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Observation of nonlinear systems via finite capacity channels: Constructive data rate limits ☆

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ABSTRACT

The paper deals with observation of nonlinear and deterministic, though maybe chaotic, discrete-time systems via finite capacity communication channels. We introduce several minimum data-rate limits associated with various types of observability, and offer new tractable analytical techniques for their both upper and lower estimation. Whereas the lower estimate is obtained by following the lines of the Lyapunov's linearization method, the proposed upper estimation technique is along the lines of the second Lyapunov approach. As an illustrative example, the potential of the presented results is demonstrated for the system which describes a ball vertically bouncing on a sinusoidally vibrating table. For this system, we provide an analytical computation of a closed-form expression for the threshold that separates the channel data rates for which reliable observation is and is not possible, respectively. Another illustration is concerned with the celebrated Hénon system. The offered sufficient data rate bound is accompanied with a constructive observer that works whenever the channel capacity fits this bound.

1. Introduction

Recent advances in communication technology have created the possibility of large-scale control systems, where the control tasks are distributed over many agents orchestrated via a communication network; a particular example can be found in modern industrial systems, where the components are often connected over digital band-limited serial communication channels (Gao, Chen, & Lam, 2008; Liu et al., 2014; Postoyan, van de Wouw, Nešić, & Heemels, 2014; Wang, Gao, & Qiu, 2015). This motivated development of a new chapter of control theory, where control and communication issues are integrated; see e.g., Antsaklis and Baillieul (2007), Mahmoud (2014), Matveev and Savkin (2009), Murray (2002), Postoyan and Nešić (2012), Postoyan et al. (2014) and Yüksel and Basar (2013).

In this area, a basic question is about the smallest communication data rate required to achieve a certain control objective for a given plant. This fundamental parameter has been studied in a variety of settings (Baillieul, 2004; de Persis, 2005; de Persis & Isidori, 2004; Liberzon & Hespanha, 2005; Matveev & Savkin, 2009; Nair, Evans, Mareels, & Moran, 2004; Nair, Fagnani, Zampieri, & Evans, 2007; Savkin, 2006) and always found to be somewhat similar to the topological entropy, which is an ubiquitous quantitative measure of randomness, chaos, uncertainty, and complexity in dynamical systems (Donarowicz, 2011; Katok, 2007). These studies gave rise to specialized control-oriented concepts of entropy. Most close to the canonical definitions (Adler, Konheim, & McAndrew, 1965; Bowen, 1971; Dinaburg, 1970) of the topological entropy is the concept accounting for uncertainties in the plant model (Savkin, 2006). Effects caused by a feedback are reckoned in the *feedback topological entropy* (Nair et al., 2004), *invariance entropy* (Colonius & Kawan, 2009), and their modifications (Colonius & Kawan, 2011; Hagihara & Nair, 2013; Kawan, 2011a): the original two concepts are shown to be essentially the same (Colonius, Kawan, & Nair, 2013).

The mentioned studies transport the topological entropy and its recent analogs from the topical areas of pure mathematics and

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physics towards everyday practice of control and communication engineers, thus enhancing the need for tractable methods of entropy evaluation. However, the last issue is highly intricate as far as nonlinearity is concerned. One of the paper's contributions is a method for constructive estimation of the entropy for nonlinear discrete-time systems.

Comprehensive closed-form expressions for entropy-related data-rate limits are well known for linear systems and cover the basic control objectives, such as stabilization and state estimation; see, e.g., the surveys in Matveev and Savkin (2009), Nair et al. (2007), Mahmoud (2014), Yüksel and Basar (2013) and Nair (2015). The "nonlinear" realm is much less inhabited by similar samples; for the most part, they are somehow close to linear systems or are one-dimensional. In Savkin (2006), topological entropy is computed in closed form for special uncertain linear systems, and this result was applied to problems of stabilization, state estimation, and optimal control. In Nair et al. (2004), closed-form computation of the local feedback topological entropy in fact deals with restriction of a smooth nonlinear system on a tight vicinity of the equilibrium, where the system is close to linear. Constructive lower and upper bounds on the invariance entropy and some its descendants are given in Kawan (2011a,b,c); conservatism of these bounds basically remains an open issue, though a closed form computation is offered for one-dimensional systems nonlinear in state and affine in control.

Intricacies in entropy evaluation are fairly well illuminated in the general theory of nonlinear dynamics. Whereas effective general approaches are known for low dimensional systems, especially for piece-wise monotone interval mappings (see, e.g., Alsedá, Llibre, & Misiurewicz, 2000; Amigó & Giménez, 2014; de Melo & van Strien, 1993; Donarowicz, 2011; Milnor & Thurston, 1988), rigorous incomputability results are obtained for more complex cases (Delvenne & Blondel, 2004; Hurd, Kari, & Culik, 1992; Koiran, 2001; Simonsen, 2006), e.g., for piece-wise affine continuous maps and $\varepsilon \approx 0$, no program can generate a rational number in a finite time that approximates the topological entropy with precision ε or better (Koiran, 2001). Exact values of topological entropy are still unknown for even such widely-studied low-dimensional chaotic systems as Hénon map, Dufing and van der Pol oscillators, Rössler system, and bouncing ball system, though various estimates and results of approximate numerical studies are available.

The first objective of this paper is to contribute to reducing conservatism of existing estimates of topological entropy via development of new tractable analytical techniques. These techniques are in the vein of the direct Lyapunov method, and the paper means to show that its potential has not been fully employed in this area up to now. This focus on the second Lyapunov method is off the main avenue of previous research, whose prevailing preferences in evaluation of the topological entropy and the likes were for Lyapunov exponents or similar instruments related to the linearized system (the first Lyapunov method) or for approaches not associated with Lyapunov. Our interest to the second Lyapunov method is partly inspired by its wide acceptance in control practice and its support by computationally efficient and theoretically well-founded algorithms based on either Linear Matrix Inequalities or the Kalman-Szegö lemma (stated in Appendix A). The proposed techniques develop some ideas from nonlinear dynamics (Boichenko & Leonov, 1995; Boichenko, Leonov, & Reitman, 2005; Douady & Osterlé, 1980; Katok, 2007; Leonov, 1991, 2008; Temam, 1997; Yomdin, 1987) and are partly based on extending our study of continuous-time plants (Pogromsky & Matveev, 2010, 2011) on systems operating in discrete time.

On the side of control theory, the paper deals with the state estimation issue for deterministic though maybe, chaotic, nonlinear systems. This is of interest in its own right and since many control problems can be typically solved if a reliable state estimate is available. The second contribution of the paper is concerned with the question: what is the minimum bit-rate of data communication from the sensor to observer which makes it possible to protect a once achieved observation accuracy from a drastic regression, or even to improve it. This question gives rise to new entropy-like characteristics of the plant, though we do not refer to them as "entropies" by retaining the name of "data rate limits".

We show that these new characteristics enhance the topological entropy; whereas the proposed techniques in fact estimate exactly them, which throws extra light on the status and scope of the techniques themselves. Their potential is demonstrated by analytically finding a closed-form expression for the exact values of these global entropy-like data rate limits for a ball vertically bouncing on a sinusoidally vibrating table, which is among the simplest physical systems that exhibit remarkably complex chaotic behaviors (Cao, Judd, & Mees, 1997; Tufillaro, Abbott, & Reilly, 1992), including existence of strange attractors (Clark, Martin, Moore, & Jesse, 1995; Mello & Tuffilaro, 1987). This system not only enjoys much attention in general study of nonlinear dynamics as a basic test example but also is of interest by its own right for various engineering applications, e.g., those concerned with jackhammers, vibro-transporters, vibratory feeders, etc. (Tufillaro et al., 1992). Our respective computation also takes the benefit of Kalman-Szegö lemma. To the best of the knowledge of the authors, the exact value of the topological entropy of the bouncing ball system still remains an open issue even for special numerical values of the parameters. Another illustration is concerned with the celebrated Hénon system (Hénon, 1976). Here the results are not so completed, partly due to the lack of analytical knowledge about invariant sets of this system. To the best of the knowledge of the authors, the paper improves the previously known closed-form estimates of the topological entropy for these two systems.

The paper is organized as follows. Section 2 offers problem setup and introduces basic concepts. Necessary and sufficient data rate bounds are reported in Sections 3 and 4, respectively. In Sections 5 and 6, they are applied to the bouncing ball and Hénon systems, respectively. In Section 7, the so obtained bounds are experimentally verified, whereas Section 8 offers brief conclusions. A complementary material and technically demanding proofs are concentrated in appendices.

The following notations are adopted in the paper: $\mathbb{Z}$ — the set of integers; $[k_1 : k_2] := \{k \in \mathbb{Z} : k_1 \leq k \leq k_2\}$; $\mathbb{Z}_+ := \{k \in \mathbb{Z} : k \geq 0\}$; $\langle \cdot ; \cdot \rangle$ and $\| \cdot \|$ — the standard inner product and Euclidean norm in $\mathbb{R}^n$, respectively; "ε-ball" — ball with the radius ε.

2. State estimation problem and basic definitions

We consider a discrete-time invariant nonlinear system

$$x(t+1) = \phi[x(t)], \quad t \in \mathbb{Z}_+,\ x(0) \in K, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, and $\phi : \mathbb{R}^n \to \mathbb{R}^n$ and $K \subset \mathbb{R}^n$ are a given continuous map and a nonempty compact set of initial states, respectively. The objective is to generate an accurate estimate $\widehat{x}(t)$ of $x(t)$ at a remote location, where direct observation of the state is impossible.

The only way to deliver information from the sensor to this location is via a discrete communication channel. At time t, it carries a discrete-valued symbol $e(t)$. So to be transmitted, continuous-valued sensor readings $x(t)$ should be first translated into such symbols. This is done by a special device, referred to as the *coder*. Its outputs are transmitted for the unit time across the channel to a *decoder* that produces an estimate $\widehat{x}(t) \in \mathbb{R}^n$ of the current state $x(t)$; see Fig. 1. Thus the *observer* is constituted by the

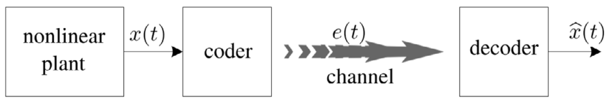

Fig. 1. State estimation over a finite capacity channel.

coder and decoder, which are described by the following respective equations:

$$e(t) = \mathcal{C}[t, x(0), \dots, x(t)|\widehat{x}(0), \delta], \quad t \geq 0,$$
$$\widehat{x}(t) = \mathcal{D}[t, e(0), \dots, e(t-1)|\widehat{x}(0), \delta], \quad t \geq 1. \tag{2}$$

Here $\widehat{x}(0)$ is an initial estimate, and $\delta > 0$ is its accuracy:

$$\|x(0) - \widehat{x}(0)\| < \delta. \tag{3}$$

We assume that both coder and decoder are initially given common values of $\widehat{x}(0)$ and δ, and are aware of $\phi(\cdot)$ and K.

A *trivial observer* is associated with the decoder that does not utilize the benefits of getting data about the state via the channel, but merely generates the trajectory of (1) starting from $\widehat{x}(0)$. Our interest is focused on the case where this solution is not satisfactory: even a very small initial discrepancy between the state and this trivial estimate may widely expand as time progresses. This for example, holds if K is an invariant chaotic set, which situation is our major interest. *How much data should be communicated per unit time to avoid a detrimental degrade of the estimation accuracy?*

To circumstantiate this question, we first note that it addresses an averaged amount of data. So we borrow the model of communication channel from Matveev and Savkin (2005), which allows for unsteady instant data rate, transmission delays and dropouts (see Matveev & Savkin, 2009, Sect. 3.4 for details). Specifically, we assume that within any time interval of duration r, no less than $b_-(r)$ but no more than $b_+(r)$ bits of data can be transmitted across the channel, and also that the respective per unit time rates converge to a common value c called the *channel capacity*:

$$r^{-1}b_-(r) \to c \quad \text{and} \quad r^{-1}b_+(r) \to c \quad \text{as } r \to \infty. \tag{4}$$

We also introduce three notions of successful observation.

