Автономная навигация мобильных роботов

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Saint Petersburg state University,

Scientific and Technological
University “Sirius”

Тема 3: Алгоритмы навигации на основе обширных данных о сцене

Общие определения и примеры

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Decomposition of the environment

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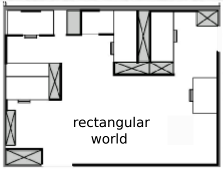

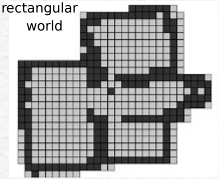

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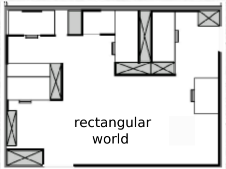

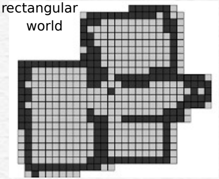

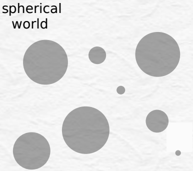

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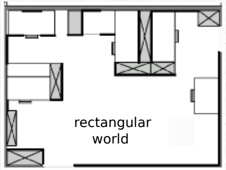

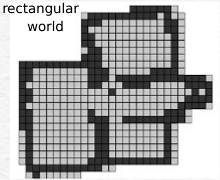

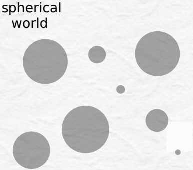

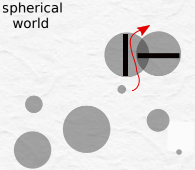

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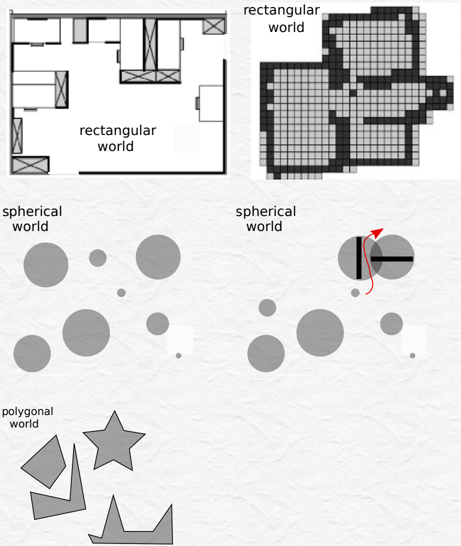

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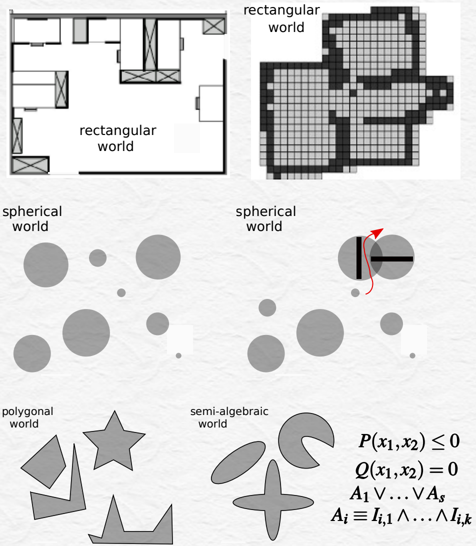

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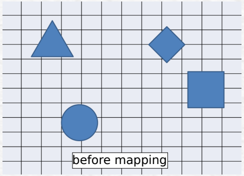

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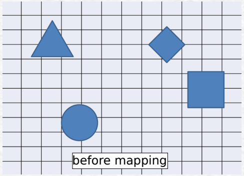

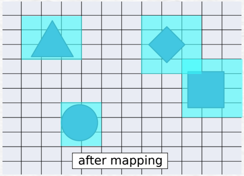

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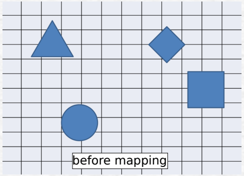

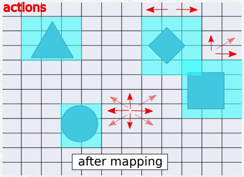

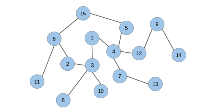

occupancy grid representation

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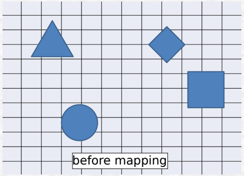

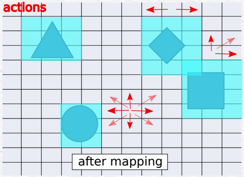

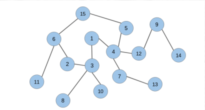

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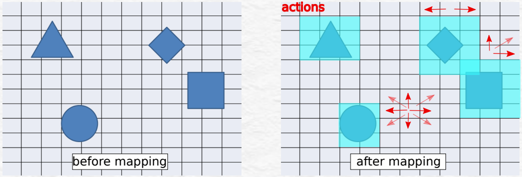

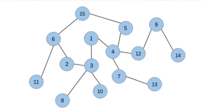

occupancy grid representation

Nodes ∼ free cells;
two nodes are linked ∼ the robot can immediately move from any of them to the other, and “knows” how
Target point ∼ target node
Path planning ∼ finding a path on the graph
Sequence of way-points (“way-cells”)

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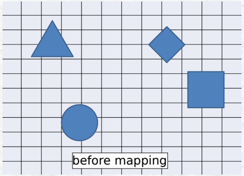

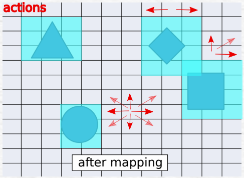

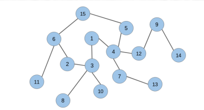

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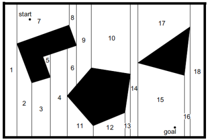
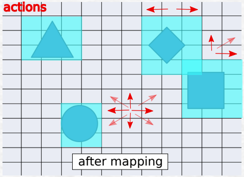
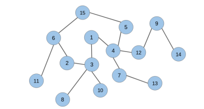


cell decomposition



after mapping



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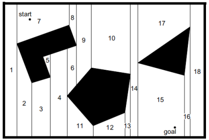
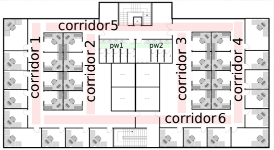


cell decomposition

34 workplaces
8 toilet cabins



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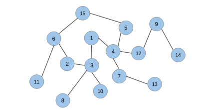

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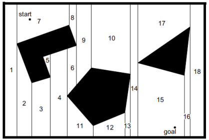

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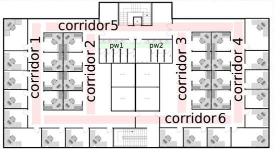

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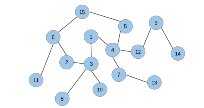

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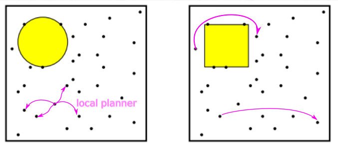

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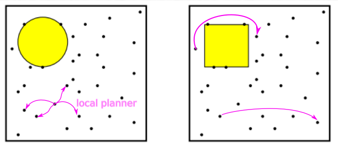

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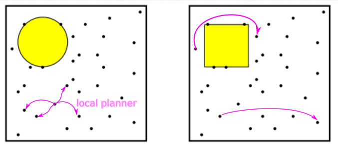

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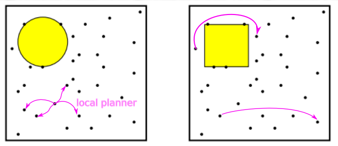

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Разбиение (мозаика) Вороного

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Definition

Let X be a metric space with the distance function $d(\cdot,\cdot)$ and let $\{x_1,\dots,x_n\} \subset X$ be a finite set of points. The Voronoi cell with the center x_i is defined to be $C_i := \{x \in X : d(x,x_i) < d(x,x_j)\ \forall j \neq i\}$.

Разбиение (мозаика) Вороного

Definition

Let X be a metric space with the distance function $d(\cdot,\cdot)$ and let $\{x_1,\dots,x_n\} \subset X$ be a finite set of points. The Voronoi cell with the center x_i is defined to be $C_i := \{x \in X : d(x,x_i) < d(x,x_j)\ \forall j \neq i\}$.

