

# ASSIGNMENT 1 SOLUTION

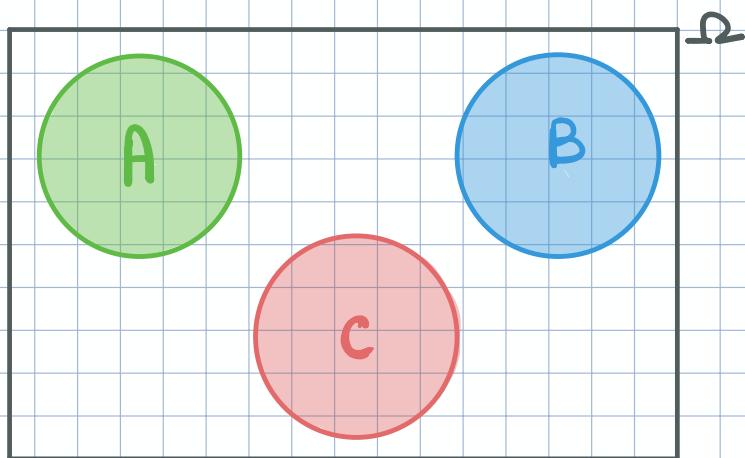
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## Excercise 1. (Random experiments)

The disjoint events  $A$ ,  $B$ , and  $C$  are defined in a sample space  $\Omega$ . Find expressions for the following probabilities in terms of sums the probabilities  $P(\emptyset), P(A), P(B), P(A \cup B), P(A \cup C), P(B \cup C), P(A \cup B \cup C)$ :  
(Example: the probability that  $A$  or  $B$  occurs:  $P(A) + P(B)$ )

- the probability that exactly two of  $A$ ;  $B$ ;  $C$  occur (*at the same time!*);
- the probability that exactly one of these events occurs;

(4 Points)



a) Probability that exactly two of  $A; B; C$  occurs : 0  
(*at the same time!*)  
(Since they are disjoint events!)

b) The probability that exactly one of these events occurs =  $1 - P(\text{two occurs at the same time})$

$$= 1 - 0$$

$$= 1,$$

**Excercise 2.** We consider the chance experiment of four tosses of a fair coin.

- Give the sample space  $\Omega$  for this experiment and define a discrete random variable  $X$  to describe the number of heads obtained in four tosses of a fair coin.
- Find the PMF and CDF of the defined random variable and plot the CDF.
- Compute the probability of the event  $X \text{ IS BIGGER THAN } 1$
- Compute the mean of the random variable.

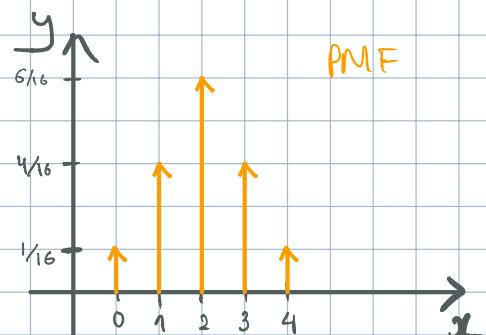
(4 Points)

a)  $\Omega = \{(H, H, H, H), (H, H, H, T), (H, H, T, H), (H, H, T, T), (H, T, H, H), (H, T, H, T), (H, T, T, H), (H, T, T, T), (T, H, H, H), (T, H, H, T), (T, H, T, H), (T, H, T, T), (T, T, H, H), (T, T, H, T), (T, T, T, H), (T, T, T, T)\}$

$X : \Omega \rightarrow \{4, 3, 2, 1, 0\}$

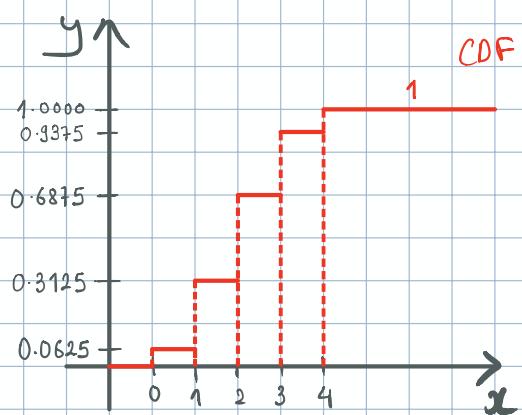
b) PMF of  $X$ :

$$\begin{aligned} p(4) &= P(X=4) = 1/16 \\ p(3) &= P(X=3) = 4/16 \\ p(2) &= P(X=2) = 6/16 \\ p(1) &= P(X=1) = 4/16 \\ p(0) &= P(X=0) = 1/16 \end{aligned}$$



