ASSIGNMENT 2 SOLUTION

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Excercise 1. (Multivariate discrete random variables)

We consider the chance experiment of a fair coin that is tossed four times.

a) Give the sample space.

b) Define the random variable X to be the number of heads, Y as the number of tails and Z as $Z = |X - Y|$.

c) Compute the joint PMF of (X, Z).

d) Compute the expectation $E(Y)$.

(4 Points)

a) $\Omega = \{(H,H,H,H), (H,H,H,T), (H,H,T,H), (H,H,T,T), (H,T,H,H),$
$(H,T,H,T), (H,T,T,H), (H,T,T,T), (T,H,H,H), (T,H,H,T), (T,H,T,H),$
$(T,H,T,T), (T,T,H,H), (T,T,H,T), (T,T,T,H), (T,T,T,T)\}$

b) $X : \Omega \to \{0, 1, 2, 3, 4\}$

$Y : \Omega \to \{0, 1, 2, 3, 4\}$

$Z : \Omega \to \{0, 1, 2, 3, 4\}$

c) Joint PMF of (X, Z)

$$p(0,0) = P(X=0, Z=0) = P(\{\omega \in \Omega \mid X(\omega)=0\} \cap \{\omega \in \Omega \mid Z(\omega)=0\})$$
$$= \frac{1}{5} \times \frac{1}{5}$$
$$= \frac{1}{25}$$

$p(0,1) = 1/25$
$p(0,2) = 1/25$
$p(0,3) = 1/25$
$p(0,4) = 1/25$
$p(1,0) = 1/25$
$p(1,1) = 1/25$
$p(1,2) = 1/25$
$p(1,3) = 1/25$
$p(1,4) = 1/25$

$p(2,0) = 1/25$
$p(2,1) = 1/25$
$p(2,2) = 1/25$
$p(2,3) = 1/25$
$p(2,4) = 1/25$
$p(3,0) = 1/25$
$p(3,1) = 1/25$
$p(3,2) = 1/25$
$p(3,3) = 1/25$
$p(3,4) = 1/25$

$p(4,0) = 1/25$
$p(4,1) = 1/25$
$p(4,2) = 1/25$
$p(4,3) = 1/25$
$p(4,4) = 1/25$

* Check : $\sum_i \sum_j p(x_i, y_j) = 1$

d) $E(Y) = \sum_{y \in R_Y} p(Y) \cdot y = 0 \times \frac{1}{5} + 1 \times \frac{1}{5} + 2 \times \frac{1}{5} + 3 \times \frac{1}{5} + 4 \times \frac{1}{5} = 0 + \frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5} = 2$ //

Excercise 2. (Conditional properties of multivariate discrete random variables)

We consider the chance experiment of a fair dice that is rolled two times. Let X_1 be the outcome of the first roll, while X_2 is the outcome of the second roll.

a) Describe the chance experiment by its sample space, and define the random variables.

b) Compute the conditional expectation $E(X_1 + X_2 | X_2 = x_2)$.

c) Compute the conditional expectation $E(X_1 X_2 | X_2 = x_2)$.

d) Compute the conditional variance $Var(X_1^2 X_2 | X_2 = x_2)$.

Hint: There might be easier and harder ways to solve these tasks.

(4 Points)

a) $\Omega = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

$\Omega = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),$
$(2,4),(2,5),(2,6),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),$
$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,1),(5,2),(5,3),$
$(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

$X_1 : \Omega \to \{1,2,3,4,5,6\}$

$X_2 : \Omega \to \{1,2,3,4,5,6\}$

b) $E(X_1 + X_2 | X_2 = x_2) = E(X_1 | X_2 = x_2) + E(X_2 | X_2 = x_2)$

$= E(X_1) + \dfrac{P(X_2 \cap X_2)}{P(X_2)} \times x_2$

$= E(X_1) + \dfrac{P(X_2)}{P(X_2)} \times x_2$

$= 3.5 + x_2$ //

c) $E(X_1 \cdot X_2 | X_2 = x_2) = E(X_1 | X_2 = x_2) \cdot E(X_2 | X_2 = x_2)$

$= E(X_1) \cdot x_2$

$= 3.5 x_2$

d) $Var(X_1^2 X_2 | X_2 = x_2) = E((X_1^2 X_2)^2 | X_2 = x_2) - E(X_1^2 X_2 | X_2 = x_2)^2$

$= E(X_1^4 X_2^2 | X_2 = x_2) - E(X_1^2 X_2 | X_2 = x_2)^2$

$= E(X_1^4 | X_2 = x_2) \cdot E(X_2^2 | X_2 = x_2) - (E(X_1^2 | X_2 = x_2) \cdot E(X_2 | X_2 = x_2))^2$

$= E(X_1^4) \cdot (x_2)^2 - (E(X_1^2) \cdot x_2)^2$

$= \left(1^4 \times \frac{1}{6} + 2^4 \times \frac{1}{6} + 3^4 \times \frac{1}{6} + 4^4 \times \frac{1}{6} + 5^4 \times \frac{1}{6} + 6^4 \times \frac{1}{6}\right) x_2^2$

$- \left(\left(1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 5^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6}\right) x_2\right)^2$