Basic properties

- $C_i \cap C_j = \emptyset\ \forall i \neq j$
- $x \notin \bigcup_{i=1}^{n} C_i \Rightarrow x \in H_\cup := \bigcup_{i \neq j} H_{i,j}$,
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  $H_{i,j} := \{x : d(x,x_i) = d(x,x_j)\}$
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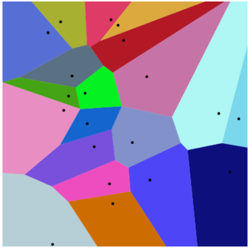

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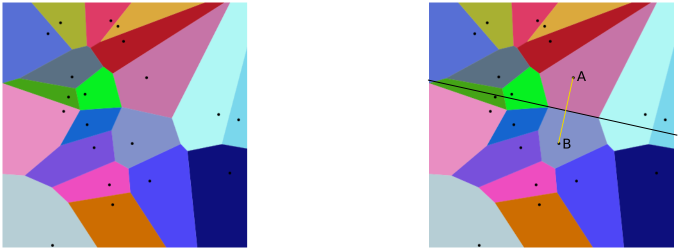

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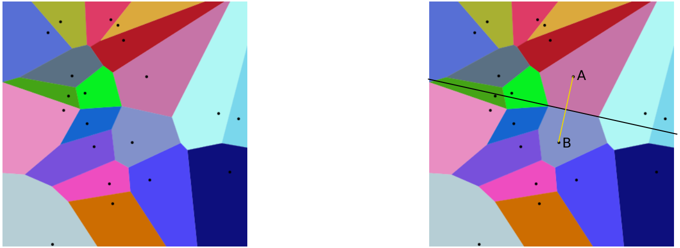

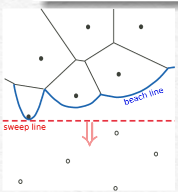

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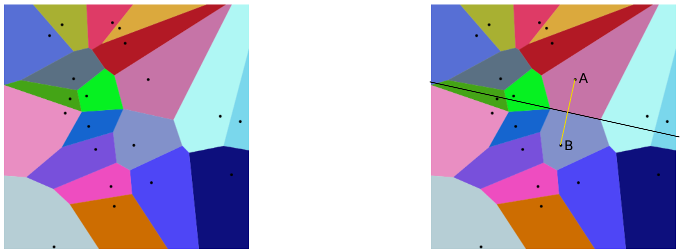

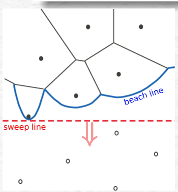

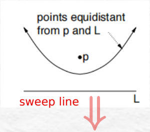

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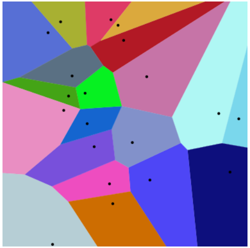
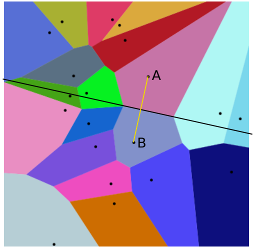
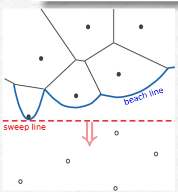
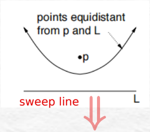
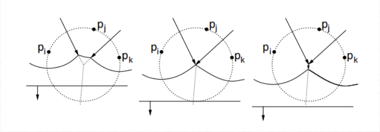


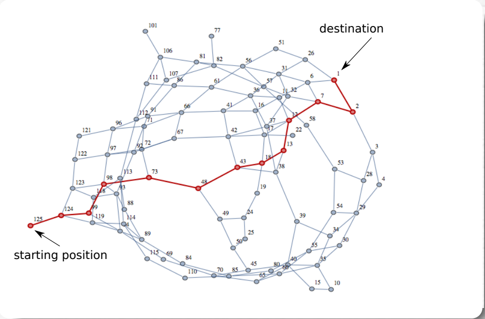








Планирование: подходы, основанные на логическом представлении



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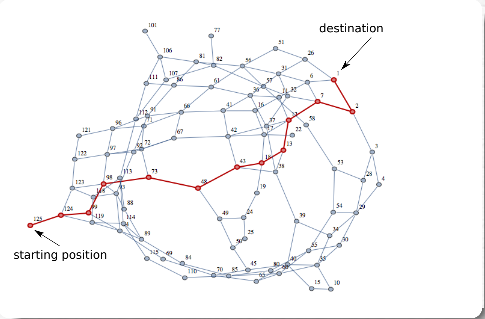
Walk over nodes: terminology



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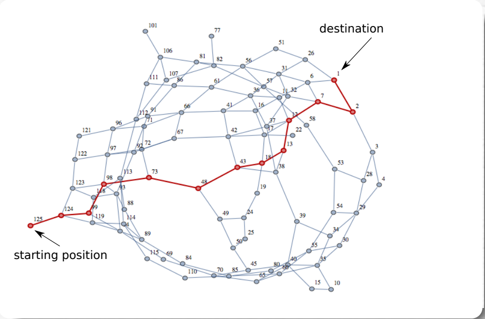

Планирование: подходы, основанные на логическом представлении

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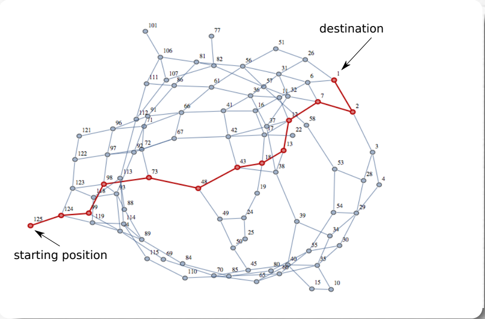

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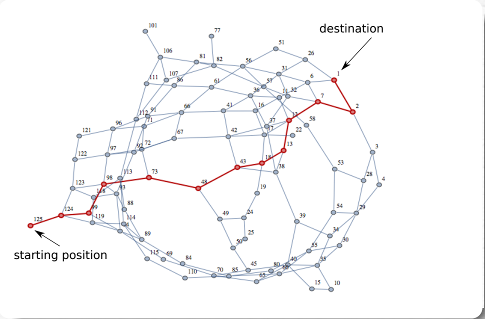

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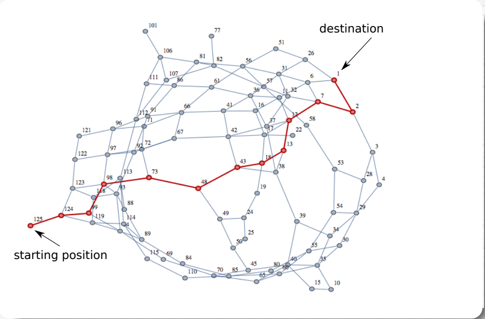

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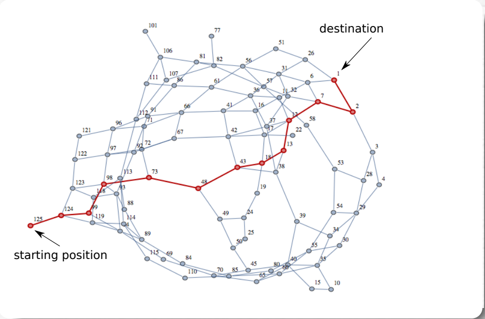

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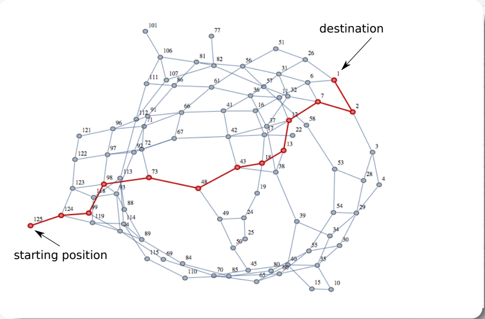

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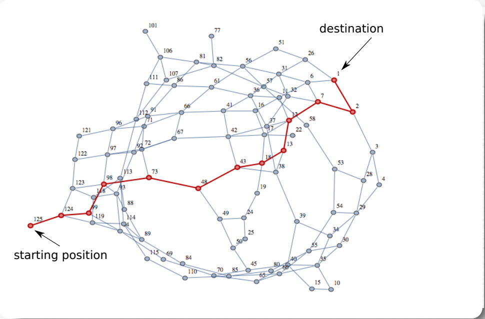

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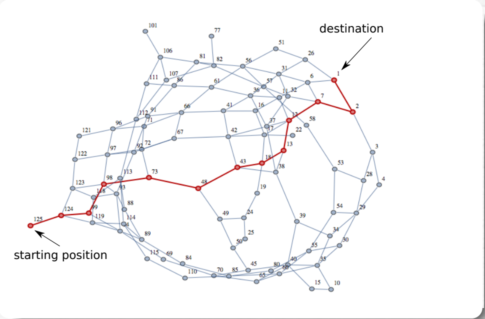

