# ASSIGNMENT 1 SOLUTION

Submitted by: Nayan Man Singh Pradhan

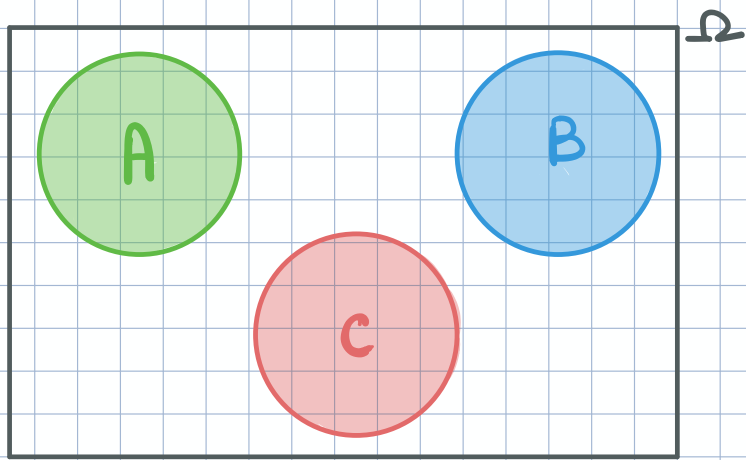
**Excercise 1.** (Random experiments)

The disjoint events $A$, $B$, and $C$ are defined in a sample space $\Omega$. Find expressions for the following probabilities in terms of sums the probabilities $P(\emptyset)$,$P(A), P(B), P(A \cup B), P(A \cup C), P(B \cup C), P(A \cup B \cup C)$:
(Example: the probability that $A$ or $B$ occurs: $P(A) + P(B)$)

a) the probability that exactly two of A; B; C occur *(at the same time!)*;

b) the probability that exactly one of these events occurs;

(4 Points)




a) Probability that exactly two of A; B; C occurs : 0
(Since they are disjoint events!)

b) The probability that exactly one of these events occurs $= 1 - P\left(\begin{matrix}\text{two occurs at} \\ \text{the same time}\end{matrix}\right)$

$= 1 - 0$

$= 1$

**Excercise 2.** We consider the chance experiment of four tosses of a fair coin.

a) Give the sample space $\Omega$ for this experiment and define a discrete random variable $X$ to describe the number of heads obtained in four tosses of a fair coin.

b) Find the PMF and CDF of the defined random variable and plot the CDF.

c) Compute the probability of the event $X$ IS BIGGER THAN 1

d) Compute the mean of the random variable.

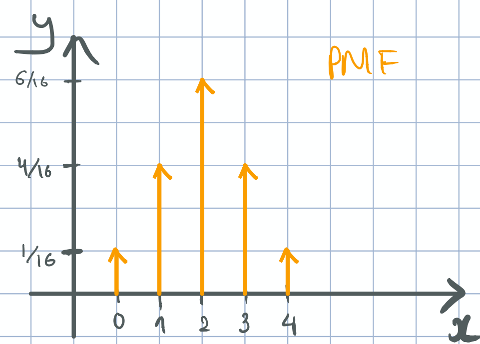
(4 Points)

a) $\Omega = \{(H,H,H,H), (H,H,H,T), (H,H,T,H), (H,H,T,T), (H,T,H,H), (H,T,H,T),$
$(H,T,T,H), (H,T,T,T), (T,T,T,T), (T,T,T,H), (T,T,H,T), (T,T,H,H),$
$(T,H,T,T), (T,H,T,H), (T,H,H,T), (T,H,H,H)\}$

$X : \Omega \rightarrow \{4, 3, 2, 1, 0\}$

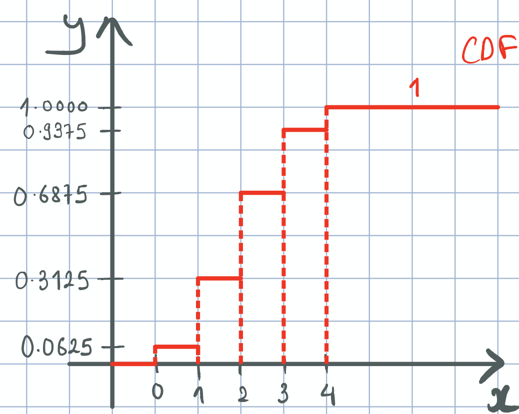
b) PMF of $X$:

$p(4) = P(X=4) = 1/16$
$p(3) = P(X=3) = 4/16$
$p(2) = P(X=2) = 6/16$
$p(1) = P(X=1) = 4/16$
$p(0) = P(X=0) = 1/16$




CDF of $X$:

$F(0) = P(X \le 0) = 1/16 = 0.0625$
$F(1) = P(X \le 1) = 5/16 = 0.3125$
$F(2) = P(X \le 2) = 11/16 = 0.6875$
$F(3) = P(X \le 3) = 15/16 = 0.9375$
$F(4) = P(X \le 4) = 16/16 = 1$




c) $P(X>1) = 1 - P(X \le 1)$
$= 1 - 5/16$
$= 11/16$
$\approx 0.6875$

d) $E(X) = \sum_{i=0}^{4} x_i \, p(x_i)$

$= 0 \times \frac{1}{16} + 1 \times \frac{4}{16} + 2 \times \frac{6}{16} + 3 \times \frac{4}{16} + 4 \times \frac{1}{16}$

$= 0 + 4/16 + 12/16 + 12/16 + 4/16$

$= 32/16$

$= 2$

**Excercise 3.** Let the function $\rho$ with $\rho(x) = \begin{cases} c|x|(1+x)(1-x) & \text{if } -1 \le x \le 1, \\ 0 & \text{else,} \end{cases}$ be given.

a) Compute the constant $c$, such that $\rho$ becomes a density of some random variable.

b) Consider a random variable $X$ with the just computed density $\rho$. Find its CDF and plot it.

c) Use the CDF to compute

$$P(X < -0.5), \quad P(X > 0.5), \quad P(-0.5 < X < 0.5)\,.$$

(4 Points)

a) $\int_{-1}^{1} c\,|x|(1+x)(1-x)\,dx = 1$

or, $c\int_{-1}^{1} |x|(1+x)(1-x)\,dx = 1$

or, $-\int_{-1}^{0} x(1-x^2)\,dx + \int_{0}^{1} x(1-x^2)\,dx = \frac{1}{c}$

or, $-\int_{-1}^{0} (x-x^3)\,dx + \int_{0}^{1} (x-x^3)\,dx = \frac{1}{c}$

or, $-\left[\int_{-1}^{0} x\,dx - \int_{-1}^{0} x^3\,dx\right] + \left[\int_{0}^{1} x\,dx - \int_{0}^{1} x^3\,dx\right] = \frac{1}{c}$

or, $-\left[\left[\frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^4}{4}\right]_{-1}^{0}\right] + \left[\left[\frac{x^2}{2}\right]_{0}^{1} - \left[\frac{x^4}{4}\right]_{0}^{1}\right] = \frac{1}{c}$

or, $-\left[-\frac{1}{2} + \frac{1}{4}\right] + \left[\frac{1}{2} - \frac{1}{4}\right] = \frac{1}{c}$

or, $-\left[-\frac{1}{4}\right] + \left[\frac{1}{4}\right] = \frac{1}{c}$

or, $\frac{1}{2} = \frac{1}{c}$

$\therefore\ c = 2$

$\therefore\ \rho(x) = \begin{cases} 2|x|(1+x)(1-x), & \text{if } -1 \le x \le 1 \\ 0, & \text{else} \end{cases}$

b) We have,

$$PMF = p(x) = \begin{cases} 2|x|(1+x)(1-x), & \text{if } -1 \le x \le 1 \\ 0 & \text{; else} \end{cases}$$

$$p(x) = \begin{cases} -2x(1+x)(1-x) & \text{; if } x < 0 \\ 2x(1+x)(1-x) & \text{; if } x \ge 0 \\ 0 & \text{; else} \end{cases}$$

We know that,

$$PMF = \frac{d}{dt} CDF \Rightarrow CDF = \int_t PMF\, dt$$

$$CDF = \int 2|x|(1-x^2)\,dx$$

When $x < 0$

$$\begin{aligned} CDF &= -\int 2x(1-x^2)\,dx \\ &= -2\int (x-x^3)\,dx \\ &= -2\left[\frac{x^2}{2} - \frac{x^4}{4} + c_1\right] \end{aligned}$$

When $x \ge 0$

$$\begin{aligned} CDF &= \int 2x(1-x^2)\,dx \\ &= 2\int (x-x^3)\,dx \\ &= 2\left[\frac{x^2}{2} - \frac{x^4}{4} + c_2\right] \end{aligned}$$