CDF of  $X$ :

$$\begin{aligned} F(0) &= P(X \leq 0) = 1/16 = 0.0625 \\ F(1) &= P(X \leq 1) = 5/16 = 0.3125 \\ F(2) &= P(X \leq 2) = 11/16 = 0.6875 \\ F(3) &= P(X \leq 3) = 15/16 = 0.9375 \\ F(4) &= P(X \leq 4) = 16/16 = 1 \end{aligned}$$



$$\begin{aligned} \text{c)} \quad P(X > 1) &= 1 - P(X \leq 1) \\ &= 1 - 5/16 \\ &= 11/16 \\ &\approx 0.6875 \end{aligned}$$

$$\begin{aligned} \text{d)} \quad E(X) &= \sum_{i=0}^4 x_i p(x_i) \\ &= 0 \times \frac{1}{16} + 1 \times \frac{4}{16} + 2 \times \frac{6}{16} + 3 \times \frac{4}{16} + 4 \times \frac{1}{16} \\ &= 0 + 4/16 + 12/16 + 12/16 + 4/16 \\ &= 32/16 \\ &= 2 \end{aligned}$$

**Excercise 3.** Let the function  $\rho$  with  $\rho(x) = \begin{cases} c|x|(1+x)(1-x) & \text{if } -1 \leq x \leq 1, \\ 0 & \text{else,} \end{cases}$  be given.

- Compute the constant  $c$ , such that  $\rho$  becomes a density of some random variable.
- Consider a random variable  $X$  with the just computed density  $\rho$ . Find its CDF and plot it.
- Use the CDF to compute

$$P(X < -0.5), \quad P(X > 0.5), \quad P(-0.5 < X < 0.5).$$

(4 Points)

$$\text{a)} \int_{-1}^1 c|x|(1+x)(1-x) dx = 1$$

$$\text{or, } c \int_{-1}^1 |x|(1+x)(1-x) dx = 1$$

$$\text{or, } - \int_{-1}^0 x(1-x^2) dx + \int_0^1 x(1-x^2) dx = \frac{1}{c}$$

$$\text{or, } - \int_{-1}^0 (x-x^3) dx + \int_0^1 (x-x^3) dx = \frac{1}{c}$$

$$\text{or, } - \left[ \int_{-1}^0 x dx - \int_{-1}^0 x^3 dx \right] + \left[ \int_0^1 x dx - \int_0^1 x^3 dx \right] = \frac{1}{c}$$

$$\text{or, } - \left[ \left[ \frac{x^2}{2} \right]_{-1}^0 - \left[ \frac{x^4}{4} \right]_{-1}^0 \right] + \left[ \left[ \frac{x^2}{2} \right]_0^1 - \left[ \frac{x^4}{4} \right]_0^1 \right] = \frac{1}{c}$$

$$\text{or, } - \left[ -\frac{1}{2} + \frac{1}{4} \right] + \left[ \frac{1}{2} - \frac{1}{4} \right] = \frac{1}{c}$$

$$\text{or, } - \left[ -\frac{1}{4} \right] + \left[ \frac{1}{4} \right] = \frac{1}{c}$$

$$\text{or, } \frac{1}{2} = \frac{1}{c}$$

$$\therefore c = 2,$$

$$\therefore \rho(x) = \begin{cases} 2|x|(1+x)(1-x), & \text{if } -1 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

b) We have,

$$\text{PMF} = p(x) = \begin{cases} 2|x| (1+x)(1-x), & \text{if } -1 \leq x \leq 1 \\ 0, & \text{else} \end{cases}$$

$$p(x) = \begin{cases} -2x(1+x)(1-x), & \text{if } x < 0 \\ 2x(1+x)(1-x), & \text{if } x \geq 0 \\ 0, & \text{else} \end{cases}$$

We know that,

$$\text{PMF} = \frac{d}{dt} \text{CDF} \Rightarrow \text{CDF} = \int_t \text{PMF} dt$$

$$\text{CDF} = \int 2|x|(1-x^2) dx$$

When  $x < 0$

$$\begin{aligned} \text{CDF} &= - \int 2x(1-x^2) dx \\ &= -2 \int (x-x^3) dx \\ &= -2 \left[ \frac{x^2}{2} - \frac{x^4}{4} + C_1 \right] \end{aligned}$$

When  $x \geq 0$

$$\begin{aligned} \text{CDF} &= \int 2x(1-x^2) dx \\ &= 2 \int (x-x^3) dx \\ &= 2 \left[ \frac{x^2}{2} - \frac{x^4}{4} + C_2 \right] \end{aligned}$$

We know that at  $x = -1$ ,  $\text{CDF} = 0$

$$\begin{aligned} 0 &= -2 \left[ \frac{1}{2} - \frac{1}{4} + C_1 \right] \\ \text{or, } 0 &= \frac{1}{4} + C_1 \\ \therefore C_1 &= -\frac{1}{4} \end{aligned}$$