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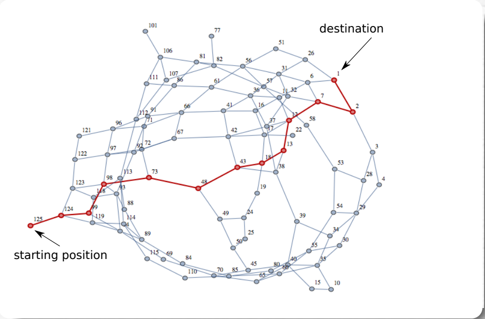

Breadth first (grass-fire front) method

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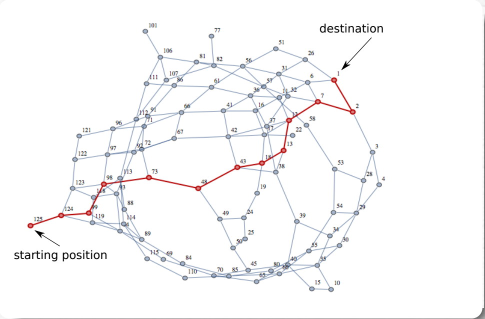

Breadth first (grass-fire front) method

- Start with the source, put in in Q

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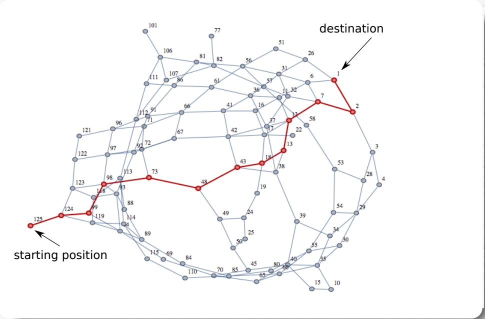

Breadth first (grass-fire front) method

- Start with the source, put in in Q
- Sort Q according to FIFO (first-in-first-out) policy

Планирование: подходы, основанные на логическом представлении

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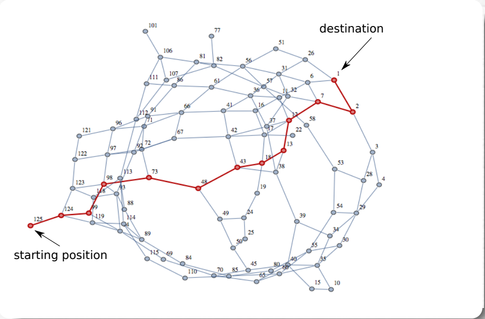
Breadth first (grass-fire front) method

- Start with the source, put in in Q
- Sort Q according to FIFO (first-in-first-out) policy
- Test only unvisited neighbors



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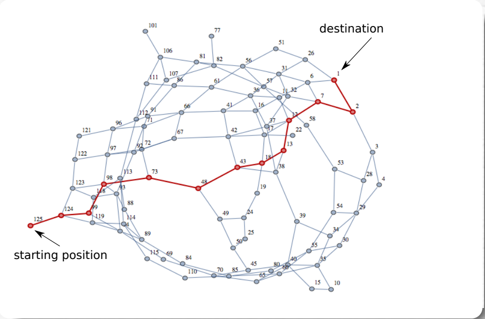

Breadth first (grass-fire front) method

- Start with the source, put in in Q
- Sort Q according to FIFO (first-in-first-out) policy
- Test only unvisited neighbors
- When arriving at the destination, terminate processing nodes

Планирование: подходы, основанные на логическом представлении

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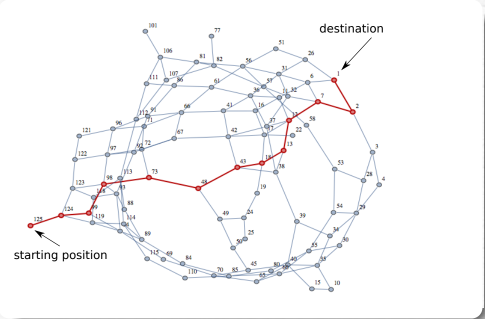

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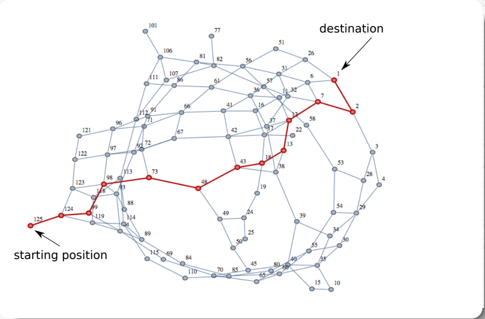

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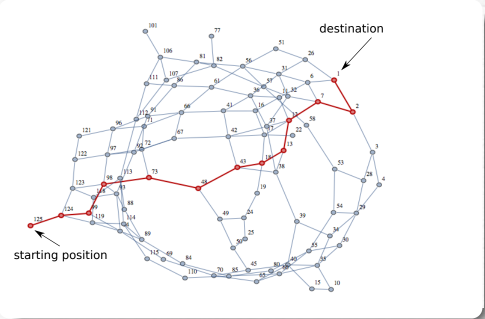

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Properties

- Degree of the node := the length of the shortest path from the source

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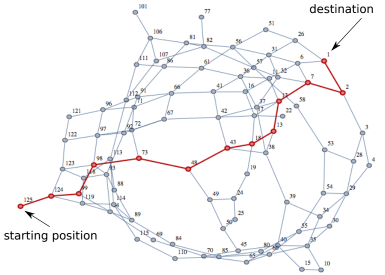

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- Test only unvisited neighbors
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Properties

- Degree of the node := the length of the shortest path from the source
- for any step of the search stage, there exists $k \geq 0$ such that Q contains only nodes of degree k or $k+1$ and all nodes of degree $< k$ are dead

Планирование: подходы, основанные на логическом представлении

Walk over nodes: terminology

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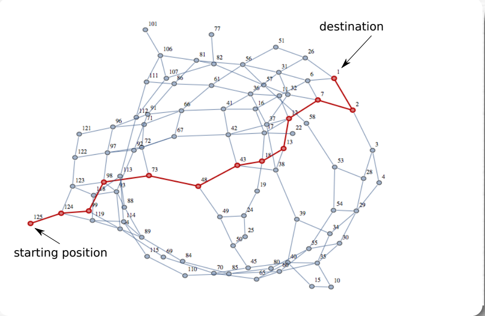

Breadth first (grass-fire front) method

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- References highlight a shortest path

Планирование: подходы, основанные на логическом представлении

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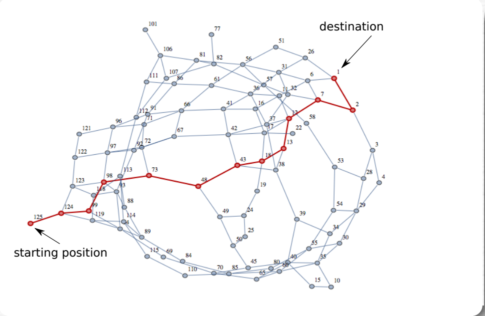

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- $O(|V| + |E|)$

Планирование: подходы, основанные на логическом представлении

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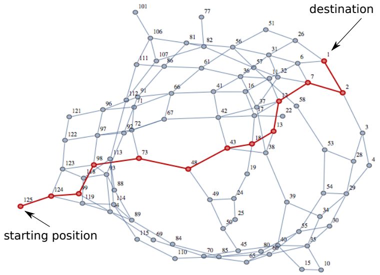

Depth first (aggressive deepening) method

- Start with the source, put in in Q
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Properties

- Degree of the node := the length of the shortest path from the source
- for any step of the search stage, there exists $k \geq 0$ such that Q contains only nodes of degree k or $k+1$ and all nodes of degree $< k$ are dead
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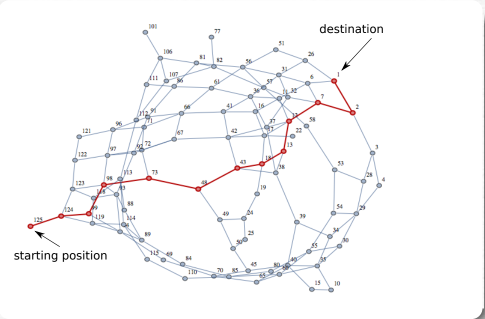
Планирование: подходы, основанные на логическом представлении

Walk over nodes: terminology

- Visited/unvisited node
- Visited dead/alive node
- Priority queue Q of the alive nodes, the dead ones are excluded
- Prioritization method $\sim$ to sort Q
- Direct walk (search) starts from the source and aims at the destination
- Reference $\sim$ indication to a parent
- Mark "dead"
- Use the references to find the path (finalizing backward walk)



