QUESTION 7

Let's go through the process step by step, starting from the beginning. I'll break it down into clear, manageable pieces to help you understand each part of the calculation.

1. Transition Probability Matrix (P)

The transition probability matrix represents the probabilities of transitioning from one vertex to another, given the edge weights between the vertices.

For each vertex x_i, the probability of moving to another vertex x_j is calculated as:

$$P(x_i \to x_j) = \frac{\text{weight of edge from } x_i \text{ to } x_j}{\text{total weight of all edges from } x_i}$$

For vertex x_1:

The edges from x_1 are:

- $x_1 \to x_2$ (weight 1)
- $x_1 \to x_4$ (weight 1)
- $x_1 \to x_3$ (weight 2)

Total weight of edges from x_1 = $1 + 1 + 2 = 4$.

So, the probabilities are:

- $P(x_1 \to x_2) = \frac{1}{4}$
- $P(x_1 \to x_4) = \frac{1}{4}$
- $P(x_1 \to x_3) = \frac{2}{4} = \frac{1}{2}$

For vertex x_2:

The edges from x_2 are:

- $x_2 \to x_1$ (weight 1)
- $x_2 \to x_3$ (weight 1)

Total weight of edges from x_2 = $1 + 1 = 2$.

So, the probabilities are:

- $P(x_2 \to x_1) = \frac{1}{2}$
- $P(x_2 \to x_3) = \frac{1}{2}$

So, the probabilities are:

- $P(x_2 \to x_1) = \frac{1}{2}$
- $P(x_2 \to x_3) = \frac{1}{2}$

For vertex x_3:

The edges from x_3 are:

- $x_3 \to x_1$ (weight 2)
- $x_3 \to x_2$ (weight 1)
- $x_3 \to x_4$ (weight 1)

Total weight of edges from x_3 = $2 + 1 + 1 = 4$.

So, the probabilities are:

- $P(x_3 \to x_1) = \frac{2}{4} = \frac{1}{2}$
- $P(x_3 \to x_2) = \frac{1}{4}$
- $P(x_3 \to x_4) = \frac{1}{4}$

For vertex x_4:

The edges from x_4 are:

- $x_4 \to x_1$ (weight 1)
- $x_4 \to x_3$ (weight 1)

Total weight of edges from x_4 = $1 + 1 = 2$.

So, the probabilities are:

- $P(x_4 \to x_1) = \frac{1}{2}$
- $P(x_4 \to x_3) = \frac{1}{2}$

The transition matrix P:

Now, let's put this all together in the transition matrix P:

$$P = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

2. **Stationary Distribution (π)**

The stationary distribution π gives the long-term probabilities of being at each vertex in a random walk. To find π, we solve the equation:

$$\pi P = \pi$$

This means that the distribution is unchanged after applying the transition matrix, i.e., the system is in equilibrium.

Symmetry of the graph:

- x_1 and x_3 are symmetric, meaning they have the same number of connections (more connections).
- x_2 and x_4 are symmetric, meaning they have fewer connections.

So, we assume:

- $\pi_1 = \pi_3$ (since x_1 and x_3 are symmetric)
- $\pi_2 = \pi_4$ (since x_2 and x_4 are symmetric)

This reduces the number of unknowns to two: $\pi_1 = \pi_3$ and $\pi_2 = \pi_4$.

Solving for π:

Let:

- $\pi_1 = \pi_3 = x$
- $\pi_2 = \pi_4 = y$

From the normalization condition $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$, we have:

$$x + y + x + y = 1 \quad \Rightarrow \quad 2x + 2y = 1 \quad \Rightarrow \quad x + y = \frac{1}{2}$$

From the balance equations, we also have:

$$x = \frac{1}{4}y \quad \text{(because of symmetry)}$$

Now, substitute $x = \frac{1}{4}y$ into $x + y = \frac{1}{2}$:

$$\frac{1}{4}y + y = \frac{1}{2} \quad \Rightarrow \quad \frac{5}{4}y = \frac{1}{2} \quad \Rightarrow \quad y = \frac{2}{5}$$

Thus:

$$x = \frac{1}{4}y = \frac{1}{4} \times \frac{2}{5} = \frac{2}{20} = \frac{1}{10}$$

So, the stationary distribution is:

$$\pi = \left[\frac{1}{10}, \frac{2}{10}, \frac{1}{10}, \frac{2}{10}\right]$$

3. Entropy Rate (H)

The entropy rate is a measure of uncertainty or randomness in the random walk. The formula for entropy rate H is:

$$H = -\sum_i \sum_j \pi_i P_{ij} \log_2 P_{ij}$$

Calculating for x_1 and x_3 (since $\pi_1 = \pi_3 = \frac{1}{10}$):

