Collection of Problems

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Contents

§ PROBLEM 1. (Analysis) *If for a function $f : \mathbb{R} \to \mathbb{R}$ image of each compact set is compact then f is continous. T/F.*

Solution. No, we can take the function

$$f = \begin{cases} \sin\left(\frac{1}{x}\right) \text{ if } x \neq 0, \\ 0 \text{ else.} \end{cases}$$

This function is discontinous at 0. □

§ PROBLEM 2. *Existence of the limit* $\lim_{n\to\infty} \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$.

Solution. Let $x_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} - log n$. Then $x_{n+1} - x_n = \frac{1}{n+1} - log(\frac{n+1}{n})$. But $\log(1+x) \geq \frac{x}{x+1}$. Thus the sequence is decreasing and we can show(!) that it is bounded below. □

§ PROBLEM 3. *What is the smallest positive real numer c such that $||x||_1 \leq c||x||_\infty$ for all $x \in \mathbb{R}^n$.*

Solution. Clearly $||x||_1 \leq n||x||_\infty$. Now, we claim that $c = n$. Let if possible $||x||_1 \leq (n-\epsilon)||x||_\infty$ for some $\epsilon > 0$, for all $x \in \mathbb{R}^n$. But for $x = (1, 1, \ldots, 1)$ we will have $||x||_1 = n, ||x||_\infty = 1$ and hence $||x|| > ||x||_\infty$. □

§ PROBLEM 4. *If a group is finitely generated then show that there exist atmost finitely many subgroup of any given index.*

Solution. Let us consider G be the group and H be its subgroup such that $[G : H] = n$. The group acts on the cosets $\{H, g_2H, \ldots, g_nH\} = \{1, 2, 3, \ldots, n\}$ and it induces a homomorphism

$$\varphi_H : G \to S_n \text{ such that } g \xrightarrow{\varphi_H} \sigma_g.$$

Now the stabilizer of the element H in G/H can be identified as $\{g \in G \mid \sigma_g = 1\}$ i.e., $\{g \in G \mid gg_iH = g_iH, 1 \leq i \leq n\}$ i.e., H. We claim that different subgroups H and H' will induce different maps. For $h \in H, h \notin H'$ we have $\varphi_H(h) = 1$ but $\varphi_{H'}(h) \neq 1$. Again there are atmost finitely many maps from G to S_n and hence as a result there can exist only finite many subgroups of index n. □

§ PROBLEM 5. *For primes $p > q > 2$, group of order pq^2 contains a subgroup of ordre pq.*

Solution. The number of sylow p subgroup n_p divides q^2 as well as $p \mid n_p - 1$. Now n_p is odd if it is equal to q or q^2. Since p is also an odd prime we can not have $p \mid n_p - 1$ in this case. Thus we must have $n_p = 1$ i.e., the sylow$-p$ subgroup, H in G is normal and has order p. Now by Cauchy's theorem there exists $b \in G$ of order q. Let $K =< b >$. Then HK is the desired subgroup of G. □

§ PROBLEM 6. *SL_n is a product of matrices of the form $E_{ij}(a) = I + a\delta_{ij}, 1 \le i \ne j \le n$.*

Solution. Clearly $E_{ij}(a) \in SL_n$ and

$$\delta_{ij}\delta_{kl} = \begin{cases} \delta_{il} & \text{if } j = k, \\ 0 & \text{else.} \end{cases}$$

implies

$$\begin{aligned} E_{ij}(a)E_{ij}(-a) &= (I + a\delta_{ij})(I - a\delta_{ij}) \\ &= I - a^2\delta_{ij}\delta_{ij} \\ &= I. \end{aligned}$$

For $A \in SL_n$, since not all entries in the first column can be zero we must have $a_{i1} \ne 0$ and $E_{1i}(1)A = (I + \delta_{1i})A = A+$ □

§ PROBLEM 7. *X be a compact metric space with atleast two points and $a \in X$. Then*

1. *either $X \setminus \{a\}$ is compact or X is connected,*
2. *but not both.*

Solution.

