \{1\}$ then G is not simple and we are done. Otherwise G is isomorphic to a subgroup of $S_N, N = \frac{n}{m}$. By Lagrange's theorem $n \mid N!$, but $N! < 2n$ implies $N! = n$ i.e., $G \cong S_N$. Therfore G is not simple as S_N

has a normal subgroup A_N for each $N \in \mathbb{N}$. $\square$

§ PROBLEM 25. (Metric Spaces) *Let X, Y be topological spaces such that Y is normal. Fruthermore for the function $f : X \to Y$ and for every continous function $\varphi : Y \to \mathbb{R}$, $\varphi \circ f$ is continous. Prove that f is continous.*

Solution. Let us consider a closed set C in Y. For a point $p \notin f^{-1}(C)$ we consider the two closed sets $\{f(p)\}$ and C. By normality of Y and Uryshon's Lemma there exists a continous function $\varphi : Y \to \mathbb{R}$ such that $\varphi(f(p)) = 0$ and $\varphi(c) = 0$ for all $c \in C$. We define $g = \varphi \circ f$. Then $g(p) = 0$ and for any $x \in f^{-1}(C)$ the image of x under g is $g(x) = \varphi(f(p)) = 1$ i.e., $g(f^{-1}(C)) = \{1\}$. Since Y is normal, there exist two disjoint open sets U and V in $\mathbb{R}$ such that $\{0\} \subseteq U$ and $\{1\} \subseteq V$. Given that g is continous. Hence $U' = g^{-1}(U)$ and $V' = g^{-1}(V)$ are two disjoint open sets in X. Clearly $p \in U'$ as $g(p) = 0 \in U$ and $f^{-1}(C) \subseteq V'$. Therefore we get a open neighbourhood U' of p such that $U' \cap f^{-1}(C) = \emptyset$ i.e., $\subseteq X \setminus f^{-1}(C)$. Hence $X \setminus f^{-1}(C)$ is open and the set $f^{-1}(C)$ is closed.



$\square$

§ PROBLEM 26. (Metric Spaces) *Let X, Y and Z are metric spaces, $f : X \to Y$ is a continous onto map and $g : Y \to Z$ is such that $g \circ f$ is continous. If X is compact prove that g is also a continous map.*

Solution. Let us consider a closed set C in Z. Now $(g \circ f)^{-1}(C)$ is closed in X and hence compact. f being a continous map implies $f((g \circ f)^{-1}(C))$ is compact in Y and hence a closed subset of Y.

We claim that $g^{-1}(C) = f((g \circ f)^{-1}(C))$. For any $y \in g^{-1}(C) \subseteq Y$ there exists x in X such that $f(x) = y$ because f is onto. Now $g(y) \in C$ i.e., $g(f(x)) \in C$. Thus $x \in (g \circ f)^{-1}(C)$ and hence

$y = f(x) \in f((g\circ f)^{-1}(C))$. Conversely for $w \in f((g\circ f)^{-1}(C))$, there exists $u \in (g\circ f)^{-1}(C)$ such that $f(u) = w$. Now $(g\circ f)(u) \in C$ implies $f(u) \in g^{-1}(C)$ i.e., $w \in g^{-1}(C)$ □

§ PROBLEM 27. *Let $\{a_i : i \in \mathbb{R}\}$ is a set of non negative real numbers in $\mathbb{R}$. If $\sup\{\sum_{i\in F} a_i \mid F \subseteq \mathbb{R}, |F| < \infty\}$ is finite. Show that except for countably many a_i's rest all are zero. Also show that the the 'countably' can not be replaces by 'finite'.*

Solution. Let $F_n = \left\{i \in \mathbb{R} \mid a_i \geq \frac{1}{n}\right\}$ and $F_0 = \cup_n F_n = \{i \in \mathbb{R} \mid a_i \neq 0\}$. Now each of the F_n must be finite, Otherwise for some N_0 and each $N \in \mathbb{N}$

$$\sup\left\{\sum_{i\in F} a_i \mid F \subseteq \mathbb{R}, |F| < \infty\right\} \geq \sup\left\{\sum_{i\in F} a_i \mid F \subseteq F_{N_0}, |F| < \infty\right\} \geq \frac{N}{N_0}.$$

As $N \to \infty$ the supremum becomes unbounded.

$a_i = \frac{1}{i^2}$ for $i \in \mathbb{N}$ and zero at other points satisfies the above condition. □

§ PROBLEM 28. *If $f : \{z \in \mathbb{C} \mid |z| > 1\}$ is defined by $f(z) = \frac{1}{z}$, show that there does not exist any entire function g such that $g = f$ on $|z| > 1$.*

Proof. If such a function exists, for $|z| > 1$ it will be bounded above by 1. Also on the comapct set $|z| \leq 1$ it will again be bounded. Thus g is an entire bounded function and hence constant by Liouville's theorem. This a contradiction to the assumption that $g(z) = \frac{1}{z}$ for $|z| > 1$. □

§ PROBLEM 29. *Any entire function f is either a polynomial or it has an essential singularity at ∞.*

Solution. Let us conisder the taylor series expansion of f about 0. If it terminated after finite terms we are done. Otherwise $g(z) = f\left\{\frac{1}{z}\right\}$ will have an essential singularity at zero i.e., f will have an essential singularity at ∞. □

> Remark. All non-constant functions that are analytic everywhere in the complex plane, $\mathbb{C}$ must be unbounded at ∞ and hence have a singularity at ∞.