Definition 1. For the observer (2), $\varepsilon > 0$ is called the anytime exactness of observation of the system (1) if there exists $\delta(\varepsilon, K) > 0$ such that

$$\|x(t) - \widehat{x}(t)\| \leq \varepsilon, \quad \forall t \in \mathbb{Z}_+ \tag{5}$$

whenever (3) holds with $\delta = \delta(\varepsilon, K) > 0$ (i.e., the initial error is small enough) and $x(0), \widehat{x}(0) \in K$.

However, this leaves room for critical degradation of accuracy: $\varepsilon \gg \delta$. Now we introduce observers for which a violent degradation is excluded: the anytime exactness is at worst proportional to the initial accuracy.

Definition 2. The observer (2) is said to *regularly observe the system* (1) if there exist $\delta_* > 0, G > 0$ such that for all $\delta < \delta_*$, all $x(0), \widehat{x}(0) \in K$ satisfying (3), and all $t \geq 0$, the estimation error obeys the inequality $\|x(t) - \widehat{x}(t)\| \leq G\delta$.

The last class is constituted by observers that not only avoid violent regress of accuracy but also restore and exponentially improve the initial accuracy as time progresses.

Definition 3. The observer is said to *finely observe the system* (1) if there exist $\delta_* > 0$, $G > 0$, and $g \in (0, 1)$ such that for all $\delta < \delta_*$ and all $x(0), \widehat{x}(0) \in K$ satisfying (3), the following inequality holds

$$\|x(t) - \widehat{x}(t)\| \leq G\delta g^t, \quad \forall t \geq 0. \tag{6}$$

Definition 4. The system (1) is said to be (i) *observable*, (ii) *regularly, and* (iii) *finely observable via a communication channel* if the following respective claims hold (where the observers are meant to operate via the channel at hands):

(i) For any $\varepsilon > 0$, there exists an observer (2) that observes the system (1) with anytime exactness ε;
(ii, iii) There exists an observer (2) that (ii) regularly or (iii) finely observes the system (1).

Clearly (i) $\Leftarrow$ (ii) $\Leftarrow$ (iii).

What channel rate c is needed for the system to be observable in each of these senses?

Though intuitively evident, the fact that the rate (4) is a comprehensive figure of merit for channel evaluation and that the larger the rate the better, will be rigorously shown (see Lemma 18). So our question is about the infimum $\mathcal{R}(\phi, K)$ of the needed rates c. Here $\mathcal{R}$ is supplied with the index $_{\text{o}}$, $_{\text{ro}}$, and $_{\text{fo}}$ in the cases (i), (ii), and (iii), respectively. Thus $\mathcal{R}_{\text{o}}(\phi, K) = \inf\{c\colon$ the system (1) is observable via any communication channel with capacity $c\}$. (By Lemma 18, "any" can be replaced by "some" in this formula.) The quantities $\mathcal{R}_{\text{ro}}(\phi, K)$ and $\mathcal{R}_{\text{fo}}(\phi, K)$ are given by the same formula, where "observable" is replaced by "regularly observable" and "finely observable", respectively. The arguments ϕ, K stress that these quantities are completely determined by the system (1), thus providing its entropy-like quantitative characterization. It is clear that

$$\mathcal{R}_{\text{o}}(\phi, K) \leq \mathcal{R}_{\text{ro}}(\phi, K) \leq \mathcal{R}_{\text{fo}}(\phi, K). \tag{7}$$

An answer to the above question will be offered in the form of separate necessary and sufficient bounds on the quantities from (7). These bounds may coincide, thus providing an exhaustive (necessary and sufficient) answer, as will be shown in Section 5 for an important classic chaotic system. However, we have no guarantees that these bounds coincide in general.

Inequality (3) means that initially the decoder knows the state $x(0)$ somewhat well. The objective is to maintain this circumstance over time, or even to improve it, in spite of possible unstable dynamics of the plant. This objective is inevitable and crucial for state estimation task. However some popular problem setups assumed less initial knowledge (see e.g., Matveev & Savkin, 2009, and the literature therein), and the additional objective to first reach a certain accuracy was set up. In this paper, we focus on only the first objective in order to highlight the relevant key phenomena by reducing technicalities in the face of paper length limit. The second objective can be presumably achieved in the vein of well-understood "zoom-out" and "zoom-in" procedures (Matveev & Savkin, 2009); elaboration of related details will be separately reported.

According to the classic information theory (Fano, 1961), a common model may describe transmission of data across both space and time. In the second incarnation, (2) may be viewed as machinery for on-line documenting an ongoing trajectory of (1) into a sequence of finite-bit records $e(t)$, where the second equation in (2) describes data restoration, and documenting involves data approximation and suppression (see e.g., Donarowicz, 2011, Sect. 0.3.5). Then the issue of this paper takes the form: *How many bits per unit time should be recorded in order that the trajectory can be accurately restored?*

Let ϕ^k stand for the kth iterate of the map ϕ, and

$$K(t) := \phi^t(K), \qquad K^\infty := \bigcup_{t \geq 0} K(t). \tag{8}$$

Remark 5. The above discussion is literally extended on the case where the system (1) acts in an arbitrary metric space $\mathfrak{M}$, not necessarily $\mathbb{R}^n$. The only alteration of the text is replacement of $\mathbb{R}^n$ by $\mathfrak{M}$ and replacement of the Euclidean distance $\|x - y\|$ by the distance (metric) $d[x, y]$ in $\mathfrak{M}$.

Consider two systems (1), the ith of them ($i = 1, 2$) is associated with $\phi_i(\cdot)$ and K_i and acts in a metric space $\mathfrak{M}_i$. These systems are said to be *Lipschitz conjugate* (Pinto, Rand, & Ferreira, 2009, Sec. 2.1) if there exists an invertible map $h : \mathfrak{M}_1 \to \mathfrak{M}_2$ such that $\phi_1 = h^{-1} \circ \phi_2 \circ h$, $K_2 = h(K_1)$, and the maps h and h^{-1} are Lipschitz on K_1^∞ and K_2^∞.

Remark 6. (i) It is easy to see by inspection that any quantity from (7) is invariant with respect to replacement of the system by its Lipschitz conjugate. In particular, they are not altered by a one-to-one change of the variable $x \mapsto y = h(x)$ if h and h^{-1} are globally Lipschitz.
(ii) Since any two norms $\|\cdot\|_1, \|\cdot\|_2$ in $\mathbb{R}^n$ are equivalent $\mu_1\|x\|_1 \le \|x\|_2 \le \mu_2\|x\|_1\ \forall x$ (where $\mu_i > 0$), (i) shows that for any system (1) in $\mathbb{R}^n$, the quantities from (7) are independent of the norm.

Let ϕ be invertible and, along with ϕ^{-1}, be Lipschitz on K^∞. Putting $h := \phi^r, \phi_1 := \phi_2 := \phi, K_1 := K, K_2 := K(r)$ in (i) of Remark 6 shows that the quantities in (7) are invariant under replacement $K \mapsto K(r)$. This is of especial interest if K is positively invariant and lies in the basin of attraction of an attractor K_* so that $K(r) \approx K_*$ for $r \approx \infty$.

For a continuous map $\phi(\cdot)$ in $\mathbb{R}^n$, a general example of a compact and positively invariant set is given by

$$K_d := \{x = x(0) : \|x(t)\| \le d,\ \forall t \ge 0\} \tag{9}$$

for any $d > 0$. This set is non-empty for at least some d if and only if (1) has bounded trajectories. Moreover, any such trajectory lies in K_d with a properly chosen d.

3. Topological entropy and necessary bounds on the channel rate

Our first necessary bound is in terms of the topological entropy of the system. So we first recall this concept. Consider a system (1) in a metric space $\mathfrak{M}$ with a metric $d(\cdot,\cdot)$. For $a \in \mathfrak{M}$, the symbol $x(t, a), t \in \mathbb{Z}_+$ denotes the solution of (1) starting from $x(0, a) = a$, and $X_k(a) := \{x(0, a), \dots, x(k, a)\} \in \mathfrak{M}^{k+1}$ is its initial piece. The second argument in $x(t, a)$ can be omitted if it is apparent from the context.

Definition 7 (*Savkin, 2006*). For $k \in \mathbb{Z}_+, \varepsilon > 0$, a set $Q \subset \mathfrak{M}^{k+1}$ is said to be (k, ε)-*spanning* for a compact set $K \subset \mathfrak{M}$ if for any $a \in K$ there exists $\{x_0^\star, \dots, x_k^\star\} \in Q$ such that

$$\max_{t=0,\dots,k} d\left[x(t, a); x_t^\star\right] < \varepsilon. \tag{10}$$

The smallest possible size (number of elements) of Q with the described property is denoted as $q(k, \varepsilon)$. Here $q(k, \varepsilon) < \infty$ since there exists a finite set Q with this property. Indeed, a standard argument based on continuity of ϕ and compactness of K suggests that the image $X_k(K)$ of K under the map $a \mapsto X_k(a)$ is compact as well. So this image is totally bounded with respect to the standard metric $d_{k+1}[X^{(1)}, X^{(2)}] := \max_{t=0,\dots,k} d[x_t^{(1)}, x_t^{(2)}], \forall X^{(i)} = \{x_0^{(i)}, \dots, x_k^{(i)}\}$ in $\mathfrak{M}^{k+1}$ (Copson, 1968, Sec. 50); in other words, $X_k(K)$ can be covered by finitely many ε-balls. It remains to note that their centers constitute the required set Q.

The *topological entropy* of the system (1) on K is defined as (Donarowicz, 2011; Katok, 2007)

$$H(\phi, K) := \lim_{\varepsilon \to 0} \overline{\lim_{k \to \infty}} \frac{1}{k+1} \log_2 q(k, \varepsilon). \tag{11}$$

Theorem 8. *The following inequality holds $H(\phi, K) \le \mathcal{R}_o(\phi, K)$. Moreover, if the set K is positively invariant,*

$$H(\phi, K) = \mathcal{R}_o(\phi, K). \tag{12}$$

The proof of this theorem will be given in Appendix D.

Computation of the topological entropy is of substantial interest in many areas. For a real n-dimensional Hilbert space $\mathfrak{M} = \mathfrak{H}$, a linear operator A in $\mathfrak{H}$, a vector $b \in \mathfrak{H}$, and a compact set K with non-empty interior, the entropy of the map $\phi(x) = Ax + b$ is given by Matveev and Savkin (2009, Ch. 2)

$$H(\phi, K) = H(A) := \sum_{i=1}^n \log_2 \max\{|\lambda_i|; 1\}, \tag{13}$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A. Unfortunately, not only computation of the topological entropy but even its estimation is a highly intricate matter for nonlinear maps, which undermines the utility of (12) as a basis for estimating $\mathcal{R}_o$. Moreover in the role of a systematic estimation tool, *formula (12) can be rather recommended for upper estimating the topological entropy based on constructive upper estimates of $\mathcal{R}_{fo}(\ge \mathcal{R}_o)$ that will be reported in Section 4.*

From (12) and results of, e.g., (Matveev & Savkin, 2009, Sect. 3.5) on exponential stabilizability of linear plants, we see that in the linear context of (13), $\mathcal{R}_o(\phi, K) = \mathcal{R}_{ro}(\phi, K) = \mathcal{R}_{fo}(\phi, K) = H(A)$.

Unlike (12), our next estimate is constructive, does not concern $\mathcal{R}_o$, and deals with a continuously differentiable system (1) in a connected Riemannian manifold $\mathfrak{M}$. The differential $\phi'(x)$ acts in the tangent space at the point x.

Theorem 9. *Let x_0 be a hyperbolic equilibrium of ϕ and let the intersection of its unstable manifold $\mathfrak{M}^{unst}(x_0)$ with a small enough vicinity of x_0 lie in K. Then*

$$\mathcal{R}_{ro}(\phi, K) \ge H\big[\phi'(x_0)\big]. \tag{14}$$

The proof of this theorem will be given in Appendix D.