1. Let us assume that $A = X \setminus \{a\}$ is not compact then we know A is not closed.
2. Let us assume that X is connected and if possible $X \setminus \{a\}$ is compact. Then $X \setminus \{a\}$ is closed. Also $\{a\}$ is a closed subset of X. This contradicts that $X = (X \setminus \{a\}) \cup \{a\}$ is connected.

   Conversely if $A = X \setminus \{a\}$ is compact then it will be closed in X and we will have $X = A \cup B$, for $B = \{a\}$. Thus X is not connected.

□

§ PROBLEM 8. *$GL_n^+(\mathbb{R})$ and $GL_n^-(\mathbb{R})$ are homeomorphic.*

Solution. We can define $\psi : GL_n^+(\mathbb{R}) \to GL_n^-(\mathbb{R})$ such that $\psi(M) = AM$, where A is a diagonal matrix such that $a_{11} = -1$ and $a_{ii} = 1$ for $1 < i \leq n$. □

§ PROBLEM 9. *Show that the General Linear group with positive determinant, $GL_n^+(\mathbb{R})$ is connected.*

Solution. We know that $GL_n^+(\mathbb{R}) = \det^{-1}((0,\infty))$ and hence it is open. If we can show that this there is some kind of homeomorphism we are through. □

§ PROBLEM 10. (Matrix, Topology) *Show that $SL_2(\mathbb{R})$ is connected.*

Solution. Here we will use the fact that the General Linear group with positive determinant, $GL_n^+(\mathbb{R})$ is path connected. With the help of this fact we can define a continous map

$$\phi : GL_n^+(\mathbb{R}) \to SL_n(\mathbb{R})$$

such that

$$\phi(A) = \frac{A}{(\det(A))^{\frac{1}{n}}}.$$

Clearly this is a surjection and hence $SL_n(\mathbb{R})$ is connected. □

§ PROBLEM 11. *$f : \mathbb{R} \to \mathbb{R}$ is continous. Then show that f is open iff it is strictly monotone.*

Solution. Let us assume that f is open and if possible there exist $a < b < c$ such that $f(a) < f(b) > f(c)$. Now if we restrict f to the interval $[a,c]$, then its supremum, M will exist and M will strictly be greater than $f(a), f(c)$ i.e., $f([a,c]) = [m,M]$. Therefore $f((a,c))$ will be a half closed interval i.e., either $f((a,c)) = [m,M]$ or $f((a,c)) = (m.M])$, contradicting our assumption that the map f is open.

Conversely WLOG let us assume that f is strictly increasing. It is sufficient to show that f maps open intetrval to open sets. Now, f being continous and strictly increasing implies $f((a,b)) = (f(a), f(b))$. □

§ PROBLEM 12. (Group Theory, Sylow Theorems) *What is the number of sylow$-p$ subgroups in $GL_n(\mathbb{F}_p)$.*

Solution. We have $|G| = |GL_n(\mathbb{F}_p)| = (p^n - 1)(p^n - p)\dots(p^n - p^{n-1})$. Therefore the cardinality of a $sylow - p$ subgroup in G is $p^{1+2+\dots+(n-1)} = p^{\frac{(n-1)n}{2}}$. Now the subgroup H of G consisting of the upper triangular matrices with diagonal entries 1 is a $sylow - p$ subgroup of G. Thus the number of $sylow - p$ subgroup is same as the index of the normalizer of H in G. We claim

$$N = \{A \in G \mid a_{ii} \neq 0, a_{ij} = 0 \text{ for } \ i < j\}$$

is equal to $N_H(G)$. $N \subseteq N_H(G)$ is obvious.

To proof the other direction we have to do some work. We have

$$N = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,n-1} & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2,n-1} & a_{2n} \\ & & & \ddots & & \\ 0 & 0 & 0 & \dots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 & a_{nn} \end{bmatrix} \mid a_{ij} \in F_p, a_{ii} \neq 0 \right\}.$$

Let us consider the subspace $V_i = \langle e_1, e_2, \dots, e_i \rangle$. It is clear that $HV_i \subseteq V_i$. **First** we claim that this are the only subspaces such that $HU \subseteq U$. If $u = (u_1, u_2, \dots, u_n)^t$ is some basis vector of U with say $u_i \neq 0$. WLOG we can assume $u_i = 1$. Now for $j \leq i$

$$(I + \delta_{ji})u = (u_1, u_2, \dots, u_j + u_i, \dots, u_n)^t.$$

Thus $(u_1, u_2, \dots, u_j + u_i, \dots, u_n) - (u_1, u_2, \dots, u_j, \dots, u_n) = (0, 0, \dots, u_i, \dots, 0) = e_j$ is contained in U. Therefore we can conclude that $U = V_j$, where j is largest index such that a basis vector has a nonzero jth entry.