§ PROBLEM 30. *Does there exist a continous surjection from $[0, 1)$ onto $\mathbb{R}$?*

Solution. Yes, $f(x) = x\sin x$. $\square$

§ PROBLEM 31. (Real Analysis, Integration) *$f:\mathbb{R}\to\mathbb{R}$ is such that $\int_{-\infty}^{\infty} f < \infty$. Then show that the function $F(x) = \int_{-\infty}^{x} f(t)dt$ is uniformly continous.*

Solution. Let us conisder $\int_{-\infty}^{\infty} f(x)dx = M < \infty$. For any $x > a$,

$$\begin{aligned}
|F(x) - F(a)| &= \left|\int_{-\infty}^{x} f(t)dt - \int_{-\infty}^{a} f(t)dt\right| \\
&= \left|\int_{a}^{x} f(t)dt\right| \\
&\leq \int_{a}^{x} |f(t)|dt.
\end{aligned}$$

Now $M = sup\{|f(t)| \mid t \in [a,x]\}$ is bounded becuase $wefwm$ $\square$

§ PROBLEM 32. (Topology, Compactness) *Let X be a topological space and $f : X \to [0,1]$ is a closed continous surjection and for each $a \in [0,1]$, $f^{-1}(a)$ is compact in X. Prove or disprove that X is compact.*

Solution. For any $a \in [0,1]$ and any nbd U containing $f^{-1}(a)$ we claim that there exists an open nbd W of a such that $f^{-1}(W) \subseteq U$. Because $f(X \setminus U)$ is closed in $[0,1]$ and hence $W = [0,1] \setminus f(X \setminus U)$ is a open set. For any $y \in f^{-1}(W)$, $f(y) \in [0,1] \setminus f(X \setminus U)$. Thus $f(y) \notin f(X \setminus U)$ and hence $y \notin X \setminus U$. This implies $y \in U$.

Now $\{U_i\}$ be an open cover for X. For each $a \in [0,1]$ there exists an open set U_a from the open cover such that $f^{-1}(a) \in U_a$. Thus we obtain a open nbd W_a of a such that $f^{-1}(W_a) \subseteq U_a$. The collection of open sets $\{W_a \mid a \in [0,1]\}$ forms an open cover for $[0,1]$. Since $[0,1]$ is compact we have $\bigcup_{i=1}^{n} W_{a_i} = [0,1]$. But f is a surjection implies $f^{-1}\left(\bigcup_{i=1}^{n} W_{a_i}\right) = X$. Thus $\bigcup_{i=1}^{n} f^{-1}(W_{a_i}) = X$ and hence $\bigcup_{i=1}^{n} U_{a_i} = X$. $\square$

§ PROBLEM 33. (Topology, Normal Spaces) *Let $f : X \to Y$ is a closed, continous, surjective map between two topological spaces. If X is normal prove that Y is also normal.*

Solution. For any two disjoint closed sets C, D in Y, the sets $A = f^{-1}(C), B = f^{-1}(D)$ are also closed in X. Moreover they are disjoint becuase $x \in f^{-1}(C) \cap f^{-1}(D)$ implies $f(x) \in C \cap D$. X being normal gives us two disjoint open sets U, V such that $A \subseteq U, B \subseteq V$. Now the sets $X \setminus U, X \setminus V$ are closed in X and so are $U' = f(X \setminus U)$ and $V' = f(X \setminus V)$ in Y because f is a closed map.

We claim that $Y \setminus U', Y \setminus V'$ are disjoint open sets and $C \subseteq Y \setminus U', D \subseteq Y \setminus V'$. The fact that they are open is straight foreward. Now

$$(Y \setminus U') \cap (Y \setminus V') = Y \setminus (U' \cup V') = Y \setminus (f(X \setminus U) \cup f(X \setminus V)).$$

But for any $f(x) = y \in Y$, $x \in U \cap V$ since $U \cap V = \phi$. i.e., $x \in X \setminus U$ or $x \in X \setminus V$. Thus $f(x) = y \in f(X \setminus U) \cup f(X \setminus V)$, which means $Y = f(X \setminus U) \cup f(X \setminus V)$. Therefore the sets $Y \setminus U'$ and $Y \setminus V'$ are disjoint open sets.

Our proof will be complete if we can show that $C \subseteq Y \setminus U'$ and $D \subseteq Y \setminus V'$. For $c \in C$, there exists $a \in A = f^{-1}(C)$ such that $f(a) = c \in C$ i.e., $a \in A \subseteq U$. If $c \notin Y \setminus f(X \setminus U)$, $c \in f(X \setminus U)$. There will be some $a' \in X \setminus U$ such that $f(a') = c$. This is a contradiction since $f^{-1}(C) \subseteq U$. □

§ Problem 34. (Group Theory) *Let G be a group with the property that for some $a \in G$, $H = G \setminus \{a\}$ is a subgroup of G. Prove that $|G| = 2$.*

Solution. For any $b \in H$, $ab \notin H$. Otherwise $a = (ab)b^{-1} \in H$. Thus for each $b \in H$, $ab = a$ i.e., $b = 1$. Since H is a subgroup $a \neq 1$. Thus there are only two elements in G, namely 1 and a. □