Theorem 9 remains true if $\mathfrak{M}^{unst}(x_0)$ is any smooth submanifold of $\mathfrak{M}$ that contains x_0 and at x_0, is tangent to the unstable subspace of $\phi'(x_0)$. By putting $\phi := \phi^k$ in (14), $\mathcal{R}_{ro}(\phi, K)$ can be estimated by means of an equilibrium x_0 of an iterate ϕ^k thanks to (B.2) from Appendix B. Such equilibria are nothing but points from periodic orbits of (1). In this case, the right-hand side of (14) is concerned with the monodromy operator of the orbit.

4. Sufficient bound on the channel rate and constructive estimate of the topological entropy

Now we disclose a constructive requirement to the channel capacity that is sufficient for fine observability of systems in $\mathbb{R}^n$. In doing so, we adopt the following.

Assumption 10. The map ϕ is continuously differentiable and its Jacobi matrix

$$A(x) := \phi'(x) \tag{15}$$

has the following properties:

(i) $A(\cdot)$ is bounded on K^∞;
(ii) $A(\cdot)$ is uniformly continuous near K^∞: for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in K^\infty, \quad \|x - y\| < \delta \Rightarrow \|A(x) - A(y)\| < \varepsilon.$$

Here (i) and (ii) are necessarily true if K^∞ is bounded; e.g., this holds if the compact set K is positively invariant.

In the following key assumption, $\Delta v(x) := v[\phi(x)] - v[x]$ is the variation of a function $v : \mathbb{R}^n \to \mathbb{R}$ caused by the one-step evolution of the system (1).

Assumption 11. There exist continuous and bounded on K^∞ functions $v_i : \mathbb{R}^n \to \mathbb{R}$, constants $\Lambda_i \geq 0$, $i \in [1 : n]$, and a positive definite $n \times n$-matrix $P = P^\top$ such that

$$\Delta v_d(x) + \sum_{i=1}^{d} \log_2 \lambda_i(x) \leq \Lambda_d, \quad \forall x \in K^\infty \tag{16}$$

for any $d \in [1 : n]$, where $\log_2 0 := -\infty$ and $\lambda_1(x) \geq \cdots \geq \lambda_n(x) \geq 0$ are the roots of the algebraic equation

$$\det \left[A(x)^\top PA(x) - \lambda P\right] = 0 \tag{17}$$

repeated in accordance with their algebraic multiplicities.

By introducing the positive definite square root $P^{1/2}$, putting $S := P^{-1/2}$, and taking into account the identity

$$\begin{aligned}\det \left[A(x)^\top PA(x) - \lambda P\right] &= \det \left(S \left[A(x)^\top PA(x) - \lambda P\right] S\right) \det P \\ &= \det P \det \left[SA(x)^\top PA(x)S - \lambda I\right],\end{aligned}$$

we see that the roots of (17) are the eigenvalues of the symmetric and nonnegative definite matrix $SA(x)^\top PA(x)S$. Hence first, these roots are real and nonnegative, and their number equals n, as was tacitly taken for granted in Assumption 11, and second, the following equation holds:

$$\begin{aligned}\lambda_1(x) \cdots \lambda_n(x) &= \det SA(x)^\top PA(x)S \\ &= \left[\det P^{-1/2}\right]^2 \det P[\det A(x)]^2 = [\det A(x)]^2. \end{aligned} \tag{18}$$

If the set K^∞ is bounded, the "boundedness" requirement can be dropped in Assumption 11, since it is necessarily fulfilled for continuous functions $v_i(\cdot)$.

The following is the main result of the section.

Theorem 12. *Let Assumptions 10 and 11 hold. Then*

$$\mathcal{R}_{fo}(\phi, K) \leq \Lambda_\star := \frac{\max_d \Lambda_d}{2}. \tag{19}$$

The description of the respective observer and the proof of this theorem will be given in Appendix E. That observer is fully applicable whenever Theorem 12 is used to justify observability, like in Sections 5 and 6. In this observer, the matrix P from Assumption 11 in fact regulates the fineness of quantization along various directions since the employed quantization is in terms of ε-balls with respect to the norm $\sqrt{\langle Px; x\rangle}$. In the opinion of the standard Euclidean space $\mathbb{R}^n$, these balls are ellipsoids whose principal axes are aligned with the eigendirections of P, with respective semi-axes being inversely proportional to the square roots of the associated eigenvalues σ_j. Thus quantization along the jth eigendirection is as fine as $\varepsilon/\sqrt{\sigma_j}$; and the fineness of quantization is inhomogeneous with respect to directions in $\mathbb{R}^n$. For linear systems, inhomogeneity of quantization fineness is known to be nearly compulsory for successful observation at data rates close to the lower limit (Baillieul, 2002). In the next section, we show that the proposed "inhomogeneous" observer is able to ensure observation of complex chaotic nonlinear systems at rates that are as close to the lower limit as desired. In a simplified form, some ideas underlying the design of the afore-discussed observer are partly illuminated in Section 7.1; however, balls with respect to the norm $\max_i |x_i|$ of the Euclidean space $\mathbb{R}^n$ (i.e., cubes) are used there.

Combining (19) with (7) and (12) yields the following new constructive estimate of the topological entropy:

$$H(\phi, K) \leq \Lambda_\star.$$

Our Theorems 8, 9 and 12 deliver a consistent message that the capacity is an exhaustive characteristics of the ability of the channel to serve successful observation of the system, whereas the other channel properties do not matter.

Assumption 11 is in terms of auxiliary functions satisfying inequalities in spirit of the method of Lyapunov functions. So utilization of the presented results shares some feature characteristics of this method. In general, there is no constructive way to find such functions, though certain approaches are known for special classes of systems. For example, for systems representable in the Lur'e form (a linear part in a feedback interconnection with a "nonlinearity" satisfying quadratic constraints), a seminal algebraic result usually referred to as the Kalman-Szegö lemma (see Appendix A) can be employed. This lemma gives constructive necessary and sufficient conditions for the solvability of linear matrix inequalities (LMI's) that arise when employing the method of Lyapunov functions. This opens the door to even closed-form analytical and exact computation by using Theorems 8 and 9, and 12 in some cases, as will be demonstrated in the next section. In any case, LMI problems can be solved to within a pre-specified error tolerance in a polynomial time (Blondel & Tsitsiklis, 2000; Boyd, Ghaoui, Feron, & Balakrishnan, 1994).

We close the section with two facts, which will be used in the next two sections. The first of them may be useful to simplify the state space of the system when estimating the quantities from (7). It states that (i) in Remark 6 partly remains true if the map h is invertible not globally but only locally. The second fact shows that regular and fine observabilities are nearly coexistent. Their proofs are given in Appendix C.

Lemma 13. *Consider two systems (1), the ith of them ($i = 1, 2$) is associated with $\phi_i(\cdot)$ and K_i and acts in a metric space $\mathfrak{M}_i$. Let there exist a Lipschitz on K_1^∞ map $h : \mathfrak{M}_1 \to \mathfrak{M}_2$ such that $\phi_2 \circ h = h \circ \phi_1$, $h(K_1) = K_2$, and any $x_2 \in K_2$ have a pre-image $x_1 \in K_1$, $h(x_1) = x_2$ such that h is one-to-one from some neighborhood $V(x_2)$ of x_1 in K_1 onto the intersection of K_2 with the δ_*-ball centered at x_2, with the inverse map being Lipschitz with the constant L. Here the constants $\delta_* > 0$ and L should not depend on x_2. Then for the second system ($i = 2$), any quantity from (7) does not exceed the respective quantity of the first system.*

Lemma 14. *If K is positively invariant $\phi(K) \subset K$, relations (7) can be specified as follows:*

$$\mathcal{R}_{ro}(\phi, K) = \mathcal{R}_{fo}(\phi, K). \tag{20}$$

5. Harmonically forced bouncing ball

This is one of the simplest physical systems that exhibit remarkably complex chaotic behaviors (Cao et al., 1997; Tufillaro et al., 1992), including existence of strange attractors (Clark et al., 1995; Mello & Tuffilaro, 1987). The system is described by the following equations (Tufillaro et al., 1992):

$$\begin{aligned} x_1(t+1) &= x_1(t) + x_2(t), \\ x_2(t+1) &= \alpha x_2(t) - \beta \cos[x_1(t) + x_2(t)]. \end{aligned} \tag{21}$$

Here $x_1(t)$ and $x_2(t)$ are the phase of the table and the scaled velocity of the ball at the tth impact, respectively, $\alpha \in (0, 1)$ is the coefficient of restitution, and $\beta = 2\omega^2(1 + \alpha)A/g > 0$, where ω and A are the angular frequency and amplitude of the table oscillation, respectively, and g is the gravitational acceleration. Eqs. (21) are invariant under the transformation $x_1 \mapsto x_1 + 2\pi k$, $k \in \mathbb{Z}$; so the phase space of the system is traditionally viewed as the cylinder $\mathfrak{C} = S_0^1 \times \mathbb{R} = \{(s, x_2)\}$. Here $S_0^1 \ni s$ is the unit circle in the plane $\mathbb{R}^2$ centered at the origin, and $s = (\cos x_1, \sin x_1)^\top$. The cylinder is treated as the two-dimensional Riemannian manifold with the standard Riemannian metric.

The system (21) has a trapping region $K_{tr} := S_0^1 \times [-\beta(1-\alpha)^{-1}; \beta(1-\alpha)^{-1}]$: this is a positively invariant compact set that attracts all trajectories. Though K_{tr} draws most interest, we examine a more general set and offer a closed-form expression of the data rates $\mathfrak{R}_{ro}, \mathfrak{R}_{fo}$ necessary and sufficient for reliable global observation of this system. In other words, we revert to the problem of state estimation posed in Section 2 and illustrated in Fig. 1 and focus on its particular case where the system (1) ("nonlinear plant" in Fig. 1) is given by (21). The following theorem answers the question: communication of how many bits per unit time across the channel in Fig. 1 is sufficient and necessary in order that reliable state estimation be feasible at the decoder site?

Theorem 15. *Let $0 < \alpha < 1$, $\beta > 0$. For the bouncing ball system in the cylindrical space $\mathfrak{C}$ and any set of the form $K = S_0^1 \times [-\eta, \eta]$, $\eta > 0$, the following equations hold:*

$$\begin{aligned}\mathfrak{R}_{ro} &= \mathfrak{R}_{fo} = \aleph \\ &:= \log_2\left(1+\alpha+\beta+\sqrt{(1+\alpha+\beta)^2-4\alpha}\right)-1. \end{aligned} \tag{22}$$

Proof. It is straightforward to check that the system has an equilibrium $\boldsymbol{e}$ associated with $x_1 = \pi/2, x_2 = 0$, where the differential of the dynamic map ϕ (acting in the respective plane tangent to the cylinder) has the following eigenvalues

$$\lambda_\pm := \left[\alpha+\beta+1 \pm \sqrt{(\alpha+\beta+1)^2-4\alpha}\right]/2.$$

It is also easy to see that $\lambda_+ > 1$, $\lambda_- \in (0,1)$ and so $H[\phi'(\boldsymbol{e})] = \aleph$ by (13). Hence Theorem 9 yields that

$$\mathfrak{R}_{ro} \geq \aleph. \tag{23}$$

To obtain a somewhat converse inequality $\mathfrak{R}_{fo} \leq \aleph$, we employ Lemma 13, where "system 2" is our bouncing ball system in the cylindrical space and "system 1" is the system (21) in the Euclidean phase space $\mathbb{R}^2$, considered in conjunction with the compact set $K_1 := [-2\pi, 2\pi] \times [-\eta, \eta]$ and the map $h(x_1, x_2) = \left[(\cos x_1, \sin x_1), x_2\right] \in S_0^1 \times \mathbb{R}$. Since the requirements of Lemma 13 are evidently fulfilled with $L = 1$ and $\delta_* = \pi/2$, transition from system 2 to system 1 does not decrease $\mathfrak{R}_{fo}$. So it suffices to justify the desired inequality for the system (21) in the Euclidean space.