Depth first (aggressive deepening) method

- Start with the source, put in in Q
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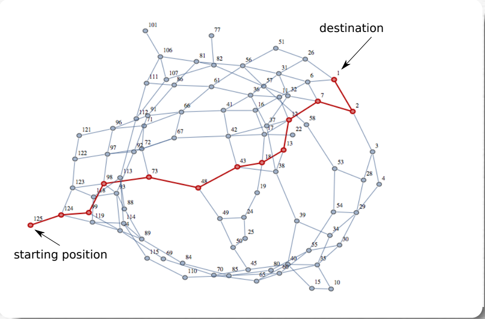

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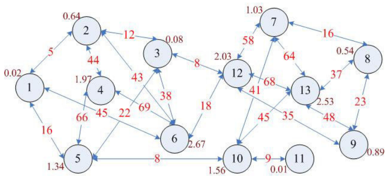
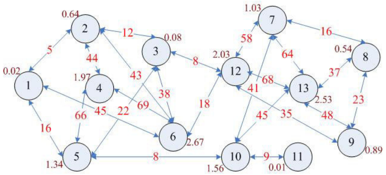


$W(n \to n') > 0$ weight of the edge

Планирование: подходы, основанные на логическом представлении

Dijkstra's algorithm



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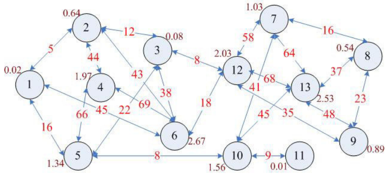

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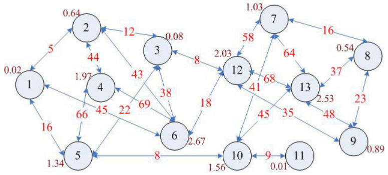

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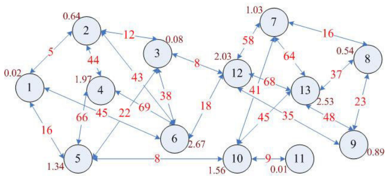

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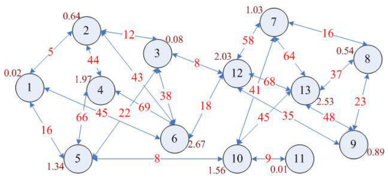

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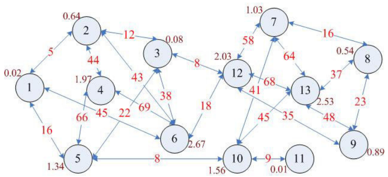

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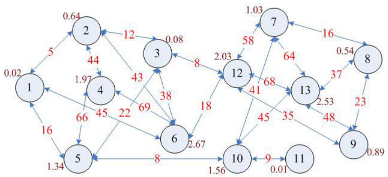

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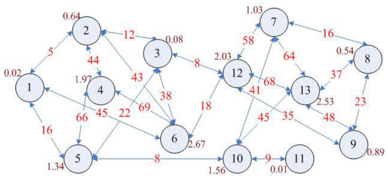

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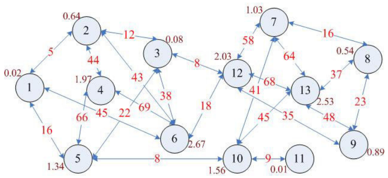

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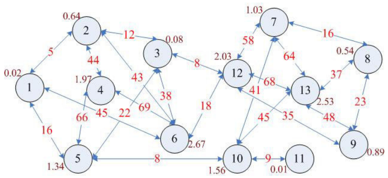

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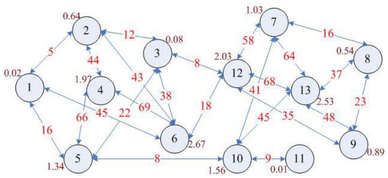

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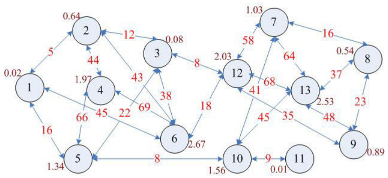

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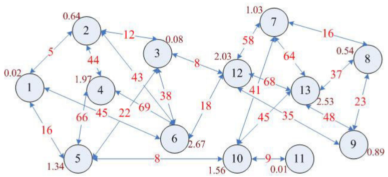

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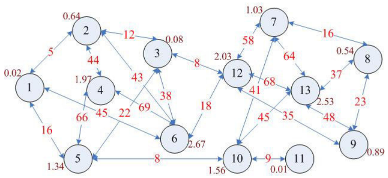

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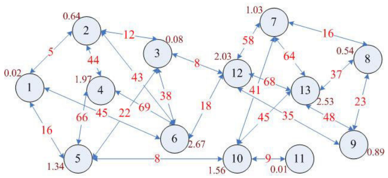

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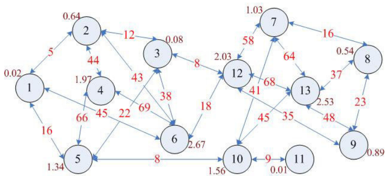

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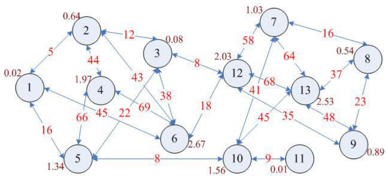

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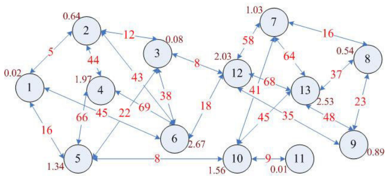

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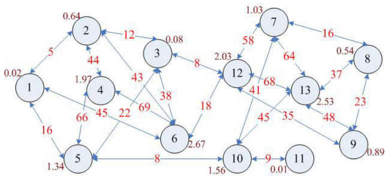

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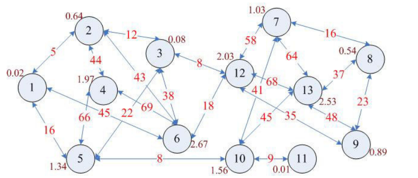

$W(n \to n') > 0$ weight of the edge

Lemma

At any step and for the node visited at this step, its label $L(n)$ is the shortest length $d(n)$ of the path from the source s to n. The final set U is the set of the nodes unreachable from s.

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$d(n_u) = d(n_v) + W(n_v \to n_u)$.

Dijkstra's algorithm

- At any step, forms a partition of the nodes into V (visited) and U (unvisited)
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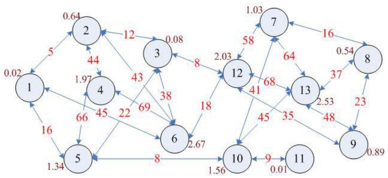

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Планирование: подходы, основанные на логическом представлении

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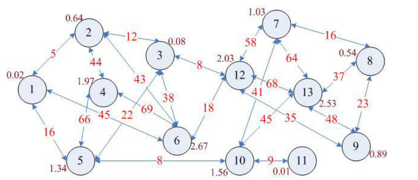

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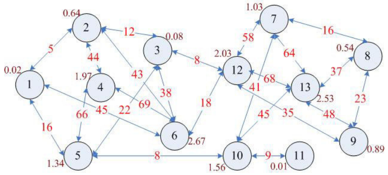
Планирование: подходы, основанные на логическом представлении



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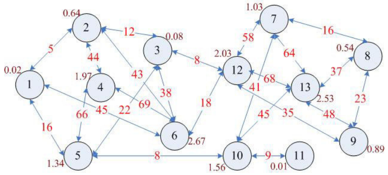

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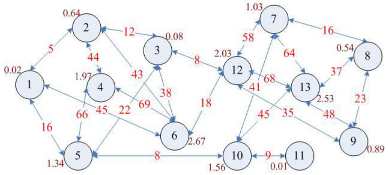

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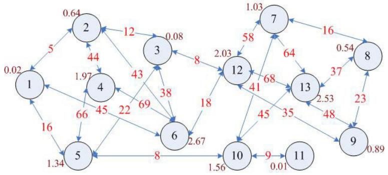

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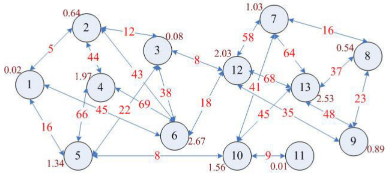

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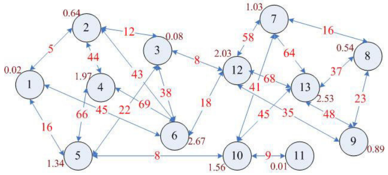

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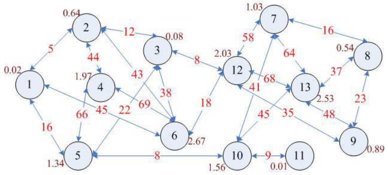

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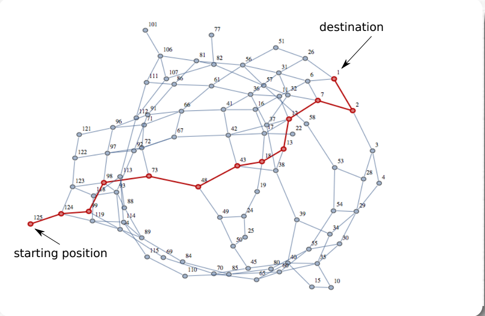
Lemma

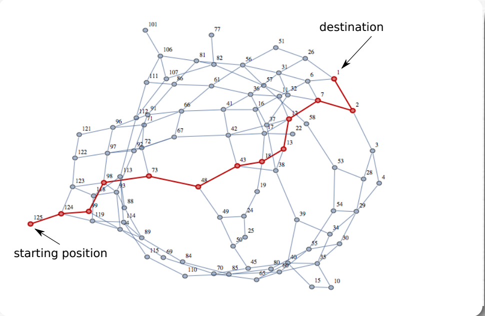
At any step and for any node, its label $L(n)$ is the shortest length $d(n)$ of the path among those that go from the source s to n through V (except for the last node). (The minimum over the empty set is defined to be ∞.) The final set U is the set of the nodes unreachable from s.