Now for any $g \in N_G(H)$ and $h \in H$, $ghg^{-1} \in H$. Therefore $gh = h'g$ for some $h' \in H$. Again we claim $hV_i = V_i$ foe each i. Since $he_i = (h_{1i}, h_{2i}, \dots, h_{ni})^t$, $he_1 = (h_{1i}, 0, \dots, 0) = e_1$. Again $he_2 = (h_{12}, 1, \dots, 0)^t = h_{12}e_1 + e_2$ i.e.,

$$he_2 - h(h_{12}e_1) = e_2 \in hV_i.$$

By this way we have $hV_i = V_i$. Therefore $ghV_i = gV_i = h(gV_i)$ i.e. $h(gV_i) \subseteq gV_i$ and $H(gV_i) \subseteq gV_i$. From our first claim we have $gV_i = V_j$ for some $1 \leq j \leq n$. Since g is invertible and it preserves rank we must have $gV_i = V_i$ for each $1 \leq i \leq n$. Thus we have $g \in N$ by simple observation. □

§ Problem 13. (Compelx Analysis) *Find the entire functions $f : \mathbb{C} \to \mathbb{R}$.*

Solution. If such an entire function $f(z)$ exists then the function $if(z)$ is also entire and so is $\exp^{if(z)}$. This gives us $| \exp^{if(z)} | = 1$ and by Liouville's theorem it is constant. Consequently $f(z)$ must be constant. □

§ PROBLEM 14. *A subgroup H of index 5 in an odd order group G is normal.*

Solution. Since $|G:H| = 5$, we get a homomorphism $\varphi : G \to S_5$ and $K = \ker(\varphi) \subseteq H$. Thus $|G:K| \geq 5$. The subgroups of odd order in S_5 can have order $3, 5$ or 15. Now $|G:K| = 5$ implies $H = K$ and hence $H \trianglelefteq G$. Otherwise G/K is a group of order 15. We know that any subgroup of S_n either contains all even permutations or exactly half of them. If there are exactly half of the elements in $G/K \cong P \subseteq S_5$ are even permutations then $\sigma : P \to \{1, -1\}$ is a surjection. This gives us $|P : \ker(\sigma)| = 2$ i.e., $2 \mid |P|$, which is a clear contradiction to the fact that $|G|$ is odd. Thus all the elements in P are even permutations i.e., $G/K \subseteq A_5$. But A_5 has no subgroup of order 15, another contradiction. □

§ PROBLEM 15. *Let G be a finite group and H a subgroup of G of prime index p. If* $\gcd(|G|, p-1) = 1$ *then* $G' \subseteq H$.

Solution. To be contd... □

§ PROBLEM 16. (Group Theory) *A finite simple group G does not have a normal subgroup of index n if $|G|$ does not divide $N!$.*

Solution. Let $|G:H| = n$ then we get a homomorphism $\varphi : G \to S_n$ induced by the action of G on the cosets of H. Now $K = \ker(\varphi) \subseteq H$ and is a normal subgroup of G. G being a simple group implies that $K = 1$ i.e., G is embedded in S_n. Thus G can be thought as a subgroup of S_n and hence $|G| \mid n!$.
Consequences- A_5 has no subgroup of order 15 and 20 since $15 \nmid 24$ and $20 \nmid 6$. □

§ PROBLEM 17. (Real Analysis, Continous Functions) *Periodic continous function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continous.*