To this end, we perform the following one-to-one change of the state variables in (21):

$$\begin{aligned} z_1 &= -\alpha x_1 + x_2 + \beta \cos x_1 \\ z_2 &= x_1, \end{aligned} \tag{24}$$

thus shaping (21) into

$$\begin{aligned} z_1(t+1) &= -\alpha z_2(t) \\ z_2(t+1) &= z_1(t) + (1+\alpha) z_2(t) - \beta \cos z_2(t). \end{aligned} \tag{25}$$

Clearly, the map (24) and its inverse are globally Lipschitz; so this change does not alter $\mathfrak{R}_{fo}$ by (i) in Remark 6.

The system (25) will be examined by means of Theorem 12. To this end, we first note that the Jacobian of the right hand side in (25) is given by

$$A(z_2) = \begin{pmatrix} 0 & -\alpha \\ 1 & 1+\alpha+\beta \sin z_2 \end{pmatrix}.$$

Since $\det A(z_2) = \alpha \in (0,1)$, (18) implies that the smallest solution λ_2 of Eq. (17) lies in $[0,1)$ for any $P > 0$. So to apply Theorem 12, it suffices to pick $\upsilon_1(\cdot) := \upsilon_2(\cdot) := 0$ and find a positive definite matrix $P = P^\top > 0$ and a z_2-invariant upper bound $\mu > 0$ on the largest solution of (17); then $\Lambda_1 := \Lambda_2 := \log_2 \mu$ in (16) and (19). By Lemma 8.1 in Pogromsky and Matveev (2011), the required bound is given by any $\mu > 0$ such that

$$\xi^\top A(z_2)^\top P A(z_2) \xi \leq \mu \xi^\top P \xi, \quad \forall \xi \in \mathbb{R}^2, z_2 \in \mathbb{R}. \tag{26}$$

Thus we are interested in finding the least μ for which there exists a positive definite matrix P such that (26) holds.

Multiplying (26) by μ^{-1} shows that this problem is equivalent to existence of a common quadratic Lyapunov function for a family of linear systems with the state $\xi \in \mathbb{R}^2$

$$\xi(t+1) = \mu^{-1/2} A(z_2) \xi(t), \tag{27}$$

where the free parameter z_2 defines a particular system. This existence implies that the system (27) is marginally stable for any z_2. So examining $z_2 := 0, \pi/2$ shows that necessarily $\mu > 1$. To make the problem tractable for such μ's, we follow the lines of Megretski and Rantzer (1997) and describe the uncertainty injected by z_2 by a quadratic constraint. To this end, we rewrite (27) as an interconnection of the "certain" linear system

$$\begin{aligned} \xi(t+1) &= A_c \xi(t) + B w(t) \\ \zeta(t) &= C \xi(t) \end{aligned} \tag{28}$$

with the scalar input $w(k)$ and output $\zeta(k)$ and

$$A_c = \mu^{-1/2} \begin{pmatrix} 0 & -\alpha \\ 1 & 1+\alpha \end{pmatrix}, \qquad B = \mu^{-1/2} \begin{pmatrix} 0 \\ \beta \end{pmatrix}, \qquad C = (0 \quad 1),$$

and the "uncertain" feedback $w(t) = \zeta(t) \sin z_2$. Then we note that the last equation implies

$$F[\zeta(t), w(t)] \geq 0, \quad \text{where } F(\zeta, w) := \zeta^2 - w^2, \tag{29}$$

and so (26) holds whenever for all ξ, w,

$$(A_c \xi + B w)^\top P (A_c \xi + B w) - \xi^\top P \xi \leq -F(C\xi, w). \tag{30}$$

By the Kalman-Szegö lemma (see Appendix A), existence of a positive definite matrix P for which (30) is true (and so (26) also holds) is equivalent to the following inequality

$$\left| \frac{\beta \mu^{-1/2} s}{s^2 - (\alpha+1)\mu^{-1/2} s + \alpha \mu^{-1}} \right| \leq 1, \quad \forall s \in S_0^1, \tag{31}$$

where the elements $s \in S_0^1$ are treated as complex numbers. Here the denominator $s^2 - (\alpha+1)\mu^{-1/2}s + \alpha\mu^{-1}$ has two real roots $s_1 = \alpha\mu^{-1/2}$ and $s_2 = \mu^{-1/2}$. Hence $|s^2 - (\alpha+1)\mu^{-1/2}s + \alpha\mu^{-1}| = |s - s_1| \cdot |s - s_2|$ for all s and therefore $|s - s_1| \cdot |s - s_2|$ attains its minimum over $s \in S_0^1$ at the point $s = 1$ since both s_1, s_2 are positive. It follows that (31) is satisfied if and only if

$$\left| \frac{\beta \mu^{-1/2}}{1 - (\alpha+1)\mu^{-1/2} + \alpha \mu^{-1}} \right| \leq 1. \tag{32}$$

It is easy to see that for $\mu > 1$, (32) is equivalent to

$$\mu \geq \mu_0 := \left[\frac{1}{2}\left(1+\alpha+\beta+\sqrt{(1+\alpha+\beta)^2-4\alpha}\right)\right]^2,$$

and the matrix A_c is Shur. Thus the Kalman-Szegö lemma (Lemma 17 in Appendix A) guarantees that (26) does hold with $\mu := \mu_0$ and some $P = P^\top > 0$. Then (19) is true with $\Lambda_1 := \Lambda_2 := \log_2 \mu_0$ by Theorem 12, which means that $\mathfrak{R}_{fo} \leq \aleph$. By invoking (7) and (23), we arrive at (22). ■

6. Hénon system

This is one of the major and popular objects of study in the general theory of dynamical systems. By its own right, the system is rather simple and described by the equations

$$\begin{cases} x(t+1) = a + b y(t) - x^2(t) \\ y(t+1) = x(t) \in \mathbb{R}, \end{cases} \tag{33}$$

where $a > 0$ and $b \in (0, 1)$ are parameters. At the same time, it produces extraordinarily complex phenomena upon iterations. Its most studied sample was introduced in Hénon (1976) and corresponds to $a = 1.4$, $b = 0.3$. In this case, computer simulations reveal both bounded and unbounded trajectories and provide an evidence that the system exhibits a chaotic behavior and has a strange attractor with a fractional Hausdorff dimension, which attractor is a fractal representable as the product of a one dimensional manifold by a Cantor set.

The system (33) has two equilibria $\boldsymbol{e}_\pm = (x_\pm, x_\pm)$, where

$$x_\pm = \frac{b-1 \pm \sqrt{(b-1)^2 + 4a}}{2}, \quad x_- < 0,\ x_+ > 0.$$

Here $\boldsymbol{e}_-$ is a saddle point, whereas $\boldsymbol{e}_+$ is a saddle point if $a > 3/4(1-b)^2$, and is stable otherwise. For the canonical parameters $a = 1.4$, $b = 0.3$, the first case holds and the strange attractor is covered by a closed-form positively invariant quadrilateral K, which in turn lies in the basin of attraction of this attractor and contains $\boldsymbol{e}_+$ in its interior (Hénon, 1976). Another example of a compact positively invariant set is concerned with arbitrary a, b and is given by (9).

The next theorem answers the question: communication of how many bits per unit time across the channel in Fig. 1 is sufficient and necessary in order that reliable state estimation be feasible at the decoder site in the case where in Fig. 1, the "nonlinear plant" is the Hénon system?

Theorem 16. *For any compact positively invariant set K of the Hénon map $\phi(\cdot)$, the following upper estimates hold:*

$$H(\phi, K) \le \mathcal{R}_{ro}(\phi, K) = \mathcal{R}_{fo}(\phi, K) \le \log_2\left(\sqrt{x_-^2 + b} - x_-\right). \tag{34}$$

If $a > \frac{3}{4}(1-b)^2$, like for the canonical parameters, and $\boldsymbol{e}_+$ lies in the interior of K, the lower estimate is also valid

$$\mathcal{R}_{ro}(\phi, K) \ge \log_2\left(\sqrt{x_+^2 + b} + x_+\right). \tag{35}$$

Finally, if the intersection of the unstable manifold of $\boldsymbol{e}_-$ with a sufficiently small vicinity of $\boldsymbol{e}_-$ lies in K, the last inequality in (34) is true with $\le$ replaced by $=$.

Proof. The Jacobi matrix $A(x) := \phi'(x) = \begin{pmatrix} -2x & b \\ 1 & 0 \end{pmatrix}$ has two real eigenvalues $\lambda_\pm(x) := -x \pm \sqrt{x^2 + b}$. An elementary analysis shows that $\lambda_-(x_-), \lambda_+(x_+) \in (0, 1)$, whereas $\lambda_+(x_-) > 1$, and $\lambda_-(x_+) < -1$ if $a > \frac{3}{4}(1-b)^2$. So (35) and the last claim of Theorem 16 are immediate from Theorem 9 provided that (34) is true. Thus in view of (20) and (12), it suffices to justify the last inequality from (34).

To this end, we use Theorem 12 with $P := \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$. It is straightforward to see that Eq. (17) has two roots

$$\lambda_1 = \left(\sqrt{x^2 + b} + |x|\right)^2 \ge \lambda_2 = \left(\sqrt{x^2 + b} - |x|\right)^2.$$

Inspired by research on the Lyapunov dimension of the Hénon system (Boichenko et al., 2005; Leonov, 2008), we pick the following functions to serve Assumption 11:

$$v_2(x, y) := v_1(x, y) := v(x, y) := \gamma(x + by),$$

where the parameter $\gamma > 0$ will be set up later on. It is easy to see that the variation Δv of $v(\cdot)$ caused by the one-step evolution of the system (33) from the state $[x(k), y(k)]$ depends only on $x(k)$, i.e., $\Delta v = \Delta v[x(k)]$, where

$$\Delta v(x) = \gamma[a + (b-1)x - x^2].$$

The left-hand side of (16) with $d = 1$ equals

$$w(x) = 2\log_2\left(\sqrt{x^2 + b} + |x|\right) + \gamma[a + (b-1)x - x^2]$$

and thanks to $-x^2$, attains a global maximum on $\mathbb{R}$. Since $b - 1 < 0$, the maximizer lies in $(-\infty, 0]$, where the derivative

$$\frac{dw}{dx} = -\frac{2}{\ln 2\sqrt{x^2 + b}} + \gamma(b - 1 - 2x)$$

strictly decreases and has a unique root. By choosing

$$\gamma = \frac{2}{\ln 2(b - 1 - 2x_-)\sqrt{x_-^2 + b}} > 0,$$

we equalize the root with x_- and make x_- the global maximizer of $w(x)$. Since $a + (b-1)x_- - x_-^2 = 0$ it follows that (16) with $d = 1$ holds for

$$\Lambda_1 := w(x_-) = 2\log_2\left(\sqrt{x_-^2 + b} - x_-\right).$$

Since $b \in (0, 1) \Rightarrow \lambda_2 \in (0, 1)$, inequality (16) is also true for $d = 2$ with $\Lambda_2 := \Lambda_1$. Theorem 12 completes the proof of the last inequality in (34). ∎

7. Experimental verification of observability bounds and the simplest observer

The observer proposed in the proof of Theorem 12 is fairly complex. In this section, we show that even a simplest observer with equally fine quantization of all scalar coordinates may occasionally perform rather well. We also use this observer in computer simulations to confirm the theoretical bounds on the data rate obtained in the previous sections.

7.1. Observation scheme

It is based on ideas from Liberzon and Hespanha (2005). Only cubes with edges parallel to the axes of the standard basis of $\mathbb{R}^n$ are considered. The symbol C^ε (with possible other indices) is used to denote a cube with the edge length ε equal to the estimation exactness.

Observer to be examined depends on ε and consists of Coder and Decoder. The both generate a common sequence of cubes $C^\varepsilon(t)$. To this end, they use a common procedure **P** to cover the image $\phi(C^\varepsilon)$ of arbitrary cube C^ε by the minimal number of disjoint cubes C_j^ε. The both are also aware of a common cube $C^\varepsilon(0)$ containing the initial state $x(0)$.