Планирование: подходы, основанные на логическом представлении



Best first

- A numerical evaluation $G_*(n)$ of "successfulness" of node n (e.g., cost-to-go)
  - Start with the source, put it in Q
  - Sort Q in the decreasing value of $G_*(n)$
  - Test only unvisited neighbors
  - When arriving at the destination, terminate processing nodes
  - Use the references to build a path



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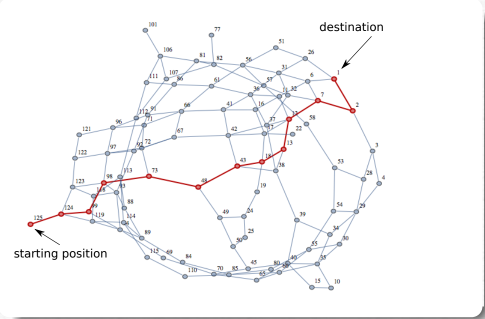
Iterative deepening

1. Put $d := 1$
2. Perform the depth-first search, with limiting it to nodes of degree $\leq d$
3. If the search is not terminated (the destination is not found)
   - Erase the results of all previous computations
   - Put $d := d + 1$
   - Go to step 2, where the search starts with the source once more



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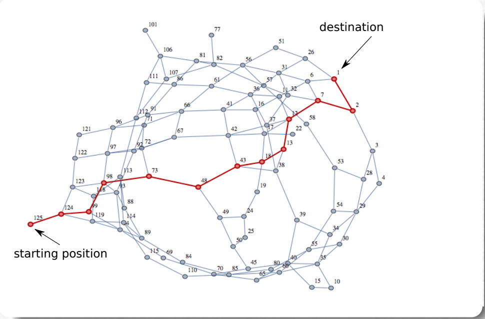

Backward search

Any of the forward search methods, where the roles of the source and destination are interchanged.

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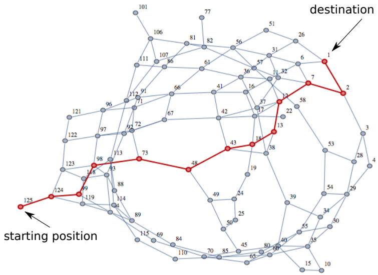

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Bidirectional search

A forward and backward search methods are run simultaneously until meeting.

Search as evolution of a dynamical system

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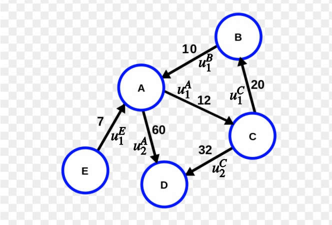

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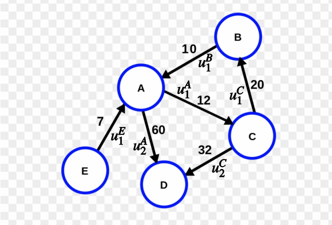

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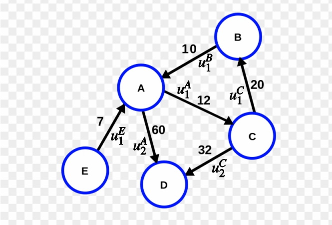

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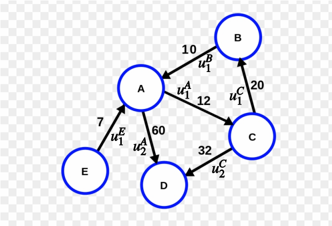

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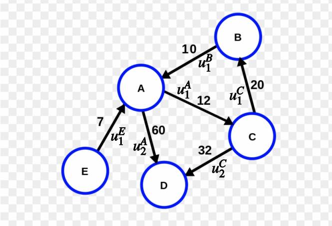

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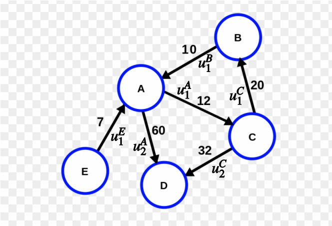
Optimal search with given number k of steps

$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [0:k) \qquad (✣)$$

$$x_0 = a, \qquad (✚)$$

$$I := \sum_{t \in [0:k)} \varphi(x_t, u_t) + \eta(x_k) \to \min \qquad (✐)$$



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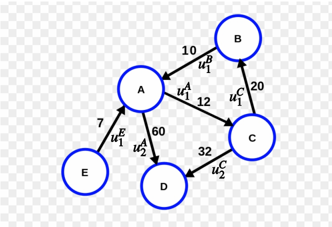

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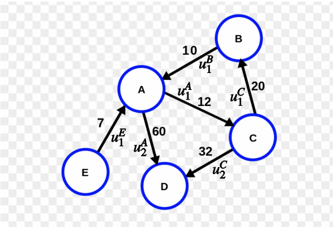

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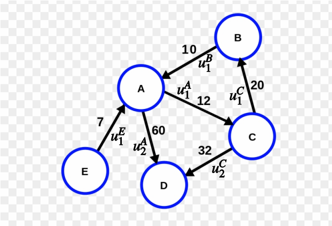

Setup hints

It is needed to

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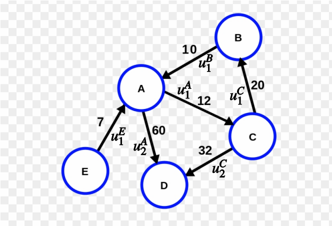

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- arrive at b: ☞ $\eta(x) := \lambda\|x - b\|, \lambda \approx \infty$

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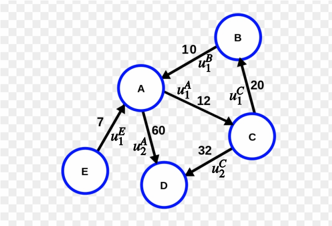

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It is needed to

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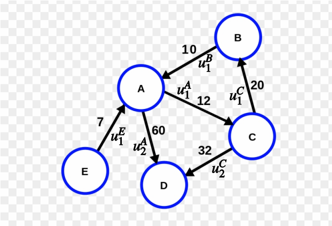

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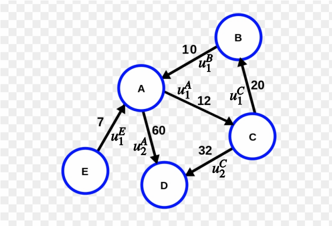

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- arrive at some of $b_1, \ldots, b_s$: ☞ $\eta(x) := \lambda \min_{i=1,\ldots,s} \|x - b_i\|, \lambda \approx \infty$

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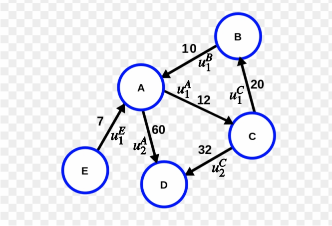

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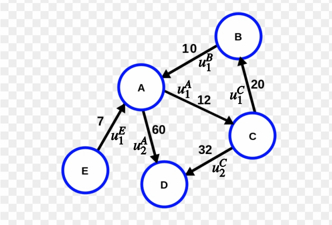

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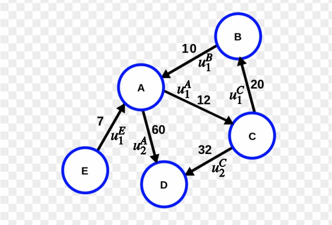

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Belman equation

$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [t_0 : k) \qquad (✣)$$

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$$V_k(\cdot) \curvearrowright V_{k-1}(\cdot) \curvearrowright \ldots \curvearrowright V_0(\cdot), \ V_t(a) \in [0, \infty);$$