$= \frac{2275}{6} x_2^2 - \left(\frac{91}{6} x_2\right)^2$

$= 149.14\, x_2^2$

Excercise 3. (Multivariate continuous random variables)

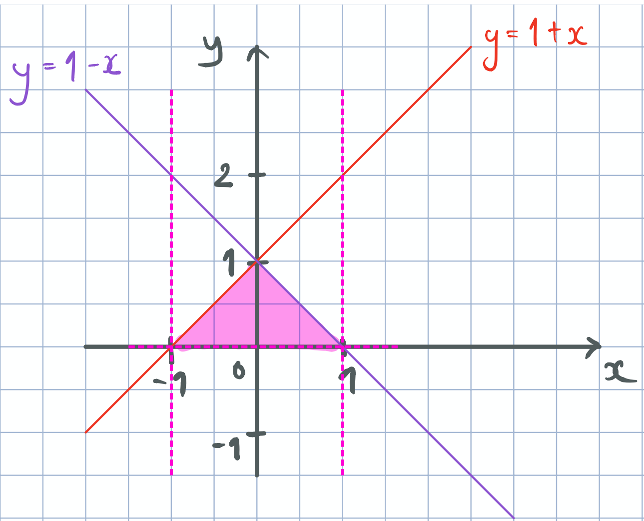
Let (X, Y) be random variables describing the two coordinates of points that are uniformly distributed in a triangle that is bound by $-1 \leq x \leq 1$, $y \geq 0$ and the two lines $y = 1 + x$ and $y = 1 - x$.

a) Find $P(X \geq -0.5)$.

b) Find $P(Y \geq 0.5)$.

c) Find the marginal densities and expexations of X and Y.

(4 Points)

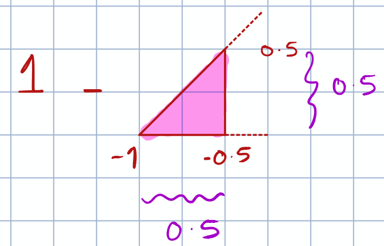


a) $P(X \geq -0.5) = 1 - P(X < 0.5)$

$= 1 - \frac{1}{2} \times 0.5 \times 0.5$

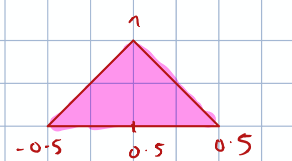
$= 1 - 0.125$

$= 0.875$



b) $P(Y \geq 0.5) = \frac{1}{2} \times 1 \times 0.5$

$= 0.250$



c) Marginal Densities :

Calculating joint density function to get the constant 'c'

$$\int_0^1 \int_{y-1}^{1-y} c \, dx \, dy = 1$$

Area of shaded region

$= \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times 2 \times 1$

$= 1$

or, $c \int_0^1 [x]_{y-1}^{1-y} \, dy = 1$

or, $c \int_0^1 1 - y - y + 1 \, dy = 1$

or, $c \int_0^1 (2 - 2y) \, dy = 1$

or, $c \int_0^1 2 \, dy - c \int_0^1 2y \, dy = 1$

$$\text{or, } c \cdot 2\,[y]_0^1 - \cancel{2}\left[\frac{y^2}{\cancel{2}}\right]_0^1 \times c = 1$$

$$\text{or, } c \cdot 2 - 1c = 1$$

$$\text{or, } 1c = 1$$

$$\therefore\ c = 1$$

For marginal densities,

$$f_X(x) = \begin{cases} \int_0^{1+x} 1\, dy = (1)[y]_0^{1+x} = (1+x), & \text{for } -1 \le x \le 0 \\ \int_0^{1-x} 1\, dy = (1)[y]_0^{1-x} = (1-x), & \text{for } 0 \le x \le 1 \end{cases}$$

$$f_Y(x) = \int_{y-1}^{1-y} 1\, dx = [x]_{y-1}^{1-y} = 1-y-y+1 = 2-2y$$

Expectations

$$E(X) = \int_x x\, f_X(x)\, dx$$

$$= \int_{-1}^{0} x\cdot(1+x)\, dx + \int_0^1 x(1-x)\, dx$$

$$= \int_{-1}^{0} (x+x^2)\, dx + \int_0^1 (x-x^2)\, dx$$

$$= \left[\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1$$

$$= 0 - \cancel{\frac{1}{2}} + 0 + \cancel{\frac{1}{3}} + \cancel{\frac{1}{2}} - 0 - \cancel{\frac{1}{3}} + 0$$

$$= 0$$

$$
\begin{aligned}
E[Y] &= \int_y y \cdot f_Y(y)\, dy \\
&= \int_0^1 y(2-2y)\, dy \\
&= 2\int_0^1 y(1-y)\, dy \\
&= 2\int_0^1 (y-y^2)\, dy \\
&= 2\left[\left[\frac{y^2}{2}\right]_0^1 - \left[\frac{y^3}{3}\right]_0^1\right] \\
&= 2\left[\frac{1}{2} - 0 - \frac{1}{3} + 0\right] \\
&= 2\left[\frac{3-2}{6}\right] \\
&= \frac{1}{3} \, //
\end{aligned}
$$