Coder carries out the following operations at time $t \ge 0$:

- **(c.1)** Covers $\phi[C^\varepsilon(t)]$ by cubes $\{C_j^\varepsilon(t)\}_{j=1}^{N(t)} := \mathbf{P}[C^\varepsilon(t)]$;
- **(c.2)** Determines the index $j = j(t)$ of the cube $C_j^\varepsilon(t)$ that contains the predicted next state $\phi[x(t)](= x(t+1))$;
- **(c.3)** Takes this cube as the next entry in the generated sequence of cubes $C^\varepsilon(t+1) := C_{j(t)}^\varepsilon(t)$;
- **(c.4)** Sends this index $j(t)$ to the decoder.

Decoder carries out the following operations at time $t \ge 1$:

- **(d.1)** Receives the index $j(t-1)$;
- **(d.2)** Constructs the set $\{C_j^\varepsilon(t-1)\}_{j=1}^{N(t-1)}$ by copying the relevant actions of the coder carried out at time $t-1$;
- **(d.3)** Generates the current entry $C^\varepsilon(t) := C_{j(t-1)}^\varepsilon(t-1)$ in the sequence of cubes;
- **(d.4)** Takes the center of this cube as the estimate $\widehat{x}(t)$ of the current state $x(t)$.

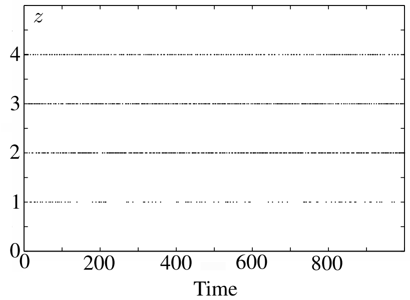

Fig. 2. Number of code symbols needed to track the Hénon system.

For (c.4) to be executable, the instant channel rate should be no less than $\log_2 N(t)$. In the next two subsections, we compare this data rate demand with the theoretical bounds from Sections 3 and 4 in the case of two classic systems from Sections 5 and 6 and by means of computer simulation tests.

The most computationally demanding procedure that should be performed in an identical way by both coder and decoder is over-approximation of the image $\phi(C^\varepsilon)$ of the cube C^ε. Numerical complexity of possible algorithms depends on properties of ϕ; its discussion is a separate issue that lies beyond the scope of this paper. Nevertheless, one can suggest that as ε becomes small enough, this overapproximation can be performed via linearization of ϕ to keep all the involved calculations relatively simple. (An initial idea about this can be drawn from the proof of the subsequent Lemma 24.)

7.2. The Hénon system

The classic parameters $a = 1.4$, $b = 0.3$ were taken in (33). For $\varepsilon = 0.1$, the size $N(t)$ of the channel alphabet requested at step (c.4) was computed for various initial states. A typical plot of $N(t)$ versus the discrete time t is shown in Fig. 2; since $N(t)$ highly oscillates over time, the plot looks like several dashed lines. According to the simulations, the maximal requested rate was 2 bits per unit time (which corresponds to 4 code symbols). This is compatible with the sufficient bound 1.7048 given by Theorem 16 ($\mathcal{R}_{fo} \leq 1.7048$) and (7) ($\mathcal{R}_o \leq \mathcal{R}_{fo}$). The bound 1.7048 is also necessary if the condition from the last sentence in Theorem 16 holds.

7.3. The bouncing ball system

The system (21) was examined for $\alpha = 0.5, \beta = 10$, and $\varepsilon = 0.1$. The maximal requested rate was $\log_2 21$ bits per unit time, as is illustrated by a typical result of computer simulation in Fig. 3. This is essentially worse than the bound 3.5181 given by Theorem 15 ($\mathcal{R}_{fo} \approx 3.5181$) and (7) ($\mathcal{R}_o \leq \mathcal{R}_{fo}$). Thus the simplest observer does not universally fit the optimal channel rate. For the system at hand, this can be partly fixed by the change (24) of the variables, after which the data rate request decayed to $\log_2 13$ bits per unit time. The final improvement down to the value $\log_2 12$ compatible with the theoretical bound was achieved via the linear change of variables $z_1 = \sqrt{\alpha}\zeta_1, z_2 = \zeta_2$, which entails non-equally fine quantization of z_1 and z_2; the respective typical result is displayed in Fig. 4. This provides a supporting evidence that intricacies injected by P into the observation scheme to be described in Appendix E (see also the discussion after Theorem 12) concern the point of the matter.

8. Conclusion and future work

In this paper, we estimated the smallest data rates required for observability of discrete-time nonlinear systems in various

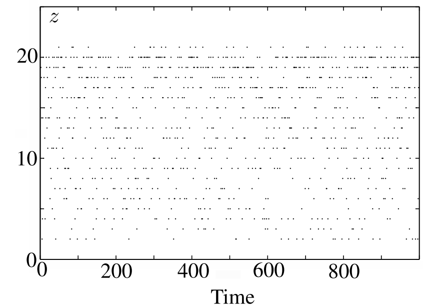

Fig. 3. Number of code symbols needed to track the system (21).

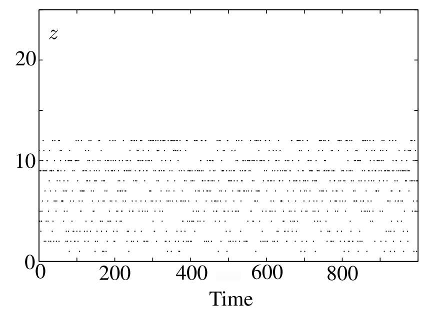

Fig. 4. Number of code symbols needed to track the system (25) with linear change $z_1 = \sqrt{\alpha}\zeta_1, z_2 = \zeta_2, \alpha = 0.5, \beta = 10$.

contexts. For the notion referred to as mere observability this smallest data rate coincides with the topological entropy of the system. The concepts of fine and regular observability introduced in the paper coincide in the most typical case where the set of admissible initial conditions is positively invariant and characterize the smallest data rate required to solve the observability problem with exponentially decaying observation error. It is shown that the value of regular (and hence, fine) observability rate admits constructive upper and lower estimates, derived via the direct and first Lyapunov methods respectively. The main results are illustrated by examples of the Hénon system and forced bouncing ball oscillator. Data rate bounds are accompanied by discussion of respective observation schemes. More examples on those data rate calculations can be found in our forthcoming publication Pogromsky and Matveev (in press).

On the technical side, the paper illuminates the issue of distributing the quantization fineness over directions in the phase space of nonlinear plants. In most closed-form schemes handling linear time-invariant systems, the fineness is assigned to the unstable modes with regard to the associated eigenvalues so that "more unstable" modes enjoy finer quantization. Moreover, this assignment becomes nearly compulsory as the channel capacity approaches the data rate limit (Baillieul, 2002). For nonlinear plants, the situation is different. On the one hand, available schemes intended to operate near the data rate limit (Colonius & Kawan, 2009; Nair et al., 2004; Savkin, 2006) use objects whose existence is merely postulated in the definitions of the concerned entropies. On the other hand, available closed-form "nonlinear" schemes are relatively indifferent to effective utilization of the channel capacity. A particular manifestation of this is that all state coordinates were typically quantized more or less homogeneously and the coordinate directions were chosen with no regard to the dynamics of the plant at hand. These expectedly result in excessive requirements to the channel data rate.

Conversely, the offered data rate bound is accompanied with a constructive observer that works whenever the channel fits this bound. This observer involves an explicit inhomogeneous assignment of quantization fineness, which is based on dynamical properties of the nonlinear system. This is indirectly made through a positive definite matrix that plays a key role in our result on those bounds. Since detailed exposition of this scheme is cumbersome, it is given in Appendix E; the body of the paper reports on its key points.

Future research includes extension of basic developments of this paper on time-varying systems. Extensions on the case of stochastic discrete memoryless communication channels (including erasure channels) by using the findings of Matveev and Savkin (2009, Ch. 6) are also on the agenda.

Appendix A. The Kalman-Szegö lemma

Here for the sake of completeness, we formulate the Kalman-Szegö lemma in a form that is used in the proof of Theorem 15.

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be real matrices of dimensions $n \times n$, $n \times m$ and $l \times n$, respectively. Let $\mathcal{F}(\zeta, w)$ be a Hermitian form with real coefficients in the vector variables $\zeta \in \mathbb{C}^l$ and $w \in \mathbb{C}^m$.

Lemma 17. *Let the pair $(\mathcal{A}, \mathcal{B})$ be controllable and $(\mathcal{A}, \mathcal{C})$ be observable. Suppose that $\mathcal{A}$ is Schur and there is $\varepsilon > 0$ such that $\mathcal{F}(\mathcal{C}\xi, 0) \geq \varepsilon\|C\xi\|^2$ for all $\xi \in \mathbb{C}^n$. For the existence of a positive definite matrix $P = P^\top > 0$ such that for any $\xi \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$ the following inequality holds*

$$(\mathcal{A}\xi + \mathcal{B}w)^\top P(\mathcal{A}\xi + \mathcal{B}w) - x^\top Px + \mathcal{F}(C\xi, w) \leq 0,$$

it is necessary and sufficient that for any $s \in \mathbb{C}^1$ such that $|s| = 1$ and for any $w \in \mathbb{C}^m$, the following inequality is satisfied

$$\mathcal{F}(\mathcal{C}(sI_n - \mathcal{A})^{-1}\mathcal{B}w, w) \leq 0.$$

Appendix B. General technical facts about the quantities in (7)

In this appendix, we present some general facts that will be used in the proofs of the main results and are of some self-interest. We assume that the map ϕ is Lipschitz on the set K^∞ defined in (8).

Lemma 18. *Observability via some channel with capacity c in any of the senses introduced in Definition 4 implies observability in the same sense via any channel with capacity $c' > c$.*

Proof. We focus on fine observability; the other kinds of observability are treated likewise. Let us consider an observer (2) that finely observes the system via the channel with capacity c, and let $b_\pm(\cdot)$ and $b'_\pm(\cdot)$ be the functions from (4) for this channel and a channel with capacity $c' > c$, respectively. By (4), $b'_-(k) \geq b_+(k)$ for a large enough k. So any data communicated across the first channel within any time interval Δ of duration k can be transmitted within Δ across the second channel as well. To switch the observer to the second channel, the coder and decoder are altered as follows. At any time $t = jk$, the coder builds the prognosis $x(jk), x(jk+1) = \phi[x(jk)], \dots, x(jk+k-1)$, instantly generates the sequence of messages $e(jk), \dots, e(jk+k-1)$, and re-shapes them into messages that can be sent across the second channel for $t \in \Delta_k := [jk : jk+k-1]$. Then the coder does send them within Δ_k. Thus at any time $t = jk$, the decoder still acquires the data required to build $\widehat{x}(jk)$, and it does this. Since $x(t) \in K(t)$ in (6), the distance from $\widehat{x}(t)$ to $K(t)$ is less than $2G\delta g^t$, and so $\widehat{x}(jk)$ can be moved to $K(jk)$ at the expense of increasing the constant $G := 3G$. For $t \in \Delta_k, t > jk$, the decoder employs the trivial estimate initiated at $t = jk$ by $\widehat{x}(jk)$.