Solution. For ease of calculation we will consider the period of f to be 1. Now f is continous on $[0, 2]$ and for given $\epsilon > 0$ there exists $\delta > 0$ such that whenever $x, y \in [0, 2]$ with

$$|x - y| < \delta \text{ implies } |f(x) - f(y)| < \epsilon.$$

For any $x, y \in \mathbb{R}$ with $x > y (> 0$, say) there exist $n, m \in \mathbb{N}$ such that $x = n + r, y = m + s$ with $0 \leq r, s < 1$. For $\delta < 1$ if $|x - y| < \delta$ we claim that $n = m$ or $n = m + 1$. Otherwise let if possible $n \geq m + 2$ then

$$\begin{aligned} x - y &= (n + r) - (m - s) \\ &\geq 2 + (r - s). \end{aligned}$$

But $0 \le r < 1$ and $0 \le s < 1$ give us $-1 \le r - s \le 1$ i.e., $x - y \ge 2 - 1 = 1$. This contradicts $|x - y| < \delta < 1$. Therefore if we choose $\delta' = \min\{\delta, 1\}$ then $|f(x) - f(y)| < \epsilon$ whenever $|x - y| \le \delta'$. $\square$

§ PROBLEM 18. (Topology, Metric Space) *The complement of a proper subspace W of $\mathbb{R}^n$ is connected if and only if* $\dim(W) \le n - 2$.

Solution. Let us consider a proper subspace W such that $\dim(W) > n-2$, therefore $\dim(W) = n-1$ and $W^{\perp}$ is of dimension 1. If $W^{\perp} = \operatorname{span}\{v\}$ then we can consider the continous function $g : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = \langle v, x \rangle$. In this case $f^{-1}(0) = W$ and $f^{-1}(\mathbb{R} \setminus \{0\}) = \mathbb{R}^n \setminus W$. Thus we obtain two open sets $A = f^{-1}((0, \infty))$ and $B = f^{-1}((-\infty, 0))$ such that $A \cup B = \mathbb{R}^n \setminus W$ i.e., the complement W is not connected. Hence for the complement of a proper subspace of $\mathbb{R}^n$ to be connected we must have $\dim(W) \le n - 2$.

Conversely let us assume that $\dim(W) \le n - 2$. We need to show that $\mathbb{R}^n \setminus W$ is connected. The idea is to project any two vectors $x, y \in \mathbb{R}^n \setminus W$ to $W^{\perp}$, which is path connected. By this we get the path $x \to x' \to y' \to y$.

$$\begin{tikzcd}
 & y \\
x \arrow[dd] & \\
 & y' \arrow[uu] \\
x' \arrow[ur, bend right] &
\end{tikzcd}$$

Let $\{e_1, e_2, \dots, e_k\}$ is an orthonormal basis for W and $\{e_{k+1}, \dots, e_n\}$is an orthonormal basis for $W^{\perp}$. The projection x' of a vector $x = \sum_{i=1}^{n} x_i e_i$ onto $W^{\perp}$ is given by $\sum_{i=k+1}^{n} \langle x, e_i \rangle e_i = \sum_{i=k+1}^{n} x_i e_i$. We claim that the straight line connecting x and x' lies on W^c. $\square$

§ PROBLEM 19. (Complex Analysis) *Entire function $f : \mathbb{C} \to \mathbb{C}$ with $\Im(f) > 0$ is constant.*

Solution. For an entire function f, $\exp^{-if(z)}$ is also an entire function and $|\exp^{-if(z)}| = |\exp^{\Im(f)}|$. Similarly $\exp^{if(z)}$ is entire and $|\exp^{if(z)}| = |\exp^{-\Im(f)}| < 1$. Therefore $\exp^{if(z)}$ is constant and so is $f(z)$ $\square$

§ PROBLEM 20. (Functional Analysis) *Let X, Y, Z are Banach spaces such that $A : X \to Y$ and $B : Y \to Z$ are linear maps. If BA, B are bounded and B is injective then A is also bounded.*

Solution. Let $x_n \to x$ and $A(x_n) \to y$. B being bounded implies $B(A(x_n)) \to B(y)$. Moreover $(BA)(x_n) \to (BA)(x)$ and B is injective. Therefore $BA(x) = B(y)$ implies $A(x) = y$ and hence A is a closed map. Hence A is a bounded linear operator. □