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Theorem

The following statements hold:

- The Bellman function is the unique solution of the Bellman equation
- $u_t \in \mathcal{U}_t(x_t)$ is the optimal control law (search rule)

Proof of the theorem

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Overall: $V_{t_0}(a) = \min\limits_{u \in U(a)} \Big\{\varphi(a,u) + V_{t_0+1}[f(a,u)]\Big\}, \qquad u^0_{t_0} = \operatorname{argmin}_{u \in U(a)}\Big\{\ldots\Big\} \Rightarrow u^0_{t_0} \in \mathfrak{U}_{t_0}[x^0_{t_0}] \Rightarrow u^0_t \in \mathfrak{U}_{t_0}[x^0_t]\ \forall t \geq t_0.$

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$$\overset{\text{Belmann equation}}{=} \sum_{t=t_0}^{k-2} \varphi[x_t, u_t] + V_\tau(a_*) = \sum_{t=t_0}^{k-2} \varphi[x_t, u_t] + V_{k-1}(x_{k-1}) = \cdots = V_{t_0}[x_{t_0}] = V_{t_0}(a)$$

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Let the process $p = [\{x_t\}_{t=t_0}^{t_1}, \{u_t\}_{t=t_0}^{t_1-1}] \in \mathfrak{M}(t_0, a)$ be generated by the dynamic programming regulator $u_t \in \mathfrak{U}_t(x_t) = \text{Arg}\,\min_{u \in U(x_t)} \{V_{t+1}[f(x_t,u)] + \varphi(x_t,u)\}$

$$V_{t+1}[\underbrace{f(x_t,u_t)}_{x_{t+1}}] + \varphi(x_t,u_t) = \min_{u \in U(x_t)} \{V_{t+1}[f(x_t,u)] + \varphi(x_t,u)\} = V_t(x_t)$$

$$I_{t_0}(p) = \sum_{t \in [t_0:k)} \varphi[x_t,u_t] = \sum_{t=t_0}^{k-1} \varphi[x_t,u_t] + V_k[x_k] = \sum_{t=t_0}^{k-2} \varphi[x_t,u_t] + \left\{\varphi_{k-1}[x_{k-1},u_{k-1}] + V_k[x_k]\right\}$$

$$\overset{\tau:=k-1}{=\!=\!=} \sum_{t=t_0}^{k-2} \varphi[x_t,u_t] + \left\{\varphi[\underbrace{x_\tau}_{=:a_*}, \underbrace{u_\tau}_{=:v \in U(a_*)}] + V_{\tau+1}[f(a_*,v)]\right\}$$

$$\overset{\text{Belmann equation}}{=} \sum_{t=t_0}^{k-2} \varphi[x_t,u_t] + V_\tau(a_*) = \sum_{t=t_0}^{k-2} \varphi[x_t,u_t] + V_{k-1}(x_{k-1}) = \cdots = V_{t_0}[x_{t_0}] = V_{t_0}(a)$$

$$V_{t_0}(a) = I_{t_0}(p)$$

Оптимальное планирование с нефиксированной длиной пути

Оптимальное планирование с нефиксированной длиной пути

Problem statement

$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [0:k) \qquad (✣)$$

$$x_0 = a, \qquad k \text{ is freely manipulable} \qquad (✚)$$

$$I := \sum_{t \in [0:k)} \varphi(x_t, u_t) + \eta(x_k) \to \min \qquad (✑)$$

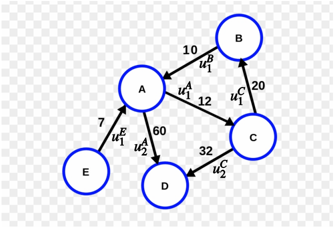
Оптимальное планирование с нефиксированной длиной пути

Problem statement

$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [0:k] \qquad (\text{✣})$$

$$x_0 = a, \qquad k \text{ is freely manipulable} \qquad (\text{✚})$$

$$I := \sum_{t \in [0:k)} \varphi(x_t, u_t) + \eta(x_k) \to \min \qquad (\text{✾})$$



Оптимальное планирование с нефиксированной длиной пути

Problem statement

$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [0:k] \qquad (✣)$$

$$x_0 = a, \qquad k \text{ is freely manipulable} \qquad (✚)$$

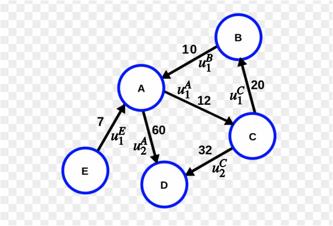
$$I := \sum_{t\in[0:k)} \varphi(x_t, u_t) + \eta(x_k) \to \min \qquad (✇)$$

Bellman equation

$V_t(a) = \min_{u\in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\}\ \forall a$

Boundary condition: $V_k(\cdot) := \eta(\cdot)$

Regulator $\mathcal{U}_t(a) :=$ Arg $\min_{u\in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\}$



Оптимальное планирование с нефиксированной длиной пути

Problem statement

$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [0:k] \qquad (✣)$$

$$x_0 = a, \qquad k \text{ is freely manipulable} \qquad (✚)$$

$$I := \sum_{t \in [0:k)} \varphi(x_t, u_t) + \eta(x_k) \to \min \qquad (✑)$$

Bellman equation

$V_t(a) = \min_{u \in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\} \ \forall a$

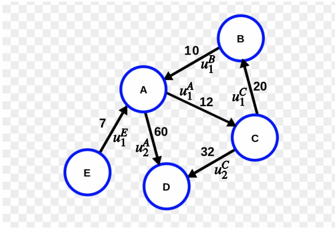
Boundary condition: $V_k(\cdot) := \eta(\cdot)$

Regulator $\mathfrak{U}_t(a) := \operatorname{Arg}\min_{u \in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\}$

The same as an iteration

$$V(\cdot) \xrightarrow{\mathfrak{F}} V_-(\cdot), \ V_-(a) := \min_{u \in U(a)} \{V[f(a,u)] + \varphi(a,u)\}$$

$$V_t(\cdot) = \mathfrak{F}[V_{t+1}(\cdot)]$$



Оптимальное планирование с нефиксированной длиной пути

Problem statement

$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [0:k] \qquad (✣)$$

$$x_0 = a, \qquad k \text{ is freely manipulable} \qquad (✚)$$

$$I := \sum_{t \in [0:k)} \varphi(x_t, u_t) + \eta(x_k) \to \min \qquad (✑)$$

Bellman equation

$V_t(a) = \min_{u \in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\} \ \forall a$

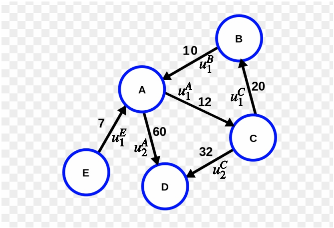
Boundary condition: $V_k(\cdot) := \eta(\cdot)$

Regulator $\mathfrak{U}_t(a) :=$ Arg $\min_{u \in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\}$

The same as an iteration

$$V(\cdot) \xrightarrow{\mathfrak{F}} V_-(\cdot), \ V_-(a) := \min_{u \in U(a)} \{V[f(a,u)] + \varphi(a,u)\}$$

$$V_t(\cdot) = \mathfrak{F}[V_{t+1}(\cdot)] \qquad t = k-1, k-2, \ldots, 0$$



Оптимальное планирование с нефиксированной длиной пути

Problem statement

$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [0:k] \qquad (✣)$$

$$x_0 = a, \qquad k \text{ is freely manipulable} \qquad (✚)$$

$$I := \sum_{t \in [0:k)} \varphi(x_t, u_t) + \eta(x_k) \to \min \qquad (✑)$$

Bellman equation

$V_t(a) = \min_{u \in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\} \ \forall a$

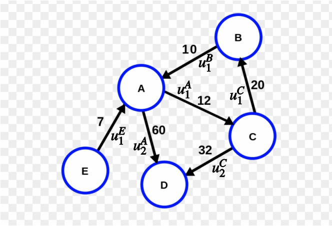
Boundary condition: $V_k(\cdot) := \eta(\cdot)$

Regulator $\mathfrak{U}_t(a) := \text{Arg} \min_{u \in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\}$

The same as an iteration

$$V(\cdot) \xrightarrow{\mathfrak{F}} V_-(\cdot), \ V_-(a) := \min_{u \in U(a)} \{V[f(a,u)] + \varphi(a,u)\}$$

$$V_t(\cdot) = \mathfrak{F}[V_{t+1}(\cdot)] \qquad t = -1, \dots, -k$$



Problem statement

$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [0:k] \qquad (✣)$$

$$x_0 = a, \qquad k \text{ is freely manipulable} \qquad (✚)$$

$$I := \sum_{t\in[0:k)} \varphi(x_t, u_t) + \eta(x_k) \to \min \qquad (✐)$$

Bellman equation

$V_t(a) = \min_{u\in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\} \ \forall a$