Let $L \geq 1$ be a common Lipschitz constant for $\phi^t, t \in [0 : k]$ on K^∞. For $t \geq 0$ and $j := \lfloor t/k \rfloor$, $i := j - jk \in [0 : k-1]$, we have

$$\|x(t) - \widehat{x}(t)\| = \left\|\phi^i[x(jk)] - \phi^i[\widehat{x}(jk)]\right\| \leq L\,\|x(jk) - \widehat{x}(jk)\|$$
$$\leq LG\delta g^{jk} = LG\delta g^{\frac{t-i}{k}} \leq LGg^{-1}\left(\sqrt[k]{g}\right)^t, \tag{B.1}$$

i.e., (6) holds with $G := LGg^{-1}$ and $g := \sqrt[k]{g} \in (0, 1)$. Definitions 3 and 4 complete the proof. ■

Lemma 19. *For any $k \geq 1$ and any index $\varsigma := o, ro, fo$,*

$$\mathcal{R}_\varsigma(\phi^k, K) = k\mathcal{R}_\varsigma(\phi, K). \tag{B.2}$$

Proof. Let $\varsigma = $ fo; for $\varsigma = $ o, ro, the proofs are similar. Given $\varepsilon > 0$, the system (1) is finely observable via a channel with capacity $c \leq \mathcal{R}_{\text{fo}}(\phi, K) + \varepsilon$. We adjust the involved observer to the *system 2* $x(t+1) = \phi^k[x(t)]$ to form the decimated sequence $\widehat{x}(jk)$, where j is the step of system 2. At step j, the altered coder fills the gaps in the decimated sequence of states by using (1) and runs the original coder to generate the compound message $e^k(j) = \{e(jk+i)\}_{i=0}^{k-1}$, which is then delivered to a new decoder during one step of system 2. At step $j+1$, this decoder feeds the received data to the original one and thus gets $\widehat{x}[(j+1)k]$. By putting $t := jk$ into (6), we see that system 2 obeys (6) with $g := g^k$ and so is finely observable via the channel transmitting $e^k(j)$'s (channel 2). Let $|e|$ be the bit size of message e. For any interval $[\rho : \rho + r - 1]$ of duration r in the time-scale of system 2,

$$\frac{1}{r}\sum_{j=\rho}^{\rho+r-1}\left|e^k(j)\right| = \frac{1}{r}\sum_{j=\rho}^{\rho+r-1}\sum_{i=kj}^{kj+k-1}|e(i)|$$
$$= \frac{1}{r}\sum_{i=k\rho}^{k(\rho+r)-1}|e(i)| \in \left[k\frac{b_-(kr)}{kr}, k\frac{b_+(kr)}{kr}\right].$$

By letting $r \to \infty$ and invoking (4), we see that the capacity of channel 2 is kc. So $\mathcal{R}_{\text{fo}}(\phi^k, K) \leq ck \leq k\mathcal{R}_{\text{fo}}(\phi, K) + k\varepsilon$. Letting $\varepsilon \to 0+$ yields that $\mathcal{R}_{\text{fo}}(\phi^k, K) \leq k\mathcal{R}_{\text{fo}}(\phi, K)$.

Now we consider a channel with capacity $c \leq \mathcal{R}_{\text{fo}}(\phi^k, K) + \varepsilon$ via which system 2 is finely observable, and the involved observer. To adjust it to the original system (1), we run the coder only at times $t = jk$ and send the respective message during k time steps in the time scale of system 1. This permits us to similarly run the decoder, thus producing the original jth estimate at time $t = jk$. This estimate is extended on $[jk+1, jk+k-1]$ via the trivial prognosis. Similarly to (B.1) (where $g := \sqrt[k]{g}$), we see that the system (1) is finely observable. Consider an interval $[\rho : \rho + r - 1]$ of duration r in the times-scale of the system (1), and put $m_r := \lfloor r/k \rfloor$. The number of bits $b'(r)$ transmitted to serve the system (1) lies in $[b_-(m_r - 2), b_+(m_r + 2)]$, where by (4),

$$\frac{b_\pm(m_r \pm 2)}{r} = \frac{m_r \pm 2}{r}\frac{b_\pm(m_r \pm 2)}{m_r \pm 2} \xrightarrow{r\to\infty} \frac{c}{k}.$$

So $\mathcal{R}_{\text{fo}}(\phi, K) \leq c/k \leq \mathcal{R}_{\text{fo}}(\phi^k, K)/k + \varepsilon/k$. Letting $\varepsilon \to 0+$ yields $\mathcal{R}_{\text{fo}}(\phi, K) \leq \mathcal{R}_{\text{fo}}(\phi^k, K)/k$. Thus (B.2) does hold. ■

Remark 20. Fine observability (6) implies that there exists an observer for which *the initial accuracy of estimation can be restored in a finite time independent of the accuracy*: there is k such that $\|x(t) - \widehat{x}(t)\| \leq \delta, \quad \forall t \geq k$ if in (3), δ is small enough and $x(0), \widehat{x}(0) \in K$. Let the system be observable in exactly this sense. Then system 2 is evidently regularly observable. So with regard to Lemmas 14 and 19, we see that $\mathcal{R}_{\text{fo}}(\phi, K)$ also gives the minimum data rate required for this "observability with restoration of the initial accuracy" on a positively invariant set K.

Appendix C. Proofs of Lemmas 13 and 14

Proof of Lemma 13. We focus on $\mathcal{R}_{\text{fo}}$; $\mathcal{R}_{\text{o}}$ and $\mathcal{R}_{\text{ro}}$ are treated likewise. It suffices to show that if system 1 is finely observable via a channel, system 2 is also finely observable via this channel. Let this premise be true, and consider the involved observer:

$$d_1[x_1(t), \widehat{x}_1(t)] \leq G\delta g^t, \quad \forall t \tag{C.1}$$

provided that $x_1(0), \widehat{x}_1(0) \in K_1$ and $d_1[x_1(0), \widehat{x}_1(0)] < \delta_\star$. Here $d_i(\cdot,\cdot)$ is the metric in $\mathfrak{M}_i$ and $\delta_\star > 0, G > 0, g \in (0,1)$. By the argument stated at the end of the first paragraph in the proof of Lemma 18, it can be assumed that $\widehat{x}_1(t) \in K_1(t), \forall t$.

Given $x_2 \in K_2$, we borrow the notation $V(x_2)$ from Lemma 13, and denote by j_{x_2} the map inverse to $h|_{V(x_2)}$. It is L-Lipschitz and is defined on the intersection of K_2 with the δ_*-ball centered at x_2. By Definition 3, it suffices to build the observer (2) for system 2 only for $\delta \approx 0$. We pick $\delta < \min\{\delta_*; L^{-1}\delta_\star\}$ and for $\widehat{x}_2(0), x_2(0) \in K_2$ such that $d_2[x_2(0), \widehat{x}_2(0)] < \delta$, equip system 2 with the following coder and decoder

$$e(t) = \mathcal{C}\left[t, x_1(0), x_1(1), \dots, x_1(t)\middle|\widehat{x}_1^0, \delta\right];$$

$$\widehat{x}_2(t) = h[\widehat{x}_1(t)], \quad \widehat{x}_1(t) := \mathcal{D}\left[t, e(0), \dots, e(t-1)\middle|\widehat{x}_1^0, \delta\right].$$

Here $\widehat{x}_1^0 := j_{\widehat{x}_2(0)}[\widehat{x}_2(0)]$ and $x_1(t)$ are computed from $x_1(0) := j_{\widehat{x}_2(0)}[x_2(0)]$ via the recursion $x_1[t+1] = \phi_1[x_1(t)]$. Since $d_1[x_1(0); \widehat{x}_1^0] \leq L d_2[x_2(0), \widehat{x}_2(0)] < L\delta \leq \delta_\star$ and $x_1(0); \widehat{x}_1^0 \in K_1$, (C.1) holds. By construction, $x_2(0) = h[x_1(0)]$. So $x_2(1) = \phi_2[x_2(0)] = \phi_2 \circ h[x_1(0)] = h \circ \phi_1[x_1(0)] = h[x_1(1)]$. By continuing likewise, we see that $x_2(t) = h[x_1(t)], \forall t \geq 0$. By invoking that h is Lipschitz on K_1^∞ with some constant L_h, we see that

$$\begin{aligned} d_2[x_2(t), \widehat{x}_2(t)] &= d_2\left[h(x_1(t)), h(\widehat{x}_1(t))\right] \leq L_h d_1[x_1(t), \widehat{x}_1(t)] \\ &\overset{(C.1)}{\leq} L_h G g^t d_1[x_1(0), \widehat{x}_1(0)] \leq L\delta L_h G g^t. \end{aligned}$$

Definition 3 completes the proof. ■

Proof of Lemma 14. By (7), $\mathcal{R}_{ro} \leq \mathcal{R}_{fo}$. To show the converse $\mathcal{R}_{fo} \leq \mathcal{R}_{ro}$, we pick $\varepsilon > 0$. By the definition of $\mathcal{R}_{ro}$ and Lemma 18, the system is regularly observable via any channel with capacity $c \in (\mathcal{R}_{ro}, \mathcal{R}_{ro} + \varepsilon)$. We pick such a channel with a k-periodically varying alphabet $E(t) \ni e(t)$; then any sequence of messages delivered across this channel during some time interval of the form $[jk : ik-1]$ can be also delivered during $[(j+m)k : (i+m)k - 1]$ with any m. We also consider the observer from Definition 2: $\|x(t) - \widehat{x}(t)\| \leq G\delta, \ \forall t \geq 0$ if $x(0), \widehat{x}(0) \in K$ and (3) holds with small enough δ. Like in the proof of Lemma 13, it can be assumed that $\widehat{x}(t) \in K(t) \subset K, \forall t$. Let us open an extra communication channel with capacity ε. By treating the two channels as a single whole, we acquire a compound channel with capacity $c + \varepsilon$.

Projection of $x \in \mathbb{R}^n$ onto K is a point $y \in K$ such that $\|x - y\| = \min_{z \in K} \|x - z\|$. Since K is compact, such a point exists, though it may be non-unique. We endow both coder and decoder with a common procedure that maps any x to its projection $y = \mathbf{Pr}_K(x)$. It is easy to see that $\|z - \mathbf{Pr}_K(x)\| \leq 2\|z - x\|, \ \forall z \in K, x \in \mathbb{R}^n$.

We pick $g \in (0,1)$. The G-ball centered at 0 can be covered by $N < \infty$ balls, each with a radius of $g/2$. We supply both coder and decoder with a common such covering, along with common labeling of its elements. For any $\delta > 0$, the $G\delta$-ball $B_{\widehat{x}}(G\delta)$ centered at $\widehat{x}$ can be covered by this covering provided that it is preliminarily δ-scaled and shifted by the vector $\widehat{x}$. By removing the center x of any covering ball to $\mathbf{Pr}_K(x)$ and doubling the radius, we cover the intersection $B_{\widehat{x}}(G\delta) \cap K$ by N balls of radius $g\delta$ with centers in K. Via multiplying k by a large enough factor, we ensure that any number $i \in [1 : N]$ and a special one-bit "failure" message ⌘ can be transmitted across the extra channel for any time interval of duration k. After this, we alter the observer.

At any time $t = jk$, the new coder builds the state prognosis $\widetilde{x}(0) := x(jk), \widetilde{x}(1) := x(jk+1) = \phi[x(jk)], \dots, \widetilde{x}(k) := \phi^k[x(jk)]$ for k steps ahead, feeds the "old" coder from (2) (where $t \in [0 : k-1], \widehat{x}(0) := \widehat{x}(jk), \delta := \delta_j := g^j\delta$) with this prognosis ($x(t) := \widetilde{x}(t)$ in (2)) to instantly generate the messages $e_{jk}(0), \dots, e_{jk}(k-1)$ and the response of the decoder, including $\widehat{x}_{jk}(k)$. If $x[(j+1)k]$ does not belong to the $G\delta_j$-ball centered at $\widehat{x}_{jk}(k)$, the coder communicates the "failure" message ⌘ across the extra channel during the forthcoming time interval $\Delta_j := [jk : jk+k-1]$. Otherwise, the coder sends the label of the $(g\delta_j/2)$-ball containing $x[(j+1)k]$. The initial channel is used to consecutively deliver the messages $e_{jk}(0), \dots, e_{jk}(k-1)$.

For $t \in [jk+1 : jk-1]$, the new decoder uses them to build $\widehat{x}(t)$ via the second equation from (2) (where $t := t - jk, \widehat{x}(0) := \widehat{x}(jk), \delta := \delta_j$). However, $\widehat{x}[(j+1)k]$ is put in use only if ⌘ is received during Δ_j. Otherwise, the decoder becomes aware of a ball with a radius of $g \cdot \delta_j = \delta_{j+1}$ centered at a point $\widehat{x} \in K$ that contains $x[(j+1)k] \in K$; and $\widehat{x}[(j+1)k]$ is removed to $\widehat{x}$, thus improving the accuracy of estimation to δ_{j+1}. At time $t = (j+1)k$, the same shift of $\widehat{x}[(j+1)k]$ is performed by the coder prior to actions described in the previous paragraph.

