ASSIGNMENT 10 SOLUTION

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Excercise 1. (K-means clustering)

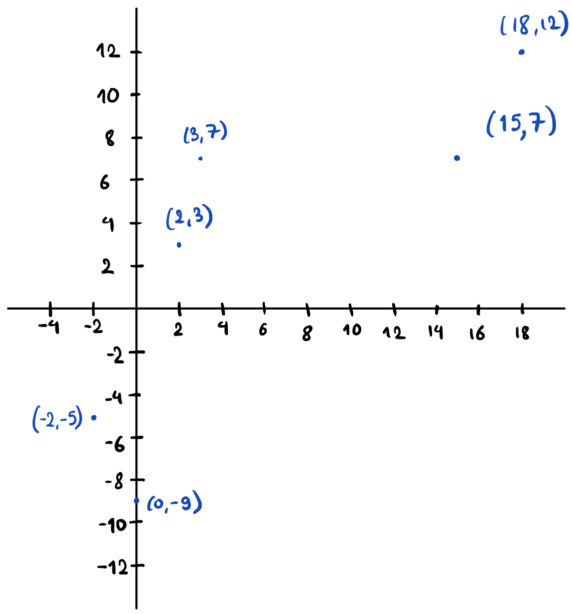
In this task, you carry out K-means clustering with $K = 3$ by paper and pencil. To this end, you are given the following data

i	x_i	$C^{(0)}(i)$
1	$(15, 7)^\top$	2
2	$(0, -9)^\top$	1
3	$(-2, -5)^\top$	1
4	$(2, 3)^\top$	0
5	$(3, 7)^\top$	2
6	$(18, 12)^\top$	0

with its initial cluster assignment. Let the clustering algorithm "run" until it finalized its assignment. Finally, draw the just computed clusters in a two-dimensional scatter plot.

(4 Points)

1)



Iteration 1

i) $N_0 = 2, \ N_1 = 2, \quad N_2 = 2$

ii) $m_0^1 = \frac{1}{N_0}\left(\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}18\\12\end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix}20\\15\end{pmatrix} = \begin{pmatrix}10\\7.5\end{pmatrix}$

$m_1^1 = \frac{1}{N_1}\left(\begin{pmatrix}0\\-9\end{pmatrix}+\begin{pmatrix}-2\\-5\end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix}-2\\-14\end{pmatrix} = \begin{pmatrix}-1\\-7\end{pmatrix}$

$m_2^1 = \frac{1}{N_2}\left(\begin{pmatrix}15\\7\end{pmatrix}+\begin{pmatrix}3\\7\end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix}18\\14\end{pmatrix} = \begin{pmatrix}9\\7\end{pmatrix}$

iii) $C^1(1) = \underset{1\leq k\leq K}{\operatorname{argmin}} \ \|x_1 - m_k^1\|^2$

$k=1, \quad \left\| \begin{pmatrix} 15 \\ 7 \end{pmatrix} - \begin{pmatrix} 10 \\ 7.5 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 5 \\ 0.5 \end{pmatrix} \right\|^2 = 5^2 + 0.5^2 = 25.25$

$k=2, \quad \left\| \begin{pmatrix} 15 \\ 7 \end{pmatrix} - \begin{pmatrix} -1 \\ -7 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 16 \\ 14 \end{pmatrix} \right\|^2 = 16^2 + 14^2 = 452$

$k=3, \quad \left\| \begin{pmatrix} 15 \\ 7 \end{pmatrix} - \begin{pmatrix} 9 \\ 7 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 6 \\ 0 \end{pmatrix} \right\|^2 = 6^2 = 36$

$\therefore C^1(1) = 1$

$C^1(2) = \underset{1 \leq k \leq K}{\operatorname{argmin}} \|x_2 - m_k^1\|^2$

$k=1, \quad \left\| \begin{pmatrix} 0 \\ -9 \end{pmatrix} - \begin{pmatrix} 10 \\ 7.5 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} -10 \\ -16.5 \end{pmatrix} \right\|^2 = 372.25$