The transition probabilities for x_1 (and similarly for x_3) are:

- $P(x_1 \rightarrow x_2) = \frac{1}{4}$
- $P(x_1 \rightarrow x_3) = \frac{1}{2}$
- $P(x_1 \rightarrow x_4) = \frac{1}{4}$

So, the entropy for x_1 (and similarly for x_3) is:

$$H_{x_1} = -\left(\frac{1}{10}\left(\frac{1}{4}\log_2\frac{1}{4} + \frac{1}{2}\log_2\frac{1}{2} + \frac{1}{4}\log_2\frac{1}{4}\right)\right)$$
$$= -\left(\frac{1}{10}(-0.5 - 0.5 - 0.5)\right) = 0.5 \text{ bits per step}$$

Calculating for x_2 and x_4 (since $\pi_2 = \pi_4 = \frac{2}{10}$):

The transition probabilities for x_2 (and similarly for x_4) are:

- $P(x_2 \rightarrow x_1) = \frac{1}{2}$
- $P(x_2 \rightarrow x_3) = \frac{1}{2}$

So, the entropy for x_2 (and similarly for x_4) is:

$$H_{x_2} = -\left(\frac{2}{10}\left(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{2}\log_2\frac{1}{2}\right)\right) = 0.1667 \text{ bits per step}$$

Total entropy rate:

Now, multiply the entropies for each vertex type by the number of occurrences and add them up:

$$H_{\text{total}} = 2 \times 0.5 + 2 \times 0.1667 = 1.33 \text{ bits per step}$$

Understanding the Transition Probability Matrix

The **transition probability matrix (P)** is a square matrix that describes the probability of moving from one vertex to another in a graph or network. Each entry P_{ij} in the matrix represents the probability of transitioning from vertex i to vertex j.

Here's a generic structure:

$$P = \begin{bmatrix} P(x_1 \to x_1) & P(x_1 \to x_2) & P(x_1 \to x_3) & P(x_1 \to x_4) \\ P(x_2 \to x_1) & P(x_2 \to x_2) & P(x_2 \to x_3) & P(x_2 \to x_4) \\ P(x_3 \to x_1) & P(x_3 \to x_2) & P(x_3 \to x_3) & P(x_3 \to x_4) \\ P(x_4 \to x_1) & P(x_4 \to x_2) & P(x_4 \to x_3) & P(x_4 \to x_4) \end{bmatrix}$$

QUESTION-8

Marginal Probabilities

P(X):

We calculate $P(X = x)$ by summing the joint probabilities over all y:

$$P(X=1) = \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{32} = \frac{7}{32}$$

$$P(X=2) = \frac{1}{16} + \frac{1}{8} + \frac{1}{32} + \frac{1}{32} = \frac{5}{32}$$

$$P(X=3) = \frac{1}{32} + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} = \frac{5}{32}$$

$$P(X=4) = \frac{1}{4} = \frac{8}{32}$$

P(Y):

Similarly, $P(Y = y)$ is the sum of the joint probabilities over all x:

$$P(Y=1) = \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{32} = \frac{5}{32}$$

$$P(Y=2) = \frac{1}{16} + \frac{1}{8} + \frac{1}{32} + \frac{1}{32} = \frac{5}{32}$$

$$P(Y=3) = \frac{1}{32} + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} = \frac{5}{32}$$

$$P(Y=4) = \frac{1}{4} = \frac{8}{32}$$

Let's walk through the full calculation for the **Joint Entropy** $H(X,Y)$, step by step.

The formula for joint entropy is:

$$H(X,Y) = -\sum_{x}\sum_{y} P(x,y)\log_2 P(x,y)$$

Now, let's calculate the **joint entropy** by applying these joint probabilities in the entropy formula.

$$H(X,Y) = -\left(\frac{1}{8}\log_2\frac{1}{8} + \frac{1}{16}\log_2\frac{1}{16} + \frac{1}{32}\log_2\frac{1}{32} + \frac{1}{32}\log_2\frac{1}{32} + \frac{1}{16}\log_2\frac{1}{16} + \frac{1}{8}\log_2\frac{1}{8} + \frac{1}{32}\log_2\frac{1}{32} + \frac{1}{32}\log_2\frac{1}{32} + \frac{1}{32}\log_2\frac{1}{32} + \frac{1}{32}\log_2\frac{1}{32} + \frac{1}{16}\log_2\frac{1}{16} + \frac{1}{16}\log_2\frac{1}{16} + \frac{1}{4}\log_2\frac{1}{4}\right)$$