It is easy to see that in fact ⌘ is never sent and $\|x(jk) - \widehat{x}(jk)\| \leq \delta_j$ for all j. For any $t \geq 0$ and $j := \lfloor t/k \rfloor$, we have

$$\|x(t) - \widehat{x}(t)\| \leq G\delta_j = G\delta g^j \leq G\delta\left(\sqrt[k]{g}\right)^t.$$

So by Definition 3, the capacity of the compound channel $c + \varepsilon \geq \mathcal{R}_{fo}$. It remains to invoke that $c \in \left(\mathcal{R}_{ro}, \mathcal{R}_{ro} + \varepsilon\right)$ and let $\varepsilon \to 0+$.

Appendix D. Proofs of Theorems 8 and 9

Proof of Theorem 8. Suppose that the system is observable via a channel with capacity c. Given $\varepsilon > 0$, Definition 4 guarantees existence of an observer (2) with anytime exactness ε. Since K is compact, there exists a subset $K_0 \subset K$ of a finite size m_ε such that any $\xi \in K$ has an element $\zeta \in K_0$ at a distance $d[\xi; \zeta] < \delta(\varepsilon, K)$. Due to Definitions 1 and 7, the set Q_k constituted by $\{\widehat{x}(0), \dots, \widehat{x}(k)\}$ as $\widehat{x}(0)$ runs over K_0 is (k, ε)-spanning for K. Since its size does not exceed $m_\varepsilon 2^{b_+(k)}$, formula (11) yields that

$$H(\phi, K) \leq \lim_{\varepsilon \to 0} \overline{\lim_{k \to \infty}} \frac{1}{k+1} \log_2\left(m_\varepsilon 2^{b_+(k)}\right) \overset{(4)}{=} c.$$

Since $\mathcal{R}_o(\phi, K)$ is the infimum of c's, $H(\phi, K) \leq \mathcal{R}_o(\phi, K)$.

To prove the converse for positively invariant sets K, we consider a channel with capacity $c > H(\phi, K)$ and $\varepsilon > 0$. Based on (4) and (11), we pick so large $k \geq 2$ that $k^{-1}\log_2 q(k-1, \varepsilon) < k^{-1}b_-(k)$. For K, there exists a (k, ε)-spanning set Q of size $q(k-1, \varepsilon)$; so its elements can be encoded and communicated across the channel for time k. For any $a \in K$, let $X^\star(a) = \{x_0^\star, \dots, x_k^\star\}$ stand for an element of Q satisfying (10). Finally since the map $a \in K \mapsto \{x(0, a), \dots, x(k-1, a)\}$ is continuous and K is compact, there exists $\delta > 0$ such that $a_1, a_2 \in K, d[a_1; a_2] \leq \delta \Rightarrow d\left[x(t, a_1); x(t, a_2)\right] < \varepsilon, \ \forall t \in [0 : k-1]$.

At any time $\tau_j = jk, j \geq 0$, let the coder encode $X^\star(\phi^k[x(\tau_j)])$ and send it across the channel during the next k time units. Within the first interval $[0 : k-1]$, let the decoder follow (1) by putting $\widehat{x}(t+1) = \phi[\widehat{x}(t)]$; the exactness ε results from $d[x(0); \widehat{x}(0)] \leq \delta$ and the choice of δ. At time $t = \tau_{j+1}$, the decoder receives $X^\star[x(t)]$. Let the decoder use the respective entries of $X^\star[x(t)]$ as estimates of $x(t), \dots, x(t+k-1)$ during the forthcoming time interval of duration k, thus ensuring the exactness ε. So by Definition 1, the system is observable via the channel at hand and so $\mathcal{R}_o(\phi, K) \leq c$. Letting $c \to H(\phi, K)$ yields $\mathcal{R}_o(\phi, K) \leq H(\phi, K)$ and thus completes the proof of (12). □

Proof of Theorem 9. Let the system (1) be regularly observable via a channel with capacity c, and consider the respective observer (2). By Definition 2, there exist $G > 0$ and $\delta_* > 0$ such that

$$d\left[x(t); \widehat{x}(t)\right] \leq G\delta, \quad \forall t \geq 0 \tag{D.1}$$

whenever $x(0), \widehat{x}(0) \in K$ and $d\left[x(0); \widehat{x}(0)\right] < \delta < \delta_*$. Here $d[\cdot;\cdot]$ is the distance function of the connected Riemannian manifold at hand, i.e., the arclength of the minimizing geodesics. Let $B_x(\varepsilon)$ denote the ε-ball centered at x, and let L be the unstable subspace of $M := \phi'(x_0)$. This subspace is invariant with respect to operator M, which is non-singular on L, i.e., $\det M|_L \neq 0$. Given $m \geq 1$, the same it true for $M^m = (\phi^m)'(x_0)$. Since L is tangent to $\mathfrak{M}^{\text{unst}}(x_0)$ at x_0, it follows that by reducing $\delta_* > 0$ if necessary, the following properties can be ensured:

[1] $\phi^m(\cdot)$ is an injective immersion of $\mathfrak{M}_* := \mathfrak{M}^{\text{unst}}(x_0) \cap B_{x_0}(\delta_*)$ into $\mathfrak{M}$, i.e., a diffeomorphism of the manifold $\mathfrak{M}_*$ onto some smooth manifold $\mathfrak{M}_\uparrow \subset \mathfrak{M}$ of the same dimension $s = \dim L$;
[2] $x_0 \in \mathfrak{M}_\uparrow$, the space L is tangent to $\mathfrak{M}_\uparrow$ at x_0, and $\mathfrak{M}_* \subset K$;
[3] $\mathbf{mes}_{\mathfrak{M}_\uparrow} \phi^m(E) \geq 1/2|\det M^m|_L|\mathbf{mes}_{\mathfrak{M}_*} E$ for any Borel set $E \subset \mathfrak{M}_*$, where **mes** is the s-dimensional Riemannian measure (volume) (Chavel, 2006, Sect. III.3) on the indicated submanifold;
[4] $\mathbf{mes}_{\mathfrak{M}_*} \mathfrak{M}(\delta) = \mu_s \delta^s [1 + \alpha_*(\delta)]$, $\forall \delta \approx 0$, where $\mathfrak{M}(\delta) := B_{x_0}(\delta) \cap \mathfrak{M}_*, \alpha_*(\delta) \to 0$ as $\delta \to 0$, and $\mu_s \delta^s$ is the s-dimensional volume of the δ-ball in the tangent space L;
[5] $\mathbf{mes}_{\mathfrak{M}_\uparrow} \left[B_y(\varepsilon) \cap \mathfrak{M}_\uparrow \right] \leq \mu_s 2^s \varepsilon^s [1 + \alpha_\uparrow(\varepsilon, \delta)]$, $\forall \varepsilon \approx 0, \delta \approx 0$, $y : \mathbf{dist}\big(y, \phi^m\left[\mathfrak{M}(\delta)\right]\big) \leq G\delta$, where $\alpha_\uparrow(\varepsilon, \delta) \to 0$ as $\varepsilon \to 0, \delta \to 0$ and $\mathbf{dist}\big(y, E\big)$ is the distance from y to the set E.

Now let $\widehat{x}(0) := x_0$ and let $x(0)$ run over $\mathfrak{M}(\delta) \subset K$, with $\delta \approx 0$ being fixed. Inequality (D.1) (where $t := m$ and $x(t) = \phi^m[x(0)]$) implies that the image $\phi^m\left[\mathfrak{M}(\delta)\right]$ is covered by $G\delta$-balls centered at the points of the form $y = \widehat{x}(m)$ and also that the distance from these points to $\phi^m\left[\mathfrak{M}(\delta)\right] \ni x(m)$ does not exceed $G\delta$. By (2), these points are completely determined by the transmissions across the channel; so their number does not exceed $2^{b_+(m)}$. Hence

$$\begin{aligned}
\mathbf{mes}_{\mathfrak{M}_\uparrow} \phi^m\left[\mathfrak{M}(\delta)\right] &\leq \sum_{y=\widehat{x}(m)} \mathbf{mes}_{\mathfrak{M}_\uparrow} \left[B_y(G\delta) \cap \mathfrak{M}_\uparrow\right] \\
&\overset{[5]}{\leq} \sum_{y=\widehat{x}(m)} \mu_s (2G)^s \delta^s [1 + \alpha_\uparrow(G\delta, \delta)] \\
&\leq \mu_s (2G)^s \delta^s 2^{b_+(m)} [1 + \alpha_\uparrow(G\delta, \delta)].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathbf{mes}_{\mathfrak{M}_\uparrow} \phi^m\left[\mathfrak{M}(\delta)\right] &\overset{[3]}{\geq} \frac{1}{2} |\det M^m|_L| \mathbf{mes}_{\mathfrak{M}_*} \mathfrak{M}(\delta) \\
&\overset{[4]}{=} \frac{1}{2} |\det M|_L|^m \mu_s \delta^s [1 + \alpha_*(\delta)].
\end{aligned}$$

Thus putting $\alpha_\uparrow^\star(\delta) := \alpha_\uparrow(G\delta, \delta)$, we see that

$$\begin{aligned}
&\frac{1}{2} |\det M|_L|^m \mu_s \delta^s [1 + \alpha_*(\delta)] \leq \mu_s (2G)^s \delta^s 2^{b_+(m)} [1 + \alpha_\uparrow^\star(\delta)] \\
&\quad \Rightarrow |\det M|_L|^m \leq 2(2G)^s 2^{b_+(m)} \frac{1 + \alpha_\uparrow^\star(\delta)}{1 + \alpha_*(\delta)} \\
&\quad \xRightarrow{\delta \to 0} m \log_2 |\det M|_L| \leq 1 + s(1 + \log_2 G) + b_+(m) \\
&\quad \Rightarrow \log_2 |\det M|_L| \leq \frac{1 + s(1 + \log_2 G)}{m} + \frac{b_+(m)}{m}.
\end{aligned}$$

It remains to let $m \to \infty$, invoke (4), and take into account that $\log_2 |\det M|_L| = H[M]$ (Matveev & Savkin, 2005). ■

Appendix E. Proof of Theorem 12

By (ii) in Remark 6, we can treat $\mathbb{R}^n$ as the Euclidean space $\mathcal{H}$ with the inner product $\langle x; y \rangle_P := y^\top P x$ and norm $\|x\|_P := \sqrt{\langle x; x \rangle_P}$, where P is taken from Assumption 11; $B_x(\varepsilon)$ denotes the ε-ball centered at x in $\mathcal{H}$. Any $n \times n$-matrix A is viewed as a linear operator in $\mathcal{H}$, and $\sigma_1(A) \geq \cdots \geq \sigma_n(A)$ are its singular values (Gohberg & Krein, 1969). We put $\omega_d(A) := \sigma_1(A) \cdots \sigma_d(A)$ for $d \in [1 : n]$.

Lemma 21. *For any $x_0 \in K^\infty$ and $d \in [1 : n]$, we have*

$$\log_2 \omega_d[(\phi^k)'(x_0)] \leq \Lambda_\star k + b, \quad \text{where } b := \sup_{\substack{x \in K^\infty \\ d \in [1:n]}} |\upsilon_d(x)|, \tag{E.1}$$

and $\upsilon_d(\cdot)$, $\Lambda_\star$ are taken from Assumption 11 and (19), respectively.