$k=2, \quad \left\| \begin{pmatrix} 0 \\ -9 \end{pmatrix} - \begin{pmatrix} -1 \\ -7 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\|^2 = 5$

$k=3, \quad \left\| \begin{pmatrix} 0 \\ -9 \end{pmatrix} - \begin{pmatrix} 9 \\ 7 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} -9 \\ -16 \end{pmatrix} \right\|^2 = 337$

$\therefore C^1(2) = 2$

$C^1(3) = \underset{1 \leq k \leq K}{\operatorname{argmin}} \|x_3 - m_k\|^2$

$k=1, \quad \left\| \begin{pmatrix} -2 \\ -5 \end{pmatrix} - \begin{pmatrix} 10 \\ 7.5 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} -12 \\ -12.5 \end{pmatrix} \right\|^2 = 300.25$

$k=2, \quad \left\| \begin{pmatrix} -2 \\ -5 \end{pmatrix} - \begin{pmatrix} -1 \\ -7 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\|^2 = 5$

$k=3, \quad \left\| \begin{pmatrix} -2 \\ -5 \end{pmatrix} - \begin{pmatrix} 9 \\ 7 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} -11 \\ -12 \end{pmatrix} \right\|^2 = 265$

$\therefore C^1(3) = 2$

$C^1(4) = \underset{1 \leq k \leq K}{\operatorname{argmin}} \|x_4 - m_k\|^2$

$k=1, \quad \left\| \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 10 \\ 7.5 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} -8 \\ -4.5 \end{pmatrix} \right\|^2 = 84.25$

$k=2, \quad \left\| \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ -7 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 3 \\ 10 \end{pmatrix} \right\|^2 = 109$

$$k=3,\ \left\| \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 9 \\ 7 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} -7 \\ -4 \end{pmatrix} \right\|^2 = 65$$

$$\therefore C^1(4) = 3$$

$$C^1(5) = \underset{1 \le k \le K}{\operatorname{argmin}} \| x_5 - m_k \|^2$$

$$k=1,\ \left\| \begin{pmatrix} 3 \\ 7 \end{pmatrix} - \begin{pmatrix} 10 \\ 7.5 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 7 \\ -0.5 \end{pmatrix} \right\|^2 = 49.25$$

$$k=2,\ \left\| \begin{pmatrix} 3 \\ 7 \end{pmatrix} - \begin{pmatrix} -1 \\ -7 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 4 \\ 14 \end{pmatrix} \right\|^2 = 212$$

$$k=3,\ \left\| \begin{pmatrix} 3 \\ 7 \end{pmatrix} - \begin{pmatrix} 9 \\ 7 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} -6 \\ 0 \end{pmatrix} \right\|^2 = 36$$

$$\therefore C^1(5) = 3$$

$$C^1(6) = \underset{1 \le k \le K}{\operatorname{argmin}} \| x_6 - m_k \|^2$$

$$k=1,\ \left\| \begin{pmatrix} 18 \\ 12 \end{pmatrix} - \begin{pmatrix} 10 \\ 7.5 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 8 \\ 4.5 \end{pmatrix} \right\|^2 = 84.25$$

$$k=2,\ \left\| \begin{pmatrix} 18 \\ 12 \end{pmatrix} - \begin{pmatrix} -1 \\ -7 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 19 \\ 19 \end{pmatrix} \right\|^2 = 722$$

$$k=3,\ \left\| \begin{pmatrix} 18 \\ 12 \end{pmatrix} - \begin{pmatrix} 9 \\ 7 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 9 \\ 5 \end{pmatrix} \right\|^2 = 106$$

$$\therefore C^1(6) = 1$$

After iteration $= 1$,

i	x_i	$C^{(1)}(i)$
1	$(15, 7)^\top$	0
2	$(0, -9)^\top$	1
3	$(-2, -5)^\top$	1
4	$(2, 3)^\top$	2
5	$(3, 7)^\top$	2
6	$(18, 12)^\top$	0

Since $C^0(i) \neq C^1(i)$, we need to iterate again!