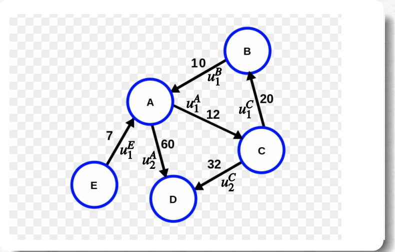
Boundary condition: $V_k(\cdot) := \eta(\cdot)$

Regulator $\mathfrak{U}_t(a) := \operatorname{Arg}\min_{u\in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\}$

The same as an iteration

$V(\cdot) \xrightarrow{\mathfrak{F}} V_-(\cdot), \ V_-(a) := \min_{u\in U(a)} \{V[f(a,u)] + \varphi(a,u)\}$

$V_t(\cdot) = \mathfrak{F}[V_{t+1}(\cdot)] \qquad t = -1, \dots, -\infty$



$V_k^{\downarrow}(a) := \min_{\theta\in[-k:0]} V_\theta(a), \qquad V_0^{\downarrow}(\cdot) = \eta(\cdot)$

Оптимальное планирование с нефиксированной длиной пути

Problem statement

$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [0:k] \qquad (✣)$$

$$x_0 = a, \qquad k \text{ is freely manipulable} \qquad (✚)$$

$$I := \sum_{t \in [0:k)} \varphi(x_t, u_t) + \eta(x_k) \to \min \qquad (✑)$$

Bellman equation

$V_t(a) = \min_{u \in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\} \ \forall a$

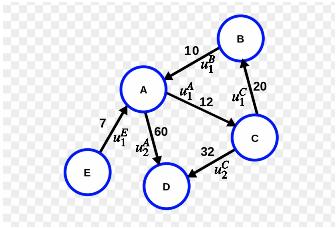
Boundary condition: $V_k(\cdot) := \eta(\cdot)$

Regulator $\mathcal{U}_t(a) :=$ Arg $\min_{u \in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\}$

The same as an iteration

$V(\cdot) \xrightarrow{\mathfrak{F}} V_-(\cdot), \ V_-(a) := \min\limits_{u \in U(a)} \{V[f(a,u)] + \varphi(a,u)\}$

$V_t(\cdot) = \mathfrak{F}[V_{t+1}(\cdot)] \qquad t = -1, \dots, -\infty$



$V_k^{\downarrow}(a) := \min_{\theta \in [-k:0]} V_\theta(a), \qquad V_0^{\downarrow}(\cdot) = \eta(\cdot)$

$$V_{-k}^{\downarrow}(a) = \min_{\theta \in [-k:0]} V_\theta(a)$$

Оптимальное планирование с нефиксированной длиной пути

Problem statement

$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [0:k] \qquad (\ast)$$

$x_0 = a,$ $\quad$ k is freely manipulable $\qquad (+)$

$$I := \sum_{t \in [0:k)} \varphi(x_t, u_t) + \eta(x_k) \to \min \qquad (\infty)$$

Bellman equation

$V_t(a) = \min_{u \in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\} \ \forall a$

Boundary condition: $V_k(\cdot) := \eta(\cdot)$

Regulator $\mathcal{U}_t(a) := \text{Arg}\min_{u \in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\}$

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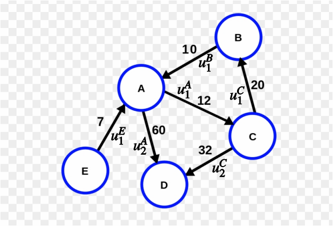
$$V(\cdot) \xrightarrow{\mathfrak{F}} V_-(\cdot), \ V_-(a) := \min_{u \in U(a)} \{V[f(a,u)] + \varphi(a,u)\}$$

$$V_t(\cdot) = \mathfrak{F}[V_{t+1}(\cdot)] \qquad t = -1, \ldots, -\infty$$



$V_k^{\downarrow}(a) := \min_{\theta \in [-k:0]} V_\theta(a), \qquad V_0^{\downarrow}(\cdot) = \eta(\cdot)$

$$V_{-k}^{\downarrow}(a) = \min_{\theta \in [-k:0]} V_\theta(a) = \min \left\{ \min_{\theta \in [-k:-1]} V_\theta(a); \eta(a) \right\}$$

Оптимальное планирование с нефиксированной длиной пути

Problem statement

$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [0:k) \qquad (✣)$$

$$x_0 = a, \qquad k \text{ is freely manipulable} \qquad (✚)$$

$$I := \sum_{t \in [0:k)} \varphi(x_t, u_t) + \eta(x_k) \to \min \qquad (✇)$$

Bellman equation

$V_t(a) = \min_{u \in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\} \ \forall a$

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The same as an iteration

$$V(\cdot) \xrightarrow{\mathfrak{F}} V_-(\cdot), \ V_-(a) := \min_{u \in U(a)} \{V[f(a,u)] + \varphi(a,u)\}$$

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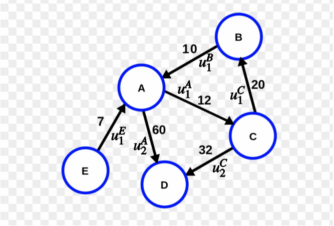

$$V_k^{\downarrow}(a) := \min_{\theta \in [-k:0]} V_\theta(a), \qquad V_0^{\downarrow}(\cdot) = \eta(\cdot)$$

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$$= \min \left\{ \min_{\theta \in [-k:-1]} \min_{u \in U(a)} \Big(V_{\theta+1}[f(a,u)] + \varphi(a,u) \Big); \eta(a) \right\}$$

Оптимальное планирование с нефиксированной длиной пути

Problem statement

$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [0:k] \qquad (✣)$$

$$x_0 = a, \qquad \text{k is freely manipulable} \qquad (✚)$$

$$I := \sum_{t\in[0:k)} \varphi(x_t, u_t) + \eta(x_k) \to \min \qquad (⚮)$$

Bellman equation

$V_t(a) = \min_{u\in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\} \;\; \forall a$

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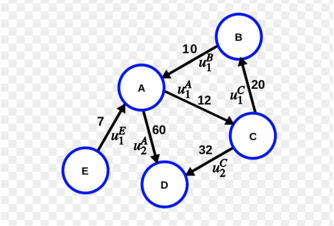
Regulator $\mathfrak{U}_t(a) := \operatorname{Arg}\,\min_{u\in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\}$

The same as an iteration

$$V(\cdot) \xrightarrow{\mathfrak{F}} V_-(\cdot), \;\; V_-(a) := \min_{u\in U(a)} \{V[f(a,u)] + \varphi(a,u)\}$$

$$V_t(\cdot) = \mathfrak{F}[V_{t+1}(\cdot)] \qquad t = -1, \ldots, -\infty$$



$$V_k^{\downarrow}(a) := \min_{\theta\in[-k:0]} V_\theta(a), \qquad V_0^{\downarrow}(\cdot) = \eta(\cdot)$$

$$V^{\downarrow}_{-k}(a) = \min_{\theta\in[-k:0]} V_\theta(a) = \min\left\{\min_{\theta\in[-k:-1]} V_\theta(a); \eta(a)\right\}$$

$$= \min\left\{\min_{u\in U(a)} \min_{\theta\in[-k:-1]} \Big(V_{\theta+1}[f(a,u)] + \varphi(a,u)\Big); \eta(a)\right\}$$

Оптимальное планирование с нефиксированной длиной пути

Problem statement

$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [0:k] \qquad (✣)$$

$$x_0 = a, \qquad k \text{ is freely manipulable} \qquad (✚)$$

$$I := \sum_{t\in[0:k)} \varphi(x_t, u_t) + \eta(x_k) \to \min \qquad (✾)$$

Bellman equation

$V_t(a) = \min_{u\in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\} \ \forall a$

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Regulator $\mathcal{U}_t(a) := \operatorname{Arg}\min_{u\in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\}$

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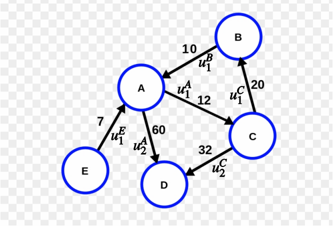
$$V_t(\cdot) = \mathfrak{F}[V_{t+1}(\cdot)] \qquad t = -1, \dots, -\infty$$



$$V_k^{\downarrow}(a) := \min_{\theta\in[-k:0]} V_\theta(a), \qquad V_0^{\downarrow}(\cdot) = \eta(\cdot)$$

$$V^{\downarrow}_{-k}(a) = \min_{\theta\in[-k:0]} V_\theta(a) = \min\left\{\min_{\theta\in[-k:-1]} V_\theta(a); \eta(a)\right\}$$

$$= \min\left\{\min_{u\in U(a)} \min_{\theta\in[-k:-1]} \Big(V_{\theta+1}[f(a,u)] + \varphi(a,u)\Big); \eta(a)\right\}$$

$$= \min\left\{\min_{u\in U(a)} \Big[\varphi(a,u) + \min V_{\theta+1}[f(a,u)]\Big]; \eta(a)\right\}$$