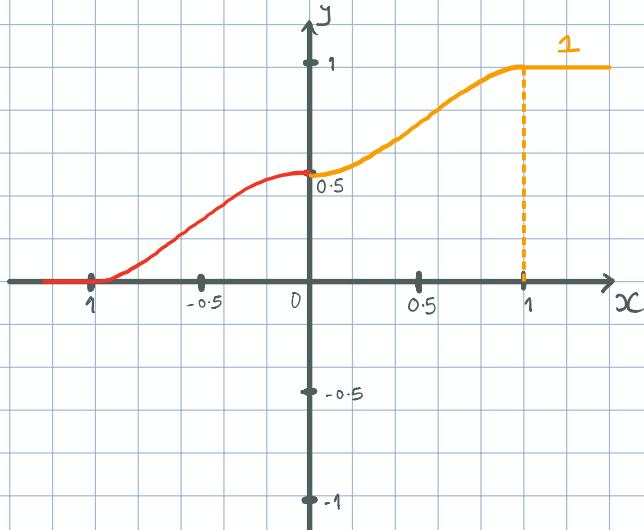
We know that at  $x = 1$ ,  $\text{CDF} = 1$

$$\begin{aligned} 1 &= 2x \left[ \frac{1}{2} - \frac{1}{4} + C_2 \right] \\ \text{or, } 1 &= 2x \left[ \frac{1}{4} + C_2 \right] \\ \text{or, } 1 &= \frac{1}{2} + 2C_2 \\ \text{or, } \frac{1}{2} &= 2C_2 \\ \therefore C_2 &= \frac{1}{4}, \end{aligned}$$

$$\therefore \text{CDF} = \begin{cases} -2 \left( \frac{x^2}{2} - \frac{x^4}{4} - \frac{1}{4} \right), & \text{if } x < 0 \\ 2 \left( \frac{x^2}{2} - \frac{x^4}{4} + \frac{1}{4} \right), & \text{if } x \geq 0 \\ 0, & \text{else} \end{cases}$$

## Plot of CDF

$$CDF = \begin{cases} -2\left(\frac{x^2}{2} - \frac{x^4}{4} - \frac{1}{4}\right), & \text{if } x < 0 \\ 2\left(\frac{x^2}{2} - \frac{x^4}{4} + \frac{1}{4}\right), & \text{if } x \geq 0 \\ 0, & \text{else} \end{cases}$$



c) Now,

$$\begin{aligned} P(X < -0.5) &= F_X(-0.5) - P(X = 0.5) \\ &= -2\left(\frac{(-0.5)^2}{2} - \frac{(-0.5)^4}{4} - \frac{1}{4}\right) - 0 \\ &= 0.28125, \end{aligned}$$

$$\begin{aligned} P(X > 0.5) &= 1 - P(X \leq 0.5) \\ &= 1 - F_X(0.5) \\ &= 1 - 2\left(\frac{(0.5)^2}{2} - \frac{(0.5)^4}{4} + \frac{1}{4}\right) \\ &= 1 - 0.71875 \\ &= 0.28125, \end{aligned}$$

$$\begin{aligned} P(-0.5 < X < 0.5) &= F_X(0.5) - P(X = 0.5) - F_X(-0.5) \\ &= 0.71875 - 0 - 0.28125 \\ &= 0.4375, \end{aligned}$$

OR

$$\begin{aligned} &= 1 - (P(X < -0.5) + P(X > 0.5) + P(X = 0.5) + P(X = -0.5)) \\ &= 1 - (0.28125 + 0.28125 + 0 + 0) \\ &= 1 - 0.5625 \\ &= 0.4375, \end{aligned}$$

**Programming Exercise 1.** In this first programming exercise, we would like to familiarize us with typical linear algebra computing tasks. To this end, we define the vectors / matrices

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}, \quad M = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Students are free to choose between the programming languages Python and C/C++. In case of Python, the module *NumPy* should be used for the linear algebra tasks. For C++, there exists a convenient numerics header-only library called *Eigen*. Classical C programmers can go for *LAPACK*. (Reference solutions will only be provided in Python.) The submission format for Python is a Jupyter notebook. The submission format for C/C++ is standard source files.

Use one of the listed programming languages with the respective library to implement the following tasks:

- a) Compute and print the inner product between  $\mathbf{a}$  and  $\mathbf{b}$ .
- b) Compute and print the matrix-vector product between  $M$  and  $\mathbf{b}$ .
- c) Compute and print the  $l_2$  norm of  $\mathbf{b}$ .
- d) Compute and print the the solution  $\mathbf{x}$  of the linear system of equations  $M\mathbf{x} = \mathbf{b}$  using the LU factorization.
- e) Compute and print the the solution  $x$  of the linear system of equations  $M\mathbf{x} = \mathbf{b}$  using the Cholesky factorization.

(4 Points)

→ Solved in python in Jupyter notebook and submitted sepeate file in Moodle.

```
In [5]: 1 ## Defining all functions used in this programming exercise
2
3 ## Function that computes inner product
4 def inner_product_calc(a, b):
5     return np.inner(a,b)
6
7 ## Function that computes matrix vector product
8 def matrix_vector_product_calc(M, b):
9     return np.dot(M, b)
10 #     return M@b ## this works too
11
12 ## Function that computes vector norm
13 def norm_calc(a):
14     return np.linalg.norm(a)
15
16 ## Function that computes solution of linear system using LU factorization
17 def LU_solver(M, b):
18     LU = linalg.lu_factor(M)
19     return (linalg.lu_solve(LU, b))
20
21 ## Function that computes solution of linear system using Cholesky factorization
22 def Cholesky_solver(M, b):
23     c, low = linalg.cho_factor(M)
24     return (linalg.cho_solve((c, low), b))
```