Iteration 2

i) $N_0 = 2, \; N_1 = 2, \; N_2 = 2$

ii) $m_0^1 = \frac{1}{N_0}\left(\begin{pmatrix}15\\7\end{pmatrix}+\begin{pmatrix}18\\12\end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix}33\\19\end{pmatrix} = \begin{pmatrix}16.5\\9.5\end{pmatrix}$

$m_1^1 = \frac{1}{N_1}\left(\begin{pmatrix}0\\-9\end{pmatrix}+\begin{pmatrix}-2\\-5\end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix}-2\\-14\end{pmatrix} = \begin{pmatrix}-1\\-7\end{pmatrix}$

$m_2^1 = \frac{1}{N_2}\left(\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}3\\7\end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix}5\\10\end{pmatrix} = \begin{pmatrix}2.5\\5\end{pmatrix}$

iii) $C^2(1) = \underset{1 \le k < K}{\operatorname{argmin}} \|x_1 - m_k\|^2$

$k=1, \; \left\|\begin{pmatrix}15\\7\end{pmatrix}-\begin{pmatrix}16.5\\9.5\end{pmatrix}\right\|^2 = \left\|\begin{matrix}-1.5\\-2.5\end{matrix}\right\|^2 = 8.5$

$k=2, \; \left\|\begin{pmatrix}15\\7\end{pmatrix}-\begin{pmatrix}-1\\-7\end{pmatrix}\right\|^2 = \left\|\begin{matrix}16\\14\end{matrix}\right\|^2 = 452$

$k=3, \; \left\|\begin{pmatrix}15\\7\end{pmatrix}-\begin{pmatrix}2.5\\5\end{pmatrix}\right\|^2 = \left\|\begin{matrix}12.5\\2\end{matrix}\right\|^2 = 160.25$

$\therefore \; C^2(1) = 1$

$C^2(2) = \underset{1 \le k \le K}{\operatorname{argmin}} \|x_2 - m_k\|^2$

$k=1, \; \left\|\begin{pmatrix}0\\-9\end{pmatrix}-\begin{pmatrix}16.5\\9.5\end{pmatrix}\right\|^2 = \left\|\begin{pmatrix}-16.5\\-18.5\end{pmatrix}\right\|^2 = 614.5$

$k=2, \; \left\|\begin{pmatrix}0\\-9\end{pmatrix}-\begin{pmatrix}-1\\-7\end{pmatrix}\right\|^2 = \left\|\begin{pmatrix}1\\-2\end{pmatrix}\right\|^2 = 5$

$k=3, \; \left\|\begin{pmatrix}0\\-9\end{pmatrix}-\begin{pmatrix}2.5\\5\end{pmatrix}\right\|^2 = \left\|\begin{pmatrix}-2.5\\-14\end{pmatrix}\right\|^2 = 202.25$

$\therefore \; C^2(2) = 2$

$C^2(3) = \underset{1 \le k \le K}{\operatorname{argmin}} \|x_3 - m_k\|^2$

$k=1, \; \left\|\begin{pmatrix}-2\\-5\end{pmatrix}-\begin{pmatrix}16.5\\9.5\end{pmatrix}\right\|^2 = \left\|\begin{pmatrix}-18.5\\-14.5\end{pmatrix}\right\|^2 = 552.5$

