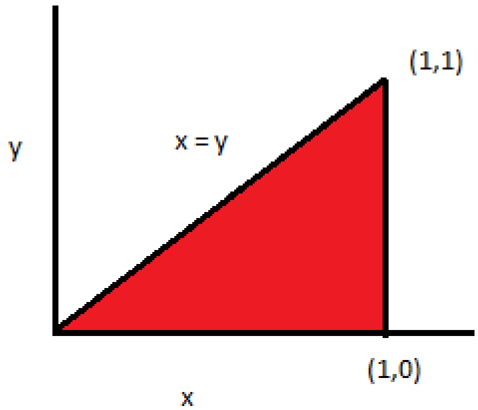
8. (20 Points)



(a)

No, X and Y are not independent (we can verify that $f(x,y) \neq f(x)f(y)$).

(b) Since, the cdf integrates to 1 we have,

$$\begin{aligned}
\int_0^1 \int_0^x f_{X,Y}(x,y)dydx &= 1 \\
\int_0^1 \int_0^x cxdydx &= \int_0^1 cx^2 dx = \frac{cx^3}{3}|_0^1 = 1 \\
\Rightarrow c &= 3
\end{aligned}$$

(c)

$$\begin{aligned}
f_Y(y) &= \int_y^1 f_{X,Y}(x,y)dx = \int_y^1 3xdx \\
&= \frac{3x^2}{2}|_y^1 \\
&= \frac{3}{2}(1-y^2)
\end{aligned}$$

(d)

$$\begin{aligned}
1 - F_U(u) &= P(X - Y \geq u) \\
&= \int_u^1 \int_0^{x-u} 3xdydx \\
&= \int_u^1 3x(x-u)dx \\
&= 3(\frac{x^3}{3})|_u^1 - 3u(\frac{x^2}{2})|_u^1 \\
&= 1 - u^3 - \frac{3}{2}u + \frac{3}{2}u^3 \\
&= 1 + \frac{1}{2}u^3 - \frac{3}{2}u
\end{aligned}$$

Therefore,

$$F_U(u) = \begin{cases} 0 & \text{if } u < 0; \\ \frac{3}{2}u - \frac{1}{2}u^3 & \text{if } 0 \leq u \leq 1; \\ 1 & \text{if } u > 1. \end{cases}$$

And, differentiating $F_U(u)$ we have

$$f_U(u) = \begin{cases} \frac{3}{2}(1 - u^2) & \text{if } 0 \leq u \leq 1; \\ 0 & \text{otherwise.} \end{cases}$$