Question 1. If for a function $f : \mathbb{R} \to \mathbb{R}$ image of each compact set is compact then f is continous. T/F.

Solution. No, we can take the function

$$f = \begin{cases} sin(\frac{1}{x}) \text{ if } x \neq 0, \\ 0 \text{ else.} \end{cases}$$

This function is discontinous at 0. □

Question 2. Existence of the limit $\lim_{n\to\infty} \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$.

Solution. Let $x_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} - log n$. Then $x_{n+1} - x_n = \frac{1}{n+1} - log(\frac{n+1}{n})$. But $log(1+x) \geq \frac{x}{x+1}$. Thus the sequence is decreasing and we can show(!) that it is bounded below. □

Question 3. What is the smallest positive real numer c such that $||x||_1 \leq c||x||_\infty$ for all $x \in \mathbb{R}^n$.

Solution. Clearly $||x||_1 \leq n||x||_\infty$. Now, we claim that $c = n$. Let if possible $||x||_1 \leq (n-\epsilon)||x||_\infty$ for some $\epsilon > 0$, for all $x \in \mathbb{R}^n$. But for $x = (1, 1, \ldots, 1)$ we will have $||x||_1 = n, ||x||_\infty = 1$ and hence $||x|| > ||x||_\infty$. □

Question 4. If a group is finitely generated then there exist atmost finitely many subgroup of any index.

Solution. Let us consider G be the group and H be its subgroup such that $[G : H] = n$. The group acts on the cosets $\{H, g_2H, \ldots, g_nH\} = \{1, 2, 3, \ldots, n\}$ and it induces a homomorphism

$$\varphi_H : G \to S_n \text{ such that } g \mapsto_{\varphi_H} \sigma_g.$$

Now the stabilizer of the element H in G/H can be identified as $\{g \in G \mid \sigma_g = 1\}$ i.e., $\{g \in G \mid gg_iH = g_iH, 1 \leq i \leq n\}$ i.e., H. We claim that different subgroups H and H' will induce different maps. For $h \in H, h \notin H'$ we have $\varphi_H(h) = 1$ but $\varphi_{H'}(h) \neq 1$. Again there are atmost finitely many maps from G to S_n and hence as a result there can exist only finite many subgroups of index n. □

Question 5. For primes $p > q > 2$, group of order pq^2 contains a subgroup of ordre pq.

Solution. The number of sylow p subgroup n_p divides q^2 as well as $p \mid n_p - 1$. Now n_p is odd if it is equal to q or q^2. Since p is also an odd prime we can not have $p \mid n_p - 1$ in this case. Thus we must have $n_p = 1$ i.e., the sylow$-p$ subgroup, H in G is normal and has order p. Now by Cauchy's theorem there exists $b \in G$ of order q. Let $K =< b >$. Then HK is the desired subgroup of G. □

Question 6. SL_n is a product of matrices of the form $E_{ij}(a) = I + a\delta_{ij}, 1 \leq i \neq j \leq n$.

Solution. Clearly $E_{ij}(a) \in SL_n$ and

$$\delta_{ij}\delta_{kl} = \begin{cases} \delta_{il} & \text{if } j = k, \\ 0 & \text{else.} \end{cases}$$

implies

$$\begin{aligned} E_{ij}(a)E_{ij}(-a) &= (I + a\delta_{ij})(I - a\delta_{ij}) \\ &= I - a^2\delta_{ij}\delta_{ij} \\ &= I. \end{aligned}$$

For $A \in SL_n$, since not all entries in the first column can be zero we must have $a_{i1} \neq 0$ and $E_{1i}(1)A = (I+\delta_{1i})A = A+$

□

Question 7. X be a compact metric space with atleast two points and $a \in X$. Then

1. either $X \setminus \{a\}$ is compact or X is connected,
2. but not both.

Solution.

1. Let us assume that $A = X \setminus \{a\}$ is not compact then we know A is not closed.
2. Let us assume that X is connected and if possible $X \setminus \{a\}$ is compact. Then $X \setminus \{a\}$ is closed. Also $\{a\}$ is a closed subset of X. This contradicts that $X = (X \setminus \{a\}) \cup \{a\}$ is connected.

   Conversely if $A = X \setminus \{a\}$ is compact then it will be closed in X and we will have $X = A \cup B$, for $B = \{a\}$. Thus X is not connected.

□