$k=2, \; \left\|\begin{pmatrix}-2\\-5\end{pmatrix}-\begin{pmatrix}-1\\-7\end{pmatrix}\right\|^2 = \left\|\begin{pmatrix}-1\\2\end{pmatrix}\right\|^2 = 5$

$$k=3, \quad \left\| \begin{pmatrix} -2 \\ -5 \end{pmatrix} - \begin{pmatrix} 2.5 \\ 5 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} -4.5 \\ -10 \end{pmatrix} \right\|^2 = 120.25$$

$$\therefore \; C^2(3) = 2$$

$$C^2(4) = \underset{1 \le k \le K}{\operatorname{argmin}} \; \| x_4 - m_k \|^2$$

$$k=1, \quad \left\| \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 16.5 \\ 9.5 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} -14.5 \\ -6.5 \end{pmatrix} \right\|^2 = 252.5$$

$$k=2, \quad \left\| \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ -7 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 3 \\ 10 \end{pmatrix} \right\|^2 = 109$$

$$k=3, \quad \left\| \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2.5 \\ 5 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} -0.5 \\ -2 \end{pmatrix} \right\|^2 = 4.25$$

$$\therefore \; C^2(4) = 3$$

$$C^2(5) = \underset{1 \le k \le K}{\operatorname{argmin}} \; \| x_5 - m_k \|^2$$

$$k=1, \quad \left\| \begin{pmatrix} 3 \\ 7 \end{pmatrix} - \begin{pmatrix} 16.5 \\ 9.5 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} -13.5 \\ -2.5 \end{pmatrix} \right\|^2 = 188.5$$

$$k=2, \quad \left\| \begin{pmatrix} 3 \\ 7 \end{pmatrix} - \begin{pmatrix} -1 \\ -7 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 4 \\ 14 \end{pmatrix}^2 \right\|^2 = 212$$

$$k=3, \quad \left\| \begin{pmatrix} 3 \\ 7 \end{pmatrix} - \begin{pmatrix} 2.5 \\ 5 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 0.5 \\ 2 \end{pmatrix} \right\|^2 = 4.25$$

$$\therefore \; C^2(5) = 3$$

$$C^2(6) = \underset{1 \le k \le K}{\operatorname{argmin}} \; \| x_6 - m_k \|^2$$

$$k=1, \quad \left\| \begin{pmatrix} 18 \\ 12 \end{pmatrix} - \begin{pmatrix} 16.5 \\ 9.5 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 1.5 \\ 2.5 \end{pmatrix} \right\|^2 = 8.5$$

$$k=2, \quad \left\| \begin{pmatrix} 18 \\ 12 \end{pmatrix} - \begin{pmatrix} -1 \\ -7 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 19 \\ 19 \end{pmatrix} \right\|^2 = 722$$

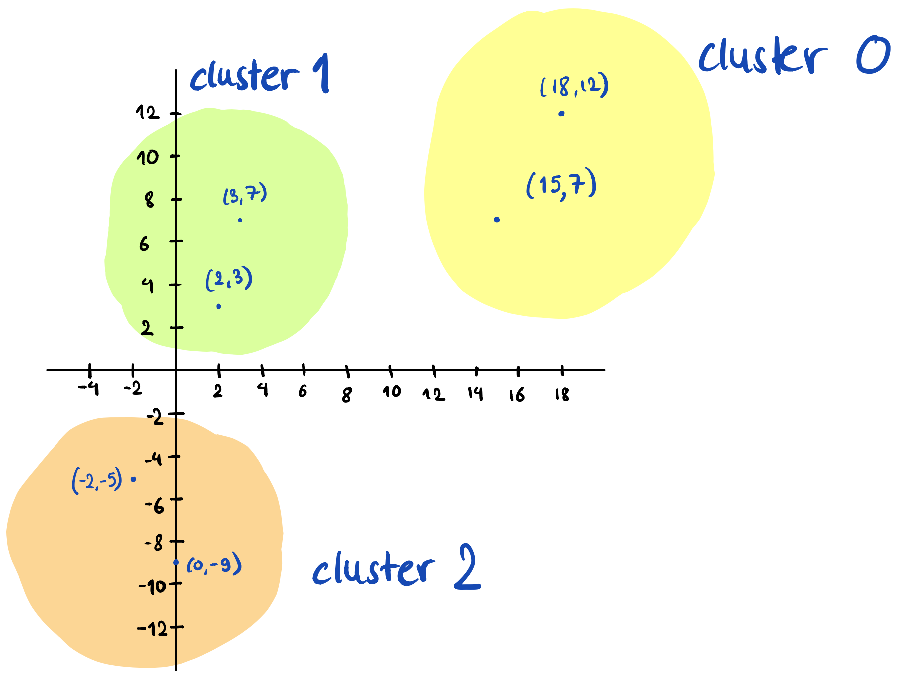
$$k=3, \quad \left\| \begin{pmatrix} 18 \\ 12 \end{pmatrix} - \begin{pmatrix} 2.5 \\ 5 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 15.5 \\ 7 \end{pmatrix} \right\|^2 = 289.25$$