We know that at $x = -1$, $CDF = 0$

$$0 = -2\left[\frac{1}{2} - \frac{1}{4} + c_1\right]$$

or, $0 = \frac{1}{4} + c_1$

$\therefore c_1 = -\frac{1}{4}$

We know that at $x = 1$, $CDF = 1$

$$1 = 2 \times \left[\frac{1}{2} - \frac{1}{4} + c_2\right]$$

or, $1 = 2 \times \left[\frac{1}{4} + c_2\right]$

or, $1 = \frac{1}{2} + 2c_2$

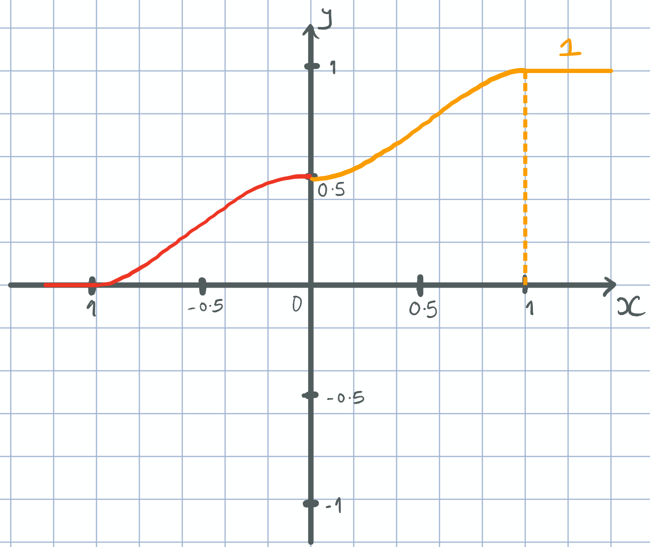
or, $\frac{1}{2} = 2c_2$

$\therefore c_2 = \frac{1}{4}$

$$\therefore CDF = \begin{cases} -2\left(\frac{x^2}{2} - \frac{x^4}{4} - \frac{1}{4}\right), & \text{if } x < 0 \\ 2\left(\frac{x^2}{2} - \frac{x^4}{4} + \frac{1}{4}\right), & \text{if } x \ge 0 \\ 0 & \text{, else} \end{cases}$$

Plot of CDF

$$\text{CDF} = \begin{cases} -2\left(\frac{x^2}{2} - \frac{x^4}{4} - \frac{1}{4}\right), & \text{if } x < 0 \\ 2\left(\frac{x^2}{2} - \frac{x^4}{4} + \frac{1}{4}\right), & \text{if } x \geq 0 \\ 0 & \text{, else} \end{cases}$$




c) Now,

$$P(X < -0.5) = F_X(-0.5) - P(X=0.5)$$
$$= -2\left(\frac{(-0.5)^2}{2} - \frac{(-0.5)^4}{4} - \frac{1}{4}\right) - 0$$
$$= 0.28125$$

$$P(X > 0.5) = 1 - P(X \leq 0.5)$$
$$= 1 - F_X(0.5)$$
$$= 1 - 2\left(\frac{(0.5)^2}{2} - \frac{(0.5)^4}{4} + \frac{1}{4}\right)$$
$$= 1 - 0.71875$$
$$= 0.28125$$

$$P(-0.5 < X < 0.5) = F_X(0.5) - P(X=0.5) - F_X(-0.5)$$
$$= 0.71875 - 0 - 0.28125$$
$$= 0.4375$$

OR

$$= 1 - \left(P(X < -0.5) + P(X > 0.5) + P(X=0.5) + P(X=-0.5)\right)$$
$$= 1 - (0.28125 + 0.28125 + 0 + 0)$$
$$= 1 - 0.5625$$
$$= 0.4375$$

**Programming Exercise 1.** In this first programming exercise, we would like to familiarize us with typical linear algebra computing tasks. To this end, we define the vectors / matrices

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}, \quad M = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Students are free to choose between the programming languages Python and C/C++. In case of Python, the module *NumPy* should be used for the linear algebra tasks. For C++, there exists a convenient numerics header-only library called *Eigen*. Classical C programmers can go for *LAPACK*. (Reference solutions will only be provided in Python.) The submission format for Python is a Jupyter notebook. The submission format for C/C++ is standard source files.

Use one of the listed programming languages with the respective library to implement the following tasks:

a) Compute and print the inner product between $\mathbf{a}$ and $\mathbf{b}$.

b) Compute and print the matrix-vector product between $M$ and $\mathbf{b}$.

c) Compute and print the $l_2$ norm of $\mathbf{b}$.

d) Compute and print the the solution $\mathbf{x}$ of the linear system of equations $M\mathbf{x} = \mathbf{b}$ using the LU factorization.

e) Compute and print the the solution $x$ of the linear system of equations $M\mathbf{x} = \mathbf{b}$ using the Cholesky factorization.

(4 Points)

→ Solved in python in Jupyter notebook and submitted seperate file in Moodle.

In [5]:

```
## Defining all functions used in this programming exercise

## Function that computes inner product
def inner_product_calc(a, b):
    return np.inner(a,b)

## Function that computes matrix vector product
def matrix_vector_product_calc(M, b):
    return np.dot(M, b)
#     return M@b ## this works too

## Function that computes vector norm
def norm_calc(a):
    return np.linalg.norm(a)

## Function that computes solution of linear system using LU factorization
def LU_solver(M, b):
    LU = linalg.lu_factor(M)
    return (linalg.lu_solve(LU, b))

## Function that computes solution of linear system using Cholesky factorization
def Cholesky_solver(M, b):
    c, low = linalg.cho_factor(M)
    return (linalg.cho_solve((c, low), b))
```