Оптимальное планирование с нефиксированной длиной пути

Problem statement

$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [0:k] \qquad (✣)$$

$$x_0 = a, \qquad \color{blue}{k \text{ is freely manipulable}} \qquad (✚)$$

$$I := \sum_{t \in [0:k)} \varphi(x_t, u_t) + \eta(x_k) \to \min \qquad (✑)$$

Bellman equation

$V_t(a) = \min_{u \in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\} \;\; \forall a$

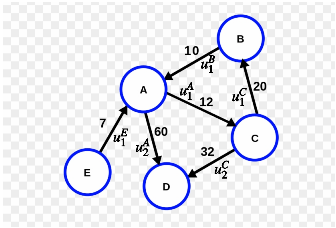
Boundary condition: $V_k(\cdot) := \eta(\cdot)$

Regulator $\mathcal{U}_t(a) := \text{Arg}\ \min_{u \in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\}$

The same as an iteration

$$V(\cdot) \xrightarrow{\mathfrak{F}} V_-(\cdot), \;\; V_-(a) := \min_{u \in U(a)} \{V[f(a,u)] + \varphi(a,u)\}$$

$$V_t(\cdot) = \mathfrak{F}[V_{t+1}(\cdot)] \qquad t = -1, \ldots, -\infty$$



$$V_k^{\downarrow}(a) := \min_{\theta \in [-k:0]} V_\theta(a), \qquad V_0^{\downarrow}(\cdot) = \eta(\cdot)$$

$$V^{\downarrow}_{-k}(a) = \min_{\theta \in [-k:0]} V_\theta(a) = \min\left\{\min_{\theta \in [-k:-1]} V_\theta(a); \eta(a)\right\}$$

$$= \min\left\{\min_{u \in U(a)} \min_{\theta \in [-k:-1]} \Big(V_{\theta+1}[f(a,u)] + \varphi(a,u)\Big); \eta(a)\right\}$$

$$= \min\left\{\min_{u \in U(a)} \left[\varphi(a,u) + \min_{\theta \in [-k:-1]} V[f(a,u)]\right]; \eta(a)\right\}$$

Оптимальное планирование с нефиксированной длиной пути

Problem statement

$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [0:k) \qquad (✣)$$

$$x_0 = a, \qquad \color{blue}{k \text{ is freely manipulable}} \qquad (✚)$$

$$I := \sum_{t\in[0:k)} \varphi(x_t, u_t) + \eta(x_k) \to \min \qquad (✑)$$

Bellman equation

$V_t(a) = \min_{u\in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\} \ \forall a$

Boundary condition: $V_k(\cdot) := \eta(\cdot)$

Regulator $\mathcal{U}_t(a) := \text{Arg} \min_{u\in U(a)} \{V_{t+1}[f(a,u)] + \varphi(a,u)\}$

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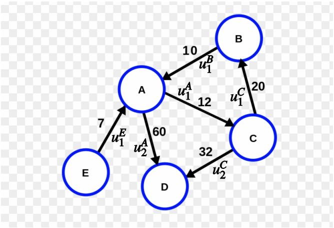
$$V_t(\cdot) = \mathfrak{F}[V_{t+1}(\cdot)] \qquad t = -1, \dots, -\infty$$



$$V_k^{\downarrow}(a) := \min_{\theta\in[-k:0]} V_\theta(a), \qquad V_0^{\downarrow}(\cdot) = \eta(\cdot)$$

$$V^{\downarrow}_{-k}(a) = \min_{\theta\in[-k:0]} V_\theta(a) = \min\left\{\min_{\theta\in[-k:-1]} V_\theta(a); \eta(a)\right\}$$

$$= \min\left\{\min_{u\in U(a)} \min_{\theta\in[-k:-1]} \Big(V_{\theta+1}[f(a,u)] + \varphi(a,u)\Big); \eta(a)\right\}$$

$$= \min\left\{\min_{u\in U(a)} \left[\varphi(a,u) + \min_{\tau\in[-k+1:0]} V_\tau[f(a,u)]\right]; \eta(a)\right\}$$

$$= \min\left\{\min_{u\in U(a)} \left[\varphi(a,u) + V^{\downarrow}_{-k+1}[f(a,u)]\right]; \eta(a)\right\}$$

Оптимальное планирование с нефиксированной длиной пути

Problem statement

$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [0:k) \qquad (✣)$$

$$x_0 = a, \qquad k \text{ is freely manipulable} \qquad (✚)$$

$$I := \sum_{t \in [0:k)} \varphi(x_t, u_t) + \eta(x_k) \to \min \qquad (\varphi(\cdot) > 0) \qquad (✇)$$

Bellman equation

$$V^{\downarrow}_{-k}(a) = \min \left\{ \min_{u \in U(a)} \left[\varphi(a, u) + V^{\downarrow}_{-k+1}[f(a, u)] \right]; \eta(a) \right\}, \; V^{\downarrow}_0(\cdot) := \eta(\cdot)$$

Оптимальное планирование с нефиксированной длиной пути

Problem statement

$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [0:k) \qquad (✣)$$

$$x_0 = a, \qquad k \text{ is freely manipulable} \qquad (✚)$$

$$I := \sum_{t\in[0:k)} \varphi(x_t, u_t) + \eta(x_k) \to \min \qquad (\varphi(\cdot) > 0) \qquad (✾)$$

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Theorem

The sequence $V^{\downarrow}_{-k}(\cdot)$ is monotonically decreasing $V^{\downarrow}_{-k}(\cdot) \le V^{\downarrow}_{-k+1}(\cdot)$ and there exists $k \le -1$ such that $V^{\downarrow}_{-k}(\cdot) \equiv V^{\downarrow}_{-k+1}(\cdot)$. For such k, we have $V^{\downarrow}(\cdot) := V^{\downarrow}_{-k}(\cdot) = V^{\downarrow}_{-\theta}(\cdot)\ \forall \theta \ge k$ and $V^{\downarrow}(\cdot)$ is the unique solution of the equation

$V(a) = \min\left\{ \min_{u\in U(a)} \left[\varphi(a,u) + V[f(a,u)]\right]; \eta(a) \right\}$ and simultaneously the Bellman function of the optimization problem.

Оптимальное планирование с нефиксированной длиной пути

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Bellman equation

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Оптимальное планирование с нефиксированной длиной пути

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$$x_{t+1} = f(x_t, u_t), \quad u_t \in U[x_t] \quad \forall t \in [0:k) \qquad (✣)$$

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Оптимальное планирование с нефиксированной длиной пути

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V-closing state $\sim V(a) = \eta(a)$
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Theorem

The set of $V^{\downarrow}$-closing states is not empty. Optimal process $\Leftrightarrow u_t \in \mathfrak{U}_V(x_t)$ and the process terminates terminates upon arrival at a closing state.

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Lemma

Let $[\{x_t\}_{t=0}^{k}, \{u_t\}_{t=0}^{k-1}]$ be an optimal process. Then the process $[\{x_t\}_{t=\theta}^{k}, \{u_t\}_{t=\theta}^{k-1}]$ is also optimal for any $\theta \in [0:k]$, the state x_k is stopping, $u_t \in \mathcal{U}_{V^\downarrow}(x_t)$, and $V^\downarrow(x_0) = \sum_{t=0}^{\theta-1} \varphi(x_t, u_t) + V^\downarrow(x_\theta)$.

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$$V^\downarrow(x_0) = \varphi(x_0, u_0) + V^\downarrow(x_1) = \varphi(a, u_0) + V^\downarrow[f(a, u_0)], \quad a := x_0$$

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Any solution of the equation $V(a) = \min\left\{\min_{u \in U(a)}\left[\varphi(a,u) + V[f(a,u)]\right]; \eta(a)\right\}$ is a lower estimate of the Bellman function.

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Lemma

Let $u_t \in \mathcal{U}_V(x_t)$ and the process terminates upon arrival at the first stopping state. Then this process is optimal.

$$V(a) = \min\left\{\min_{u \in U(a)}\Big[\varphi(a,u) + V[f(a,u)]\Big]; \eta(a)\right\}$$

$$I = \sum_{t=0}^{k-1} \varphi(x_t, u_t) + \eta[x_k] \geq \sum_{t=0}^{k-1} \varphi(x_t, u_t) + V[x_k]$$

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