$$\therefore \; C^2(6) = 1$$

After iteration $=2$,

i	x_i	$C^{(2)}(i)$
1	$(15,7)^\top$	0
2	$(0,-9)^\top$	1
3	$(-2,-5)^\top$	1
4	$(2,3)^\top$	2
5	$(3,7)^\top$	2
6	$(18,12)^\top$	0

Since $C^1 = C^2$, we can stop the algorithm. We get $C^2(i)$ as our clusters with $i = 1, 2, 3, \ldots, 6$



Excercise 2. (Principle component analysis)

You are given the data set

$$\{(1,0)^\top, (0.5,1)^\top, (1,0.5)^\top\}.$$

Manually compute the principle components of this data set by solving the associated eigenvalue problem.

(4 Points)

x	1	0.5	1
y	0	1	0.5

Mean of $x = \bar{x} = 5/6$

Mean of $y = \bar{y} = 1/2$

Ordered pairs: $(x,x), (x,y), (y,x), (y,y)$

Covariance of all ordered pairs

$$\mathrm{Cov}(x,x) = \frac{1}{N-1}\sum_{k=1}^{N}(x_{ik} - \bar{x}_i)(x_{jk} - \vec{\bar{x}_j})$$

$$= \frac{1}{3-1}\left[\left(1 - 5/6\right)^2 + \left(0.5 - 5/6\right)^2 + \left(1 - 5/6\right)^2\right]$$

$$= \frac{1}{2} \times \frac{1}{6}$$

$$= \frac{1}{12}$$

$$\mathrm{Cov}(x,y) = \frac{1}{3-1}\left[\left(1-\frac{5}{6}\right)\left(0-\frac{1}{2}\right) + \left(0.5-\frac{5}{6}\right)\left(1-\frac{1}{2}\right) + \left(1-\frac{5}{6}\right)\left(0.5-\frac{1}{2}\right)\right]$$

$$= \frac{1}{2} \times -\frac{1}{4}$$

$$= -\frac{1}{8}$$

$$\mathrm{Cov}(y,x) = \mathrm{Cov}(x,y) = -\frac{1}{8}$$

$$\mathrm{Cov}(y,y) = \frac{1}{3-1}\left[\left(0-\frac{1}{2}\right)^2 + \left(1-\frac{1}{2}\right)^2 + \left(0.5-\frac{1}{2}\right)^2\right]$$

$$= \frac{1}{4}$$

$$\text{Covariance Matrix} = \begin{bmatrix} cov(x,x) & cov(x,y) \\ cov(y,x) & cov(y,y) \end{bmatrix}$$

$$C = \begin{bmatrix} 1/12 & -1/8 \\ -1/8 & 1/4 \end{bmatrix}$$

Now, we find the eigenvalues of C

$$(C - \lambda I) = \begin{pmatrix} 1/12 & -1/8 \\ -1/8 & 1/4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\det(C - \lambda I) = \det \begin{pmatrix} 1/12 - \lambda & -1/8 \\ -1/8 & 1/4 - \lambda \end{pmatrix}$$