Proof. For the trajectory $x(0) = x_0, x(1), \ldots$ of (1), $(\phi^k)'(x_0) = A[x(k-1)]A[x(k-2)] \cdots A[x(0)]$, where $A(x)$ is given by (15). By the generalized Horn inequality (see e.g., Lemma 8.3 in Pogromsky and Matveev (2011)),

$$\begin{aligned}
&\omega_d \left[(\phi^k)'(x_0) \right] \leq \prod_{t=0}^{k-1} \omega_d \left(A[x(t)] \right) \\
&\qquad \Rightarrow \log_2 \omega_d \left[(\phi^k)'(x_0) \right] \leq \sum_{t=0}^{k-1} \log_2 \omega_d \left(A[x(t)] \right) \\
&\qquad = \sum_{t=0}^{k-1} \sum_{j=1}^{d} \log_2 \sigma_j \left(A[x(t)] \right).
\end{aligned}$$

Meanwhile by Lemma 8.1 in Pogromsky and Matveev (2011), $\sigma_j[A(x)] = \sqrt{\lambda_j(x)}$, where $\lambda_j(x)$ are defined in Theorem 12. So

$$\begin{aligned}
\log_2 \omega_d \left[(\phi^k)'(x_0) \right] &\leq \frac{1}{2} \sum_{t=0}^{k-1} \sum_{j=1}^{d} \log_2 \lambda_j[x(t)] \\
&\overset{(16)}{\leq} \frac{1}{2} \sum_{t=0}^{k-1} \left(\Lambda_d - \Delta \upsilon_d[x(t)] \right) \xRightarrow{(19)} \text{(E.1)}. \quad ■
\end{aligned}$$

Lemma 22. *The claims* (i) *and* (ii) *from Assumption 10 are true with $\phi := \phi^k$ for any $k \geq 1$.*

Proof. Evidently $\phi^r(K^\infty) \subset K^\infty$, $\forall r \geq 0$. So (i) with $\phi := \phi^k$ is immediate from (i) of Assumption 10. Claim (ii) will be proven via induction on k. For $k = 1$, it is given by Assumption 10. Suppose that this claim is true for some k. Then for any ε_i, there exists $\delta_i > 0$ such that whenever $x_0 \in K^\infty$,

$$\begin{aligned}
&\|x - x_0\|_P < \delta_1 \Rightarrow \|\phi'(x) - \phi'(x_0)\|_P < \varepsilon_1, \\
&\|x - x_0\|_P < \delta_2 \Rightarrow \|(\phi^k)'(x) - (\phi^k)'(x_0)\|_P < \varepsilon_2.
\end{aligned} \tag{E.2}$$

Since (i) is true for ϕ and ϕ^k, there exist $G_i \in (0, \infty)$ such that

$$\|\phi'(x_0)\| \leq G_1, \qquad \|(\phi^k)'(x_0)\| \leq G_2, \quad \forall x_0 \in K^\infty. \tag{E.3}$$

Given $\varepsilon > 0$, we take $\varepsilon_1 := \varepsilon/(2G_2)$, $\varepsilon_2 := \varepsilon/[2(G_1 + \varepsilon_1)]$. Let $x_0 \in K^\infty$ and $\|x - x_0\|_P < \delta := \min\{\delta_2; \delta_1/(G_2 + \varepsilon_2)\}$. Thanks to (E.2),

$$\left\| \phi^k(x) - \phi^k(x_0) - (\phi^k)'(x_0)[x - x_0] \right\|_P \leq \varepsilon_2 \|x - x_0\|_P \tag{E.4}$$

and so $\|y - y_0\|_P < (G_2 + \varepsilon_2)\delta \leq \delta_1$, where $y := \phi^k(x), y_0 := \phi^k(x_0) \in K^\infty$. Hence

$$\begin{aligned}
&\|(\phi^{k+1})'(x) - (\phi^{k+1})'(x_0)\| \\
&\quad = \|\phi'(y)(\phi^k)'(x) - \phi'(y_0)(\phi^k)'(x_0)\| \\
&\quad \leq \|[\phi'(y) - \phi'(y_0)][(\phi^k)'(x) - (\phi^k)'(x_0)]\| \\
&\qquad + \|[\phi'(y) - \phi'(y_0)](\phi^k)'(x_0)\| \\
&\qquad + \|\phi'(y_0)[(\phi^k)'(x) - (\phi^k)'(x_0)]\| \\
&\quad < \varepsilon_1 \varepsilon_2 + \varepsilon_1 G_2 + G_1 \varepsilon_2 \leq \varepsilon,
\end{aligned}$$

i.e., (ii) does hold for ϕ^{k+1}. ■

Remark 23. It follows from (E.3) and (E.4) that any map ϕ^j is locally Lipschitz continuous in a vicinity of K^∞: there exist $\eta_j > 0$ and $L_j > 0$ such that $\|\phi^j(x) - \phi^j(x_0)\|_P \le L_j \|x - x_0\|_P$ whenever $x_0 \in K^\infty$ and $\|x - x_0\|_P < \eta_j$.

Lemma 24. *Given $k \ge 1$, $\nu \in (0,1)$, there is $\delta_k > 0$ such that for any $\delta \in (0, \delta_k)$, $r \ge 0$, $x_0 \in K(r)$, the image $\phi^k[B_{x_0}(\delta) \cap K(r)]$ is covered by M balls of radius $\nu\delta$ centered in $K(r+k)$, where*

$$M \le \mathcal{M}(k) := \max\left\{2^{\Lambda_\star k + b_\star}; 1\right\} \tag{E.5}$$

and $b_\star := b + 3n + \frac{n}{2}\log_2 n - n\log_2 \nu$.

Proof. We pick $\varkappa \in (0, 1/2)$ and then, based on Lemma 22, $\delta_k > 0$ such that (E.4) with $\varepsilon_2 := \varkappa$ holds for all $x_0 \in K^\infty$ and $x \in B_{x_0}(\delta_k)$. Let $\delta \in (0, \delta_k)$ and $x_0 \in K(r) \subset K^\infty$. Then

$$\phi^k(x) - \left(\phi^k\right)'(x_0)[x - x_0] - \phi^k(x_0) \in B_0(\varkappa\delta), \quad \forall x \in B_{x_0}(\delta).$$

By Pogromsky and Matveev (2011, Prop. 8.5) (with $\eta := \varkappa\delta$, $\varepsilon := 1/2\nu\delta$), M balls of the radius $1/2\nu\delta$ are enough to cover $\phi^k[B_{x_0}(\delta) \cap K(r)]$, where

$$\begin{aligned} M &= \prod_{i=1}^{n}\left\lceil \frac{2\sqrt{n}}{\nu\delta}\left\{\delta\sigma_i[\left(\phi^k\right)'(x_0)] + \delta\varkappa\right\}\right\rceil \\ &= \prod_{i=1}^{n}\left\lceil \frac{2\sqrt{n}}{\nu}\left\{\sigma_i[\left(\phi^k\right)'(x_0)] + \varkappa\right\}\right\rceil \end{aligned}$$

and $\lceil a \rceil$ is the integer ceiling of a. Since $\phi^k[B_{x_0}(\delta) \cap K(r)] \subset K(r+k)$, the centers of these balls can be deposed to $K(r+k)$ at the expense of doubling the radius $1/2\nu\delta \mapsto \nu\delta$. By Pogromsky and Matveev (2011, Lemma 8.8),

$$M \le \max\left\{\frac{2^{2n} n^{\frac{n}{2}}}{\nu^n} \max_{d \in [1:n]} \underbrace{\prod_{j=1}^{d}\left\{\sigma_i[\left(\phi^k\right)'(x_0)] + \varkappa\right\}}_{\varrho_d}; 1\right\}.$$

Here $\max_d$ can be confined to d such that $\sigma_i[\left(\phi^k\right)'(x_0)] + \varkappa > 1, \ \forall i \in [1:d]$. Since $\varkappa < 1/2$, then $\sigma_i[\left(\phi^k\right)'(x_0)] > 1/2, \ \forall i \in [1:d]$. The proof of (E.5) is completed by (E.1) and noting that

$$\begin{aligned} \varrho_d &= \prod_{i=1}^{d}\left\{1 + \frac{\varkappa}{\sigma_i[\left(\phi^k\right)'(x_0)]}\right\}\sigma_i[\left(\phi^k\right)'(x_0)] \\ &\le 2^d \prod_{i=1}^{d}\sigma_i[\left(\phi^k\right)'(x_0)] \le 2^n \omega_d[\left(\phi^k\right)'(x_0)]. \quad \blacksquare \end{aligned}$$

Proof of Theorem 12. We consider a channel with capacity $c > \Lambda_\star$ and pick $\nu \in (0,1)$. By (4) and (E.5), there is $k \ge 2$ such that $\log_2 \mathcal{M}(k) + 1 < b_-(k)$. Then we take δ_k from Lemma 24 and η_j from Remark 23. Then assuming that $0 < \delta < \delta_k, \eta_1, \ldots, \eta_k$, we build an observer (2) to satisfy Definition 3.[1] Its operation will be organized in *epochs* $E_j := [jk : (j+1)k - 1]$, $j \in \mathbb{Z}_+$ of duration k. Based on Lemma 24, the coder and decoder are endowed, for any j, with a common procedure of covering the image $\phi^k[B_{\widehat{x}}(\delta) \cap K(jk)]$ with any $\widehat{x} \in K(jk)$ by $\mathcal{M}(k)$ enumerated $\nu\delta$-balls centered in $K[(j+1)k]$. Our decoder should generate estimates $\widehat{x}(t) \in K(t)$.

At epoch j, the coder carries out the following operations:

(c.1) At $t = jk$, puts $\delta_j := \delta\nu^j$ and covers $\phi^k[B_{\widehat{x}(jk)}(\delta_j) \cap K(jk)]$ with $\mathcal{M}(k)$ $\nu\delta_j$-balls B with centers in $K[(j+1)k]$;

(c.2) If $x(jk) \in B_{\widehat{x}(jk)}(\delta_j) \cap K(jk)$, determines a ball B containing $\phi^k[x(jk)]$ and communicates its encoded index across the channel during the epoch E_j;

(c.3) If $x(jk) \notin B_{\widehat{x}(jk)}(\delta_j) \cap K(jk)$, sends a special one-bit "failure" message across the channel during the epoch E_j;

(c.4) Copycats the decoder to compute $\widehat{x}[(j+1)k]$.

Transmissions from (c.2) and (c.3) are feasible due to the inequality $\log_2 \mathcal{M}(k) + 1 < b_-(k)$; (c.4) is possible since the decoder is driven by the signals sent from the encoder and the shared data $\widehat{x}(0)$, δ.

At epoch j, the decoder carries out the following operations:

(d.1) At the beginning of the epoch $t = jk$
- If $j > 0$ and the message μ_j received within the preceding epoch E_{j-1} does not mean "failure", decodes μ_j, thus determining the encoded ball, and takes its center as $\widehat{x}(jk)$;
- Otherwise, puts $\widehat{x}(jk) := \phi[\widehat{x}(jk-1)]$ if $j > 0$ and does nothing if $j = 0$;

(d.2) During the epoch, recursively generates the estimate by putting $\widehat{x}(jk+i+1) := \phi[\widehat{x}(jk+i)]$ for $i \in [0 : k-2]$.

The step (c.3) was introduced only for formal logical completeness: it is easy to see that due to (3), it is always inactive and

$$\|x(jk) - \widehat{x}(jk)\| \le \nu^j\delta, \ \forall j, \quad \widehat{x}(t) \in K(t), \ \forall t.$$

By Remark 23, we have for any $t \ge 0$ and $j := \lfloor t/k \rfloor$, $i := j - jk \in [0 : k-1]$, $L := \max\{L_0 := 1, L_1, \ldots, L_{k-1}\}$,

$$\begin{aligned} &\|x(t) - \widehat{x}(t)\| = \left\|\phi^i[x(jk)] - \phi^i[\widehat{x}(jk)]\right\| \le L_i \|x(jk) - \widehat{x}(jk)\| \\ &\le L\delta\nu^j = L\delta\nu^{\frac{t-i}{k}} \le L\nu^{-1}\left(\sqrt[k]{\nu}\right)^t, \end{aligned}$$

i.e., (6) does hold with $G := L/\nu$ and $g := \sqrt[k]{\nu} \in (0,1)$. By Definition 3, the system is finely observable over the channel at hand. So the channel capacity $c \ge \mathcal{R}_{\mathrm{fo}}(\phi, K)$, and letting $c \to \Lambda_\star +$ completes the proof.

[1] The design extends recipes of Matveev and Savkin (2009, Ch. 3,6) on nonlinear plants.

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