Now let's calculate each term separately:

1. $\frac{1}{8}\log_2\frac{1}{8} = \frac{1}{8}\times(-3) = -\frac{3}{8}$
2. $\frac{1}{16}\log_2\frac{1}{16} = \frac{1}{16}\times(-4) = -\frac{4}{16} = -\frac{1}{4}$
3. $\frac{1}{32}\log_2\frac{1}{32} = \frac{1}{32}\times(-5) = -\frac{5}{32}$
4. $\frac{1}{32}\log_2\frac{1}{32} = -\frac{5}{32}$
5. $\frac{1}{16}\log_2\frac{1}{16} = -\frac{1}{4}$
6. $\frac{1}{8}\log_2\frac{1}{8} = -\frac{3}{8}$
7. $\frac{1}{32}\log_2\frac{1}{32} = -\frac{5}{32}$
8. $\frac{1}{32}\log_2\frac{1}{32} = -\frac{5}{32}$
9. $\frac{1}{32}\log_2\frac{1}{32} = -\frac{5}{32}$
10. $\frac{1}{32}\log_2\frac{1}{32} = -\frac{5}{32}$
11. $\frac{1}{16}\log_2\frac{1}{16} = -\frac{1}{4}$
12. $\frac{1}{16}\log_2\frac{1}{16} = -\frac{1}{4}$
13. $\frac{1}{4}\log_2\frac{1}{4} = \frac{1}{4}\times(-2) = -\frac{2}{4} = -\frac{1}{2}$

Now, summing all these values together:

$$H(X,Y) = -\left(\frac{3}{8} + \frac{1}{4} + \frac{5}{32} + \frac{5}{32} + \frac{1}{4} + \frac{3}{8} + \frac{5}{32} + \frac{5}{32} + \frac{5}{32} + \frac{5}{32} + \frac{1}{4} + \frac{1}{4} + \frac{1}{2}\right)$$

Now let's simplify:

$$H(X,Y) = -\left(\frac{3}{8} + \frac{3}{8} + \frac{5}{32} + \frac{5}{32} + \frac{5}{32} + \frac{5}{32} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{2}\right)$$

First, convert everything to the same denominator (32):

$$H(X,Y) = -\left(\frac{6}{32} + \frac{5}{32} + \frac{5}{32} + \frac{5}{32} + \frac{5}{32} + \frac{8}{32} + \frac{8}{32} + \frac{8}{32} + \frac{8}{32} + \frac{16}{32}\right)$$

Now, sum the numerators:

$$H(X,Y) = -\left(\frac{6+5+5+5+5+8+8+8+8+16}{32}\right)$$

$$H(X,Y) = -\left(\frac{75}{32}\right)$$

$$H(X,Y) = 3.375\,\text{bits}$$

So the **Joint Entropy $H(X,Y)$ is 3.375 bits**.

This is the full calculation. Let me know if any step is unclear!

Marginal Entropies:

$H(X)$:

$$H(X) = -\left(\frac{7}{32}\log_2\frac{7}{32} + \frac{5}{32}\log_2\frac{5}{32} + \frac{5}{32}\log_2\frac{5}{32} + \frac{8}{32}\log_2\frac{8}{32}\right)$$

The value of $H(X)$ is **1.750 bits**.

$H(Y)$:

$$H(Y) = -\left(\frac{5}{32}\log_2\frac{5}{32} + \frac{5}{32}\log_2\frac{5}{32} + \frac{5}{32}\log_2\frac{5}{32} + \frac{8}{32}\log_2\frac{8}{32}\right)$$

The value of $H(Y)$ is **2.000 bits**.

Conditional Entropies:

$H(Y|X)$:

$$H(Y|X) = H(X,Y) - H(X) = 3.375 - 1.750 = 1.625\,\text{bits}$$

$H(X|Y)$:

$$H(X|Y) = H(X,Y) - H(Y) = 3.375 - 2.000 = 1.375\,\text{bits}$$

Mutual Information:

Finally, we calculate the mutual information:

$$I(X;Y) = H(X) + H(Y) - H(X,Y) = 1.750 + 2.000 - 3.375 = 0.375\,\text{bits}$$

Final Results:

- **Joint Entropy** $H(X, Y)$ = 3.375 bits
- **Marginal Entropy** $H(X)$ = 1.750 bits
- **Marginal Entropy** $H(Y)$ = 2.000 bits
- **Conditional Entropy** $H(Y|X)$ = 1.625 bits
- **Conditional Entropy** $H(X|Y)$ = 1.375 bits
- **Mutual Information** $I(X; Y)$ = 0.375 bits