$$= \left(\frac{1}{12} - \lambda\right)\left(\frac{1}{4} - \lambda\right) - \left(-\frac{1}{8}\right)\left(-\frac{1}{8}\right)$$

$$= \frac{1}{12}\left(\frac{1}{4} - \lambda\right) - \lambda\left(\frac{1}{4} - \lambda\right) - \frac{1}{64}$$

$$= \frac{1}{48} - \frac{1}{12}\lambda - \frac{1}{4}\lambda + \lambda^2 - \frac{1}{64}$$

$$0 = \lambda^2 - \frac{1}{3}\lambda + \frac{1}{192}$$

$$\lambda_1 = 0.316 \quad , \quad \lambda_2 = 0.016$$

Eigenvector corresponding to eigenvalue 0.316 is

$$\begin{pmatrix} 1/12 - \lambda_1 & -1/8 \\ -1/8 & 1/4 - \lambda_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.232 & -0.125 \\ -0.125 & -0.066 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$ Eigenvector of λ_1 is $\begin{bmatrix} 2 - \sqrt{13} \\ 3 \end{bmatrix}$

Eigenvector corresponding to eigenvalue 0.016 is

$$\begin{pmatrix} 1/12 - \lambda_2 & -1/8 \\ -1/8 & 1/4 - \lambda_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0.067 & -0.125 \\ -0.125 & 0.234 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$ Eigenvector of λ_2 is $\begin{bmatrix} 2+\sqrt{13} \\ 3 \end{bmatrix}$

$\therefore$ Principle components are $\left\{ \begin{bmatrix} 2-\sqrt{13} \\ 3 \end{bmatrix}, \begin{bmatrix} 2+\sqrt{13} \\ 3 \end{bmatrix} \right\}$

Excercise 3. (Principle component analysis for data compression)

A well-known application of principle component analysis is lossy data compression. In this application, you are given a large data set $\{\mathbf{x}\}_{i=1}^N$ with $\mathbf{x}_i \in \mathbb{R}^D$ and reduce it to a data set $\{\tilde{\mathbf{x}}\}_{i=1}^N$ with $\tilde{\mathbf{x}}_i \in \mathbb{R}^d$ where $d < D$, while storing a matrix that allows to reconstruct an approximation to the $\mathbf{x}_i$ from the vectors $\tilde{\mathbf{x}}_i$.

Develop and give a compression and a decompression algorithm which carry out the above described data compression and decompression tasks by using principle component analysis. You can either try to develop the idea by yourself or do a research in the internet. In the latter case, please quote the source from which you took the information.

(4 Points)

Input : $\{x\}_{i=1}^N$ where $x_1, x_2, x_3, x_4, \ldots, x_N \in \mathbb{R}^D$

Step 1 : Preprocessing / mean normalization

$$\mu_j = \frac{1}{m} \sum_{i=1}^{m} x_j^{(i)}$$

Mean normalized dataset $(\bar{x})$

$$\bar{x}_j = x_j - \mu_j$$

Step 2 : Compute covariance matrix (Σ)

$$\Sigma = \frac{1}{m} \sum_{i=1}^{n} (x^{(i)})(x^{(i)})^T$$

Step 3 : Compute "eigenvectors" of matrix Σ using SVD

$$[U, S, V] = svd(\Sigma)$$

We get $U = \begin{bmatrix} | & | & | & | & & | \\ u^{(1)} & u^{(2)} & u^{(3)} & u^{(4)} & \cdots & u^{(n)} \\ | & | & | & | & & | \end{bmatrix} \in \mathbb{R}^{n \times n}$

Now in order to compress $\{x\}_{i=1}^N$ with $x_i \in \mathbb{R}^D$ and reduce it to $\{\tilde{x}\}_i^N$, we simply take the first N columns of U and store it into a new vector $U_{reduce} = U(:, 1:k)$.

$$z = U_{reduce}^T \cdot x$$

Reference Video: https://www.youtube.com/watch?v=rng04VJxUt4